Assignment due date: Friday, March 3, 11:59pm

**Hand-in and code to be submitted to the
CDF server by the above due date**

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I hereby affirm that all the solutions I provide, both in writing and in code, for this assignment are my own. I have properly cited and noted any reference material I used to arrive at my solution, and have not shared my work with anyone else.

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Signature

(note: -3 marks penalty for not completing properly the above section)

1. Surfaces of Revolution

a) In general, if we have a parametric curve in the xz plane defined by $x(\lambda)$ and $z(\lambda)$, then the surface of revolution can be expressed as:

$$\vec{f}(u,v) = [x(u)\cos(v),\ x(u)\sin(v),\ z(u)]$$

(From Lecture 3)

In our case, this gives:

$$\vec{p}(u,v) = \left[(a + u^2\cos u)\cos(v),\ (a+u^2\cos u)\sin(v),\ \frac{u}{b}\right]$$

for $0 \leq u \leq 2\pi$
and $0 \leq v \leq 2\pi$

I think $a = 50$ and $b = \frac{1}{70}$ gives a nice shape.

b) Let's first find the tangents of the coordinate curves:

$$\frac{\partial \vec{p}}{\partial u} = \left[\frac{\partial}{\partial u}(a+u^2\cos u)\cos(v),\ \frac{\partial}{\partial u}(a+u^2\cos u)\sin(v),\ \frac{\partial}{\partial u}\left(\frac{u}{b}\right)\right]$$

$$= \left[\cos(v)(2u\cos u - u^2\sin u),\ \sin(v)(2u\cos u - u^2\sin u),\ \frac{1}{b}\right]$$

$$\frac{\partial \vec{p}}{\partial v} = \left[\frac{\partial}{\partial v}(a+u^2\cos u)\cos(v),\ \frac{\partial}{\partial v}(a+u^2\cos u)\sin(v),\ \frac{\partial}{\partial v}\left(\frac{u}{b}\right)\right]$$

$$= \left[-\sin(v)(a+u^2\cos u),\ \cos(v)(a+u^2\cos u),\ 0\right]$$

A plane can be defined by a point on the plane, and two vectors that span the plane. We can use the tangents of the coordinate curves as the two vectors.

Thus, we can write a parametric equation for the tangent plane at point $\vec{p}(u_0, v_0)$ as:

$$\vec{f}(\alpha, \beta) = \vec{p}(u_0, v_0) + \alpha \frac{\partial \vec{p}}{\partial u}(u_0, v_0) + \beta \frac{\partial \vec{p}}{\partial v}(u_0, v_0)$$

(where $\vec{p}$ is as defined in part (a) and $\frac{\partial \vec{p}}{\partial u}$ and $\frac{\partial \vec{p}}{\partial v}$ are as defined on the previous page).

c) Since the normal to the surface is perpendicular to all curves through the surface, it must be perpendicular to both of the tangents of the coordinate curves. So we can find it by taking the cross product of $\frac{\partial \vec{p}}{\partial u}$ and $\frac{\partial \vec{p}}{\partial v}$. To make it a unit normal, we can divide the result by its own magnitude. Finally, to ensure the normal is outward-facing, we take the right-hand-rule into account and make sure to cross the vectors in the right order:

$$\vec{n} = \frac{\frac{\partial \vec{p}}{\partial v} \times \frac{\partial \vec{p}}{\partial u}}{\left\| \frac{\partial \vec{p}}{\partial v} \times \frac{\partial \vec{p}}{\partial u} \right\|}$$

2. Camera Transformations

a)

$$\vec{d} = \frac{(\vec{c} - \vec{e}_m) \times \vec{t}}{\|(\vec{c} - \vec{e}_m) \times \vec{t}\|}$$

b) $\vec{e}_R = \vec{e}_m + \frac{s}{2}\vec{d}$

$\vec{e}_L = \vec{e}_m - \frac{s}{2}\vec{d}$

c) For the left camera:

- the camera is looking towards $\vec{c}$, so the gaze direction is $\vec{g}_L = \vec{c} - \vec{e}_L$
- the view up vector is $\vec{t}$
- Thus, our basis vectors are (from Lecture 4):

$$\vec{u}_L = \frac{\vec{t} \times \vec{g}_L}{\|\vec{t} \times \vec{g}_L\|} = \frac{\vec{t} \times (\vec{c} - \vec{e}_L)}{\|\vec{t} \times (\vec{c} - \vec{e}_L)\|}$$

$$\vec{v}_L = \frac{\vec{g}_L \times \vec{u}_L}{\|\vec{g}_L \times \vec{u}_L\|} = \frac{(\vec{c} - \vec{e}_L) \times \vec{u}_L}{\|(\vec{c} - \vec{e}_L) \times \vec{u}_L\|}$$

$$\vec{w}_L = \frac{-\vec{g}_L}{\|\vec{g}_L\|} = \frac{\vec{e}_L - \vec{c}}{\|\vec{c} - \vec{e}_L\|}$$

Similarly, for the right camera:

$$\vec{u}_R = \frac{\vec{t} \times \vec{g}_R}{\|\vec{t} \times \vec{g}_R\|} = \frac{\vec{t} \times (\vec{c} - \vec{e}_R)}{\|\vec{t} \times (\vec{c} - \vec{e}_R)\|}$$

$$\vec{v}_R = \frac{\vec{g}_R \times \vec{u}_R}{\|\vec{g}_R \times \vec{u}_R\|} = \frac{(\vec{c} - \vec{e}_R) \times \vec{u}_R}{\|(\vec{c} - \vec{e}_R) \times \vec{u}_R\|}$$

$$\vec{w}_R = \frac{-\vec{g}_R}{\|\vec{g}_R\|} = \frac{\vec{e}_R - \vec{c}}{\|\vec{c} - \vec{e}_R\|}$$

d) We can break down the transformation into a rotation and a translation, i.e. $M_{LR} = RT$

Let's find the translation first. We just need to move the origin from e_L to e_R, so we write:

$$T = \begin{bmatrix} 1 & 0 & 0 & (x_{e_L} - x_{e_R}) \\ 0 & 1 & 0 & (y_{e_L} - y_{e_R}) \\ 0 & 0 & 1 & (z_{e_L} - z_{e_R}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notation note: x_{e_L} means the x component of $\vec{e}_L$. Similarly for other entries.

Now the rotation. We can perform this rotation in two parts: $R = R_2 R_1$. First we rotate from e_L's coordinate system to the world coordinate system (R_1). Then we rotate from the world coordinate system to e_R's coordinate system. We write:

$$R_1 = \begin{bmatrix} x_{uL} & x_{vL} & x_{wL} & 0 \\ y_{uL} & y_{vL} & y_{wL} & 0 \\ z_{uL} & z_{vL} & z_{wL} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notation note: x_{uL} means the x component of $\vec{u}_L$. Similarly for the other entries.

(From lecture 4, slide 57)

and $R_2 = \begin{bmatrix} x_{uR} & y_{uR} & z_{uR} & 0 \\ x_{vR} & y_{vR} & z_{vR} & 0 \\ x_{wR} & y_{wR} & z_{wR} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Thus, the full matrix is:

$$M_{LR} = R_2 R_1 T$$

$$= \begin{bmatrix} x_{uR} & y_{uR} & z_{uR} & 0 \\ x_{vR} & y_{vR} & z_{vR} & 0 \\ x_{wR} & y_{wR} & z_{wR} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{uL} & x_{vL} & x_{wL} & 0 \\ y_{uL} & y_{vL} & y_{wL} & 0 \\ z_{uL} & z_{vL} & z_{wL} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & (x_{eL}-x_{eR}) \\ 0 & 1 & 0 & (y_{eL}-y_{eR}) \\ 0 & 0 & 1 & (z_{eL}-z_{eR}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we expand this matrix out fully, we get:

$$M_{LR} = \begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where
$a = u_R \cdot u_L$
$b = u_R \cdot v_L$
$c = u_R \cdot w_L$
$d = v_R \cdot u_L$
$e = v_R \cdot v_L$
$f = v_R \cdot w_L$
$g = w_R \cdot u_L$
$h = w_R \cdot v_L$
$i = w_R \cdot w_L$
$t_x = (x_{eL}-x_{eR})(u_R \cdot u_L) + (y_{eL}-y_{eR})(u_R \cdot v_L) + (z_{eL}-z_{eR})(u_R \cdot w_L)$
$t_y = (x_{eL}-x_{eR})(v_R \cdot u_L) + (y_{eL}-y_{eR})(v_R \cdot v_L) + (z_{eL}-z_{eR})(v_R \cdot w_L)$
$t_z = (x_{eL}-x_{eR})(w_R \cdot u_L) + (y_{eL}-y_{eR})(w_R \cdot v_L) + (z_{eL}-z_{eR})(w_R \cdot w_L)$

e) If $\vec{n}$ is the outward-facing normal and $\vec{p}$ is a point on the polygon face, then the culling criteria for each camera are:

left camera: cull if $(\vec{p} - \vec{e_L}) \cdot \vec{n} > 0$
right camera: cull if $(\vec{p} - \vec{e_R}) \cdot \vec{n} > 0$

Thus, we can cull from both cameras if

$$(\vec{p} - \vec{e_L}) \cdot \vec{n} > 0 \quad \text{AND} \quad (\vec{p} - \vec{e_R}) \cdot \vec{n} > 0$$

3 - Camera Coordinates and Coordinate Conversion

a) First, let's find the orthonormal basis vectors for the camera coordinate system.

Our gaze vector is: $\vec{g} = \vec{p_{wc}} - \vec{e_{wc}} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -5 \end{bmatrix}$

Let's find the unnormalized vectors first:

$$\vec{u_0} = \vec{t} \times \vec{g} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ -4 \\ -5 \end{bmatrix} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -2 & -4 & -5 \end{vmatrix} = \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix}$$

$$\vec{v_0} = \vec{g} \times \vec{u_0} = \begin{bmatrix} -2 \\ -4 \\ -5 \end{bmatrix} \times \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -4 & -5 \\ -5 & 5 & -2 \end{vmatrix} = \begin{bmatrix} 8+25 \\ -(4-25) \\ -10-20 \end{bmatrix} = \begin{bmatrix} 33 \\ 21 \\ -30 \end{bmatrix}$$

$$\vec{w_0} = -\vec{g} = -\begin{bmatrix} -2 \\ -4 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

Let's next normalize the basis vectors:

$$\vec{u} = \frac{\vec{u}_0}{\|\vec{u}_0\|} = \frac{[-5,\ 5,\ -2]^T}{\sqrt{(-5)^2+5^2+(-2)^2}} = \begin{bmatrix} -0.6804 \\ 0.6804 \\ -0.2722 \end{bmatrix}$$

$$\vec{v} = \frac{\vec{v}_0}{\|\vec{v}_0\|} = \frac{[33,\ 21,\ -30]^T}{\sqrt{33^2+21^2+(-30)^2}} = \begin{bmatrix} 0.6694 \\ 0.4260 \\ -0.6086 \end{bmatrix}$$

$$\vec{w} = \frac{\vec{w}_0}{\|\vec{w}_0\|} = \frac{[2,\ 4,\ 5]^T}{\sqrt{2^2+4^2+5^2}} = \begin{bmatrix} 0.2981 \\ 0.5963 \\ 0.7454 \end{bmatrix}$$

Once we have these basis vectors, the world-to-camera coordinate-aligning matrix is:

$$R = \begin{bmatrix} x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ x_w & y_w & z_w & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.6804 & 0.6804 & -0.2722 & 0 \\ 0.6694 & 0.4260 & -0.6086 & 0 \\ 0.2981 & 0.5963 & 0.7454 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The world-to-camera translation matrix is:

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_e \\ 0 & 1 & 0 & -y_e \\ 0 & 0 & 1 & -z_e \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the total matrix is:

$$M = RT = \begin{bmatrix} -0.6804 & 0.6804 & -0.2722 & 0.6806 \\ 0.6694 & 0.4260 & -0.6086 & 1.5216 \\ 0.2981 & 0.5963 & 0.7454 & -5.2177 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) To go from camera back to world, we translate in the opposite direction and apply the inverse rotation matrix:

$$M^{-1} = (RT)^{-1} = T^{-1} R^{-1}$$

Since T is a rotation matrix, its inverse is simply a translation with the signs flipped:

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since R is orthonormal, its inverse is simply its transpose:

$$R^{-1} = R^{T} = \begin{bmatrix} -0.6804 & 0.6694 & 0.2981 & 0 \\ 0.6804 & 0.4260 & 0.5963 & 0 \\ -0.2722 & -0.6086 & 0.7454 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the complete matrix is:

$$M^{-1} = T^{-1} R^{-1} = \begin{bmatrix} -0.6804 & 0.6694 & 0.2981 & 1 \\ 0.6804 & 0.4260 & 0.5963 & 2 \\ -0.2722 & -0.6086 & 0.7454 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$