Lecture 16 Solutions: Auctions and Other Mechanisms

Stanford University — Econ 51 — Prof. Christopher Makler

Unit V: Mechanism Design

Ex 16.1: Room allocation

Three Stanford undergraduates, Alex Vickrey, Billy Clarke, and Casey Groves have grown tired of the modest housing options available on Stanford campus. By asking very nicely, they convince their parents to buy a 3-bedroom apartment off-campus for them to live in.

The students now face the problem of how to allocate the 3 bedrooms amongst themselves. Let's call the rooms Room 1, Room 2, and Room 3. We will use the following notation to describe the allocations: {A,B,C} means Alex gets Room 1, Billy gets Room 2, and Casey gets Room 3. $v_A(\{A, B, C\})$ is the value Alex gets from the allocation {A,B,C} and so forth.

Each student has a different valuation of the potential bedroom allocations, and these valuations are **private information**. In particular, Room 1 is bigger than Room 2 and Room 3, so the students generally attach the highest valuation to Room 1. However, each student cares not only about which room he or she gets, but about the rooms the other students get. This is because Casey suffers from night terrors, occasionally waking up screaming in the middle of the night. Hence, Alex and Billy prefer rooms further away from Casey, all else equal.

In the matrix below, each column shows the valuations of every student (in hundreds of dollars) for a given room allocation. For example, column 1 says that if the allocation is {A,B,C}, Alex gets a value of \$800, Billy gets a value of \$300, and Casey gets a value of \$500.

	{A,B,C}	{A,C,B}	{B,A,C}	{B,C,A}	{C,A,B}	{C,B,A}
v_A	8	7	5	5	5	6
v_B	3	3	9	5	7	3
v_C	5	5	5	5	8	8

(a) Which room allocation maximizes total welfare?

Total welfare is simply the sum of each student's value, which is added as the last row in the updated matrix below. The allocation which maximizes welfare is {C,A,B}.

	{A,B,C}	{A,C,B}	{B,A,C}	{B,C,A}	{C,A,B}	{C,B,A}
v_A	8	7	5	5	5	6
v_B	3	3	9	5	7	3
v_C	5	5	5	5	8	8
$v_A + v_B + v_C$	16	15	19	15	20	17

(b) Suppose Alex were to say he didn't care who got which room: that is, he declared his valuation to be $\phi = [0, 0, 0, 0, 0, 0]$. Would that change which room maximized total welfare? In other words, is Alex **pivotal**?

If Alex reported ϕ, the allocation which maximizes welfare is still {C,A,B}. Therefore Alex is not pivotal.

	{A,B,C}	{A,C,B}	{B,A,C}	{B,C,A}	{C,A,B}	{C,B,A}
v_A	0	0	0	0	0	0
v_B	3	3	9	5	7	3
v_C	5	5	5	5	8	8
$v_A + v_B + v_C$	8	8	14	10	**15**	11

(c) Repeat part (b) for Billy and Casey. Are they pivotal?

If Billy reported a vector of zeros, the allocation which maximizes welfare would become {C,B,A}. Therefore Billy is pivotal.:

	{A,B,C}	{A,C,B}	{B,A,C}	{B,C,A}	{C,A,B}	{C,B,A}
v_A	8	7	5	5	5	6
v_B	0	0	0	0	0	0
v_C	5	5	5	5	8	8
$v_A + v_B + v_C$	13	12	10	10	13	**14**

Likewise, if Casey reported a vector of zeros, the allocation which maximizes welfare would become {B,A,C}. Therefore Casey is also pivotal:

	{A,B,C}	{A,C,B}	{B,A,C}	{B,C,A}	{C,A,B}	{C,B,A}
v_A	8	7	5	5	5	6
v_B	3	3	9	5	7	3
v_C	0	0	0	0	0	0
$v_A + v_B + v_C$	11	10	**14**	10	12	9

(d) The Vickrey-Clarke-Groves pivot mechanism in this setting states that given the three students' vectors of reported valuations, the outcome that maximizes reported value will be chosen, and each player n should make a payment equal to the sum of all the other players' values from the chosen outcome, minus the sum of all the other players' values from the outcome that would be chosen if player n professed no preference: mathematically,

$$p_n(k^*(v), v_{-n}) = \sum_{j \neq n} v_j(k^*(v)) - \sum_{j \neq n} v_j(k^*(\phi, v_{-n}))$$

In practice, this means that for each of the players that's pivotal, you should find the difference between the value to the other players from the outcome you found in part (a), and the value those other players would get from the outcomes you found in parts (b) and/or (c). (For non-pivotal players, the two terms are identical since the outcome doesn't change as a result of their report; so their payment is zero.)

What transfers would this mechanism entail?

As we found in parts (b) and (c), Billy and Casey are pivotal, but Alex is not. Therefore Alex has no payment.

When Billy is reporting accurately, the VCG mechanism chooses {C,A,B}, which gives Alex and Casey a total utility of $5 + 8 = 13$. Without Billy's report the VCG mechanism would select {C,B,A} which yields the highest payoff to Alex and Casey: namely, $6 + 8 = 14$. By reporting his true value, Billy therefore inflicts \$100 total external damage on Alex and Casey, and so must pay that amount. In equations:

$$\begin{aligned} p_B &= [v_A(\{C, A, B\}) + v_C(\{C, A, B\})] - [v_A(\{C, B, A\}) + v_C(\{C, B, A\})] \\ &= [5 + 8] - [6 + 8] \\ &= -1 \end{aligned}$$

So Billy's transfer is -1, or a payment of \$100.

Likewise, when Casey is reporting accurately, the VCG mechanism chooses {C,A,B}, which gives Alex and Billy a total utility of $5 + 7 = 12$. Without Casey's report the VCG mechanism would select {B,A,C} which yields the highest payoff to Alex and Billy: namely, $5+9 = 14$. By reporting her true value, Casey therefore inflicts \$200 total external damage on Alex and Billy, and so must pay that amount. In equations:

$$\begin{aligned} p_C &= [v_A(\{C, A, B\}) + v_B(\{C, A, B\})] - [v_A(\{B, A, C\}) + v_B(\{B, A, C\})] \\ &= [5 + 7] - [5 + 9] \\ &= -2 \end{aligned}$$

So Casey's transfer is -2, or a payment of \$200.

(e) Verify that Casey has no incentive to misreport her valuation of any room allocation.

Casey's net utility from the VCG mechanism is the value she gets from the room (8) minus her transfer payment (2), for a net value of 6:

$$\begin{aligned} U_C &= v_C(k^*(v)) + p_C(k^*(v), v_{-n}) \\ &= 8 + (-2) \\ &= 6 \end{aligned}$$

To see whether she has an incentive to misreport her valuation of any room allocation, let's observe that she could select any outcome simply by saying she has a valuation of \$1 million for that outcome. (Note that she would never have to pay the million dollars, because her payment is only based on Alex and Billy's reported valuations; but she could use that kind of astronomical valuation to have the mechanism select any outcome she wanted.)

Remember that without Casey's report the VCG mechanism would select {B,A,C} and give Alex and Billy a total value of $5 + 9 = 14$. Therefore, no matter which outcome she said was worth \$1 million, her payment would be the sum of Alex and Billy's reports for that outcome, minus the 14 they would get from {B,A,C}; that is,

$$p_c = v_A + v_B - 14$$

For each of the six cases, therefore, let's see what would happen if Casey lied and said her valuation was \$1 million when her true value was v_C, thereby getting $U_C = v_C + p_C$ of actual payoff:

	{A,B,C}	{A,C,B}	{B,A,C}	{B,C,A}	{C,A,B}	{C,B,A}
v_A	8	7	5	5	5	6
v_B	3	3	9	5	7	3
$v_A + v_B$	11	10	14	10	12	9
$p_C = v_A + v_B - 14$	-3	-4	0	-4	-2	-5
v_C	5	5	5	5	8	8
$U_C = v_C + p_C$	2	1	5	1	6	3

Since v_C is Casey's actual value, she maximizes her utility by reporting in such a way that selects {C,A,B}. Since this is what is selected when she reports her true value, she has no incentive to misreport her valuation of any room allocation.