APEC Math Review
Khun Tucker

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Motivation

- We cannot simplify all problems to problems where we can use Lagrange's Theorem
- In particular, some inequality constraints cannot be replaced by an open set
- So we need another theorem: Khun Tucker
- We will see feasible sets of the form: $D = U \cap \{x \in \mathbb{R}^n : h_i(x) \geq 0, i = 1, ..., l\}$
- the Theorem of Khun Tucker describes necessary conditions for local optima in such problems.
- We say that the constraint $h_i(x) \geq 0$ is effective at a point x^* if the constraint holds with equality at x^*

Theorem of Khun and Tucker

- Let $f : \mathbb{R}^n \to \mathbb{R}$, and $h_i : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions.
- Suppose x* is a local maximum of f on $D = U \cap \{x \in \mathbb{R}^n : h_i(x) \geq 0, \forall i = 1, ... l\}$, where U is an open set.
- Let $E \subset \{1, ..., l\}$ denote the set of effective constraints at x*.
- Let $h_E = (h_i)_{i \in E}$
- Suppose $\rho(Dh_E(x^*)) = |E|$

Then, there exists a vector $\lambda^* = (\lambda_1^*, ... \lambda_l^*) \in \mathbb{R}^l$ such that the following conditions are met:

1. $\lambda_i^* \geq 0$ and $\lambda_i^* h_i(x^*) = 0, \forall i = 1, ..., l$
2. $Df(x^*) + \sum_{i=1}^{L} \lambda_i Dh_i(x^*) = 0$

- The first condition is called the complimentary slackness (at most one inequality is slack)
- These are necessary conditions: not sufficient! Here is an example:

$f(x) = x^3$, $g(x) = x$. $x^* = \lambda^* = 0$ is not a maximum

The Constraint Qualification

- Note that we only require it for effective constraints at the maximum.
- Example of when the constraint qualification is not met: $f(x, y) = -(x^2 + y^2)$, $h(x, y) = (x - 1)^3 - y^2$
- What does the set D look like?
- Inspect the function, where is it maximized?

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- What does the set D look like?
- Inspect the function, where is it maximized?
- $(x^*, y^*) = (1, 0)$
- The problem's single constraint is effective at this point
- What is $Dh(x^*, y^*)$
- The assumptions of the theorem fail, and so does its conclusion.

The Khun Tucker Multiplier

- KT multipliers also measure the sensitivity of the objective function at x* to relaxations of its various constraints.
- This intuition is clear with the complementary slackness condition.

Khun Tucker Cookbook Procedure

For Maximization:

1. Set up the lagrangean
2. Find all solutions to the following set of equations:
   - $\frac{\delta L}{\delta x_j}(x, \lambda) = 0, \forall j = 1, ..., n$
   - $\frac{\delta L}{\delta \lambda_i}(x, \lambda) \geq 0, \; \lambda_i \frac{\delta L}{\delta \lambda_i}(x, \lambda) = 0, \; \lambda_i \geq 0, \forall i = 1, ...l$

   The points that solve those equations are called critical points of L, note that it is not the same set as with equality constraints
3. Compute the value of f at every critical point of L.

What about minimization??

$\rightarrow$ *Examples*

General Case: Mixed Constraints

- $D = U \cap \{x \in \mathbb{R}^n : g(x) = 0, h(x) \geq 0$
- U is open
- $g : \mathbb{R}^n \to \mathbb{R}^k$
- $h : \mathbb{R}^n \to \mathbb{R}^l$
- We can define ϕ_i to be the collection of all g and h functions.
- We make a lagrangean with all ϕ functions and apply Khun-Tucker.

6.6 Exercises

- Why did we spend so much time with convexity?

Convex Structures in Optimization Theory

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Convex Structures in Optimization Theory

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- We will say that a maximization problem is a convex maximization problem if it has a convex constraint set and a concave objective function.
- A minimization problem is said to be convex if the constraint set is convex and the objective function is convex
  $\rightarrow$ "the optimization problem is convex"
- Strictly concave (convex) objective function in a convex maximization (minimization) problem also gives us uniqueness.
- This result is extremely useful in economics

Implications of convexity

1. Every concave or concave function has to be continuous on the interior of its domain
2. Every concave or concave function must have some differentiability properties (in particular, every directional derivative must be well-defined at all points in the domain)
   - A concave or convex function defined over an open set of $\mathbb{R}^n$ must be differentiable "almost anywhere" in its domain.
   - With respect to the Lebesgue measure
3. The concavity or convexity of an everywhere differentiable function can be completely characterized by the behavior of its derivative. Concavity or convexity of a C^2 function can be completely characterized by its second derivative

- Theorem 7.9: Let D be an open and convex set in $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be differentiable on D. Then, f is concave if and only if $Df(x)(y - x) \geq f(y) - f(x)$ for all $x, y \in D$
- Theorem 7.10: Let $f : D \to \mathbb{R}$ be a C^2 function. Then
    1. f is concave on D IFF $D^2 f(x)$ is NSD for all x in D
    2. f is convex on D IFF $D^2 f(x)$ is PSD for all x in D
    3. If $D^2 f(x)$ is ND for all x in D, f is strictly concave on D
    4. If $D^2 f(x)$ is PD for all x in D, f is strictly convex on D

3 and 4 are only sufficient, not necessary: for example $f(x) = x^{-4}$, $g(x) = x^4$
Example: $f(x, y) = x^a y^b$. Check concavity/convexity both ways.

Convexity and Optimization

- Theorem 7.13: Suppose $D \subset \mathbb{R}^n$ is convex and $f : D \to \mathbb{R}$ is concave. Then:
    1. Any local maximum of f is a global maximum of f
    2. The set $argmax\{f(x)|x \in D\}$ of maximizers of f over D is either empty or convex
- Theorem 7.14 $D \subset \mathbb{R}^n$ is convex and $f : D \to \mathbb{R}$ is strictly concave. Then $argmax\{f(x)|x \in D\}$ is either empty or it contains only one element.

What about convex functions? What about minimization?

- Theorem 7.15 $D \subset \mathbb{R}^n$ is convex and $f : D \to \mathbb{R}$ is concave and differentiable. Then x is an unconstrained maximum of f if and only if $Df(x) = 0$

Convexity and Optimization

Theorem 7.16 (KT under Convexity): Let f be a concave C^1 function mapping $U \subset \mathbb{R}^n$ onto the real numbers, where U is open and convex. For i=1,...,l let $h_i : U \to \mathbb{R}$ also be concave C^1 functions. Suppose $\exists \bar{x} \in U$ such that $h_i(\bar{x}) > 0, i = 1, ..., l$ (Slater's condition, for necessity, replaces the rank condition). Then, x^* maximizes f over $D = \{x \in U : h_i(x) \geq 0, i = 1, ..., l\}$ If and only if $\exists \lambda^* \in \mathbb{R}^k$ such that the Khun-Tucker FOC are satisfied:

- $Df(x*) + \sum_{i=1}^{l} \lambda_i^* Dh_i(x^*) = 0$
- $\lambda^* \geq 0, \sum_{i=1}^{l} \lambda_i^* h_i(x^*) = 0$

Example showing the relevance of Slater's condition: $f(x) = x$, $h(x) = -x^2$, $x \in \mathbb{R}$

max f(x) subject to $x \in D = \{z in U : h_i(z) \geq 0, i = 1, ..., l\}$

- functions f, h_i are C^1, concave on D, and U is open.
- $L(x, \lambda) = f(x) + \sum_{i=1}^{l} \lambda_i h_i(x)$
- The critical points of L are characterized by
  1. $Df(x) + \lambda_i Dh_i(x) = 0$
  2. $\lambda_i \geq 0, h_i(x) \geq 0, \lambda_i h_i(x) = 0$ i=1,...,l

Then, we have two cases:

- If Slater's condition is met, Theorem 7.16 we can use the Khun-Tucker cookbook straight away, without checking for the conditions of proposition 6.5 (optimum exists, and constraint qualification)

- The above is true because Slater's condition + concavity of f and h imply that the KT conditions are necessary at an optimum, but also because they imply that KT conditions are sufficient.

- $\Rightarrow$ If L has no critical points, no solution exists, and If $(*x^*, \lambda^*)$ is a critical point of L, x^* is a solution to the maximization problem.

- We don't have to compare the value of solutions: they are all global max

- If Slater's condition is not met, every critical point of L identifies a solution to the problem, since KT conditions are still sufficient
- (note: for sufficiency concavity replaced the constraint qualification)
- However, they are no longer necessary
- In the absence of a critical point of L we cannot conclude that a solution to the problem does not exist

Previous example: $f(x) = x, h(x) = -x^2$ Slater's condition fails, lagrangean has no critical points but the solution to the problem exists (x=0)

A failure of Slater's condition is a big deal, but it is somewhat rare

Final Remark

- Note that when all $h_i(x)$ functions are concave, the constraint set D is *convex*, so if the objective function is strictly concave, there is at most one solution to the problem (T7.14).
- Under these circumstances, if you find one critical point of L, you are done!

Exercises: S7.8 (page 198)

Optimization and Quasiconcavity

- How about a convex D and a quasiconcave f? Can you use the previous results?
- $u(x) = x_1^{\alpha_1}...x_n^{\alpha_n}$ is only concave when alphas add up to 1
- In general, quasi-concavity does not imply continuity, and FOC are not sufficient to identify global optima.
- But they are sufficient if we add another regularity condition.
- Remember quasiconcavity (upper contour set is convex) IFF
  $f(\lambda x + (1 - \lambda)y) \geq min\{f(x), f(y)\}$

Implications of Quasi-convexity

1. QC functions are not necessarily continuous in the interior of their domains
2. QC functions can have local maxima that are not global maxima (example on the board)
3. FOC are not sufficiento to identify even *local* optima uner quasiconvexity/quasiconcavity of f

Quasi-Convexity and Optimization

- Theorem: Let $f : D \to \mathbb{R}$ de strictly quasi-concave, $D \subset \mathbb{R}^n$ convex. Then any local maximum of f on D is also a global maximum on D. Moreover, the set $argmax\{f(x)|x \in D\}$ is either empty or has a single element. Proof.

Khun-Tucker under Quasi-concavity

Theorem 8.13:

- Let f and h_i be C^1 quasiconcave functions mapping $U \subset \mathbb{R}^n$ into the reals. Where U is open and convex.
- Define $D = \{x \in U : h_i(x) \geq 0, i = 1, ..., k\}$
- Suppose there exists $x^* \in D, \lambda \in \mathbb{R}^k$ such that the KT conditions are met
    1. $\lambda_i^* \geq 0$ and $\lambda_i^* h_i(x^*) = 0, \forall i = 1, ..., k$
    2. $Df(x^*) + \sum_{i=1}^k \lambda_i Dh_i(x^*) = 0$

    Then x* maximizes F over D, provided that at least one of the following conditions holds:
    1. $Df(x^*) \neq 0$
    2. f is concave