A Note on Error Bounds For the Dynamic Liquidity Term

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Abstract

This is a follow up note building off of our first paper around dynamically varying the liquidity parameter in DODO exchange's Proactive Market Maker (PMM) [1]. The focus for this analysis is around understanding in practice how accurate our estimate of the k parameter needs to be for it to be dependable and what distributional process has the best tradeoff of accuracy vs. computational cost.

Derivation of Error Bounds

For interested readers it is suggested to refer to the original article linked above but we will again hit upon the main base assumptions around how order flow in a DEX can be used to dictate the needed curvature for an automated market maker. We will continue to focus on DODO's PMM given the uniqueness of it's approach and the ease with which one could potentially update the concentration of swappable liquidity on chain [2].

The Proactive Market Maker framework provides an interesting opportunity to have DEXes better adjust to the order flow dynamics in their marketplaces. Before focusing too much on what algorithms would be the most efficient to update the k parameter within each LP's smart contract it is helpful to first understand how accurrate we actually need to be for this value to be achieved. The bulk of this paper will be diving into some analysis around estimating error bounds for the dynamic liquidity terms.

Theorem 1 (Dynamic Liquidity Term Error Bounds)**.** *For an error rate r, order arrival rate per period of λ and the true dynamic liquidity term k_T the acceptable error amount c of our k estimate is bounded as follows for the PMM:*

$$\frac{-r}{\sqrt{\frac{1}{\lambda}+1}}\Big(\frac{1}{\frac{\beta_0^2}{\beta_1^2}-1}+k_T\Big) \leq c \leq \frac{r}{\sqrt{\frac{1}{\lambda}+1}}\Big(\frac{1}{\frac{\beta_0^2}{\beta_1^2}-1}+k_T\Big)$$

Proof. We first revisit the basics of the PMM which has a price function as follows: $Price = iR$ with $Price$ being the effective swap price, i being the oracle price and R being an offset term that is defined as follows:

1. if $B < B_0, R = 1 - k + (\frac{B_0}{B})^2 k$
2. if $Q < Q_0, R = (1 - k + (\frac{Q_0}{Q})^2)^{-1} k$
3. else $R = 1$

B is defined as the current number of "base" tokens in the pool. Q is defined as the current number of "quote" tokens in the pool. B_0 is the originally deposited number of base tokens supplied by the LPs. And Q_0 is the originally deposited number of quote tokens supplied by LPs. The state of the LP can only ever be in one three aforementioned price definitions.

We focus without loss of generality on the situation where $B < B_0$ for this proof. Leveraging the price curve we arrive at the formula for determining how many Q tokens one receives (ΔQ) when adding B tokens (ΔB): $i(B_2 - B_1)(1 - k + k\frac{B_0^2}{B_1 B_2})$. From here we simplify the notation to substitute B_2 for $B_1 + \Delta B$ as all incremental trades work from the current amount of token B i.e. B_1. This then reduces the swap formula down to $S(\Delta B, k) = i\Delta B(1 - k + k\frac{B_0^2}{B_1(B_1+\Delta B)})$ for a predetermined k value.

Our next step is to then define our error function which will be the Normalized Root Mean Squared Deviation [5] weighted by the probability distribution f of ΔB. The error function will be focused on measuring the amount of "incorrect" swapped quantities of token Q vs the "correct" swapped amount. We additionally assume that ΔB follows a compound Poisson process with orders following a Poisson distribution and having an arrival rate of λ per period. Each order is assumed to be an independent and identically distributed random variable following an exponential distribution with size θ [3]:

$$\frac{\sqrt{\int_0^\infty (S(\Delta B, k_T) - S(\Delta B, k_E))^2 f(\Delta B) d\Delta B}}{\int_0^\infty S(\Delta B, k_T) f(\Delta B) d\Delta B} \tag{1}$$

$S(\Delta B, k_T)$ represents the true amount of Q that should be swapped for an exchanging quantity of ΔB. $S(\Delta B, k_E)$ represents the swapped amount based upon an internal approximation of k_E that is the estimated k individual to the order flow of each LP. Solving first the denominator:

$$\int_0^\infty S(\Delta B, k_T) f(\Delta B) d\Delta B = \int_0^\infty i\Delta B(1 - k_T + k_T \frac{B_0^2}{B_1(B_1 + \Delta B)}) f(\Delta B) d\Delta B \tag{2}$$

Recognizing that $\frac{B_0^2}{B_1(B_1+\Delta B)}$ is difficult if not impossible to integrate for common probability distributions we simplify the equation by dropping ΔB in the denominator. This simplificiation allows for a much easier analysis and is equivalent with the full formula for almost all circumstances. See Appendix A for more details around this section of the analysis.

$$\begin{aligned} \int_0^\infty i\Delta B(1 - k_T + k_T \tfrac{B_0^2}{B_1(B_1+\Delta B)}) f(\Delta B) d\Delta B &\approx \int_0^\infty i\Delta B(1 - k_T + k_T \frac{B_0^2}{B_1^2}) f(\Delta B) d\Delta B \\ &= i\lambda\theta(1 - k_T + k_T \frac{B_0^2}{B_1^2}) \end{aligned} \tag{3}$$

Now tackling the numerator we arrive at

$$\begin{aligned} \sqrt{\int_0^\infty (S(\Delta B, k_T) - S(\Delta B, k_E))^2 f(\Delta B) d\Delta B} &= \sqrt{\int_0^\infty i^2 \Delta B^2 \big((k_E - k_T) + (k_T - k_E) \frac{B_0^2}{B_1(B_1 + \Delta B)}\big)^2 f(\Delta B) d\Delta B} \\ &= \sqrt{\int_0^\infty i^2 \Delta B^2 \big((k_E - k_T) + (k_T - k_E) \frac{B_0^2}{B_1^2}\big)^2 f(\Delta B) d\Delta B} \\ &= \sqrt{i^2(\lambda\theta^2 + \lambda^2\theta^2)\big((k_E - k_T) + (k_T - k_E) \frac{B_0^2}{B_1^2}\big)^2} \\ &= i\theta\sqrt{\lambda + \lambda^2}(\frac{B_0^2}{B_1^2} - 1)\big|k_E - k_T\big| \end{aligned} \tag{4}$$

Combining our numerator and denominator together we can further simplify to

$$\frac{\sqrt{1 + \frac{1}{\lambda}}(\frac{B_0^2}{B_1^2} - 1)\big|k_E - k_T\big|}{(1 - k_T + k_T \frac{B_0^2}{B_1^2})} \tag{5}$$

Substituting the $\left|k_E - k_T\right|$ term for c and bounding our fraction by $\pm r$ we get

$$
\begin{gathered}
-r \leq \frac{\sqrt{1+\frac{1}{\lambda}}(\frac{B_0^2}{B_1^2}-1)c}{(1-k_T+k_T\frac{B_0^2}{B_1^2})} \leq r \\
\frac{-r}{\sqrt{\frac{1}{\lambda}+1}}\left(\frac{1}{\frac{\beta_0^2}{\beta_1^2}-1}+k_T\right) \leq c \leq \frac{r}{\sqrt{\frac{1}{\lambda}+1}}\left(\frac{1}{\frac{\beta_0^2}{\beta_1^2}-1}+k_T\right)
\end{gathered}
\tag{6}
$$

Which now gives us the original forumla above [1].

□

Corollary 0.1. *It should be noted that if instead we assumed that the order flow is only exponentially distributed we can quickly arrive at that version of the error bounds by setting $\lambda = 1$. Effectively the order flow's distribution has no impact on the error bounds of our estimate of k_E as it reduces down to*

$$
\frac{-r}{\sqrt{2}}\left(\frac{1}{\frac{\beta_0^2}{\beta_1^2}-1}+k_T\right) \leq c \leq \frac{r}{\sqrt{2}}\left(\frac{1}{\frac{\beta_0^2}{\beta_1^2}-1}+k_T\right)
\tag{7}
$$

Analysis of Error Bounds

We now look to understand the ramifications of this error bound we've just derived. Despite it's simplicity there are a number of things that become apparent around how important the accuracy of k_E is:

- As $\frac{\beta_0^2}{\beta_1^2}$ approaches 1 from the PMM rebalancing closer towards the initial token balance, the impact of a missestimation of k_T has less impact. Once $\frac{\beta_0^2}{\beta_1^2} = 1$ the risk of missestimating disapears (on an infinitessimal basis).
- For larger expected average order flow (larger λ) the risk of missestimating k_T decreases too. Conversely, the less frequent trades occur the larger the impact (all else being equal) of having the wrong value of k_T.
- Unsurprisingly, a constant value of missestimation c has a higher impact all else being equal on situations when k_T is lower vs when it is higher. This can be seen by taking $\frac{\partial}{\partial k_T}\frac{\sqrt{1+\frac{1}{\lambda}}(\frac{B_0^2}{B_1^2}-1)c}{(1-k_T+k_T\frac{B_0^2}{B_1^2})}$ which equals $\frac{\sqrt{1+\frac{1}{\lambda}}(\frac{B_0^2}{B_1^2}-1)^2c}{(1-k_T+k_T\frac{B_0^2}{B_1^2})^2}$. Since $\frac{\beta_0^2}{\beta_1^2} \geq 1$ then the denominator of the partial derivative will always be increasing as k_T increases, lowering the total ratio. But even though lower curvature PMM's (smaller k_T) are more susceptible to incorrect liquidity term estimations, the relative impact of this is muted by $\frac{\beta_0^2}{\beta_1^2}$.
- Estimation of the true dynamic liquidity term will matter most under the following circumstances: 1) one side of the pool has been substantially depleted vs it's original staked balance, 2) there is a small number of transactions per period and 3) the true dynamic liquidity term is closer to 0. Under most other situations a reasonable approximation of the k_T parameter should be sufficient. Below is a graph showing different combinations of $\frac{\beta_0^2}{\beta_1^2}$ (referred to as "B Ratio") and k_T with the associated error term c that would result in a 2.5% error. It can be seen that given the ramp like shape there is more dramatically more leeway around acceptable error rates on an absolute basis (i.e. higher values) for lower B Ratios and higher k_T.

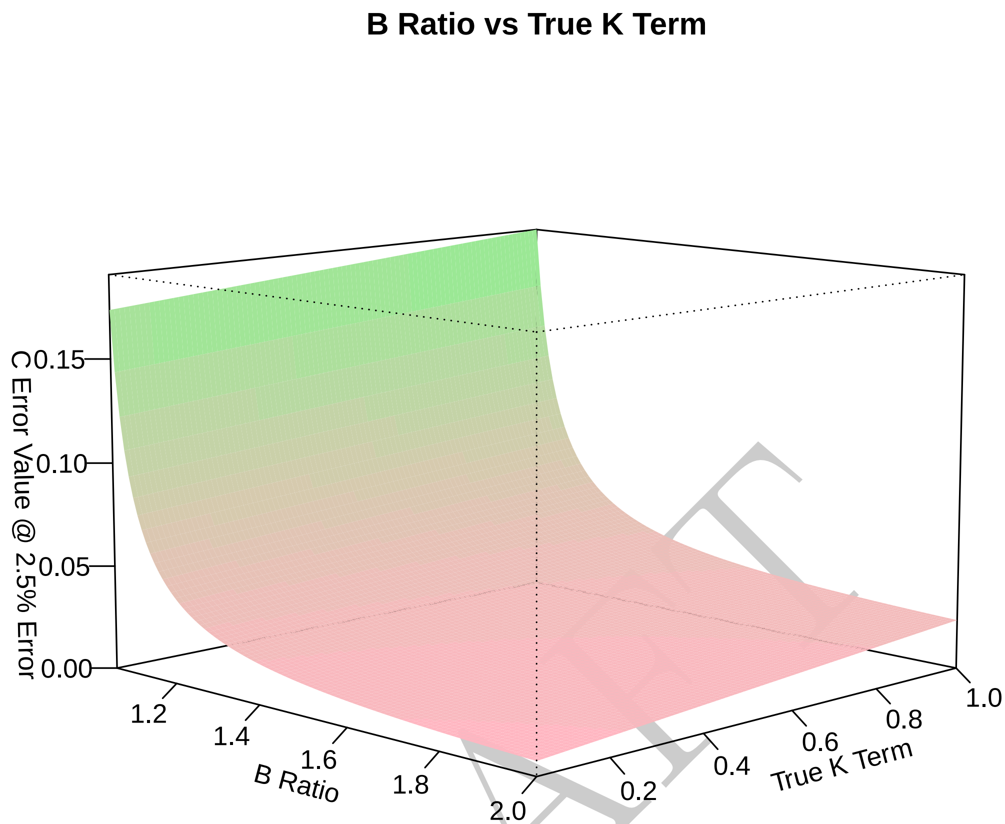

Figure 1: For $\lambda = 1$ and $r = 0.025$ different implied error rates c across combinations of $\frac{\beta_0^2}{\beta_1^2}$ and k_T

Comparison of Accuracy for Different Distributions

Given that we have derived our error bounds for missestimating k_T the natural next area to explore is which version of the Hellinger distance formula gives us the best trade off of accuracy vs. computational cost [4]. In this section we will build off of the order flow based framework established in our earlier work [1]. Our main assumption is that the full order flow based compound Poisson process is our baseline "truth" and the two competing options are an exponential model and a Poisson model. Across these three processes the parameters will be consistent in that the compound Poisson process' θ and λ variables are used in the exponential and Poisson processes respectively.

The compound Poisson process is approximated by a normal distribution in our original work and while it hypothetically captures the full dynamics of both incoming/outgoing swap frequency and size, the downside is that it requires a significant amount of floating point math which could be prohibitive on certain blockchains. Below we will walk through some analysis around trying to identify which of the Poisson or exponential versions work best and under what conditions that is the case.

Reusing the notation from our earlier work we establish that for each LP pool there are bidrectional swaps occuring that follow a compound Poisson process. Buy transaction volume is defined as a swap from token **A** to token **B** and denoted as $B(t)$ with t being a predefined time period:

$$\begin{aligned}
B(t) &= \sum_{j=1}^{BN(t)} BT_j \\
BN(t) &\sim Poisson(\lambda_B t) \\
BT_j &\sim Exponential(\theta_B) : j \geq 1
\end{aligned} \tag{8}$$

Thus buy transaction volume over t is denoted as $BN(t)$ and is assumed to follow a compound Poisson process where the number of transactions per time period follow an arrival rate of λ_B. This implies that the mean number of transactions over a time period t is equal to $\lambda_B t$. Each buy transaction (BT_i) is assumed to be an independent and identically distributed random variable following an exponential distribution with size θ_B [3]. $BN(t)$ is assumed to be independent of BT. This compound Poisson process has the following properties:

$$\begin{aligned}
\mathrm{E}[B(t)] &= \mathrm{E}[\mathrm{E}[B(t)|BN(t)]] \\
&= \mathrm{E}[BN(t)]\mathrm{E}[BT(t)] \\
&= \lambda_B t \theta_B \\
\mathrm{Var}[B(t)] &= \mathrm{E}[\mathrm{Var}[B(t)|BN(t)]] + \mathrm{Var}[\mathrm{E}[B(t)|BN(t)]] \\
&= \mathrm{E}[BN(t)]\mathrm{E}[BT(t)^2] \\
&= 2\lambda_B t \theta_B^2
\end{aligned} \tag{9}$$

Additionally we assumed the same process for sale transactions.

Sell transaction volume is defined as a swap from token **B** to token **A** and denoted as $Q(t)$ with t being a predefined time period:

$$\begin{aligned}
Q(t) &= \sum_{j=1}^{QN(t)} QT_j \\
QN(t) &\sim Poisson(\lambda_Q t) \\
QT_j &\sim Exponential(\theta_Q) : j \geq 1
\end{aligned} \tag{10}$$

Sale transaction volume over t is similarly denoted as $QN(t)$ and also is assumed to follow a compound Poisson process where the number of transactions per time period follow an arrival rate of λ_Q. Each sale transaction is assumed to be an independent and identically distributed random variable following an exponential distribution with size $\theta_Q^{'}$. $QN(t)$ is assumed to be independent of QT. $\theta_Q^{'}$ is the rebased size in B's analogous units to allow for a more direct comparison of order flow size. So accounting for our oracle price of i we have $\theta_Q^{'} = \theta_Q / i$. This compound Poisson process has the following properties:

$$\begin{aligned}
\mathrm{E}[Q(t)] &= \mathrm{E}[\mathrm{E}[Q(t)|QN(t)]] \\
&= \mathrm{E}[QN(t)]\mathrm{E}[QT(t)] \\
&= \lambda_Q t \theta_Q^{'} \\
\mathrm{Var}[Q(t)] &= \mathrm{E}[\mathrm{Var}[Q(t)|QN(t)]] + \mathrm{Var}[\mathrm{E}[Q(t)|QN(t)]] \\
&= \mathrm{E}[QN(t)]\mathrm{E}[QT(t)^2] \\
&= 2\lambda_Q t \theta_Q^{'2}
\end{aligned} \tag{11}$$

We will be comparing the mean absolute error for each of our two distributions' expected swap amount vs. the "true" normal distribution's swap amount given an identical ΔB. This will allow us to understand the tracking error of using simpler distributions to estimate the dynamic liquidity parameter k and make a direct comparison between the Poisson and exponential processes.

Leveraging the simplified PMM swap formula we have for distribution D_1 and distribution D_2 the mean absolute error:

$$
\begin{aligned}
MAE[\Delta B, k_{D_1}, k_{D_2}] &= \left|S(\Delta B, k_{D_1}) - S(\Delta B, k_{D_2})\right| \\
&= \left|i\Delta B(1 - k_{D_1} + k_{D_1}\frac{B_0^2}{B_1(B_1+\Delta B)}) - i\Delta B(1 - k_{D_2} + k_{D_2}\frac{B_0^2}{B_1(B_1+\Delta B)})\right|
\end{aligned} \tag{12}
$$

Now comparing the Poisson distribution and exponential distribution (denoted D_P and D_E respectively) vs the Normal distribution's (D_N) Hellinger distance and seeing which is larger gives us

$$
\begin{gathered}
MAE[\Delta B, k_{D_P}, k_{D_N}] \overset{?}{\geq} MAE[\Delta B, k_{D_E}, k_{D_N}] \\
\left|i\Delta B(k_{D_N} - k_{D_P} + (k_{D_P} - k_{D_N})\frac{B_0^2}{B_1(B_1+\Delta B)})\right| - \left|i\Delta B(k_{D_N} - k_{D_E} + (k_{D_E} - k_{D_N})\frac{B_0^2}{B_1(B_1+\Delta B)})\right| \overset{?}{\geq} 0
\end{gathered} \tag{13}
$$

Assuming that we are only focused on situations with $\Delta B \geq 0$ and recalling that i will always be positive we can further simplify

$$
\begin{gathered}
\left|k_{D_N} - k_{D_P} + (k_{D_P} - k_{D_N})\frac{B_0^2}{B_1(B_1+\Delta B)}\right| - \left|k_{D_N} - k_{D_E} + (k_{D_E} - k_{D_N})\frac{B_0^2}{B_1(B_1+\Delta B)}\right| \overset{?}{\geq} 0 \\
\left|(k_{D_P} - k_{D_N})(\frac{B_0^2}{B_1(B_1+\Delta B)} - 1)\right| - \left|(k_{D_E} - k_{D_N})(\frac{B_0^2}{B_1(B_1+\Delta B)} - 1)\right| \overset{?}{\geq} 0 \\
\left|k_{D_P} - k_{D_N}\right| - \left|k_{D_E} - k_{D_N}\right| \overset{?}{\geq} 0
\end{gathered} \tag{14}
$$

The final step follows from how $B_1 \leq B_2$ and $\Delta B \geq 0$.

Now establishing the Hellinger distance formulas for each of our three distributions we have

$$
\begin{aligned}
H_{D_N}(t, \lambda_Q, \lambda_B, \theta_Q', \theta_B) &= \sqrt{1 - \sqrt{\frac{2\sqrt{\lambda_Q\lambda_B}\theta_Q'\theta_B}{\lambda_Q\theta_Q'^2 + \lambda_B\theta_B^2}}\, e^{-\frac{t}{8}\frac{(\lambda_B\theta_B - \lambda_Q\theta_Q')^2}{\lambda_Q\theta_Q'^2 + \lambda_B\theta_B^2}}} \\
H_{D_P}(t, \lambda_Q, \lambda_B) &= \sqrt{1 - e^{-0.5(\sqrt{\lambda_Q} - \sqrt{\lambda_B})^2 t}} \\
H_{D_E}(\theta_Q', \theta_B) &= \sqrt{1 - \frac{2\sqrt{\theta_Q'\theta_B}}{\theta_Q' + \theta_B}}
\end{aligned} \tag{15}
$$

Unfortunaely substituting the Hellinger distance formulas into [14] does not allow for an easy simplification of our inequality. Abuse of Taylor series approximations can make things a bit easier to work with but it is still unfruitful with regards to finding a parsimonious formula. We can however do a dimension reduction and run some scenario analysis to come up with some rough rule of thumbs around which distribution is superior. The dimension reduction will consist of replacing λ_B with $\lambda_Q c_\lambda$ where c_λ is bounded from below by 0 and represents the ratio of $\frac{\lambda_B}{\lambda_Q}$. Additionally we will replace θ_B with $\theta_Q' c_\theta$ where c_θ is also bounded from below by 0 and represents the ratio of $\frac{\theta_B}{\theta_Q'}$. Now plugging in these new definitions we get

$$
\begin{aligned}
H_{D_N}(t, \lambda_Q, \lambda_B, \theta_Q', \theta_B) &= \sqrt{1 - \sqrt{\frac{2\sqrt{c_\lambda}c_\theta}{1 + c_\theta^2 c_\lambda}}\, e^{-\frac{t}{8}\frac{\lambda_Q(c_\lambda c_\theta - 1)^2}{1 + c_\lambda c_\theta}}} \\
H_{D_P}(t, \lambda_Q, \lambda_B) &= \sqrt{1 - e^{-0.5\lambda_Q(1 - \sqrt{c_\lambda})^2 t}} \\
H_{D_E}(\theta_Q', \theta_B) &= \sqrt{1 - \frac{2\sqrt{c_\theta}}{1 + c_\theta}}
\end{aligned} \tag{16}
$$

Regrettably not all of the base terms drop out but now instead of having t, λ_Q, λ_B, $\theta_Q^{'}$ and θ_B we now have t, λ_Q, c_λ and c_θ. We can further reduce this by noting that for a scenario analysis λ_Q is always proportional to the time period t and the t term can be set equal to 1. With just three dimensions we can now compare different combinations of λ_Q, c_λ and c_θ. For the purposes of our analysis we choose $\{c_\lambda \in \mathbb{R} : c_\lambda > 0 \text{ and } c_\lambda \leq 2\}$, $\{c_\theta \in \mathbb{R} : c_\theta > 0 \text{ and } c_\theta \leq 2\}$ and $\lambda_Q \in \{0.05, 0.5, 1.0, 5.0, 10.0, 20.0\}$. The reduced dimensions will allow us to visually understand the combinations of parameters that lead to the Poisson Hellinger being superior/inferior to exponential version. Below are six different scenarios.

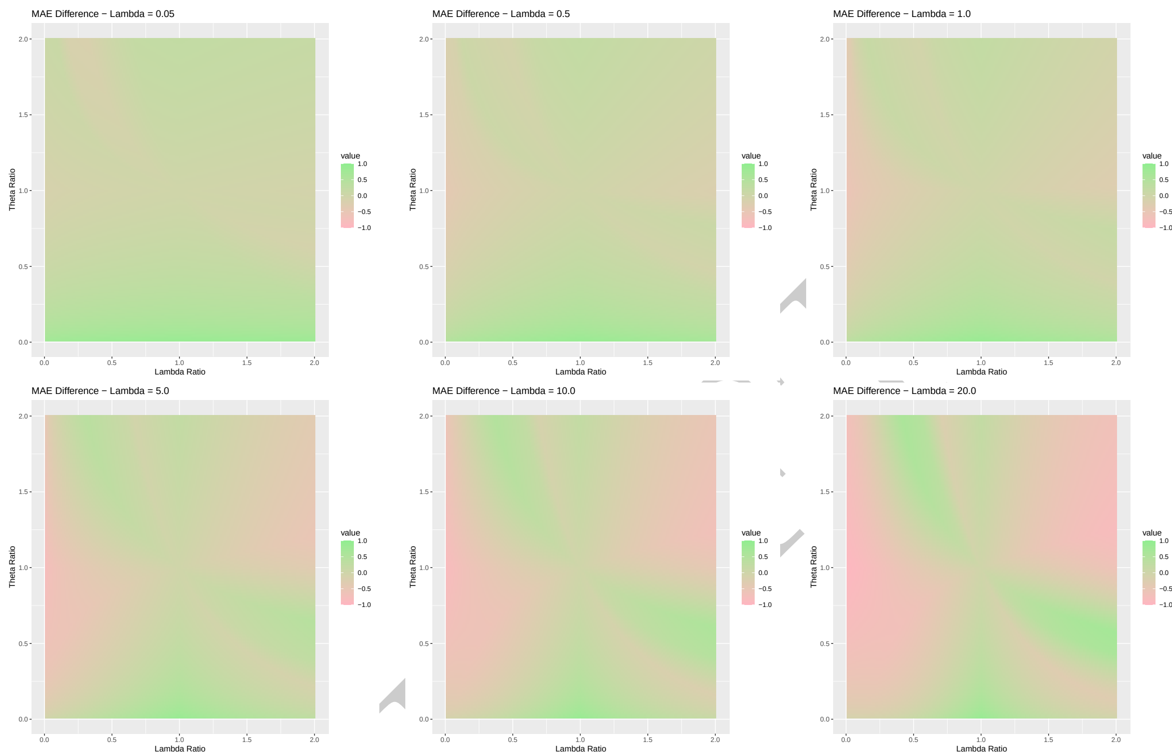

Figure 2: Mean absolute error Hellinger distances of |Poisson - Normal| - |Exponential - Normal| under varying λ_Q. Green indicates Poisson's error rate is larger while pink indicates exponential's error rate is larger.

For figures 1 and 2 we can see that under situations with lower frequency swaps per period the MAE for the Poisson Hellinger is almost always larger then the MAE for the exponential version. This holds across almost all combinations of λ pairs and θ pairs and is characterized by the bulk of the surface charts being in the positive (green) range. So for pools with relatively infrequent swaps an exponential based Hellinger dynamic liquidity term will be superior to a Poisson version. As λ_Q increases this takeaway does not hold though. As the order flow frequency per period picks up unless the order flow is balanced in someway (with c_λ or c_θ) then the bulk of the heatmap is below 0 (i.e. is pink). This indicates that the Poisson based Hellinger liquidity term does a better job of matching the full Normal distribution's liquidity term. As the success of each AMM is derived from attracting the most swap volume, it is likely a better bet to aim for using a Poisson based Hellinger in anticipation of that increased volume rather then accepting low volume (and an early death of the AMM protocol) with the exponential Hellinger.

Conclusion

Within this article we have identified a simple framework to quantify the degree of error around missestimating the dynamic liquidity term of the PMM and under what situations this error will be most pronounced. For token pairs that usually require modest to high curvature the risk of having an incorrect liquidity term estimation is relatively mild. Additionally, as long as the PMM's "B ratio" is close to 1 then even only decent estimates of k will suffice. We also conducted analysis to identify under what situations a Poisson or exponential Hellinger distance formula is superior. For high volume pairs the Poisson Hellinger does a better job while for lower volume pairs the exponential distribution is superior. Our hope is that this research helps make it simpler to implement dynamic concentrated liquidity algorithms on chain as AMM's continue to evolve in capability and efficiency.

Appendices

A Numeric Integration Analysis

To showcase how the simplification of the integral is a valid approach we have ran numeric integration on a number of different scenarios for $\int_0^\infty i\Delta B(1 - k_T + k_T \frac{B_0^2}{B_1(B_1+\Delta B)})f(\Delta B)d\Delta B$. The different combination of parameters are meant to highlight how the error in the approximated integral is indeed larger when B_0 is smaller but even then the tracking error is not very substantial. "Full Integral" in the table below shows the true integrated value of price while "Simplified Integral" shows the approximation where $\frac{B_0^2}{B_1(B_1+\Delta B)}$ is replaced by $\frac{B_0^2}{B_1^2}$ (using the composite trapezoid approximation with one million subsections). "Error Rate" is defined as $\frac{FullIntegral}{SimplifiedIntegral} - 1$. To simplify this section of the analysis we used an exponential distribution with parameter θ as our density function.

It can be quickly recognized that even instances with a tiny amount of initial tokens (100) that are very depleted (75), the error rate is almost non existant. As B_0 grows or $\frac{B_0}{B_1}$ approaches 1 the error rate shrinks even more.

B_0	B_1	θ	i	k	Full Integral	Simplified Integral	Error Rate
100	75	5	1.00	1.00	0.353600	0.355481	-0.00529
100	75	5	1.00	0.75	0.315190	0.316601	-0.00446
100	95	5	1.00	1.00	0.220633	0.221560	-0.00419
100	75	5	1.00	0.50	0.276779	0.277720	-0.00339
100	95	5	1.00	0.75	0.215465	0.216160	-0.00322
100	75	10	1.00	1.00	0.177158	0.177630	-0.00266
100	75	10	1.00	0.75	0.157847	0.158201	-0.00224
100	95	5	1.00	0.50	0.210296	0.210759	-0.00220
100	95	10	1.00	1.00	0.110478	0.110711	-0.00210
100	75	5	1.00	0.25	0.238369	0.238839	-0.00197
100	75	10	1.00	0.50	0.138537	0.138773	-0.00170
100	95	10	1.00	0.75	0.107838	0.108012	-0.00161
100	95	5	1.00	0.25	0.205127	0.205359	-0.00113
100	95	10	1.00	0.50	0.105198	0.105314	-0.00110
100	75	10	1.00	0.25	0.119227	0.119345	-0.00099
100	95	10	1.00	0.25	0.102557	0.102615	-0.00057
1000	750	50	1.00	1.00	0.034822	0.034824	-0.00005
1000	750	50	1.00	0.75	0.031014	0.031015	-0.00005
1000	950	50	1.00	1.00	0.021704	0.021705	-0.00004
1000	750	50	1.00	0.50	0.027205	0.027206	-0.00003
1000	950	50	1.00	0.75	0.021175	0.021176	-0.00003
1000	750	100	1.00	1.00	0.016367	0.016368	-0.00003
1000	750	100	1.00	0.75	0.014577	0.014577	-0.00002
1000	950	100	1.00	1.00	0.010201	0.010201	-0.00002
1000	950	50	1.00	0.50	0.020646	0.020647	-0.00002
1000	750	50	1.00	0.25	0.023397	0.023397	-0.00002
1000	750	100	1.00	0.50	0.012787	0.012787	-0.00002
1000	950	100	1.00	0.75	0.009953	0.009953	-0.00002
1000	950	100	1.00	0.50	0.009704	0.009704	-0.00001
1000	950	50	1.00	0.25	0.020117	0.020118	-0.00001
1000	750	100	1.00	0.25	0.010997	0.010997	-0.00001
1000	950	100	1.00	0.25	0.009455	0.009455	-0.00001

References

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