Universal Enveloping Algebra for Yang-Baxter equations

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Abstract

A handout for my seminar on Oct 8. The aim of this seminar is to define universal enveloping algebras (UEA) and other foundational knowledge needed for understanding the quantization of classical Yang-Baxter equation, Hopf algebras and integrable systems over the few subsequent meetings. We will establish the uniqueness and existence of the UEA, which together with its universal property, makes the extension $g \to U(g)$ a functor.

Motivation

For a Lie algebra g over field F, the classical Yang-Baxter equation (cYBE) is defined for $r \in g \otimes g$ as

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

where $[\cdot, \cdot]$ is the extension of the Lie bracket in g to $g^{\otimes 3}$, and r_{ij} are embeddings of r into $g^{\otimes 3}$ with (i, j) specifying the tensor factors r acts on. For example, if $r = \sum_i a_i \otimes b_i$, then $r_{13} = \sum_i a_i \otimes 1 \otimes b_i$.

To extend the Lie bracket to tensor products of Lie algebras, and later, to understand quantification of cYBE, knowledge of universal enveloping algebra is required and will be covered in this seminar.

Definition (Lie algebra)**.** *A Lie algebra over field F is a vector space g over F equipped with a Lie bracket, a bilinear map $[\cdot, \cdot] : g^2 \to g$ satisfying $\forall x, y \in g$*

$$[x, x] = 0 \qquad (\textit{Alternating property})$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad (\textit{Jacobi identity})$$

Let the adjoint action of an element $x \in g$ be the linear map $\mathrm{ad}_x = [x, \cdot] : g \to g$. The following two properties are immediate

Properties. *For all $x, y, z \in g$,*

$$[x, y] = -[y, x] \qquad (\textit{Antisymmetry})$$
$$ad_x[y, z] = [ad_x y, z] + [y, ad_x z] \qquad (\textit{Derivation property})$$

Proof. Antisymmetry results from bilinearity and alternating property, as $[x, y] + [y, x] = [x, y + x] + [y, y + x] = [x + y, y + x] = 0$. Using antisymmetry, the derivation property can be rearranged into Jacobi identity. ☐

The derivation property measures how much the algebra fails to be associative, in the sense that the difference

$$[x, [y, z]] - [[x, y], z] = [y, [x, z]]$$

may or may not be zero.

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Definition (General linear algebra)**.** *A general linear algebra $gl(V)$ of a vector space V is the Lie algebra* $(\operatorname{End} V, \textit{comm.})$ *whose Lie bracket is the commutator*

$$[x, y]_c = x \circ y - y \circ x \quad \forall\, x, y \in \operatorname{End} V$$

As for why the commutator is a natural choice, there may be deeper reasons, but one motivation is that it is the only choice that makes adjoint actions of g a representation on g itself, as will be discussed.

Lie algebra morphism is a linear map $\phi : g_1 \to g_2$ between two Lie algebras that preserves the Lie bracket. That is, for all $x, y \in g_1$, $\phi([x, y]) = [\phi(x), \phi(y)]$. Representations can now be introduced.

Definition (Lie algebra representation)**.** *A Lie algebra representation (ϕ, V) is a F-vector space V together with a Lie algebra morphism $\phi : g \to gl(V)$.*

Theorem (Adjoint representation)**.** *The map* $\operatorname{ad} : g \to gl(g)$ *defined by* $x \mapsto \mathrm{ad_x} \in \operatorname{End} g$ *is a representation of g.*

Proof. The map is linear because Lie brackets are bilinear. Rearranging the derivation property gives, for all $x, y, z \in g$,

$$\begin{aligned} \operatorname{ad}_{[x,y]}(z) &:= [[x, y], z] = [x, [y, z]] - [y, [x, z]] \\ &= (\operatorname{ad}_x \circ \operatorname{ad}_y - \operatorname{ad}_y \circ \operatorname{ad}_x)(z) \end{aligned}$$

The commutator in the last line is precisely the Lie bracket, by definition, of $gl(g)$, and thus $\operatorname{ad}_{[x,y]} = [\operatorname{ad}_x, \operatorname{ad}_y]$. □

An equivalent definition of representations can be given through Lie modules.

Definition (Lie module)**.** *A Lie module $(V, \cdot)$ over g is a F-vector space V equipped with a bilinear map $\cdot : g \times V \to V$, $(x, v) \mapsto xv$, called the module action, satisfying*

$$[x, y]v = x(yv) - y(xv)$$

for all $x, y \in g$ and $v \in V$.

Any Lie modules give a Lie algebra representation (ϕ, V) if ϕ is defined by $\phi(x)v := xv$ for all $v \in V$ and $x \in g$. Conversely, any representations (ϕ, V) defines a Lie module if the module action is $xv := \phi(x)v$. The reason is that the above module condition becomes equivalent to checking whether ϕ is a Lie algebra representation.

Theorem. *The map* $\operatorname{Lie} : \mathrm{AssocAlg}_F \to \mathrm{LieAlg}_F$ *from the category of associative algebra over F to the category of Lie algebra over F, if acts as identity on morphisms and equip any associative algebra with the commutator as Lie bracket, is a functor.*

Proof. For all unital associative algebra morphisms ϕ, $\operatorname{Lie}(\phi) = \phi$ and

$$\operatorname{Lie}(\phi)([a, b]) = \phi(ab - ba) = \phi(a)\phi(b) - \phi(b)\phi(a) = [\phi(a), \phi(b)] \quad \forall\, a, b \in \operatorname{Dom}(\phi)$$

shows that $\operatorname{Lie}(\phi)$ is a Lie algebra morphism. Preservations of identity morphisms and composition are trivial because Lie acts as identity on morphisms. □

An enveloping algebra (U, ϕ) of g is a unital associative algebra U with a Lie algebra morphism $\phi : g \to \operatorname{Lie}(U)$. They are generalizations of Lie algebra representations, in the sense that every Lie algebra representations (ϕ, V) of g defines an enveloping algebra of g of the form $(\operatorname{End} V, \phi)$. This can be seen by noting $gl(V) = \operatorname{Lie}(\operatorname{End} V)$, and $\operatorname{End} V$ is a unital associative algebra with the id as unit.

Definition (Universal enveloping algebra)**.** *The universal enveloping algebra $(U(g), \Phi)$ of g is an enveloping algebra such that for all enveloping algebra (U, ϕ) of g, there exists a unique unital associative algebra morphism $f : U(g) \to U$ satisfying $f \circ \Phi = \phi$. This condition is called the universal property, and ϕ is said to factors through $U(g)$ uniquely.*

Theorem (Uniqueness theorem)**.** *The universal enveloping algebra of a Lie algebra is unique up to associative algebra isomorphisms.*

Proof. Let $(U_0(g), \Phi_0)$ and $(U_1(g), \Phi_1)$ be universal enveloping algebras of g. By universal property, there exists $f_0 : U_0(g) \to U_1(g)$ and $f_1 : U_1(g) \to U_0(g)$ such that

$$\begin{cases} f_0 \circ \Phi_0 = \Phi_1 \\ f_1 \circ \Phi_1 = \Phi_0 \end{cases}$$

This implies

$$\begin{cases} f_0 \circ f_1 \circ \Phi_1 = \Phi_1 \\ f_1 \circ f_0 \circ \Phi_0 = \Phi_0 \end{cases}$$

The universal property also states, for each $i = 0, 1$, that there is only one map f satisfying $f \circ \Phi_i = \Phi_i$, and the fact that f can be id means that $f_0 \circ f_1 = \mathrm{id}$ and $f_1 \circ f_0 = \mathrm{id}$. f_0 is then invertible and gives an associative algebra isomorphism $U_0(g) \cong U_1(g)$. □

We now proceed with an existence theorem. The proof is constructive, providing a tangible object for later use.

Definition. *The tensor algebra $T(g)$ of g is the free (unital) associative algebra generated by g. Specifically,*

$$T(g) = \bigoplus_{n=0}^{\infty} g^{\otimes n} = \mathrm{span}_F \left\{ \prod_{n=1}^{d} x_n \middle| x : [1, d] \to g,\ d \in \mathbb{N}_0 \right\}$$

where the empty product is the unit in F.

It has a grading structure $(\mathbb{N}_0, +)$ since $g^{\otimes m} g^{\otimes n} \subset g^{\otimes (m+n)}$ for $m, n \in \mathbb{N}_0$.

Theorem (Existence theorem)**.** *The universal enveloping algebra $(U(g), \Phi)$ exists for all Lie algebras g. It is given by $U(g) = T(g)/\sim$ where the equivalence relation is $xy - yx \sim [x, y]$ for all $x, y \in g$.*

Proof. To enforce the equivalence relation consistently, consider the two-sided ideal

$$J(g) = \langle xy - yx - [x, y] | x, g \in g \rangle$$

, which by definition absorbs all linear combinations, left and right concatenation of tensors. Note that concatenation by nonzero tensors could only increase the degree. The unit, being degree zero, cannot lies in $J(g)$, and thus the associative algebra $U(g) = T(g)/J(g)$ is unital.

$$\begin{aligned} \Phi([x, y]) &= [x, y] + J(g) = [x, y] + xy - yx - [x, y] + J(g) \\ &= xy - yx + J(g) \\ &= (x + J(g))(y + J(g)) - (y + J(g))(x + J(g)) = \Phi(x)\Phi(y) - \Phi(y)\Phi(x) \end{aligned}$$

for all $x, y \in g$ implies that Φ is a Lie algebra morphism to $\mathrm{Lie}\, U(g)$, and hence $(U(g), \Phi)$ is an enveloping algebra.

For any enveloping algebra (U, ϕ) of g, define $f : T(g) \to U$ by setting $f(\prod_{n=1}^{d} x_n) = \prod_{n=1}^{d} \phi(x_n)$ for all sequences $\{x_n\} \subset g$ and extending f linearly to the whole of $T(g)$. One can check that this extension is consistent with the defined values on monomials due to the linearity of ϕ. For all $x, y \in g$,

$$f(xy - yx - [x, y]) = f(xy) - f(yx) - f([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x) - \phi([x, y]) = 0$$

because ϕ gives a Lie algebra morphism to $\mathrm{Lie}\, U$. Then, $f(J(g)) = \{0\}$ means that f descends to $T(g)/J(g)$ as some $\bar{f}$ satisfying $f = \bar{f} \circ q$, where q is the quotient map induced by $\sim$. Let $i : g \to T(g)$ denotes the canonical inclusion, then

$$\phi = f \circ i = \bar{f} \circ q \circ i = \bar{f} \circ \Phi$$

To visualize these morphisms, they correspond to all oriented paths $g \to U$ in the commutative diagram

$$\begin{tikzcd}
g \arrow[r, "\phi"] \arrow[d, "\Phi"'] \arrow[dr, "i"' near start] & U \\
T(g)/J(g) \arrow[ur, "\bar{f}" near end] & T(g) \arrow[u, "f"'] \arrow[l, "q"]
\end{tikzcd}$$

Let $\{\beta_n\}$ be a basis of g. For any unital associative algebra $f_0 : T(g)/J(g) \to U$ satisfying $f_0 \circ \Phi = \phi$,

$$f_0(\beta_n + J(g)) = (f_0 \circ \Phi)(\beta_n) = (\bar{f} \circ \Phi)(\beta_n) = \bar{f}(\beta_n + J(g))$$

Unitarity of the morphisms also implies that $f_0(1) = 1 = \bar{f}(1)$. Since all monomials of $\{\beta_n\}$ together with the unit is a basis of $T(g)$, all monomials of $\{\beta_n + J(g)\}$ together with the unit spans $U(g)$, and therefore $f_0 = \bar{f}$. $\square$