MTH 222: ORDINARY DIFFERENTIAL EQUATION

Course Outline

First-order-ordinary differential equations. Existence and uniqueness of solution. Second-order ordinary differential equations with constant coefficients. General theory of nth-order linear ordinary differential equations. The Laplace transform. Solution of initial boundary-value problems by Laplace transform method. Simple treatment of partial differential equations in two independent variables. Application of ordinary and partial differential equations to physical, life and social sciences.

1. FIRST-ORDER ORDINARY DIFFERENTIAL EQUATION

1.1 INTRODUCTION

An equation containing derivatives of the unknown function is called a differential equation. We have two types of differential equations. The classification is based on whether the unknown function depends on a single independent variable or on several independent variables. If the unknown function depends on a single independent variable, only ordinary derivatives $(\frac{d}{dx})$ appear in the differential equation and hence the equation is called ordinary differential equation (O.D.E.). On the other hand, the derivatives are partial derivatives $(\frac{\partial}{\partial x})$ and the equation is called a partial differential equation (P.D.E.).

E.g. of O.D.E.: $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + kx = 0$

E.g. of P.D.E.: $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x-2y$

1.2 ORDER OF ORDINARY DIFFERENTIAL EQUATION

The order of an ordinary differential equation is the order of the highest derivative that appears in the equation. For ease of notation

$$\frac{dy}{dx} = y', \qquad \frac{d^2y}{dx^2} = y'' \text{ etc.}$$

Take, for instance

$$y''' + 2e^{x}y'' + yy' = x^{4}$$

is a third-order o.d.e. At times, other letters will be used in place of y and the meaning will be the same.

Other Examples

1.	$x^{2}y'' + xy' + 2y = \sin x$	second order
2.	$(1-y^{2})y'' + xy' + y = e^{x}$	second order
3.	$y^{iv} + y''' + y'' + y' + y = 1$	Fourth order
4.	$y' + xy^{2} = 0$	First order.

Note that in this chapter, we are interested in the first-order o.d.e.

1.3 SEPARABLE EQUATIONS

Definition

If the right hand side of the equation

$$\frac{dy}{dx} = f(x,y) \qquad (1)$$

can be expressed as a function that depends only on x times a function that depends only on y, then the differential equation is called separable.

That is, if (1) can be written in the form

$$\frac{dy}{dx} = g(x)p(y) \qquad (2)$$

Take for example, the equation

$$\frac{dy}{dx} = \frac{2x+xy}{y^{2}+1}$$

is separable because

$$\frac{2x+xy}{x^{2}+1} = \frac{x(2+y)}{y^{2}+1} = g(x)p(y).$$

Note that the equation $y' = 1+xy$ admits no such factorization of the right-hand side, and so it is not separable.

1.3.1 METHOD OF SOLUTION TO SEPARABLE EQUATIONS

To obtain solution to separable equation, we multiply through the equation (2) by dx to obtain

$$h(y)dy = g(x)dx \qquad (3)$$

where $h(y) = \frac{1}{p(y)}$.

We then integrate both sides of (3) to get

$$\int h(y)\,dy = \int g(x)\,dx$$

$$= H(y) = G(x) + C \qquad (4)$$

Equation (4) is the implicit solution.

Example

1. Solve $y' = \frac{x-5}{y^2}$. (5)

Solution

We separate the variables and rewrite (5) in the form

$$y^2 dy = (x-5)dx \qquad (6)$$

Integrating both sides of (6) we get

$$\int y^2 dy = \int (x-5)\,dx$$

$$\Longrightarrow \frac{y^3}{3} = \frac{x^2}{2} - 5x + c$$

Solving for y gives

$$y = \left(\frac{3x^2}{2} - 15x + 3c\right)^{1/3}$$

i.e. $y = \left(\frac{3x^2}{2} - 15x + k\right)^{1/3}$ since $3c \equiv$ constant.

2. Solve this initial value problem (IVP)

$$y' = \frac{y-1}{x+3}, \quad y(-1) = 0 \qquad (7)$$

Solution

Separating the variables, we have

$$\frac{dy}{y-1} = \frac{dx}{x+3}$$

Integrating both sides, we get

$$\int \frac{dy}{y-1} = \int \frac{dx}{x+3}$$

i.e. $\ln(y-1) = \ln(x+3)+c$ (8)

$$y-1 = e^{(\ln(x+3)+c)} = e^c(x+3)$$

$$\Longrightarrow y = 1+c_1(x+3) \quad (\text{where } c_1 = e^c.)$$

Applying initial condition, we have

$$0 = 1+c_1(-1+3) = 1+2c_1$$

$$\Longrightarrow c_1 = -\frac{1}{2}$$

So the solution to IVP is

$$y = 1-\frac{1}{2}(x+3) = -\frac{1}{2}(x+1)$$

Exercise 1.3

A. Determine whether the given ode is separable

1. $y' = y^3+y$
2. $y' = \sin(x+y)$
3. $y' = \dfrac{3e^{x+y}}{x^2+2}$
4. $s^2 + \dfrac{ds}{dt} = \dfrac{s+1}{st}$

B. Solve the equation

1. $y' = \dfrac{x^2-1}{y^2}$
2. $\dfrac{dy}{dx} = \dfrac{1}{xy^3}$
3. $\dfrac{dy}{dx} = 3x^2(1+y^2)$
4. $\dfrac{dy}{dx} + y^2 = y$
5. $x\dfrac{dv}{dx} = \dfrac{1-4v^2}{3v}$
6. $\dfrac{dy}{dx} = y(2+\sin x)$

C. Solve these IVP

1. $x^2dx + 2y\,dy = 0$ $y(0) = 2$
2. $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2y+1}$ $y(0) = -1$
3. $\dfrac{dy}{dx} = x^2(1+y)$ $y(0) = 3$
4. $\sqrt{y}\,dx + (1+x)dy = 0$ $y(0) = 1$

1.4 EXACT EQUATIONS

The first order differential equation

$$y' = f(x,y) \tag{1}$$

may be expressed in the differential form

$$M(x,y)dx + N(x,y)dy = 0 \tag{2}$$

For example, the equation

$$\frac{dy}{dx} = \frac{3x^2-y}{x-1} \tag{3}$$

can be expressed as

$$(y-3x^2)dx + (x-1)dy = 0 \tag{4}$$

Comparing (2) and (4) we have

$$M(x,y) = y-3x^2 \quad \text{and} \quad N(x,y) = x-1.$$

1.4.1 Exact Differential Form

Definition

The differential form

$$M(x,y)dx + N(x,y)dy \qquad (5)$$

is said to be exact in a rectangle R, if there is a function F(x,y) such that

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x,y) = N(x,y)$$

for all (x,y) in R.

i.e. the total differential of F(x,y) satisfies

$$df(x,y) = M(x,y)dx + N(x,y)dy$$

If $M(x,y)dx + N(x,y)dy$ is an exact differential form, then the equation

$$M(x,y)dx + N(x,y)dy = 0 \qquad (6)$$

is called an exact equation.

1.4.2 Test For Exactness

Theorem

Suppose the first partial derivatives of M(x,y) and N(x,y) are continuous in a rectangle R, then

$$M(x,y)dx + N(x,y)dy = 0$$

is an exact equation in R if and only if

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$

Example 1

Show that

$$(1+e^x y+xe^x y)dx + (xe^x+2)dy = 0 \qquad (7)$$

is an exact equation.

Solution

In this case

$$M(x,y) = 1+e^x y+xe^x y \quad \text{and} \quad N(x,y) = xe^x+2$$

$$\frac{\partial M(x,y)}{\partial y} = e^x+xe^x \qquad (1)$$

and

$$\frac{\partial N(x,y)}{\partial x} = e^x+x^e x \qquad (2)$$

Since (1) = (2), it follows that (8) is an exact equation.

Example 2

Show that

$$(x+3x^3 \sin y)dx + (x^4 \cos y)dy = 0 \qquad (8)$$

is not an exact equation.

Solution

In this equation

$$M(x,y) = x+3x^3 \sin y, \qquad N(x,y) = x^4 \cos y.$$

$$\frac{\partial M(x,y)}{\partial y} = 3x^3 \cos y \qquad (1)$$

$$\frac{\partial N(x,y)}{\partial x} = 4x^3 \cos y \qquad (2)$$

Since (1) $\neq$ (2), equation (8) is not an exact equation.

1.4.3 Method For Solving Exact Equations

(a) If $M(x,y)dx + N(x,y)dy = 0$ is an exact equation, then $\frac{\partial f}{\partial x} = M(x,y)$.

We then integrate this last equation with respect to x to get

$$F(x,y) = \int M(x,y)dx + g(y) \qquad (9)$$

(b) To determine g(y), we take the partial derivative with respect to y of both sides of (9) and substitute N for $\frac{\partial f}{\partial y}$. We then solve for $g'(y)$

(c) Integrate $g'(y)$ to get $g(y)$ up to a numerical constant. Substituting $g(y)$ into equation (9) gives $F(x,y)$.

(d) The solution to $M(x,y)dx + N(x,y)dy = 0$ is then given implicitly by

$$F(x,y) = c. \qquad (10)$$

Example 1

Solve

$$(2xy-\sec^2 x)dx + (x^2+2y)dy = 0 \qquad (11)$$

Solution

Here $M(x,y) = 2xy-\sec^2 x$, and $N(x,y) = x^2+2y$ since

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

equation (11) is an exact one.

To find $F(x,y)$, we begin by integrating $M(x,y)$ with respect to x to get

$$F(x,y) = \int(2xy-\sec^2 x)dx + g(y) \quad (12)$$

$$= x^2y - \tan x + g(y)$$

Next, we take the partial derivative of (12) with respect to y and substitute x^2+2y for $N(x,y)$

$$\frac{\partial F}{\partial y}(x,y) = N(x,y)$$

i.e. $x^2+g'(y) = x^2+2y$. Thus,

$$g'(y) = 2y \quad (13)$$

Note that the choice of constant of integration is not important at this stage. So, we have on integrating (13) that

$$g(y) = y^2.$$

Substituting for $g(y)$ in (12) we get

$$F(x,y) = x^2y - \tan x + y^2.$$

So the solution to equation (11) is given implicitly by

$$x^2y - \tan x + y^2 = C.$$

Example 2

Solve

$$(1+e^xy+xe^xy)dx + (xe^x+2)dy = 0 \quad (14)$$

Solution

Here $M(x,y) = 1+e^xy+xe^xy$ and $N(x,y) = xe^xy+2$. Since

$$\frac{\partial M}{\partial y} = e^x+xe^x = \frac{\partial N}{\partial x}$$

equation (14) is an exact one.

Integrating $N(x,y)$ with respect to y, we get

$$F(x,y) = \int N(x,y)dy + g(x)$$

$$= (xe^x+2)dy = xe^xy+2y+g(x) \quad (15)$$

We now take partial derivative of (15) with respect to x and substitute for M to get

$$\frac{\partial F(x,y)}{\partial x} = M(x,y)$$

i.e.

$$e^x y + xe^x y + g'(x) = 1 + e^x y + xe^x y$$

Thus $\qquad g'(x) = 1 \implies g(x) = x.$

Hence $F(x,y) = xe^x y + 2y + x$. So the solution to equation (14) is implicitly given as

$$xe^x y + 2y + x = c$$

Note that in this case, we can solve explicitly for y to get

$$y = \frac{c-x}{xe^x+2}$$

<u>Example 5</u>

Show that

$$(x+3x^3\sin y)dx + (x^4\cos y)dy = 0 \qquad (16)$$

is not exact, but that multiplying this equation by a factor x^{-1} yields an exact equation. Use this fact to solve (16).

<u>Solution</u>

In this equation

$$M(x,y) = x+3x^3\sin y \quad \text{and} \quad N(x,y) = x^4\cos y$$

$$\frac{\partial M(x,y)}{\partial y} = 3x^3\cos y \qquad (1)$$

and

$$\frac{\partial N(x,y)}{\partial x} = 4x^3\cos y \qquad (2)$$

(1) ≠ (2), so equation (16) is not exact. multiplying (16) by x^{-1} we get

$$(1+3x^2\sin y)dx + (x^3\cos y)dy = 0 \qquad (17)$$

From this new equation (17) we have

$$M(x,y) = 1+3x^2\sin y \quad \text{and} \quad N(x,y) = x^3\cos y.$$

Now

$$\frac{\partial M}{\partial y} = 3x^2\cos y = \frac{\partial N}{\partial x}.$$

So (17) is an exact equation.

Integrating $M(x,y)$ with respect to x, we get

$$F(x,y) = \int M(x,y)\,dx + g(y)$$

$$= x + x^3 \sin y + g(y) \qquad (18)$$

Taking partial derivative of (18) with respect to y and substituting for $N(x,y)$ we get

$$x^3\cos y + g'(y) = x^3\cos y$$

$$\Longrightarrow g'(y) = 0 \Longrightarrow g(y) = c.$$

From (18), we get that

$$F(x,y) = x+x^3\sin y + c.$$

Then the solution to (17) is

$$x + x^3\sin y + c = k.$$

$$\therefore x + x^3\sin y = p \quad (\text{where } p = k-c).$$

But equations (16) and (17) differ only by a factor of x. Then any solution of one will be a solution for the other; whenever $x \neq 0$.

So the solution to the problem is given implicitly by

$$x + x^3\sin y = P$$

NOTE:

In Example 3 above, the function $\mu = x^{-1}$ is called an integrating factor for equation (16). Generally, any factor $\mu(x,y)$ which changes a non exact equation into an exact one is called an integrating factor. We shall treat this in detail later.

Exercises

A. Determine whether or not each of the following equations is exact. If it is exact, find the solution.

1. $(2x+3)dx + (2y-2)dy = 0$
2. $(2x+4y)dx + (2x-2y)dy = 0$
3. $(9x^2+y-1) - (4y-x)dy = 0$
4. $(2xy^2+2y)dx + (2x^2y+2x)dy = 0$
5. $(e^x\sin y-2y\sin x)dx + (e^x\cos y+2\cos x)dy = 0$
6. $(e^x\sin y+3y) - (3x-e^x\sin y)y' = 0$
7. $(x\ln y+xy)dx + (y\ln x+xy)dy = 0, \quad x > 0, \ y > 0$
8. $-(ax+by)dx + (bx-cy)dy = 0.$

B. Find the value of b for which each of the following equations is exact, and then solve it using that value of b.

1. $(xy^2 + bx^2y)dx + (x+y)x^2 dy = 0$

2. $(ye^{2xy} + x)dx + bxe^{2xy} dy = 0$

1.5 LINEAR EQUATIONS

A linear first order ode is an equation that can be expressed in the form

$$a_1(x)\frac{dy}{dx} + a_o(x)y = b(x) \qquad (1)$$

where $a_1(x)$, $a_o(x)$ and $b(x)$ depend only on the independent variable x, not on y.

E.g. $$(\sin x)\frac{dy}{dx} = x^2\sin x - (\cos x)y$$

is linear because it can be written in the form

$$(\sin x)\frac{dy}{dx} + (\cos x)y = x^2\sin x$$

and $\sin x = a_1(x)$, $\cos x = a_o(x)$ and $x^2\sin x = b(x)$, are functions of only the independent variable x.

However $y\frac{dy}{dx} + (\sin x)y^3 = e^x + 1$ is not linear, because it cannot be put in the form of equation (1). This is due to the presence of the y^3 and $y\frac{dy}{dx}$ terms.

1.5.1 Standard Form of Linear O.D.E.

Let us assume that the functions $a_1(x)$, $a_o(x)$ and $b(x)$ are continuous on an interval and that $a_1(x) \neq 0$ on that interval, then by dividing (1) by $a_1(x)$ we get

$$\frac{dy}{dx} + p(x)y = Q(x) \qquad (2)$$

where $p(x) = \frac{a_o(x)}{a_1(x)}$ and $Q(x) = \frac{b(x)}{a_1(x)}$.

Equation (2) is called the standard form of first-order linear o.d.e.

1.5.2 Method of Solution of First-Order Linear O.D.E.

Let us express equation (2) in the form

$$[P(x)y - Q(x)]dx + dy = 0 \qquad (3)$$

If we test this equation (3) for exactness, we find that

$$\frac{\partial M}{\partial y} = P(x) \quad \text{and} \quad \frac{\partial N}{\partial x} = 0.$$

Consequently, (3) is exact only when $P(x) = 0$. So we now look for a factor called an integrating factor, μ, which depends only on x, for the equation (3).

So, let us multiply (3) by a function $\mu(x)$ and try to determine $\mu(x)$ so that the resulting equation

$$[\mu(x)P(x)y - \mu(x)Q(x)]dx + \mu(x)dy = 0 \quad (4)$$

is exact. From equation (4), we have

$$\frac{\partial M(x,y)}{\partial y} = \mu(x)p(x)$$

and

$$\frac{\partial N(x,y)}{\partial x} = \frac{d\mu(x)}{dx}$$

We see that equation (4) is exact if $\mu(x)$ satisfies the differential equation

$$\frac{d\mu}{dx}(x) = p(x)\mu(x)$$

Equation (5) is separable, so

$$\frac{d\mu(x)}{\mu(x)} = p(x)dx$$

$$\Longrightarrow \ln \mu(x) = p(x)dx$$

$$\Longrightarrow \mu(x) = e^{\int p(x)dx} \quad (6)$$

This is our needed integrating factor. So we can now proceed to solve equation (4) using our already known method.

OR, briefly, multiplying equation (2) by $\mu(x)$ obtained using (6) we have

$$\mu(x)\frac{dy}{dx} + p(x)\mu(x)y = \mu(x)Q(x) \quad (7)$$

From (5), we know that $\frac{d\mu}{dx} = p(x)\mu(x)$ and so (7) can be written in the form

$$\underbrace{\mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y}_{} = \mu(x)Q(x)$$

$$\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x) \quad (8)$$

This is as a result of product rule of differentiation.

Integrating (8) with respect to x, we get

$$\mu(x)y = \int \mu(x)Q(x)dx + c$$

Solving for y gives

$$y = \mu(x)^{-1}\left(\int \mu(x)Q(x)\,dx + c\right)$$

1.5.3 Initial Value Problem

A differential equation $y'+ay = 0$ together with an initial condition $y(o) = y_0$ form an initial value problem. So with an initial condition, the value of arbitrary constant of integration c can be determined.

Examples

1. Find the solution of the initial value problem

$$y'+2y = e^{-x}, \quad y(o) = 3 \tag{9}$$

Comparing (9) with (2), we see that

$$p(x) = 2 \quad \text{and} \quad Q(x) = e^{-x}$$

So by (6)

$$\mu(x) = e^{\int 2dx} = e^{2x}$$

Multiplying (9) by e^{2x} we get

$$e^{2x}y' + 2e^{2x}y = e^{2x}e^{-x}$$

i.e. $(e^{2x}y)' = e^{x}$.

Integrating both sides with respect to x we get

$$e^{2x}y = \int e^{x}dx = e^{x} + c$$

$$\therefore y = e^{-2x}(e^{x}+c) = e^{-x}+ce^{-2x}$$

$$\therefore y = e^{-x} + ce^{-2x} \tag{10}$$

Using the initial condition, we get

$$y(o) = 1+c = 3 \implies c = 2.$$

So the solution to ivp (9) is

$$y = e^{-x} + 2e^{-2x}$$

Exercises

A. Obtain a general solution to the equations

1. $y'-y = e^{3x}$
2. $xy'+2y = e^{-3}$
3. $y' = \frac{y}{x} + 2x + 1$
4. $(x^2+1)y'+xy = x.$
5. $xy'+3y+2x^2 = x^3+4x$

B. Solve the following initial value problems

1. $y'+4y-e^{-x} = 0 \qquad y(0) = \frac{4}{3}$

2. $y' + \frac{3y}{x} + 2 = 3x \qquad y(1) = 1$

3. $\sin x\, y' + y\cos x = x\sin x \qquad y(\frac{\pi}{2}) = 2$

4. $x^3 y'+3x^2 y = x, \quad y(2) = 0$

1.5.4 Special Integrating Factor

Consider an equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

If this equation is not already exact, we will try to choose a function μ, which may depend on both x and y so that the equation

$$\mu M(x,y)dx + \mu N(x,y)dy = 0 \tag{2}$$

is exact. This function μ is called an integrating factor. Equation (2) can then be solved by the method already treated, and we note that the solution to (2) is also a solution of the original equation (1).

Theorem

If $\dfrac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N}$ is continuous and depends only on x, then

$$\mu(x) = \exp\left(\int \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} dx\right) \tag{3}$$

is an integrating factor for (1).

If $\dfrac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M}$ is continuous and depends only on y, then

$$\mu(y) = \exp\left(\int \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M} dy\right) \tag{4}$$

is an integrating factor for (1).

METHOD FOR FINDING SPECIAL INTEGRATING FACTOR

If $Mdx + Ndy = 0$ is neither separable nor linear, compute $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the equation is exact.

If it is not exact, consider

$$\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} \tag{5}$$

If (5) is a function of just x, then an integrating factor is given by (3).

If not, consider

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M} \qquad (6)$$

If (6) is just a function of only y, then an integrating factor is given by (4).

Examples

Solve $(2x^2+y)dx + (x^2y-x)dy = 0$ (7)

Solution

By inspection, of (7), we shall see that (7) is neither separable, nor linear. We also note that

$$\frac{\partial M}{\partial y} = 1 \neq (2xy-1) = \frac{\partial N}{\partial x}$$

So (7) is not an exact equation. We compute

$$\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} = \frac{1-(2xy-1)}{x^2y-x} = \frac{2(1-xy)}{-x(1-xy)} = -\frac{2}{x}$$

Since we obtain a function of only x, then an integrating factor for (7) is given by (3). That is

$$\mu(x) = \exp\left(\int \frac{-2}{x}\,dx\right) = x^{-2}$$

Then, multiplying (7) by x^{-2}, we get an exact equation

$$(2+yx^{-2})dx + (y-x^{-1})dy = 0 \qquad (8)$$

Solving (8) using already discussed method, we get an implicit solution

$$2x-yx^{-1} + \frac{y^2}{2} = C$$

Exercises

A. Identify the equation as separable, linear, exact or having an integrating factor that is a function of either x alone or y alone.

1. $(2x+yx^{-1})dx + (xy-1)dy = 0$
2. $(y^2+2xy)dx - x^2dy = 0$
3. $(2x+y)dx + (x-2y)dy = 0$
4. $(2y^2x-y)dx + xdy = 0$

B. Solve the following equations

1. $(3x^2+y)dx + (x^2y-x)dy = 0$

2. $(x^4-x+y)dx - xdy = 0$

3. $(y^2+2xy)dx - x^2dy = 0$

4. $(2xy)dx + (y^2-3x^2)dy = 0$

C. Find an integrating factor for these equations and hence solve the equation using the factor.

1. $(12+5xy)dx + (6xy^{-1}+3x^2)dy = 0$

2. $(2y^2-6xy)dx + (3xy-4x^2)dy = 0$

1.6 NON-LINEAR DIFFERENTIAL EQUATIONS

Consider the differential equation of the form

$$y' = f(x,y) \tag{1}$$

$$y(x_0) = y_0 \tag{2}$$

We have seen cases when $f(x,y)$ is linear. We now consider the case when $f(x,y)$ is not linear. In this case, our earlier treatment no longer applies. In this section, we will discuss, in a general way, some features of non-linear initial value problems. In particular, we will note some important differences between the non-linear problem (1), (2) and linear problem consisting of $y'+P(x)y = q(x)$ and the initial condition (2).

The first order linear differential equations are relatively simple because there is a formula for solution of such equation in all cases. In contrast, there is no corresponding general method for solving first order non-linear equations.

In fact, the analytic determination of the solution $y = \phi(x)$ of a non-linear equation is usually very difficult and often impossible.

The lack of a general formula for the solution of a non-linear equation made us to look for methods which may yield approximate, perhaps numerical solutions and qualitative information about solutions. We differ this till later

1.6.1 Homogeneous Equations

Definition

If the right-hand side of the equation

$$\frac{dy}{dx} = f(x,y) \tag{1}$$

can be expressed as a function of the ratio $\frac{y}{x}$ alone, then the equation is said to be homogeneous.

NOTE: We have another meaning for homogeneity and so its meaning here has to be reserved as is applied in this case.

Consider, for example, the equation

$$(x-y)dx + xdy = 0 \tag{2}$$

This can be written in the form

$$\frac{dy}{dx} = \frac{y-x}{x} = \frac{y}{x} - 1$$

Since we have expressed $\frac{(y-x)}{x}$ as a function of the ratio $\frac{y}{x}$, then equation (2) is homogeneous.

The equation

$$(x-2y+1)dx + (x-y)dy = 0 \tag{3}$$

cannot be written as a function of $\frac{y}{x}$ alone because of $\frac{1}{x}$ in the numerator. Hence equation (3) is not homogeneous.

1.6.2 Test for Homogeneity

To test if equation (1) is homogeneous, replace x by tx and y by ty. Then (1) is homogeneous if and only if

$$f(tx,ty) = f(x,y) \tag{4}$$

METHOD OF SOLUTION TO HOMOGENEOUS EQUATIONS

To solve a homogeneous equation, we make a known substitution. That is, we set $V = \frac{y}{x}$. The homogeneous equation then takes the form

$$\frac{dy}{dx} = G(v) \tag{5}$$

and all we need is to express $\frac{dy}{dx}$ in terms of x and v. Since $v = \frac{y}{x}$, then $y = vx$. Keeping in mind that both v and y are functions of x, we use product rule for differentiation to get from $y = vx$. That is

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{6}$$

Substituting the above expression for $\frac{dy}{dx}$ into equation (5) gives

$$v + x\frac{dv}{dx} = G(v) \qquad (7)$$

Fortunately, the new equation (7) is separable, and we can obtain its implicit solution from

$$\int \frac{dv}{G(v)-v} = \int \frac{dx}{x}$$

What remains is then to express the solution in terms of the original variables x and y!

Examples

1. Solve

$$(xy+y^2+x^2)\,dx - x^2dy = 0 \qquad (8)$$

Solution.

A check will show that equation (8) is not separable, exact nor linear. If we express (8) in the derivative form we have

$$\frac{dy}{dx} = \frac{xy+y^2+x^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1 \qquad (9)$$

So, we see that the right hand side of (9) is a function of $\frac{y}{x}$.

Thus the equation is homogeneous.

Now let $v = \frac{y}{x}$ and we know that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

With this substitution in equation (9) we get

$$v + x\frac{dv}{dx} = v + v^2 + 1$$

The above equation is separable and on separating the variables and integrating, we obtain

$$\int \frac{dv}{v^2+1} = \int \frac{dx}{x}$$

$$\Longrightarrow \arctan v = \ln x + c$$

Hence $v = \tan(\ln x + c)$.

Finally, we substitute $\frac{y}{x}$ for v and solve for y to get

$$y = x\tan(\ln x + c),$$

as an explicit solution to (8).

Exercises

A. Show that the following equations are homogeneous, and find their solutions

1. $y' = \frac{(x+y)}{x}$ 2 $y' = \frac{4y-3x}{2x-y}$

3. $2ydx - xdy = 0$ 4. $y' = \frac{4x+3y}{2x+y}$

5. $(x^2+3xy+y^2)dx - x^2dy = 0$

B. Find the solution of the equation

1. $\frac{dy}{dx} = \frac{2y-x+5}{2x-y-4}$ 2. $\frac{dy}{dx} = \frac{2y-x}{2x-y}$

1.7 EXISTENCE AND UNIQUENESS OF SOLUTION

We shall here state a fundamental theorem giving conditions under which an initial value problem for a first order linear equation will always have one and only one solution.

Theorem 1

If the function p and q are continuous on an open interval $\alpha < x < \beta$ containing the point $x = x_0$, then there exists a unique function $y = \phi(x)$ that satisfies the differential equation

$$y' + p(x)y = g(x) \tag{1}$$

for $\alpha < x < \beta$ and that also satisfies the initial condition

$$y(x_0) = y_0 \tag{2}$$

The following fundamental existence and uniqueness theorem for non-linear equations is analogous to Theorem 1 for linear equation.

Theorem 2

Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < x < \beta$, $\gamma < y < \delta$, containing the point (x_0, y_0). Then in some interval $x_0 - h < x < x_0 + h$ contained in $\alpha < x < \beta$, there is a unique solution $y = \phi(x)$ of the initial value problem (1) and (2).

$$y' = f(x,y), \quad y(x_0) = y_0$$

It can be shown by means of examples that some conditions on f are essential in order to obtain the result stated in the theorem. For instance, the following example shows that the initial value problem (1) and (2) may have more than one solution if the hypothesis of Theorem 2 is violated.

Example

Find the solution of

$$y' = y^{1/3}, \qquad y(0) = 0, \quad (x \geq 0) \qquad (3)$$

We can show that $y = \phi(x) = (\frac{2}{3}x)^{3/2}$, $x \geq 0$ satisfies (3). On the other hand, the function

$$y = \phi(x) = 0$$

is also a solution of the given initial value problem. Hence, this problem does not have a unique solution.

This fact does not contradict the existence and uniqueness theorem since

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^{1/3}) = \frac{1}{3} y^{-2/3}$$

and this function is not continuous, or even defined, at any point where $y = 0$.

1.7.1 General Solution

One way in which linear and non-linear equations differ is in connection with the concept of a general solution. For a first order linear equation, it is possible to obtain a solution containing one arbitrary constant, from which all possible solutions follow by specifying values for this constant.

For non-linear equations, this may not be the case; though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by giving values to this constant.

Example

Consider

$$y' + y^3 = 0 \qquad c = \text{arbitrary} \qquad (4)$$

$y = \phi(x) = [2(x+c)]^{-1/2}$ where c constant is a solution to (4).

Note that $y = 0$ is also a solution to (4) but this solution can not be obtained by assigning values to c in $y = \phi(x)$.

Thus, we will use the term "general solution" only when discussing linear equations.

2. SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

2.1 INTRODUCTION

The general second order differential equation is an equation of the form

$$f(x,y,y',y'') = 0 \qquad (1)$$

We are interested in the equations which can be solved for y'', that is, equations which can be written in the form

$$y'' = f(x,y,y') \qquad (2)$$

Note that for first order differential equation $y' = f(x,y)$ we found that there is a solution containing one arbitrary constant. So, a second order equation involves a second derivative, and hence, roughly speaking, two integrations are required to find a solution. It is natural then to expect a solution containing two arbitrary constants.

We expect the solution of a second order equation to contain two arbitrary constants. In order to obtain a unique solution, it will be necessary to specify two conditions; for example, the value of the solution y_0 and its derivative y_0' at a point x_0. These conditions are referred to as initial conditions.

2.2 SECOND ORDER LINEAR EQUATIONS

The general second order linear equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \qquad (3)$$

where $a_0(x)$, $a_1(x)$, $a_2(x)$ and $b(x)$ are continuous functions of x on an interval I. When a_0, a_1 and a_2 are constants, we say that the equation has constant coefficients. otherwise, it is said to have a variable coffficients.

STANDARD FORM OF SECOND ORDER LINEAR EQUATION

If $a_2(x) \neq 0$, we can rewrite (3) in the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x) \qquad (4)$$

where

$$p(x) = \frac{a_1(x)}{a_2(x)}, \quad q(x) = \frac{a_0(x)}{a_2(x)} \quad \text{and} \quad g(x) = \frac{b(x)}{a_2(x)}$$

are continuous on I. Equation (4) is called the standard form of second order linear equations.

Associated with (4) is the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \qquad (5)$$

which is obtained from (4) by replacing $g(x)$ with zero. We say that (4) is a non homogeneous equation and that (5) is the corresponding homogeneous equation. Note that the meaning of homogeneous is not related to the use of the term for the first order equations.

2.2.1 Linear Differential Operator

Let us consider the expression on the left-hand side of equation (5), which we can write with prime notation as

$$y''(x) + p(x)y'(x) + q(x)y(x) \qquad (6)$$

Given any function y with continuous derivative on the interval I, then (6) generates a new function that will be denoted by $L[y]$. That is

$$L[y](x) = y'' + py' + qy \qquad (7)$$

The term function in the general sense means a mapping that associates with each element in the domain a unique element in the range.

So L can be interpreted as a function that is defined on a set of functions. The domain is the collection of functions with a continuous second derivative; the range consists of continuous functions; and the rule of correspondence is given by (7).

Such mappings are called operators. Since L involves differentiation, we refer to L as a differential operator.

Example

Let $p(x) = x$, and $q(x) = e^x$, then

$$L[y](x) = y''(x) + xy'(x) + (e^x)y(x) \qquad (8)$$

1. If $y_1(x) = x^3$, we get

$$L[y_1](x) = 6x-x(3x^2)+(x-1)x^3$$
$$= x^4+2x^3+6x$$

So L maps x^3 to x^4+2x^3+6x.

2. If $y_2(x) = \sin 2x$, then we have from (8)

$$L[y_2](x) = -4\sin 2x + x(2\cos 2x) + (x-1)\sin 2x$$
$$= x\sin 2x - 5\sin 2x + 2x\cos 2x \qquad (9)$$

So L maps $\sin 2x$ to

$$x\sin 2x-5\sin 2x+2x\cos 2x$$

LINEARITY OF DIFFERENTIAL OPERATOR L

Lemma

Let $L[y] := y''+py'+qy$ and y_1 and y_2 are two functions with continuous second derivatives on the interval I, and if c is any constant, then

$$L[y_1+y_2] = L[y_1] + L[y_2] \qquad (10)$$

$$L[cy_1] = cL[y_1] \qquad (11)$$

Proof: Exercise

Note that an operator which satisfies (10) and (11) for any constant c and any functions y_1 and y_2 in its domain is called a linear operator.

Example

Show that the operator T defined by

$$T[y](x) := y''(x) + \sin(y(x)) \qquad (12)$$

where y is any function whose second derivative is continuous for all real x is nonlinear.

Solution

To show that T is a nonlinear operator, it suffices to show that property (11) is not always satisfied.

Let us choose $y_1(x) = x$. Since we have $y_1''(x) = 0$, we have

$$T[cy_1](x) = \sin(cx)$$

and

$$cT[y_1](x) = c\sin x.$$

In general $\sin(cx) \neq c\sin x$.

[e.g. Take $c = 2$, and $x = \frac{\pi}{2}$] So, (11) is violated. Hence T is a nonlinear operator.

3.2.2 Linear Combinations of Solutions

Theorem

Let y_1 and y_2 be solutions to the homogeneous equation

$$y'' + py' + qy = 0 \tag{13}$$

Then any linear combination $c_1y_2 + c_2y_2$ of y_1 and y_2, where c_1 and c_2 are constants is also a solution of (13).

Example

Given that $y_1(x) = e^{2x}\cos 3x$ and $y_2(x) = e^{2x}\sin 3x$ are solutions to the homogeneous equation

$$y''-4y'+13y = 0 \tag{14}$$

find a solution to (14) that satisfies the initial conditions

$$y(o) = 2 \quad \text{and} \quad y'(o) = -5 \tag{15}$$

Solution

As a consequence of the above theorem, any linear combination

$$y(x) = c_1e^{2x}\cos 3x + c_2e^{2x}\sin 3x \tag{16}$$

with c_1 and c_2 arbitrary constants will be a solution to (14).

Thus, we try to select c_1 and c_2 so as to satisfy the initial conditions (15).

On differentiating ~~(16) we get~~

$$y_1'(x) = c_1(2e^{2x}\cos 3x-3e^{2x}\sin 3x) + c_2(2e^{2x}\sin 3x+3e^{2x}\cos 3x) \tag{17}$$

Substituting (16) and (17) into the initial conditions (15), we get a system of equations

$$c_1 = 2, \qquad 2c_1+3c_2 = -5$$

whose solution is given by $c_1 = 2$, $c_2 = -3$. Hence the solution to (14) that satisfies (15) is

$$y(x) = 2e^{2x}\cos 3x - 3e^{2x}\sin 3x$$

Exercises

1. Let $L[y](x) = y''(x) - 4y'(x) + 3y(x)$. Compute

   (a) $L[x^2]$ (b) $L[e^{3x}]$

2. Show that T defined by

$$T[y](x) = y''(x) - y'(x) + y^2(x)$$

   where y is any function whose second derivative is continuous for all real x is a nonlinear operator.

3. Given that $y_1(x) = e^{2x}\cos x$ and $y_2(x) = e^{2x}\sin x$ are solutions to the homogeneous equation

$$y'' - 4y' + 5y = 0$$

   find solutions to this equation that satisfy the following initial conditions

   (a) $y(0) = 1, \quad y'(0) = -1$

   (b) $y(\pi) = 4e^2, \quad y'(\pi) = 5e^2$

4. Given that $y_1(x) = e^{2x}$ and $y_2(x) = e^{-x}$ are solutions to the homogeneous equation

$$y'' - y' - 2y = 0$$

   find solutions to this equation that satisfy the following initial conditions

   (a) $y(0) = -1, \quad y'(0) = 4$

   (b) $y(0) = \frac{3}{2}, \quad y'(0) = 0.$

2.2.3 Fundamental Solution of Homogeneous Equation

We now discuss those properties of homogeneous equations that help us to obtain all solutions of these equations. Consider this particular equation

$$L[y] = \frac{d^2y}{dx^2} - y = 0 \qquad (1)$$

It is easy to verify that the two functions $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions to this homogeneous equation. Since L is linear, we know that

$$y = c_1e^x + c_2e^{-x}$$

is also a solution for any choice of constants c_1 and c_2. We now ask if all the solutions of (1) can be represented by $c_1e^{x} + c_2e^{-x}$ with appropriate choices for c_1 and c_2? The answer lies in applying the existence and uniqueness theorem, as we now show.

Let $\phi(x)$ be a solution to (1), and let x_0 be a fixed real number. If we choose c_1 and c_2 so that

$$\phi(x_0) = c_1e^{x_0} + c_2e^{-x_0} \tag{2}$$

$$\phi'(x_0) = c_1e^{x_0} - c_2e^{-x_0} \tag{3}$$

Then since $\phi(x)$ and $c_1e^{x} + c_2e^{-x}$ satisfy the same initial conditions at x_0, the uniqueness conclusion guarantees that

$$\phi(x) = c_1e^{x} + c_2e^{-x} \quad \text{for all } x.$$

To solve (2) and (3) for c_1 we add the two equations and then divide by $2e^{x_0}$ to get

$$c_1 = \frac{\phi(x_0) + \phi'(x_0)}{2e^{x_0}} \tag{4}$$

Subtracting (3) from (2) and dividing by $2e^{x_0}$ we get

$$c_2 = \frac{\phi(x_0) - \phi'(x_0)}{2e^{x_0}} \tag{5}$$

Notice that since e^{x_0} and e^{-x_0} are never zero, we can always carry out the above method to get c_1 and c_2.

So by uniqueness, we have

$$\phi(x) = c_1e^{x} + c_2e^{-x} \quad \text{for all } x$$

where c_1 and c_2 are given by (4) and (5) respectively.

So we conclude that every solution of (1) can be expressed as a linear combination of two particular solutions of (1). Note that similar result holds for general linear homogeneous equations, provided that the two solutions y_1 and y_2 satisfy a certain property.

Theorem

Let y_1 and y_2 denote two solutions on (a,b) of

$$y'' + p(x)y' + q(x)y = 0 \qquad (6)$$

where p and q are continuous on (a,b). If at some point x_0 in (a,b) these solutions satisfy

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0 \qquad (7)$$

then every solution of (6) on (a,b) can be expressed in the form

$$y(x) = c_1y_1(x) + c_2y_2(x) \qquad (8)$$

where c_1 and c_2 are constants.

Note that the linear combination of y_1 and y_2 in (8) is referred to as a "general solution" to (6).

Definition

For any two differentiable functions y_1 and y_2, the function

$$W[y_1,y_2](x) := y_1'(x)y_2'(x) - y_1'(x)y_2'(x) \qquad (9)$$

is called the Wronskian of y_1 and y_2.

A convenient way of writing the Wronskian $W[y_1,y_2]$ is in terms of the determinant

$$W[y_1,y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \qquad (10)$$

FUNDAMENTAL SOLUTION SET

Definition

A pair of solution $\{y_1,y_2\}$ of $y''+py'+qy = 0$ on (a,b) is called a fundamental solution set if

$$W[y_1,y_2](x_0) \neq 0 \qquad (11)$$

for some $x_0 \in (a,b)$.

Example

$y_1 = e^x$ and $y_2 = e^{-x}$ of (1) form a fundamental solution set of (1) on $(-\infty,\infty)$ because

$$W[y_1,y_2] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^xe^{-x} - e^xe^{-x}$$

$$= -2 \neq 0 \text{ for any } x.$$

2.2.4 Method of Solution for Homogeneous Equations

To determine all solutions to $y''+py'+qy = 0$ we do the followings:

A. Find two solutions y_1, y_2 that form a fundamental solution set,

B. A general solution is

$$y(x) = c_1y_1(x) + c_2y_2(x) \tag{12}$$

where c_1 and c_2 are arbitrary constants.

Example

1. Given that $y_1(x) = \cos 3x$ and $y_2(x) = \sin 3x$ are solutions to

$$y'' + 9y = 0 \tag{13}$$

on $(-\infty, \infty)$, find a general solution to (13).

Solution

We first verify that $\{\cos 3x, \sin 3x\}$ is a fundamental solution set. We can easily verify that y_1 and y_2 are actually solutions of (13). We now show that $W[y_1,y_2] \neq 0$ for some x in $(-\infty, \infty)$.

$$W[y_1,y_2] = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x$$

$$= 3[\cos^2 3x + \sin^2 3x] = 3 \neq 0.$$

Thus $\{\cos 3x, \sin 3x\}$ forms a fundamental solution set, and a general solution to (13) is

$$y(x) = c_1\cos 3x + c_2\sin 3x.$$

2.2.5 Linear Independence of Functions

Definition

Two functions y_1 and y_2 are said to be linearly dependent on an interval I, if there exist constants c_1 and c_2, not both zero, such that $c_1y_1(x) + c_2y_2(x) = 0$ for all x in I.

If two functions are not linearly dependent, they are said to be linearly independent. We note that linear dependence can be extended in a natural way to more than two functions. In the case of just two functions, linear dependence on I is equivalent to one function being a constant multiple of the other function on I.

Example

Determine whether the following pairs of functions y_1 and y_2 are linearly dependent on (-5,5).

(a) $y_1(x) = e^{3x}$, $y_2(x) = x+1$

(b) $y_1(x) = \sin 2x$, $y_2(x) = \cos x \sin x$.

Solution

(a) A quick look at the functions $y_1 = e^{3x}$ and $y_2(x) = x+1$ indicates that neither is a constant multiple of the other. Indeed if a constant c exists such that

$$e^{3x} = c(x+1) \text{ for all } x \text{ in } (-5,5) \text{ then}$$

we have a contradiction by setting x = 0 and x = 1. Thus

$$e^0 = c(0+1) \Longrightarrow c = 1$$

$$e^3 = c(1+1) \Longrightarrow c = \frac{e^3}{2} \neq 1$$

Hence e^{3x} and x+1 are linearly independent.

(b) Since $y_1(x) = \sin 2x = 2\sin x \cos x$. We see that $y_1(x) = 2y_2(x)$. Hence $y_1(x)$ and $y_2(x)$ are linearly dependent on (-5,5).

FUNDAMENTAL SETS AND LINEAR INDEPENDENCE

Corollary 1

Let y_1 and y_2 be solutions to $y''+py'+qy = 0$ on (a,b). Then $\{y_1, y_2\}$ is a fundamental solution set on (a,b) if and only if the functions y_1 and y_2 are linearly independent on (a,b).

Proof: Later.

A PROPERTY OF THE WRONSKIAN OF SOLUTION

Corollary 2

Let y_1, y_2 be solutions to $y''+py'+qy = 0$ on (a,b). Then the Wronskian $W[y_1, y_2](x)$ of the two solutions is either identically zero or never zero on (a,b). Furthermore, the Wronskian of two functions is identically zero if and only if the solutions are linearly dependent.

Example

Show that $y_1(x) = x^{-1}$ and $y_2(x) = x^3$ are solutions to

$$x^2y''-xy'-3y = 0 \qquad (*)$$

on the interval (o, ∞) and give a general solution.

Solution

The verification that y_1 and y_2 are solutions to (*) is straight forward. Substituting $y = x^{-1}$ and $y = x^3$ in (*) gives respectively, the identities

$$x^2(2x^{-3}) - x(-x^{-2}) - 3(x^{-1}) = 0$$

and

$$x^2(6x) - x(3x^2) - 3(x^3) = 0$$

Furthermore, the solution functions x^{-1} and x^3 are linearly independent on (o, ∞). Hence by Corollary 1, $\{x^{-1}, x^3\}$ is a fundamental solution set on (o, ∞) and so a general solution is

$$y = c_1x^{-1} + c_2x^3.$$

Exercises

1. Verify that e^x and e^{-2x} and the linear combination $c_1e^x + c_2e^{-2x}$, where c_1 and c_2 are arbitrary constants are solutions of the differential equation

$$y''+y'-2y = 0$$

2. In problem 1 above, (a) find the unique solution of the differential equation that satisfies the initial conditions $y(o) = 1$, $y'(o) = 0$.

   (b) What is the unique solution of this problem if the initial conditions are $y(1) = 0$, $y'(1) = 0$?

3. Compute the Wronskians of the following pairs of functions

   (a) e^{mx}, e^{nx}, where m and n are integers and $m \neq n$.

   (b) x, xe^x (c) $e^x\sin x$, $e^x\cos x$

   (d) $\cos^2x$, $1+\cos 2x$.

4. In the following problems

   (a) Verify that the functions y_1 and y_2 are linearly independent solutions of the given differential equation.

(b) Find a general solution to the given differential equation.

(c) Find the solution that satisfies the given initial conditions

(i) $y''-5y'+6y = 0;\quad y_1(x) = e^{2x},\quad y_2(x) = e^{3x},$
$y(0) = -1,\quad y'(0) = -4.$

(ii) $y''-5y' = 0;\quad y_1(x) = 2,\quad y_2(x) = e^{5x},$
$y(0) = 2,\quad y'(0) = 5.$

(iii) $x^2y''-2y = 0;\quad y_1(x) = x^2,\quad y_2(x) = x^{-1}$
$y(1) = -2,\quad y'(1) = -7.$

(iv) $xy''-(x+2)y'+2y = 0;\quad y_1(x) = e^x,\quad y_2(x) = x^2+2x+2$
$y(1) = 0,\quad y'(1) = 1.$

2.3 REDUCTION OF ORDER

If one solution of a second order linear homogeneous differential equation is known, a second linearly independent solution can be determined.

The procedure is usually referred to as the method of reduction of order. Suppose we know one solution y_1, not identically zero of

$$y''+p(x)y'+q(x)y = 0 \tag{1}$$

Thus, cy_1, where c is any constant, is also a solution of (1). This suggests the following question: Can we determine a function v, not a constant, such that $y = v(x)y_1(x)$ is a solution of (1)? The answer is yes. V(x) can be determined in a straight forward manner. If we set

$$y = v(x)y_1(x) \tag{2}$$

Then

$$y' = v(x)y_1'(x) + v'(x)y_1(x)$$

$$y'' = v(x)y_1''(x) + 2v'(x)y_1'(x) + v''(x)y_1(x)$$

Substituting for y, y', and y'' in (1) and collecting like terms gives

$$v(y_1''+py_1'+qy_1) + v'(2y_1'+py_1) + v''y_1 = 0 \tag{3}$$

Since y_1 is a solution of (1), the quantity in the first bracket is zero. In any interval in which y_1 is not zero, we can divide by y_1 to obtain

$$v'' + (p + \frac{y_1'}{y_1})v' = 0 \qquad (4)$$

Equation (4) is a first-order linear equation for v' and can be solved. The solution is

$$v'(x) = c \exp\left[- \int (p(s) + \frac{2y_1'(s)}{y_1(s)}) ds\right]$$

$$= c\, u(x)$$

where c is an arbitrary constant and

$$u(x) = \frac{1}{[y_1(x)]^2} \exp\left[- \int p(s)ds\right]$$

Then $v = c \int^x u(t)dt + k$

where k is also an arbitrary constant. However, we can omit the constant k. Since

$$y_1 = v(x)y_1(x)$$

$$= cy_1 \int u(t)dt + ky_1(x)$$

Thus two solutions of (1) are

$$y = y_1(x) \qquad \text{and} \qquad y = y_1(x)\int^x u(t)dt$$

NOTE:

The method of reduction of order does not tell us how to find the first solution of equation (1). It is encouraging to know that we have reduced the problem of solving equation (1) to that of finding just one solution.

2.3.1 Summary: Reduction of Order Procedures

Given a nontrivial solution $y_1(x)$ to $y''+py'+qy = 0$, a second linearly independent solution y(x) can be determined in either of the following ways:

1. Set $y(x) = v(x)y_1(x)$ and substitute for y, y' and y" in the given equation. This gives a separable equation for v'. Solve for v' and integrate v' to obtain v. The desired second solution is then given as $v(x)y_1(x)$.

Again, we set $c_2 = 0$. This gives the same linearly independent solution as before

$$y = xe^x$$

Example 2

Given that $y_1(x) = x$ is a solution to

$$y''-2xy'+2y = 0 \tag{7}$$

determine a second linearly independent solution.

Solution

Here $p(x) = -2x$ and so

$$-\int p(x)dx = \int 2xdx = x^2$$

where we have taken the constant of integration to be zero.

The reduction of order formula (5) gives

$$y(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{[y_1(x)]^2}dx$$

$$= x\int \frac{e^{x^2}}{x^2}dx \tag{8}$$

as a second linearly independent solution. (8) is called integral representation for a second linearly independent solution.

Note that we can not evaluate (8) in terms of elementary functions.

One approach to this is to obtain a power series expansion for the solution (8). We defer this till later.

Exercises

1. Find a second solution of the following differential equation by the method of reduction of order.

(a) $y''-4y'-12y = 0$; $y_1(x) = e^{6x}$.

(b) $y''+2y'+y = 0$; $y_1(x) = e^{-x}$

(c) $x^2y''+2xy'-2y = 0$; $y_1(x) = x$ $x>0$

(d) $x^2y''-2xy'-4y = 0$; $x > 0$, $y_1(x) = x^{-1}$

(e) $xy''+(1-2x)y'+(x-1)y = 0$; $x > 0$, $y_1(x) = e^x$.

2. Given that $f(x) = x$ is a solution to

$$y''-xy'+y = 0$$

Obtain an integral representation for a second linearly independent solution.

2.4 HOMOGENEOUS LINEAR EQUATION WITH CONSTANT COEFFICIENTS

Let us consider the homogeneous linear second order differential equation with constant coefficients.

$$ay'' + by' + cy = 0 \tag{1}$$

where $a\ (\neq 0)$, b and c are real constants.
Since constant functions are continuous every where (1) has solutions which are defined for all x in $(-\infty, \infty)$.

If we can find two linearly independent solutions of (1), say y_1 and y_2, then we can express a general solution of (1) in the form

$$y = c_1y_1 + c_2y_2$$

where c_1 and c_2 are arbitrary constants. A method of solving equation (1) can be found by reading the equation i.e. what function $y = \phi(x)$ satisfies the relationship that a times its second derivative plus b times its first derivative plus c times itself adds up to zero for all x?

With only constant coefficients, it is natural first to consider functions $y = \phi(x)$ such that y, y', y'' differ only by constant multiplicative factor.

One function that has the desired property is the exponential function e^{rx}. Hence we shall try to find solutions of (1) of the form e^{rx} for suitably chosen values of r. Substituting $y = e^{rx}$ in (1) leads to the equation

$$L[e^{rx}] = a(e^{rx})'' + b(e^{rx})' + c(e^{rx}) = 0 \tag{2}$$

$$= e^{rx}(ar^2 + br + c) = 0 \tag{3}$$

Since e^{rx} is never zero, we must have

$$ar^2 + br + c = 0 \tag{4}$$

Equation (4) is called the characteristic or auxiliary equation. If r is a root of (4), then e^{rx} is a solution of (1).

Notice that the coefficients in the characteristic equation (4) are the same as those in the differential equation (1). The roots r_1 and r_2 of (4) are given by

$$r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \tag{5a}$$

$$r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} \tag{5b}$$

The nature of the solutions of (1) clearly depends on the values of r_1 and r_2; which in turn depend on the constant coefficients in the differential equations through the relations (5a) and (5b).

Just as in elementary algebra, we examine separately the case b^2-4ac positive, zero and negative.

CASE 1: REAL AND UNEQUAL ROOTS

For $b^2-4ac > 0$, (5a) and (5b) give two real, unequal values of r_1 and r_2. Hence, e^{r_1x} and e^{r_2x} are solutions of (1). It can also be verified that $W[e^{r_1x}, e^{r_2x}]$ is no where zero. Hence the general solution of (1) is

$$y = c_1e^{r_1x} + c_2e^{r_2x} \tag{6}$$

Example

Find the solution of $y''+5y'+6y = 0$ satisfying the initial conditions $y(o) = 0$, $y'(o) = 1$.

Solution

Substituting $y = e^{rx}$ into the equation leads to

$$r^2+5r+6 = 0$$

i.e. $(r+3)(r+2) = 0$

Hence $r = -2$ and -3. So the general solution is

$$y = c_1e^{-2x} + c_2e^{-3x}$$

To satisfy the initial conditions at $x = 0$, we must have

$$c_1+c_2 = 0, \qquad -2c_1-3c_2 = 1$$

Solving these simultaneous equation, we get $c_1 = 1$ and $c_2 = -1$. Hence the solution of the differential equation satisfying the prescribed conditions is

$$y = e^{-2x} - e^{-3x}.$$

CASE II: REAL AND EQUAL ROOTS

When $b^2-4ac = 0$, it follows from (5a), (5b) that $r_1 = r_2 = \frac{-b}{2a}$, and we have only one solution $(e^{(-b/2a)x}$.

However, we can use the method of reduction of order to reduce the order of the equation and find a second solution. Let

$$y = v(x)e^{(-b/2a)x}$$

Then,

$$y' = v'(x)e^{(-b/2a)x} - \frac{b}{2a}v(x)e^{-(b/2a)x},$$

$$y'' = \left[v''(x) - \frac{b}{a}v'(x) + \frac{b^2}{4a^2}v(x)\right]e^{-(b/2a)x}$$

Substituting for y, y' and y'' in (1) and dividing by the common factor $e^{-(b/2a)x}$ will give the following equation for v.

$$a(v'' - \frac{b}{a}v' + \frac{b^2}{4a^2}v) + b(v' - \frac{b}{2a}v) + cv = 0$$

After collecting like terms together, we find that

$$av'' - (\frac{b^2}{4a} - c)v = 0.$$

Since $b^2-4ac = 0$, the last term drops out, and we get

$$v'' = 0$$

Hence $v(x) = c_1x + c_2$. where c_1 and c_2 are arbitrary constants. Consequently, a second solution of (1) is

$$y(x) = (c_1x+c_2)e^{-(b/2a)x}.$$

In particular, corresponding to $c_2 = 0$; $c_1 = 1$ we get

$$y(x) = xe^{-(b/2a)x}.$$

From our knowledge of the method of reduction of order, $e^{-(b/2a)x}$ and $xe^{-(b/2a)x}$ are linearly independent solutions of (1) [i.e. by finding their Wronskian]. Hence, when $b^2-4ac = 0$, the general solution of (1) is

$$y = c_1e^{r_1x} + xc_2e^{r_1x}, \quad r_1 = \frac{-b}{2a} \qquad (7)$$

Example

Find the general solution of $y''+4y'+4y = 0$.

Solution

Substituting $y = e^{rx}$ in the equation leads to

$$r^2+4r+4 = 0$$

$$\text{i.e. } (r+2)(r+2) = 0$$

Hence $r = -2$ is a repeated root of the characteristic equation.

One solution is e^{-2x} and the second linearly independent solution is xe^{-2x}. Thus, the general solution is

$$y = c_1e^{-2x} + c_2xe^{-2x}$$

CASE III: COMPLEX ROOTS

We consider the case when $b^2-4ac < 0$. The roots r_1 and r_2 of (1) as obtained from (15a) and (15b) are complex numbers of the form $\lambda + i\mu$, where λ and μ are real. Since a, b and c are real, the roots occur in conjugate pairs.

$$r_1 = \lambda + i\mu \qquad \text{and} \qquad r_2 = \lambda - i\mu$$

Hence the general solution to (1) is

$$y = c_1e^{(\lambda+i\mu)x} + c_2e^{(\lambda-i\mu)x} \tag{8}$$

Meaning of e^{ix} and e^{-ix}

Note that

$$e^{ix} = \cos x + i\sin x \tag{9}$$

$$e^{-ix} = \cos x - i\sin x \tag{10}$$

So the functions $\cos x$ and $\sin x$ can be expressed in terms of e^{ix} and e^{-ix} by adding and subtracting (9) and (10) respectively. We obtain

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

So,

$$e^{(\lambda+i\mu)x} = e^{\lambda x}.e^{i\mu x} = e^{\lambda x}(\cos\mu x + i\sin\mu x) \tag{11}$$

Since $e^{(\lambda+i\mu)x}$ and $e^{(\lambda-i\mu)x}$ are solutions of (1), their sum and difference are also solutions; thus

$$\begin{aligned} e^{(\lambda+i\mu)x} + e^{(\lambda-i\mu)x} &= e^{\lambda x}(\cos\mu x + i\sin\mu x) \\ &\quad + e^{\lambda x}(\cos\mu x - i\sin\mu x) \\ &= 2e^{\lambda x}\cos\mu x, \end{aligned}$$

and

$$\begin{aligned} e^{(\lambda+i\mu)x} - e^{(\lambda-i\mu)x} &= e^{\lambda x}(\cos\mu x + i\sin\mu x) \\ &\quad - e^{\lambda x}(\cos\mu x - i\sin\mu x) \\ &= 2ie^{\lambda x}\sin\mu x. \end{aligned}$$

are solutions of (1). Hence neglecting the constant multipliers 2 and 2i respectively, the functions

$$e^{\lambda x}\cos\mu x \quad \text{and} \quad e^{\lambda x}\sin\mu x$$

are real-valued solutions of (1). It is easy to show that $W[e^{\lambda x}\cos\mu x,\ e^{\lambda x}\sin\mu x] = \mu e^{2\lambda x}$ and since this is never zero, these two functions form a fundamental set of real-valued solutions. So we can express the general solution of (1) in the form

$$y = c_1e^{\lambda x}\cos\mu x + c_2e^{\lambda x}\sin\mu x,$$

where c_1 and c_2 are arbitrary constants.

Example

Find the general solution of

$$y'' + y' + y = 0 \qquad (12)$$

Solution

The characteristics equation is $r^2+r+1 = 0$, and the roots are

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

Hence the general solution of (12) is

$$y = e^{-x/2}(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2})$$

SUMMARY

1. DISTINCT REAL ROOTS

If the auxiliary equation has distinct real roots r_1 and r_2, then $e^{r_1 x}$ and $e^{r_2 x}$ are linearly independent solution of (1). Therefore, a general solution of (1) is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

where c_1 and c_2 are arbitrary constant.

2. REPEATED REAL ROOTS

If the auxiliary equation has a repeated root r, then two linearly independent solutions to (1) are e^{rx} and xe^{rx} and the general solution is

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}.$$

where c_1 and c_2 are arbitrary constants.

3. COMPLEX CONJUGATE ROOTS

If the auxiliary equation has complex conjugate roots

then two linearly independent solutions to (1) are:

$$e^{\lambda x}\cos\mu x \quad \text{and} \quad e^{\lambda x}\sin\mu x$$

and a general solution is

$$y(x) = c_1 e^{\lambda x}\cos\mu x + c_2 e^{\lambda x}\sin\mu x$$

where c_1 and c_2 are arbitrary constants.

Exercises

1. In each of the following problems, determine the general solution of the differential equations:

1. $y''+2y'-3y = 0$	2. $6y''-y'-y = 0$
3. $y''-y = 0$	4. $y''+5y' = 0$
5. $y''-2y'-2y = 0$	6. $4y''+4y'+y = 0$
7. $y''-2y'+y = 0$	8. $y''-9y'+9y = 0$

B. Find the general solution of the given differential equations satisfying the initial conditions given

1. $y''+y'-2y = 0,$ $\quad y(0) = 1,\ y'(0) = 1$
2. $y''-6y'+9y = 0$ $\quad y(0) = 0,\ y'(0) = 2$
3. $y''+8y'-9y = 0$ $\quad y(1) = 1,\ y'(1) = 0$
4. (see next page)

4. $y''+2y'-8y = 0$ $\quad y(0) = 3,\ y'(0) = -12$

5. $y''-4y'+4y = 0$ $\quad y(1) = 1,\ y'(1) = 1$

6. $y''-4y'-5y = = 0$ $\quad y(-1) = 3,\ y'(-1) = 9$

C. Find a general solution for the given differential equations

1. $y''+y' = 0$ $\qquad$ 2. $y''+4y = 0$

3. $y''+4y'+6y = 0$ $\qquad$ 4. $y''-4y'+7y = 0$

5. $4y''+4y'+y = 0$ $\qquad$ 6. $y''-10y'+26y = 0$

D. Solve the following initial value problems

1. $y''+2y'+2y = 0$ $\quad y(0) = 2,\ y'(0) = 1$

2. $y''+2y'+17y = 0$ $\quad y(0) = 1,\ y'(0) = -1$

3. $y''+9y = 0$ $\quad y(0) = 1,\ y'(0) = 1$

4. $y''-2y'+y = 0$ $\quad y(0) = 1,\ y'(0) = -2$

5. $y''-4y'+5y = 0$ $\quad y(0) = 1,\ y'(0) = 6$

6. $y''-2y'+2y = 0$ $\quad y(\pi) = e,\ y'(\pi) = 0$

2.5 METHOD OF UNDETERMINED COEFFICIENTS

We give a simple procedure for finding a particular solution to a nonhomogeneous linear equation with constant coefficients when the nonhomogeneous term $g(x)$ is of a special type. Let us look at a few examples.

Example 1

Find a particular solution to

$$L[y](x) := y''+3y'+2y = 3x+1 \tag{1}$$

Solution

We want to find a function $y_p(x)$ such that $L[y_p](x) = 3x+1$. Notice that if L is applied to any linear function $= Ax+B$, we obtain the linear function

$$L[y_p](x) = 0+3A+2(Ax+B) = 2Ax+(3A+2B)$$

Therefore $2Ax+(3A+2B) = 3x+1$.

Since two polynomials are equal when their corresponding coefficients are equal, we get

$$2A = 3 \quad \text{and} \quad 3A+2B = 1.$$

Solving this system, we get $A = \frac{3}{2}$ and $B = \frac{-7}{4}$. Thus the function

$$y_p(x) = \frac{3}{2}x - \frac{7}{4}$$

is a particular solution to equation (1).

Note that Example 1 suggests the following method for finding a particular solution to the equation

$$L[y](x) = P_n(x) \qquad (2)$$

where $P_n(x)$ is a polynomial of degree n. We set

$$y_p(x) = A_n x^n + \dots + A_1 x + A_0$$

and solve $L[y_p](x) = P_n(x)$ for the coefficients $A_0, A_1, \dots, A_n$. This procedure involves solving n+1 linear equations in the n+1 unknowns $A_0, \dots, A_n$, and usually we have solutions.

This technique is called the method of undetermined coefficients.

Example 2

Find the particular solution to

$$L[y](x) := y'' + 3y' + 2y = e^{3x} \qquad (3)$$

Solution

Here we seek for a solution y_p such that

$$L[y_p](x) = e^{3x}$$

If we try $y_p(x) = Ax^{3x}$, where A is a constant, then

$$L[y_p](x) = 9Ae^{3x} + 3(3Ae^{3x}) + 2(Ae^{3x}) = e^{3x}$$

$$\Longrightarrow \quad 20Ae^{3} = e^{3x}.$$

Solving for A, we get that $A = \frac{1}{20}$. Hence

$$y_p(x) = \frac{1}{20}e^{3x} = \frac{e^{3x}}{20},$$

is a particular solution to equation (3).

Example 3

Find a particular solution to

$$L[y](x) := y'' - y' - y = \sin x \qquad (4)$$

Solution

This time, we look for a function y_p that satisfies $L[y_p](x) = \sin x$. Our initial guess might be to try letting $y_p(x) = A\sin x$, in which we get

$$L[y_p](x) = -2A\sin x - A\cos x$$

However, since the right-hand side of (4) involves only sinx, this choice of solution would force A (and hence y_p) to be zero, which is absurd.

So to compensate for the cosx term, we try

$$y_p(x) = A\cos x + B\sin x$$

so that

$$L[y_p](x) = (-2A-B)\cos x + (A-2B)\sin x$$

Then setting

$$(-2A-B)\cos x + (A-2B)\sin x = \sin x$$

and equating the coefficients of cosx and sinx, we get

$$-2A-B = 0 \quad \text{and} \quad A-2B = 1$$

Solving for A and B gives $A = \frac{1}{5}$ and $B = \frac{-2}{5}$. Hence

$$y_p(x) = \frac{1}{5}\cos x - \frac{2}{5}\sin x$$

is a particular solution of equation (4).

Example 4

Find a particular solution to

$$L[y](x) = y''+y' = 5$$

Solution (5)

Since the right-hand side of (5) is the constant polynomial $P_0(x) = 5$. Example 1 suggests that we let $y_p(x) = A$ and set $L[y_p](x) = 5$. But then, we get $L[y_p](x) = 0 \neq 5$. This unfortunate result occurs because any constant function is a solution to the corresponding homogeneous equation $L[y](x) = 0$. To overcome this difficulty, we temporarily leave the method of undetermined coefficients and observe that we can integrate both sides of (5) to obtain a linear first order equation

$$y'+y = 5x+c_1 \tag{6}$$

where c_1 is an arbitrary constant. Using the method for solving linear first order equations, we find that

$$y(x) = 5x-5+c_1+c_2e^{-x} \tag{7}$$

is the general solution to (6) and hence to (5).

Notice that with $c_1 = 5$ and $c_2 = 0$ in (7) we get $y_p(x) = 5x$, as a particular solution to (5).

This suggests that in our original attempt to use undetermined coefficients, we would have taken $y_p(x) = Ax$ instead of $y_p(x) = A$. Indeed, with $y_p = Ax$, setting $L[y_p](x) = 5$ immediately gives $A = 5$, and we get

$$y_p(x) = 5x.$$

Example 5

Determine the form of a particular solution to

$$y''-y'-12y = e^{4x} \tag{8}$$

Solution

The auxiliary equation for the corresponding homogeneous equation

$$y''-y'-12y = 0 \tag{9}$$

has roots $r_1 = 4$ and $r_2 = -3$. A general solution of (9) is therefore

$$y_c(x) = c_1e^{4x} + c_2e^{-3x}.$$

Now, since the right-hand side of (8) is e^{4x}, we first let $y_p(x) = Ae^{4x}$. Because e^{4x} is a solution to equation (9), we replace this choice with $y_p = Axe^{4x}$.

Since xe^{4x} is not a solution to (9), there exists a particular solution to (8) of the form

$$y_p = Axe^{4x}.$$

TABLE (*) SUMMARY

Forms of particular solution $y_p(x)$ of $L[y](x) = g(x)$, when $L[y]$ has constant coefficients.

Type	$g(x)$	$y_p(x)$
1.	$P_n(x) = a_n x^n + \dots + a_1 x + a_0$	$x^s P_n(x) = x^s(A_n x^n + \dots + A_1 x + A_0)$
2.	$ae^{\ x}$	$x^s Ae^{\ x}$
3.	$a\cos\ x + b\sin\ x$	$x^s\ A\cos\ x + B\sin\ x$
4.	$P_n(x)e^{\ x}$	$x^s P_n(x)e^{\ x}$
5.	$P_n(x)\cos\ x + q_m(x)\sin\ x$ where $q_m(x) = b_m x^m + \dots + b_1 x + b_0$	$x^s\ P_N(x)\cos\ + Q_N(x)\sin\ x$ where $Q_N(x) = B_N x^N + \dots + B_0$ and $N = \max\ n, m$
6.	$ae^{\ x}\cos\ x + be^{\ x}\sin\ x$	$x^s\ Ae^{\ x}\cos\ x + Be^{\ x}\sin\ x$
7.	$P_n(x)e^{\ x}\cos\ x + q_m(x)\ ^{x}$ $+ q_m(x)e^{\ x}\sin\ x$	$x^s e^{\ x}\ P_N(x)\cos\ x +$ $Q_N(x)\sin\ x$ where $N = \max\ n, m$

The non negative integer s is chosen to be the smallest integer so that no term in the particular solution $y_p(x)$ is a solution to the corresponding homogeneous equation $L[y](x) = 0$.

For linear second order equation, s will be either be 0, 1 or 2.

Example 6

Using the above table, find the form for a particular solution y_p to

$$y'' + 2y' - 3y = g(x) \tag{10}$$

where $g(x)$ equals

(a) $7\cos 3x$ (b) $5e^{-3x}$ (c) $x^2\cos\pi x$

(d) $2xe^x\sin x - e^x\cos x$ (e) $x^2e^x + 3xe^x$ (f) $\tan x$

Solution

The auxiliary equation for the corresponding homogeneous equation

$$y''+2y'-3y = 0 \qquad (11)$$

has roots $r_1 = 1$ and $r_2 = -3$. Thus .. general solution for (11) is

$$y_c(x) = c_1e^x + c_2e^{-3x}$$

then referring to Table (*)

(a) The function $g(x) = 7\cos 3x$ is of type 3.

With $a = 7$, $b = 0$ and $\beta = 3$. Hence y_p has the form

$$y_p(x) = A\cos 3x + B\sin 3x.$$

(b) The function $g(x) = 5e^{-3x}$ is of type 2, and so y_p has the form x^sAe^{-3x}. Here we take $s = 1$ since e^{-3x} is a solution to (11) and xe^{-3x} is not. Thus

$$y_p(x) = Axe^{-3x}.$$

(c) The function $g(x) = x^2\cos\pi x$ is of type 5 with $P_2(x) = x^2$, $q_o(x) = 0$. Since $N = \max\{2,0\} = 2$

$$y_p(x) = (A_2x^2+A_1x+A_o)\cos\pi x + (B_2x^2+B_1x+B_o)\sin\pi x \qquad (12)$$

where we have taken $s = 0$ because no term in (12) is a solution to the homogeneous equation (11).

(d) The function $g(x) = 2xe^x\sin x - e^x\cos x$ is of type 7 with $P_o(x) = -1$ and $q_1(x) = 2x$. For these polynomials, $N = \max\{0,1\} = 1$, so

$$x^se^x\{(A_1x+A_o)\cos x + (B_1x+B_o)\sin x\}$$

is the form for particular solution. Since none of the terms $e^xx\cos x$, $e^xx\sin x$ and $e^x\sin x$ is a solution to the homogeneous equation (11), we set $s = 0$ and get

$$y_p(x) = e^x\{(A_1x+A_o)\cos x + (B_1x+B_o)\sin x\}$$

(c) Here $g(x) = x^2e^x + 3xe^x = (x^2+3x)e^x$ is of type 4. With $P_2(x) = x^2+3x$. Thus

$$y_p(x) = x(A_2x^2+A_1x+A_0)e^x$$

where we have taken $s = 1$ since the term A_0e^x in $(A_2x^2+A_1x+A_0)e^x$ is a solution to (11).

(f) Unfortunately, $g(x) = \tan x$ is not one of the forms for which the method of undetermined coefficients can be used.

2.5.1 Summary: Method of Undetermined Coefficients

To determine a particular solution to $L[y] = g$, we

(a) check that the linear equation has constant coefficients and that the non homogeneous term $g(x)$ is one of the type for which the method can be applied.

(b) solve the corresponding homogeneous equation $L[y] = 0$.

(c) Based on the form of the non homogeneous term $g(x)$, determine the appropriate term of the particular solution $y_p(x)$. Remember that no term in the trial expression for $y_p(x)$ may be a solution to the corresponding homogeneous equation.

(d) Since we want $L[y_p](x) = g(x)$, set the corresponding coefficients from both sides of this equation equal to each other to form a system of linear equations.

(e) Solve this system of linear equations for the coefficients of $y_p(x)$.

Example 7

Find a general solution to the equation

$$y''-y = 2e^{-x} - 4xe^{-x} + 10\cos 2x \qquad (13)$$

Solution

We first solve the associated homogeneous equation $y''-y = 0$ and obtain as a general solution

$$y_c(x) = c_1e^x + c_2e^{-x}$$

superposition principle and consider separately the equations

$$y''-y = 2e^{-x} - 4xe^{-x} \tag{14}$$

and

$$y''-y = 10\cos 2x \tag{15}$$

A particular solution to (14) has the form $y_q(x) = x^s(A_1x+A_0)e^{-x}$.

Since A_0e^{-x} is a solution to the corresponding homogeneous equation, we take $s = 1$. Then

$$y_q(x) = (A_1x^2+A_0x)e^{-x}.$$

Substituting $y_q(x)$ for y in (14) gives

$$-4A_1xe^{-x} + (2A_1-2A_0)e^{-x} = -4xe^{-x} + 2e^{-x} \tag{16}$$

When we equate the coefficients of xe^{-x} and e^{-x} on both sides of (16) we get the system

$$-4A_1 = -4; \quad 2A_1-2A_0 = 2.$$

Solving, we get $A_1 = 1$ and $A_0 = 0$. Thus

$$y_q(x) = x^2e^{-x}$$

is a particular solution to (14).

Next, we consider equation (15). A particular solution to this equation has the form

$$y_r(x) = A\cos 2x + B\sin 2x.$$

[Here we take $s = 0$]. Substituting y_r in (15) gives

$$-5A\cos 2x - 5B\sin 2x = 10\cos 2x,$$

which gives $A = -2$, and $B = 0$. Thus

$$y_r(x) = -2\cos 2x.$$

It follows from the superposition principle that a particular solution to the original equation (13) is given by the sum y_q+y_r. That is

$$y_p(x) = x^2e^{-x} - 2\cos 2x.$$

Hence a general solution to equation (13) given by the sum of the general solution to the corresponding homogeneous equation and a particular solution is

$$y(x) = c_1e^{x} + c_2e^{-x} + x^2e^{-x} - 2\cos 2x.$$

Exercises

A. Find the particular solution to the following differential equations:

1. $y''+2y'-y = 10$
2. $y''+y = 5e^{2x}$
3. $y''+y'+y = 2\cos 2x - 3\sin 2x$
4. $y''-5y'+6y = xe^{x}$
5. $y''-4y' = x^2-1$
6. $y''-6y'+9y = x^2+e^{x}$
7. $y''-y = x\sin x$
8. $y''-4y'-5y = 2e^{-x}$
9. $y''-2y'+y = 8e^{x}$
10. $y''+2y' = 3+4\sin 2x$

B. Find a general solution to the following differential equation

1. $y''-y = -11x+1$
2. $y''+y'-2y = x^2-2x+3$
3. $y''-3y'+2y = e^{x}\sin x$
4. $y''+y = 7x+5$
5. $y''-7y' = x^2$
6. $y''-3y = x^2-e^{x}$
7. $y''+y'+y = \cos x-x^2e^{x}$
8. $y''+4y'+5y = e^{-x}-\sin 2x$

C. Find the solution to the following initial value problems

1. $y''+y = 2e^{-x}$; $y(o) = 0$, $y'(o) = 0$
2. $y''-y = e^{x}-e^{-x}+2$; $y(o) = 0$, $y'(o) = 0$
3. $y''-y = \sin x-e^{2x}$; $y(o) = 1$, $y'(o) = -1$
4. $y''-7y'+10y = x^2-4+e^{x}$; $y(o) = 3$, $y'(o) = -3$
5. $y'' = 6x$; $y(o) = 3$, $y'(o) = -1$
6. $y''+9y = 27$; $y(o) = 4$, $y'(o) = 6$
7. $y''+2y'+y = x^2+1-e^{x}$; $y(o) = 0$, $y'(o) = 2$
8. $y''-2y'+y = xe^{x}+4$; $y(o) = 1$, $y'(o) = 1$

2.6 THE METHOD OF VARIATION OF PARAMETERS

We have discussed a simple method of determining particular solutions of nonhomogeneous differential equations with constant coefficients, provided the nonhomogeneous term is of a suitable form.

Here, we will consider a general method of determining a particular solution of the equation

$$y''+p(x)y'+q(x)y = g(x) \qquad (1)$$

where the functions p, q and g are continuous on the interval of interest. In order to use this method, known as method of variation of parameters, it is necessary to know the fundamental set of solutions of the corresponding homogeneous equation

$$y''+p(x)y'+q(x)y = 0 \qquad (2)$$

Suppose y_1 and y_2 are linearly independent solutions of the homogeneous equation (2). Then the general solution of (2) is

$$y_c(x) = c_1y_1(x) + c_2y_2(x)$$

The method of variation of parameters involves the replacement of the constants c_1 and c_2 by functions u_1 and u_2 so that

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \qquad (3)$$

satisfies the nonhomogeneous differential equation (1). The importance of this method is due to the fact that it is possible to determine the functions u_1 and u_2 in a simpler manner.

In order to determine u_1 and u_2, two conditions are required. One condition on u_1 and u_2 comes from the fact that y_p must satisfy (1). A second condition can be imposed arbitrarily, and will be selected so as to facilitate the calculations. So differentiating (3) gives

$$y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2') \qquad (4)$$

As was indicated earlier, it is required that u_1 and u_2 satisfy

$$u_1'y_1 + u_2'y_2 = 0 \qquad (5)$$

With this condition on u_1' and u_2', y_p'' is given by

$$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' \qquad (6)$$

Notice that as a result of imposing condition (5), only first derivatives of u_1 and u_2 appear in the expression for y_p''. Substituting for y_p, y_p' and y_p'' in (1) gives

$$u_1(y_1''+py_1'+qy) + u_2(y_2''+py_2'+qy) + u_1'y_1' + u_2'y_2' = g(x)$$

The term in brackets will be zero because y_1 and y_2 are solutions of the homogeneous equation (2), hence the condition that y_p satisfies (1) leads to the requirement

$$u_1'y_1' + u_2'y_2' = g \tag{7}$$

Rewriting (5) and (7), we have the following system of two equations

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= g \end{aligned} \tag{8}$$

for the two unknown functions u_1' and u_2'. Solving this system of equation gives

$$u_1' = \frac{-y_2 g}{W(y_1,y_2)}, \qquad u_2' = \frac{y_1 g}{W(y_1,y_2)} \tag{9}$$

where $W(y_1;y_2) = y_1y_2' - y_1'y_2$. This division is possible since $W(y_1;y_2) \neq 0$ in the interval. Integrating (9) and substituting in (3) gives a particular solution of the nonhomogeneous equation (1).

Note that we finally obtain

$$u_1(x) = \int \frac{-y_2(x)g(x)}{W(y_1,y_2)}dx, \qquad u_2(x) = \int \frac{y_1(x)g(x)}{W(y_1,y_2)}dx \tag{10}$$

Theorem

If the functions p, q and g are continuous on $\alpha < x < \beta$ and if the functions y_1 and y_2 are linearly independent solution of the homogeneous equation associated with the differential equation (1)

$$y''+p(x)y'+q(x)y = g(x)$$

then a particular solution of (1) is given by

$$y_p(x) = -y_1(x)\int^x \frac{y_2(t)g(t)}{W(y_1;y_2)}dt + y_2(x)\int^x \frac{y_1(t)g(t)}{W(y_1;y_2)}dt \tag{11}$$

NOTE

For (11) to be used in determining a particular solution for a differential equation, care must be taken to ensure that it is written in the standard form (1).

We emphasize also that in computing a particular solution by the method of variation of parameters, it is not necessary that the coefficients in the differential equation be constants; all that is required is that we know two linearly independent solutions of the homogeneous equation.

2.6.1 Summary: Determination of Particular Solution By Method of Variation of Parameters

To determine a particular solution to

$$y''+py'+qy = G$$

we do the followings:

1. Find a fundamental solution set $\{y_1(x), y_2(x)\}$ for the corresponding homogeneous equation and take

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

2. Determine $u_1(x)$ and $u_2(x)$ by using the formula (10) or by solving the system (8) for $u_1'(x)$ and $u_2'(x)$ and integrating.
3. Substitute $u_1(x)$ and $u_2(x)$ into the expression for $y_p(x)$ to obtain a particular solution.

Example 1

Determine the general solution of the differential equation

$$y''+y = \sec x \qquad 0 < x < \frac{\pi}{2} \tag{12}$$

Solution

Two linearly independent solutions of the homogeneous equation are $y_1(x) = \cos x$ and $y_2(x) = \sin x$. Even though (12) has constant coefficients and we know $y_c(x)$, we cannot use the method of undetermined coefficients because the right hand side of (12) is not of the form $e^{\alpha x}p_n(x)\cos\beta x$ or $e^{\alpha x}p_n(x)\sin\beta x$. Instead, we use the method of variation of parameters. So we write

$$y_p(x) = u_1(x)\cos x + u_2(x)\sin x.$$

Then

$$y_p'(x) = [-u_1(x)\sin x + u_2(x)\cos x] + [u_1'(x)\cos x + u_2'(x)\sin x].$$

Setting the second term in brackets equal to zero, differentiating again and substituting in (12) we get

$$u_1'(x)\cos x + u_2'\sin x = 0$$

$$-u_1'(x)\sin x + u_2'\cos x = \sec x$$

Solving we obtain

$$u_1'(x) = -\tan x; \qquad u_2'(x) = 1$$

So, we have

$$u_1(x) = \ln \cos x; \qquad u_2(x) = x$$

Hence, a particular solution of (12) is

$$y_p(x) = \cos x \ln \cos x + x \sin x.$$

and the general solution of (12) is

$$y(x) = c_1\cos x + c_2\sin x + x\sin x + (\cos x)\ln\cos x$$

Example 2

Find a particular solution on $(-\frac{\pi}{2}, \frac{\pi}{2})$ to

$$y''+y = \tan x + 3x-1 \tag{13}$$

Solution

With $g(x) = \tan x + 3x-1$, the variation of parameter procedure will lead to a solution of (13). But it is simpler to consider separately the equations

$$y''+y = \tan x \tag{14}$$

$$y''+y = 3x-1 \tag{15}$$

and then use the superposition principle. The fundamental solution set for the homogeneous equation $y''+y = 0$ is $\{\cos x, \sin x\}$. We set

$$y_p(x) = u_1(x)\cos x + u_2(x)\sin x \tag{16}$$

Referring to (8) we solve the system

$$\cos x\, u_1'(x) + (\sin x)u_2'(x) = 0$$

$$-\sin x\, u_1'(x) + (\cos x)u_2'(x) = \tan x$$

for $u_1'(x)$ and $u_2'(x)$. This gives

$$u_1'(x) = -\tan x \sin x$$

$$u_2'(x) = \tan x \cos x = \sin x.$$

Integrating, we get

$$u_1(x) = -\int \tan x \sin x\,dx = -\int \frac{\sin^2 x}{\cos x}dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}dx = \int (\cos x - \sec^2 x)dx$$

$$= \sin x - \ln(\sec x + \tan x) + c_1$$

$$u_2(x) = \int \sin x\,dx = -\cos x + c_2.$$

Since we need only one particular solution, we take c_1 and c_2 to be zero. Hence plugging u_1 and u_2 back into (16) we get

$$y_q(x) = [\sin x - \ln(\sec x + \tan x)]\cos x - \cos x \sin x.$$

which simplifies to

$$y_q(x) = -(\cos x)\ln(\sec x + \tan x)$$

For the equation (15), the method of undetermined coefficient can be used. On seeking a solution to (15) of the form

$$y_r(x) = Ax+B \quad \text{we get} \quad y_r(x) = 3x-1$$

Finally, we apply the superposition principle to get

$$y_p(x) = y_q(x) + y_r(x)$$

$$= -(\cos x)\ln(\sec x + \tan x) + 3x - 1$$

as a particular solution for (15).

Exercises

A. Determine a particular solution of the following differential equation using the method of variation of parameters; and then the general solution of the equation.

1. $y''-5y'+6y = 2e^x$
2. $y''-y'-2y = 2e^{-x}$
3. $y''+y = 3\cos 2x$
4. $y''+9y = \sec^2 3x$
5. $y''+16y = \sec 4x$
6. $y''-y = 2x+4$
7. $y''+4y = \tan 2x$
8. $2y''-2y'-4y = 2e^{3x}$.

B. Find a general solution to the following

1. $y''+y = \tan x + e^{3x} - 1$
2. $y''+y = \sec^3 x$
3. $y''+y = 3\sec x - x^2 + 1$
4. $y''-5y'+6y = 18x^2$
5. $y''-6y'+9y = x^{-3}e^{3x}$
6. $\frac{1}{2}y''+2y = \tan 2x - \frac{1}{2}e^x$.

3. HIGHER ORDER LINEAR EQUATIONS

3.1 INTRODUCTION

In this chapter, we will extend the theory of second order linear equations studied already to higher order linear equation. An nth order linear differential equation is an equation of the form

$$P_0(x)\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = g(x) \quad (1)$$

We will assume that $P_0, \dots, P_n$ and g are continuous real-valued functions on some interval $\alpha < x < \beta$ and that P_0 is nowhere zero in the interval. Then dividing (1) by $P_0(x)$, we get

$$L[y] = \frac{d^n y}{dx} + P_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = G(x) \quad (2)$$

The linear differential operator L is also introduced in (2). The mathematical theory associated with (2) is completely the same to that of second order equation. So we ~~www~~ shall in most part, state simply the results for nth order linear equation.

Since (2) involves nth derivative of y with respect to x, it will require n integrations to solve (2). Each of these integrations introduces an arbitrary constant. So to obtain a unique solution, we expect that it is necessary to specify n initial conditions say

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \dots, y^{n-1}(x_0) = y_0^{n-1} \quad (3)$$

where x_0 may be any point in the interval $\alpha < x < \beta$ and $y_0, y'_0, \dots, y_0^{n-1}$ is any set of prescribed constants. The existence of such unique solution is assured by the following existence and uniqueness theorem.

Theorem 1

If the functions $P_1, P_2, \dots, P_n$ and g are continuous on an open interval $\alpha < x < \beta$, then there exists one and only one function $y = \phi(x)$ satisfying (2) on the interval $\alpha < x < \beta$ and the prescribed initial conditions (3).

Proof: Later.

3.2 BASIC THEORY OF nth ORDER LINEAR DIFFERENTIAL EQUATIONS

If the functions $y_1, y_2, \ldots, y_n$ are solutions of the nth order linear homogeneous equation

$$L[y] = y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_n(x)y = 0 \quad (1)$$

it follows by direct computation that the linear combination

$$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) \quad (2)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants, is also a solution of (1). It is known that every solution of (1) can be expressed as a linear combination of $y_1, y_2, \ldots, y_n$ regardless of the initial conditions

$$y(x_o) = y_o, \; y'(x_o) = y'_o, \ldots, y^{(n-1)}(x_o) = y_o^{(n-1)} \quad (3)$$

Specifically, for any choice of the point x_o in $\alpha < x < \beta$ and for any choice of $y_o, y'_o, \ldots, y_o^{(n-1)}$, we must be able to determine $c_1, c_2, \ldots, c_n$ so that the equations

$$\begin{aligned} c_1y_1(x_o) + \ldots + c_ny_n(x_o) &= y_o \\ c_1y'_1(x_o) + \ldots + c_ny'_n(x_o) &= y'_o \\ \vdots \qquad\qquad & \quad \vdots \\ c_1y_1^{(n-1)}(x_o) + \ldots + c_ny_n^{(n-1)}(x_o) &= y_o^{(n-1)} \end{aligned} \quad (4)$$

are satisfied. Equation (4) can always be solved for the constants $c_1, \ldots, c_n$ provided the determinant of the coefficients does not vanish. So, a necessary and sufficient condition for the existence of a solution of (4) for arbitrary values of $y_o, y'_o, \ldots, y_o^{(n-1)}$ is that Wronskian

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y'_1 & y'_2 & \ldots & y'_n \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix} \quad (5)$$

does not vanish at x = x

Theorem 2

Let y_1, y_2, ..., y_n be n solutions on (a,b) of

$$y^{(n)}(x) + P_1(x)y^{(n-1)}(x) + \dots + P_n(x)y(x) = 0 \qquad (6)$$

where $P_1, \dots, P_n$ are continuous functions on (a,b). If at some point x_0 in (a,b) these solution satisfy

$$W(y_1, y_2, \dots, y_n) \neq 0 \qquad (7)$$

Then every solution of (6) on (a,b) can be expressed in the form

$$y(x) = c_1y_1(x) + c_2y_2(x) + \dots + c_ny_n(x) \qquad (8)$$

where c_1, c_2, ..., c_n are constants.

Example

Given that $y_1(x) = x$, $y_2(x) = x^2$ and $y_3(x) = x^{-1}$ are solutions to

$$x^3y''' + x^2y'' - 2xy' + 2y = 0 \qquad (9)$$

find a general solution to (9).

Solution

We first show that $\{y_1, y_2, y_3\}$ is a fundamental solution set for equation (9) on $(0, \infty)$. Since we are told that y_1, y_2 and y_3 satisfy (9) we need only consider

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{vmatrix} = \begin{vmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{vmatrix}$$

Evaluating this determinant, we get

$$W(y_1, y_2, y_3)(x) = 6x^{-1} \neq 0 \quad \text{for } x > 0.$$

Thus $\{y_1, y_2, y_3\}$ is a fundamental solution and hence a general solution is

$$y(x) = c_1x + c_2x^2 + c_3x^{-1}.$$

3.3 LINEAR DEPENDENCE OF FUNCTIONS

Definition

The n functions f_1, ..., f_n are said to be linearly dependent on an interval I if there exist constants c_1, c_2, ..., c_n not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0 \qquad (10)$$

for all x in I. If the functions $f_1, \dots, f_n$ are not linearly dependent on I, they are said to be linearly independent on I.

Example 1

Show that the functions $f_1(x) = e^x$, $f_2(x) = e^{-2x}$ and $f_3(x) = 3e^x - 2e^{-2x}$ are linearly dependent on $(-\infty, \infty)$.

Solution

Notice that $f_3(x)$ is a linear combination of $f_1(x)$ and $f_2(x)$. That is

$$f_3(x) = 3f_1(x) - 2f_2(x) = 3e^x - 2e^{-2}$$

Therefore, we have

$$3f_1(x) - 2f_2(x) - f_3(x) = 0$$

for all x in $(-\infty, \infty)$. Consequently f_1, f_2 and f_3 are linearly dependent on $(-\infty, \infty)$.

Example 2

Show that the functions $f_1(x) = x$, $f_2(x) = x^2$ and $f_3(x) = 1-2x^2$ are linearly independent on $(-\infty, \infty)$.

Solution

Assume that c_1, c_2 and c_3 are constants for which

$$c_1 x + c_2 x^2 + c_3(1-2x^2) = 0 \qquad (11)$$

holds for every x. If we can prove that (11) implies that $c_1 = c_2 = c_3 = 0$ then linearly independence follows.

Let us set x = 0, 1, -1 in equation (11), respectively to get

For x = 0 $\qquad c_3 = 0$

For x = 1, $\qquad c_1 + c_2 - c_3 = 0$

For x = -1, $\qquad -c_1 + c_2 - c_3 = 0$

Solving this system, we find that the only possible solution is $c_1 = c_2 = c_3 = 0$. Consequently, the functions f_1, f_2 and f_3 are linearly independent on $(-\infty, \infty)$.

Theorem 3

Let $y_1, \ldots, y_n$ be n solutions to

$$y^{(n)} + P_1 y^{(n-1)} + \ldots + P_n y = 0 \quad \text{on} \quad (a,b).$$

Then $y_1, y_2, \ldots, y_n$ is a fundamental solution set on (a,b) if and only if these functions are linearly independent on (a,b).

3.3.1 Representation of Solutions (Nonhomogeneous Case)

Theorem 4

Let $y_p(x)$ be a particular solution to the nonhomogeneous equation

$$y^{(n)}(x) + P_1(x)y^{(n-1)}(x) + \ldots + P_n(x)y(x) = g(x) \tag{12}$$

on the interval (a,b) and let $\{y_1, \ldots, y_n\}$ be a fundamental solution set on (a,b) for the corresponding homogeneous equation

$$y^{(n)}(x) + P_1{}^{(n-1)}(x) + \ldots + P_n(x)y(x) = 0 \tag{13}$$

then every solution of (12) on the interval (a,b) can be expressed in the form

$$y(x) = y_p(x) + c_1 y_1(x) + \ldots + c_n y_n(x) \tag{14}$$

Example 3

Given that $y_p(x) = x^2$ is a particular solution to

$$y''' - 2y'' - y' + 2y = 2x^2-2x-4 \tag{15}$$

on $(-\infty, \infty)$ and that $y_1(x) = e^{-x}$, $y_2(x) = e^x$ and $y_3(x) = e^{2x}$ are solutions to the corresponding homogeneous equation, find a general solution to (15).

Solution

We note that the functions e^{-x}, e^x and e^{2x} are linearly independent because the exponents -1, 1 and 2 are distinct. Since each of these functions is a solution of the corresponding homogeneous equation, then e^{-x}, e^x, e^{2x} is a fundamental solution set. It then follows from the representation theorem for nonhomogeneous equation that a general solution is

$$y(x) = x^2 + c_1e^{-x} + c_2e^{x} + c_3e^{2x}.$$

Exercises

A. Determine, whether the given functions are linearly dependent or linearly independent on the specified interval, then compute their Wronskian.

1. $\{e^{3x}, e^{5x}, e^{-x}\}$ on $(-\infty, \infty)$

2. $\{x^2, x^2, -1, 5\}$ on $(-\infty, \infty)$

3. $\{\sin^2 x, \cos^2 x, 1\}$ on $(-\infty, \infty)$

4. $\{\sin x, \cos x, \tan x\}$ on $(-\infty, \infty)$

5. $\{x, x^2, x^3, x^4\}$ on $(-\infty, \infty)$

6. $\{x, xe^{x}, 1\}$ on $(-\infty, \infty)$

B. Verify that the given functions form a fundamental solution set for the given differential equation, and find a general solution

1. $y''' + 2y'' - 11y' - 12y = 0$; $\{e^{3x}, e^{-x}, e^{-4x}\}$.

2. $y''' - y'' + 4y' - 4y = 0$ $\{e^{x}, \cos 2x, \sin 2x\}$

3. $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$, $x > 0$, $\{x, x^2, x^3\}$

4. $y^{(4)} - y = 0$ $\{e^{x}, e^{-x}, \cos x, \sin x\}$

C. In the following problems, a particular solution and a fundamental solution set to the nonhomogeneous equation and its corresponding homogeneous equation respectively are given.

(a) Find a general solution to the nonhomogeneous equation.

(b) Find the solution that satisfies the specified initial conditions.

1. $y''' + y'' + 3y' - 5y = 2 + 6x - 5x^2$, $y(0) = 1$, $y'(0) = 1$, $y''(0) = -3$, $y_p = x^2$ $\{e^{x}, e^{-x}\cos 2x, e^{-x}\sin 2x\}$

2. $xy''' - y'' = -2$; $y(1) = 2$, $y'(1) = -1$, $y''(1) = -4$
$y_p = x^2$; $\{1, x, x^3\}$.

3. $y^{(v)} + 4y = 5\cos x$; $y(0) = 2$, $y'(0) = 1$, $y''(0) = -1$, $y'''(0) = -2$. $y_p = \cos x$; $\{e^{x}\cos x, e^{x}\sin x, e^{-x}\cos x, e^{-x}\sin x\}$

4. $x^3y''' + xy' - y = 3-\ln x$; $y(1) = 3$, $y'(1) = 3$, $y''(1) = 0$.

$y_p = \ln x$; $\{x, x\ln x, x(\ln x)^2\}$

3.4 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

In this section, we discuss the homogeneous linear nth order differential equation

$$a_0y^{(n)}+a_1y^{(n-1)} +\dots+ a_{n-1}y'+a_ny = 0 \qquad (1)$$

where $a_0, a_1, \dots, a_n$ are real constant with $a_0 \neq 0$. Since constant functions are everywhere continuous, (1) has solutions defined for all x in $(-\infty, \infty)$. If we can find n linearly independent solutions to (1) in $(-\infty, \infty)$, say $y_1, y_2, \dots, y_n$; then we can express a general solution to (1) in the form

$$y(x) = c_1y_1(x) + c_2y_2(x) +\dots+ c_ny_n(x) \qquad (2)$$

where $c_1, \dots, c_n$ are arbitrary constants.

To find these n linearly independent solutions, we shall use our knowledge with second order equations. Namely, experience suggests that we begin by trying a function of the form $y = e^{rx}$.

If we let L to be the differential operator defined by the left hand side of (1), that is

$$L[y]: = a_0y^{(n)} + a_1y^{(n-1)} +\dots+ a_{n-1}y' + a_ny \qquad (3)$$

then we can write (1) in operator form

$$L[y](x) = 0 \qquad (4)$$

For $y = e^{rx}$, we find

$$L[e^{rx}](x) = e^{rx}(a_0r^n + a_1r^{n-1} +\dots+ a_{n-1}r + a_n) = e^{rx}p(r) \qquad (5)$$

where $P(r)$ is a polynomial $a_0r^n + a_1r^{n-1} +\dots+ a_{n-1}r + a_n$. Thus e^{rx} is a solution to (4) provided that r is a root of the characteristic equation

$$P(r) = a_0r^n + a_1r^{n-1} +\dots+ a_{n-1}r + a_n = 0 \qquad (6)$$

A polynomial of degree n has n zeros, say $r_1, r_2, \ldots, r_n$; which may be either real or complex. We expect to have three types of roots (1) Distinct real roots (2) Repeated roots and (3) Complex roots. We take them one by one.

1. DISTINCT REAL ROOTS

If the roots of (6) are real and no two of them are equal, then we have n distinct solutions $e^{r_1x}, e^{r_2x}, \ldots, e^{r_nx}$ of (4).

It can be shown that the general solution of (4) is of the form

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x} + \ldots + c_ne^{r_nx} \tag{7}$$

Since it can also be shown that the functions $e^{r_1x}, \ldots, e^{r_nx}$ are linearly independent on the interval $-\infty < x < \infty$

Example

Find a general solution to

$$y''' - 2y'' - 5y' + 6y = 0 \tag{8}$$

Solution

The auxiliary equation is

$$r^3 - 2r^2 - 5r + 6 = 0, \tag{9}$$

Solving for r in (9), we get $r_1 = 1$, $r_2 = -2$, $r_3 = 3$. Since these roots are real and distinct, a general solution to (8) is

$$y(x) = c_1e^{x} + c_2e^{-2x} + c_3e^{3x}.$$

2. REPEATED ROOTS

If the roots of the auxiliary equation are not distinct, that is, if some of the root are repeated, then the solution (7) is clearly not the general solution of (1). Recalling that if r_1 was a repeated root for a second order linear equation $a_0y'' + a_1y' + a_2y = 0$, then the two linearly independent solutions were e^{r_1x} and xe^{r_1x}. It seems reasonable to expect that if a root of $P(r) = 0$ say $r = r_1$ is repeated s times $(s \leq n)$ then

$$e^{r_1x}, \; xe^{r_1x}, \; x^2e^{r_1x} \; \ldots \; x^{s-1}e^{r_1x} \tag{10}$$

are solutions of (1). Then the general solution to (1) is

$$y(x) = c_1e^{r_1x} + c_2xe^{r_1x} + c_3x^2e^{r_1x} + \dots + c_sx^{s-1}e^{r_1x} \qquad (11)$$

Example

Find a general solution to

$$y^{(4)} - y''' - 3y'' + 5y' - 2y = 0 \qquad (12)$$

Solution

The auxiliary equation to (12) is

$$r^4 - r^3 - 3r^2 + 5r - 2 = (r-1)^3(r+2) = 0$$

which has roots $r_1 = 1$, $r_2 = 1$, $r_3 = 1$ and $r_4 = -2$. Since the root of 1 has multiplicity 3, a general solution is

$$y(x) = c_1e^x + c_2xe^x + c_3x^2e^x + c_4e^{-2x}.$$

3. COMPLEX ROOTS

If $\alpha + i\beta$ (α, β real) is a complex root of the auxiliary equation (6), then so is its complex conjugate $\alpha - i\beta$, since the coefficients of $P(r)$ are real-valued. If we accept complex valued functions as solutions, then both $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ are solutions to (1).

If there are no repeated roots, then a general solution to (1) is given by (7). To find two real-valued solutions, corresponding to the roots $\alpha \pm i\beta$, we take the real and imaginary parts of $e^{(\alpha+i\beta)x}$. That is, since

$$e^{(\alpha+i\beta)x} = e^{\alpha x}\cos\beta x + ie^{\alpha x}\sin\beta x \qquad (13)$$

then two linearly independent solutions to (1) are

$$e^{\alpha x}\cos\beta x, \quad e^{\alpha x}\sin\beta x. \qquad (14)$$

In fact, using these solutions in place of $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ in (7) preserves the linear independence of the set of n solutions. Thus, treating each of the conjugate pairs of roots in this manner, we obtain a real-valued solution to (1).

Finally, if a complex root $\alpha + i\beta$ is repeated s times, the complex conjugate $\alpha - i\beta$ will also be repeated s times. Corresponding to these 2s complex-valued solutions, we can find 2s real-valued solutions by noting that the real and imaginary parts of $e^{(\alpha+i\beta)x}$, $xe^{(\alpha+i\beta)x}$, ..., $x^{(s-1)}e^{(\alpha+i\beta)x}$ are also linearly independent solutions.

$$e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x,\quad xe^{\alpha x}\cos\beta x,\quad xe^{\alpha x}\sin\beta x,\ \ldots,$$

$$x^{s-1}e^{\alpha x}\cos\beta x,\quad x^{s-1}e^{\alpha x}\sin\beta x.$$

Hence the general solution of (1) can always be expressed as a linear combination of n real-valued solutions.

Example 1

Find the general solution of

$$y^{(4)} + 2y'' + y = 0 \tag{15}$$

Solution

The auxiliary equation is

$$r^4+2r^2+1 = (r^2+1)(r^2+1) = 0$$

The roots are i, i, -i, -i and the general solution of (15) is

$$y(x) = c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x$$

Example 2

Find a general solution to

$$y^{(4)}-8y'''+26y''-40y'+25y = 0 \tag{16}$$

Solution

The auxiliary equation of (16) is

$$r^4-8r^3+26r^2-40r+25 = (r^2-4r+5)^2 = 0 \tag{17}$$

(17) has repeated complex roots

$$r_1 = 2+i,\quad r_2 = 2+i,\quad r_3 = 2-i,\quad r_4 = 2-i$$

Hence a general solution is

$$y(x) = c_1e^{2x}\cos x + c_2e^{2x}\sin x + c_3xe^{2x}\cos x + c_4xe^{2x}\sin x.$$

Exercises

A. Determine the general solution of the following differential equations:

1. $y'''-y''-y'+y = 0$
2. $2y'''-4y''-2y'+4y = 0$
3. $y'''-3y''-y'+3y = 0$
4. $y'''+2y''-8y' = 0$
5. $6y'''+7y''-y'-2y = 0$
6. $y'''+5y''-13y'+7y = 0$

7. $y'''+5y''+3y'-9y = 0$ 8. $y^{(4)}+4y'''+6y''+4y'+y = 0$

9. $y'''+3y''-4y'-6y = 0$ 10. $y'''-y''+2y = 0$

B. Find a general solution to the linear homogeneous differential equation with constant coefficients, whose auxiliary equation is given

1. $(r-1)^2(r+3)(r^2+2r+5)^2 = 0$
2. $(r+1)^2(r-6)^3(r+5)(r^2+1)(r^2+4) = 0$
3. $(r-1)^3(r-2)(r^2+r+1)(r^2+6r+10)^3 = 0$
4. $r^5(r+4)(r-3)(r+2)^3(r^2+4r+5)^2 = 0$

C. Solve the following initial value problems

1. $y'''+7y''+14y'+8y = 0;$ $y(o) = 1,$ $y'(o) = -3,$ $y''(o) = 13.$
2. $y'''-y''-4y'+4y = 0;$ $y(o) = -4,$ $y'(o) = -1,$ $y''(o) = -19$
3. $y'''-4y''+7y'-6y = 0;$ $y(o) = 1,$ $y'(o) = 0,$ $y''(o) = 0.$

3.5 THE METHOD OF VARIATION OF PARAMETERS

The method of variation of parameters for determining a particular solution of the nonhomogeneous nth order linear differential equation

$$L[y] = y^{(n)}+P_1(x)y^{(n-1)}+\dots+P_{n-1}(x)y'+P_ny = g(x) \qquad (1)$$

is a direct extension of the theory for the second order differential equation. In order to use the method of variation of parameters, it is first very necessary to solve the corresponding homogeneous equation.

Suppose that we know a fundamental set of solutions $y_1, y_2, \dots, y_n$ of the homogeneous equation, then

$$y_c(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \dots + u_n(x)y_n(x) \qquad (2)$$

The method of variation of parameters for determining a particular solution of (1) rests on the possibility of determining n functions $u_1, u_2, \dots, u_n$ such that $y_p(x)$ is of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \dots + u_n(x)y_n(x) \qquad (3)$$

As we have n functions to determine, we have to specify n conditions, One of these is that y_p satisfy (1). The other (n-1) conditions are chosen so as to facilitate the calculations, like imposing conditions to suppress the terms that will lead to higher derivatives of u. From (3), we get

$$y_p' = (u_1y_1'+u_2y_2'+\dots+u_ny_n') + (u_1'y_1+u_2'y_2+\dots+u_n'y_n) \qquad (4)$$

Thus the first condition that we impose on the u_i is that

$$u_1'y_1 + u_2'y_2 + \dots + u_n'y_n = 0 \qquad (5)$$

Continuing this process in a similar manner through n-1 derivatives of y_p gives

$$y_p^{m} = u_1y_1^{m} + u_2y_2^{m} + \dots + u_ny_n^{m} \qquad (6)$$

$m = 1,2,\dots, n-1$, and the following (n-1) conditions on the functions $u_1, u_2, \dots, u_n$.

$$u_1'y_1^{(m-1)} + u_2'y_2^{(m-1)} + \dots + u_n'y_n^{(m-1)} = 0 \qquad (7)$$

$$m = 1, 2, \dots, n-1.$$

The nth derivatives of y_p is

$$y_p^{(n)} = (u_1y_1^{n}+\dots+u_ny_n^{n}) + (u_1'y_1^{(n-1)}+\dots+u_n'y_n^{(n-1)}) \qquad (8)$$

Finally, we impose the condition that y_p be a solution of (1). On substituting for the derivatives of y_p, collecting like terms and making use of the fact that $L(y_i) = 0$, $i = 1,2,\dots,n$; we obtain

$$u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \dots + u_n'y_n^{(n-1)} = g \qquad (9)$$

Equation (9) and (n-1) equations (7) give n simultaneous linear nonhomogeneous equations for $u_1', u_2', \dots, u_n'$

$$\left.\begin{array}{lllll}
y_1u_1' + y_2u_2' & +\dots+ & y_nu_n' & = & 0 \\
y_1'u_1' + y_2'u_2' & +\dots+ & y_n'u_n' & = & 0 \\
y_1''u_1' + y_2''u_2' & +\dots+ & y_n''u_n' & = & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' & +\dots+ & y_n^{(n-1)}u_n' & = & g
\end{array}\right\} \qquad (10)$$

A sufficient condition for the existence of a solution of the system of equations (10) is that the determinant of the coefficient is nonzero for each value of x. However the determinant of the coefficients is precisely $W(y_1, \ldots, y_n)$ and it is nowhere zero since $y_1, y_2, \ldots, y_n$ are linearly independent solutions of the homogeneous equation.

Hence it is possible to determine $u_1', \ldots, u_n'$. Using Cramer's rule, we find that the solution of the system of equation (10) is

$$u_m' = \frac{g(x)W_m(x)}{W(x)} \qquad m = 1, 2, \ldots, n \tag{11}$$

Here, $W(x) = W(y_1, y_2, \ldots, y_n)(x)$ and $W_m(x)$ is the determinant obtained from $W(y_1, \ldots, y_n)$ by replacing the mth column by the $(0, 0, \ldots, 0, 1)^T$.

With this notation, a particular solution of (1) is given by

$$y_p(x) = \sum_{m=1}^{n} y_m(x) \int^{x} \frac{g(t)W_m(t)}{W(t)}\,dt \tag{12}$$

Example

Find a general solution to the equation

$$x^3y''' + x^2y'' - 2xy' + 2y = x^3 \sin x, \quad x > 0 \tag{13}$$

given that $\{x, x^{-1}, x^2\}$ is a fundamental solution set to the corresponding homogeneous equation.

Solution

We first divide (13) by x^3 to obtain the standard form

$$y''' + \frac{1}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = \sin x \tag{14}$$

From which we see that $g(x) = \sin x$.

Since $\{x, x^{-1}, x^2\}$ is a fundamental solution set, we can obtain a particular solution of the form is nonzero for each value of x. However the determinant of the coefficients is precisely $W(y_1, \ldots, y_n)$ and it is nowhere zero since $y_1, y_2, \ldots, y_n$ are linearly independent solutions of the homogeneous equation.

Hence it is possible to determine $u_1', \ldots, u_n'$. Using Cramer's rule, we find that the solution of the system of equation (10) is

$$u_m' = \frac{g(x)W_m(x)}{W(x)} \qquad m = 1,2,\ldots,n \qquad (11)$$

Here, $W(x) = W(y_1, y_2, \ldots, y_n)(x)$ and $W_m(x)$ is the determinant obtained from $W(y_1, \ldots, y_n)$ by replacing the mth column by the $(0,0,\ldots,0,1)^T$.

With this notation, a particular solution of (1) is given by

$$y_p(x) = \sum_{m=1}^{n} y_m(x) \int^{x} \frac{g(t)W_m(t)}{W(t)}dt \qquad (12)$$

Example

Find a general solution to the equation

$$x^3y''' + x^2y'' - 2xy' + 2y = x^3\sin x, \quad x > 0 \qquad (13)$$

given that $\{x, x^{-1}, x^2\}$ is a fundamental solution set to the corresponding homogeneous equation.

Solution

We first divide (13) by x^3 to obtain the standard form

$$y''' + \frac{1}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = \sin x \qquad (14)$$

From which we see that $g(x) = \sin x$.

Since $\{x, x^{-1}, x^2\}$ is a fundamental solution set, we can obtain a particular solution of the form

$$y_p(x) = u_1(x)x + u_2(x)x^{-1} + u_3(x)x^2 \qquad (15)$$

To use (12) we first evaluate the determinant

$$W(x, x^{-1}, x^2)(x) = \begin{vmatrix} x & x^{-1} & x^2 \\ 1 & -x^{-2} & 2x \\ 0 & 2x^{-3} & 2 \end{vmatrix} = -6x^{-1}$$

$$W_1(x) = (-1)^{3-1} W(x^{-1}, x^2)(x) = \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix} = 3$$

$$W_2(x) = (-1)^{3-2} \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = -x^2$$

$$W_3(x) = (-1)^{3-3} \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1},$$

Substituting the above expressions in (12) we get

$$y_p(x) = x \int \frac{3\sin x}{-6x^{-1}} dx + x^{-1} \int \frac{(-x^2)\sin x}{-6x^{-1}} dx + x^2 \int \frac{(-2x^{-1})\sin x}{-6x^{-1}} dx$$

$$= x \int \frac{1}{2} x \sin x dx + x^{-1} \int \frac{1}{6} x^3 \sin x \, dx + x^2 \int \frac{1}{3} \sin x dx.$$

which simplifies to

$$y_p(x) = \cos x - x^{-1} \sin x + c_1 x + c_2 x^{-1} + c_3 x^2.$$

Exercises

A. Use the method of variation of parameters to determine a particular solution to the given differential equation

1. $y''' - 2y'' + y' = x$
2. $y''' - 3y'' + 4y = e^{2x}$
3. $y''' + 3y'' - 4y = e^{2x}$
4. $y''' + y' = \tan x, \ 0 < x < \frac{\pi}{2}.$

B. If x, x^2 and x^3 are solutions of the homogeneous equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0, \qquad x > 0$$

find a formula involving integral for a particular solution of the differential equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = g(x) \qquad x > 0.$$

2.6 THE METHOD OF UNDETERMINED COEFFICIENTS

A particular solution of the nonhomogeneous nth order linear equation with constant coefficients

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = g(x) \qquad (1)$$

can be obtained by the method of undetermined coefficients provided that $g(x)$ is of an appropriate form. While the method of undetermined coefficients is not as general as the method of variation of parameters described earlier, it is usually much easier to use.

Just as for the second order linear equation, it is clear that when the constant coefficient linear differential operator L is applied to a polynomial $A_0x^n + A_1x^{(n-1)} + \dots + A_n$, an exponential function $e^{\alpha x}$ or a sine function $\sin \beta x$, or a cosine function $\cos \beta x$, the result is a polynomial, an exponential function or linear combination of sine and cosine functions respectively.

Hence if $g(x)$ is a sum of polynomials exponential sines and cosine or even products of such functions, we can expect that it is possible to find $y_p(x)$ by choosing a suitable combination of polynomials, exponentials, etc with a number of undetermined constants. The constants are then determined so that (1) is satisfied.

Example

Find a particular solution of

$$y''' - 4y' = x + 3\cos x + e^{-2x} \qquad (2)$$

Solution

We first solve the homogeneous equation. The characteristic equation is $r^3 - 4r = 0$ and the roots are 0, ± 2. So,

$$y_c(x) = c_1 + c_2e^{2x} + c_2e^{-2x}.$$

Using the superposition principle, we can write a particular solution of (2) as the sum of particular solutions to the differential equations

$$y''' - 4y' = x, \quad y''' - 4y' = 3\cos x, \quad y''' - 4y' = r^{-2x}.$$

Our original guess for particular solution to y_{p_1} of the first equation is $A_0x + A_1$. But since a constant is a solution of the homogeneous equation, we multiply by x then,

$$y_{p_1}(x) = x(A_0x+A_1) = A_0x^2 + A_1x$$

For the second equation, we guess

$$y_{p_2}(x) = B\cos x + C\sin x$$

[No modification of this since neither $\cos x$ nor $\sin x$ is a solution to the homogeneous equation].

For the third equation, since e^{-2x} is a solution of the homogeneous equation, we assume that

$$y_{p_3} = Exe^{-2x}.$$

The constants are then determined by substituting into the individual differential equation to get

$$A_0 = \frac{-1}{8}, \quad A_1 = 0, \quad B = 0, \quad C = \frac{-3}{5}, \quad \text{and} \quad E = \frac{1}{8}.$$

Hence a particular solution of (2) is

$$y_p(x) = \frac{-1}{8}x^2 - \frac{3}{5}\sin x + \frac{1}{8}xe^{-2x}.$$

Note: The method of undetermined coefficients can be used whenever it is possible to guess the correct form of $y_p(x)$.

Exercises

A. Determine a suitable form for $y_p(x)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

1. $y'''-2y''+y' = x^3+2e^x$
2. $y'''-y' = xe^{-x} + 2\cos x$
3. $y^{iv}-2y''+y = e^x+\sin x$
4. $y^{iv}-4y'' = \sin 2x + xe^x + 4.$

B. Determine the general solution of the given differential equations

1. $y'''-y''-y'+y = 2e^{-x}+3$
2. $y'''+y''+y'+y = e^{-x}+4x$
3. $y^{iv}-4y'' = x^2+e^x$
4. $y^{iv}+2y''+y = 3+\cos 2x$

C. Determine the general solution of the following differential equation satisfying the given initial conditions

1. $y'''+4y' = x$; $y(0) = y'(0) = 0$. $y''(0) = 1$
2. $y^{iv}+2y''+y = 3x+4$; $y(0) = y'(0) = 0$. $y''(0) = y'''(0) = 1$
3. $y'''-3y''+2y' = x+e^x$; $y(0) = 1$, $y'(0) = -\frac{1}{4}$, $y''(0) = \frac{-3}{2}$
4. $y'''-2y''+5y' = -24e^{3x}$; $y(0) = 4$, $y'(0) = -1$, $y''(0) = -5$
5. $y'''+2y''-9y'-18y = -18x^2-18x+22$. $y(0) = -2$, $y'(0) = -8$, $y''(0) = -12$.
6. $y'''-2y''-3y'+10y = 34xe^{-2x} - 16e^{-2x} - 10x^2 + 6x + 34$

   $y(0) = 3$, $y'(0) = 0$, $y''(0) = 0$.

11 x 12

MTH 2[illegible] [illegible] DIFFERENTIAL EQUATIONS I

NAME [illegible] Reg. No. [illegible] Dept. [illegible] S/No. [illegible]

INSTRUCTIONS:

[illegible] the following questions in the spaces provided in this paper and attach [illegible] on a separate sheet of paper.

1. The solution of [illegible] equation $y' = \frac{[illegible]-1}{y^2}$ is

2. The solution to $([illegible] + 2y)\,dx + (2x^2y + 2x)\,dy = 0$ is

3. $y_1(x) = e^{2x}\cos x$ and $y_2(x) = e^{2x}\sin x$ are solutions to $y'' - 4y' + 5y = 0$.

   The solution that satisfy $y(0) = 1$, $y'(0) = -1$ is

4. The Wronskian of the solutions $y_1(x) = \cos 3x$ and $y_2(x) = \sin 3x$ is

5. The general solution to $y'' - 3y' + 2y = e^x \sin x$ is

6. The Wronskian to $\{e^{3x}, e^{5x}, e^{-x}\}$ on $(-\infty, \infty)$ for a given third-order ODE is

7. The solution to $y''' + 7y'' + 14y' + 8y = 0$, $y(0) = 1$, $y'(0) = -3$, $y''(0) = 13$ is

8. The suitable form for $y_p(x)$ if the method of undetermined coefficients is to the be used for $y''' - 2y'' + y' = x^3 + 2e^x$ is

9. The Wronskian of the solutions $y_1(x) = e^x \sin x$ and $y_2(x) = e^x \cos x$ is

10. The solution to $y'' + 8y' - 9y = 0$, $y(1) = 1$, $y'(1) = 0$ is