Why is Chebyshev interpolation optimal?

Consider a function $f(z)$ analytic in some neighborhood of $z=[-1+0i, 1+0i]$. There is an elegant theory of the error in a polynomial interpolation to $f(z)$ using nodes x_j, $j=0,..,N$: Let $p_N(z)$ be the interpolating polynomial.

$p_N(z)$ can be written in terms of cardinal functions:

$$p_N(z) = \sum_{k=0}^{N} f(x_k)\, F_k(z)$$

where $F_k(z)$ is an N'th order poly. such that $F_k(x_j) = \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & j\neq k \end{cases}$

By construction,

$$F_k(z) = \prod_{\substack{j=0 \\ j\neq k}}^{N} (z-x_j) \Big/ \prod_{\substack{j=0 \\ j\neq k}}^{N} (x_k - x_j)$$

Define

$$\omega(z) = \prod_{j=0}^{N} (z-x_j)$$

as the $N+1$'st order polynomial with zeros at all the interpolation notes.

Note that

$$\omega'(z) = \sum_{k=0}^{N} \prod_{\substack{j=0 \\ j\neq k}}^{N} (z-x_j)$$

so

$$\omega'(x_k) = \prod_{\substack{j=0 \\ j\neq k}}^{N} (x_k - x_j)$$

Thus we can write

$$F_k(z) = \frac{\omega(z)}{(z-x_k)\,\omega'(x_k)}$$

Now consider the remainder (or interpolation error)

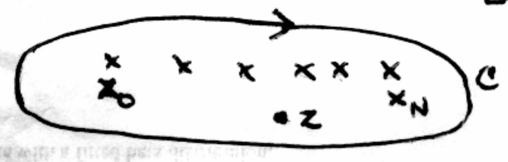
$$R_N(z) = f(z) - p_N(z) = f(z) - \sum_{k=0}^{N} f(x_k)\, F_k(z)$$

$$= f(z) - \left[\sum_{k=0}^{N} \frac{f(x_k)}{(z-x_k)\,\omega'(x_k)}\right] \omega(z)$$

This can be elegantly and usefully written using complex analysis.

We consider the function

$$g(t) = \frac{f(t)}{(t-z)\,\omega(t)}$$



of a complex variable t, where z is treated as a parameter. Let C be a counterclockwise contour enclosing z, all the $\{x_k\}$ and s.t. $f(t)$ is an analytic fn. of t in and on C. Then, by Residue Thm,

$$\int_C g(t)\,dt = 2\pi i \sum \mathrm{Res}_g$$

The singularities of $g(t)$ inside C are at $t=z$ and $t=x_k$, $k=0,\ldots,N$:

$$\mathrm{Res}_g(z) = \frac{f(z)}{\omega(z)}$$

$$\mathrm{Res}_g(x_k) = \frac{f(x_k)}{(x_k-z)\,\omega'(x_k)} = -\frac{f(x_k)}{(z-x_k)\,\omega'(x_k)}$$

Hence

$$\frac{1}{2\pi i}\int_C g(t)\,dt = \frac{f(z)}{\omega(z)} - \sum_{k=0}^{N} \frac{f(x_k)}{(z-x_k)\,\omega'(x_k)} = \frac{R_N(z)}{\omega(z)}$$

We conclude

$$R_N(z) = \frac{\omega(z)}{2\pi i}\int_C \frac{f(t)}{\omega(t)(t-z)}\,dt \quad \text{(Hermite's formula)}.$$

Now we can use this formula to estimate $R_N(z)$. The only part of $R_N(z)$ depending on the interpolation nodes is $\omega(z)$. We can estimate

$$|\omega(z)| = \prod_{k=0}^{N} |z-x_k| = \exp\left[-N\phi_N(z)\right]$$

where

$$\phi_N(z) = -\frac{1}{N}\sum_{k=0}^{N} \ln|z-x_k| .$$

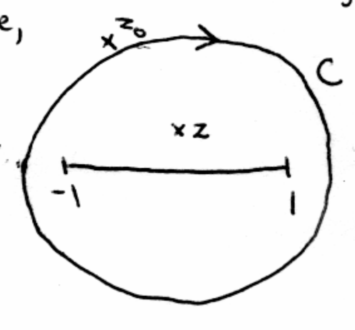
Now consider the limit of $N\to\infty$, when the nodes are closely spaced with density $\left|\frac{dk}{dx_k}\right| = N\mu(x)$. For instance, for evenly spaced nodes on $[-1,1]$, $x_k = -1+\frac{2k}{N} \Rightarrow \frac{dx_k}{dk} = \frac{2}{N}$, $\mu(x) = \frac{1}{2}$.

For Chebyshev nodes, $x_k = \cos\frac{k\pi}{N}$, $\left|\frac{dx_k}{dk}\right| = \frac{\pi}{N}\sin\frac{k\pi}{N} = \frac{\pi}{N}(1-x_k^2)^{\frac{1}{2}} \Rightarrow \mu(x) = \frac{1}{\pi(1-x^2)^{\frac{1}{2}}}$.

Now return to the remainder

$$R_N(z) = \frac{\omega(z)}{2\pi i}\int_C \frac{f(t)}{\omega(t)(t-z)}\,dt \qquad (z \text{ inside } C).$$

We choose C to be an equipotential of $\phi(t)$ that encloses $[-1,1]$ but lies just inside the nearest singularity z_0 of $f(t)$. Such a curve may not exist; in this case $R_N(z)$ will diverge. Let us suppose that $\phi(\ ) = \phi(z_0) + \varepsilon$ on C. ($\phi(t)$ monotonically decreases away from the distributed charge, so $\varepsilon > 0$). Then, on C Lt



$$|\omega(t)| \approx |e^{-N\phi(t)}| = e^{-N\phi(z_0)}e^{-N\varepsilon}$$

$$\frac{1}{2\pi}\int_C \left|\frac{f(t)}{t-z}\right||dt| \leq c(\varepsilon) \text{ independent of } N$$

$$|\omega(z)| \approx e^{-N\phi(z)}$$

so

$$|R_N(z)| \leq \frac{|\omega(z)|}{2\pi}\int_C \left|\frac{f(t)}{t-z}\right||dt|\,|\omega(t)|$$

$$\leq c(\varepsilon)\cdot e^{N\varepsilon}\cdot e^{N(\phi(z_0)-\phi(z))}$$

$$\ln|R_N(z)| \leq N\left[\phi(z_0)-\phi(z)\right] + \ln c(\varepsilon)e^{N\varepsilon}$$

Now, taking $\varepsilon \to 0$ as $N \to \infty$, we can connive that $\ln[c(\varepsilon)e^{N\varepsilon}] = o(N)$ so

$$\ln|R_N(z)| \leq N[\phi(z_0)-\phi(z)] + o(N)$$

i.e. the error decreases exponentially with N, with exponent proportional to $\phi(z)-\phi(z_0)$. For instance, for $f(z) = \frac{1}{1+16z^2}$ and Chebyshev interpolation $z_0 = \pm\frac{i}{4}$ and $\phi(z_0) = -\ln\frac{|z_0-\sqrt{z_0^2-1}|}{2} = -\ln\frac{|\frac{i}{4}(1+\sqrt{17})|}{2} = -\ln\frac{\sqrt{17}+1}{8} = 0.44$
which on the real axis, $\phi(x+0i) = \ln 2 = .69$ so

$$\ln|R_N(x+0i)| \leq N[.44-.69] = -0.25N + o(N)$$, i.e. $R_N \sim e^{-0.25N}$ for all x in $[-1,1]$.

On the other hand, consider equispaced interpolation. Now

$$\phi(z_0) = 1 - \frac{1}{2}\,\mathrm{Re}\left\{\left(\frac{i}{4}+1\right)\ln\left(\frac{i}{4}+1\right) - \left(\frac{i}{4}-1\right)\ln\left(\frac{i}{4}-1\right)\right\}$$

$$= 1 - \frac{1}{2}\ln\frac{17}{16} - \frac{1}{4}\tan^{-1}4$$

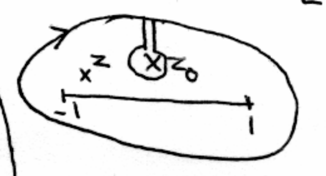
$$= 0.64$$

Now since $\phi(\pm 1 + 0i) = 1 - \ln 2 = 0.31 < \phi(z_0)$, there is no contour along an equipotential that encloses $[-1, 1]$ but not the singularity z_0.

In this case, we can indent C around the singularity and use a larger equipotential ϕ_1 that does enclose $[-1, 1]$

Now

$$\oint \frac{1}{2\pi i}\int_C \frac{f(t)}{\omega(t)(t-z)}\,dt$$



$$= \frac{1}{2\pi i}\left[\int_{C_{\text{equipot'l}}} \frac{f(t)}{\omega(t)(t-z)}\,dt - \underbrace{2\pi i\,\mathrm{Res}_{\frac{f}{\omega(t-z)}}(z_0)}_{\text{contribution from indentation.}}\right]$$

$$\left|\frac{1}{2\pi i}\int_C \frac{f(t)}{\omega(t)(t-z)}\,dt\right| \le \underbrace{\left|\frac{1}{2\pi i}\int_{C_{\text{equipot'l}}} \frac{f(t)}{\omega(t)(t-z)}\,dt\right|}_{c_1 e^{N\phi_1}} + \underbrace{\left|\frac{\mathrm{Res}_f(z_0)}{\omega(z_0)(z_0-z)}\right|}_{c_0 e^{N\phi}}$$

Since $\phi_1 < \phi_0$, the second term dominates. Hence, again

$$|R_N(z)| \le \underbrace{|\omega(z)|}_{e^{-N\phi(z)}} \cdot c_0 e^{N\phi(z_0)} = O\left(e^{N(\phi(z_0)-\phi(z))}\right)$$

Thus equispaced interpolation converges along the real axis for those x such that

$$\phi(x) = 1 - \frac{1}{2}\left\{(1+x)\ln(1+x) + (1-x)\ln(1-x)\right\} > \phi(z_0) = 0.64$$

This is true for $|x| < 0.79$. For $|x| > 0.79$, equispaced interpolation of $\frac{1}{1+16x^2}$ diverges ... exactly the Runge phenomenon we saw earlier.

In this case, for z not too close to any x_k, the sum in $\phi_N(z)$ can be regarded as a Riemann sum for an integral:

$$\phi(z) = \lim_{N\to\infty} \phi_N(z) = \lim_{N\to\infty} -\frac{1}{N}\sum_{k=0}^{N} \ln|z-x_k| \underbrace{\Delta k}_{=1}$$

$$\approx \lim_{N\to\infty} -\frac{1}{\cancel{N}}\sum_{k=0}^{N} \ln|z-x_k| \cancel{N}\mu(x_k)\underbrace{\Delta x_k}_{(x_{k+1}-x_k)}$$

$$= -\int_{-1}^{1} \ln|z-x|\,\mu(x)\,dx.$$

Physically, this is the electrical potential associated with a charge density $\mu(x)$ along the line $[-1,1]$. This function can be calculated:

Uniform node spacing: $\phi(z) = -\int_{-1}^{1} \ln|z-x|\cdot\frac{1}{2}dx$

$$= 1 - \frac{1}{2}\mathrm{Re}\left\{(z+1)\ln(z+1) - (z-1)\ln(z-1)\right\}.$$

Chebyshev spacing: $\phi(z) = -\int_{-1}^{1} \frac{\ln|z-x|}{\pi(1-x^2)^{\frac{1}{2}}}dx$ (set $x=\cos\theta$)

$$= -\ln\frac{|z-\sqrt{z^2-1}|}{2}.$$

Note that

$$\omega(z) = \prod_{k=0}^{N}(z-z_k) \sim e^{-N\phi(z)}$$

is smaller near maxima of the potential $\phi(z)$. For Chebyshev spacing the real line segment $[-1,1]$ is an <u>equipotential</u> $\phi(z) = \ln 2$ so $\omega(x)$ oscillates by similar amounts for all x on $[-1,1]$, while for equispaced interpolation, $\phi(0) = 1$ and $\phi(\pm 1) = 1 - \ln 2$, so $\omega(x)$ has much larger magnitude near $x = \pm 1$ than near $x=0$.

It is that Chebyshev spacing makes the line segment $[-1,1]$ an equipotential that renders it optimal for polynomial interpolation and guarantees that it converges exponentially with N for any $f(z)$ analytic on $[-1+0i, 1+0i]$, as we see next. Physically, if $N+1$ equal charges were constrained to $[-1,1]$ they would end up at the Chebyshev points, since then $-\nabla\phi = 0$ for all of them.