APEC Math Review

Functions

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Relations

Definition: A relation on a set A is a subset $R \subseteq AxA$. We can denote this set by $(x, y) \in R$ or xRy.

- A relation is a set.
- Preference relations are the precursors of utility functions.
- Example: let $a = \{1, 2, 3, 4\}$ and consider the following set:
  $R = \{(1,1), (2,1), (2,2), (3,3), (3,2), (3,1), (4,4), (4,3), (4,2), (4,1)\} \subseteq AxA$
- Let S be a relation on AxA, defined by "having the same parity as"
- How about the "at least as good as¿

Properties of Relations

Let R be a relation on A

- R is *reflexive* if xRx for every $x \in S$
- R is *symmetric* if $xRy \Rightarrow yRx$ for all $x, y \in A$
- R is *transitive* if whenever xRy and yRz then xRz

Examples: $\geq$, $=$, $\neq$, are they reflexive, symmetric, transitive?

The Book of Proof has a deeper discussion of relations, including diagrams and a lot of examples. Go through chapter 11 (sections 1 and 2) if you feel like you need to.

We can generalize the notion of relations on A to relations from A to B

Functions

Definition: suppose A and B are sets. a function f from A to B (f:A$\rightarrow$ B) is a relation $f \subseteq AxB$ satisfying the property that $\forall a \in A$ the relation f contains exactly one ordered pair of form (a,b).
We abbreviate $(a, b) \in f$ to f(a)=b

- not all relations are functions
- Draw figure 12.2 on the board
- domain, codomain, range
- A function is really just a special kind of set.
- $f = g \iff f(x) = g(x) \forall x \in A$, where A is the domain of f and g

Injective and Surjective Functions

A function $f : A \to B$ is:

- Injective (one to one) if $\forall a, a' \in A, a \neq a'$ implies $f(a) \neq f(a')$
- Surjective (or onto B) if $forallb \in B, \exists a \in Astf(a) = b$
- Bijective if f is both injective and surjective

Visual description on the board, some examples

Composition of Functions

- if $f : A \to B$ and $g : B \to C$, then $gof(x) = g(f(x)) : A \to C$
- Theorem: Composition of functions is associative: $(h \circ g) \circ f = h \circ (g \circ f)$
- If f and g are surjective, then $g \circ f$ is also surjective.

Inverse Functions

- Definition: if $f : A \to B$ is bijective, then its inverse is the function $f^{-1} : B \to A$. The two functions obey the equations $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(x) = x$
- Theorem: $f : a \to B$ is bijective if and only if it's inverse f^{-1} is a function from B to A

Image and Preimage

Definition: Let $f : A \to B$ be a function.

1. If $X \subseteq A$, the image of X is the set $f(X) = \{f(x) : x \in X\} \subseteq B$
2. if $Y \subseteq B$, the preimage of Y is the set $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$.

Theorem: given $f : A$, let $W, X \subseteq A$, and $Y, Z \subseteq B$. Then:

1. $f(W \cap X) \subseteq f(W) \cap f(X)$
2. $f(W \cup X) = f(W) \cup f(X)$
3. $X \subseteq f^{-1}(f(X))$
4. $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$
5. $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$
6. $f(f^{-1}(Y)) \subseteq Y$

Homogeneity

Consider functions $f(x_1, x_2, ..., x_N)$ defined for all $(x_1, x_2, ..., x_n) \geq 0$

Definition: a function $f(x_1, x_2, ..., x_N)$ is homogeneous of degree $r \in \mathbb{Z}$ if $\forall t > 0$ we have

$$f(tx_1, tx_2, ..., tx_N) = t^r f(x_1, x_2, ..., x_n)$$

- Examples: $f(x_1, x_2) = x_1/x_2$, $f(x_1, x_2) = (x_1 x_2)^{1/2}$
- Note: what happens if I take $t = 1/x_1$ to a homogeneous function of degree 0? 1?

Homogeneity

Theorem M.B.1: if $f(x_1, x_2, ..., x_N)$ is homogeneous of degree $r \in \mathbb{Z}$, then $\forall n \in \mathbb{N}$, the partial derivative function $(x_1, x_2, ..., x_N)/\partial x_n$ is homogeneous of degree r-1

Level sets and Homogeneity

- Definition: a level set of function f(.) is a set of the form $\{x \in \mathbb{R}_+^N : f(x) = k$ for some real number k
- A radial expansion of this set is obtained by multiplying each vector x by some positive scalar.
- If f(.) is homogeneous of any degree, then $f(x_1, x_2, ..., x_N) = f(x_1', x_2', ..., x_N')$ implies $f(tx_1, tx_2, ..., tx_N) = f(tx_1', tx_2', ..., tx_N')$ for any positive t
- ie, a radial expansion of a level set of f(.) gives another level set of f(.)
- See that this implies that the slopes of the level sets of f are unchanged along any ray that passes through the origin.

Q: Why do we care about this?

Homothetic Functions

Definition: Let f() be homogeneous of some degree and h() be an increasing function of one variable. Then $h(f(x_1, ...x_N))$ is called homothetic.

- Note that f and h have the same level sets.
- Monotonic transformations of homogeneous functions

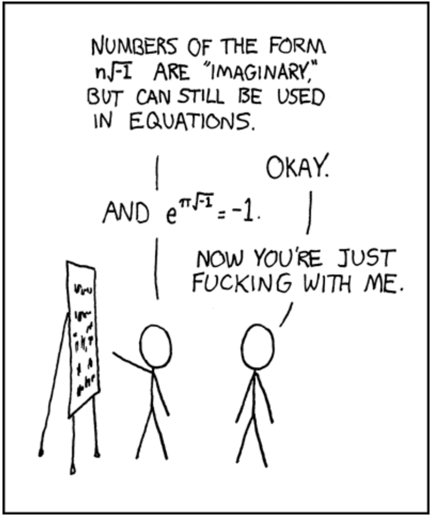
Theorem M.B.2: (Euler's Formula) Let $f(x_1, x_2, ..., x_N)$ be homogeneous of degree r, and differentiable. Then at any $(\bar{x}_1, ..., \bar{x}_N)$ we have:

$$\Sigma_{n=1}^{N} \frac{\partial f(\bar{x}_1, ..., \bar{x}_N)}{\partial x_n} \bar{x}_n = rf(\bar{x}_1, ..., \bar{x}_N)$$

In matrix notation: $\nabla f(\bar{x}) \cdot \bar{x} = rf(\bar{x}_1, ..., \bar{x}_N)$

What does this mean for a function that is homogeneous of degree 0?

Euler


NUMBERS OF THE FORM $n\sqrt{-1}$ ARE "IMAGINARY," BUT CAN STILL BE USED IN EQUATIONS.
OKAY.
AND $e^{\pi\sqrt{-1}} = -1$.
NOW YOU'RE JUST FUCKING WITH ME.

Concavity

We will consider functions of N variables, defined on a domain A that is convex, $A \subseteq \mathbb{R}^N.//$
We denote $x = (x_1, ..., x_N)$

- Definition: The function $f : A \to \mathbb{R}$ is concave if

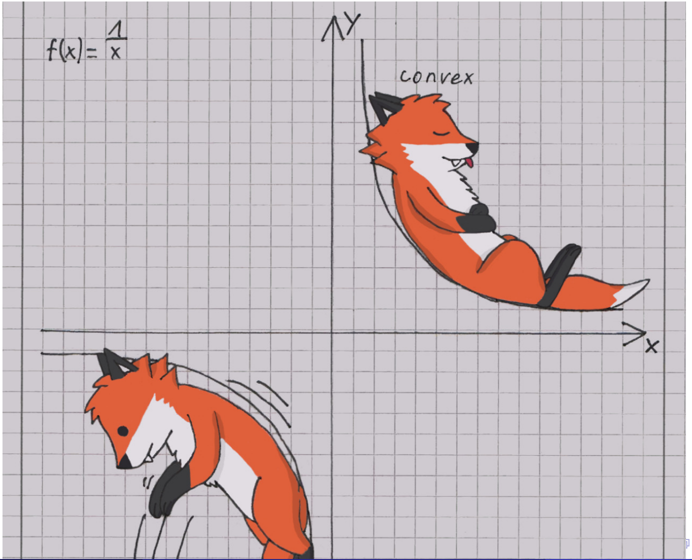
$$f(\alpha x' + (1-)x) \geq \alpha f(x')(1-\alpha)f(x)$$

  for all $x, x' \in A, \alpha \in [0,1]$. If the inequality is stric, we say that f is strictily concave.
- Examples on the board
- The straight line connecting two points lies entirely below the graph of a function (strict concavity)

Concavity

- Note that the previous condition is equivalent to $f(\alpha_1 x^1 + ...\alpha_K x^K) \geq \alpha_1 f(x^1) + ... + \alpha_k f(x^k)$ for any collection of vectors in A and numbers α_i that add up to one
- It is easier if we think of f(A) as a set.
- One example: (Jensen's Inequality) $f(\int x dF) \geq \int f(x) dF$ for any distribution function $F : \mathbb{R}[0,1]$
- The properties of convexity and strict convexity are defined analogously (inequality reversed)
- f is concave if and only if -f is convex

Concavity and Convexity



Alternative Characterization of Concavity

Let $f : A \to \mathbb{R}$ be continuously differentiable. It is concave if and only if

$$f(x+z) \leq f(x) + (x) * z$$

For all $x, z \in \mathbb{R}^N$ (with $x + z \in A$)

- Graph on the board
- Any tangent to the graph must lie weakly above the graph

Continuity

Fixed Point Theorems