

# APEC Math Review

## Proofs

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Theorem: a mathematical statement that is true and can be (and has been) verified as true

A proof of a theorem is a written verification that shows that the theorem is definitely and unequivocally true.

- Understandable
- Convincing
- Unambiguous

Definition: exact, unambiguous explanation of the meaning of a mathematical word or phrase

# Theorems

*Theorem:* Let  $f$  be differentiable on an open interval  $I$  and let  $c \in I$ . If  $f(c)$  is the maximum or minimum value of  $f$  on  $I$ , then  $f'(c) = 0$

*Theorem:* Every absolutely convergent series converges.

*Theorem:* Suppose each consumer's preferences are locally non-satiated. Then any allocation  $(x^*, y^*)$  that with prices  $p^*$  forms a competitive equilibrium is Pareto optimal

Note: all of these are in the conditional form or can be written in it.

# Definitions

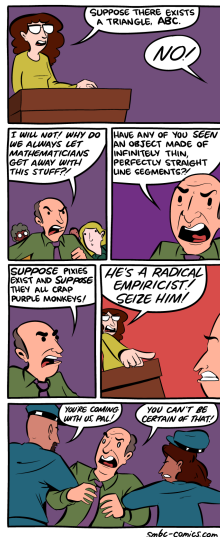
- An integer  $n$  is even if  $n = 2a$  for some integer  $a \in \mathbb{Z}$
- An integer  $n$  is odd if  $n = 2a + 1$  for some integer  $a \in \mathbb{Z}$
- A number  $n \in \mathbb{N}$  is prime if it has exactly two positive divisors: 1 and  $n$ . If  $n$  has more than two positive divisors, it is called a composite (Thus  $n$  is composite if and only if  $n = ab$  for  $1 < a, b < n$ .)
- A feasible allocation  $(x_1, \dots, x_I, y_1, \dots, y_J)$  is Pareto Optimal if there is no other feasible allocation  $(x'_1, \dots, x'_I, y'_1, \dots, y'_J)$  such that  $u_i(x'_i) \leq u_i(x_i)$  for all  $i=1, \dots, I$  and  $u_i(x'_i) > u_i(x_i)$  for some  $i$ .

# Other types of statements

- Theorems: statements that have been proven to be true
- Proposition, lemma : A statement that is true but not as significant as a theorem
- Corollary: a result that is an immediate consequence of a theorem

# Direct Proof

- Let  $P, Q$  be statements
- Proposition:  $P \Rightarrow Q$
- Direct proofs require us to construct a chain of implications  $R_1, R_2, \dots, R_n$  such that:  
 $P \Rightarrow R_1, R_1 \Rightarrow R_2, \dots, R_n \Rightarrow Q$
- Transitivity holds for conditional statements.
- Draw the truth table
- Remember: we only care about those cases when  $P$  is true



# Direct Proof

- We are interested in the logical implications of  $P$
- and statements that imply  $Q$
- The idea is to work forward from  $P$  and backwards from  $Q$ , and connect the chain of implications
- Sometimes we may have to either strengthen  $P$  (add assumptions) or weaken  $Q$
- Example on the board:

Theorem 1: The square of an odd integer is also odd



# Direct Proof

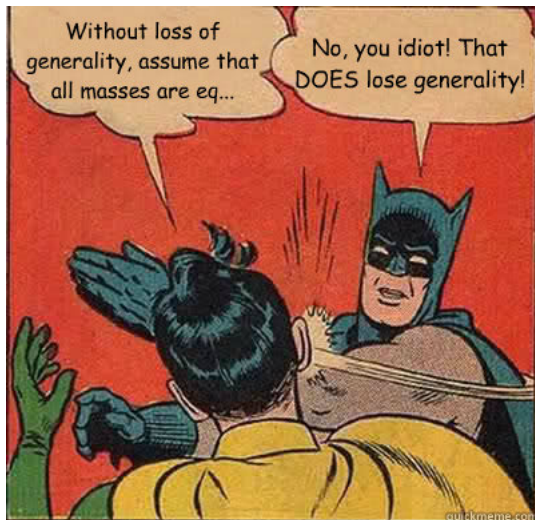
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- Example on the board:

Theorem 1: The square of an odd integer is also odd

Use the definition of odd

## Other examples of direct proof

- Proposition: Let  $x, y$  be positive real numbers. If  $x \leq y$ , then  $\sqrt{x} \leq \sqrt{y}$
- Proposition: Let  $x, y$  be positive real numbers, then  $2\sqrt{xy} \leq x + y$
- Proposition: If  $n \in \mathbb{N}$ , then  $1 + (-1)^n(2n - 1)$
- Proposition: Every multiple of 4 equals  $1 + (-1)^n(2n - 1)$  for some  $n \in \mathbb{N}$
- Proposition: If two integers have opposite parity, their sum is odd



# Indirect Proof

Some propositions are difficult to prove directly. We will study two ways of indirect proofs: Proof by contrapositive and proof by contradiction.

- Proof by contrapositive: instead of proving  $P \Rightarrow Q$ , we prove its logical equivalent:  
 $\neg Q \Rightarrow \neg P$
- Proof by contradiction: instead of proving  $P \Rightarrow Q$ , we show that  $(P \wedge \neg Q)$  implies a contradiction.

# Prove by Contrapositive

- Draw the truth table that verifies that a statement is logically equivalent to its contrapositive
- The outline of proofs by contrapositive is:
  - 1 Suppose  $\neg Q$
  - 2 ...
  - 3 Therefore  $\neg P$
- Compare to the outline of a direct proof.
- An example: Proposition: if  $n$  is an integer, and  $n^2$  is even, then  $n$  is also even
- Proofs by contrapositive are convenient when the universal quantifier is present, because the contrapositive will include the existence quantifier.

# Proof by Contrapositive: Examples

- Suppose  $x \in \mathbb{Z}$ . If  $7x + 9$  is even, then  $x$  is odd (prove both ways)
- Suppose  $x \in \mathbb{Z}$ . If  $x^2 - 6x + 5$  is even, then  $x$  is odd (try both)
- 
- Let  $a \geq 0, \in \mathbb{R}$ . If  $\forall \epsilon > 0$  it is true that  $0 \leq a < \epsilon$  then  $a = 0$
- If  $m, n$  are natural numbers such that  $m + n \geq 20$  then either  $m \geq 10$  or  $n \geq 10$

# Mathematical Writing

## Hammack's style guidelines for mathematical writing

- 1 Never begin a sentence with a mathematical symbol (capitalization)
- 2 End each sentence with a period
- 3 Separate mathematical symbols and expressions with words (to avoid confusion)
- 4 Avoid misuse of symbols (!!)
- 5 Avoid unnecessary symbols
- 6 Use the first person plural (in math)
- 7 Use the active voice
- 8 Explain each new symbol
- 9 Watch out for "it" (!!)
- 10 Since, because, as for, so
- 11 Thus, hence, therefore, consequently

# Suggested Exercises

Either odd or even exercises for Chapters 4 and 5 of Hammack



# Proof by Contradiction

- We can use this to prove all kinds of statements, not just conditional ones.
- Idea: assume not, and get to nonsense
- Sometimes called reduction to absurdity
- A contradiction is a statement that cannot be true
- We will use the fact that if  $C$  is a contradiction, then  $P \wedge \neg Q \Rightarrow C$ , and  $P \Rightarrow Q$  are logically equivalent

# Proof by Contradiction

- You begin by saying “suppose  $P$  but not  $Q$ ”
- You make sound logical steps
- if you arrive to a contradiction, then your initial assumption\* must be wrong.
- Small detail: we do not necessarily know what the contradiction will be
- Example: The number  $\sqrt{2}$  is irrational
- Example: Euclid's Theorem: There are infinitely many prime numbers
- Let  $a > 0$ , a real number. Then  $1/a > 0$

# Combining techniques

- Every non-zero rational number can be expressed as the product of two irrational numbers.
- (By contradiction and then by contrapositive) Suppose  $a \in \mathbb{Z}$ . if  $a^2 - 2a + 7$  is even then  $a$  is odd

# Suggested Exercises

Either even or odd exercises from Chapter 6 of Hammack

# More on Proofs: Biconditional statements

- If and only if
- Prove a conditional statement and its converse
- Example: an integer  $n$  is odd if and only if  $n^2$  is odd
- That's it!

# Equivalent statements

*Theorem* Suppose  $A$  is an  $n \times n$  matrix. The following statements are equivalent

- 1 The matrix  $A$  is invertible
- 2 The equation  $A\mathbf{x} = \mathbf{b}$  has a unique solution for every  $\mathbf{b} \in \mathbb{R}^n$
- 3  $\det(A) \neq 0$
- 4 The matrix  $A$  does not have 0 as an eigenvalue

The theorem says that these are all either true or all false. How do we prove these?

# Existence, Uniqueness

- We have been proving conditional statements, which are universally quantified statements
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# Existence, Uniqueness

- We have been proving conditional statements, which are universally quantified statements
- How would you prove an existentially quantified statement?
- all we need is an example, as we saw last time
- Example: There exists an even prime number
- Example: There is an integer that can be expressed as the sum of two perfect cubes in two different ways
- Uniqueness statements assert that there is exactly one example  $x$  for which  $P(x)$  is true.
- It exists and it is unique



# Uniqueness Example

To show uniqueness, an example is not enough, you must show that there are no others

- Given  $a, b, c$  be real numbers. There is only one real number  $x$  that satisfies  $a + bx = c$

# Uniqueness Example

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- Given  $a, b, c$  be real numbers. There is only one real number  $x$  that satisfies  $a + bx = c$
- We need something else here. What is it?
- Show existence for the example above

# Constructive vs non constructive proofs

Existence proofs are either constructive or non-constructive

- Constructive proofs display an explicit example
- Non-constructive proofs prove an example exists without providing it
- There exist irrational numbers  $x$  and  $y$  for which  $x^y$  is rational

# Suggested Exercises

Hammack Section 7 (either even or odd)

# Proofs Involving Sets

- Show  $a \in A$ , when  $A = \{x \in S : P(x)\}$
- Then show that  $a \in S$  and  $P(a)$  holds
- Show  $A \subseteq B$
- The you must show that  $\forall a \in A, a \in B$
- directly, by contrapositive, contradiction, ...

# Examples: proofs involving sets

- Prove that if  $A$  and  $B$  are sets, then  $\wp(A) \cup \wp(B) \subseteq \wp(A \cup B)$
- Let  $A$  and  $B$  be sets, show that if  $\wp(A) \subseteq \wp(B)$  then  $A \subseteq B$
- Let  $A, B, C$  be sets, and  $C \neq \emptyset$ . Show that if  $A \times C = B \times C$  then  $A=B$

# Disproof: universal statements

→ proving that something is not true.

- How do you disproof? You cannot just say “there is no proof”
- Statements = { Are known to be true , truth unknown, known to be false }
- $\neg P$
- Disproving universal statements: counterexamples
- $\forall x \in S, P(x)$  can be disproved by proving  $\exists x \in S : \neg P(x)$
- Conjecture: for every integer  $n$ ,  $f(n) = n^2 - n + 11$  is prime

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- Conjecture: for every integer  $n$ ,  $f(n) = n^2 - n + 11$  is prime
- Consider  $n=11$
- Conjecture: If  $A, B, C$  are sets then  $A - (B \cap C) = (A - B) \cap (A - C)$



# Disproof: existence statements

- $\exists x \in S, P(x)$
- To prove, an example is enough. To disprove it is not.
- The negation of the above statement is universally quantified
- Conjecture: There is a real number  $x$  for which  $x^4 < x < x^2$

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- Prove the negation by contradiction
- Conjecture: There exist three different integers  $x, y, z$  greater than 1 such that  $x^y = y^z$

# Disproof by contradiction

- Disprove  $P$
- equivalent to proving  $\neg P$
- to prove that by contradiction, I begin assuming  $\neg\neg P$
- so assume  $P$ , work, get to a contradiction, done!

# Induction



Professor Schmidt demonstrates  
the concept of proof by induction.

# Finally: Induction

- Induction: from particular to general
- Useful for statements that involve natural numbers.
- We use the property of good order of natural numbers.
- Any nonempty set of the natural numbers has a smallest element

# Induction

- Conjecture: the sum of the first  $n$  odd natural numbers equals  $n^2$  (table)
- From the table: sequential statements  $S_n$
- Outline:
  - ① Show that it holds for  $n=1$  (basis)
  - ② Show that if it holds for  $n > 1$  (inductive hypothesis), it holds for  $n+1$  (inductive)
- Convinced? Want to see the proof? B&S pg 15
- note:  $n=1$  does not have to be the first statement.
- note: does not work for  $n$  in integers (why?)
- Prove the above conjecture using mathematical induction.

# Induction examples

- If  $n$  is a non negative integer, then 5 is a factor of  $(n^5 - n)$  (notation:  $5|(n^5 - n)$ )
- If  $n \in \mathbb{Z}$  and  $n \geq 0$ , then  $\sum_{i=0}^n i \cdot i! = (n+1)! - 1$
- For each  $n \in \mathbb{N}$ ,  $2^n \leq 2^{n+1} - 2^{n-1} - 1$



# Strong Induction

- Sometimes, it is not easy to show  $s_{k+1}$  from  $S_k$
- strong induction has a different inductive step:
- Given any integer  $k \geq 1$ , prove  $(S_1 \wedge S_2 \wedge S_3 \wedge \dots \wedge S_k) \Rightarrow S_{K+1}$
- Is strong induction equivalent to induction?

# Proof by Smallest Counterexample

- Hybrid of induction and counterexample
  - ① Check that  $S_1$  is true
  - ② For the sake of contradiction, suppose not every  $S_n$  is true
  - ③ Let  $k > 1$  be the smallest integer for which  $S_k$  is false
  - ④ Then  $S_{k-1}$  is true and  $S_k$  is false
  - ⑤ find a contradiction
- Example in page 165 of The Book of Proof
- Fibonacci example
- Suggested induction exercises in section 10 of Hammack and Section 1.2 of Bartle and Sherbert