Probing Topological Phase Transitions with String-net Models and Their Tensor Network Representations

Chow Yin Hei
A presentation for course PHYS3999 Directed studies in physics
Project supervised by Prof. Chenjie Wang

Agenda

- Introduce topological orders, and why people study them
- Mathematical framework – monoidal category
- Method – String-net model & PEPS
- Elaborate on the method
- Numerical results
- Future directions

Topological orders

- **Anyons.** Topological orders in 2+1d are characterized by their anyon data in terms the modular matrices describing their topological spins (self statistics) and mutual statistics.

- **Microscopic Origin.** Topologically ordered phases arise due to the presence of long-range entanglements

- These long-range entanglements provide topological protection to the anyons, in that the anyons are robust against any local perturbations (or finite compositions of piece-wise local unitary transformations to be precise)

- **Application to Quantum Computation.** Therefore, anyon models attract ample of attention from the field of quantum computation ever since topological protection has intimated a clear path towards realizing quantum memory and universal quantum computation that are stable against any local noises from the environment - perform computations with anyons in a gapped system.

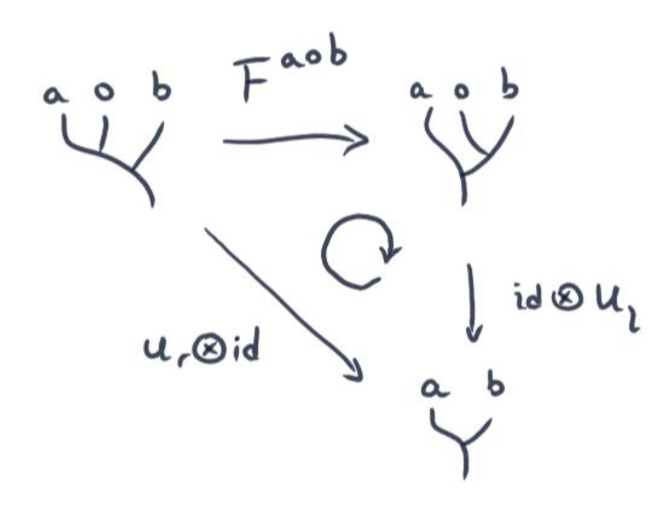
Mathematical framework – monoidal category

- 2+1d topological orders are described by UMTC, some of which are Drinfeld centers of UFC
- **Fusion.** The fusion category is a vector-enriched monoidal category, where the monoidal product describes fusion. (Collective behavior of two quasiparticles).

$$a \times b = \sum_c N_{ab}^c c$$

- **Associators.** A naturally isomorphism measuring how badly associativity is violated.
- **Braiding.** Hexagon relation on three objects

$$\Phi : X \otimes (-) \to (-) \otimes X$$



$$F : ((\cdot) \times (\cdot)) \times (\cdot) \xmapsto{\cong} (\cdot) \times ((\cdot) \times (\cdot))$$

$$F^{fcd}_{egl} F^{abl}_{efk} = \sum_h F^{abc}_{gfh} F^{ahd}_{egk} F^{bcd}_{khl}$$

String-net model – fixed point Hamiltonian

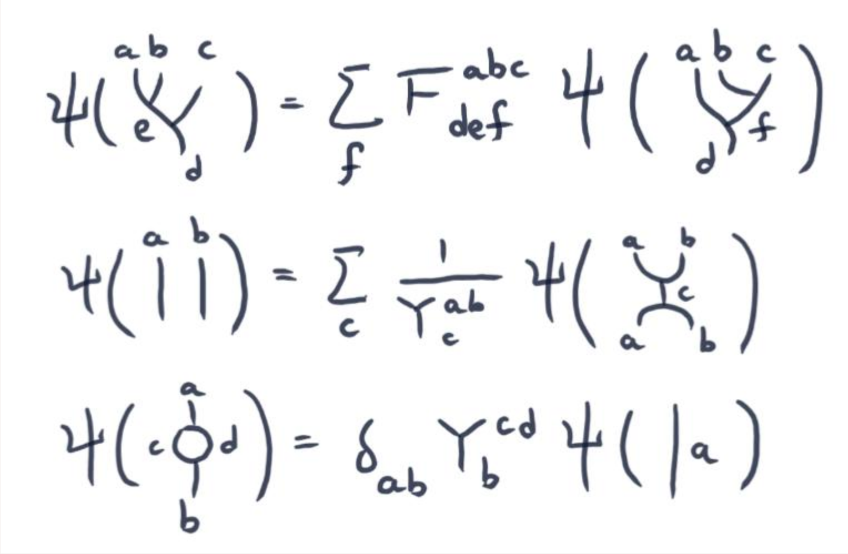
- **Hilbert Space.** A string-net configuration is a trivalent graph defined on some time-orientable surface. Links are labelled by string-types in the UFC, and the fusion rules of these string type are satisfied on each vertex. These configurations are motivated by worldlines on 1+1d. Define Hilbert space as their linear span and define inner product so that string-net configurations are orthonormal.

- **Ground-state.** Determined by local rules. Coherence relations.

$$|GS\rangle = \sum_{s.n.} \psi(s.n.)|s.n.\rangle$$

- **Parent Hamiltonian.** Commuting projectors / stabilizers

$$H = -J_Q \sum_{v\in\Lambda} Q_v - J_B \sum_{p\in\Lambda^*} B_p$$



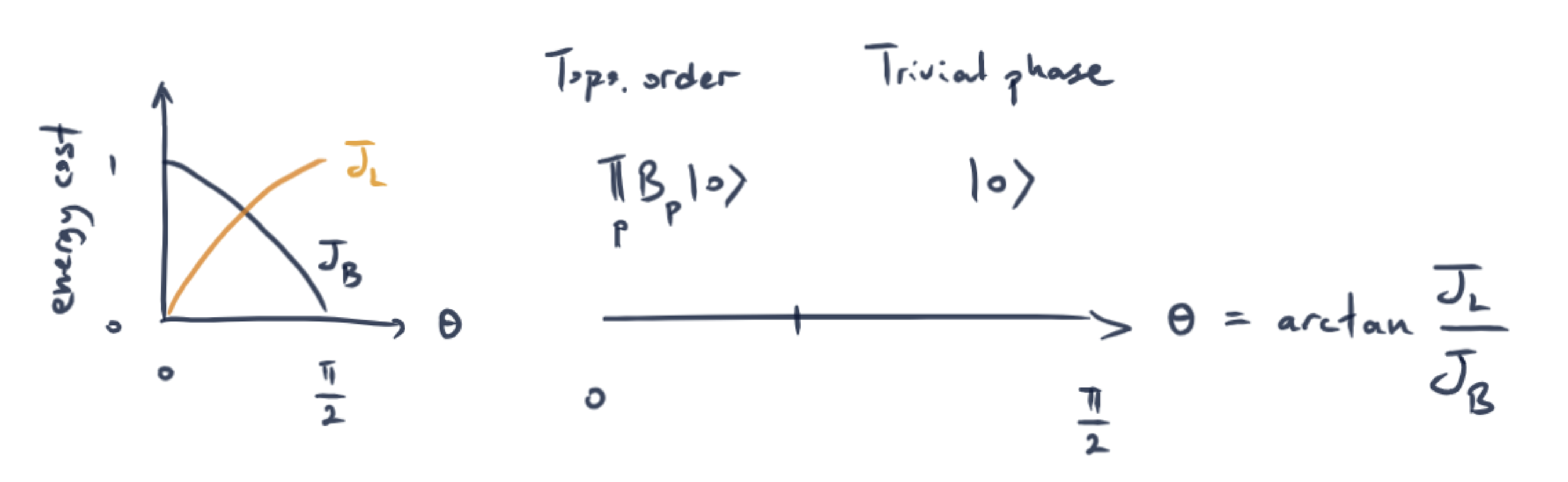
$$\psi(0) = 1$$

Charge-free string-net model with string-tension

- Vertex operators are local symmetries of the ground-state. Enforce all of them as gauge constraints

$$H = -J_Q \sum_{v\in\Lambda} (Q_v - 1) - J_B \sum_{p\in\Lambda^*} B_p \; - J_l \sum_{l} L_l$$

- Introduce string-tension. It is a non-commuting term with flux operator
- **Question becomes:** Is the phase transition continuous or discontinuous?

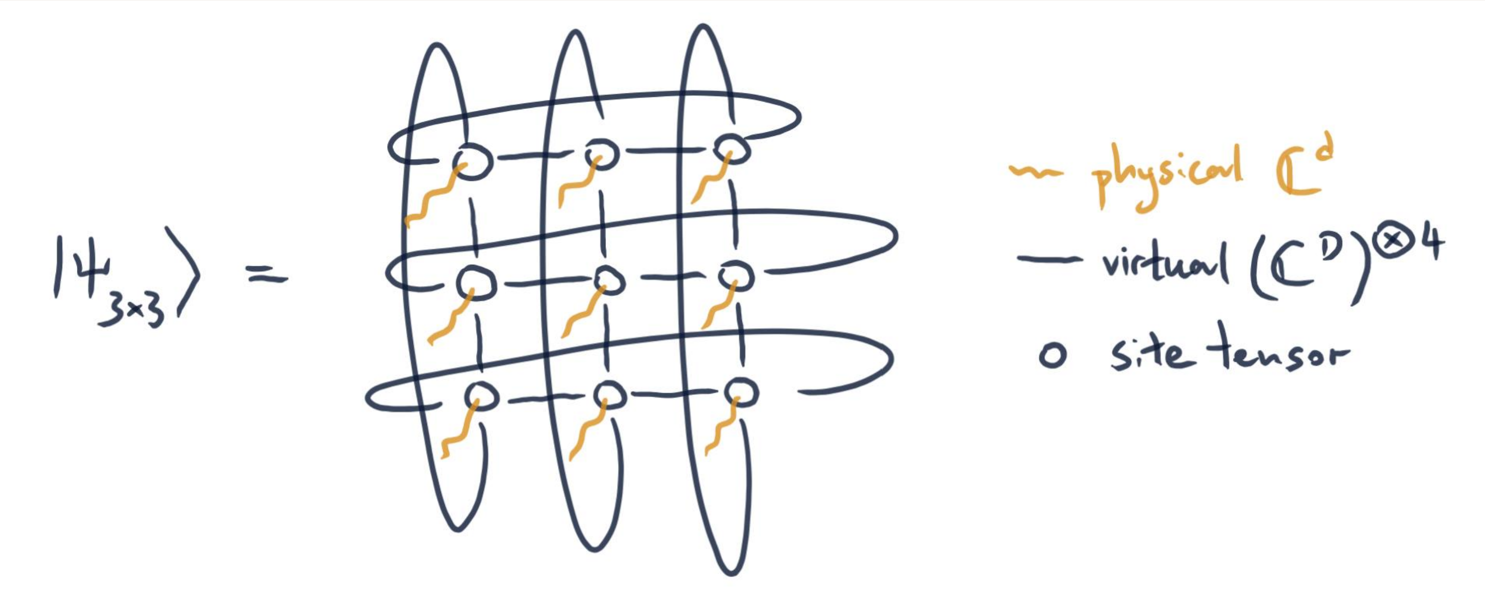


Projected entangled pair states (PEPS)

- Projected entangled pair state (PEPS) extends the matrix product state (MPS) into two-dimension.

- It is a decomposition of the wavefunction into a tensor contraction of site tensors. The tensor network is usually taken to reflects the geometry of the underlying lattice, so the bond dimensions can be interpreted as entanglement degrees of freedom. Site tensors are then linear maps from the virtual level to the physical level, or the local Hilbert space of site. For example, on a square lattice they read

$$T_i : (\mathbb{C}^D)^{\otimes 4} \to \mathbb{C}^d$$



Ground-state ansatz & its theoretical properties

- **First-order Ansatz.** Non-parametric / general tensor network method fails because of the 12-site interaction.

$$|\alpha, \beta, \eta\rangle := \mathcal{N} \prod_l e^{\beta L_l} \prod_p e^{-\eta B_p} \prod_p (1 + \alpha Z_p)|0\rangle$$

- When α=1, β=0, η=0, we recover the exactly soluble wavefunction $Z_p = 2B_p - 1$
- When α=0, β=0, η=0, we get the vacuum state
- When β=0, η=0, we get a mean-field ansatz. No flux interactions. $\langle\alpha| \prod_p B_p |\alpha\rangle = \prod_p \langle\alpha|B_p|\alpha\rangle$
- Parameter α is not simply a linear interpolation. Inside the topological order iff α=1. Topological order parameter is non-zero iff α≠1.

$$\prod_p B_p|GS\rangle = \prod_p B_p \prod_p B_p|0\rangle = \prod_p B_p|0\rangle = |GS\rangle$$

$$\langle 0| \prod_p B_p |0\rangle = \langle 0|GS\rangle = 0$$

Improvement upon existing works

- **Existing works.** a. Exact diagonalization shows continuous phase transition (by suffer from finite-size effect). b. Mean-field ansatz shows discontinuous phase transition (by not exact). c. A work tried to work with a simplified version of the above ansatz but only reproduce the result in the mean-field ansatz.

- **Problem.** This term is dropped without proper justification $B = \prod_p e^{-\eta B_p}$
But the idea of using tensor network is worth pursuing.

- **Solution.** The term can be incorporated as a rescaling of quantum dimensions

$$\mathcal{N} \prod_l e^{\beta L_l} \prod_p e^{-\eta B_p} \prod_p (1 + \alpha Z_p) |0\rangle \; = \widetilde{\mathcal{N}} \prod_l e^{\beta L_l} \prod_p \widetilde{B_p} |0\rangle$$

$$\gamma := \frac{A(\alpha, \eta)}{(1-\alpha) D^2 + A(\alpha, \eta)}$$

$$\widetilde{B_p} := \sum_s \frac{\widetilde{d}_s}{D^2} B_p^s \text{ , with } \widetilde{d}_s := \begin{cases} \gamma d_s \text{ , } & s \neq 0 \\ d_s \text{ , } & s = 0 \end{cases}$$

Proof

$$A(\alpha,\eta) \coloneqq (1+\alpha)e^{-\eta} + \alpha - 1$$

$$\gamma \coloneqq \frac{A(\alpha,\eta)}{(1-\alpha)D^2 + A(\alpha,\eta)}$$

$$\widetilde{\mathcal{N}} \coloneqq \frac{\mathcal{N}}{\big[(1-\alpha)D^2 + A(\alpha,\eta)\big]^{N_{\text{plaq}}}}$$

$$\widetilde{B_p} \coloneqq \sum_s \frac{\widetilde{d}_s}{D^2} B_p^s \text{ , with } \widetilde{d}_s \coloneqq \begin{cases} \gamma d_s \ , \ s \neq 0 \\ d_s \ , \ s = 0 \end{cases}$$

$$\begin{aligned}
|\alpha,\beta,\eta\rangle &= \mathcal{N} \prod_l e^{\beta L_l} \prod_p e^{-\eta B_p} \prod_p (1+\alpha Z_p)|0\rangle \\
&= \mathcal{N} \prod_l e^{\beta L_l} \prod_p \left[1 + (-\eta + \frac{1}{2}\eta^2 - \ldots)B_p\right] \prod_p (1+\alpha Z_p)|0\rangle \\
&= \mathcal{N} \prod_l e^{\beta L_l} \prod_p \left[1 + (e^{-\eta} - 1)B_p\right] \prod_p (1-\alpha+2\alpha B_p)|0\rangle \\
&= \mathcal{N} \prod_l e^{\beta L_l} \prod_p \left\{1 - \alpha + \left[(1+\alpha)e^{-\eta} + \alpha - 1\right] B_p\right\}|0\rangle \\
&= \mathcal{N} \frac{1}{D^{2N_{\text{plaq}}}} \prod_l e^{\beta L_l} \prod_p \left[(1-\alpha)D^2 + A(\alpha,\eta) \sum_s d_s B_p^s\right]|0\rangle \\
&= \frac{1}{D^{2N_{\text{plaq}}}} \frac{\mathcal{N}}{\big[(1-\alpha)D^2 + A(\alpha,\eta)\big]^{N_{\text{plaq}}}} \prod_l e^{\beta L_l} \prod_p (1 + \gamma \sum_{s\neq 0} d_s B_p^s)|0\rangle \\
&= \widetilde{\mathcal{N}} \prod_l e^{\beta L_l} \prod_p \widetilde{B_p}|0\rangle
\end{aligned}$$

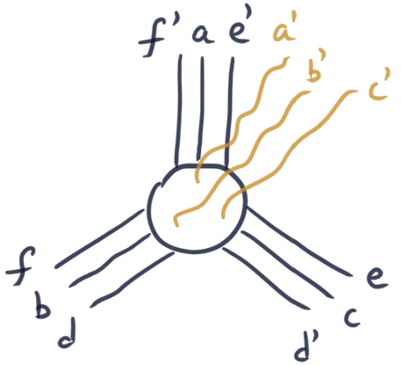
Tensor network representation of ground-state

- **Tensor network decomposition.** Start with the ansatz deduced, we decompose it by applying the associator, splitting any phases and normalizing factors even among vertices.

$$|\alpha,\beta,\eta\rangle = \tilde{N} \prod_l e^{\beta L_l} \prod_p \tilde{B}_p |0\rangle$$

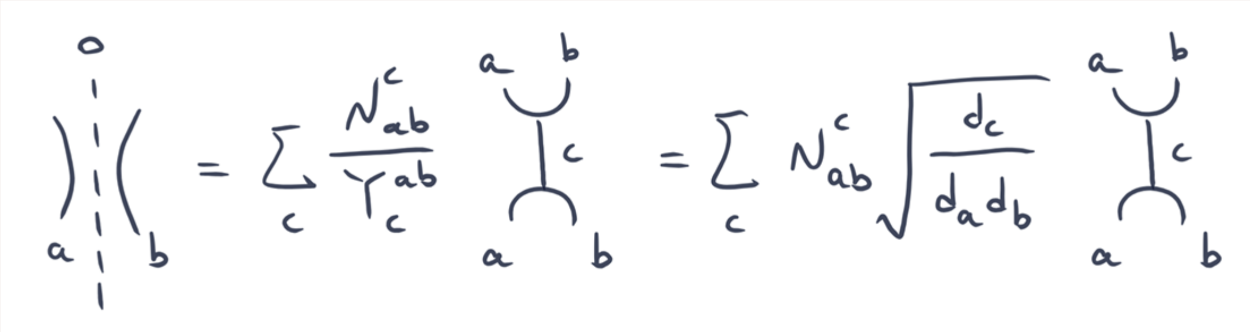
$$= \frac{\tilde{N}}{D^{2N_{plaq}}} \prod_l e^{\beta L_l} \sum_\mu \prod_p \tilde{d}_{\mu_p} \Big| \quad \Big\rangle_\mu$$

$$T_i : (\mathbb{C}^D)^{\otimes 4} \to \mathbb{C}^d$$

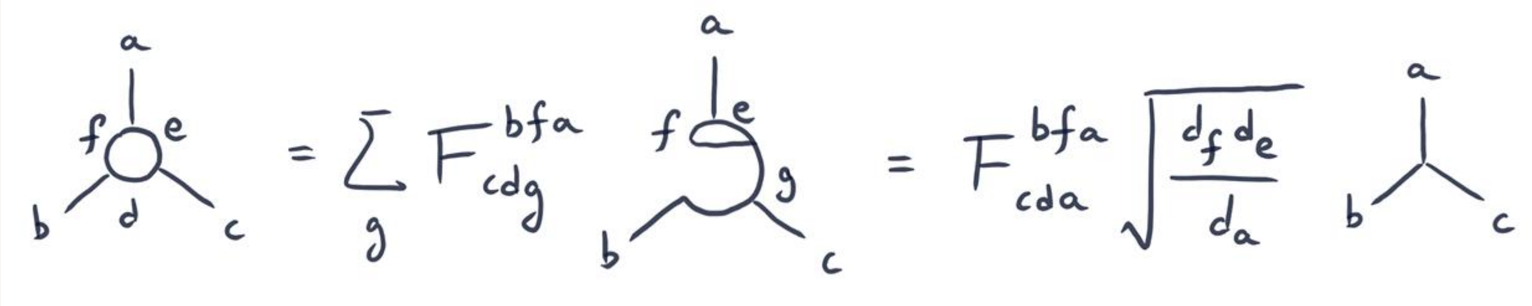


$$= \delta_{aa'}\delta_{bb'}\delta_{cc'}\delta_{dd'}\delta_{ee'}\delta_{ff'}(\tilde{d}_d\tilde{d}_e\tilde{d}_f)^{\frac{1}{6}}(d_a d_b d_c)^{\frac{1}{4}} G^{abc}_{def}$$

- **Link contribution.** Absorb a square root of the normalizing factor incurred into neighboring vertices



- **Vertex contribution.** Get rid of two-third of the sites (This can be interpreted as an RG move)



$$G_{def}^{abc} \cdot \sqrt{d_f d_e d_d}$$

$$G_{def}^{abc} = F_{cda}^{bfa} \cdot \frac{1}{\sqrt{d_a d_d}}$$

- **Plaquette contribution.** In the summation, split the pre-factor rescaled quantum dimension among six vertices touching the plaquette. Each vertex receives this from three plaquettes.

$$(\widetilde{d}_d \widetilde{d}_e \widetilde{d}_f)^{\frac{1}{6}}$$

Action of Hamiltonian on entanglement level

$$H_p = -\frac{sin\theta}{2}\sum_{l\in\partial p} L_l - cos\theta B_p$$

- **Problem.** The Hamiltonian consist of a 12-site interaction and writing it as a matrix acting on the 12-site local Hilbert space require too much storage space to be computed
- **Solution.** Pull back the Bp operator through the PEPS-map onto the virtual level
- Decompose it into a matrix product operator (MPO) acting on the entanglement degrees of freedom. In general, this is possible when the PEPS constructed is an injective linear map
- We find that the plaquette operator Bp acts on the PEPS simply by scalar multiplication, and this scalar depends not on the physical indices, but on a virtual index.

$$\Phi(\mu_p) = \frac{1}{D^2}\sum_{ab} N_{ab}^{\mu_q}\frac{d_a\widetilde{d}_b}{\widetilde{d}_{\mu_q}}$$

Proof

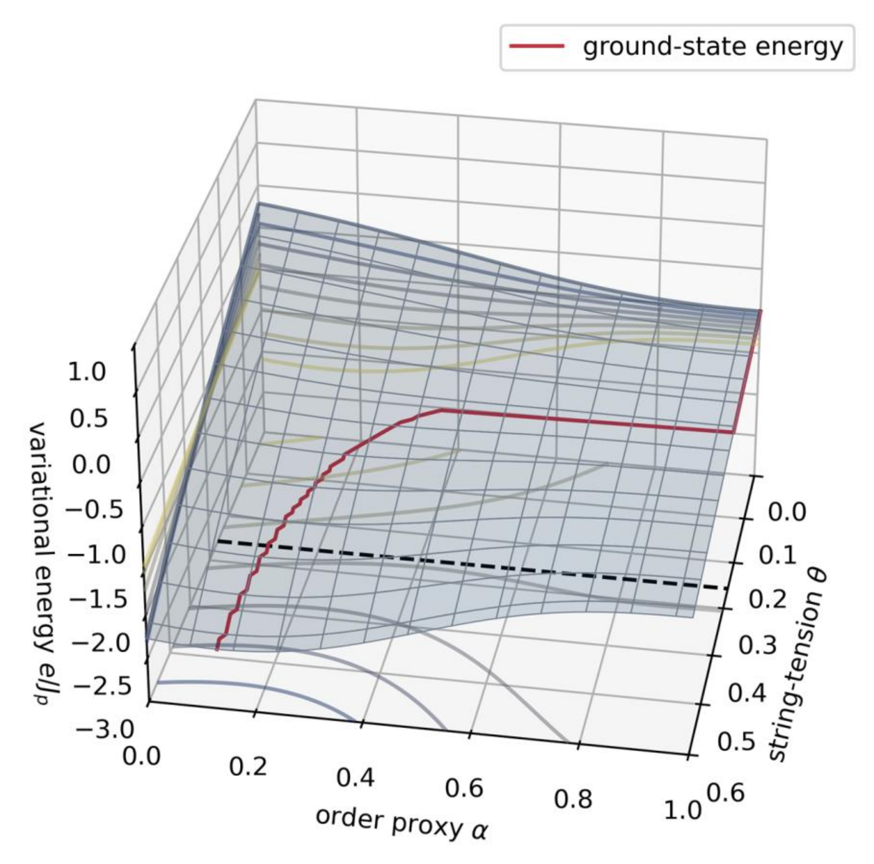
- Apply the fusion algebra on loops, then swap the domain codomain virtual indices

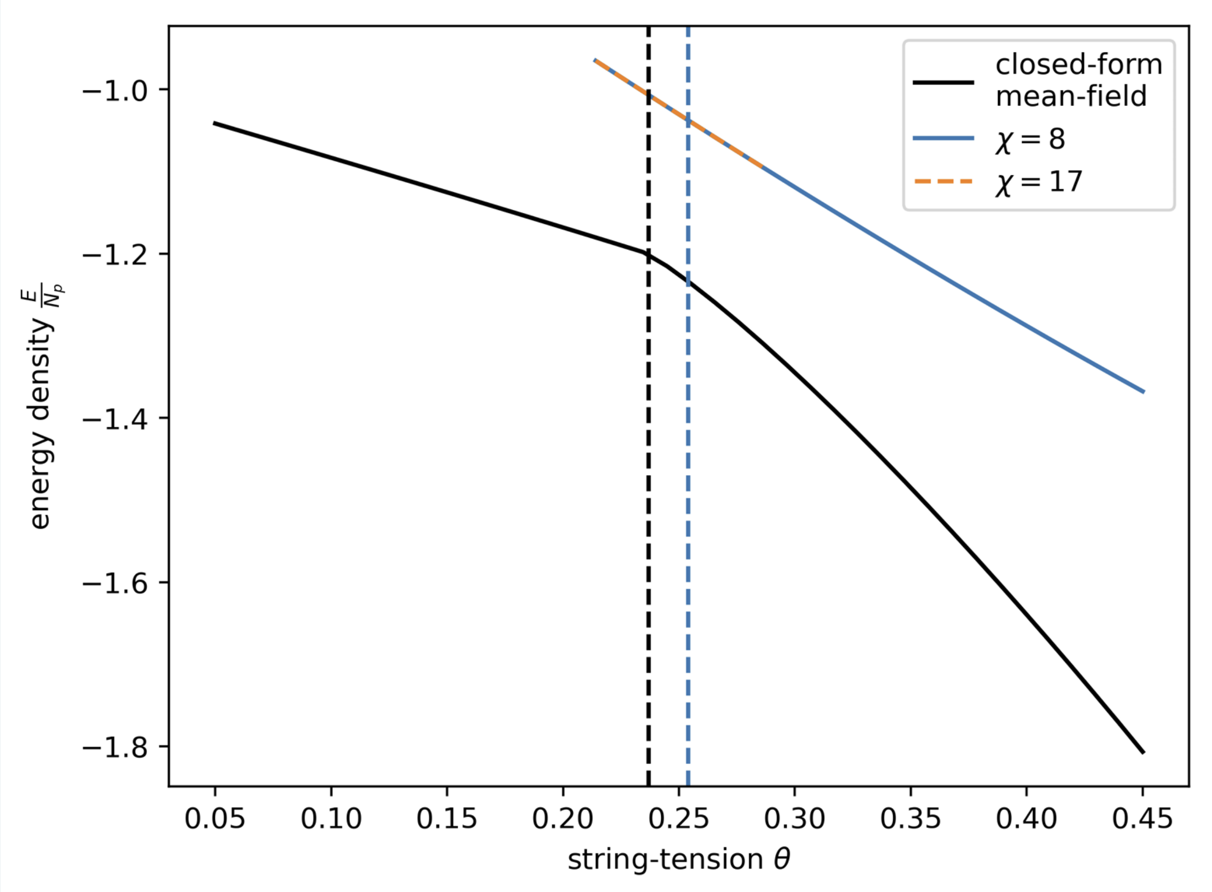
For the plaquette operator B_q, swapping out the virtual index μ_p gives

$$
\begin{aligned}
B_q|\alpha,\beta,\eta\rangle &= \sum_s \frac{d_q}{D^2} B_q^s \cdot \widetilde{N} \prod_l e^{\beta L_l} \sum_\mu \prod_p \frac{\widetilde{d}_{\mu_p}}{D^2} B_p^{\mu_p}|0\rangle \\
&= \sum_{s\mu} \frac{d_s}{D^2} \cdot \widetilde{N} \prod_l e^{\beta L_l} \left(\prod_p \frac{\widetilde{d}_{\mu_p}}{D^2}\right)\left(\prod_{p\neq q} B_p^{\mu_p}\right)\left(\sum_{\widetilde{\mu}_q} N_{s\mu_q}^{\widetilde{\mu}_q} B_q^{\widetilde{\mu}_q}\right)|0\rangle \text{ , by fusion algebra,} \\
&= \sum_{s\mu\widetilde{\mu}_q} \frac{1}{D^2} \frac{d_s \widetilde{d}_{\widetilde{\mu}_q}}{\widetilde{d}_{\mu_q}} N_{s\widetilde{\mu}_q}^{\mu_q} \cdot \widetilde{N} \prod_l e^{\beta L_l} \prod_p \frac{\widetilde{d}_{\mu_p}}{D^2} B_p^{\mu_p}|0\rangle \quad \text{, by swapping } \mu_q \leftrightarrow \widetilde{\mu}_q \text{ ,} \\
&= \frac{1}{D^2} \sum_{ab} N_{ab}^{\mu_q} \frac{d_a \widetilde{d}_b}{\widetilde{d}_{\mu_q}} |\alpha,\beta,\eta\rangle
\end{aligned}
$$

Result & discussion

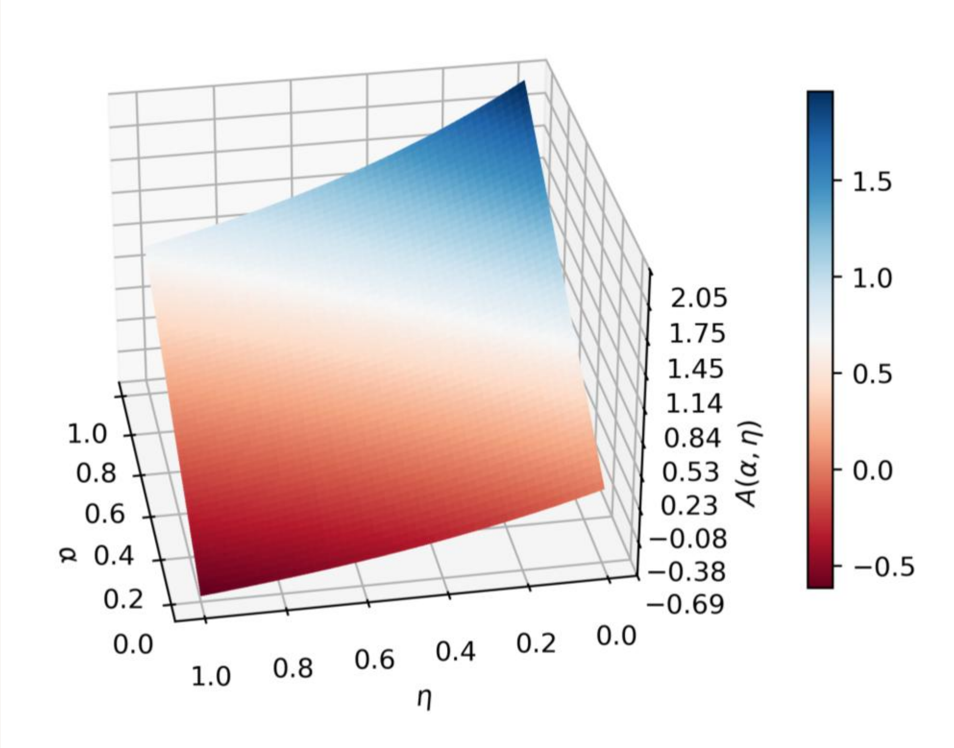
- **Model validation fails.** Fix β=0, η=0, then there are analytical results for benchmarking the convergence of the tensor network algorithm. Failure to reproduce analytic results.

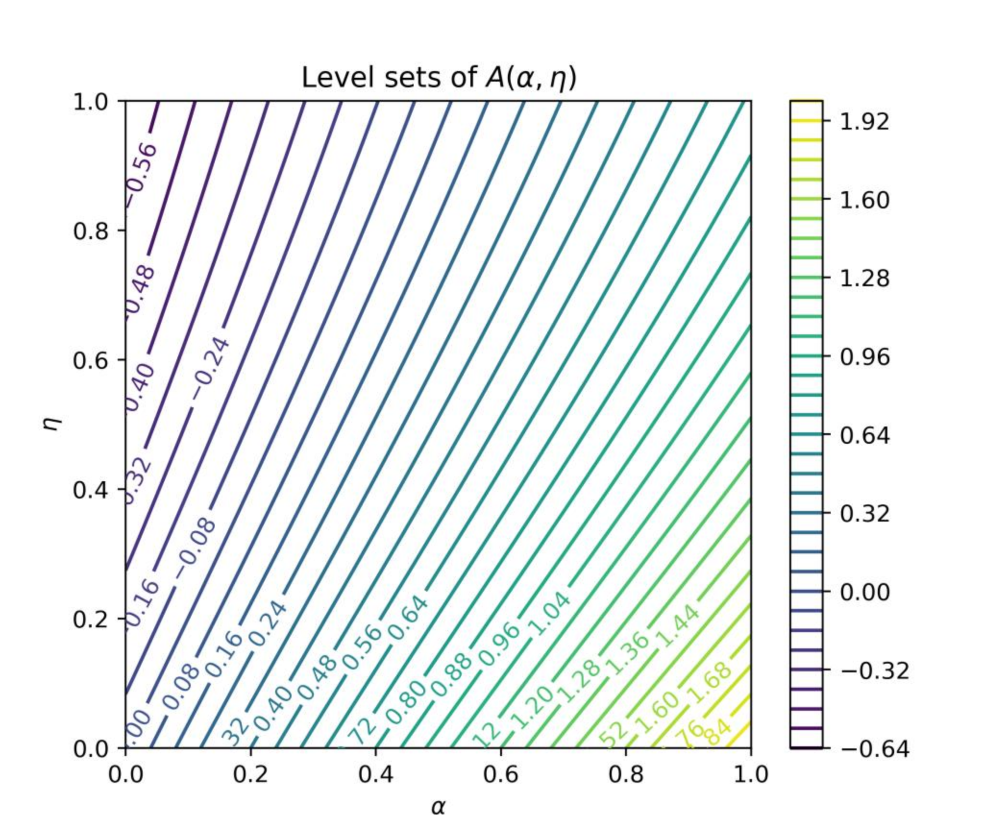




First-order correction

Interesting, we know the qualitative difference that the first-order correction will bring to the prediction of α. The phase transition point will occur at a higher string-tension, but whether this postponement will close the discontinuity cannot be known without the numerics.





Future directions

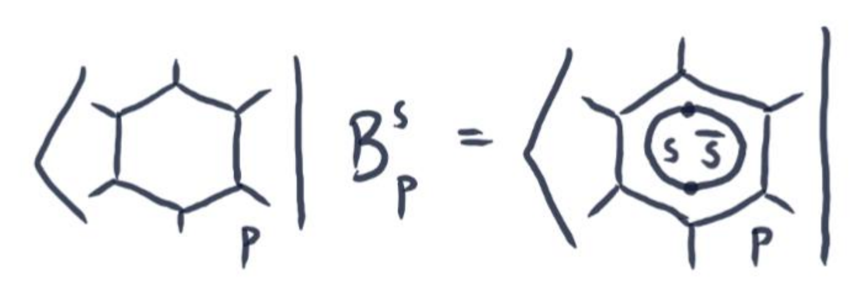
- **Bond dimension.** Increase environment bond dimension in the Fibonacci fusion rules until the benchmark shows convergence. Then add first-order corrections.
- **Models without an extended interaction term.** Decomposing the Hamiltonian into MPOs will not help. As shown in results, a run on TY3 with minimal environment bond dimension give memory error. This numerical scheme that theoretically works for all string-net model is in practice already pushed to its limit at 4 string-types.
- **Automatic differentiation.** Instead of the current grid search method over the variational parameters α, β, η, see whether the variational energy can be written as a directed acyclic graph (DAG) to generate gradients. [see vari-PEPS and differentiable programming tensor networks].Note that power method cannot be used.

Conclusion

- Topological orders, their significance to quantum computations, and their description using monoidal category
- Introduced the string-net model with string-tension as general tool for studying large class of topological phase transitions in 2+1d
- Improved upon existing works to obtain a tensor network ansatz that incorporates all first-order correction without increasing computational complexity
- Showed some results and suggested three future research directions

Extra slides for Q&A

$$B_p = \sum_s \frac{(Y_0^{s\bar{s}})^*}{D^2} B_p^s$$



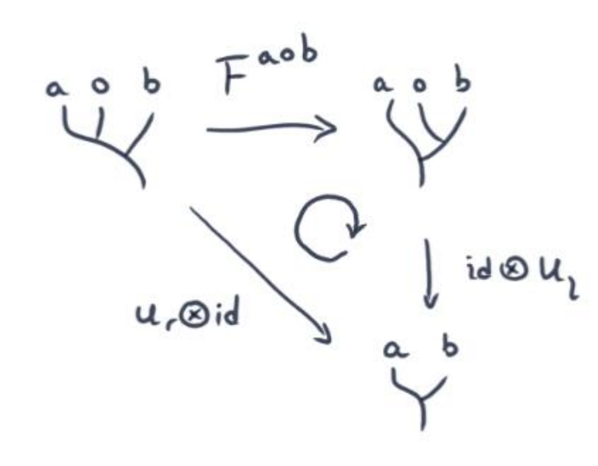
Parent Hamiltonian further information

Tambara-Yamagami category for $\mathbb{Z}_3$ is a $\mathbb{Z}_2$-graded fusion category defined by the set of simple objects $D = \mathbb{Z}_3 \sqcup \{\sigma\}$ with Z_3 corresponding to the identity element in the group grading. On top of the usual rules within $\mathbb{Z}_3$ and the grading structure, the non-invertible object σ has the following fusion data: $\forall a, b \in \mathbb{Z}_3$,

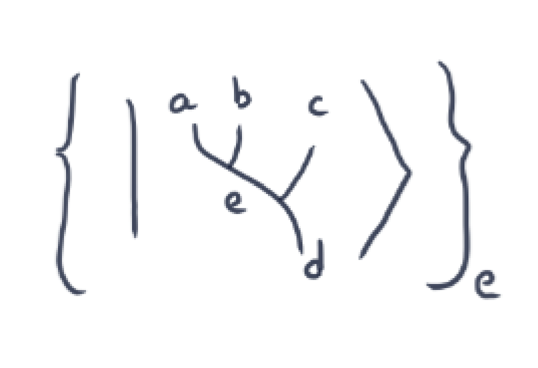
$$F^{a\sigma b}_{\sigma\sigma\sigma} = F^{\sigma a \sigma}_{b\sigma\sigma} = \chi(a,b)$$
$$F^{\sigma\sigma\sigma}_{\sigma ab} = \frac{p}{\sqrt{3}} \chi^*(a,b)$$
$$\sigma \times \sigma = \bar{0} + \bar{1} + \bar{2}$$
$$Y^{\sigma\sigma}_{a} = d_\sigma$$

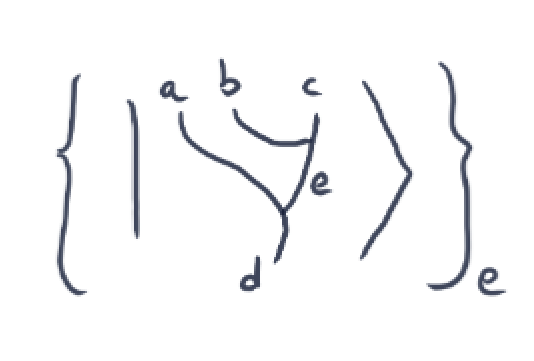
and other F, Y symbols, if defined, are all ones. $\chi(a,b) = e^{i2\pi ab/3}$ for the cyclic group. One also obtains $a \times \sigma = \sigma$ $\forall a \in \mathbb{Z}_3$ purely from the grading structure.

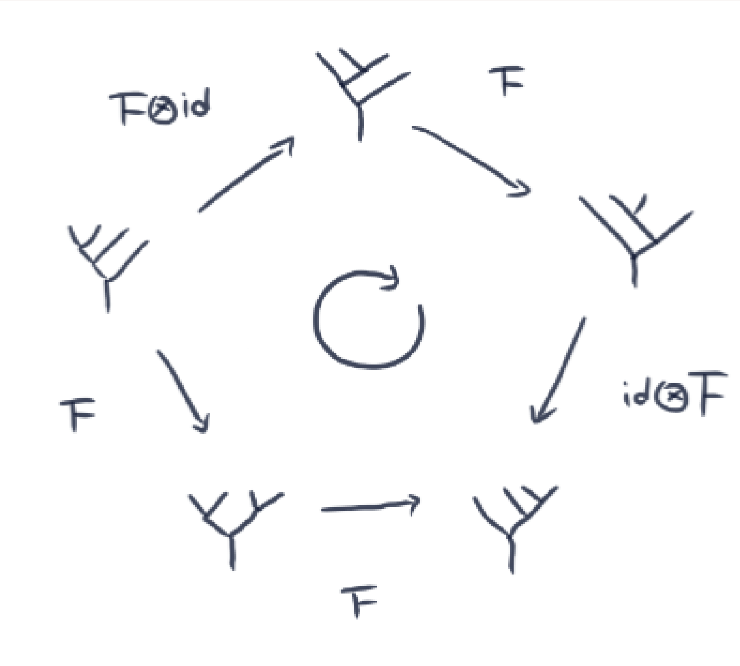
Further information



$$F : ((\cdot) \times (\cdot)) \times (\cdot) \overset{\cong}{\longmapsto} (\cdot) \times ((\cdot) \times (\cdot))$$







Definition 2.3. Let $(\mathcal{C}, \otimes)$ be a monoidal category. Its ***Drinfeld center*** is a monoidal category $Z(\mathcal{C})$ whose

- objects are pairs (X, Φ) of an object $X \in \mathcal{C}$ and a natural isomorphism (braiding morphism)

$$\Phi : X \otimes (-) \to (-) \otimes X$$

  such that for all $Y \in \mathcal{C}$ we have

$$\Phi_{Y \otimes Z} = (\mathrm{id} \otimes \Phi_Z) \circ (\Phi_Y \otimes \mathrm{id})$$

- morphisms are given by

$$\mathrm{Hom}((X,\Phi),(Y,\Psi)) = \left\{ f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \mid (\mathrm{id} \otimes f) \circ \Phi_Z = \Psi_Z \circ (f \otimes \mathrm{id}), \forall Z \in \mathcal{C} \right\}.$$

- the tensor product is given by

$$(X,\Phi) \otimes (Y,\Psi) = (X \otimes Y, (\Phi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Psi)).$$

Source: https://ncatlab.org/nlab/show/Drinfeld+center