A: Lines, Points, Vectors, Dot-Products

1a) We have:

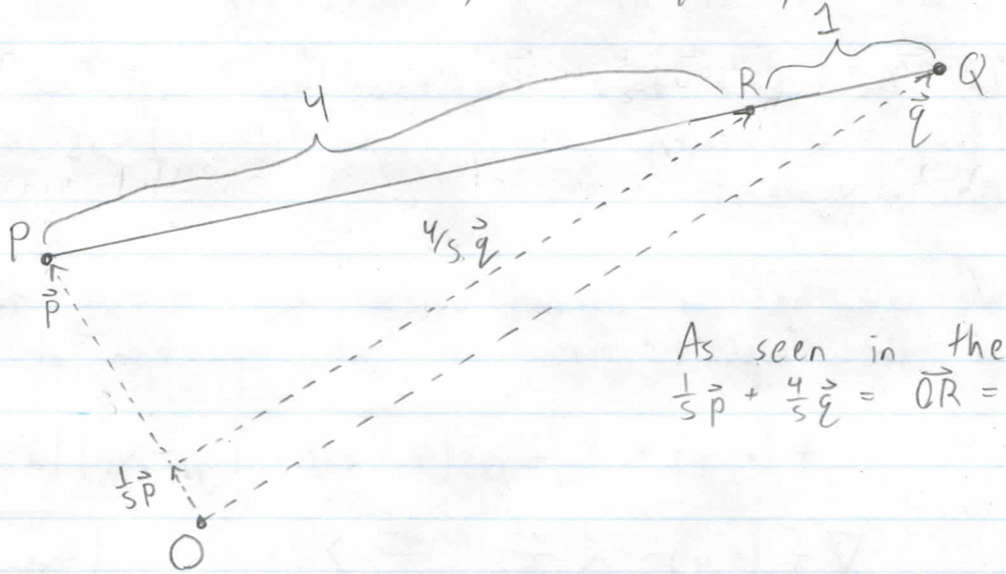
$$\vec{p} = \overrightarrow{OP}$$
$$\vec{q} = \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} \quad \text{(by properties of vector addition)}$$
$$\vec{r} = \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} \quad \text{(again, by vector addition)}$$
$$= \overrightarrow{OP} + \tfrac{4}{5}\overrightarrow{PQ} \quad \left(\overrightarrow{PR} = \tfrac{4}{5}\overrightarrow{PQ} \text{ by definition of } R\right).$$

We want to show: $\vec{r} = \frac{1}{5}\vec{p} + \frac{4}{5}\vec{q}$

$$\text{L.H.S.} = \vec{r} = \overrightarrow{OP} + \tfrac{4}{5}\overrightarrow{PQ}$$

$$\text{R.H.S.} = \tfrac{1}{5}\vec{p} + \tfrac{4}{5}\vec{q} = \tfrac{1}{5}(\overrightarrow{OP}) + \tfrac{4}{5}(\overrightarrow{OP} + \overrightarrow{PQ}) = \overrightarrow{OP} + \tfrac{4}{5}\overrightarrow{PQ}.$$

Since L.H.S. = R.H.S., the equality is verified.



As seen in the diagram, $\frac{1}{5}\vec{p} + \frac{4}{5}\vec{q} = \overrightarrow{OR} = \vec{r}$

1b) From lecture notes, we know that the parametric form of a line from $\vec{p}_0$ to $\vec{p}_1$ is:

$$\vec{f}(\lambda) = \vec{p}_0 + \lambda(\vec{p}_1 - \vec{p}_0) \quad \text{with } 0 \le t \le 1$$

In our case, this leads to:

$$\vec{f}(t) = \vec{p} + t(\vec{q} - \vec{p}) \quad \text{with } 0 \le t \le 1,\ \vec{p} = \overrightarrow{OP},$$
$\vec{q} = \overrightarrow{OQ}$, and point O is the origin of the coordinate system.

1c) We can use the implicit form of a line. From lecture notes, we define:

$$f(x,y) = (y-y_0)(x_1-x_0) - (y_1-y_0)(x-x_0)$$

If $f(x,y)=0$, then the point (x,y) is on the line defined by (x_0, y_0) and (x_1, y_1). To further check that the point is on the line <u>segment</u>, we need to check if it is between the two endpoints. We verify:

$$\min(x_0, x_1) \le x \le \max(x_0, x_1)$$

and $$\min(y_0, y_1) \le y \le \max(y_0, y_1)$$

If the above two conditions are satisfied, and if $f(x,y) = 0$, then the point (x,y) is on the line segment.

2a) We can get a normal vector by taking the gradient of the implicit form of the equation of the line.

$$f(x,y) = (y-y_0)(x_1-x_0) - (y_1-y_0)(x-x_0)$$

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$= \langle -(y_1-y_0), (x_1-x_0) \rangle$$

Assuming:
$P = \langle x_0, y_0 \rangle$
$Q = \langle x_1, y_1 \rangle$

To get a unit normal, divide by magnitude:

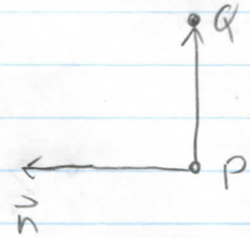
$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle y_0 - y_1, x_1 - x_0 \rangle}{\sqrt{(y_0-y_1)^2 + (x_1-x_0)^2}}$$

$$= \left\langle \frac{y_0-y_1}{\sqrt{(y_0-y_1)^2+(x_1-x_0)^2}}, \frac{x_1-x_0}{\sqrt{(y_0-y_1)^2+(x_1-x_0)^2}} \right\rangle$$

Now let's just verify that the direction of $\vec{n}$ is correct. If $P = \langle 0, 0 \rangle$ and $Q = \langle 0, 1 \rangle$, then:

$$\vec{n} = \left\langle \frac{0-1}{\sqrt{1^2+0^2}}, \frac{0-0}{\sqrt{1^2+0^2}} \right\rangle$$

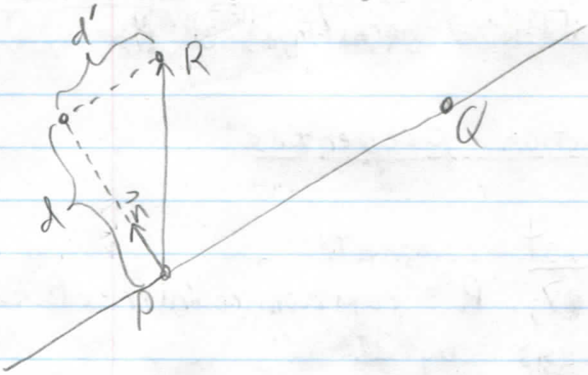
$$= \langle -1, 0 \rangle$$



Yes, $\vec{n}$ has the correct direction. Thus, the final expression is:

$$\vec{n} = \left\langle \frac{y_0 - y_1}{\sqrt{(y_0-y_1)^2+(x_1-x_0)^2}}, \frac{x_1 - x_0}{\sqrt{(y_0-y_1)^2+(x_1-x_0)^2}} \right\rangle$$

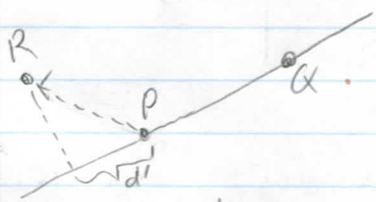
2b) First let's compute distance between (x, y) and the (infinte) line defined by P and Q.



We are trying to find the distance d. This is precisely the length of $\overrightarrow{PR}$ projected onto $\vec{n}$. Since $\vec{n}$ is a unit vector, we can find this with a simple dot product:

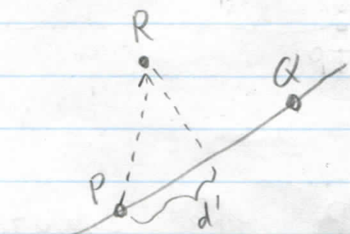
$$d = \overrightarrow{PR} \cdot \vec{n} = \langle x - x_0, y - y_0 \rangle \cdot \vec{n}$$

Now let's consider the fact that $\overrightarrow{PQ}$ is a finite-length segment. Now the distance d' matters. There are three cases:



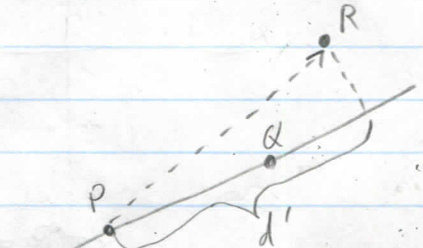
Case 1: $d' < 0$

Shortest distance: $\|\overrightarrow{PR}\|$



Case 2: $0 \leq d' \leq \|\overrightarrow{PQ}\|$

Shortest distance: $\overrightarrow{PR} \cdot \vec{n}$



Case 3: $d' > \|\overrightarrow{PQ}\|$

Shortest distance: $\|\overrightarrow{QR}\|$

We can calculate d' by noting that it is the length of the projection of $\vec{PR}$ onto $\vec{PQ}$. This is:

$$d' = \frac{\vec{PR} \cdot \vec{PQ}}{\|\vec{PQ}\|} = \frac{(x-x_0)(x_1-x_0) + (y-y_0)(y_1-y_0)}{\sqrt{(x_1-x_0)^2 + (y_1-y_0)^2}}$$

Thus our final answer is:

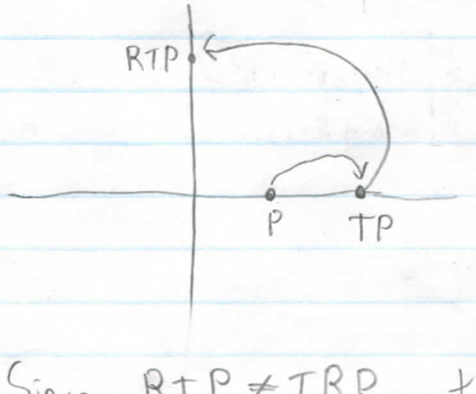
$$\text{distance from } R \text{ to } \overline{PQ} = \begin{cases} \vec{PR} \cdot \vec{n} & \text{if } 0 \le d' \le \|\vec{PQ}\| \\ \|\vec{PR}\| & \text{if } d' < 0 \\ \|\vec{QR}\| & \text{if } d' > \|\vec{PQ}\| \end{cases}$$

where d' is defined as above, $\vec{n}$ is defined as in 2a, and double bars represent the 2-norm (e.g. $\|\vec{AB}\| = \sqrt{(A_x - B_x)^2 + (A_y - B_y)^2}$)

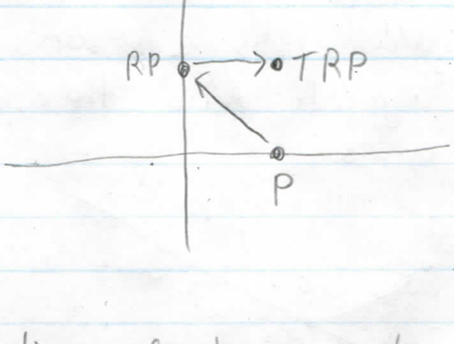
B: Transformations and transformation properties

1) Rotations and Translations: do not commute.
Counterexample: Let $P = \langle 1, 0 \rangle$, R = rotation counter-clockwise by 90°, and T = translate right by 1.

$RTP = \langle 0, 2 \rangle$



$TRP = \langle 1, 1 \rangle$



Since $RTP \neq TRP$, the operations don't commute.

2) Rotations and Rotations: do commute.

Proof: Let $P = \begin{bmatrix} x \\ y \end{bmatrix}$, $R_1 = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix}$,

$$R_2 = \begin{bmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{bmatrix}$$

We want to show $R_1 R_2 P = R_2 R_1 P$. Since matrix multiplication is associative, it is sufficient to show that $R_1 R_2 = R_2 R_1$

$$\begin{aligned} L.H.S. &= R_1 R_2 \\ &= \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} \begin{bmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 & -\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2 \\ \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 & -\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2 \end{bmatrix} \end{aligned}$$

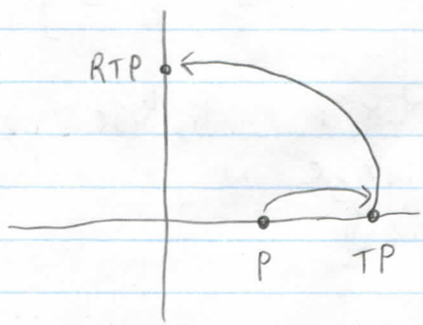
$$\begin{aligned} R.H.S. &= R_2 R_1 \\ &= \begin{bmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{bmatrix} \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 & -\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2 \\ \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 & -\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2 \end{bmatrix} \end{aligned}$$

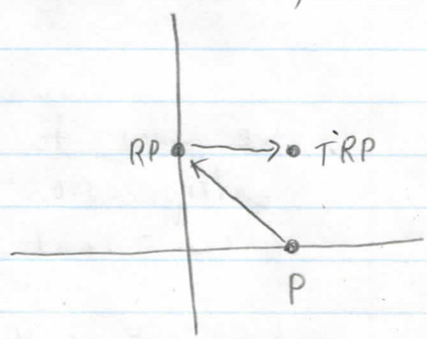
Since L.H.S. = R.H.S., the transformations commute.

3) Translations and reflections: do not commute

Counterexample: Let $P = \langle 1, 0 \rangle$, T = translate right by one, R = reflect across $y = x$

$RTP = \langle 0, 2 \rangle$ $\qquad\qquad$ $TRP = \langle 1, 1 \rangle$





Since $RTP \neq TRP$, the operations don't commute.

4) Shear w.r.t. x-axis and uniform scaling: do commute

Proof: Let $P = \begin{bmatrix} x \\ y \end{bmatrix}$, $S_1 = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$ for $s \neq 0$, and (scale)

$S_2 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ for $t \neq 0$ (shear)

We want to show $S_1 S_2 P = S_2 S_1 P$. Since matrix multiplication is associative, it is sufficient to show $S_1 S_2 = S_2 S_1$

$$\text{L.H.S.} = S_1 S_2 = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s & st \\ 0 & s \end{bmatrix}$$

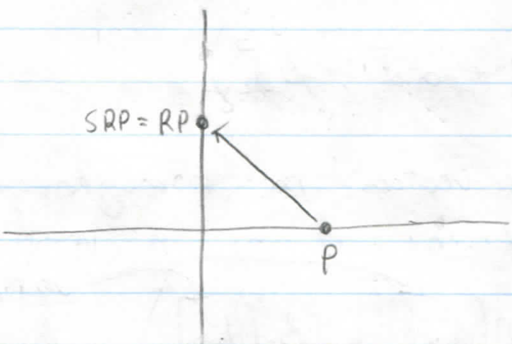
$$\text{R.H.S.} = S_2 S_1 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

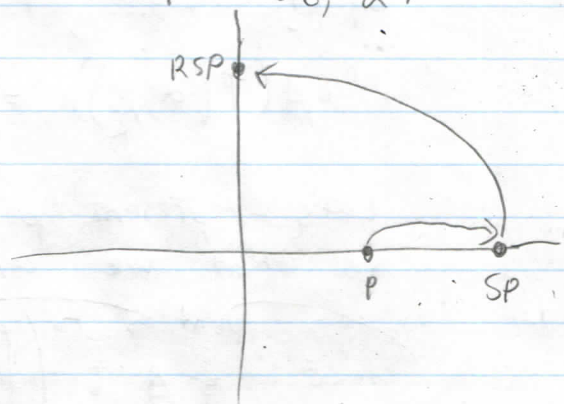
$$= \begin{bmatrix} s & st \\ 0 & s \end{bmatrix}$$

Since L.H.S. = R.H.S., the operations commute.

5) Rotation, non-uniform scaling: <u>do not</u> commute
Counter example: Let $P = \langle 1, 0 \rangle$, R = rotate counter-clockwise by 90°, and S = scale by 2 in <u>x direction only</u>.

$SRP = \langle 0, 1 \rangle$ $\qquad\qquad$ $RSP = \langle 0, 2 \rangle$





Since $SRP \neq RSP$, the operations <u>do not</u> commute.

<u>C: Affine transformation properties</u>

1) ① $S(2/3, 1/2)$ - scale by $2/3$ in the x direction and by $1/2$ in the y direction.

② Re - reflect about the line $y = x$

③ $R(45^\circ)$ - rotate counterclockwise by 45°

④ $T(5, 1.5)$ - translate right 5 units and up 1.5 units

$$① = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad ② = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad ③ = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$④ = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{bmatrix}$$

2) Call the invertible affine transform A.
The set of all points in the triangle is:

$$P = \{ T(\alpha, \beta) \text{ for } \alpha, \beta \geq 0 \text{ and } \alpha + \beta \leq 1 \}$$

Let's look at how a point on the triangle is transformed by A.

$$A(T(\alpha,\beta)) = A(\vec{p}_1 + \alpha \vec{d}_1 + \beta \vec{d}_2)$$

Let's represent points & vectors in homogeneous coordinates, so that we can represent A as a matrix multiplication:

point $\qquad\qquad\qquad\qquad\qquad$ vectors

$$= A \cdot \left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} d_{1x} \\ d_{1y} \\ 0 \end{bmatrix} + \beta \begin{bmatrix} d_{2x} \\ d_{2y} \\ 0 \end{bmatrix} \right)$$

Because of the linearity of matrix multiplication, we get:

$$= A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} + \alpha A \begin{bmatrix} d_{1x} \\ d_{1y} \\ 0 \end{bmatrix} + \beta A \begin{bmatrix} d_{2x} \\ d_{2y} \\ 0 \end{bmatrix}$$

Further, since A is an affine transform, it's not just an arbitrary matrix:

$$= \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{1y} \\ 0 \end{bmatrix} + \beta \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{2y} \\ 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}}_{\text{Call this } \vec{p_1}'} + \alpha \underbrace{\begin{bmatrix} ad_{1x} + bd_{1y} \\ dd_{1x} + ed_{1y} \\ 0 \end{bmatrix}}_{\text{Call this } \vec{d_1}'} + \beta \underbrace{\begin{bmatrix} ad_{2x} + bd_{2y} \\ dd_{2x} + ed_{2y} \\ 0 \end{bmatrix}}_{\text{Call this } \vec{d_2}'}$$

$$= \vec{p_1'} + \alpha \vec{d_1'} + \beta \vec{d_2'}$$

This is just another triangle!
If we let $T'(\alpha,\beta) = \vec{p_1'} + \alpha \vec{d_1'} + \beta \vec{d_2'}$, then the full set of transformed points is just:

$$A(\{T(\alpha,\beta) \text{ for } \alpha,\beta \geq 0 \text{ and } \alpha+\beta \leq 1\})$$

$$= \{T'(\alpha,\beta) \text{ for } \alpha,\beta \geq 0 \text{ and } \alpha+\beta \leq 1\}$$

Thus, the triangle maps to another triangle.

3) Yes, we can represent affine transforms in Cartesian coordinates. A general affine transform that maps (x,y) to (x',y') can be written as:

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

(the transform is defined by scalars a, b, c, d, e, f)

The advantage of homogeneous coordinates is that transformations become simple matrix multiplications. Matrix multiplications have nice properties:

① They are <u>associative</u>, so I can multiply multiple transformations together to get a single aggregate transformation. This can make it much more efficient to apply a cascade of transformations to a set of points.
② They are (usually) <u>invertible</u>, so I can invert a transformation just by inverting its matrix
③ They are <u>linear</u>, which makes them much easier to reason about (e.g. see previous question - our proof relies on linearity)