

APEC Math Review

Functions

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Definition: A relation on a set A is a subset $R \subseteq A \times A$. We can denote this set by $(x, y) \in R$ or xRy .

- A relation is a set.
- Preference relations are the precursors of utility functions.
- Example: let $a = \{1, 2, 3, 4\}$ and consider the following set:
$$R = \{(1, 1), (2, 1), (2, 2), (3, 3), (3, 2), (3, 1), (4, 4), (4, 3), (4, 2), (4, 1)\} \subseteq A \times A$$
- Let S be a relation on $A \times A$, defined by "having the same parity as"
- How about the "at least as good as"

Properties of Relations

Let R be a relation on A

- R is *reflexive* if xRx for every $x \in S$
- R is *symmetric* if $xRy \Rightarrow yRx$ for all $x, y \in A$
- R is *transitive* if whenever xRy and yRz then xRz

Examples: \geq , $=$, \neq , are they reflexive, symmetric, transitive?

The Book of Proof has a deeper discussion of relations, including diagrams and a lot of examples. Go through chapter 11 (sections 1 and 2) if you feel like you need to.

We can generalize the notion of relations on A to relations from A to B

Definition: suppose A and B are sets. a function f from A to B ($f:A \rightarrow B$) is a relation $f \subseteq A \times B$ satisfying the property that $\forall a \in A$ the relation f contains exactly one ordered pair of form (a,b) .

We abbreviate $(a,b) \in f$ to $f(a)=b$

- not all relations are functions
- Draw figure 12.2 on the board
- domain, codomain, range
- A function is really just a special kind of set.
- $f = g \iff f(x) = g(x) \forall x \in A$, where A is the domain of f and g

Injective and Surjective Functions

A function $f : A \rightarrow B$ is:

- Injective (one to one) if $\forall a, a' \in A, a \neq a'$ implies $f(a) \neq f(a')$
- Surjective (or onto B) if *forall* $b \in B, \exists a \in A$ *st* $f(a) = b$
- Bijective if f is both injective and surjective

Visual description on the board, some examples

Composition of Functions

- if $f : A \rightarrow B$ and $g : B \rightarrow C$, then $g \circ f(x) = g(f(x)) : A \rightarrow C$
- Theorem: Composition of functions is associative: $(h \circ g) \circ f = h \circ (g \circ f)$
- If f and g are surjective, then $g \circ f$ is also surjective.

Inverse Functions

- Definition: if $f : A \rightarrow B$ is bijective, then its inverse is the function $f^{-1} : B \rightarrow A$. The two functions obey the equations $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(x) = x$
- Theorem: $f : a \rightarrow B$ is bijective if and only if it's inverse f^{-1} is a function from B to A

Image and Preimage

Definition: Let $f : A \rightarrow B$ be a function.

- ① If $X \subseteq A$, the image of X is the set $f(X) = \{f(x) : x \in X\} \subseteq B$
- ② if $Y \subseteq B$, the preimage of Y is the set $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$.

Theorem: given $f : A \rightarrow B$, let $W, X \subseteq A$, and $Y, Z \subseteq B$. Then:

- ① $f(W \cap X) \subseteq f(W) \cap f(X)$
- ② $f(W \cup X) = f(W) \cup f(X)$
- ③ $X \subseteq f^{-1}(f(X))$
- ④ $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$
- ⑤ $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$
- ⑥ $f(f^{-1}(Y)) \subseteq Y$

Homogeneity

Consider functions $f(x_1, x_2, \dots, x_N)$ defined for all $(x_1, x_2, \dots, x_n) \geq 0$

Definition: a function $f(x_1, x_2, \dots, x_N)$ is homogeneous of degree $r \in \mathbb{Z}$ if $\forall t > 0$ we have

$$f(tx_1, tx_2, \dots, tx_N) = t^r f(x_1, x_2, \dots, x_n)$$

- Examples: $f(x_1, x_2) = x_1/x_2$, $f(x_1, x_2) = (x_1 x_2)^{1/2}$
- Note: what happens if I take $t = 1/x_1$ to a homogeneous function of degree 0? 1?

Theorem M.B.1: if $f(x_1, x_2, \dots, x_N)$ is homogeneous of degree $r \in \mathbb{Z}$, then $\forall n \in \mathbb{N}$, the partial derivative function $(x_1, x_2, \dots, x_N)/\partial x_n$ is homogeneous of degree $r-1$

Level sets and Homogeneity

- Definition: a level set of function $f(\cdot)$ is a set of the form $\{x \in \mathbb{R}_+^N : f(x) = k \text{ for some real number } k$
- A radial expansion of this set is obtained by multiplying each vector x by some positive scalar.
- If $f(\cdot)$ is homogeneous of any degree, then $f(x_1, x_2, \dots, x_N) = f(x'_1, x'_2, \dots, x'_N)$ implies $f(tx_1, tx_2, \dots, tx_N) = f(tx'_1, tx'_2, \dots, tx'_N)$ for any positive t
- ie, a radial expansion of a level set of $f(\cdot)$ gives another level set of $f(\cdot)$
- See that this implies that the slopes of the level sets of f are unchanged along any ray that passes through the origin.

Q: Why do we care about this?

Homothetic Functions

Definition: Let $f()$ be homogeneous of some degree and $h()$ be an increasing function of one variable. Then $h(f(x_1, \dots, x_N))$ is called homothetic.

- Note that f and h have the same level sets.
- Monotonic transformations of homogeneous functions

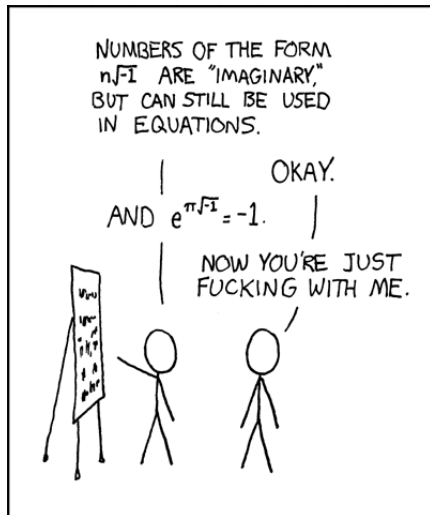
Euler's Formula

Theorem M.B.2: (Euler's Formula) Let $f(x_1, x_2, \dots, x_N)$ be homogeneous of degree r , and differentiable. Then at any $(\bar{x}_1, \dots, \bar{x}_N)$ we have:

$$\sum_{n=1}^N \frac{\partial f(\bar{x}_1, \dots, \bar{x}_N)}{\partial x_n} \bar{x}_n = r f(\bar{x}_1, \dots, \bar{x}_N)$$

In matrix notation: $\nabla f(\bar{x}) \cdot \bar{x} = r f(\bar{x}_1, \dots, \bar{x}_N)$

What does this mean for a function that is homogeneous of degree 0?



Convex Structures and Optimization

(Sundaram Chapter 7: read the intro!)

- Why do we study convex things?
- Turns out that when optimization problems meet certain convexity properties, first order conditions are not only necessary but they are sufficient.
- Whoa, what? Yes, even global optima. Yes, even unique.
- So, we study all sort of convex objects. Remember the initial questionnaire?
- convexity assumptions are pretty strong. Why are we ok with these?
- convex constraints and concave objective function imply FOC are sufficient (and necessary) for global maxima.
- convex constraints and convex objective function...

Concave and Convex Functions

Definition: Let $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$, and let D be a convex set. The subgraph of f and the epigraph of f are defined by:

$$\text{sub}f = \{(x, y) \in D \times \mathbb{R} : f(x) \geq f(y)\}$$

$$\text{epi}f = \{(x, y) \in D \times \mathbb{R} : f(x) \leq f(y)\}$$

A function f is said to be concave on D if $\text{sub} f$ is a convex set. f is said to be convex on D if $\text{epi} f$ is a convex set.

Concave and Convex Functions

Theorem 7.1 A function $f : D \rightarrow \mathbb{R}$ is concave on D if and only if for all x, y in D , and $\alpha \in (0, 1)$, it is the case that:

$$f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y)$$

How do we modify this theorem for convexity?

Can a function be concave and convex? can a function be neither? Take $f(x) = x^3$ in the reals, $x=-2$, $y=2$, $\alpha = 1/4$

How about strict convexity and concavity? take $f(x) = x^\lambda$

Strict Concavity, Strict Convexity

- A function f is (strictly convex) in D if and only if $-f$ is (strictly concave) in D
- We will skip the rest of Sundaram Chapter 7 for now!
- Just know that convexity has important implications for: continuity, differentiability.
- now MWG

MWG on Concavity

We will consider functions of N variables, defined on a domain A that is convex, $A \subseteq \mathbb{R}^N$.//

We denote $x = (x_1, \dots, x_N)$

- Definition: The function $f : A \rightarrow \mathbb{R}$ is concave if

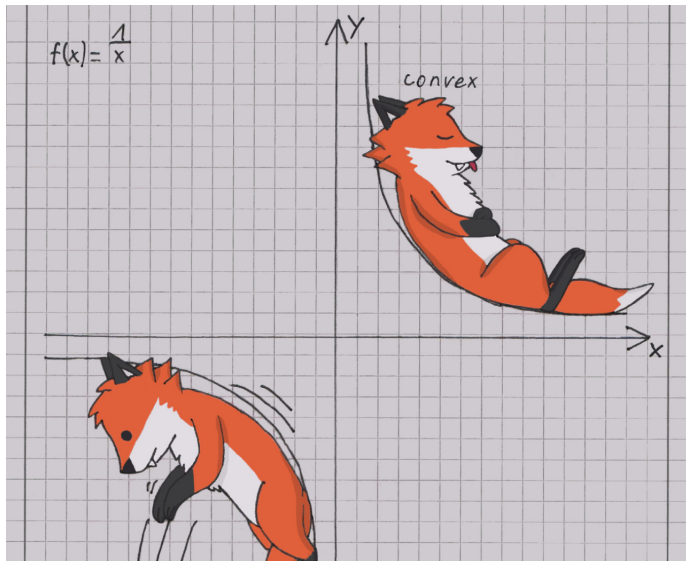
$$f(\alpha x' + (1-\alpha)x) \geq \alpha f(x') + (1-\alpha)f(x)$$

for all $x, x' \in A, \alpha \in [0, 1]$. If the inequality is strict, we say that f is strictly concave.

- Examples on the board
- The straight line connecting two points lies entirely below the graph of a function (strict concavity)

- Note that the previous condition is equivalent to $f(\alpha_1 x^1 + \dots \alpha_K x^K) \geq \alpha_1 f(x^1) + \dots + \alpha_K f(x^K)$ for any collection of vectors in A and numbers α_i that add up to one
- It is easier if we think of $f(A)$ as a set.
- One example: (Jensen's Inequality) $f(\int x dF) \geq \int f(x) dF$ for any distribution function $F : \mathbb{R}[0, 1]$
- The properties of convexity and strict convexity are defined analogously (inequality reversed)
- f is concave if and only if $-f$ is convex

Concavity and Convexity



Alternative Characterization of Concavity

Let $f : A \rightarrow \mathbb{R}$ be continuously differentiable. It is concave if and only if

$$f(x + z) \leq f(x) + \nabla f(x) * z$$

For all $x \in A, z \in \mathbb{R}^N$ (with $x + z \in A$)

- Graph on the board MC3
- Any tangent to the graph must lie weakly above the graph

Aside: Matrices

The $N \times N$ matrix M is negative semidefinite if

$$z \cdot Mz \leq 0$$

$\forall z \in \mathbb{R}^N$. If the inequality is strict for all non-zero z , then the matrix M is negative definite. Reversing the inequality we get the concept of positive semidefinite and positive definite matrices. (We will go deeper into this later on!)

- Theorem: The (twice continuously differentiable) function $f : A \rightarrow \mathbb{R}$ is concave if and only if $D^2f(x)$ is negative semidefinite $\forall x \in A$. If $D^2f(x)$ is negative definite $\forall x \in A$, then the function is strictly concave.
- What does this mean for $N=1$?

Sundaram Chapter 8

- Convexity might be quite restrictive
- We can weaken the assumption by replacing it with another one: quasi-convexity
- Some trade-offs: not sufficient for global optima
- Some useful properties remain, especially for Khun-Tucker, which we will see later on.

Quasi-concave and Quasi-convex functions

Let $f : D \rightarrow \mathbb{R}$, D a convex subset of \mathbb{R}^n . The upper contour set of f at $a \in \mathbb{R}$, denoted $Uf(a)$ is defined as:

$$Uf(a) = \{x \in D : f(x) \geq a\}$$

while the lower contour set, denoted $Lf(a)$ is defined as:

$$Lf(a) = \{x \in D : f(x) \leq a\}$$

- The function f is said to be quasiconcave on D if $Uf(a)$ is a convex set for all a
- The function f is said to be quasiconvex on D if $Lf(a)$ is a convex set for all a

Quasi-concave Functions

Theorem 8.1: $f : D \rightarrow \mathbb{R}$ is quasi-concave on D if and only if for all x, y elements of D , $\lambda \in (0, 1)$, it is the case that:

$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$$

Theorem 8.2: The function $f : D \rightarrow \mathbb{R}$ is (strictly) quasi-concave on D if and only if $-f$ is (strictly) quasi-convex in D .

Quasi-convexity as a Generalization of Convexity

From Theorem 8.1 it is easy to see that all concave functions are also quasi-concave (why?, prove it)

- The converse is not true, take $f(x) = x^3$
- Theorem 8.3: Any monotone transformation of a concave function results in a quasi-concave function.
- Note that:
 - ① quasi-concave and quasi-convex functions need not be continuous in the interior of their domains
 - ② They can have local optima that are not global optima
 - ③ FOC are not sufficient to identify even local optima under quasi-convexity

Example: page 209 of Sundaram.

Quasi-convex and Quasi-concave Functions

Theorem 8.8: let $f : D \rightarrow \mathbb{R}$ be a C^1 function, where $D \subset \mathbb{R}^n$ is convex and open. Then f is a quasi-concave function on D if and only if it is the case that for any $x, y \in D$,

$$f(x) \geq f(y) \Rightarrow Df(x)(y - x) \geq 0$$

Quasi-convex and Quasi-concave Functions

Theorem 8.9: let $f : D \rightarrow \mathbb{R}$ be a C^2 function, where $D \subset \mathbb{R}^n$ is convex and open. Then:

- ① if f is quasi-concave on D , we have $(-1)^k |C_k(x)| \geq 0$ for $k=1, \dots, n$
- ② if $(-1)^k |C_k(x)| > 0$ for all $k \in \{1, 2, \dots, m\}$ then f is quasi-concave on D .

Sundaram Exercises:

From Chapter 7: 1-13, Chapter 8: 1-10

Definition: The function $f : A \rightarrow \mathbb{R}$, defined on the convex set $A \subseteq \mathbb{R}^N$ is quasiconcave if its upper contour sets $\{x \in A : f(x) \geq t\}$ are convex sets, that is if $f(x) \geq t$ and $f(x') \geq t$ implies that $f(\alpha x + (1 - \alpha)x') \geq t$ for any real number t , $x, x' \in A$, $\alpha \in [0, 1]$

Analogously, we say that a function is quasiconvex if its lower contour sets are convex

Note that f is quasiconcave IFF $-f$ is quasiconvex

- Level sets of a quasiconcave function: what do they look like?

- Level sets of a quasiconcave function: what do they look like?
- Ok, what are level sets?
- Figure M.C.4

Ok, now see that from the definition of quasiconcavity it follows that all concave functions are quasiconcave. The converse is not true: concavity is a stronger property than quasiconcavity (what does stronger mean?)

Theorem M.C.3: The (continuously differentiable) function $f : A \rightarrow \mathbb{R}$ is quasiconcave IFF

$$\nabla f(x) \cdot (x' - x) \geq 0$$

whenever $f(x') \geq f(x)$, $\forall x, x' \in A$. The strict inequality depicts strict quasiconcavity

Theorem M.C.4 Theorem: The (twice continuously differentiable) function $f : A \rightarrow \mathbb{R}$ is quasiconcave if and only if $\forall x \in A$, the Hessian matrix $D^2f(x)$ is negative semidefinite in the subspace $\{z \in \mathbb{R}^N : \nabla f(x) \cdot z = 0\}$, that is if and only if $z \cdot D^2f(x)z \leq 0$ whenever $\nabla f(x) \cdot z = 0$