APEC Math Review

Functions

Natalia Ordaz Reynoso

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Relations

Definition: A relation on a set A is a subset $R \subseteq AxA$. We can denote this set by $(x, y) \in R$ or xRy.

- A relation is a set.
- Preference relations are the precursors of utility functions.
- Example: let $a = \{1, 2, 3, 4\}$ and consider the following set: $R = \{(1, 1), (2, 1), (2, 2), (3, 3), (3, 2), (3, 1), (4, 4), (4, 3), (4, 2), (4, 1)\} \subseteq AxA$
- Let S be a relation on AxA, defined by "having the same parity as"
- How about the "at least as good as¿

Properties of Relations

Let R be a relation on A

- R is *reflexive* if xRx for every $x \in S$
- R is *symmetric* if $xRy \Rightarrow yRx$ for all $x, y \in A$
- R is *transitive* if whenever xRy and yRz then xRz

Examples: $\geq$, $=$, $\neq$, are they reflexive, symmetric, transitive?

The Book of Proof has a deeper discussion of relations, including diagrams and a lot of examples. Go through chapter 11 (sections 1 and 2) if you feel like you need to.

We can generalize the notion of relations on A to relations from A to B

Functions

Definition: suppose A and B are sets. a function f from A to B (f:A$\rightarrow$ B) is a relation $f \subseteq AxB$ satisfying the property that $\forall a \in A$ the relation f contains exactly one ordered pair of form (a,b).
We abbreviate $(a, b) \in f$ to f(a)=b

- not all relations are functions
- Draw figure 12.2 on the board
- domain, codomain, range
- A function is really just a special kind of set.
- $f = g \iff f(x) = g(x) \forall x \in A$, where A is the domain of f and g

Injective and Surjective Functions

A function $f : A \rightarrow B$ is:

- Injective (one to one) if $\forall a, a' \in A, a \neq a'$ implies $f(a) \neq f(a')$
- Surjective (or onto B) if $forall b \in B, \exists a \in A st f(a) = b$
- Bijective if f is both injective and surjective

Visual description on the board, some examples

Composition of Functions

- if $f : A \to B$ and $g : B \to C$, then $gof(x) = g(f(x)) : A \to C$
- Theorem: Composition of functions is associative: $(h \circ g) \circ f = h \circ (g \circ f)$
- If f and g are surjective, then $g \circ f$ is also surjective.

Inverse Functions

- Definition: if $f : A \to B$ is bijective, then its inverse is the function $f^{-1} : B \to A$. The two functions obey the equations $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(x) = x$
- Theorem: $f : a \to B$ is bijective if and only if it's inverse f^{-1} is a function from B to A

Image and Preimage

Definition: Let $f : A \to B$ be a function.

1. If $X \subseteq A$, the image of X is the set $f(X) = \{f(x) : x \in X\} \subseteq B$
2. if $Y \subseteq B$, the preimage of Y is the set $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$.

Theorem: given $f : A$, let $W, X \subseteq A$, and $Y, Z \subseteq B$. Then:

1. $f(W \cap X) \subseteq f(W) \cap f(X)$
2. $f(W \cup X) = f(W) \cup f(X)$
3. $X \subseteq f^{-1}(f(X))$
4. $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$
5. $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$
6. $f(f^{-1}(Y)) \subseteq Y$

Homogeneity

Consider functions $f(x_1, x_2, ..., x_N)$ defined for all $(x_1, x_2, ..., x_n) \geq 0$

Definition: a function $f(x_1, x_2, ..., x_N)$ is homogeneous of degree $r \in \mathbb{Z}$ if $\forall t > 0$ we have

$$f(tx_1, tx_2, ..., tx_N) = t^r f(x_1, x_2, ..., x_n)$$

- Examples: $f(x_1, x_2) = x_1/x_2$, $f(x_1, x_2) = (x_1 x_2)^{1/2}$
- Note: what happens if I take $t = 1/x_1$ to a homogeneous function of degree 0? 1?

Homogeneity

Theorem M.B.1: if $f(x_1, x_2, ..., x_N)$ is homogeneous of degree $r \in \mathbb{Z}$, then $\forall n \in \mathbb{N}$, the partial derivative function $(x_1, x_2, ..., x_N)/\partial x_n$ is homogeneous of degree r-1

Level sets and Homogeneity

- Definition: a level set of function f(.) is a set of the form $\{x \in \mathbb{R}^N_+ : f(x) = k$ for some real number k
- A radial expansion of this set is obtained by multiplying each vector x by some positive scalar.
- If f(.) is homogeneous of any degree, then $f(x_1, x_2, ..., x_N) = f(x'_1, x'_2, ..., x'_N)$ implies $f(tx_1, tx_2, ..., tx_N) = f(tx'_1, tx'_2, ..., tx'_N)$ for any positive t
- ie, a radial expansion of a level set of f(.) gives another level set of f(.)
- See that this implies that the slopes of the level sets of f are unchanged along any ray that passes through the origin.

Q: Why do we care about this?

Homothetic Functions

Definition: Let f() be homogeneous of some degree and h() be an increasing function of one variable. Then $h(f(x_1, ...x_N))$ is called homothetic.

- Note that f and h have the same level sets.
- Monotonic transformations of homogeneous functions

Euler's Formula

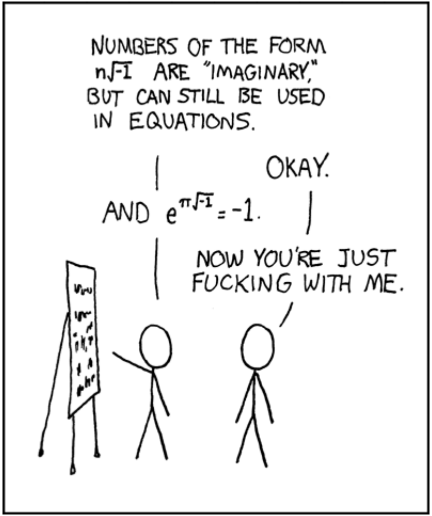
Theorem M.B.2: (Euler's Formula) Let $f(x_1, x_2, ..., x_N)$ be homogeneous of degree r, and differentiable. Then at any $(\bar{x}_1, ..., \bar{x}_N)$ we have:

$$\Sigma_{n=1}^{N} \frac{\partial f(\bar{x}_1, ..., \bar{x}_N)}{\partial x_n} \bar{x}_n = rf(\bar{x}_1, ..., \bar{x}_N)$$

In matrix notation: $\nabla f(\bar{x}) \cdot \bar{x} = rf(\bar{x}_1, ..., \bar{x}_N)$

What does this mean for a function that is homogeneous of degree 0?

Euler


NUMBERS OF THE FORM $n\sqrt{-1}$ ARE "IMAGINARY," BUT CAN STILL BE USED IN EQUATIONS.
OKAY.
AND $e^{\pi\sqrt{-1}} = -1$.
NOW YOU'RE JUST FUCKING WITH ME.

Convex Structures and Optimization

(Sundaram Chapter 7: read the intro!)

- Why do we study convex things?
- Turns out that when optimization problems meet certain convexity properties, first order conditions are not only necessary but they are sufficient.
- Whoa, what? Yes, even global optima. Yes, even unique.
- So, we study all sort of convex objects. Remember the initial questionnaire?
- convexity assumptions are pretty strong. Why are we ok with these?
- convex constraints and concave objective function imply FOC are sufficient (and necessary) for global maxima.
- convex constraints and convex objective function...

Concave and Convex Functions

Definition: Let $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$, and let D be a convex set. The subgraph of f and the epigraph of f are defined by:

$$subf = \{(x, y) \in Dx\mathbb{R} : f(x) \geq f(y)$$

$$epif = \{(x, y) \in Dx\mathbb{R} : f(x) \leq f(y)$$

A function f is said to be concave on D if *sub f* is a convex set. f is said to be convex on D if *epi f* is a convex set.

Concave and Convex Functions

Theorem 7.1 A function $f : D \to \mathbb{R}$ is concave on D if and only if for all x, y in D, and $\alpha \in (0,1)$, it is the case that:

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$$

How do we modify this theorem for convexity?

Can a function be concave and convex? can a function be neither? Take $f(x) = x^3$ in the reals, x=-2, y=2, $\alpha = 1/4$

How about strict convexity and concavity? take $f(x) = x^\lambda$

Strict Concavity, Strict Convexity

- A function f is (strictly convex) in D if and only if -f is (strictly concave) in D
- We will skip the rest of Sundaram Chapter 7 for now!
- Just know that convexity has important implications for: continuity, differentiability.
- now MWG

MWG on Concavity

We will consider functions of N variables, defined on a domain A that is convex, $A \subseteq \mathbb{R}^N$.//
We denote $x = (x_1, ..., x_N)$

- Definition: The function $f : A \to \mathbb{R}$ is concave if

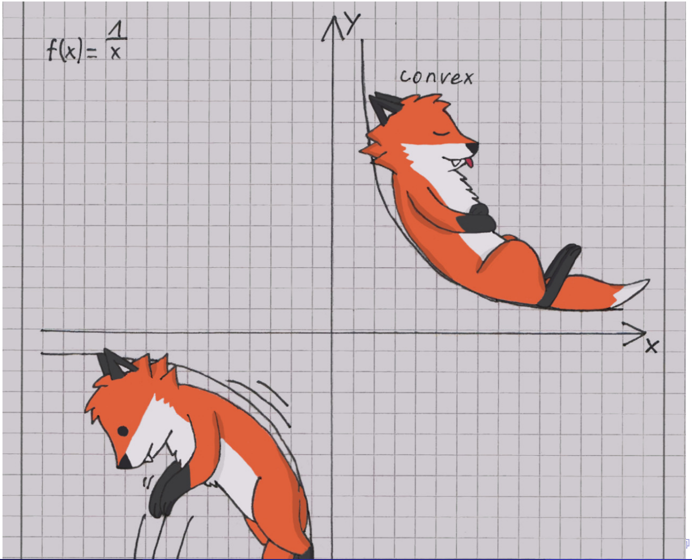
$$f(\alpha x' + (1-)x) \geq \alpha f(x')(1-\alpha)f(x)$$

  for all $x, x' \in A, \alpha \in [0,1]$. If the inequality is stric, we say that f is strictily concave.
- Examples on the board
- The straight line connecting two points lies entirely below the graph of a function (strict concavity)

Concavity

- Note that the previous condition is equivalent to $f(\alpha_1 x^1 + ...\alpha_K x^K) \geq \alpha_1 f(x^1) + ... + \alpha_k f(x^k)$ for any collection of vectors in A and numbers α_i that add up to one
- It is easier if we think of f(A) as a set.
- One example: (Jensen's Inequality) $f(\int xdF) \geq \int f(x)dF$ for any distribution function $F : \mathbb{R}[0,1]$
- The properties of convexity and strict convexity are defined analogously (inequality reversed)
- f is concave if and only if -f is convex

Concavity and Convexity



Alternative Characterization of Concavity

Let $f : A \to \mathbb{R}$ be continuously differentiable. It is concave if and only if

$$f(x+z) \leq f(x) + \nabla f(x) * z$$

For all $x \in A, z \in \mathbb{R}^N$ (with $x + z \in A$)

- Graph on the board MC3
- Any tangent to the graph must lie weakly above the graph

Aside: Matrices

The N x N matrix M is negative semidefinite if

$$z \cdot Mz \leq 0$$

$\forall z \in \mathbb{R}^N$. If the inequality is strict for all non-zero z, then the matrix M is negative definite. Reversing the inequality we get the concept of positive semidefinite and positive definite matrices. (We will go deeper into this later on!)

- Theorem: The (twice continuously differentiable) function $f : A \to \mathbb{R}$ is concave if and only if $D^2 f(x)$ is negative semidefinite $\forall x \in A$. If $D^2 f(x)$ is negative definite $\forall x \in A$, then the function is strictly concave.
- What does this mean for N=1?

Quasi-Convexity

Sundaram Chapter 8

- Convexity might be quite restrictive
- We can weaken the assumption by replacing it with another one: quasi-convexity
- Some trade-offs: not sufficient for global optima
- Some useful properties remain, especially for Khun-Tucker, which we will see later on.

Quasi-concave and Quasi-convex functions

Let $f : D \to \mathbb{R}$, D a convex subset of $\mathbb{R}^n$. The upper contour set of f at $a \in \mathbb{R}$, denoted $Uf(a)$ is defined as:

$$Uf(a) = \{x \in D : f(x) \geq a\}$$

while the lower contour set, denoted $Lf(a)$ is defined as:

$$Lf(a) = \{x \in D : f(x) \leq a\}$$

- The function f is said to be quasiconcave on D if $Uf(a)$ is a convex set for all a
- The function f is said to be quasiconvex on D if $Lf(A)$ is a convex set for all a

Quasi-concave Functions

Theorem 8.1: $f : D \to \mathbb{R}$ is quasi-concave on D if and only if for all x, y elements of D, $\lambda \in (0, 1)$, it is the case that:

$$f(\lambda x + (1 - \lambda)y) \geq min\{f(x), f(y)\}$$

Theorem 8.2: The function $f : D \to \mathbb{R}$ is (strictly) quasi-concave on D if and only if -f is (strictly) quasi-convex in D.

Quasi-convexity as a Generalization of Convexity

From Theorem 8.1 it is easy to see that all concave functions are also quasi-concave (why?, prove it)

- The converse is not true, take $f(x) = x^3$
- Theorem 8.3: Any monotone transformation of a concave function results in a quasi-concave function.
- Note that:
  1. quasi-concave and quasi-convex functions need not be continuous in the interior of their domains
  2. They can have local optima that are not global optima
  3. FOC are not sufficient to identify even local optima under quasi-convexity

  Example: page 209 of Sundaram.

Quasi-convex and Quasi-concave Functions

Theorem 8.8: let $f : D \to \mathbb{R}$ be a C^1 function, where $D \subset \mathbb{R}^n$ is convex and open. Then f is a quasi-concave function on D if and only if it is the case that for any $x, y \in D$,

$$f(y) \geq f(x) \Rightarrow Df(x)(y - x) \geq 0$$

Theorem 8.9: let $f : D \to \mathbb{R}$ be a C^2 function, where $D \subset \mathbb{R}^n$ is convex and open. Then:

1. if f is quasi-concave on D, we have $(-1)^k|C_k(x)| \geq 0$ for k=1, ..., n
2. if $(-1)^k|C_k(x)| > 0$ for all $k \in \{1, 2, ..., m\}$ then f is quasi-concave on D.

Sundaram Exercises:

From Chapter 7: 1-13, Chapter 8: 1-10

MWG on Quasi-concavity

Definition: The function $f : A \to \mathbb{R}$, defined on the convex set $A \subseteq \mathbb{R}^N$ is quasiconcave if its upper contour sets $\{x \in A : f(x) \geq t\}$ are convex sets, that is if $f(x) \geq t$ and $f(x') \geq t$ implies that $f(\alpha x + (1-\alpha)x') \geq t$ for any real number t, $x, x' \in A$, $\alpha \in [0, 1]$

Analogously, we say that a function is quasiconvex if its lower contour sets are convex

Note that f is quasiconcave IFF -f is quasiconvex

Level sets

- Level sets of a quasiconcave function: what do they look like?

Level sets

- Level sets of a quasiconcave function: what do they look like?
- Ok, what are level sets?
- Figure M.C.4

Ok, now see that from the definition of quasiconcavity it follows that all concave functions are quasiconcave. The converse is not true: concavity is a stronger property than quasiconcavity (what does stronger mean?)

Quasiconcavity

Theorem M.C.3: The (continuously differentiable) function $f : A \rightarrow \mathbb{R}$ is quasiconcave IFF

$$\nabla f(x) \cdot (x' - x) \geq 0$$

whenever $f(x') \geq f(x)$, $\forall x, x' \in A$. The strict inequality depicts strict quasiconcavity

Theorem M.C.4 Theorem: The (twice continuously differentiable) function $f : A \rightarrow \mathbb{R}$ is quasiconcave if and only if $\forall x \in A$, the Hessian matrix $D^2 f(x)$ is negative semidefinite in the subspace $\{z \in \mathbb{R}^N : \nabla f(x) \cdot z = 0\}$, that is if and only if $z \cdot D^2 f(x) z \leq 0$ whenever $\nabla f(x) \cdot z = 0$

- Definition 5.1.1: Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}, c \in A$ Then f is continuous at c if given any $\epsilon > 0$, exists $\delta > 0$ such that $\forall x \in A$ that satisfies $|x - c| < \delta$ it is the case that $|f(x) - f(c)| < \epsilon$

Continuity (B&S)

- Definition 5.1.1: Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}, c \in A$ Then f is continuous at c if given any $\epsilon > 0$, exists $\delta > 0$ such that $\forall x \in A$ that satisfies $|x - c| < \delta$ it is the case that $|f(x) - f(c)| < \epsilon$
- Reminds us of the definition of limit (which we will see later!)
- f is continuous in a set A if it is continuous at all points of that set.
- adding, multiplying, compositions of continuous functions remain continuous.

Theorem of the maximum-minimum. if $f : I \to \mathbb{R}$, where I is closed and bounded (compact) and f is continuous, then f(x) has a maximum and a minimum in I

Theorem of the Intermediate Value (Bolzano

Let I be an interval and $f : I \to \mathbb{R}$ continuous in I. If $a, b \in I$, $k \in \mathbb{R}$ satisfies that $f(a) < k < f(b)$ then there is $c \in I$ between a and b, such that $f(c) = k$

Continuity: Sequence Criterion

Definition M.F.1: The sequence $\{x_m\}$converges to $x \in \mathbb{R}^N$, written as $lim_{m \to \infty} x^m = x$ or $x_m \to x$, if for every $\epsilon > 0$ there is an integer $M_\epsilon$such that $||x^m - x|| < \epsilon$ whenever $n > M_\epsilon$

Definition M.F.2: consider a domain $X \subset \mathbb{R}^N$. A function $f : X \to \mathbb{R}$ is continuous if $\in X$, and every sequence $x^m \to x$ in X, we have that $f(x^m)(x)$ Examples

Theorem M.F.2: Suppose that $f : X \to \mathbb{R}^k$ defined in a nonempty subset of $\mathbb{R}^N$ is continuous, then:

1. The image of a compact set under f(x) is compact
2. Let k=1 and X be compact, then f(x) has a maximizer.$\in X : f(x) \geq f(x') \forall x' \in X$

Upper-hemicontinuity

Definition M.H.1: Given a set $A \subset \mathbb{R}^N$, a correspondence $f : A \to \mathbb{R}^k$ is a rule that assigns a set $f(x) \subset \mathbb{R}^k$ to every x in A

Definition M.H.2: Given a set $A \subset \mathbb{R}^N$ and the closed set $Y \subset \mathbb{R}^k$, the correspondence $f : A \to Y$ has a closed graph if for any two sequences $x^m \to x \in A, y^m \to y$, with $x^m \in A$ and $y^m \in f(x^m) \forall m$, we have $y \in f(x)$

$\to$ Note that this is the definition of closedness relative to AxY applied to the set $\{(x, y) \in AxY : y \in f(x)\}$

$\to$ Why don't we have an equivalent definition for functions?

Upper-hemicontinuity

Definition M.H.3: Given a set $A \subset \mathbb{R}^N$, and the closed set $Y \subset \mathbb{R}^k$, the correspondance $f : A \rightarrow Y$ is upper-hemicontinuous (uhc) if it has a closed graph and the images of compact sets are bounded, that is for every compact set $B \subset A$ the set
$f(B) = \{y \in Y : y \in f(x) for some x \in B\}$
In many cases, the range space Y is already compact, so we only need the closed graph condition.

Fixed Point Theorems

Theorem M.I.1 (Brouwer's Fixed Point Theorem) Let $A \subset \mathbb{R}^N$ be nonempty, compact and convex, $f : a \to A$. is a continuous function from A onto itself. Then f(.) has a fixed point, that is $\exists x \in A$ such that $f(x) = x$

Theorem M.I.2 (Kakutani's Fixed Point Theorem). Theorem M.I.1 (Brouwer's Fixed Point Theorem) Let $A \subset \mathbb{R}^N$ be nonempty, compact and convex, $f : a \to A$. is an upper-hemicontinuous correspondence from A onto itself such that the set $f(x) \subseteq A$ is nonempty and convex for every x in A. Then f has a fixed point

What is the difference between these two?

Suggested Exercises

Sundaram: Chapter 1: 46, 47, 48, 49, 51, 53, 54 Bartle and Sherbert Chapter 5.1: 1, 3, 6, 10, 11 Chapter 5.2: 2, 3, 8