APEC Math Review

Some Important Results

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Intermediate Value Theorem

Intermediate Value Theorem in $\mathbb{R}^n$

Let $D := [a, b]$, $f : D \to \mathbb{R}$ be a continuous function. If $f(a) < f(b)$ and c is a real number such that $f(a) < c < f(b)$ then $\exists x \in (a, b)$ such that $f(x) = c$
Fig 1.7 shows an example.

- What is the difference between this theorem and Weistrass' that we saw last time?
- The intermediate value theorem does not characterize continuous functions. It is only one side of the implication. The converse is actually false.

Intermediate Value Theorem for the Derivative

Even though not all differentiable functions have to be continuously differentiable, they do need to retain some form of continuity. No jump discontinuities

Let $D = [a, b]$, $f : D \rightarrow \mathbb{R}$ differentiable everywhere on D. If $f'(a) < f'(b)$ and if c is a real number such that $f'(a) < c < f'(b)$, then there is a point $x \in (a, b)$ such that $f'(x) = c$

Note that this does not assume that f is C^1. If it did, this would be a trivial application of the intermediate value theorem.

The Mean Value Theorem

Let $D = [a, b]$, $f : D \to \mathbb{R}$ continuous. Suppose f is differentiable on (a,b). Then, there is an x in (a,b) such that:

$$f(b) - f(a) = (b - a)f'(x)$$

A generalization of the Mean Value Theorem called Taylor's Theorem can be found on page 62 of Sundaram. It talks about how a many times differentiable function can be approximated by a polinomial.

The Intermediate Value Theorem in $\mathbb{R}^n$

Let $D \subset \mathbb{R}^n$ be a convex set and let $f : D \to \mathbb{R}^n$ be continuous on D. Suppose a and b are in D such that $f(a) < f(b)$ then, for any c such that $f(a) < c < f(b)$ $\exists \hat{\lambda} \in (0,1)$ such that $f((1-\hat{\lambda})a + \hat{\lambda} b) = c$

The Mean Value Theorem in $\mathbb{R}^n$

Let $D \subset \mathbb{R}^n$ be an open convex set and let $f : D \to \mathbb{R}^n$ be differentiable everywhere on D. Then $\forall a, b \in D$, $\exists \hat{\lambda} \in (0, 1)$ such that $f(b) - f(a) = Df((1 - \hat{\lambda})a + \hat{\lambda}b) \cdot (b - a)$

Implicit Function Theorem

Let $F : S \subset \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ be a C^1 function, where S is open. Let (x^*, y^*) be a point in S such that $DF_y(x^*, y^*)$ is invertible, and let $F(x^*, y^*) = c$. Then there is an neighborhood $U \subset \mathbb{R}^m$ of x^* and a C^1 function $g : U \rightarrow \mathbb{R}^n$ such that:

- $(x, g(x)) \in S \forall x \in U$
- $g(x^*) = y^*$
- $F(x, g(x)) = x \forall x \in U$

The derivative of g at any point x in U may be obtained from the chain rule:

$$Dg(x) = (DF_y(x, y))^{-1} \cdot DF_x(x, y).$$

L'Hopital Rule

For indeterminate forms ONLY

$$lim_{x \to c} \frac{f(x)}{g(x)} = lim_{x \to c} \frac{f'(x)}{g'(x)}$$

What else do we need for this?
Exercises: Chapter 6 Bartle and Sherbert.