Course Title: Mathematical Methods I
Course Code: MTH 242
Credit Units: 3
Course Content:
Real-valued functions of real variable. Review of differentiation and integration and applications. Mean-value theorem. Tayor series. Real-valued functions of two variables or three variables. Partial derivatives. Chain-rule, extrema, Lagrange's multipliers, increments, differentials, and linear approximations. Evaluation of line-integrals. Multiple integrals.

CHAPTER ONE- Review of Elementary Calculus

FUNCTIONS

REAL VALUED FUNCTIONS OF A REAL VARIABLE

A real valued function of a real variable is a function that reasigns to each $x \in D(f) \subseteq \mathbb{R}$ a unique element $y \in \mathbb{R}$ denoted by $y = f(x)$ i.e. $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$.
The domain of the function, denoted by D_f, is the set of all variables $x \in D(f) \subseteq \mathbb{R}$ such that $y = f(x)$ is defined in R. i.e. $D_f = \{x \in D(f) \subseteq \mathbb{R} : y = f(x) \in \mathbb{R}\}$.
The range of the function, denoted by R_f, is the set of all $y \in \mathbb{R}$ such that $y = f(x)$ for some $x \in D_f$ i.e. $R_f = \left\{y \in \mathbb{R} : y = f(x) \text{ for some } x \in D_f\right\}$.
Example 1: $f(x) = \frac{x}{x-1}$. For the function to be undefined, $x = 1$. Since the function must be defined, $x \neq 1$. Therefore $D_f = \{x \in \mathbb{R} : x \neq 1\}$.
To find the range of this function, let $y = \frac{x}{x-1}$. Make x the subject of the formula as follows:

$$y(x-1) = x \Rightarrow yx - y = x \Rightarrow yx - x = y \Rightarrow x(y-1) = y \Rightarrow x = \frac{y}{y-1}$$

For the range, the function must be defined in $\mathbb{R}$. i.e. $y - 1 \neq 0$. $\therefore R_f = \{y \in \mathbb{R} : y - 1 \neq 0\} = \{y \in \mathbb{R} : y \neq 1\}$
Example 2: $f(x) = \sqrt{x+1}$. Remember the domain must be a real number. If $x + 1$ is less than zero, $f(x)$ would be a complex number. Therefore

$$x + 1 \geq 0 \Rightarrow x \geq -1. \, D_f = \{x \in \mathbb{R} : x \geq -1\}.$$

To find the range, let $y = \sqrt{x+1} \Rightarrow y^2 = x + 1 \Rightarrow x = y^2 - 1$
$\because x \geq -1$ for domain implies $y^2 - 1 \geq -1$, by substituting for x. $\Rightarrow y^2 \geq 0$. therefore $R_f = \{y \in \mathbb{R} : y \geq 0\}$.

Exercises

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined at $x \in \mathbb{R}$ by $(a)\ f(x) = \frac{x}{x-1}$ $(b)\ f(x) = \sqrt{x+3}$ $(c)\ f(x) = \sqrt{x-2}$

$$(d)\ f(x) = \frac{x^2 + x}{(x-1)(x-2)} \qquad (e)\ f(x) = \frac{x^2 + x}{(x+a)(x+b)}.$$

Find the domains and ranges of $(a) - (c)$. At what points are (d) and (e) undefined. Hence find the domains of (d) and (e)

DIFFERENTIATION

The Derivative: Given a function $y = f(x)$, which is continuous for all x in the domain of f, then the derivative of $f(x)$ with respect to $(wrt)\, x$ is the function $f'(x) = \frac{dy}{dx}$ defined by
$f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, $\Delta x \neq 0$.
E.g. (1) $f(x) = x^3$.

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right], (\Delta x \neq 0) = \lim_{\Delta x \to 0}\left[\frac{(x+\Delta x)^3 - x^3}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^3 - x^3}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{3x^3 \cdot \Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[3x^2 + 3x \cdot \Delta x + \Delta x\right] = 3x^2 + 0 + 0 = 3x^2.$$

E.g.(2) $f(x) = x^2$. $f'(x) = \lim_{\Delta x \to 0}\left[\frac{(x+\Delta x)^2 - x^2}{\Delta x}\right] (\Delta x \neq 0) = \lim_{\Delta x \to 0}\left[\frac{x^2+2x\cdot\Delta x+(\Delta x)^2-x^2}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{2x\cdot\Delta x(\Delta x)^2}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0}[2x + \Delta x] = 2x + 0 = 2x$$

E.g. (3) $f(x) = x^2 + 3$.

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{(x+\Delta x)^2+3-(x^2+3)}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{x^2+2x\cdot\Delta x+(\Delta x)^2+3-x^2-3}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{2x\cdot\Delta x(\Delta x)^2}{\Delta x}\right]$$
$$= \lim_{\Delta x \to 0}[2x + \Delta x] = 2x + 0 = 2x.$$

From examples one and two, we observe that if $f(x) = x^n$ is a continuous function defined for all x in the domain of f, then the derivative of $f(x)$ denoted by $\frac{dy}{dx} = f'(x) = nx^{n-1}$.
Example 3 shows that if $f(x) = x^n + a$, where 'a' is an arbitrary constant, then $f'(x) = nx^{n-1}$.
From example (3), $f'(x) = \frac{dy}{dx} = \frac{d}{dx}(x^2+3) = \frac{d}{dx}(x^2) + \frac{d}{dx}(3) = 2x + 0 = 2x$.
Thus $\frac{d}{dx}(a) = 0$, a being an arbitrary constant.
Example: $f(x) = x^2 + x^3 \Rightarrow f'(x) = \frac{d(x^2+x^3)}{dx} = 3x + 3x^2 = \frac{d(x^2)}{dx} + \frac{d(x^3)}{dx}$ This gives a meaning to our having distributed $\frac{d}{dx}$ over x^2+3 above. It further gives a general rule that says that if $h(x)$ and $g(x)$ are both defined and continuous at a point x of the domain, then

$$\frac{d[h(x)+g(x)]}{dx} = \frac{d}{dx}[h(x)] + \frac{d}{dx}[g(x)]$$

It is easy to see that if $f(x)$ is defined and continuous for some x in the domain of f, then

$$\frac{d}{dx}(kf(x)) = k\frac{df(x)}{dx}$$

As a matter of fact, if $f(x) = ax^n$ then $\frac{dy}{dx} = f'(x) = anx^{n-1}$, $'a'$ being constant.
Differentiating the function e^x, yields the same result. i.e. $\frac{de^x}{dx} = e^x$. Generally, differentiating a function $f(x) = e^{ax}$, we have $\frac{de^{ax}}{dx} = ae^{ax}$. To differentiate a function a^x, where a is an arbitrary constant cannot be done directly. We first change to the base $'e'$ and then carry out the differentiation. We proceed as follow $a^x = e^z$, where z is an unknown constant.

$$\ln a^x = \ln e^z \Rightarrow x\ln a = z\ln e \Rightarrow x\ln a = z. \therefore a^x = e^{x\ln a}. \text{ Thus } \frac{da^x}{dx} = \frac{de^{\ln a}}{dx} = \ln a\, e^{x\ln a}$$

Product Rule
Let $u(x)$ and $v(x)$ be two continuous functions of x. We aim at differentiating the product $u(x) \bullet v(x)$. This is differentiated as follows $\frac{d(u(x)\bullet v(x))}{dx} = v(x)\frac{du(x)}{dx} + u(x)\frac{dv(x)}{dx}$.
E.g. Let $u(x) = 2x$ and $v(x) = 4x + 1$. $\frac{d}{dx}((4x+1)(2x)) = (4x+1)\frac{d}{dx}(2x) + 2x\frac{d}{dx}(4x+1) = (4x+1)2 + 2x \times 4$

$$= 2(4x + 1 + 4x) = 2(8x + 1).$$

Quotient Rule
Again let $u(x)$ and $v(x)$ be two continuous functions of x. We aim at differentiating the quotient $\frac{u(x)}{v(x)}$ *wrt* x. We proceed thus $\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)\frac{d}{dx}u(x) - u(x)\frac{d}{dx}v(x)}{v(x)^2}$
e.g. Let $u(x) = x^2$ and $v(x) = 2x$

$$\frac{du(x)}{dv(x)} = \frac{d}{dx}\left(\frac{x^2}{2x}\right) = \frac{2x\frac{dx^2}{dx} - x^2\frac{d}{x}(2x)}{(2x)^2} = \frac{2x \bullet 2x - 2x^2}{dx^2} = \frac{4x^2 - 2x^2}{4x^2} = \frac{2x^2}{4x^2} = \frac{1}{2}.$$

Another special differentiation is that of ln x: $\frac{d}{dx}\ln x = \frac{1}{x}$

Function of a Function (Chain Rule)

If $f(g(x))$, f is a function of $g(x)$ and $g(x)$ is itself a function of x, then $f(g(x))$ is also a function of x. We aim at differentiating such functions. To do this, first differentiate the outer function f wrt g and then the inner functiong (x) wrt x, and multiply the results to get the required result.

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \bullet g'(x).$$

e.g. let $f(g(x)) = (2x+1)^3$. Notice this is a function of a function the inner function $2x+1$ is $g(x)$. Then by raising it to the power 3, we get another function altogether, which is $f(g(x))$.

$$\therefore \frac{d}{dx}(2x+1)^3 = 3(2x+1)^2 \times 2 = 6(2x+1)^2.$$

Trigonometric Differentiation

We take our example from first principle and then quote the other results, which could be obtained in a similar way.

Let $f(x) = \sin x$. We differentiate from first principles as follows:

Recall that $f'(x) = \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right] \therefore \frac{d}{dx}\sin x = \lim_{\Delta x\to 0}\left[\frac{\sin(x+\Delta x)-\sin x}{\Delta x}\right] =$

$\lim_{\Delta x\to 0}\left[\frac{2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x}\right] = \lim_{\Delta x\to 0}\left[\cos\left(x+\frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right]\lim_{\Delta x\to 0}\left[\frac{2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x}\right] = \lim_{\Delta x\to 0}\left[\cos\left(x+\frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right] =$

$\lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right)\cdot\left(\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right) = \cos x\cdot 1 = \cos x$ (x is measured in rad).

Similarly, $\frac{d}{dx}\cos x = -\sin x$; $\frac{d}{dx}\tan x = \sec^2 x$.

To differentiate a trigonometric function of power, say n, we treat it as "a function of a function". E.g. $f(x) = \sin^n x$. Observe that $\sin x$ is the inner function and when we raise it to a power, we obtain the outer function.

$$\frac{d}{dx}\sin^n x = n\sin^{n-1}x\cos x. \text{ E.g. } f(x) = \cos^3 x. \quad \frac{d}{dx}\cos^3 x = -3\cos^2 x\sin x.$$

To differentiate $\sin nx$, we first differentiate the inner function nx and then $\sin nx$ and multiply the results. E.g. (1) Differentiate $y = \sin 3x$

$\frac{dy}{dx} = 3\cos 3x$. E.g. (2) Differentiate $y = 4\cos 2x \Rightarrow \frac{dy}{dx} = -2\times 4\ \sin 2x = -8\sin 2x$.

Differentiating Inverse Trigonometric Functions

DERIVATIVE OF INVERSE TRIGNOMETRIC FUNCTIONS:

(i) $y = \sin^{-1}\frac{x}{a}$, $y' = \frac{1}{\sqrt{a^2-x^2}}$.

VERIFICATION: $y = \sin^{-1}\frac{x}{a} \Rightarrow \frac{x}{a} = \sin y \Rightarrow x = a\sin y$ (x a function of y).

$$\therefore \frac{dx}{dy} = a\cos y \text{ (since } x \text{ is now a function of } y\text{, we differentiate } x \text{ wrt } y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a\cos y}\left(\text{taking recprocal of both sides as we are looking for } \frac{dx}{dy}\right)$$

$$= 1 \div a\sqrt{1-\sin^2 y}\left(\sin^2 y + \cos^2 y = 1 \Rightarrow \cos y = \sqrt{1-\sin^2 y}\right) = 1 \div \left(a\sqrt{1-\frac{x^2}{a^2}}\right) = 1 \div \left(a\sqrt{\frac{a^2-x^2}{a^2}}\right)$$

$$= 1 \div \cancel{a}\frac{\sqrt{a^2-x^2}}{\cancel{a}} = 1\times\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-x^2}} \quad \square$$

(ii) Similarly $y = \cos^{-1}\frac{x}{a}$, $y' = -\frac{1}{\sqrt{a^2-x^2}}$

(iii) $y = \tan^{-1}\frac{x}{a},\ y' = \frac{a}{a^2+x^2}$
VERIFICATION:$y = \tan^{-1}\frac{x}{a} \Rightarrow \frac{x}{a} = \tan y \Rightarrow x = a\tan y$ (x a function of y)

$$\therefore \frac{dx}{dy} = a\sec^2 y \text{ (since } x \text{ is now a function of } y\text{, we differentiate } x \text{ wrt } y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a\sec^2 y}\left(\text{taking recprocal of both sides as we are looking for } \frac{dx}{dy}\right)$$

$$= 1 \div a\left(1+\tan^2 y\right)\left(\text{since } \sec^2 y = 1+\tan^2 y\right) = 1\div\left[a\left(1+\frac{x^2}{a^2}\right)\right] = 1\div\left[\cancel{a}\left(\frac{a^2+x^2}{a^{\cancel{2}1}}\right)\right] = 1\times\frac{a}{a^2+x^2} = \frac{a}{a^2+x^2}\ \square$$

Derivative of Reciprocal of Inverse Trigonometric Functions.
(iv) $y = \sec^{-1}\frac{x}{a},\ \ y' = \frac{a}{x\sqrt{x^2-a^2}}$.
VERIFICATION: $y = \sec^{-1}\frac{x}{a} \Rightarrow \frac{x}{a} = \sec y \Rightarrow x = a\sec y \Rightarrow \frac{dx}{dy} = a\sec y\tan y$

$$= 1\div\left(x\frac{\sqrt{x^2-a^2}}{a}\right) = 1\times\frac{a}{x\sqrt{x^2-a^2}} = \frac{a}{x\sqrt{x^2-a^2}}\ \square$$

(v) Similarly $y = \csc^{-1}\frac{x}{a},\ \ y' = -\frac{a}{x\sqrt{x^2-a^2}}$ (csc = cosec)
(vi) Similarly, using (iii) $y = \cot^{-1}\frac{x}{a},\ y' = -\frac{a}{a^2+x^2}$
Example: Find derivative of the following: (i) $\sin^{-1}4x$ (ii) $\cos^{-1}\frac{x}{3}$ (iii) $\tan^{-1}\frac{2x}{x^2-1}$ (iv) $\cot^{-1}(1-3x)$ (v) $\sec^{-1}\frac{x^2+1}{x^2-1}$ (vi) $\text{cosec}^{-1}\frac{1}{2x^2-1}$.
SOLUTION: (i) $y = \sin^{-1}4x \equiv \sin^{-1}\frac{x}{a} \Rightarrow 4 = \frac{1}{a} \Rightarrow a = \frac{1}{4}$. But $y = \sin^{-1}\frac{x}{a} \Rightarrow y' = \frac{1}{\sqrt{a^2-x^2}}$

$$\therefore\ y = \sin^{-1}4x \Rightarrow y' = \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2 - x^2}} = 1\div\sqrt{\frac{1}{16}-x^2} = 1\div\sqrt{\frac{16-x^2}{16}} = 1\div\frac{\sqrt{16-x^2}}{4} = 1\times\frac{4}{\sqrt{16-x^2}} = \frac{4}{\sqrt{16-x^2}}$$

$$\text{(ii) } y = \cos^{-1}\frac{x}{3} \equiv \cos^{-1}\frac{x}{a} \Rightarrow \frac{1}{3} = \frac{1}{a} \Rightarrow a = 3. \text{ But } y = \cos^{-1}\frac{x}{a} \Rightarrow y' = -\frac{1}{\sqrt{a^2-x^2}}$$

$$\therefore\ y = \cos^{-1}\frac{x}{3} \Rightarrow y' = -\frac{1}{\sqrt{3^2-x^2}} = -\frac{1}{\sqrt{9-x^2}}$$

(iii) $y = \tan^{-1}\frac{2x}{1-x^2} \equiv \tan^{-1}u$ where $u = \frac{2x}{1-x^2} \Rightarrow u' = \frac{(1-x^2)\frac{d}{dx}(2x) - 2x\frac{d}{dx}(1-x^2)}{(1-x^2)^2}$ (quotient rule)
$= \frac{2-2x^2+4x^2}{(1-x^2)^2} = \frac{2+2x^2}{(1-x^2)^2} = \frac{2(1+x^2)}{(1-x^2)^2}$. But $y = \tan^{-1}\frac{x}{a} \Rightarrow y' = \frac{1}{a^2+x^2}$

$$\Rightarrow \frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}\left(\text{Since } \tan^{-1}\frac{x}{a} \equiv \tan^{-1}x \Rightarrow a = 1\right) \Rightarrow \frac{d}{du}\tan^{-1}u = \frac{1}{1+u^2}$$

$$\therefore y = \tan^{-1}\frac{2x}{1-x^2} \equiv \tan^{-1}u \Rightarrow y' = \frac{dy}{du}\bullet\frac{du}{dx} = \left(\frac{1}{1+u}\right)\left(\frac{2\left(1+x^2\right)}{(1-x^2)^2}\right) = \left[1\div\left(1+\left(\frac{2x}{1-x^2}\right)^2\right)\right]\left(\frac{2\left(1+x^2\right)}{(1-x^2)^2}\right)$$

$$= \left[1\div\left(1+\frac{4x^2}{(1-x^2)^2}\right)\right]\left(\frac{2\left(1+x^2\right)}{(1-x^2)^2}\right) = \left[1\div\left(\frac{\left(1-x^2\right)^2+4x^2}{(1-x^2)^2}\right)\right]\left(\frac{2\left(1+x^2\right)}{(1-x^2)^2}\right) = \left[1\div\left(\frac{1+2x^2+x^4}{(1-x^2)^2}\right)\right]\left(\frac{2\left(1+x^2\right)}{(1-x^2)^2}\right)$$

$$= \left[1\div\left(\frac{\left(1+x^2\right)^2}{(1-x^2)^2}\right)\right]\left(\frac{2\left(1+x^2\right)}{(1-x^2)^2}\right) = 1\times\left(\frac{\left(1-x^2\right)^2}{(1+x^2)^{\cancel{2}1}}\right)\left(\frac{2\left(1+x^2\right)}{(1-x^2)^2}\right) = \frac{2}{1+x^2}.$$

$$\text{(iv) } y = \cot^{-1}(1-3x) \equiv \cot^{-1}u \text{ where } u = (1-3x) \Rightarrow u' = -3. \text{ But } y = \cot^{-1}\frac{x}{a} \Rightarrow y' = -\frac{1}{a^2+x^2}$$

$$\Rightarrow \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}\left(\text{Since } \cot^{-1}\frac{x}{a} \equiv \cot^{-1} x \Rightarrow a = 1\right) \Rightarrow \frac{d}{du}\left(\cot^{-1} u = -\frac{1}{1+u^2}\right).$$

$$\therefore y = \cot^{-1}(1-3x) \equiv \cot^{-1} u \Rightarrow y' = \frac{dy}{du} \bullet \frac{du}{dx} = \left(-\frac{1}{1+u}\right) \bullet -3 = \left(-\frac{1}{1+(1-3x)^2}\right) \bullet -3$$

$$= \left(-\frac{1}{1+1-6x+9x^2}\right) \bullet -3 = \frac{3}{2-6x+9x^2}$$

(v) $y = \sec^{-1}\frac{x^2+1}{x^2-1} \equiv \sec^{-1} u$ where $u = \frac{x^2+1}{x^2-1} \Rightarrow u' = \frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$ (quotient rule)

$$= \frac{(x^2-1)2x-(x^2+1)(2x)}{(x^2-1)^2} = \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2} = \frac{-4x^2}{(x^2-1)^2}.$$

But $y = \sec^{-1}\frac{x}{a} \Rightarrow y' = \frac{a}{x\sqrt{x^2-a^2}} \Rightarrow \frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}\left(\text{Since } \sec^{-1}\frac{x}{a} \equiv \sec^{-1} x \Rightarrow a = 1\right)$

$$\Rightarrow \frac{d}{du}\sec^{-1} u = \frac{1}{u\sqrt{u^2-1}} \therefore y = \sec^{-1}\frac{x^2+1}{x^2-1} \equiv \sec^{-1} u \Rightarrow y' = \frac{dy}{du} \bullet \frac{du}{dx} = \frac{1}{u\sqrt{u^2-1}} \bullet \frac{-4x}{(x^2-1)^2}$$

$$= \left[1 \div \left(\frac{x^2+1}{x^2-1}\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2 - 1}\right)\right] \bullet \frac{-4x}{(x^2-1)^2} = \left[1 \div \left(\frac{x^2+1}{x^2-1}\sqrt{\frac{(x^2+1)^2}{(x^2-1)^2} - 1}\right)\right] \bullet \frac{-4x}{(x^2-1)^2}$$

$$= \left[1 \div \left(\frac{x^2+1}{x^2-1}\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2-1)^2}}\right)\right] \bullet \frac{-4x}{(x^2-1)^2} = \left[1 \div \left(\frac{x^2+1}{x^2-1}\sqrt{\frac{x^4+2x^2+\cancel{1}-(x^4-2x^2-\cancel{1})}{(x^2-1)^2}}\right)\right] \bullet \frac{-4x}{(x^2-1)^2}$$

$$= \left[1 \div \left(\frac{x^2+1}{x^2-1}\sqrt{\frac{4x^2}{(x^2-1)^2}}\right)\right] \bullet \frac{-4x}{(x^2-1)^2} = \left[1 \div \left(\frac{x^2+1}{x^2-1} \bullet \frac{2x}{x^2-1}\right)\right] \bullet \frac{-4x}{(x^2-1)^2} = 1 \times \frac{(x^2-1)^2}{2x(x^2+1)} \times \frac{-4x2}{(x^2-1)^2} = \frac{-2}{(x^2+1)}.$$

(vi) $y = \csc^{-1}\frac{1}{2x^2-1} \equiv \csc^{-1} u$ where $u = \frac{1}{2x^2-1} \Rightarrow u' = \frac{d}{dx}\left(2x^2-1\right)^{-1} = (-1)\left(2x^2-1\right)^{-2}(4x) = -\frac{4x}{(2x^2-1)^2}.$

But $y = \csc^{-1}\frac{x}{a} \Rightarrow y' = -\frac{a}{x\sqrt{x^2-a^2}} \Rightarrow \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}\left(\text{Since } \csc^{-1}\frac{x}{a} \equiv \csc^{-1} x \Rightarrow a = 1\right)$

$$\Rightarrow \frac{d}{du}\csc^{-1} u = -\frac{1}{u\sqrt{u^2-1}} \therefore y = \csc^{-1}\frac{1}{2x^2-1} \equiv \csc^{-1} u \Rightarrow y' = \frac{dy}{du} \bullet \frac{du}{dx} = -\frac{1}{u\sqrt{u^2-1}} \bullet \frac{-4x}{2x^2-1}$$

$$= \left[1 \div \left(\frac{1}{2x^2-1}\sqrt{\left(\frac{1}{2x^2-1}\right)^2 - 1}\right)\right] \bullet \frac{4x}{(x^2-1)^2}\left[1 \div \left(\frac{1}{2x^2-1}\sqrt{\frac{1}{(2x^2-1)^2} - 1}\right)\right] \bullet \frac{4x}{(x^2-1)^2}$$

$$= \left[1 \div \left(\frac{1}{2x^2-1}\sqrt{\frac{1-(2x^2-1)^2}{(2x^2-1)^2}}\right)\right] \bullet \frac{4x}{(2x^2-1)^2} = \left[1 \div \left(-\frac{1}{2x^2-1}\sqrt{\frac{\cancel{1}-(4x^4-4x^2+\cancel{1})}{(2x^2-1)^2}}\right)\right] \bullet \frac{4x}{(2x^2-1)^2}$$

$$= \left[1 \div \left(\frac{1}{2x^2-1}\sqrt{\frac{4x^2(1-x^2)}{(2x^2-1)^2}}\right)\right] \bullet \frac{4x}{(2x^2-1)^2} = \left[1 \div \left(\frac{1}{2x^2-1} \bullet \frac{2x\sqrt{1-x^2}}{2x^2-1}\right)\right] \bullet \frac{4x}{(2x^2-1)^2}$$

$$= 1 \times \frac{(2x^2-1)^2}{2x\sqrt{1-x^2}} \times \frac{4x2}{(2x^2-1)^2} = \frac{2}{\sqrt{1-x^2}}$$

Differentiate the following: (i) $\sin^{-1}\frac{x}{2}$ (ii) $\cos^{-1} 2x^2$ (iii) $\tan^{-1}\sqrt{x}$ (iv) $\sec^{-1}\frac{1}{x}$ (v) $\sin^{-1}(3x-1)$ (vi) $\cot^{-1}\left(\frac{x-1}{x+1}\right)$ (vii) $\sec^{-1}\frac{b}{\sqrt{b^2-x^2}}$ (viii) $\operatorname{cosec}^{-1}\sqrt{2x}$ (ix) $\cot^{-1} 5x$ (x) $\cos^{-1}(\cos x)$ (xi) $\sin^{-1}\sqrt{\sin x}$

Derivative of Hyperbolic Functions

$$\sinh x = \frac{e^x - e^x}{2},\ \cosh x = \frac{e^x + e^x}{2},\ \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{2} \div \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{\cancel{2}} \times \frac{\cancel{2}}{e^x + e^{-x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},\ \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}},\ \coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^x}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{\left(e^{2x} + e^{-2x}\right)}{2} = \cosh 2x$$

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^x}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1$$

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x}\left(\text{dividing through by } \sinh^2 x\right) \Rightarrow \coth^2 x - 1 = \text{cosech}^2 x$$

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}\left(\text{dividing through by } \cosh^2 x\right) \Rightarrow 1 - \tanh^2 x = \text{sech}^2 x$$

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x,\ \frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh x\left(\frac{d}{dx}\sinh x\right) - \sinh x\left(\frac{d}{dx}\cosh x\right)}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \text{sech}^2 x$$

$$\frac{d}{dx}\coth x = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh x\left(\frac{d}{dx}\cosh x\right) - \cosh x\left(\frac{d}{dx}\sinh x\right)}{\sinh^2 x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\text{cosech}^2 x$$

$$\frac{d}{dx}\text{sech} x = \frac{d}{dx}\left(\frac{1}{\cosh x}\right) = \frac{\cosh x\left(\frac{d}{dx}(1)\right) - (1)\frac{d}{dx}\cosh x}{\cosh^2 x} = \frac{-\sinh x}{\cosh^2 x} = -\frac{1}{\cosh x} \bullet \frac{\sinh x}{\cosh x} = -\text{sech} x \tanh x$$

$$\frac{d}{dx}\text{cosech} x = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = \frac{\sinh x\left(\frac{d}{dx}(1)\right) - (1)\frac{d}{dx}\sinh x}{\sinh^2 x} = \frac{-\cosh x}{\sinh^2 x} = -\frac{1}{\sinh x} \bullet \frac{\cosh x}{\sinh x} = -\text{cosech} x \coth x$$

Inverse and Reciprocal of Inverse Hyperbolic Functions

(i) $y = \sinh^{-1}\frac{x}{a},\ y' = \frac{1}{\sqrt{x^2+a^2}}$

Verification:

$$y = \sinh^{-1}\frac{x}{a} \Rightarrow \frac{x}{a} = \sinh y \Rightarrow x = a\sinh y \Rightarrow \frac{dx}{dy} = a\cosh y \Rightarrow \frac{dy}{dx} = \frac{1}{a\cosh y}$$

$$= 1 \div a\sqrt{1 + \sinh^2 y} = 1 \div \sqrt{1 + \frac{x^2}{a^2}} = 1 \div a\sqrt{\frac{a^2 + x^2}{a^2}} = 1 \div \cancel{a}\frac{\sqrt{a^2 + x^2}}{\cancel{a}} = 1 \times \frac{1}{\sqrt{a^2 + x^2}} = \frac{a}{\sqrt{a^2 + x^2}}$$

(ii) Similarly, $y = \cosh^{-1}\frac{x}{a},\ y' = \frac{1}{\sqrt{x^2-a^2}}$

(iii) $y = \tanh^{-1}\frac{x}{a},\ y' = \frac{a}{x^2-a^2}$

Verification:

$$y = \tanh^{-1}\frac{x}{a} \Rightarrow \frac{x}{a} = \tanh y \Rightarrow x = a\tanh y \Rightarrow \frac{dx}{dy} = a\text{sech}^2 y \Rightarrow \frac{dy}{dx} = \frac{1}{a\text{sech}^2 y}.$$

$$= 1 \div a\left(1 - \tanh^2 y\right) = 1 \div a\left(1 - \left(\frac{x}{a}\right)^2\right) = 1 \div \cancel{a}\left(\frac{a^2 - x^2}{a^{\cancel{2}1}}\right) = 1 \times \frac{a}{a^2 - x^2} = \frac{a}{a^2 - x^2}$$

(iv) Similarly, $y = \coth^{-1}\frac{x}{a},\ y' = -\frac{a}{x^2-a^2}$

(v) $y = \text{sech}^{-1}\frac{x}{a},\ y' = -\frac{a}{x\sqrt{a^2-x^2}}$

Verification:

$$y = \text{sech}^{-1}\frac{x}{a} \Rightarrow \frac{x}{a} = \text{sech} y \Rightarrow x = a\text{sech} y \Rightarrow \frac{dx}{dy} = -a\,\text{sech} y \tanh y \Rightarrow \frac{dy}{dx} = -\frac{1}{a\,\text{sech} y \tanh y}$$

$$= -1 \div \cancel{a}\left(\frac{x}{\cancel{a}}\sqrt{1 - \frac{x^2}{a^2}}\right) = -1 \div \left(x\sqrt{\frac{a^2 - x^2}{a^2}}\right) = -1 \div \left(x\frac{\sqrt{a^2 - x^2}}{a}\right) = -1 \times \left(\frac{a}{x\sqrt{a^2 - x^2}}\right) = -\frac{a}{x\sqrt{a^2 - x^2}}.$$

(vi) Similarly, $y = \text{cosech}^{-1}\frac{x}{a},\ y' = -\frac{a}{x\sqrt{a^2+x^2}}.$

Example: Differentiate: (i) $\sinh 2x$ (ii) $\cosh^3 x$ (iii) $\tanh 2x^2$ (iv) $\sinh^{-1}\frac{x}{2}$ (v) $\cosh^{-1}(4x+1)$ (vi) $\tanh^{-1}\frac{1}{1+x}$.

Solution: (i) $y = \sinh 2x$. Let $u = 2x \Rightarrow u' = 2$. Thus $y = \sinh u \Rightarrow y' = \frac{dy}{du} \bullet \frac{du}{dx} = \cosh u.2 = 2\cosh u = 2\cosh 2x$.

(ii) $y = \cosh^3 x$. Let $u = \cosh x \Rightarrow u' = \sinh x$. Thus $y = \sinh u \Rightarrow y' = \frac{dy}{du} \bullet \frac{du}{dx} = 3u^2.\sinh x = 3\cosh^2 x.\sinh x = 3(1+\sinh^2 x)\sinh x = 3\sinh x + 3\sinh^3 x$

(iii) $y = \tanh 2x^2$. Let $u = 2x^2 \Rightarrow u' = 4x \therefore y = \tanh u \Rightarrow y' = \frac{dy}{du} \bullet \frac{du}{dx} = \text{sech}^2 u.4x = 4x\text{sech}^2 2x^2$.

(iv) $y = \sinh^{-1}\frac{x}{2}$. Let $= u = \frac{x}{2} \Rightarrow u' = \frac{1}{2}$. $\therefore y = \sinh^{-1} u \Rightarrow y' = \frac{dy}{du}\bullet\frac{du}{dx} = \frac{1}{\sqrt{u^2+1}}\bullet\frac{1}{2} = \frac{1}{2}\bullet\frac{1}{\sqrt{\left(\frac{x}{2}\right)^2+1}}$

$$= \frac{1}{2}\bullet\frac{1}{\sqrt{\frac{x^2}{4}+1}} = \frac{1}{2}\bullet\frac{1}{\sqrt{\frac{x^2+4}{4}}} = \frac{1}{\cancel{2}}\bullet\frac{1}{\frac{\sqrt{x^2+4}}{\cancel{2}}} = \frac{1}{\sqrt{x^2+4}}.$$

(v) $\cosh^{-1}(4x+1)$. Let $= u = 4x+1 \Rightarrow u' = 4$. $\therefore y = \cosh^{-1} u \Rightarrow y' = \frac{dy}{du} \bullet \frac{du}{dx} = \frac{1}{\sqrt{u^2-1}} \times 4$
$= \frac{4}{\sqrt{(4x+1)^2-1}} = \frac{4}{\sqrt{16x^2+8x}} = \frac{4}{\sqrt{4(4x^2+2x)}} = \frac{2}{\sqrt{2x(2x+1)}}$.

(vi) $\tanh^{-1}\frac{1}{1+x}$. Let $= u = \frac{1}{1+x} \Rightarrow u' = \frac{-1}{(1+x)^2}$. $\therefore y = \tanh^{-1} u \Rightarrow y' = \frac{dy}{du} \bullet \frac{du}{dx} = \frac{1}{1-u^2} \bullet \frac{-1}{(1+x)^2}$
$= \frac{-1}{(1+x)^2} \bullet \left[1 \div \left(1 - \left(\frac{1}{1+x}\right)^2\right)\right] = \frac{-1}{(1+x)^2} \bullet \left[1 \div \left(1 - \frac{1}{(1+x)^2}^2\right)\right] = \frac{-1}{(1+x)^2} \bullet \left[1 \div \left(\frac{1+2x+x^2-1}{(1+x)^2}^2\right)\right]$
$= \frac{-1}{(1+x)^2} \bullet \frac{(1+x)^2}{2x+x^2} = \frac{-1}{x(2+x)}$.

Exercise. Find the derivative of the following functions: (a) $\cosh\frac{x}{3}$ (b) $\tanh ax$ (c) $\sinh(ax+b)$

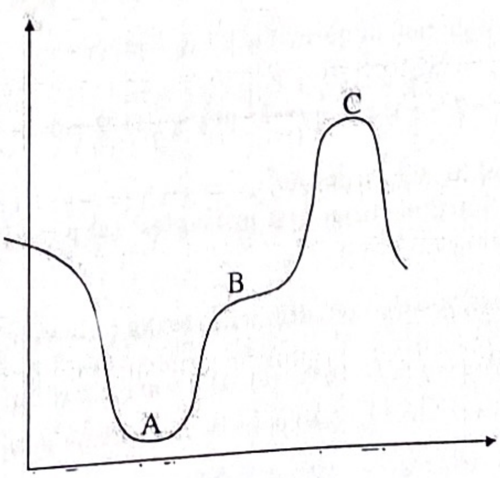
(d) $\cosh^{-1} 2x$ (e) $\sinh^{-1}\tan x$ (f) $\cosh^{-1}\sec x$ (g) $\tanh^{-1}\tan\frac{1}{2}x$ (h) $\tan^{-1}\tanh\frac{1}{2}x$

Application of differentiation

Differentiation could be used to: determine the maximum and minimum values of a function, obtain the equations of tangents and normal to curves, obtain Taylors (Maclaurins') series' expansion of functions, etc.

Turning Points (Minimum, Maximum, Saddle Points)

Turning points are those points along the graph of the function which have a zero gradient, or the points with horizontal tangents to the graph of the function. Consider the figure show (Fig. 1).



Turning points occur at A, B and C respectively. The point A is a minimum turning point (the graph passes from a negative gradient through zero to a positive gradient). The point B is a saddle point (the graph passes from a positive gradient through zero to a positive gradient). The point C is a maximum turning point (the graph passes from a positive gradient through zero to a negative gradient).

If $y = f(x)$ is the graph of a function, then at all turning points, $\frac{dy}{dx} = f'(x) = 0$. The notion of maximum and minimum turning points are very useful; in areas such as engineering, economics, etc.

An entrepreneur may need to minimize the cost of producing a certain commodity, while at the same time he wishes to maximize the profit he gets from producing the commodity. What he needs is the production and profit functions. When the situation has been properly modelled and he has the required functions, he could then find the maximum profit level by taking the first derivative of the profit function and equating to zero, and then solving for the values. When he has gotten them, he tests to ensure that what he got was in fact the maximum profit level.

Similarly to minimize costs, he takes the first derivative of the production function and equate to zero, after which he solves for the minimum costs levels.

We already know that at turning points, $f'(x) = \frac{dy}{dx} = 0$. We need a further test to determine whether this turning point is a minimum or maximum turning point. A common test is to take the second derivative and check if it is negative or positive at this point. If it is negative, then the point is a maximum turning point but if it is positive, then it is a minimum turning point.

Example: Find the turning point of $y = f(x.) = x^2 - 3x + 2$.

Solution: $y = x^2 - 3x + 2$; $\frac{dy}{dx} = 2x - 3$. At turning point, $\frac{dy}{dx} = 0 \Rightarrow 2x - 3 = 0 = x = \frac{3}{2}$.

$y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 2 = \frac{9}{4} - \frac{18}{4} + \frac{8}{4} = -\frac{1}{4}$. Therefore turning point is $\left({}^3/_2, -{}^1/_4\right)$.

To see if this is a minimum or maximum turning point we check the

$$\text{sign of } \frac{d^2y}{dx^2}; \frac{dy}{dx} = 2x - 3; \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(2x-3) = 2 > 0.$$

Therefore, the point is a minimum turning point.

Example3: Find the equation of the tangent and normal to the curve $y = x^2 - 3x + 2$ at the point where it cuts the y-axis.

Solution: $y = x^2 - 3x + 2$ cuts the y-axis when x=0. Therefore y=0 – 0 + 2 = 2. The point is (0, 2). The gradient of the tangent at this point is the value of $\frac{dy}{dx}$ when $x = 0$. $\frac{dy}{dx} = 2x - 3$. At $x = 0, \frac{dy}{dx} = -3$. The equation of tangent is given by $y - y_0 = m(x - x_0)$ where m = gradient and $(x_0, y_0) = (0, 2)$. Therefore equation of tangent is $y - 2 = -3(x - 0)$; $y - 2 = -3x$; $y = -3x + 2$; or $y + 3x - 2 = 0$. To find the equation of normal, we recall that $m \bullet m_1 = -1$ where m = gradient of tangent and m_1 = gradient of normal.

$\Rightarrow m_1 = \frac{-1}{m} = \frac{-1}{-3} = \frac{1}{3}$. Therefore equation of normal is $y - y_0 = m_1(x - x_0)$

$$y - 2 = \frac{1}{3}(x - 0);\ y - 2 = \frac{1}{3}x;\ y = \frac{1}{3}x + 2;\ \text{ or } y - \frac{1}{3}x - 2 = 0;\ \text{or } 3y - x - 6 = 0.$$

Exercise on Differentiation and its Applications

1. Differentiate the following functions from first principles: $(a)\ y = f(x) = x^2 + 5$; $(b)\ y = f(x) = \cos x$ $(c)\ y = f(x) = x^2 + a$, $'a'$ being a consant.

1. With the known rules of differentiation, differentiate the following functions.

   (a) $(i)\ y = f(x) = 10^x$; $(ii)\ y = f(x) = 10^{2x}$ (b) $(i)\ y = arc\ \tan x$; $(ii)\ y = arc\ \cos x$
   (c) $(i)\ y = x^2 \sin x$; $(ii)\ y = x^2 \cos 2x$; $(iii)\ y = (\cos x)^2$ (d) $(i)\ y = \left(2x^2 + 2x + 4\right)^5$; $(ii)\ y = x \ln x$;
   $(iii)\ \tan x$ (e) $(i)\ \frac{x^2}{\sin x}$; $(ii)\ \frac{\sin x}{\cos x}$; $(iii)\ y = \sqrt{x}$

2. (a) What do you understand by turning (stationary) points? Briefly distinguish between the various turning points. (b) Name two applications of differentiation.

3. (a) What are the necessary and sufficient tests for a function of a single variable to have (i) a maximum turning point; (ii) a saddle point.

(b) What are the n.a.s.c. for a bivariate function to have (i) a minimum turning point (ii) a saddle point. What is the information provided by $(f_{xy})^2 - f_{xx}f_{yy} = 0$?

1. A monopolist's revenue function is $r = 240x + 57x^2 - x^3$. Find the output level that gives maximum revenue. What is the maximum revenue?

2. What is the difference between

   (a) A local and an absolute minimum

   (b) An absolute and a relative maxima

3. (a) Find the equation of the tangent to the curve $y = x^2 + 5x - 2$ at the point where the curve cuts the line $x = 4$.

(b) Find the equations of the tangent and normal to $y = x^2 = 3x + 2$ which has a gradient of $\frac{1}{2}$.
(c) Find the value of k for which$y = 2x + k$ is a normal to $y + 2x^2 - 3$.

Integration

This is the reverse of differentiation. In differentiation, we find $\frac{dy}{dx} = f'(x)$ where $f(x) = y$ is given. In integration, we find $y = f(x)$ when $\frac{dy}{dx} = f'(x)$ is given. We need to recognize differential forms to be able to integrate.

E.g. $\frac{dx^3}{dx} = 3x^2 \Rightarrow d\left(x^3\right) = 3x^2dx$.

Then $\int d\left(x^3\right) = \int 3x^2dx; \Rightarrow x^3 + c = \int 3x^2dx \; \frac{x^3}{3} + k = \int x^2dx$.

The result is obvious, because when x^3 is differentiated w.r.t x, the derived function is $3x^2$. Conversely, given that an unknown function has a derived function $3x^2$ it is clear that the unknown function could be x^3. This process of finding an unknown function from its derived function is called integration; and it reverses the operation of differentiation.

The Constant of Integration

Consider the functions x^3+ 8 and x^3 – 1.$\frac{d}{dx}\left(x^3 + 8\right) = 3x^2$ and $\frac{d}{dx}\left(x^3 - 1\right) = 3x^2$.

So to integrate $3x^2$, we have that the result is $x^3 + k$; k being a constant. This is because the derived function of $x^3 + k$ – any k – would yield $3x^2$. Thus k is the constant of integration.

Rules for Integration

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1.$ (2) $\int dx = \int dx = x + c$ (3) If $n = -1$, then $\int x^n dx = \int \frac{1}{x}dx = \ln x + c$
(4) $\int e^x dx = e^x + c$ (5) $\int e^{ax}dx = \frac{e^{ax}}{a} + c$

6) $\int [a\, f(x) + b\, g(x)]\, dx = a\int f(x)\, dx + b\int g(x)\, dx$ (7) $\int \sin x\, dx = -\cos x + c$
(8) $\int \cos x\, dx = \sin x + c$ (9) $\int sex^2x\, dx = \tan x + c$ (10) $\int cos\,ec^2x\ dx = -\cot x + x$

Exercise: Evaluate

i) $\int \left(\frac{3}{x^2} + x^2\right)dx$ (ii) $\int 2x\sqrt{x^2+1}\,dx$ (iii) $\int \left(2t^{7/2} - 3t^{-11/7} + 4t^5\right)dt$ (iv) $\int \left(w^{1/2} - \frac{1}{w^2}\right)dw.$

Integration by Simple Change of Variable (Substitution)

Examples: (*i*) $\int 2x\sqrt{x^2+1}dx$. Let $u = x^2+1;\ du = 2xdx \Rightarrow dx = \frac{du}{2x}\ \therefore \int 2x\sqrt{x^2+1dx} = \int \cancel{2}\, x\sqrt{u}\frac{du}{\cancel{2x}} = \int u^{\frac{1}{2}}du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c = \frac{2}{3}u^{\frac{3}{2}}+c = \left(x^2+1\right)^{\frac{3}{2}}+c$

(*ii*) $\int \cos t\sin^3 t\, dt$. Let $u = \sin t;\ du = \cos t\, dt\ \ \therefore \int \cos t\sin^3 t\, dt = u^3\, du = \frac{u^4}{4}+c = \frac{\sin^4 t}{4}+c$

Exercise (*i*) $\int \sec^2 x\tan x\, dx$ (*ii*) $\int \cos 3t\sqrt{2+\sin 3t\, dt}$ (*iii*) $\int \cos^6 x\sin x\, dx$.
To evaluate $\int (ax+b)^n dx$, where a, b and n are real numbers; we recall

$$\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}\times a = an(ax+b)^{n-1}$$
$$\therefore \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}+c;\ n\neq -1$$

Examples (*i*) $\int (3x-1)^4 dx = \frac{(3x-1)^5}{3\times 5}+c$ (*ii*) $(2-5s)^{3/2}ds = \frac{(2-5s)^{\frac{5}{2}}}{\frac{5}{2}\times -5} = \frac{-2}{25}(2-5s)^{5/2as}+c$

Exercises: Evaluate (1) $\int \sqrt{3-x}\, dx$ (2) $\int (9+x)^{3/2}dx$ (3) $\int \frac{t}{(3t^2+4)^3}dt$

Exponential Form: $\int e^{ax}dx = \frac{e^{ax}}{a}$ *E.g.* $\int e^{-\frac{1}{2}}dx = \frac{e^{-\frac{1}{2}}}{-\frac{1}{2}} = -2e^{-\frac{1}{2}}$

2) $\int \frac{e^{\sqrt{1+x}}}{\sqrt{1+x}}dx$. Let $u = \sqrt{1+x} = (1+x)^{1/2};\ du = \frac{1}{2}(1+x)^{-1/2} = \frac{1}{2\sqrt{1+x}}dx$
$\therefore \int 2\frac{e^{\sqrt{1+x}}}{2\sqrt{1+x}}dx = 2\int e^u du = 2e^u + c = 2e^{\sqrt{1+x}}+c$

Recall $\frac{d}{du}\ln u = \frac{1}{u} \Rightarrow d\ln u = \frac{1}{u}du\ \therefore \int \frac{du}{u} = \ln u + c$ or $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$

E.g. (1) $\int \frac{2x}{1+x^2}dx = \int \frac{f'(x)}{f(x)}dx = \ln(1+x^2)+c\left(\text{where } f(x) = 1+x^2 \text{ and } f'(x) = 2x\right)$

2) $\int \frac{x}{3x^2+4}dx$. Let $u = 3x^2+4;\ du = 6xdx\ \ \therefore \int \frac{x}{3x^2+4} = \frac{1}{6}\int \frac{6x}{3x^2+4}dx = \frac{1}{6}\ln\left|3x^2+4\right|+c$
3) $\int \tan x\, dx = -\int -\frac{\sin x}{\cos x} = -\ln|\cos x + c| = \ln|\cos x| + c = \ln|\sec x| + c$
4) $\int \cot x\, dx = \int \frac{\cos x}{\sin x} = \ln|\sin x| + c.$

Exercises: Evaluate (*i*) $\int \frac{dx}{e^{3x}}$ (*ii*) $\int \frac{\cos ec^2 2t}{\sqrt{1+\cot 2t}}dt$ (*iii*) $\int x^{1/3}\sqrt{x^{4/3}-1}\, dx$ (*iv*) $\int \frac{x}{4+x^2}dx$ (*v*) $\int \frac{dx}{4+x^2}$.

Integration by Substitution Continued

This is a device used for converting integrals into standard forms. It is the converse of the method of differentiation of a function of a function; e.g. $y = \int f(x)dx$ and it is required to change the variable from x to u where u is a given function of x. Since

$$y = \int f(x)dx, \text{and } \frac{dy}{dx} = f(x)\ \ \therefore y = \int f(x)\bullet\frac{dx}{du}\bullet du$$

Interchanging x and u, we obtain $y = \int f(u)\frac{du}{dx}dx = \int f(u)du$. This means that an integral of the product of $f(u)$ and $\frac{du}{dx}$ w.r.t x can be written as integral of $f(u)$ w.r.t. u.

E.g. $\int \tan^3 x\sec^2 x\, dx = \int (\tan x)^3\sec^2 x\, dx$. Let $u = \tan x \Rightarrow \frac{du}{dx} = \sec^2 x$.
$\therefore \int (\tan x)^3\sec^2 x dx = \int u^3 du = \frac{u^4}{4}+c = \frac{(\tan x)^4}{4}+c$
E.g. $\int x(2-x)^4 dx$. Let $u = 2-x \Rightarrow x = 2-u;\ \frac{du}{dx} = -1 \Rightarrow -du = dx$.

$$\int x(2-x)^4 dx = -\int (2-u)u^4 du = -\int \left(2u^4-u^5\right)du = -\frac{2}{5}u^5 - \frac{u^6}{6}+c = -\frac{2}{5}(2-x)^5 + \frac{(2-x)^6}{6}+c$$

$$= -(2-x)^5\left(\frac{2}{5}-\frac{(2-x)}{6}\right)$$

$$E.g.\quad \int \frac{x^2}{\sqrt{x+2}}dx.\ \text{Let } x+2=u^2 \Rightarrow x=u^2-2;\ \frac{dx}{du}=2u \Rightarrow dx=2udu$$

$$\therefore \int \frac{x^2}{\sqrt{x+2}} = \int \frac{\left(u^2-2\right)^2}{u}\bullet 2ud = 2\int \left(u^2-2\right)^2 du = 2\int \left(u^4-4u+4\right)du = 2\left[\frac{u^5}{5}-\frac{4}{3}u^3+4u\right]+c$$

$$\therefore \int \frac{x^2}{\sqrt{x+2}}dx = 2\left[\frac{(x+2)^{5/2}}{5}-\frac{4}{3}(x+2)^{3/2}+4(x+2)^{1/2}\right]+c$$

Exercises: Evaluate

$$1)\ \int x\sqrt{x+1dx}\qquad (2)\ \int \frac{x}{(2x+1)}dx\qquad (3)\ \int \frac{2x+1}{\sqrt{x+1}}$$

Integrating $\sin ax$, $\cos ax$, etc give the following results. (1) $\int \sin ax\, dx = -\frac{\cos ax}{a}+c$ (2) $\int \cos ax\, dx = \frac{\sin ax}{a}+c$

Examples (1) $\int \sin 3x\, dx = -\frac{\cos 3x}{3}+c$ (2) $\int \cos \frac{1}{2}x\, dx = 2\sin\frac{1}{2}x+c$

$$(3)\quad \int \frac{\sin\frac{1}{x}}{\frac{1}{x^2}}dx.\ \text{Let } u=\frac{1}{x};\ du=-\frac{1}{x^2}dx \Rightarrow dx=-x^2du \Rightarrow \int \frac{\sin\frac{1}{x}}{\frac{1}{x^2}} = -\int \frac{\sin u\, du\, \cancel{x}^{-2}}{\cancel{x}^{-2}}$$

$$= -\int \sin u\, du = \cos u + c = \cos\frac{1}{x}+c$$

Integrating Powers of Trigonometric Functions

This involves the integration of the following types of functions: $\int \cos^n x dx$, $\int \sin^n x dx$, $\int \cos^n x \sin^m x dx$. For $\int \cos^n x\, dx$ and $\int \sin^n x\, dx$, the trick is to convert to its equivalent compound angle form; and release one factor before transforming in this way, when n is odd. After having gotten the equivalent compound angle function, we proceed by substitution when n is odd. Otherwise we integrate directly.

Example

1) $\int \cos^2 x dx = \int\left(\frac{1}{2}+\frac{1}{2}\cos^2 x\right)dx = \frac{1}{2}x+\frac{1}{4}\sin 2x+c.$
2) $\int \sin^2 x dx = -\int\left(\frac{1}{2}\cos^2 x-\frac{1}{2}\right)dx = -\left(\frac{1}{4}\sin 2x-\frac{1}{2}x\right)+c.$
3) $\int \cos^5 xex = \int \cos^4 x\cos x dx = \int\left(1-\sin^2 x\right)^2\cos x dx = \int\left(1-2\sin^2 x+\sin^4 x\right)\cos x dx$

$= \int \cos x dx - 2\int \sin^2 x\cos x dx + \int \sin^4 x\cos x dx = \sin x - \frac{2}{3}\sin^3 x + \frac{\cos^5 x}{5}+c.$

Integral of the form $\int \cos^n x\sin^m x\, dx$ can be evaluated by release of factor; transformation to its equivalence using $\sin^2 x+\cos^2 x = 1$ of the type whose factor was released and then carryout integration by substitution. This happens especially when the type whose factor was released is of odd power. E.g.

$$\int \cos^2 x\sin^3 x dx = \int \cos^2 x\sin^2 x\sin x dx = \int \cos^2 x\left(1-\cos^2 x\right)\sin x dx = \int \cos^2 \sin x - \cos^4 x\sin x dx.$$

$$\text{Let } u=\cos x;\ du=-\sin x \Rightarrow -\int \cos^2 x d(\cos x) + \int \cos^4 x d(\cos x) = -\int u^2 du + \int u^4 du = -\frac{u^3}{3}+\frac{u^5}{5}+c$$

$$= \frac{u^5}{5}-\frac{u^3}{3}+c = \frac{\cos^5 x}{5}-\frac{\cos^3 x}{3}+c$$

Exercise: Evaluate

$$(i)\int \sin^2 x\cos^3 x dx \quad (ii)\int \sin^3 x\cos^3 x dx \quad (iii)\int \cos^4 x dx \quad (iv)\int \sin^4 x dx \quad (v)\int \sin^5 x dx.$$

Integrals of form $\int \cos nx \ \sin mx \, dx$.
For integrals of this form, we use any of the following transformation.

$$\sin mx \cos nx = \tfrac{1}{2}\sin(m+n)x + \sin(m-n)x$$
$$\cos mx \cos nx = \tfrac{1}{2}\cos(m+n)x + \cos(m+n)x$$
$$\sin nx \sin mx = \tfrac{1}{2}\cos(m-n)x - \cos(m+n)x$$

E.g. (1) $\int \sin 7x \cos 3x dx = \frac{1}{2}\int \sin 10x + \sin 4x dx = \frac{1}{2}\left[-\frac{1}{10}\cos 10x - \frac{1}{4}\cos 4x\right] + c.$

(2) $\int \sin 3x \cos 7x dx = \frac{1}{2}\int [\sin 10x + \sin(-4x)] dx = \frac{1}{2}\int (\sin 10x - \sin 4x) dx = \frac{1}{20}\cos 10x + \frac{1}{8}\cos 4x + c.$

Exercises: Evaluate (*i*) $\int \sin 2x \sin x dx$ (*ii*) $\int \sin 2x \sin^2 x dx$ (*iii*) $\int \cos 2x \sin^2 x \, dx.$

INTEGRALS OF INVERSE OF TRIGONOMETRIC FUNCTIONS AND THEIR RECIPROCALS

$$\text{(i)}\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c \quad \text{(ii)}\int \frac{-dx}{\sqrt{a^2-x^2}} = \cos^{-1}\frac{x}{a} + c \quad \text{(iii)}\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\text{(iv)}\int \frac{-dx}{a^2+x^2} = \frac{1}{a}\cot^{-1}\frac{x}{a} + c \quad \text{(v)}\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} \quad \text{(vi)}\int \frac{-dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\text{cosec}^{-1}\frac{x}{a}$$

Verification: (i) Let $x = a\sin\theta \Rightarrow \frac{dx}{d\theta} = a\cos\theta \Rightarrow dx = a\cos\theta d\theta$

$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2\cos^2\theta}} = \int \frac{a\cos\theta d\theta}{a\cos\theta} = \int d\theta = \theta + c = \sin^{-1}\frac{x}{a} + c.$

$\left(x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\frac{x}{a}\right)$

(ii) Similarly, $\int \frac{-dx}{\sqrt{a^2-x^2}} = \cos^{-1}\frac{x}{a} + c$ (by sustituting $x = a\cos\theta$)

(iii) Let $x = a\tan\theta \Rightarrow \frac{dx}{d\theta} = a\sec^2\theta \Rightarrow dx = a\sec^2\theta d\theta \therefore \int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta d\theta}{a^2+a^2\tan^2\theta} = \int \frac{a\sec^2\theta d\theta}{a^2(1+\tan^2\theta)}$

$= \int \frac{\cancel{a}\sec^2\theta d\theta}{a^{\cancel{2}1}\sec^2\theta} = \frac{1}{a}\int d\theta = \frac{\theta}{a} + c = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\left(x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a} \Rightarrow \theta = \tan^{-1}\frac{x}{a}\right).$

(iv) Similarly, $\int \frac{-dx}{a^2+x^2} = \frac{1}{a}\cot^{-1}\frac{x}{a} + c$ (by substituting $x = a\cot\theta$).

(v) Let $x = a\sec\theta \Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta \Rightarrow dx = a\sec\theta\tan\theta d\theta. \therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}}$

$= \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{a^2(\sec^2\theta-1)}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{a^2\tan^2\theta}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta a\tan\theta} = \frac{1}{a}\int d\theta = \theta + c = \frac{1}{a}\sec^{-1}\frac{x}{a}.$

(vi) Similarly, $\int \frac{-dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\text{cosec}^{-1}\frac{x}{a}$ (by substituting $x = a\text{cosec}\theta$).

Integral of Hyperbolic Functions

$$\int \sinh x dx = \int \frac{e^x - e^{-x}}{2} dx = \frac{e^x + e^{-x}}{2} = \cosh x + c, \int \cosh x dx = \int \frac{e^x + e^{-x}}{2} dx = \frac{e^x - e^{-x}}{2} = \sinh x + c.$$

$$\int \tanh x dx = \int \frac{\sinh x}{\cosh x} dx = \ln\cosh x + c \left(\frac{d}{dx}\cosh x = \sinh x\right), \text{ similarly, } \int \coth x dx = \ln\sinh x + c$$

It can be shown using method similar the one used in immediate (i), (iii) and (v) above that:

(i) $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + c$ (by substituting $x = a\sinh\theta$)
(ii) $\int \frac{dx}{\sqrt{x^2-a^2}} = \sinh^{-1}\frac{x}{a} + c$ (by substituting $x = a\cosh\theta$)
(iii) $\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + c$ (by substituting $x = a\tanh\theta$)
(iv) $\int \frac{-dx}{a^2-x^2} = \frac{1}{a}\coth^{-1}\frac{x}{a} + c$ (by substituting $x = a\coth\theta$)
(v) $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\text{sech}^{-1}\frac{x}{a} + c$ (by substituting $x = a\text{sech}\theta$)
(vi) $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\text{cosech}^{-1}\frac{x}{a} + c$ (by substituting $x = a\text{cosech}\theta$)

Exercise: Evaluate (i) $\int \sinh 2x dx$ (ii) $\int \cosh\frac{ax}{2}dx$ (iii) $\int \tanh 4x dx$ (iv) $\int \frac{dx}{\sqrt{4-x^2}}$ (v) $\int \frac{dx}{\sqrt{4+x^2}}$

(vi) $\int \frac{dx}{9+x^2}$ (vii) $\int \frac{dx}{9-x^2}$ (viii) $\int \frac{dx}{x^2-25}$ (ix) $\int \frac{dx}{x\sqrt{x^2-4}}$ (x) $\int \frac{dx}{x\sqrt{x^2+4}}$ (xi) $\int \frac{dx}{x\sqrt{4-x^2}}$

(xii) $\int \frac{dx}{4x^2+25}$ (xiii) $\int \frac{dx}{\sqrt{2-x^2}}$ (xiv) $\int \frac{-3dx}{\sqrt{9-x^2}}$

✱Integration by Partial Fractions (review partial fractions)

This method is used for integrating rational fractions (i.e. functions of the form $r(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where $p(x)$ and $q(x)$ are polynomials which are not in any known standard form.
Before breaking into partial fractions we make sure that the degree of $p(x) \leq$ degree of $q(x)$ e.g
1) $\int \frac{x+1}{x^2-3x+2}dx$ is in proper form so we can break into partial fractions but
2) $\int \frac{x^2+1}{x^2-3x+2}dx$ is not in proper form, so we do not integrate directly. If degree of $q(x) \leq$ degree of $p(x)$ then we use long division to get a polynomial plus a proper fraction i.e. divide to get

$$\frac{p(x)}{q(x)} = s(x) + \frac{t(x)}{q(x)},$$ where $s(x)$ is a polynomial and degree of $t(x) \leq$ degree of $q(x)$

2) $\Rightarrow \dfrac{x^2+1}{x^2-3x+2} = 1 + \dfrac{3x-1}{x^2-3x+2}$ (by long division) $\therefore \displaystyle\int \frac{x^2+1}{x^2-3x+2}dx = \int\left[1 + \frac{3x-1}{x^2-3x+2}\right]dx.$

We can now express the rational term in partial fractions. If $r(x) = \frac{p(x)}{q(x)}$ is such that degree of $p(x) <$ degree of $q(x)$, we simplify the process of integration by breaking $r(x)$ into partial fractions which will be of the standard form and so easily evaluated.
First Step is to factorise the denominator $q(x)$. The factors, will be either linear or quadratic; they can be distinct or repeated factors.

Distinct Linear Factors

$$q(x) = (a_1x+b_1)(a_2x+b_2)(a_3x+b_3)\ \dots \text{ then } \frac{p(x)}{q(x)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \frac{C}{a_3x+b_3} + \dots$$

Where A, B, C, ... are constants whose values are to be determined.
e.g. $\int \frac{x+1}{x^2-3x+2}dx = \int \frac{x+1}{(x-1)(x-2)}dx$, let

$$\frac{x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)} \Rightarrow x+1 = (A+B)x - 2A - B$$

Comparing coefficients, we have

$$A+B = 1 \quad (i), \quad -2A-B = 1 \quad (ii) \text{ then } (i)+(ii) \Rightarrow A = -2.$$

Substituting for A in equation (i), we have $-2+B = 1 \Rightarrow B = 3$. $\therefore \int \frac{x+1}{x^2-3x+2} = \int\left(\frac{-2}{x-1} + \frac{3}{(x-2)}\right)dx =$ $-2\int\frac{dx}{x-1} + 3\int\frac{dx}{x-2}$

$$= -2\ln|x-1| + 3\ln|x-2| + c = \ln\left|\frac{(x-2)^3}{(x-1)^2}\right| + c$$

Repeated Linear Factors

$$q(x) = (a_1x+b_1)^r(a_2x+b_2)^n \dots \text{ so that } \frac{p(x)}{q(x)} = \frac{p(x)}{(a_1xb_1)^r(a_2x+b_2)^n}\cdots$$

Each factor must be repeated as many times as it appears.

$$\frac{p(x)}{q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_1x+b_1)} + \cdots + \frac{A_r}{(a_1x+b_1)^r} + \frac{B_1}{(a_2x+b_2)} + \cdots + \frac{B_n}{(a_2x+b_2)^n}$$

Where $A_1, A_2, \cdots, A_r, B_1, B_2, \cdots, B_n$ are constants.

$$E.g.\ \int \frac{x}{(x-4)^3}dx = \int\left(\frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{(x-4)^3}\right)dx$$

$$i.e.\ \frac{x}{(x-4)^3} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{(x-4)^3} = \frac{A(x-4)^4+B(x-4)+C}{(x-4)^3}$$

$$\Rightarrow 0x^2+1x+0 = A\left(x^2-8x+16\right)+B(x-4)+C$$

Equating coefficients, we have

$$A=0,\ \ 8A+B=1 \Rightarrow B=1, \qquad 16A-4B+C=0 \Rightarrow C=4.$$

$$\therefore \int \frac{x}{(x-4)^3}dx = \int\left(\frac{1}{(x-4)^2} + \frac{4}{(x-4)^3}\right)dx = \int \frac{1}{(x-4)}dx + 4\int \frac{1}{(x-4)^3}dx = \frac{-1}{(x-4)} - \frac{2}{(x-4)^2} + C$$

$\int \frac{p(x)}{q(x)}dx$ where $q(x)$ has quadratic factors which cannot be reduced. In this case, each quadratic factor of $q(x)$ must have a linear numerator.

$$E.g.\ \int \frac{p(x)}{q(x)}dx = \int \frac{p(x)}{(x+a)(ax^2+bx+c)}dx.\ \text{Let } \frac{p(x)}{(x+a)(ax^2+bx+c)} = \frac{A}{x+a} + \frac{Bx+C}{ax^2+bx+c}$$

where ax^2+bx+c is not factorisable(reducible)

We complete by computing the values of A, B and C and then substituting into the partial fraction before integrating.

Exercises: Evaluate (1) $\int \frac{1}{(x^2+4)(x^2+8)}$ (2) $\int \frac{dx}{x(x+1)^2}$ (3) $\int \frac{x^2}{(x+1)(1-x)^2}dx$

4) $\int \frac{x}{x^2+4x-5}dx$ (5) $\int \frac{x^2}{(x-2)(x^2+1)}dx$ (6) $\int \frac{(x-2)^2}{x^2+1}$

Integration by Parts:

Recall : If $y = u(x)v(x)$, then $y' = [u(x)v(x)]' = v(x)u'(x) + u(x)v'(x)$ (product rule) order of choice of u.

$$\therefore \frac{d}{dx}(u(x)v(x)) = v(x)\frac{du}{dx} + u(x)\frac{dv}{dx} \Rightarrow \int \frac{d}{dx}(u(x)v(x))\,dx = \int\left(v(x)\frac{du}{dx} + u(x)\frac{dv}{dx}\right)dx$$

Example: Evaluate (i) $\int \ln x\,dx$ (ii) $\int e^{ax}\cos bx\,dx$ (iii) $\int e^{ax}\sin bx\,dx$ (iv) $\int xe^{-2x}dx$.

Solution: (i) $\int \ln x\,dx$. Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx$. And $dv = dx \Rightarrow v = \int dx = x + c$.

But $\int u\,dv = uv - \int v\,du \Rightarrow \int \ln x\,dx = x\ln x - \int \cancel{x} \bullet \frac{1}{\cancel{x}}dx = x\ln x - \int dx = x\ln x - x + c = x(\ln x - 1) + c$

(ii) $\int e^{ax}\cos bx\,dx$. Let $u = e^{ax} \Rightarrow \frac{du}{dx} = ae^{ax} \Rightarrow du = ae^{ax}dx$. And $dv = \cos bx\,dx \Rightarrow \int dv = \int \cos bx\,dx$

$\Rightarrow v = \frac{1}{b}\sin bx + c.$ But $\int u\,dv = uv - \int v\,du \Rightarrow \int e^{ax}\cos bx = \frac{a}{b}e^{ax}\sin bx - \left[\frac{a}{b}\int e^{ax}\sin bx\,dx\right]$

Integrating $\int e^{ax}\sin bx\,dx$ by parts again: $\int e^{ax}\sin bxdx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$

$$\therefore \int e^{ax}\cos bx = \frac{e^{ax}}{b}\sin bx - \left[\frac{a}{b}\left(-\frac{e^{ax}}{b}\cos bx + \frac{a}{b}\int e^{ax}\cos bxdx\right)\right] = \frac{e^{ax}}{b}\sin bx + \frac{ae^{ax}}{b^2}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bxdx$$

$$\text{Let } \int e^{ax}\cos bxdx = I \Rightarrow I = \frac{e^{ax}}{b}\sin bx + \frac{ae^{ax}}{b^2}\cos bx - \frac{a^2}{b^2}I + c$$

$$\Rightarrow I + \frac{a^2}{b^2}I = \frac{a^2+b^2}{b^2}I = \frac{e^{ax}}{b}\sin bx + \frac{ae^{ax}}{b^2}\cos bx + c$$

$$\Rightarrow I = \frac{b^{\cancel{2}1}}{a^2+b^2}\bullet\frac{1}{\cancel{b}}e^{ax}\sin bx + \frac{b^2}{a^2+b^2}\bullet\frac{a}{b^2}e^{ax}\cos bx + \frac{c}{a^2+b}$$
$$= \frac{e^{ax}}{a^2+b}(b\sin bx + a\cos bx) + \frac{c}{a^2+b} = \int e^{ax}\cos bxdx$$

$$\text{Similarly, } \int e^{ax}\cos(bx+c) = \frac{e^{ax}}{a^2+b}[b\sin(bx+c) + a\cos(bx+c)]$$

(iii) $\int e^{ax}\sin bxdx$. Let $u = e^{ax} \Rightarrow \frac{du}{dx} = ae^{ax} \Rightarrow du = ae^{ax}dx$. And $dv = \sin bxdx \Rightarrow \int dv = \int \sin bxdx$

$\Rightarrow v = -\frac{1}{b}\cos bx + c$. But $\int udv = uv - \int vdu \Rightarrow \int e^{ax}\sin bx = -\frac{a}{b}e^{ax}\cos bx - \left[-\frac{a}{b}\int e^{ax}\cos bxdx\right]$

Integrating $\int e^{ax}\cos bxdx$ by parts again: $\int e^{ax}\cos bxdx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bxdx$

$$\therefore \int e^{ax}\sin bx = -\frac{e^{ax}}{b}\cos bx + \left[\frac{a}{b}\left(\frac{e^{ax}}{b}\sin bx - \frac{a}{b}\int e^{ax}\sin bxdx\right)\right] = \frac{-e^{ax}}{b}\cos bx + \frac{ae^{ax}}{b^2}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bxdx$$

$$\text{Let } \int e^{ax}\sin bxdx = I \Rightarrow I = \frac{-e^{ax}}{b}\cos bx + \frac{ae^{ax}}{b^2}\sin bx - \frac{a^2}{b^2}I + c$$

$$\Rightarrow I + \frac{a^2}{b^2}I = \frac{a^2+b^2}{b^2}I = \frac{-e^{ax}}{b}\cos bx + \frac{ae^{ax}}{b^2}\sin bx + c$$

$$\Rightarrow I = \frac{b^{\cancel{2}1}}{a^2+b^2}\bullet\frac{-e^{ax}}{\cancel{b}}\cos bx + \frac{b^2}{a^2+b^2}\bullet\frac{ae^{ax}}{b^2}\sin bx + \frac{c}{a^2+b}$$
$$= \frac{e^{ax}}{a^2+b}(a\sin bx - b\cos bx) + \frac{c}{a^2+b} = \int e^{ax}\cos bxdx$$

$$\text{Similarly, } \int e^{ax}\sin(bx+c) = \frac{e^{ax}}{a^2+b}[a\sin(bx+c) - b\cos(bx+c)] + k$$

(iv) Let $u = x \Rightarrow du = dx$. Let $dv = e^{-2x} \Rightarrow v = -\frac{1}{2}e^{-2x}$. Substituting in $\int udv = uv - \int vdu$, we have $\int xe^{-2x}dx = -\frac{xe^{-2x}}{2} - \int -\frac{1}{2}e^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx = \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c$.
$\therefore \int xe^{-2x}dx = \frac{e^{-2x}}{4}(2x+1) + c$.

Note: In choosing u and dv, we also make sure that the integral we get on the right hand side is simpler than the given integral (on the left hand side).

There may be need to apply the formula more than once before arriving at the final answer as in (ii) and (iii) above and other integrals of the form $\int x^n e^{ax}dx$, $\int x^n\cos bx\,dx$, $\int e^{ax}\cos nx\,dx$.

Example: Evaluate $\int x^2e^{3x}dx$. Let $u = x^2 \Rightarrow du = 2xdv = e^{3x} \Rightarrow v = \frac{e^{3x}}{3}$

$$\int x^2e^{3x}dx = x^2\bullet\frac{e^{3x}}{3} - \int 2x\bullet\frac{1}{3}e^{3x}dx = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}dx \qquad ...(1)$$
$$\int udv = \int xe^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$$

Substituting back in (1) to get $\int x^2e^{3x}dx = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c\right) = \frac{1}{3}e^{3x}\left(x^2 - \frac{2}{3}x + \frac{2}{9}\right) + c$

Exercises: Evaluate

(1) $\int_0^4 x^2e^{x/4}dx$ (2) $\int_0^\pi x\sin x\,dx$ (3) $\int e^{3x}\sin 4x\,dx$ (4) $\int \sin 2x\cos 3x\,dx$

$$(5) \int x\ln x\,dx \quad (6) \int \cos\sqrt{x}\,dx \quad (7) \int \sin(\ln x)\,dx \tag{1}$$

Improper Integrals

Here, we discuss those integrals that demand extra care in evaluation. Recall that if $\int F(x)$ is a continuous function over the closed interval $[a,b]$ where $b > a$, then $F(x)dx$ exists. The definite integral (i) $\int F(x)dx$ always exists under 'suitable' conditions on $[a,b]$ and F. When conditions are 'not suitable' then the integral (i) is called 'improper'.

Definition: An integral $\int F(x)dx$ is said to be 'improper' if any (or both) of the following conditions hold: The interval of integration is infinite such as $\int_a^{+\infty} F(x)dx$, $\int_{-\infty}^{b} F(x)dx$, $\int_{-\infty}^{+\infty} F(x)dx$

1. The integrand becomes infinite within the interval of integration E.g. $\int_0^2 \frac{1}{(x-1)^2}dx$

Definition: If F is continuous on the interval $(a,+\infty)$ we define the improper integral $\int_a^{+\infty} F(x)dx$, as follows:

1. $\int_a^{+\infty} F(x)dx = \lim\limits_{L\to+\infty}\int_a^L F(x)dx$, provided the limit exist.

If the limit does not exist, then the improper integral is said not to exist. Also

1. $\int_{-\infty}^{b} F(x)dx = \lim\limits_{L\to-\infty}\int_L^b F(x)dx$ provided the limit exist
2. $\int_{-\infty}^{+\infty} F(x)dx = \int_{-\infty}^{0} F(x)dx + \int_0^{+\infty} F(x)dx = \lim\limits_{L\to-\infty}\int_L^0 F(x)dx + \lim\limits_{L\to+}\int_0^L F(x)dx.$

And $\int_{-\infty}^{+\infty} F(x)dx$ is said to exist when both limits on the RHS of c exist. E.g. $\int_1^{+\infty}\frac{dx}{x^3}$.

Solution $\int_1^{+\infty}\frac{1}{x^3}dx = \lim\limits_{L\to\infty}\int_1^L \frac{1}{x^3}dx = \lim\limits_{L\to\infty}\left[-\frac{1}{2x^2}\right]_1^L = \lim\limits_{L\to\infty}\left[-\frac{1}{2L^2}+\frac{1}{2}\right] = 0+\frac{1}{2} = \frac{1}{2}$

E.g. 2: Evaluate $\int_1^{+\infty}\frac{2}{x}dx = 2\int_1^{+\infty}\frac{1}{x}dx = \lim\limits_{L\to\infty}2\int_1^L\frac{1}{x}dx = \lim\limits_{L\to\infty}[2\ln x]_1^L = 2\lim\limits_{L\to\infty}[\ln x]_1^L$

$$= 2\lim_{L\to\infty}\{\ln L - \ln 1\} = \lim_{L\to\infty}\{\ln L - 0\} = +\infty \therefore \int_1^{+\infty}\frac{2}{x}dx \text{ does not exist.}$$

Exercises: Evaluate (*i*) $\int_0^{+\infty}\frac{3}{1+x^2}dx$ (*ii*) $\int_{-\infty}^{+\infty}\frac{x}{(x^2+3)^2}dx$ (iii) $\int_0^{\infty}\frac{dx}{x^2+4}$ (iv) $\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}}$

Applications of Integration Integration is a very useful tool for calculating the area under the graph of a function, volumes of certain solids or regions in space, and length, work done by force, centroids of plane regions and curves, moments and centroids of curves etc.

We take examples of some of these applications for a better understanding of the concepts.

Example: Suppose the velocity of a moving particles is given by $V = 30 - 2t\,m/s$. Find both the net distance and the total distance it travels between the time $t = 20$ seconds.

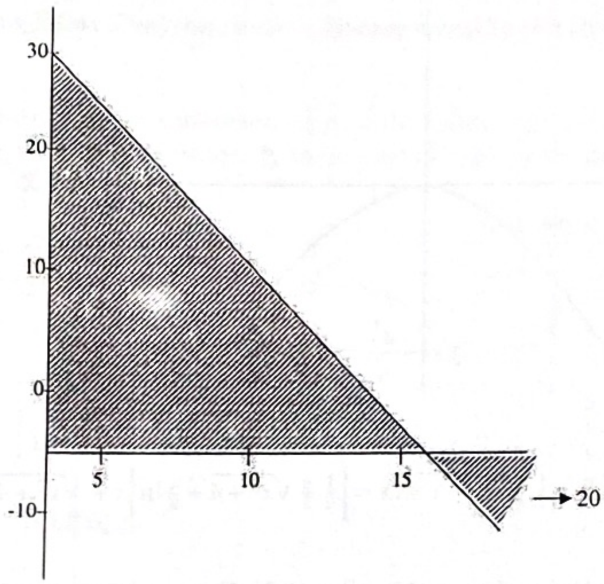
Solution

The solution of this problem may be begun by sketching the velocity time graph that depicts this motion. We see that the total distance is the shaded region in the graph. While the net distance is the shaded region for all positive velocity. To calculate this areas, we have, for net distance.

$$S = \int_0^{20}(30-2t)dt = \left[30t - t^2\right]_0^{20} = 200m.$$

To find total distance, we note that v is positive if $t = 15$ while v is negative for $t = 20$. Total distance is the total area of the shaded region on the graph. i.e. $v = -(30-2t)$.

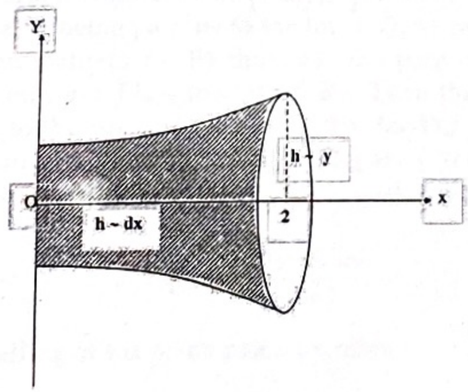
$$S = \int_0^{15}(30-2t)dt + \int_{15}^{20}(2t-30)dt = \left[30t-t^2\right]_0^{15} + \left[t^2-30t\right]_{15}^{20} = 225+25 = 250m.$$



Observe that the distances are obtained by integrating for the area under the graph.

Example: Find the volume generated by rotating $y = e^x$ about the x-axis, bounded by $x = 2$.

Solution: The first step is to sketch the graph and indicate the required solid of revolution. Note that generally the formula for volume is $V = r^2h$. Notice that $r = y$ and $h = dx$. Therefore the volume generated is given by



$$v = \int_0^2 y^2 dx = \int_0^2 e^{2x} dx = \left[\frac{e^{2x}}{2}\right]_0^2 = \frac{e^4}{2} - \frac{1}{2} = \frac{1}{2}\left(e^4 - 1\right) \text{ units.}$$

Integration is also very useful in finding the length of curves in the plane. Suppose $f(x)$ is a curve in the Cartesian plane and it is required to find the length of the curve between two limits say $x = a$ and $x = b$, we apply a well established formula for the length of arc. $s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

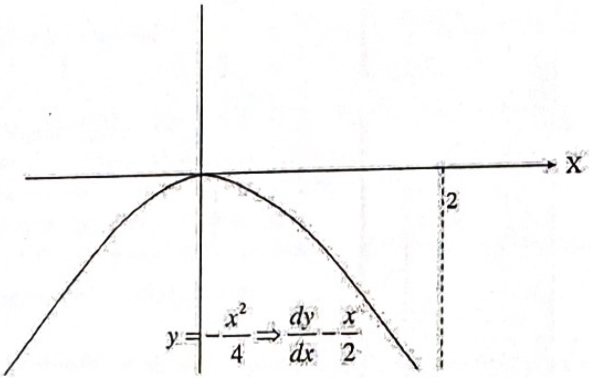
Example: Find the arc length of the parabola $x^2 + 4y = 0$, from the vertex to the point $x = 2$.

Solution: $x^2 + 4y = 0$ or $y = -\frac{x^2}{4} \Rightarrow \frac{dy}{dx} = -\frac{x}{2}$.

Apply the formula for arc length.

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^2 \sqrt{1 + \left(-\frac{x}{2}\right)^2} dx = \int_0^2 \sqrt{1 + \frac{x^2}{4}} dx = \int_0^2 \sqrt{\frac{4+x^2}{4}} dx = \frac{1}{2}\int_0^2 \sqrt{4 + x^2} dx$$

$$\int_0^2 \sqrt{u^2 + a^2} du = \frac{u}{2}\sqrt{u^2 + a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 + a^2}\right|$$



This is a standard integral, $S=\frac{1}{2}\int_0^2\sqrt{x^2+4}dx=\left[\frac{1}{2}\frac{x}{2}\sqrt{x^2+4}+\frac{4}{2}\ln\left|x+\sqrt{x^2+4}\right|_0^2\right]$.

Exercises

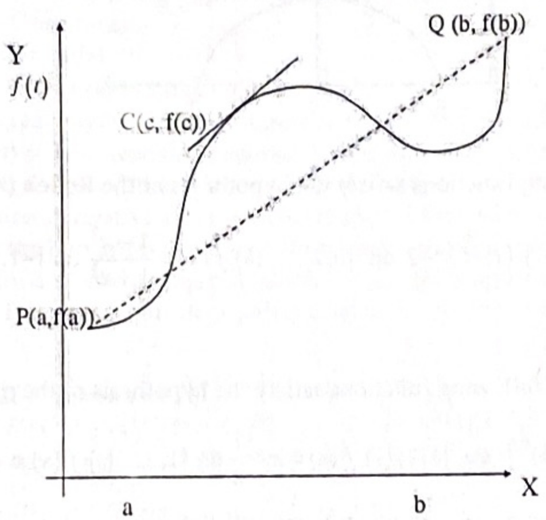
1. The area enclosed by the curve $y=4x-x^2$ and the line $y=3$ is rotated about $y=3$. Find the volume of the solid generated.
2. The area defined by the inequalities $y\geq x^2+1, x\geq 0, y\leq 2$ is rotated completely about the y-axis. Find the volume of the solid generated.
3. Evaluate the length of the curve $x=\left(1-y^{2/3}\right)^{3/2}$; $0\leq y\leq 2$.

CHAPTER TWO - **Mean Value Theorem, Rolle's Theorem and Taylor Series.**

Mean Value Theorem

Motivation

As an introduction to the mean value theorem, we pose the following question. Suppose P and Q are two points in the plane, with Q lying generally to the east of P, as on the diagram below.



Is it possible to fly a plane from P to Q, flying always roughly east, without ever (even for an instant) flying in the exact direction from P to Q? That is can we fly from P to Q without even an instantaneous line of motion ever being parallel to the line OQ, no matter which path we choose. We can rephrase this situation mathematically thus: Let the path of the plane be a differentiable function $y = f(x)$, with and point $p(a, f(a))$ and $Q(b, f(b))$. Then there must be some point on this graph where the tangent line to the curve is parallel to the line PQ joining its end points. Let this point be C; then slope of the tangent line at the point $(c, f(c))$ is $f'(c)$, while the slope of the line PQ is $\frac{f(b)-f(a)}{b-a}$. Since the tangent line to the curve and PA are parallel, then we have the same slope.

$$\therefore f'(c) = \frac{f(b)-f(a)}{b-a}.$$

This is the mathematical modelling of the mean value theorem.

The Mean value Theorem

Suppose that the function f is continuous on $[a, b]$ and differentiable on (a,b). Then $f'(c) = \frac{f(b)-f(a)}{b-a}$, for some number c in (a, b).

Rolle's Theorem

Suppose on [a, b], a and b are the zeros of f i.e. $f(a) = 0, f(b) = 0$, then the mean value theorem reduces to $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0$ i.e $f'(c) = 0$; which is known as the Rolle's Theorem.

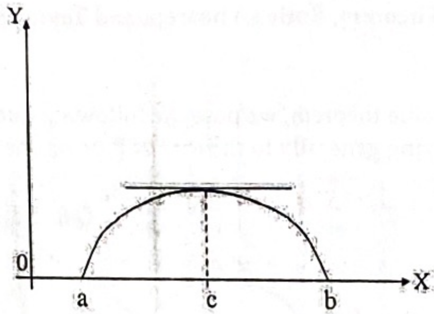
The Rolle's Theorem

Suppose that the function f is continuous on [a,b] and is differentiable on (a, b); if $f(a) = f(b) = 0$, then $f'(0) = 0$ for some c in (a, b).

Examples

1. Show that the following functions satisfy the hypotheses of the mean value theorem.

$$(a)\ f(x) = x^3 \text{ on } [-1,1] \quad (b)\ f(x) = 3x^2 + 6x - 5 \text{ on } [-2,1]$$



2. Show that the following functions satisfy the hypotheses of the Rolle's theorem.

$$(a)\ f(x) = x^2 - 2 \text{ on } [0,2] \qquad (b)\ f(x) = \frac{1-x^2}{1+x^2} \text{ on } [-1,1].$$

Exercises

1. Show that each of the following functions satisfy the hypothesis of the mean value theorem.

$$(a)\ f(x) = (x-1)^{2/3} \text{ on } [1,2]\ (b)\ f(x) = x + \frac{1}{x} \text{ on } [1,2] \quad (c)\ f(x) = \sqrt{x-1} \text{ on } [2,5]$$

2. Show that each of the following functions satisfy the hypotheses of the Rolle's theorem.

$$(a)\ f(x) = 9x^2 - x^4 \text{ on } [-3,3] \qquad (b)\ f(x) = 5x^{2/3} - x^{5/3} \text{ on } [0,5]$$

Solutions to Examples

$1a)\ f(x) = x^3$ on $[-1,1]$. $f'(x) = 3x^2$. $f(a) = f(-1) = -1,\ f(b) = f(1) = 1,\ f'(c) = 3c^2$.

By means of value theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 3c^2 = \frac{1-(-1)}{1-(-1)} = \frac{2}{2} \Rightarrow c^2 = \frac{1}{3} = c = \frac{1}{\sqrt{3}}.$$

$\therefore \frac{1}{\sqrt{3}}$ is in the interval $[-1,1]$, and therefore the hypothesis of the mean value theorem is satisfied.

$(b)\ f(x) = x^2 + 6x - 5$ on $[-2,1]$, $f'(x) = 2x - 6$, $f'(c) = 2c + 6$, $f(a) = f(-2) = 4 - 12 - 6 = -13$
$f(b) = f(1) = 1 + 6 - 6 = 2$

By mean value theorem, $f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 2c + 6 = \frac{2-(-13)}{1-(-3)} = \frac{15}{3} = 5 \Rightarrow 2c + 6 = 5 = c = -\frac{1}{2}$.
$-\frac{1}{2}$ is in the interval $[-2,1]$, therefore the hypothesis of the mean value theorem is satisfied.

$$2a)\ f(x) = x^2 - 2x \text{ on } [0,2],\quad f'(x) = 2x - 2 \Rightarrow f'(c) = 2c - 2.$$

By Rolle's theorem $f'(c) = 0 \Rightarrow 2c - 2 = 0 \Rightarrow 2c = 2 \Rightarrow c = 1$.
c = 1 is in the interval [0, 2] and therefore the hypothesis of the Rolle's theorem is satisfied.

Taylor's Series

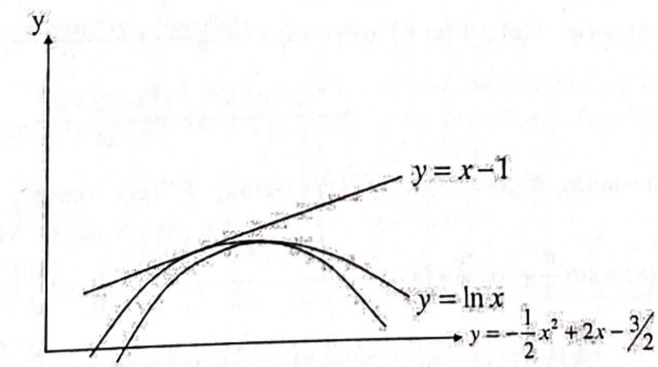
The definitions of various abstract functions leave to unclear how to compute their values, precisely, except at a few isolated points. Examples of such functions are $\ln x$, e^x, $x^{1/2}$ (except x is the square of a specific rational number). Obviously $\ln 1 = 0$; but no other value of $\ln x$ is obvious. Also, at $x = 0$, $e^0 = 1$, but no other value of e^x is obvious.

On the other hand any value of a polynomial $p(x) = a_0 + a_1 + a_2x^2 + \cdots + a_nx^n$, with known coefficients $a_0, a_1, a_2, \cdots, a_n$ is easily computed since we just substitute for x in the polynomial. The Taylor's series of a function is such an equivalent polynomial of the function; which aids in the numerical evaluation of the function at any argument, $x = n$ (say).
To obtain the equivalent polynomial for any function f at $x = a$ (say), we proceed thus. The simplest polynomial approximation is the linear approximation (the equation of the tangent to the curve at the point). $f(x) \cong f(a) + f'(a)(x-a)$
hence the first degree polynomial.
Polynomial approximation of $f(x)$ is $p_1(x) = f(a) + f'(a)(x-a)$.
Notice that this first degree polynomial $p_1(x)$ agrees with $f(x)$ and its first derivative at $x = a$; that is $p_1(a) = f(a)$ and $p_1'(a) = f'(a)$ (convince yourself). For example, suppose that $f(x) = \ln xt$ and suppose $a = 1$, then $p_1(x) = -1 + x$. Then $f(1) = \ln 1 = 0 = p_1(1)$, $\quad f'(1) = \frac{1}{1} = p_1(1) = 1$
Therefore, $f(x)$ and its first derivative agree with $p_1(x)$ and its first derivative at $x = 1$.
We however reason that the best approximation to $\ln x$ is not a linear approximation but a quadratic one. We therefore look for a second degree polynomial $p_2(x)$ which approximates $\ln x$.
$p_2(x) = a_0 + a_1x + a_2x^2$. This is an equivalent polynomial to $f(x) = \ln x$ at the point $x = 1$ iff (if and only if).

$$f(1) = p_2(1) \quad \text{but } f(x) = \ln x \Rightarrow f(1) = \ln 1 = 0,$$
$$f'(1) = p_2'(1),\ f'(x) = \tfrac{1}{x} \Rightarrow f'(1) = 1,\ f''(1) = p_2''(1),\ f''(x) = -\tfrac{1}{x^2} \Rightarrow f''(1) = 1$$
$$p_2(1) = a_2 + a_1 + a_0 = 0 = f'(1),\ p_2'(1) = 2a_2a_1 = 1 = f'(1),\ p_2''(1) = 2a_2 = -1 = f''(1)$$

These give $a_0 = -\frac{3}{2}$, $a_1 = 2$, $a_2 = -\frac{1}{2}$. So, $p_2(x) = -\frac{3}{2} + 2x - \frac{x^2}{2}$.
The tangent line (first degree polynomial approx.) and the parabola (second degree polynomial approximation) illustrate the general approach to polynomial approximation.
We therefore reason that in order to approximate the function $f(x)$ near $x = a$ we look for an nth degree polynomial $P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ such that its values at $x = a$ and the values of its first n derivatives at 'a' agree with the corresponding values of f and its first n derivatives i.e.

$$\left.\begin{aligned} P_n(a) &= f(a) \\ P_{n'}(a) &= f'(a) \\ P_{n''}(a) &= f''(a) \\ &\vdots \\ P_n^{(n)}(a) &= f^{(n)}(a) \end{aligned}\right\} \cdots (1)$$



These n + 1 conditions are used to determine the values of the n + 1 coefficients $a_0, a_1, a_2, \cdots, a_n$.
We reason that he algebra would be much simpler if we express $P_n(x)$ as an nth degree polynomial of powers of $x - a$ rather than powers of x. i.e.

$$P_n(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n$$

By substituting $x = a$ into $P_n(x)$, we get $P_n(a) = b_0 = f(a)$ from (1) above. i.e. $b_0 = f(a)$; $P_{n'}(x) = b_1 + 2b_2(x-a) + \cdots + bb_n(x-a)^{n-1}$
By substituting $x = a$ into $p_n'(x)$, we get

$$P_n'(a) = b_1 = f'(a) \text{ from (1) again}$$

$$P_n''(x) - 2b_2 + 3\times 2\, b_{-3}(x-a) + \cdots + n(n-1)\, b_n (x-a)^{n-2}$$

Substituting $x = a$ into $P_n''(x)$ we get

$$P_n''(a) = 2b_2 = f''(a) \text{ from (1) again} \Rightarrow b_2 = \frac{f''(a)}{2}$$

We continue in this way to find $b_3, b_4, \cdots, b_n$. In general, the constant term in $P_n^{(k)}(a)$ is $k!b_k$

$$\therefore k!b_k = P_n^{(k)}(a) = f^{(k)}(a) \Rightarrow b_k = \frac{f^{(k)}(a)}{k!}$$

Therefore equation (2) reduces to

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

(by substituting the coefficient)
i.e. $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ written more compactly.
This is the Taylor's series expansion of $f(x)$ in terms of a polynomial $P_n(x)$ about an argument $x = a$.
Named after Brook Taylor who first published it.
If however $a = 0$, such that the expansion is about $x = 0$, then the Taylor series reduce to the Maclaurin series.

(1) (a) $f(x) = e^x$ about $x = 0$; (b) $f(x) = \sin x$, $x = \frac{\pi}{2}, n = 5$; (c) $f(x) = \cos x$, $x = \frac{\pi}{2}, n = 4$;

(d) $f(x) = \ln(1+x), x = 1, n = 4$.

(2) Use the result of (b) to find $\sin\frac{\pi}{2}$, $\sin\pi$, $\sin\frac{\pi}{3}$,

Solution (b) $f(x) = \sin x$, $a = x = \frac{\pi}{2}, n = 5$.
By Taylor's series $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f^{(''')}(a)(x-a)^3}{3!}$

$$\frac{f^{iv}(a)(x-a)^4}{4!} + \frac{f^{v}(a)(x-a)^5}{5!}$$

$$f(x) = \sin x,\quad f'(x) = \cos x,\quad f''(x) = -\sin x,\quad f'''(x) = -\cos x,\quad f^{iv}(x) = \sin x,\quad f^{v}(x) = \cos x$$

$$\therefore f(x) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2}\bullet\left(x-\frac{\pi}{2}\right) - \frac{\left(\sin\frac{\pi}{2}\bullet\left(x-\frac{\pi}{2}\right)^2\right)}{2!} - \frac{1}{3!}\cos\frac{\pi}{2}\left(x-\frac{\pi}{2}\right)^3 + \frac{1}{4!}\left(\sin\frac{\pi}{2}\bullet\left(x-\frac{\pi}{2}\right)^4\right)$$
$$+\frac{1}{5!}\cos\frac{\pi}{2}x\left(x-\frac{\pi}{2}\right)^5 = \sin\frac{\pi}{2} + \left(x-\frac{\pi}{2}\right)\cos\frac{\pi}{2} - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2\sin\frac{\pi}{2} - \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3\cos\frac{\pi}{2}$$
$$+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4\sin\frac{\pi}{2} + \frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5\cos\frac{\pi}{2}.\therefore f(x) = 1 - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4.$$

Others: Exercise.

CHAPTER THREE-Partial Derivatives

Real-valued Functions of two or three Variables

Recall $f : \mathrm{R} \Rightarrow \mathrm{R}$ is a real-valued function of a real variable. As we noted earlier in the work, the domain and range of f are both subsets of R.

Generally, if $g : \mathrm{R}^n \to \mathrm{R}$, then we have a real-valued function of real variables (an n-tuple). If $n = 2$, then $g : \mathrm{R} \to \mathrm{R}$, a function from the plane (say xy-plane) to R; if $n = 3$, then we have $g : \mathrm{R}^3 \to \mathrm{R}$,a function from the 3-dimensional space (xyz-space) say to R.

Space

1. $Z = \left(x^2 + y^2\right)^{1/2}$ is a function of two variables x, y (from the XY-plane) (independence variables) to Z (dependent variable). Here the domain of Z is a subset of the XY-plane, while the range is a subset of R.

2. Again $w = \left(x^2 + y^2 + z^2\right)^{1/2}$ is a function of three variable x, y, z (from XYZ-space) (independence variables) to W (dependence variable). Here the domain of W is a subset of the XYZ-space, while the range is a subject ofR.

Domain

Let $F : R^2 \to R$ be defined by $Z = f(x,y)$. The domain of f are all the pairs (x, y) R^2 such that $Z = f(x,y)$ is defined (real) i.e. $D = \{(x,y), \in R^2 : f(x,y) = z \in \mathrm{R}\}$ is defined (real).

e.g. Find the domain of $Z = \frac{x-1}{\sqrt{x^2-y^2}}$.

Solution

$$D = \left\{(x,y) \in R^2 : Z = \frac{x-1}{\sqrt{x^2-y^2}} \text{ is real}\right\}.$$

Now, Z is real if it is defined. So for it to be defined,

$$x^2 - y^2 > 0 \Rightarrow x^2 > y^2 \Rightarrow |x| > |y|$$

Thus $D = \{(x,y) \in \mathrm{R}^2 : |x| > |y|\}$.

Let $g : R^3 \to R$ be defined by $W = g(x,y,z)$. The domain of g are all the triples $(x,y,z) \in R^3$ such that $w = g(x,y,z)$ is defined (real). i.e. $D\{(x,y,z) \in R^3 : g(x,y,z) = W \text{ is defined}\}$. E.g. Find the domain of $W = \left(x^2 + y^2 + z^2\right)^{1/2}$.

Solution: $D = \{(x,y,z) : g(x,y,z) = W \text{ is real}\}$. For it to be real, $x^2 + y^2 + z^2 \geq 0$.

Therefore, $D = \{(x,y,z) : x^2 + y^2 + z^2 \geq 0\}$

Range: Let $f : R^2 \to R$ be defined by $Z = f(x,y)$. The range of f is the set of all $z \in \mathrm{R}$ such that $f(x,y) = z$. i.e. $R_f = \{z \in \mathrm{R} : f(x,y) = z\}$

Let $g : R^3 \to R$ be defined by $w = g(x,y,z)$. The range of g is the set of all $w \in \mathrm{R}$ such that $g(x,y,z) = w$. The range of g is the set of all $W \in \mathrm{R}$ such that $g(x,y,z) = w$.

i.e. $R_g = \{w \in \mathrm{R} : g(x,y,z) = w\}$

Partial Derivatives

The partial derivative of $f(x,y)$ $w.r.t.$ x is defined as $\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x,y)-f(x,y)}{\delta x}$, if the limit exists.

Similarly, the partial derivative of $f(x,y)$ $w.r.t.$ y is defined as $\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x,y+\delta y)-f(x,y)}{\delta y}$, if the limit exist. If $U = f(x,y)$ then $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x}, \frac{\partial u}{\partial y} = \frac{\partial f}{\partial y}$.

Other notations for partial derivatives include $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} = u_x = f_x$ and $\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} = u_y = f_y$.

Examples: Find f_x, f_y, f_z if : (1) $f(x,y) = 3x - y^2$ (2) $f(x,y) = \sin^2 x \cos y + \frac{x}{y^2}$ (3) $f(x,y,z) = e^{2z} \cos(xy)$

Solution: (1) $f(x,y) = 3x - y^2 f_x = 3,\ f_y = -2y,\ f_z = 0$

(2) $f(x,y) = \sin^2 x \cos y + \frac{x}{y^2},\ \frac{\partial f}{\partial x} = f_x = 2\sin x \cos x \cos y + \frac{x}{y^2},\ \frac{\partial f}{\partial y} = f_y = -\sin^2 x \sin y - \frac{2x}{y^3}$

(3) $f(x,y,z) = e^{2z}\cos(xy),\ \frac{\partial f}{\partial x} = f_x = -ye^{2z}\sin(xy),\ \frac{\partial f}{\partial y} = f_y = -xe^{2z}\sin(xy),\ \frac{\partial f}{\partial z} = f_z = 2e^{2z}\cos(xy).$

It should be noted that the partial derivative of a function of several variables is identical to the ordinary derivative of the same function w.r.t. the same variable, when the other variables are treated as constants. Further, all rules of differentiation hold.

Exercise

1) Find u_x and u_y where $u = xye^{\lambda y}$ (2) Find f_x, f_y if $f(x,y) = \sin^2 x \cos y + \frac{x}{y^2}$

3) Find f_x, f_y where $f(x,y) = \tan^{-1}\frac{y}{x}$. (4) If $\bar{U} = \cos xyi + (3xy - 2x^2)j - (3x + 2y)k$, find $\bar{U}_x, \bar{U}_y$.

CHAIN RULE (Function of a Function)

Let $f = f(u)$, where $U = U(x)$. Then $\frac{df}{dx} = \frac{df}{du} \times \frac{du}{dx}$ i.e. for one independent variable x. Similarly, $f = f(u)$, where $U = U(x,y)$. Then $\frac{\partial f}{\partial x} = f_x = \frac{\partial f}{\partial u} \bullet \frac{\partial u}{\partial x}$ and $\frac{\partial f}{\partial y} = f_y = \frac{\partial f}{\partial u} \bullet \frac{\partial u}{\partial y}$

Example: Let $f = f(u) = (x^2 + y^2)^{1/2}$, i.e. $u = x^2 + y^2$.

$\therefore f = U^{1/2} \Rightarrow \frac{\partial f}{\partial u} = \frac{1}{2}U^{-1/2},\ \frac{\partial u}{\partial x} = 2x$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \bullet \frac{\partial u}{\partial x} = \frac{1}{2}u^{-1/2} \bullet 2x = \frac{1}{2}(x^2 + y^2)^{-1/2} \bullet 2x = \frac{x}{(x^2 + y^2)^{1/2}}$$

Similarly, $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \bullet \frac{\partial u}{\partial y},\ \frac{\partial u}{\partial y} = 2 = \frac{1}{2}u^{-1/2} \bullet 2y = \frac{1}{2}(x^2 + y^2)^{-1/2},\ 2y = \frac{y}{x^2+y^2}$.

Exercises(1) $F = (x^2 y + y^2)^3$. Find U_x, U_y. (2) $U = (\cos xy)^2$. Find U_x and U_y. (3) $W = (s^3 + t)^2$. Find W_s, W_t.

Higher Order Partial Derivatives

If U is a function of "x, y, z, ...", then $U_x, U_y, U_z, \cdots$ are themselves function of x, y, z, ... and may therefore be differentiable. This further derivation of $U_x, U_y, U_z, \cdots$ yield second order partial derivatives. The four second order partial derivatives of U w.r.t x and y are:

$$U_{xx} = (U_x)_x = \frac{\partial}{\partial x}U_x = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2},\quad U_{yy} = (U_y)_y = \frac{\partial}{\partial y}(U_y) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

$$U_{xy} = (U_x)_y = \frac{\partial}{\partial y}(U_x) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x},\quad U_{yx} = (U_y)_x = \frac{\partial}{\partial x}(U_y) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

Higher order derivatives are obtained in a similar way. Second order mixed partials are equal (i.e. $U_{xy} = U_{yx}$) especially when they are constants (sufficient condition).

Example: Find the four second partial derivatives w.r.t. the arguments of the function.

$$1)\quad f(x,y) = \cot^{-1}\frac{y}{x}.$$

Solution: $f(x,y) = \cot^{-1}\left(\frac{y}{x}\right)$. Let $u = \frac{y}{x}$. $u_x = \frac{d}{dx}\left(\frac{y}{x}\right) = y\frac{d}{dx}(x^{-1}) = \frac{-y}{x^2}$ and $u_y = \frac{d}{dy}\left(\frac{y}{x}\right) = \frac{1}{x}\frac{d}{dy}(y) = \frac{1}{x}$

$$\therefore f_x = \frac{\partial f}{\partial u} \bullet \frac{\partial u}{\partial x} = \frac{\partial}{\partial u}\cot^{-1}(u) \bullet \frac{\partial u}{\partial x} = -\frac{1}{1+u^2} \bullet \frac{-y}{x^2} = \left[1 \div \left(1 + \frac{y^2}{x^2}\right)\right] \bullet \frac{y}{x^2} = \left[1 \div \left(\frac{x^2+y^2}{x^2}\right)\right] \bullet \frac{y}{x^2}$$

$$= 1 \times \frac{x^2}{x^2+y^2} \times \frac{y}{x^2} = \frac{y}{x^2+y^2}$$

$$\therefore f_y = \frac{\partial f}{\partial u} \bullet \frac{\partial u}{\partial y} = \frac{\partial}{\partial u}\cot^{-1}(u) \bullet \frac{\partial u}{\partial y} = -\frac{1}{1+u^2} \bullet \frac{1}{x} = \left[1 \div -\left(1 + \frac{y^2}{x^2}\right)\right] \bullet \frac{1}{x} = \left[1 \div \left(-\frac{x^2+y^2}{x^2}\right)\right] \bullet \frac{1}{x}$$

$$= 1 \times -\frac{x^{21}}{x^2+y^2} \times \frac{1}{x} = -\frac{x}{x^2+y^2}$$

$$\therefore f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = y\frac{\partial}{\partial x}(x^2+y^2)^{-1} = \frac{-y}{(x^2+y^2)^2}$$

$$\therefore f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{x}{x^2+y^2}\right) = -x\frac{\partial}{\partial y}\left(x^2+y^2\right)^{-1} = \frac{2xy}{(x^2+y^2)^2}$$

$$\therefore f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{\left(x^2+y^2\right)\frac{\partial}{\partial y}(y) - (y)\frac{\partial}{\partial y}\left(x^2+y^2\right)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right) = \frac{\left(x^2+y^2\right)\frac{\partial}{\partial y}(-x) - -x\frac{\partial}{\partial x}\left(x^2+y^2\right)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Notice that the mixed partials U_{xy} and U_{yx} are equal.

Observe that $U_{xx} + U_{yy} = 0$. This is known as the Laplace equation in two variables. Any function $U = f(x, y)$ satisfying the Laplace's equation $U_{xx} + U_{yy} = 0$ is called a harmonic or potential function.

Exercises

1. If $\bar{C}_1$ and $\bar{C}_2$ are constant vectors and λ is a constant scalar, show that $\bar{H} = e^{-\lambda x}\left(\bar{C}_1 \sin xy + \bar{C}_2 \cos xy\right)$ satisfies the partial differential equation $\frac{\partial^2 \bar{H}}{\partial x^2} + \frac{\partial^2 \bar{H}}{\partial x^2} = 0$.

2. Find u_{xx}, u_{yy}, u_{yy}, u_{xy} and u_{yx} if (a) $u(x,y) = \cos xy + 3xy - 2x^2 - 3x + 2y$ (b) $\cot^{-1}\left(\frac{x}{y}\right)$ (c) $\tan^{-1}\left(\frac{x}{y}\right)$ (d) $\tan^{-1}\left(\frac{y}{x}\right)$. Are the second mixed partials equal?

Chain Rule: Total Partial Derivatives (Chain Rule).

Let $U = U(x, y)$, where $x = x(s,t)$, $y = y(s,t)$. Since $U(x,y)$ is function of the variables $x(s,t)$ and $y(s,t)$ then it is also a function of the variables s and t. $U(x,y)$ therefore has two total partial derivatives w.r.t. s and t respectively (U_s and U_t).

$$\partial u = \frac{\partial u}{\partial x}\partial x + \frac{\partial u}{\partial y}\partial y \therefore \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \text{ and as } \partial s \to 0$$

We have $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = U_s$. Similarly, $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \bullet \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \bullet \frac{\partial y}{\partial t} = U_t$

Generally, if $U = U(x_1, x_2, x_3, \cdots, x_n)$ and $x_1, x_2, \cdots, x_n$ are each a function of the variables $t_1, t_2, t_3, \cdots, t_r$; then

$$U_{t_1} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$
$$U_{t_2} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_n}$$
$$\vdots$$
$$U_{t_r} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_r} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_r} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_r}$$

Example: $U = U(x,y)$, $x = s^2 - t^2$, $y = 2st$ (1)

Prove that $sU_s - tU_t = 2\left(s^2 + t^2\right)U_x$.

Solution

$U_s = \frac{\partial u}{\partial x} \bullet \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = U_x\frac{\partial x}{\partial s} + U_y\frac{\partial y}{\partial s}$ ... (2) $U_t = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = U_x\frac{\partial x}{\partial t} + U_y\frac{\partial y}{\partial t}$ ... (3)

From (1), $\frac{\partial x}{\partial s} = 2s$, $\frac{\partial x}{\partial t} = -2t$, $\frac{\partial y}{\partial t} = 2s$.

Substituting these in (2) and (3) we get.

$$U_s = 2sU_x + 2tU_y, \qquad U_t = -2tU_x + 2sU_y$$
$$sU_s = 2s^2U_x + 2stU_y, \qquad tU_y = -2t^2U_x + 2stU_y \Rightarrow sU_s - tU_t = 2\left(s^2+t^2\right)U_x$$

Example: Suppose that $z = f(u,v)$, $u = 2x + y$; $v = 3x - 2y$ and given $\frac{\partial z}{\partial u} = 3$, $\frac{\partial z}{\partial u} = -2$ at the point $u = 3$, $v = 1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial u}$ at the point $(x,y) = (1,1)$.

Solution: $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$

$$u = 2x + y \Rightarrow \frac{\partial u}{\partial x} = 2,\ v = 3x - 2y \Rightarrow \frac{\partial v}{\partial x} = 3 \ \therefore \frac{\partial z}{\partial x} = 3(2) + (-2)3 = 0$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} u = 2x + y \Rightarrow \frac{\partial u}{\partial y} = 1,\ v = 3x - 2y \Rightarrow \frac{\partial v}{\partial y} = -2 \ \therefore \frac{\partial z}{\partial y} = 3(1) + (-2) = 7$$

Exercises

1. Let $w = f(x,y)$ and let $x = r\cos\theta$ and $y = r\sin\theta$. Calculate $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial \theta}$ and $\frac{\partial^2 w}{\partial r^2}$ in terms of r and θ and the partial derivatives of w w.r.t. x and y.

2. Let $W = \ln\left(x^2 + y^2 + z^2\right)$, $x = s - t$, $y = s + t$, $z = 2\sqrt{st}$. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

3. In the following problems, find $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$ and $\frac{\partial r}{\partial z}$

   (a) $r = e^{u+v+w}$, $u = yz$, $v = xz$, $w = xy$

   (b) $r = uvw - u^2 - v^2$, $u = y + z$, $v = x + z$, $w = x + y$

Extrema (Turning Points Brief Review)

Recall: if $f(x)$ is a function of a single variable x, defined in an interval I, then the following are true:

1. f is said to have relative minimum (or local minimum) at $x_0 \in I$ if $f(x) \geq f(x_0)$ in the some neighbourhood of x_0. If however, $f(x) \geq f(x_0) \forall x \in I$, then f is said to have an absolute minimum at x_0.

2. f is said to have a relative maximum (or local maximum) at $x_0 \in I$ if $f(x) \leq f(x_0)$ in some neighbourhood of x_0.

If however, $f(x) \leq f(x_0) \forall x \in I$, then f is said to have an absolute maximum at x_0.
Minimum and maximum points are otherwise called **extrema**. Thus the local maxima and local minima of a function f are called its local extrema.
The most useful and applicable concept of extrema points (relative or absolute) is that $f'(x) = 0$ at these points (x say).
Theorem: if f is a continuous function defined on an interval I, then

1. The local extrema of f (if they exist) occur only at the critical (turning) points of f belonging to I, where $f' = 0$ or f' does not exist.

2. $f' = 0$ at absolute maxima and minima.

The tests for extrema have already been treated under turning points.
These same concepts for functions of a single variable are transferable to functions of several variables. The only thing that alters is that we now deal with several variables and use differentials to test for extrema as opposed to the usage of 'derivatives' in the single variable situation.
Let $f = f(x,y)$ be a function of two variables x and y in a region R of the xy-plane.

1. $f(x,y)$ is said to have a relative (local) maximum at a point (x_0, y_0) in R if $f(x,y) \leq f(x_0, y_0)$ for all points (x,y) in some neighbourhood of (x_0, y_0). If however $f(x,y) \leq f(x_0, y_0)$ for all point (x,y) in R, then $f(x,y)$ is said to have an absolute maximum at (x_0, y_0).

2. $f(x,y)$ is said to have a relative (local) minimum at a point (x_0, y_0) in R if $f(x,y) \geq f(x_0, y_0)$ for all points (x,y) in some neighbourhood of (x_0, y_0). If however $f(x,y) \geq f(x_0, y_0)$ for all points (x,y) in R, then $f(x,y)$ is said to have an absolute minimum at (x_0, y_0).

The maxima or minima of $f(x,y)$ are called its extrema. It is worth noting that at extrema points.

$$f_x(x,y) = 0, \quad f_y(x,y) = 0$$

The tests for extrema points for functions of several variables is given below:
If however, $f(x,y)$ is a bivariate function we are dealing with, then to check for maximum or minimum points we, check the sign of $f_{xx}f_{yy} - \left(f_{xy}\right)^2$.

$$(a) \quad \left(f_{xy}\right)^2 - f_{xx}f_{yy} < 0, \begin{cases} f_{xx} \text{ and } f_{yy} < 0 \Rightarrow \text{maximum point} \\ f_{xx} \text{ and } f_{yy} > 0 \Rightarrow \text{minimum point} \end{cases}$$

(b) $(f_{xy})^2 - f_{xx}f_{yy} > 0 \Rightarrow$ saddle point

(c) $(f_{xy})^2 - f_{xx}f_{yy} = 0 \Rightarrow$ yields no information

Example 1: Check $f(x,y) = x^2 + y^2$ for minimum or maximum points.
Solution: $f(x,y) = x^2 + y^2$, $f_x = 2x$, $f_{xx} = 2$, $f_y = 2y$, $f_{yy} = 2$, $f_{xy} = 0$

$(f_{xy})^2 - f_{xx}f_{yy} = (0 - 2\times 2) = (0-4) = -4 < 0$

Notice : f_{xx} or $f_{yy} > 0$ and $(f_{xy})^2 - f_{xx}f_{yy} < 0$. This implies the equation yields a minimum point.

Example 2: What is the nature of the turning point of $f(x,y) = x^2 + y^2 - 4xy$ at $(0, 0)$.
Solution: $f_x = 2x - 4y$, $f_{xx} = 2$

$f_y = 2y - 4x$, $f_{yy} = 2$, $f_{xy} = -4$ $\therefore (f_{xy})^2 - f_{xx}f_{yy} = 16 - 2\times 2 = 12 > 0$

Notice : f_{xx} or $f_{yy} > 0$ and $(f_{xy})^2 - f_{xx}f_{yy} > 0$. Therefore the point is a saddle.

We have given example of turning points of bivariate functions. Differentiation can also be used to determine the equation of the tangents and normal to curves. This is better illustrated by an example.

Exercises

(1) Discuss the turning point of the function $f(x,y) = x^2 + xy - x - y^4 + 1$ in the region $x^2 + y^2 = 1$. State clearly which is minimum and which is maximum, if they exist.

(2) Find and classify the extrema points of: (i) $x^3 + y^2 - 6xy$ (ii) $x^3 - 3x^2 + xy^2$

Constraints in Extrema – Language Multipliers

We now consider the problem of determining extrema of a function of several variables which has some constraints. A constraint in a function is a pre-condition which must be satisfied by the function. In general, we denote the function as $f(x,y)$ and the constraint as $g(x,y)$.

This concept can be better clarified by an example.

Example: What are the dimensions of the rectangle of perimeter L with the largest area.

SolutionLet x, y be the dimensions. Then the area is

$$A = f(x,y) = xy, \; x,y > 0 \qquad (1)$$

Since $L = 2x + 2y \Rightarrow 2x + 2y - L = 0$ or $x + y - \frac{L}{2} = 0$

Thus the constraint is $g(x,y) = x + y - \frac{L}{2} = 0$ (2)

We wish to find the extrema of (1) subject to (2). The constraint (2) can be used to eliminate x or y.

from (2), $y = \frac{L}{2} - x$. Thus (1) becomes $h(x) = f\left(x, \frac{L}{2} - x\right) = \left(\frac{L}{2} - x\right)\cdot x$, $x > 0$ (3)

The extrema of (3) i.e. $h(x)$ corresponds with those of (1) i.e. $f(x,y)$.

Thus we can solve (3) instead $h'(x) = \left(\frac{L}{2} - x\right) - x$. At extrema, $h'(x) = 0 \Rightarrow \left(\frac{L}{2} - x\right) - x = 0 \Rightarrow x = \frac{L}{4}$

$h'(x) = \frac{L}{2} - 2x$ $h''(x) = -2 \Rightarrow$ local maximum.$\therefore$ maximum area occurs at $x = \frac{L}{4}$.

And since $y = \frac{L}{2} - x \Rightarrow y = \frac{L}{4}$ also.

Hence for maximum area subject to (2), the dimensions of the rectangle are $\left(\frac{L}{4}, \frac{L}{4}\right)$ i.e. a square.

Lagrange's Multipliers

Let $f = f(x,y)$ be a function which is to be maximized or minimized subject to the constraint $\psi(x,y) = 0$. The technique of solution is to introduce a parameter λ called the Lagrange's multiplier.

We therefore form a new function F called the Lagrangian i.e.

$$F = f(x,y) + \lambda\psi(x,y)$$

An extrema points; we have for first order conditions that

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial \lambda} = 0.$$

Furthermore, we have for second order conditions that if

(i) $\frac{\partial^2 F}{\partial x^2} > 0$, $\frac{\partial^2 F}{\partial y} > 0$ and $\left(F_{xy}\right)^2 - F_{xx}F_{yy} < 0$, then a minimum point is implied.

(ii) $\frac{\partial^2 F}{\partial x^2} < 0$, $\frac{\partial^2 F}{\partial y^2} < 0$ and $\left(F_{xy}\right)^2 - F_{xx}F_{yy} < 0$, then a maximum point is obtained.

(iii) $\frac{\partial^2 F}{\partial x^2} < 0$, $\frac{\partial^2 F}{\partial y^2} < 0$ $\left(\text{or } \frac{\partial^2 F}{\partial x^2} > 0, \frac{\partial^2 F}{\partial y^2} > 0\right)$ and

$\left(F_{xy}\right)^2 - F_{xx}F_{yy} > 0$, then a saddle point is obtained.

Example: Find the extrema values of $f(x,y) = x^2 + 4y^2 + 6$ subject to $2x - 8y = 20$.
SolutionConstraint is $2x - 8y - 20 = 0$ which is $\psi(x,y)$. We then form the Lagrangian as

$$F = x^2 + 4y^2 + 6 + \lambda(2x - 8y - 20), \quad \frac{\partial F}{\partial x} = (2x + 2)\lambda, \quad \frac{\partial F}{\partial y} = (8y - 8)\lambda, \quad \frac{\partial F}{\partial \lambda} = 2x - 8y - 20$$

At extrema points, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial \lambda} = 0$

$$i.e.\ 2x + 2\lambda = 0 \ \cdots \ (i) \quad 8y - 8\lambda = 0 \ \cdots \ (ii) \quad 2x - 8y - 20 = 0 \ \cdots \ (iii)$$

From (i), $x = -\lambda$, from (ii) $y = \lambda$. Substitute for x and y in (iii)

$$-2\lambda - 8 - 20 = 0 \Rightarrow -10\lambda - 20 = 0 \Rightarrow 10\lambda = -20 \Rightarrow \lambda = -2 \Rightarrow x = 2,\ y = -2.$$

$$\frac{\partial^2 F}{\partial x^2} = 2, \ \frac{\partial^2 F}{\partial y^2} = 8, \quad \frac{\partial^2 F}{\partial x \partial y} = 0. \quad \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 - \frac{\partial^2 F}{\partial x^2} \bullet \frac{\partial^2 F}{\partial y^2} = 0 - 2 \times 8 = -16 < 0$$

Notice $\frac{\partial^2 F}{\partial x^2} > 0$, $\frac{\partial^2 F}{\partial y^2} > 0$, $\left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 - \frac{\partial^2 F}{\partial x^2} \bullet \frac{\partial^2 F}{\partial y^2} < 0$

Therefore (2, -2) is a minimum extreme point. The extreme value of $f(x,y) = x^2 + 4y^2 + 6$ subject to $2x - 8y = 20$ occurs at $(2,-2) \therefore f(x,y)\big|_{(2,-2)} = f(2,-2) = 4 + 16 + 6 = 26$

Example 2: Let utility function be $U(x_1,x_2) = x_1^3 x_2^3$ and the budget be $I = p_1x_1 + p_2x_2$. Find or maximize the utility function subject to the budget constraint given that $p_1 = 2$, $p_2 = 3$, $I = 48$.

Solution $U(x_1,x_2) = x_1^3 x_2^3$ Constrain is $48 - 2x_1 - 3x_2 = 0 = \psi(x_1,x_2)$.
We form the Lagrangian $F = U(x_1,x_2) + \lambda\psi(x_1,x_2) = x_1^3x_2^3 + \lambda(48 - 2x_1 - 3x_2)$.

$$\frac{\partial F}{\partial x_1} = 3x_1^2x_2^3 - 2; \quad \frac{\partial F}{\partial x_2} = 3x_1^3x_2^2 - 3, \quad \frac{\partial F}{\partial \lambda} = 48 - 2x_1 - 3x_2$$

At extrema $\frac{\partial F}{\partial x_1} = 0$, $\frac{\partial F}{\partial x_2} = 0$, $\frac{\partial F}{\partial \lambda} = 0$.
This imply

$$3x_1^3x_2^3 - 2\lambda = 0 \ \cdots \ (1) \quad 3x_1^3x_2^2 - 3\lambda = 0 \ \cdots \ (2) \quad 48 - 2x_1 - 3x_2 = 0 \ \cdots \ (3)$$

From (1) $\lambda = \frac{3}{2}x_1^2x_2^3$ From (2) $\lambda = x_1^3x_2^2$
Equating the two, we get $\frac{3}{2}x_1^2x_2^3 = x_1^3x_2^2 \Rightarrow \frac{3}{2}x_2 = x_1$ or $x_2 = \frac{2}{3}x_1$
Substituting this result in (3), we get

$$48 - 2x_1 - 3\left(\frac{2}{3}x_1\right) = 0 \Rightarrow 48 - 2x_1 - 2x_1 = 0 \Rightarrow 48 = 4x_1 \Rightarrow x_1 = \frac{48}{4} = 12$$

Therefore $x_2 = \frac{2}{3}x_1 = \frac{2}{3} \times 12 = 8$ So the critical point is $(12,8) = (x_1, x_2)$.
To check whether this is a minimum or maximum point, we proceed to the second order test.

$\frac{\partial^2 F}{\partial x_1^2}\Big|_{(12,8)} = 6x_1x_2^3\Big|_{(12,8)} = 6 \times 12 \times 8^3 > 0 = 36864$, $\quad \frac{\partial^2 F}{\partial x_2^2}\Big|_{(12,8)} = 6x_1^3x_2 = 6\left(12^3\right) \times 8 > 0 = 82944$

$\frac{\partial^2 F}{\partial x_1 \partial x_2}\Big|_{(12,8)} = 9x_1^2x_2^2\Big|_{12,8} = 9 \times 12^2 \times 8^2 = 8294$ $\quad \left(\frac{\partial^2 F}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 F}{\partial x_1^2} \times \frac{\partial^2 F}{\partial x_2^2} = 3822059520 > 0$

This implies the utility function at the critical point is a saddle point.

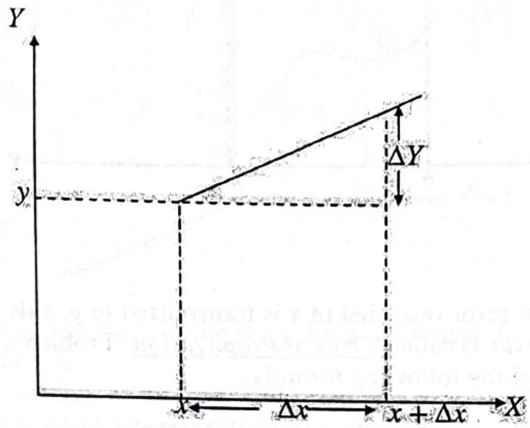
Exercises

1) Find and classify the extrema of the function $f(x,y) = 2x^2 + 10xy + 3y^2$ on the boundary of the circle $x^2 + y^2 = 5$.

1. By the Lagrange's method of otherwise, find the dimensions of the box of maximum volume which can just be fitted into the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. $F=8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$, $x=y=z=\frac{a}{\sqrt{3}}$

Differentials, Increments and Linear Approximation

Let $y=f(x)$be a single variable function. Usually, a change in x denoted by Δx induces a change in y denoted by $\Delta y$are referred to as increments.



Traditionally, we define the derivative of y with respect to x as follows

$$f'(x)=\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \quad (1)$$

From (1), we have

$$dy=f'(x)dx \quad (2)$$

The symbols dy, dx are called differentials, and are different from increments Δy, Δx. Clearly

$$\Delta y=f(x+\Delta x)-f(x) \quad (3)$$

We define $\frac{dy}{dx}=f'(x)=m_t$ (say) (4)

Where m_t is the gradient of the tangent to the curve $y=f(x)$ at x.

Hence we see that whereas Δy denote change in y, dy denote change in the tangent to the curve $y=f(x)$. This is shown in the diagram below

Example: Determine dy and Δy if $y=-\sqrt{x}$, $x=4$, $dx=\Delta x=2$.

Solution$f(x)=y=-\sqrt{x}\Rightarrow f'(x)=\frac{dy}{dx}=-\frac{1}{2\sqrt{x}}\Rightarrow dy=-\frac{1}{2\sqrt{x}}dx=\frac{-1}{2\sqrt{x}}\bullet 2=\frac{-1}{2}=-0.5$

From (3), $\Delta y=f(x+\Delta x)-f(x)=-\sqrt{x+\Delta x}-\left(-\sqrt{x}\right)=\sqrt{x}-\sqrt{x+\Delta x}=2-\sqrt{6}=2-2.4=0.4$

This is a linear (tangent line) approximation of the function$y=f(x)$.

Alternately, this is written as:$f(x)=f'(x_0)(x-x_0)+f(x_0)$

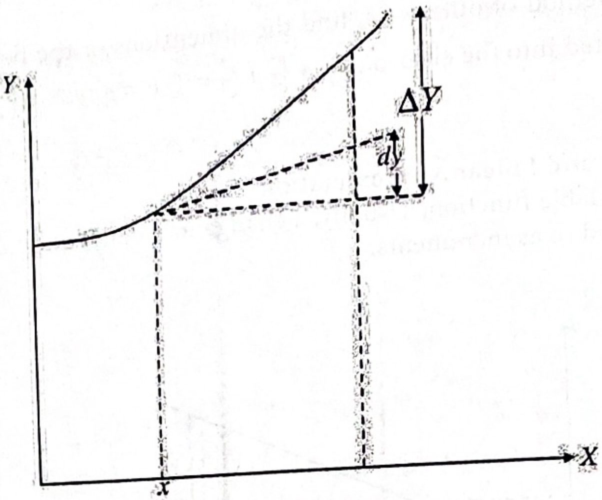
Define $\Delta x=x-x_0$ so that $x=\Delta x+x_0$. We get $f(x)=f(x_0+\Delta x)=f(x_0)+f'(x_0)\Delta x$

This equation is called the linear approximation of $f(x)$ near x_0. At times it enables us to calculate $f(x_0+\Delta x)$ which is difficult or tedious to calculate whereas $f(x_0)$ and $f'(x_0)$ are easy to compute.

Example: Determine the appropriate value of $\sqrt{4.1}$

Solution: Let $f(x)=\sqrt{x}$. Set $x_0=4$. $\Delta x=0.1$. Now $f(x)=\sqrt{x}\Rightarrow f'(x)=\frac{1}{2\sqrt{x}}$. from $f(x_0+\Delta x)=f(x_0)+f'(x_0)\Delta x$, we have $f(4.1)=f(4)+f'(4)(0.1)=\sqrt{4.1}=\sqrt{4}+\frac{1}{2\sqrt{4}}(0.1)=2+\frac{0.1}{4}=2+0.25$.

$\therefore \sqrt{4.1}=2.025$



1. **Error Propagation**

Suppose $y = f(x)$, then an error recorded in x is transmitted to y. this implies y has an error as well. Such transmission of error is called ***Error of Propagation***. Problems in error propagation are usually handled by the help of the following formula.

$$\Delta y \cong dy \qquad (1)$$

1. Follows easily from the definition of derivatives, for as $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$, if $\Delta x = dx$, we have $\lim_{dx \to 0} \frac{\Delta y}{dx} = \frac{dy}{dx}. \Rightarrow \Delta y \cong dy$ as required.

Definition: If the exact quantity of an object is P, and the measured or calculated error is ΔP, then we define the ***relative error*** in calculation as dP. Thus Relative Error = $\frac{\Delta P}{P} = \frac{dP}{P}$.
Thus Relative Error = $\frac{dP}{P}$. Expressed as a percentage, this is called the percentage relative error.
Example: Suppose the radius of a sphere is measured as 20cm and has a probable measurement error of ±0.05cm. Find (i) the possible error in the volume, (ii) the relative error in the radius; and (iii) the percentage error in the volume. (Take $\pi = 3.1$).
Solution: If V is the volume of sphere of radius r then $V = \frac{4}{3}\pi r^3$.

1. $\Delta v = dv = \frac{4}{3}\pi . 3r^2 dr = 4\pi r^2 \bullet dr = 4(3.1) \times 400(\pm 0.05) = \pm 248cm^3$. Possible error of value = $+248\text{cm}^3$.

2. For radius, relative error = $\frac{dr}{r} = \pm\frac{0.05}{20} = \pm 0.0025cm$

3. Relative error in volume = $\frac{dv}{v} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r} = \frac{3(\pm 0.05)}{20} \pm 3(0.0025) = \pm 0.0075$.

Percentage error = 0.75%
Exercises

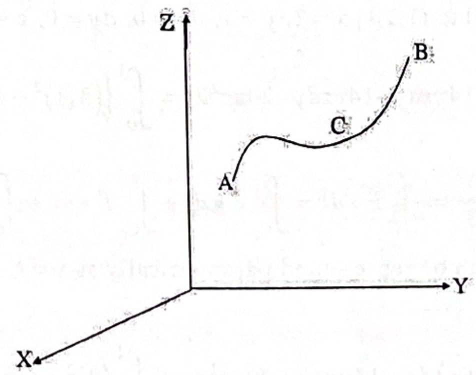
1. Let $y = \frac{1}{x}$. (a) Find Δy if $\Delta x = 0.5$ and the initial value of x is 1. (b) Find dy if $dx = 0.5$ and the initial value of is x= 1. (c) Make a sketch of $y = \frac{1}{x}$ showing Δy and dy respectively.

2) The side of a square is measured to be 10cm with a possible error of ±0.1cm. (a) Use differentials to estimate the error in the calculated area. (b) Estimate the % error in the side and area.

CHAPTER FOUR – Evaluation of Line and Multiple Integrals.

Line Integrals

Motivation: Let's imagine a thin wire shaped like the smooth curve C in the figure shown. Suppose that the wire has variable density given at the point (x,y,z) by the known continuous function $f(x,y,z)$ in units such as grams per (linear) centimetre.



We wish to compute the total mass of the wire along the length A to B. Line integration is our working tool in such a situation.

Let $\bar{r}(t) = x(t)i + y(t)j + z(t)k$, where $\bar{r}$ is the position vector of (x,y,z); defines a curve C joining points P_1 and P_2 where $t = t_1$ and $t = t_2$ respectively. Let $F(x,y,z) = f_1 i + f_2 j + f_3 k$ be a vector function of position (such as the density in one illustration above) defined and continuous long C. then the line integral of F over C is given by

$\int_c \bar{F}(x) \bullet d\bar{r} = \int_{P_1}^{P_2} F \bullet dr = \int_c (f_1 dx + f_2 dy + f_3 dz)$ where $X = (x,y,z)$

If C is piecewise smooth but not wholly smooth, then C is made up of a number of 'sections' each of which is smooth since C is a continuous; these sections are joined (i.e. C is composed of a finite number of curves for each of which $r(t)$ has continuous derivatives. If C is made up of n smooth curves $C_1, C_2, C_3, \cdots, C_n$, then

$$\int \bar{F}(r) \bullet d\bar{r} = \int_{C_1} \bar{F} \bullet d\bar{r} + \int_{C_2} \bar{F} \bullet d\bar{r} + \int_{C_3} \bar{F} \bullet d\bar{r} + \cdots + \int_{C_n} \bar{F} \bullet d\bar{r}$$

Example: $F = \left(3x^2 + 6y\right)i - 14yzi + 20xz^2$. Evaluate $\int \bar{F} \bullet d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths:

1. $x = t,\ y = t^2,\ z = t^3$.
2. The straight lines from (0,0,0) to (1,0,0) then to (1,1,0) and then to (1,1,1).
3. The straight line joining (0,0,0) to (1,1,1)

Solution

$\int \bar{F} \bullet d\bar{r} = \int_C \left[\left(3x^2 + 6y\right)i - 14yzj + 20xz^2 k\right] = \int_C \left[\left(3x^2 + 6y\right)dx - 14zdy + 20xz^2 dz\right]$

(a) $x = t,\ y = t^2,\ z = t^3$. $\therefore \int_{C_1} \bar{F} \bullet d\bar{r} = \int_{t_1=C}^{t=1} \left[\left(3t^2 + 6t^2\right)dt - 14\left(t^2\right)\left(t^3\right)d\left(t^2\right) + 20(t)\left(t^3\right)^2 d\left(t^3\right)\right]$

$$= \int_0^1 \left(9t^2 - 28t^6 + 60t^9\right)dt = \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1 = 5$$

b) (i) Along the line from (0,0,0) to (1,0,0); $y = 0, z = 0, dy = 0, dz = 0$; while x varies from 0 to 1.

$\int_{C_1} \bar{F} \bullet d\bar{r} = \int_{C_1} \left(3x^2 + 6(y)\right)dx - 14yzdy + 20xz^2 dz = \int_{C_1} \left[\left(3x^2 + 6(0)\right)dx - 14(0)(0)(0) + 20x(0)^2(0)\right]$
$= \int_0^1 3x^2 dx = \left[x^3\right]_0^1 = 1$

(ii) Along the line from (1,0,0) to (1,1,0) $x = 1$ constant, $z = 0$, while y varies from 0 to 1 ; $dx = 0$, $dz = 0$.

$$\therefore \int_{C_2} \bar{F} \bullet d\bar{r} = \int_{C_2} \left(3x^2 + 6y\right) dx - 14yzdy + 20xz^2 dz = \int_0^1 \left(3(1)^2 + 6(1)y\right) \times 0 - 14y(0)\,dy + 20(1)(0)^2 0 = 0$$

(iii) Along the line from (1,1,0) to (1,1,1) $x = 1$, $y = 1$, $dx = 0$, $dy = 0$, z varies from 0 to 1.

$$\int_{C_3} \bar{F} \bullet d\bar{r} = \int_{z=0}^1 \left(3x^2 + 6y\right) dx - 14yzdy - 14yzdy + 20xz^2 dz = \int_0^1 \left[\left(3(1)^2 - 6(1) \times 0 - 14(1)(0) + 20(1)z^2 dz\right)\right]$$

$$= \int_0^1 20z^2 dz = \left[\frac{20z^3}{3}\right]_0^1 = \frac{20}{3} \Rightarrow \int_C \bar{F} \bullet d\bar{r} = \int_{C_1} \bar{F} \bullet d\bar{r} + \int_{C_2} \bar{F} \bullet d\bar{r} + \int_{C_3} \bar{F} \bullet d\bar{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

(c) From (0,0,0) to (1,1,1). C can be represented parametrically as $x = t$, $y = t, z = t$.

$$\int_C \bar{F} \bullet d\bar{r} = \int_{z=0}^1 \left(3x^2 + 6y\right) dx - 14yzdy + 20xz^2 dz = \int_0^1 \left(3t^2 + 6t\right) dt - 14t^2 dt + 20t^3 dt$$

$$= \left[t^3 + 3t^2 - \frac{14}{3}t^3 + 5t^4\right]_0^1 = 1 + 3 - \frac{14}{3} + 5 = 9 - \frac{14}{3} = \frac{27}{3} - \frac{14}{3} = \frac{13}{3} = \frac{13}{3}.$$

Theorem: If $F = \nabla\phi$ for some single valued scalar function ϕ in some region R of space defined by $a_1 \le x \le a_2$, $b_1 \le y \le b_2$, $c_1 \le z \le c_2$, and $\nabla\phi$ has a continuous derivative in R, then

1. $\int_{P_1}^{P_2} \bar{F} \bullet d\bar{r}$ as independent of the path C in R joining P_1 and P_2
2. $\bar{F} \bullet d\bar{r} = 0$ around any closed curve C in R.

In such a case, F is called a conservative vector field and ϕ its scalar potential.
Definition: A vector field is conservative if and only if $\nabla \wedge \bar{F} = \bar{0}$ or equivalently $\bar{F} = \nabla\phi$ and $\bar{F} \bullet d\bar{r} = f_1 dx + f_2 dy + f_3 dz = d\phi$. We can rephrase of a function ϕ, then

1. For any piecewise smooth curve C, the line integral (a) $\int_C \bar{F} \bullet d\bar{r}$ is independent of the path and (b) for any piecewise smooth curve C in D starting at x_0 and ending at x_1. $\int \bar{F} \bullet d\bar{r} = \phi(x_1) - \phi(x_0)$; where $\bar{F} = \nabla\phi$. i.e. the value of the integral depends on the end points of C.

Corollary: Let $F = P(x,y)i + Q(x,y)j$. Thus if $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are all continuous in an open disk containing C, and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (i.e. second mixed partials of ϕ) then $\int_C \bar{F} \bullet d\bar{r}$ is independent of the path C. i.e. $\bar{F}$ is a conservative vector field.
Recall: $\bar{F}$ is a conservative vector field if $\nabla \wedge \bar{F} = \bar{0}$.
Example: Let $\bar{F}(x,y) = \left(4x^3y^3 + \frac{1}{x}\right)i + \left(3x^4y^2 - \frac{1}{y}\right)j$. Calculate $\int_C \bar{F} \bullet d\bar{r}$for any smooth curve from (1, 1) to (2, 3).
Solution: $\frac{\partial P}{\partial y} = 12x^3y^2$ and $\frac{\partial Q}{\partial x} = 12x^3y^2 = \frac{\partial P}{\partial y}$. So $\bar{F}$ is a conservative vector field. We find ϕ such that $\bar{F} = \nabla\phi$. i.e. $\frac{\partial \phi}{\partial x} = P$ and $\frac{\partial \phi}{\partial y} = Q$

$$\phi = \int \left(4x^3y^3 + \frac{1}{x}\right) dx + g(y) \Rightarrow \phi = x^4y^3 + \ln|x| + g(y) \quad \cdots \quad (1)$$

Now $\frac{\partial \phi}{\partial y} = 3x^4y^2 + g'(y) = Q$.

Equating to Q to given equation, we have

$$3x^4y^2 + g'(y) = 3x^4y^2 - \frac{1}{y} \Rightarrow g'(y) = -\frac{1}{y} \Rightarrow g(y) = -\ln y = \ln\left|y^{-1}\right| = \ln\left|\frac{1}{y}\right|$$

From (1), $\phi = x^4y^3 + \ln|x| + \ln\left|\frac{1}{y}\right|$

$$\phi = x^4y^3 + \ln\left|\frac{x}{y}\right|$$

$$\therefore \int_C \bar{F} \bullet d\bar{r} = x^4y^3 + \ln\left|\frac{x}{y}\right| = \left(2^4\right)\left(3^3\right) + \ln\frac{2}{3} - 1^4 1^3 + \ln 1 = 16 \times 27 + \ln\frac{2}{3} - 1 = 431 + \ln\frac{2}{3}.$$

Note: If $\bar{F}$ is a force of a particle moving along C, the line integral represents the work done by the force. If C is a closed curve i.e. a curve which does not intersect itself, the integral around C is often denoted by $\int_C \bar{F} \bullet d\bar{r} = \int_C (f_1 dx + f_2 dy + f_3 dz)$

Exercises

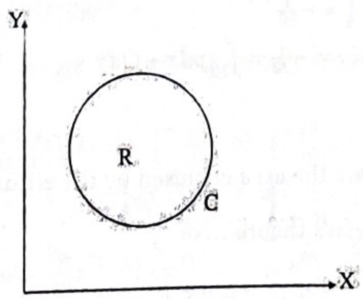
1. (a) Show that $\bar{F} = \left(2xy + z^3\right)i + x^2 j + 3xxz^2 k$ is a conservative force field.

(b) Find the scalar potential (ϕ)

(c) Find the work done by the force in moving an object in this field for $(1, -2, 1)$ to $(3, 1, 4)$.

1. Calculate $\int \bar{F} \cdot d\bar{r}$ where $\bar{F}(x, y) = xyi + ye^x j$ and Cis the rectangle joining (0, 0), (2, 0), (2, 1) and (0, 1) if C is traversed in the counter clockwise direction.
2. Prove that $\bar{F} = y^2(\cos x)i + (2y\sin x - 4)j + \left(3x^2z + 2\right)k$ is a conservative force field.
3. Find the scalar potential of $\bar{F}$
4. Find the work done in moving an object from (0, 1, -1) to $\left(\frac{\pi}{2}, -1, 2\right)$

Green's Theorem: This gives an interesting relationship between line integrals and double integrals.

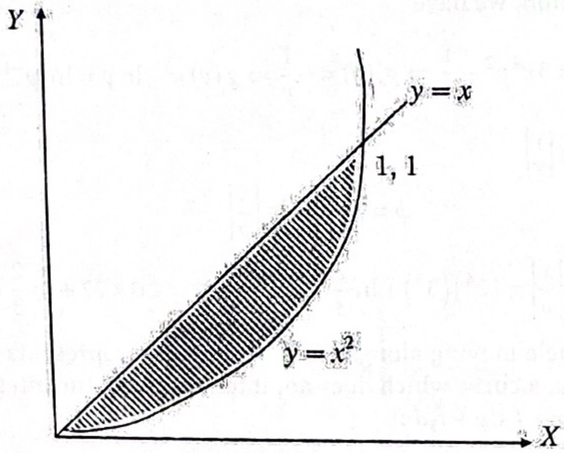


Theorem: Let R be a region in the xy-plane and let C be the boundary of R with C piecewise smooth. Let P and Q be continuous first partial derivatives in an open disk contained in R. Then

$$\oint_C \bar{F} \bullet d\bar{r} = \oint_C (Pdx + Qdy) = \int\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

i.e. under suitable conditions, the line integral around C $\left(\int_C pdx + qdy\right)$ in the xy-plane is equal to the double integral $\int\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$ over the region R that lies inside C.

Example: Verify Green's theorem for $\left(xy + y^2\right)dx + x^2 dy$. C is the close curve of the region bounded by $y = x$ and $y = x^2$.



Solution: $y = x$ and $y = x^2$ intersect when $x^2 = x$ i.e. $x^2 - x = 0 = x(x-1) = 0,\ x = 0,\ x = 1$. when $x = 0,\ y = 0$, when $x = 1,\ y = 1$.

Along $y = x^2$, the line integral is $\int_C \bar{F} \bullet d\bar{r} = \int_0^1 \left[\left(x \bullet x^2 + x^4\right)dx + x^2 \bullet 2x\ dx\right] = \int_0^1 \left(3x^3 + x^4\right)dx$ $= \left[\frac{3}{4}x^4 + \frac{x^5}{5}\right]_0^1 = \frac{19}{20}$

Along $y = x$ from $(1, 1)$ to $(0, 0)$ the line integral is

$$\int_C \bar{F} \bullet d\bar{r} = \int_{01}^{0}\left(\left(x \bullet x + x^2\right)dx + x^2 dx\right) = \int_{01}^{0}\left[3x^2 dx = x^3\right]_0^1 = -1 \therefore \int_C = \int_{C_1} + \int_{C_2} = \oint\left[\left(xy + y^2\right)dx + x^2 dy\right]$$

$$= \frac{19}{20} - 1 = -\frac{1}{20}$$

$$P = xy + y^2 \Rightarrow \frac{\partial P}{\partial y} = x + 2y, \quad Q = x^2 \Rightarrow \frac{\partial Q}{\partial x} = 2x$$

$$\iint_R 2x - (x + 2y)\,dxdy = \int_0^1 \int_{x^2}^{x} (x - 2y)\,dydx = \int_0^1 \int_y^{\sqrt{y}} (x - 2y)\,dxdy$$

$$I = \int_0^1 \left[xy - y^2\right]_{x_2}^{x} dx = \int_0^1 \left(x^2 - x^2 - x^3 - x^4\right)dx = \int_0^1 \left(x^4 - x^3\right)dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

$$\therefore \iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = -\frac{1}{20} = \oint_{20} p dx + Q dy$$

Exercises

1. Use Green's theorem to calculate the area enclosed by the ellipse $\frac{x^2}{x^2} + \frac{y^2}{b^2} = 1$.
2. Find the line integral using Green's theorem, of
   (a) $3ydx + 5xdy : C = (x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1$
   (b) $\oint e^x \cos y\ dx + e^x \sin y\ dy$: C is the region enclosed by the triangle with vertices at (0, 0), (1, 0), (0, 1).

2009-2010 MTH 242 EXAMINATION QUESTIONS AND SOLUTION

1. (*a*) Differentiate $f(x) = x^2 - 5x$ from the first principle.

ANS: $f(x) = x^2 - 5x,\ f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \frac{(x+h)^2 - 5(x+h) - (x^2-5x)}{h}$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 5x - 5h - x^2 + 5x}{h} = \lim_{h \to 0} \frac{h(2x+h-5)}{h} = \lim_{h \to 0} 2x - 5 + h = 2x - 5.$$

(*b*) Find the equation of normal to $f(x) = 2x^3 - x^2 + 3x - 1$ at $x = 0$.

ANS: Equation of normal to $f(x) = 2x^3 - x^2 + 3x - 1$ at $x = 0$.

At $x = 0,\ y = f(x) = -1. \Rightarrow (0, -1) = (x_0, y_0),\ f'(x) = 3x^2 - 2x + 3,\ f'(0) = 3 =$ slope of the tangent.

$\therefore$ Slope of the normal $= -\frac{1}{3}$. Equation of normal is $y - y_1 = m(x - x_1)$

$$\Rightarrow y - (-1) = \frac{-1}{3}(x - 0) \Rightarrow y + 1 = \frac{-1}{3}(x - 0) = \frac{-1}{3}x \Rightarrow 3y + x + 3 = 0 \text{ or } y = \frac{-1}{3}x - 1.$$

(*c*) Find the turning point and its nature for (i) $f(x) = 3x - 2x^2 - 3$ (ii) $f(x, y) = xy - x^2 - y^2$.

ANS: (i) Turning point and its nature of $f(x) = 3x - 2x^2 - 3$. $f'(x) = 3 - 4x = 0 \Rightarrow x = \frac{3}{4}$.

$$y = f(x) = 3\left(\frac{3}{4}\right) - 2\left(\frac{3}{4}\right)^2 - 3 = \frac{-15}{8}.\ \therefore \text{Turning point } = \left(\frac{3}{4}, \frac{-15}{8}\right).$$

Nature of Turning point: $f''(x) = -4 < 0$. Therefore $\left(\frac{3}{4}, \frac{-15}{8}\right)$ is a maximum turning point.

$f(x, y) = xy - x^2 - y^2,\quad f_x = y - 2x,\quad f_y = x - 2y.$
At extreme point, $f_x = 0 = f_y$ i.e. $y - 2x = 0,\ -2y + x = 0$.

Solving simultaneously, we get $x = 0,\ y = 0$. So the function has extreme value at (0,0). Extreme value is $f(0, 0) = 0$.

For the Nature: $f_{xx} = -2 = f_{yy} < 0,\ f_{xy} = 1$.

$$(f_{xy})^2 - f_{xx}f_{yy} = 1^2 - (-2)(-2) = -3 < 0.\ \therefore (0,0) \text{ is a maximum value of } f(x, y).$$

2. Evaluate: (a) $\int 2xe^{3x}dx$ (b) $\int_1^3 \frac{x+1}{x^2-4}dx$ (c) $\int_1^\infty \frac{dx}{x^4}$ (d) $\int_0^4 \int_0^5 (3x + y)dydx$.

ANS: (*a*) $\int 2xe^{3x}dx$. Let $u = 2x,\ dv = e^{3x}dx \Rightarrow du = 2dx,\ v = \frac{1}{3}e^{3x}$.

$$\therefore \int u dv = uv - \int v du = \frac{2x}{3}e^{3x} - \int \frac{1}{3}e^{3x} 2dx = \frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + c = \frac{2}{3}e^{3x}\left[x - \frac{1}{3}\right] + c.$$

(b) $\displaystyle\int_1^3 \frac{x+1}{x^2-4}dx.\ \frac{x+1}{x^2-4} = \frac{x+1}{(x+2)(x-2)} = \frac{A}{x-2} + \frac{B}{x+2} \Rightarrow x + 1 = A(x+2) + B(x-2)$

when $x = 2,\ 4A = 3 \Rightarrow A = \frac{3}{4}$, when $x = -2,\ -4B = -1 \Rightarrow B = \frac{1}{4}$.
$\int_1^3 \frac{x+1}{x^2-4}dx = \int_1^3 \frac{3dx}{4(x-2)} + \int_1^3 \frac{dx}{4(x+2)} = \frac{3}{4}\ln(x-2)|_1^3 + \frac{1}{4}\ln(x+2)|_1^3 = \frac{3}{4}[\ln 1 - \ln(-1)] + \frac{1}{4}[\ln 5 - \ln 3]$
$= \frac{1}{4}\ln\left(\frac{5}{3}\right)$.

(c) $\int_1^\infty \frac{dx}{x^4} = \lim\limits_{L \to \infty} \int_1^L x^{-4}dx = \lim\limits_{L \to \infty} \frac{-x^{-3}}{3}\Big|_1^L = \lim\limits_{L \to \infty}\left[\frac{-L^{-3}}{3} - \frac{(-1)^{-3}}{3}\right] = \frac{1}{3}$.

(d) $\int_0^4 \int_0^5 (3x + y)dydx = \int_0^4 \left(3xy + \frac{y^2}{2}\Big|_0^5\right)dx = \int_0^4 \left[15x + \frac{25}{2} - 0\right]dx = \frac{15x^2}{2} + \frac{25x}{2}\Big|_0^4$
$= \frac{15 \times 16}{2} + \frac{25 \times 4}{2} - 0 = 120 + 50 = 170.$

3. (a) If $f(x,y) = f(x,y) = \cos x \sin y + \frac{y^2}{x^2}$, find (i) f_x (ii) f_y (iii) f_{xy}.

ANS: $f(x,y) = \cos x \sin y + \frac{y^2}{x^2}$, $f_x = -\sin x \sin y - \frac{2y^2}{x^3}$, $f_y = \cos x \cos y + \frac{2y}{x^2}$ $f_{yx} = -\sin x \cos y - \frac{4y}{x^3}$

(b) Extremize $f(x,y) = x^2 + 2y^2 + 3$, subject to the constraint $2x - 6y = 11$.

ANS:

$$f(x,y) = x^2 + 2y^2 + 3,\ g(x,y) = 2x - 6y - 11$$
$$\therefore F(x,y,\lambda) = x^2 + 2y^2 + 3 + \lambda(2x - 6y - 11)$$
$F_x = 2x + 2\lambda$, $F_y = 4y - 6\lambda$, $F_\lambda = 2x - 6y - 11$. At extreme point, $F_x = 0 = F_y = F_\lambda$

$$\text{So} \begin{cases} 2x + 2\lambda = 0 & ----------(1) \\ 4y - 6\lambda = 0 & --------(2) \\ 2x - 6y - 11 = 0 & ---------(3) \end{cases}$$

From (1) and (2), $x = -\lambda$, $y = \frac{3}{2}\lambda$. Substitute for x and y in (3), we get

$2(-\lambda) - 6(\frac{3}{2}\lambda) - 11 = 0 \Rightarrow \lambda = -1$. $\therefore x = 1, y = \frac{-3}{2}$(for both x and y).
The extreme value is $f(1, \frac{-3}{2}) = 1^2 + 2(\frac{-3}{2})^2 + 3 = 8\frac{1}{2}$.

(c) (i) State Rolle's theorem

ANS: Rolle's theorem states that if $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b), and if $f(a) = f(b)$, then there exists a point c in (a,b) such that $f'(c) = 0$.

(ii) $f(x) = 2-3x^2+3x$ is continuous on $[0,1] \equiv [a,b]$, $f'(x) = -6x+3$ exists on (0,1). $f(0) = 2 = f(1)$.
$\therefore$ $f(x)$ Satisfies the hypothesis of Rolle's theorem.

$$(iii)\ f'(c) = -6c + 3 = 0 \Rightarrow c = \frac{1}{2}$$

4. (i)

$$F(x,y,z) = (2y^2 + 4x)i - 14xyzj + 2xyk,\ r = xi + yj + zk,\ dr = dxi + dyj + dzk$$
$$F.dr = (2y^2 + 4x)dx - 14xyzdy + 2xydz.$$

$\int_C F \cdot dr$ for C = (0,0,0) to (2,0,0) to (2,2,0) and then to (2,2,2).
Let $C = C_1 + C_2 + C_3$, where C_1 joins (0,0,0) to (2,0,0), C_2 joins (2,0,0) to (2,2,0), C_3 joins (2,2,0) to (2,2,2).

$$\int_{C_1} F \cdot dr = \int_{C_1} 4xdx = \int_0^2 4xdx = 2x^2\Big|_0^2 = 8,\ \int_{C_2} F \cdot dr = \int_{C_2} 0dy = 0$$
$$\int_{C_3} F \cdot dr = \int_0^2 2 \cdot 2 \cdot 2 \cdot dz = 8z\Big|_0^2 = 16$$
$$\therefore \int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr + \int_{C_3} F \cdot dr = 8 + 0 + 16 = 24.$$

(ii) Along the line from (0,0,0) to (1,1,1) represented by $x = 2t$, $y = 2t$, $z = 2t \Rightarrow dx = 2dt = dy = dz$.

For limits : $x = 0, y = 0, z = 0 \Rightarrow t = 0$ $\left(\text{i.e } 2t = 0 \Rightarrow t = \frac{0}{2} = 0\right)$
$x = 1, y = 1, z = 1 \Rightarrow t = \frac{1}{2}$ $\left(\text{i.e } 2t = 1 \Rightarrow t = \frac{1}{2}\right)$.

$$\therefore \int_C F \cdot dr = \int \left(2y^2 + 4x\right)dx + \int -14xyzdy + \int 2xydz$$

$$\int_C F \cdot dr = \int_0^{\frac{1}{2}} \left(2(2t)^2 + 4(2t)\right)2dt - \int_0^{\frac{1}{2}} 14(2t)(2t)(2t)\,2dt + \int_0^{\frac{1}{2}} 2(2t)(2t)\,2dt$$

$$\int_C F \cdot dr = \int_0^{\frac{1}{2}} \left(16t^2 + 16t - 224t^3 + 16t^2\right)dt = \int_0^{\frac{1}{2}} \left(32t^2 + 16t - 224t^3 +\right)dt$$

$$\left[\frac{32}{3}t^3+8t^2-56t^4\right]_0^{\frac{1}{2}}=\frac{32}{3}\left(\frac{1}{2}\right)^3+8\left(\frac{1}{2}\right)^2-56\left(\frac{1}{2}\right)^4=\frac{324}{3}\cdot\frac{1}{\cancel{8}}+\cancel{8}2\cdot\frac{2}{\cancel{4}}-456\cdot\frac{1}{16}$$

$$=\frac{4}{3}-2=\frac{4-6}{3}=-\frac{2}{3}.$$

5. (a) **Taylor series expansion of $f(x)=\cos x$ about $x_0=\frac{\pi}{2}$; $n=6$.**

ANS: $f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\dots$

$$f(x)=\cos x,\ f\left(\tfrac{\pi}{2}\right)=\cos\left(\tfrac{\pi}{2}\right)=0;f'(x)=-\sin x,\ f'\left(\tfrac{\pi}{2}\right)=-\sin\left(\tfrac{\pi}{2}\right)=-1$$

$$f''(x)=-\cos x,\ f''\left(\tfrac{\pi}{2}\right)=-\cos\left(\tfrac{\pi}{2}\right)=0;f'''(x)=\sin x,\ f'''\left(\tfrac{\pi}{2}\right)=\sin\left(\tfrac{\pi}{2}\right)=1$$

$$f^{iv}(x)=\cos x,\ f^{iv}\left(\tfrac{\pi}{2}\right)=\cos\left(\tfrac{\pi}{2}\right)=0;f^{v}(x)=-\sin x,\ f^{v}\left(\tfrac{\pi}{2}\right)=-\sin\left(\tfrac{\pi}{2}\right)=-1$$

$$f^{vi}(x)=-\cos x,\ f^{vi}\left(\tfrac{\pi}{2}\right)=-\cos\left(\tfrac{\pi}{2}\right)=0$$

$$f(x)=0-\left(x-\frac{\pi}{2}\right)+0+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3+0-\frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5+\dots=-\left(x-\frac{\pi}{2}\right)+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3-\frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5+\dots$$

(b) $f(x)=\sqrt{x}$, $f(5)=\sqrt{5}$. Let $x=4$ (the immediate perfect square before 5)

$$f'(x)=\frac{1}{2\sqrt{x}}\Rightarrow f'(x_0)=\frac{1}{2\sqrt{x_0}}$$

$$f(x)=f(x_0+\Delta x)=f(x_0)+f'(x_0)\Delta x$$
$$f(5)=\sqrt{5}=\sqrt{4+1}=f(4)+f'(4)\cdot 1$$
$$\sqrt{4}+\tfrac{1}{2\sqrt{4}}=2+\tfrac{1}{4}=2\tfrac{1}{4}\text{ or }2.25$$

(c) $v=\frac{4}{3}\pi r^3$. $\quad\Delta v=\frac{4}{3}\cdot 3\pi r^2dr=4\cdot(??)\cdot 100\cdot\pm 0.5=\pm 620\text{cm}^3$

$yy=e^{3x}\ x1k=10yy=e^{3x}\ x1k=10$. Relative error in radius$=\frac{dr}{r}=\frac{0.5}{10}=\pm 0.05$

6. (a) $v=\frac{180}{360}\int_0^4\pi y^2dx=\frac{\pi}{2}\int_0^4\left(e^{3x}\right)^2dx$

$$=\frac{\pi}{2}\int_0^4 e^{6x}dx=\frac{\pi}{2}\left[\frac{1}{6}e^{6x}\right]_0^4=\frac{\pi}{12}\left[e^{24}-1\right].$$

(b) $\int_C\left(y^2xdx+x^2ydy\right)=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$

$$Q=x^2y\Rightarrow\frac{\partial Q}{\partial x}=2xy,\ P=xy^2\Rightarrow\frac{\partial P}{\partial y}=2xy.$$

$$\therefore\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\iint_R(2xy-2xy)\,dxdy=0.$$

(c) $w=f(x,y)=\left(x^2+y^2\right)^{\frac{1}{2}}$, $x=r\cos\theta$, $y=r\sin\theta$.

(i) $\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{2x}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}\cdot\cos\theta+\frac{2y}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}\cdot\sin\theta$

$$=\left(x^2+y^2\right)^{-\frac{1}{2}}(x\cos\theta+y\sin\theta)=\left(x^2+y^2\right)^{-\frac{1}{2}}\left(\frac{x^2}{r}+\frac{y^2}{r}\right)=\left(x^2+y^2\right)^{-\frac{1}{2}}\left(\frac{x^2+y^2}{r}\right)=\frac{\left(x^2+y^2\right)^{\frac{1}{2}}}{r}=\frac{r}{r}=1.$$

(ii) $\frac{\partial w}{\partial\theta}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial\theta}=\frac{2x}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}\cdot-r\sin\theta+\frac{2y}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}\cdot r\cos\theta$

$$=\left(x^2+y^2\right)^{-\frac{1}{2}}(-xr\sin\theta+ry\cos\theta)=\left(x^2+y^2\right)^{-\frac{1}{2}}\left(-r^2\cos\theta\sin\theta+r^2\sin\theta\cos\theta\right)=0.$$

2006-2007 MTH 242 EXAMINATION QUESTIONS AND SOLUTION

1. (a) When do we say that a function $f(x,y)$ is harmonic?

ANS: A function $f(x,y)$ is harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

(ii) Show that $f(x,y) = \sin(nx)e^{-ny}$ is a harmonic function.

ANS: $f(x,y) = \sin(nx)e^{-ny}$, $f_x = n\cos(nx)e^{-ny}$, $f_{xx} = -n^2\sin(nx)e^{-ny}$, $f_y = -n\sin(nx)e^{-ny}$, $f_{yy} = n^2\sin(nx)e^{-ny}$.

$$\therefore f_{xx} + f_{yy} = -n^2\sin(nx)e^{-ny} + n^2\sin(nx)e^{-ny} = 0$$
$$\Rightarrow f(x,y) \text{ is harmonic.}$$

(b) Find and determine the nature of the turning points of the following functions:

(i) $f(x) = 3x^2 + 12x - 8$ (ii) $f(x,y) = x^2 + xy - x + y^2$

ANS: (i) $f(x) = 3x^2 + 12x - 8$, $f'(x) = 6x - 12$. At turning point, $f'(x) = 0$

$$\Rightarrow 6x + 12 = 0 \Rightarrow 6x = -12 \Rightarrow x = -2.$$
$$f(-2) = 3(-2)^2 + 12(-2) - 8 = 12 - 24 - 8 = -20.$$
$$\therefore \text{turning point is at } (-2,-20).$$

Nature of turning point: $f''(x) = 6 \Rightarrow f''(-2) = 6 > 0 \Rightarrow (-2,-20)$ is a minimum point.

(ii) $f(x,y) = x^2 + xy - x + y^2$, $f_x = 2x + y - 1$, $f_y = x + 2y = 0$.

At turning point $\nabla f = 0 \Rightarrow \begin{cases} 2x + y - 1 = 0 \cdots \text{(i)} \\ x + 2y = 0 \quad \cdots \text{(ii)} \end{cases}$

From (ii) $x = -2y$, substituting in (i) $\Rightarrow 2(-2y) + y - 1 = 0 \Rightarrow -4y + y = 1 \Rightarrow -3y = 1 \Rightarrow y = -\frac{1}{3}$.

From (ii) $x = -2y = -2\left(-\frac{1}{3}\right) = \frac{2}{3}$ $\therefore$ the turning point is $\left(\frac{2}{3}, -\frac{1}{3}\right)$.

$$f_{xx} = 2 > 0,\ f_{yy} = 2 > 0,\ f_{xy} = 1$$

Nature of turning point: $\therefore \left(f_{xy}\right)^2 - f_{xx}\cdot f_{yy} = 1^2 - 2\cdot 2 = 1 - 4 = -3 < 0$

$$\Rightarrow \left(\frac{2}{3}, -\frac{1}{3}\right) \text{ is a minimum point.}$$

(c) Find the equation of (i) the tangent (ii) the normal to the curve $y = x^2 + 5x - 2$ at $x = 4$.

ANS: $y = x^2 + 5x - 2$ at $x = 4$ $y' = 2x + 5 \Rightarrow y'(4) = 2(4) + 5 = 13$, which is gradient of tangent i.e. m. Also $x_1 = 4 \Rightarrow y_1 = {x_1}^2 + 5x_1 - 2 = 4^2 + 5(4) - 2 = 34$.

(i) Equation of tangent is that of a straight line given by $y - y_1 = m(x - x_1) \Rightarrow y - 34 = 13(x-4) = 13x - 52 \Rightarrow y = 13x - 52$.

(ii) Equation of normal: $m = -\frac{1}{13}$ (since normal is perpendicular to tangent)

$$y - y_1 = m(x - x_1) \Rightarrow y - 34 = -\frac{1}{13}(x-4) \Rightarrow 13(y-34) = -x + 4 \Rightarrow 13y - 442 = x + 4 \Rightarrow 13y + x - 446 = 0$$

2. (a) (i) State the Mean-Value theorem.

ANS: If the function f is continuous on $[a,b]$ and differentiable on (a,b). Then $\exists\, c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

(ii) Does $f(x) = (1-x)^{\frac{2}{3}}$ satisfy the hypothesis of the Mean-Value theorem on $[0,2]$

ANS: (o) $f(x) = (1-x)^{\frac{2}{3}}$ *is continuous on* $[0,2]$ since $f(c) = \lim_{x\to c} f(x)$ for all points $c \in [0,2]$

(oo) f is diiferentiable on $(0,2)$ since $f'(x) = -\frac{2}{3}(1-x)^{-\frac{1}{3}}$ exists $\forall$ points $x \in (0,2)$

$(\circ\circ\circ)$ $\therefore \exists\ c \in (0,2) \ni f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0} \Rightarrow -\frac{2}{3}(1-c)^{-\frac{1}{3}} = \frac{(1-2)^{\frac{2}{3}}-(1)^{\frac{2}{3}}}{2} = 0$

$\Rightarrow \frac{2}{3}(1-c)^{-\frac{1}{3}} = 0 \Rightarrow 1-c=0 \Rightarrow c=1 \in (0,2)$, so f satisfies the hypothesis of MVT

(b) Find the Taylor's Series expansion of $f(x)=e^{\frac{x}{3}}$ about $x=3$. Take $n=4$.

ANS: Taylor series expansion of $f(x)=e^{\frac{x}{3}}$ about $x=3$; $n=4$.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

$$f(x)=e^{\frac{x}{3}},\ f(3)=e^{\frac{3}{3}}=e;\ f'(x)=\frac{1}{3}e^{\frac{x}{3}}\ f'(3)=\frac{1}{3}e^{\frac{3}{3}}=\frac{1}{3}e$$

$$f''(x)=\frac{1}{3^2}e^{\frac{x}{3}},\ f''(3)=\frac{1}{3^2}e^{\frac{3}{3}}=\frac{1}{3^2}e;f'''(x)=\frac{1}{3^3}e^{\frac{x}{3}},\ f'''(3)=\frac{1}{3^3}e^{\frac{3}{3}}=\frac{1}{3^3}e$$

$$f^{IV}(x)=\frac{1}{3^4}e^{\frac{x}{3}},\ f^{IV}(3)=\frac{1}{3^4}e^{\frac{3}{3}}=\frac{1}{3^4}$$

$$\therefore f(x) = e + \frac{e}{3}(x-3) + \frac{e}{3^2}\frac{(x-3)^2}{2!} + \frac{e}{3^3}\frac{(x-3)^3}{3!} + \frac{e}{3^4}\frac{(x-3)^4}{4!} + \cdots$$

(c) Evaluate(i) $\int \frac{x}{x^2+4}dx$ (ii) $\int \frac{x+1}{x^2-3x+2}dx$

ANS:(i) $\int \frac{x}{x^2+4}dx$. Let $u=x^2+4 \Rightarrow \frac{du}{dx}=2x \Rightarrow dx=\frac{du}{2x}$

$$\text{Substituting} : \Rightarrow \int \frac{x}{x^2+4}dx = \int \frac{\cancel{x}}{u}\frac{du}{2\cancel{x}} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u + c = \frac{1}{2}\ln\left(x^2+4\right)+c$$

(ii) $\int \frac{x+1}{x^2-3x+2}dx = \int \frac{x+1}{(x-1)(x-2)}dx$. But $\frac{x+1}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$

$\Rightarrow x+1 = A(x-2)+B(x-1)\ \cdots (*)$, $x=2$ in $(*) \Rightarrow B=1$, $x=1$ in $(*) \Rightarrow A=-2$

$$\int \frac{x+1}{x^2-3x+2}dx = \int \frac{x+1}{(x-1)(x-2)}dx = \int\left[\frac{A}{(x-1)}+\frac{B}{(x-2)}\right]dx = \int\left[\frac{-2}{(x-1)}+\frac{3}{(x-2)}\right]dx$$

$$-2\int \frac{1}{x-1}dx + 3\int\frac{1}{x-2}dx = -2\ln(x-1)+3\ln(x-2)+c = \ln(x-1)^{-2}+\ln(x-2)^3+c = \ln\frac{(x-2)^3}{(x-1)^2}+c$$

3. (a) The area enclosed by the curve $y=4x-x^2$ and x-axis from 0 to $x=3$ is rotated about x-axis through 270°. Find the volume of solid generated.

ANS: $y=4x-x^2=0 \Rightarrow x(4-x)=0 \Rightarrow x=0$ or 4

$$y\,x10y\,x10v = \frac{270}{360}\int_0^4 \pi y^2 dx = \frac{3\pi}{4}\int_0^3 \left(4x-x^2\right)^2 dx$$

$$= \frac{3\pi}{4}\int_0^3\left(16x^2-8x^3+x^4\right)dx$$

$$= \frac{3\pi}{4}\left[\left(\frac{16}{3}x^3-2x^4+\frac{x^5}{5}\right)\right]_0^3 = \frac{3\pi}{4}\left(\frac{16}{3}\cdot 3^3-2\cdot 3^4+\frac{3^5}{5}\right)$$

$$\frac{3\pi}{4}\left(\frac{16}{\cancel{3}}\cdot 3^{\cancel{3}2}-2\cdot 81+\frac{243}{5}\right) = \frac{3\pi}{4}\left(\frac{720-810+243}{5}\right) = \frac{459\pi}{20}\text{cubic units}$$

(b) Evaluate (i) $\int_{-\infty}^{\infty}\frac{xdx}{(x^2+3)^2}dx$ (ii) $\int_1^2 xe^{3x}dx$

ANS: (i) $\int_{-\infty}^{\infty} \frac{xdx}{(x^2+3)^2}dx$. Let $u = x^2 + 3 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$.

Substituting:$\Rightarrow \int_{-\infty}^{\infty} \frac{xdx}{(x^2+3)^2}dx = \int_{-\infty}^{\infty} \frac{\cancel{x}}{u^2}\frac{du}{2\cancel{x}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{du}{u^2} = \frac{1}{2}\left[\lim_{L\to\infty}\int_0^L \frac{du}{u^2} + \lim_{L\to-\infty}\int_L^0 \frac{du}{u^2}\right]$

$$= \frac{1}{2}\left[\lim_{L\to\infty}\int_0^L u^{-2}du + \lim_{L\to-\infty}\int_L^0 u^{-2}du\right] = \frac{1}{2}\left[\lim_{L\to\infty} -\frac{1}{u}\Big|_0^L + \lim_{L\to-\infty} -\frac{1}{u}\Big|_L^0\right] = 0.$$

(ii) $\int_1^2 xe^{3x}dx$. Let $u = x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$. $dv = e^{3x}dx \Rightarrow \int dv = \int e^{3x}dx \Rightarrow v = \frac{1}{3}e^{3x} + c$

But $\int udv = uv - \int vdu \Rightarrow \frac{x}{3}e^{3x}\Big|_1^2 - \frac{1}{3}\int_1^2 e^{3x}dx = \left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_1^2 = \frac{2}{3}e^6 - \frac{1}{9}e^6 - \frac{1}{3}e^3 + \frac{1}{9}e^3$

$$= \frac{6e^6 - e^6}{9} + \frac{e^3 - 3e^3}{9} = \frac{1}{9}\left(5e^6 - 2e^3\right)$$

4. (a) If $f(x,y) = \sin^2 x\cos y + \frac{x}{y^2}$, find (i) f_x (ii) f_y (iii) f_{xy}

ANS: (i) $f(x,y) = \sin^2 x\cos y + \frac{x}{y^2}$, $f_x = \frac{\partial}{\partial x}\left(\sin^2 x\cos y + \frac{x}{y^2}\right) = 2\sin x\cos x\cos y + \frac{1}{y^2}$.

$f_y = \frac{\partial}{\partial y}\left(\sin^2 x\cos y + \frac{x}{y^2}\right) = -\sin^2 x\sin y - \frac{2x}{y^3}$,

$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2\sin x\cos x\cos y + \frac{1}{y^2}\right) = -2\sin x\cos x\sin y - \frac{2}{y^3}$

(b) (i) Find the point (x,y) that makes $f(x,y) = x^2 + 4y^2 + 6$ subject to the constraint $2x - 8y = 0$ to be extreme. (ii) Find the extreme value and nature of it.

ANS: $f(x,y) = x^2 + 4y^2 + 6$, $g(x,y) = 2x - 8y$

$$F(x,y,\lambda) = f(x,y) - \lambda g(x,y) = x^2 + 4y^2 + 6 - \lambda(2x - 8y)$$

$F_x = 2x - 2\lambda$, $F_y = 8y + 8\lambda$, $F_\lambda = 2x - 8y$. At critical point $\nabla F = 0$

$$\Rightarrow \begin{cases} 2x - 2\lambda = 0 \cdots (1) \Rightarrow x = \lambda = 0 \\ 8y + 8\lambda = 0 \cdots (2) \Rightarrow y = -\lambda = 0 \\ 2x - 8y = 0 \cdots (3) \Rightarrow 10\lambda = 0 \Rightarrow \lambda = 0 \end{cases}$$

So the point $(x,y) = (0,0)$ is the extreme point.The extreme value of $f(x,y)$ is $0^2 + 4.0^2 + 6 = 6$

Nature of extreme point: $f(x,y) = x^2 + 4y^2 + 6$, $f_x = 2x$, $f_{xx} = 2 > 0$; $f_y = 8y$, $f_{yy} = 8 > 0$

$f_{xy} = 0 = f_{yx}$. $\therefore \left(f_{xy}\right)^2 - f_{xx}\cdot f_{yy} = 0 - 2\cdot 8 = -16 < 0 \Rightarrow (0,0)$ is a minimum point.

(c) Evaluate (i) $\int_0^1\int_0^{\sin x} ydydx$ (ii) $\int_0^4\int_0^5 (x+y)dxdy$

ANS: (i) $\int_0^1\int_0^{\sin x} ydydx = \int_0^1\left(\frac{y^2}{2}\Big|_0^{\sin x}\right)dx = \frac{1}{2}\int_0^1 \sin^2 xdx = \frac{1}{2}\left[\frac{1}{2}\int_0^1 (1 - \cos 2x)dx\right] = \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^1$
$= \frac{1}{4}(1 - \sin 2)$

(ii) $$\int_0^4\int_0^5 (x+y)dxdy = \int_0^4\left(\left[\frac{x^2}{2} + xy\right]_0^5\right)dy = \int_0^4\left(\frac{25}{2} + 5y\right)dy$$

$$= \left[\frac{25y}{2} + \frac{5y^2}{2}\right]_0^4 = \frac{25\cdot 4}{2} + \frac{5\cdot 4^2}{2} = \frac{180}{2} = 90$$

5. (a) Show that $F(x,y,z) = (2xy+z^3)i + x^2 j + 3xz^2 k$ is a conservative force field.

ANS: $F$is conservative if $\nabla \times F = 0$. $\nabla \times F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$

$$= i\left(\frac{\partial}{\partial y}(2xy+z^3) - \frac{\partial}{\partial z}(x^2)\right) - j\left(\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy+z^3)\right) + k\left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy+z^3)\right)$$

$$i(0) - j(3z^2 - 3z^2) + k(2x - 2x) = 0 \Rightarrow F \text{ is conservative}$$

(b) Find the scalar potential Φ.

ANS: $F = \nabla\Phi \Rightarrow (2xy+z^3)i + x^2 j + 3xz^2 k = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} \Rightarrow \begin{cases} \frac{\partial\Phi}{\partial x} = 2xy+z^3 & \cdots (1) \\ \frac{\partial\Phi}{\partial y} = x^2 & \cdots (2) \\ \frac{\partial\Phi}{\partial z} = 3xz^2 & \cdots (3) \end{cases}$

$$\text{From (1)}: \int \frac{\partial\Phi}{\partial x} dx = \int (2xy+z^3)dx \Rightarrow \Phi = \int (2xy+z^3)dx = x^2y + xz^3 + g(y,z)$$

$$\frac{\partial\Phi}{\partial y} = x^2 + g_y(y,z) = x^2 \Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z) = h(z) \therefore \Phi = x^2y + xz^3 + h(z)$$

$$\frac{\partial\Phi}{\partial z} = 3z^2x + h'(z) = 3z^2x \Rightarrow h'(z) = 0 \Rightarrow h(z) = c \therefore \Phi(x,y,z) = x^2y + xz^3 + c$$

(c) Find the work done by the force in moving a particle in the field from $(1,-2,1)$ to $(3,1,4)$

ANS: Work done from $(1,-2,1)$ to $(3,1,4)$ equal to

$$\int_{(1,-2,1)}^{(3,1,4)} F \cdot dr = \Phi(x,y,z)\Big|_{(1,-2,1)}^{(3,1,4)} = (x^2y + xz^3)\Big|_{(1,-2,1)}^{(3,1,4)} = (3^2 \cdot 1 + 3 \cdot 4^3) - ((1^2 \cdot -2) + 1 \cdot 1^3) = 202 \text{ Joules.}$$

6. Let $F(x,y,z) = (3x^2+6y)i - 14yzj + 20xz^2k$. Evaluate $\int_C F \cdot dr$ from $(0,0,0)$ to $(2,2,2)$ along

(i) the straight lines joining $(0,0,0)$ to $(2,0,0)$ then to $(2,2,0)$ to $(2,2,2)$

ANS: $F(x,y,z) = (3x^2+6y)i - 14yzj + 20xz^2k$. $\int_C F \cdot dr$ from $(0,0,0)$ to $(2,2,2)$

$$\int_C F \cdot dr = \int_{(0,0,0)}^{(2,2,2)} \left((3x^2+6y)dx - 14yzdy + 20xz^2dz\right) = \int_0^2 3x^2dx + \int_0^2 0 \cdot dy + \int_0^2 40z^2dz$$
$$= x^3\Big|_0^2 + \frac{40z^3}{3}\Big|_0^2 = 8 + \frac{320}{3} = \frac{344}{3}$$

(ii) the straight line represented parametrically by $x = 2t,\ y = 2t,\ z = 2t$

ANS: $x = 2t,\ y = 2t,\ z = 2t,\ dx = 2dt = dy = dz$. Also, $x = 0 = y = z, t = 0$ and $x = 2 = y = z, t = 1$

$$\therefore \int_C F \cdot dr = \int_{(0,0,0)}^{(2,2,2)} \left((3x^2+6y)dx - 14yzdy + 20xz^2dz\right)$$
$$= \int_0^1 \left[(3(2t)^2 + 6 \cdot 2t)2dt - 14 \cdot 2t \cdot 2t \cdot 2dt + 20 \cdot 2t \cdot (2t)^2 \cdot 2dt\right]$$
$$= \int_0^1 \left[24t^2 + 24t - 112t^2 + 320t^3 2\right]dt = \int_0^1 \left[24t - 88t^2 + 320t^3 2\right]dt$$
$$= \left[12t^2 - \frac{88}{3}t^3 + 80t^4\right]_0^1 = 12 - \frac{88}{3} + 80 - 0 = \frac{36 - 88 + 240}{3} = \frac{188}{3}.$$

(b) Use Green theorem to evaluate $\int_C \left(x^2ydx + xy^2dy\right)$, where C is a circle $x^2 + y^2 = 1$ traversed in the anticlockwise direction.

ANS: (b) $\int_C \left(y^2xdx + x^2ydy\right) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$. $Q = x^2y \Rightarrow \frac{\partial Q}{\partial x} = 2xy$, $P = xy^2 \Rightarrow \frac{\partial P}{\partial y} = 2xy$.

$$\therefore \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_R (2xy - 2xy)\,dxdy = 0.$$

2013-2014 MTH 242 EXAMINATION QUESTIONS AND SOLUTION

1. (a) From the first principle, find the derivative of $y = 5x^2$.

ANS: $y = f(x) = 5x^2 \Rightarrow y + \Delta y = f(x + \Delta x) = 5(x + \Delta x)^2 = 5\left(x^2 + 2x\Delta x + (\Delta x)^2\right)$

$$= 5x^2 + 10x\Delta x + 5(\Delta x)^2 \Rightarrow \Delta y = 5x^2 + 10x\Delta x + 5(\Delta x)^2 - y \Rightarrow \frac{\Delta y}{\Delta x} = 10x + 5\Delta x.$$

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}(10x + 5\Delta x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} 10x + \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} 5\Delta x = 10x + 0 = 10x.$$

(b) What are the necessary and sufficient condition for a bivariate function $f(x, y)$ to have:

ANS: (i) a minimum turning point: $f_{xx} > 0$, $f_{yy} > 0$ and $\left(f_{xy}\right)^2 - f_{xx}f_{yy} < 0$.

(ii) a maximum turning point: $f_{xx} < 0$, $f_{yy} < 0$ and $\left(f_{xy}\right)^2 - f_{xx}f_{yy} < 0$.

(iii) a saddle point: $f_{xx} < 0$, $f_{yy} < 0$ (or $f_{xx} > 0$, $f_{yy} > 0$) and $\left(f_{xy}\right)^2 - f_{xx}f_{yy} > 0$.

(c) Determine whether or not the function $f(x) = x^3$ satisfies the hypothesis of Mean Value Theorem on $[-1, 1]$.

ANS: $f(x) = x^3$ is continuous on $[-1, 1] \equiv [a, b]$ and differentiable on $(-1, 1) \equiv (a, b)$ since it is a polynomial as polynomials are continuous and differentiable everywhere in $\mathbb{R}$. $\therefore \exists\, c \in (-1, 1) \ni f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2 = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - (-1)}{1 - (-1)} = \frac{2}{2} = 1$$

$$\Rightarrow c^2 = \frac{1}{3} \Rightarrow c = \pm\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}} \notin (-1, 1)\right).$$

So $f(x) = x^3$ satisfies the hypothesis of Mean Value Theorem on $[-1, 1]$

(d) Determine whether or not the function $f(x) = x^2 - 2x$ satisfies the hypothesis of Rolle's on $[0, 2]$.

ANS: $f(x) = x^2 - 2x$is continuous on $[0, 2] \equiv [a, b]$ and differntiable on $(0, 2) \equiv (a, b)$ since it is a polynomial as polynomials are continuous and differentiable everywhere in R

$f(0) = 0 = f(2)$ $\therefore \exists\, c \in (0, 2) \ni f'(c) = 0$. $f'(x) = 2x - 2$. $\therefore f'(c) = 0 \Rightarrow 2c - 2 = 0 \Rightarrow 2c = 2 \Rightarrow c = 1 \in (0, 2)$ So $f(x) = x^2 - 2x$ satisfies the hypothesis of Rolle's Theorem on $[0, 2]$

(d) What is the nature of turning point of $f(x, y) = x^2 + y^2$

ANS: $f(x, y) = x^2 + y^2$, $f_x = 2x$, $f_{xx} = 2 > 0$, $f_y = 2y$, $f_{yy} = 2 > 0$, $f_{xy} = 0$

$\left(f_{xy}\right)^2 - f_{xx}f_{yy} = 0^2 - 2 \cdot 2 = 0 - 4 = -4 < 0$, a minimum turning point

2. Evaluate the following integral: (a) $\int_1^\infty \frac{10}{x^2}dx$ (b) $\int x^2e^{2x}dx$ (c) $\int \cot dx$ (d) $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}$.

ANS: (a) $\int_1^\infty \frac{10}{x^2}dx = 10\int_1^\infty x^{-2}dx = 10\lim_{T\to\infty}\int_1^T x^{-2}dx = 10\lim_{T\to\infty}\left(-\frac{1}{x}\Big|_1^T\right) = 10\lim_{T\to\infty}\left(-\frac{1}{T}+\frac{1}{1}\right) = 10(0+1) =$ 10

(b) $\int x^2e^{2x}dx$. Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2xdx$. Then $dv = e^{2x}dx \Rightarrow \int dv = \int e^{2x}dx$

$\Rightarrow v = \frac{1}{2}e^{2x} + c$.But $\int udv = uv - \int vdu \Rightarrow \int x^2e^{2x}dx = \frac{x^2e^{2x}}{2} - \int \frac{1}{\cancel{2}}e^{2x}\,\cancel{2}xdx$

$= \frac{x^2e^{2x}}{2} - \int xe^{2x}dx$ and $\int xe^{2x}dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$ (integrating by parts)

$$\therefore \int x^2e^{2x}dx = \frac{x^2e^{2x}}{2} - \left(\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c\right) = \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + k = \frac{e^{2x}}{2}\left(x^2 - x - \frac{1}{2}\right) + k$$

(c) $\int \cot dx = \int \frac{\cos x}{\sin x}dx = \ln|\sin x| + c$ (since $\frac{d}{dx}(\sin x) = \cos x$)

(d) $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx$. Let $u = \sqrt{x+1} = (x+1)^{\frac{1}{2}} \Rightarrow \frac{du}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+1}} \Rightarrow dx = 2\sqrt{x+1}du$

Substituting : $\Rightarrow \int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx = \int \frac{e^u}{\sqrt{x+1}}2\sqrt{x+1}du = 2\int e^u du = 2e^u + c = 2e^{\sqrt{x+1}} + c$

3. (a) Find the equation of tangent and normal to the curve $y = 3x^2 - 5x + 2$ at the point $x = 1$

ANS: $y = 3x^2 - 5x + 2$ at the point $x = 1$

$y = 6x^2 - 5x + 2$ at $x = 1$ $y' = 6x - 5 \Rightarrow y'(1) = 6(1) - 5 = 1$, which is gradient of tangent i.e. m.

Also $x_1 = 1 \Rightarrow y_1 = 3x_1{}^2 - 5x_1 + 2 = 3(1^2) - 5(1) - 2 = 0$

(i) Equation of tangent is that of a straight line given by $y - y_1 = m(x - x_1) \Rightarrow y - 0 = 1(x-1) = x - 1 \Rightarrow y = x - 1$

(ii) Equation of normal: $m = -1$ (since normal is perpendicular to tangent)

$$y - y_1 = m(x - x_1) \Rightarrow y - 0 = -1(x-1) \Rightarrow y = -x + 1 \Rightarrow y + x = 1$$

(b) Evaluate the following double integrals: (i) $\int_0^1\int_0^x dydx$ (ii) $\int_0^1\int_0^{\cos x} dydx$

ANS: (i) $\int_0^1\int_0^x dydx = \int_0^1\left[\int_0^x dy\right]dx = \int_0^1\left[y|_0^x\right]dx = \int_0^1 xdx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

(ii) $\int_0^1\int_0^{\cos x} dydx = \int_0^1\left[\int_0^{\cos x} dy\right]dx = \int_0^1\left[y|_0^x\right]dx = \int_0^1 \cos xdx = \sin x|_0^1 = \sin 1 - \sin 0 = \sin 1$

4. (a) Let $w = \ln(x^2 + y^2 + z^2)$, $x = s + t$, $y = s - t$, $z = 2\sqrt{st}$. Find (i) $\frac{\partial w}{\partial s}$ (ii) $\frac{\partial w}{\partial t}$

ANS: $w = \ln(x^2 + y^2 + z^2)$, $x = s + t$, $y = s - t$, $z = 2\sqrt{st}$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(\ln(x^2 + y^2 + z^2)\right) = \frac{2x}{x^2 + y^2 + z^2}, \frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left(\ln(x^2 + y^2 + z^2)\right) = \frac{2y}{x^2 + y^2 + z^2}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left(\ln(x^2 + y^2 + z^2)\right) = \frac{2z}{x^2 + y^2 + z^2}, \frac{\partial x}{\partial t} = 1 = \frac{\partial x}{\partial s}, \frac{\partial y}{\partial t} = -1, \frac{\partial y}{\partial s} = 1, \frac{\partial z}{\partial t} = \sqrt{\frac{s}{t}}, \frac{\partial z}{\partial s} = \sqrt{\frac{t}{s}}$$

(i) $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{2x}{x^2+y^2+z^2}\cdot 1 + \frac{2y}{x^2+y^2+z^2}\cdot 1 + \frac{2z}{x^2+y^2+z^2}\cdot\sqrt{\frac{t}{s}}$

$$= \frac{2}{x^2+y^2+z^2}\left(x+y+z\sqrt{\frac{t}{s}}\right)$$

(ii) $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = \frac{2x}{x^2+y^2+z^2}\cdot 1 + \frac{2y}{x^2+y^2+z^2}\cdot -1 + \frac{2z}{x^2+y^2+z^2}\cdot\sqrt{\frac{s}{t}}$

$$= \frac{2}{x^2+y^2+z^2}\left(x-y+z\sqrt{\frac{s}{t}}\right)$$

(b) Find the extrema values of $f(x,y) = x^2+4y^2+6$ subject to $2x-8y=20$.

ANS:Constraint is $2x-8y-20=0$ which is $\psi(x,y)$. We then form the Lagrangian as

$$F(x,y,z) = x^2+4y^2+6+\lambda(2x-8y-20),\quad \frac{\partial F}{\partial x} = (2x+2)\lambda,\quad \frac{\partial F}{\partial y} = (8y-8)\lambda,\quad \frac{\partial F}{\partial \lambda} = 2x-8y-20$$

At extrema points, $\frac{\partial F}{\partial x}=0,\quad \frac{\partial F}{\partial y}=0,\quad \frac{\partial F}{\partial \lambda}=0$

$$i.e.\ 2x+2\lambda = 0 \ \cdots\ (i)\quad 8y-8\lambda=0\ \cdots\ (ii)\quad 2x-8y-20=0\ \cdots\ (iii)$$

From (i), $x=-\lambda$, from (ii) $y=\lambda$. Substitute for x and y in (iii)

$$-2\lambda-8-20=0 \Rightarrow -10\lambda-20=0 \Rightarrow 10\lambda=-20 \Rightarrow \lambda=-2 \Rightarrow x=2,\ y=-2.$$

$$\frac{\partial^2 F}{\partial x^2}=2,\quad \frac{\partial^2 F}{\partial y^2}=8,\quad \frac{\partial^2 F}{\partial x\partial y}=0.\quad \left(\frac{\partial^2 F}{\partial x\partial y}\right)^2 - \frac{\partial^2 F}{\partial x^2}\cdot\frac{\partial^2 F}{\partial y^2} = 0-2\times 8 = -16<0$$

Notice $\frac{\partial^2 F}{\partial x^2}>0,\quad \frac{\partial^2 F}{\partial y^2}>0,\quad \left(\frac{\partial^2 F}{\partial x\partial y}\right)^2 - \frac{\partial^2 F}{\partial x^2}\cdot\frac{\partial^2 F}{\partial y^2}<0$

Therefore (2, -2) is a minimum extreme point. The extreme value of $f(x,y)=x^2+4y^2+6$ subject to $2x-8y=20$ occurs at $(2,-2)$.$\therefore f(x,y)\big|_{(2,-2)} = f(2,-2) = 4+16+6 = 26$

5. (a) Let $F(x,y,z) = (3x^2+6y)i - 14yzj + 20xz^2k$. Evaluate $\int_C F\cdot dr$ from $(0,0,0)$ to $(1,1,1)$ along

$$x=t,\ y=t^2,\ z=t^3$$

ANS: $x=t,\ y=t^2,\ z=t^3, dx=dt, dy=2tdt, dz=3t^2$. Also, $x=0=y=z, t=0$ and $x=1=y=z, t=1$

$$\therefore \int_C F\cdot dr = \int_{(0,0,0)}^{(1,1,1)}\left((3x^2+6y)dx - 14yzdy + 20xz^2dz\right)$$

$$= \int_0^1\left[(3t^2+6t^2)dt - 14t^2\cdot t^3\cdot 2tdt + 20t\cdot t^6\cdot 3t^2dt\right]$$

$$= \int_0^1\left[9t^2-28t^6+60t^9\right]dt = \left[3t^3-4t^7+6t^{10}\right]_0^1 = 5.$$

(b) If $u=(\cos xy)^2$, evaluate (i) u_x (ii) u_y (iii) u_{xy}

ANS: Let $v=xy \Rightarrow v_x = y$ and $v_y = x$. $\therefore u=\cos^2 v$. Let $w=\cos v \Rightarrow w_v = -\sin v$. $\therefore u = w^2$.

(i) $u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = 2w(-\sin v)y = 2\cos(xy)(-\sin(xy))y = -2y\sin(xy)\cos(xy)$

(ii) $u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = 2w(-\sin v)x = 2\cos(xy)(-\sin(xy))y = -2x\sin(xy)\cos(xy)$

(iii) $$\begin{aligned} u_{xy} &= \frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial y}(-2y\sin(xy)\cos(xy)) \\ &= -2\sin(xy)\cos(xy)\frac{\partial}{\partial y}(y) - 2y\cos(xy)\frac{\partial}{\partial y}(\sin(xy)) - 2y\sin(xy)\frac{\partial}{\partial y}(\cos(xy)) \\ &= -2\sin(xy)\cos(xy)\cdot 1 - 2y\cos(xy)\cdot x\cos xy - 2y\sin(xy)\cdot(-x\sin(xy)) \\ &= -2\sin(xy)\cos(xy) - 2xy\cos^2(xy) + 2yx\sin^2(xy) \\ &= -2\sin(xy)\cos(xy) - 2xy\left(\cos^2(xy) - \sin^2(xy)\right) \\ &= -2\sin(xy)\cos(xy) - 2xy\left(2\cos^2(xy) - 1\right) \\ &= 2xy - 4xy\cos^2(xy) - \sin 2(xy) \end{aligned}$$