Notes for chapter 4 of Applied Function Analysis

Topology, through filters

Contents

Introduction

Chapter starts with a discussion about the equivalence of concepts in mathematics:

Example 1. Equivalence relations and partitions are conceptually equivalent

- An equivalence relation $\sim$ *induces* a partition $\mathcal{P}$, i.e., there exists a natural mapping from equivalence relations to partitions. The induced partition is the family of equivalence classes of $\sim$. Call it f for a second.
- Viceversa from the other side. The induced equivalence relation is described as

  $$x \sim y \iff \text{x and y belong to the same partition}$$

  Call it g for 2 seconds.
- $g \circ f = Id_{\sim}$
- $f \circ g = Id_{\mathcal{P}}$

Author goes on to describe how topology can be introduced from a bunch of different (but equivalent) concepts, *a savoir* : open & closed sets, the interior/closure of a set, neighborhoods of points. Author chooses *neighborhoods of points*.

1 Basic notions

We start out by defining the equivalence relation $\succ$ on the power set $\mathcal{P}(X)$: for 2 families of subsets of X, $\mathcal{A}$ & $\mathcal{B}$,

$$\mathcal{A} \succ \mathcal{B} \iff \forall B \in \mathcal{B} \quad \exists A \in \mathcal{B} : A \subset B$$

It is said that $\mathcal{A}$ is stronger than $\mathcal{B}$ If we also have $\mathcal{B} \succ \mathcal{A}$, $\mathcal{A}$ & $\mathcal{B}$ are *equivalent*.

Definition 1.1: Bases and filters

A non-empty class $\mathcal{B}$ is called a **base** iff

1. $\varnothing \notin \mathcal{B}$
2. $\forall A, B \in \mathcal{B}, \ \exists C \in \mathcal{B} : C \subset A \cap B$

A non-empty class of sets $\mathcal{F}$ is called a **filter** iff

1. $\varnothing \notin \mathcal{F}$
2. $\forall A, B \in \mathcal{F}$, $C = A \cap B \in \mathcal{F}$
3. All supersets of an element are contained, i.e. $A \in \mathcal{F}, A \subset D \implies D \in \mathcal{F}$

Every filter is a base. We define the filter *generated* by base $\mathcal{B}$ as

$$C \in \mathcal{F} \iff \exists B \in \mathcal{B} : B \subset C$$

We call $\mathcal{F}$ the *filter of base* $\mathcal{B}$.

Example 1. Consider $X = \{1, 2, 3, 4\}$

- $\mathcal{B}_1 = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$ is a base.
- $\mathcal{B}_2 = \{\{1\}\}$
- $\mathcal{F}(\mathcal{B}_1) = \mathcal{F}(\mathcal{B}_2) = \{\{2\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$

Proposition 1.1

Let $\mathcal{B}, \mathcal{C}$ denote two bases

$$\mathcal{B} \succ \mathcal{C} \iff \mathcal{F}(\mathcal{B}) \supset \mathcal{F}(\mathcal{C})$$

Moreover, $\mathcal{B}$ & $\mathcal{C}$ are *equivalent* with respect to $\succ$ iff $\mathcal{F}(\mathcal{B}) = \mathcal{F}(\mathcal{C})$.

Proof Consider $M \in \mathcal{F}(\mathcal{B})$. By definition, $\exists B \in \mathcal{B} \mid B \subset M$. $\mathcal{C} \succ \mathcal{B} \implies \exists C \in \mathcal{C} \mid C \subset B \subset M \implies M \in \mathcal{F}(\mathcal{C})$ ■

The classical approach consists in *introducing a topology from open sets*:

Definition 1.2: Open sets

Consider a class $\mathcal{T} \subset \mathcal{P}(X)$ such that

i. $\varnothing$ & $X \in \mathcal{T}$

ii. $\bigcup_{U \in \mathcal{T}} U \in \mathcal{T}$ i.e. arbitrary union belongs to $\mathcal{T}$

iii. $\bigcap_{U \in \mathcal{T}}^{N} U \in \mathcal{T}$ i.e. finite intersection belongs to $\mathcal{T}$

The elements of $\mathcal{T}$ are called **open sets**. $\mathcal{T}$ itself is frequently called the topology on X. $(X, \mathcal{T})$ is called a *topological space*.

Another approach is to *introduce the topology through neighborhoods*

Definition 1.3: Neighborhoods

For each $x \in X$, consider a filter $\mathcal{F}_x$ such that

i. $\forall Y \in \mathcal{F}_x, x \in Y$

ii. $A, B \in \mathcal{F}_x \implies A \cap B \in F_x$

iii. For each element, its supersets are contained aswell.

iv. $A \in \mathcal{F}_x \implies \mathring{A} := \{z \in A : A \in \mathcal{F}_x\} \in \mathcal{F}_x$ i.e. the *interior of* A also belongs to $\mathcal{F}_x$

The elements of $\mathcal{F}_x$ are called ***neighborhoods of x***. The mapping

$$\begin{aligned} \mathcal{T} : X &\to \mathcal{P}(X) \\ x &\mapsto \mathcal{F}_x \end{aligned}$$

is referred to as *a topology on* X, and $(X, \mathcal{T})$ is referred to as a *topological space*. Let us now explicitly define the **interior operation** referred to in property *iv*. Consider $A \subset X$:

- A point x is an **interior point** of A iff$A \in \mathcal{F}_x$
- Set of interior points of A, *aka* **interior of set** A : $\mathrm{int} A := \{y \in A : A \in \mathcal{F}_y\} (= \mathring{A} big)$

We could have started the previous construction from a base and generated the previous filter from there, in the following fashion: let $\mathcal{B}_x$ be a class such that:

i. $\forall A \in \mathcal{B}_x, x \in A$

ii. $\forall A, B \in \mathcal{B}_x, \ \exists C \in \mathcal{B}_x \ | \ C \subset A \cap B$

iii. $\forall B \in \mathcal{B}_x, \ \exists C \in \mathcal{B}_x \ | \ \forall y \in C, \ \exists D \in \mathcal{B}_y \ | \ D \subset B$

This base is called *base of neighborhoods of x*.

STATEMENT 1. A *base of neighborhoods of x* indeed generates a filter of *neighborhoods of x*.

PROOF The first two conditions define a base, thus $\mathcal{F}(\mathcal{B}_x)$ has the first three properties of $\mathcal{F}_x$. We will show that property *iii* of $\mathcal{B}_x$ is equivalent to $\mathcal{F}(\mathcal{B}_x)$ having property *iv* of $\mathcal{F}_x$

$$\begin{gathered} \forall B \in \mathcal{B}_x, \ \exists C \in \mathcal{B}_x \ | \ \forall y \in C, \ \exists D \in \mathcal{B}_y \ | \ D \subset B \\ \Updownarrow \\ \forall B \in \mathcal{B}_x, \ \exists C \in \mathcal{B}_x \ | \ \forall y \in C, B \in \mathcal{F}(\mathcal{B}_y) \\ \Updownarrow \\ B \in \mathcal{B}_x \implies \mathring{B} := \{y \in B : B \in \mathcal{F}(\mathcal{B}_y)\} \in \mathcal{F}(\mathcal{B}_x) \\ \Updownarrow^1 \\ A \in \mathcal{F}(\mathcal{B}_x) \implies \mathring{A} := \{y \in A : A \in \mathcal{F}(\mathcal{B}_y)\} \in \mathcal{F}(\mathcal{B}_x) \end{gathered}$$

[1] $\Downarrow$ follows immediately from $\mathcal{B}_x \subset \mathcal{F}(\mathcal{B}_x)$. $\Uparrow$ follows from $B \subset A \implies \mathring{B} \subset \mathring{A}$ ■

Our goal in the next few lines is to show that the *filter topology* and the *open set topology* are mathematically equivalent concepts, in the sense of the introductory discussion. We shall construct an *open topology* starting from neighborhoods, a *filter topology* starting from open sets & show that by going forth and forth again we return to our starting point.

Definition 1.4: Alternative definitions of open sets & neighborhoods

Neighborhoods, from open sets : Let $(X, \mathcal{T}^o)$ be a space with an open set topology. Consider an arbitrary point $x \in X$. The *base of open neighborhoods* is defined as follows

$$\mathcal{B}_x^o := \{U \in \mathcal{T}^o \quad | \quad x \in U\}$$

Neighborhoods of x are defined as the elements of the $\mathcal{F}(\mathcal{B}_x^o)$.

Open sets, from neighborhoods : Let $(X, \mathcal{T}^f)$ be a space with a filter topology. U is an *open set* iff $U = \text{int } U$

Let us check that these definitions are compatible with the previous ones:

- $\mathcal{B}_x^o$ (definition 1.4) is a base of neighborhoods of x (definition 1.3):
  - $\mathcal{B}_x^o$ is a base (i.e. $\mathcal{F}(\mathcal{B}_x^o)$ is a filter) : by definition of $\mathcal{B}_x^o$ and open sets, x is an element of all members of $\mathcal{B}_x^o$ and the intersection of members is also a member. ✓
  - By definition of $\mathcal{B}_y^o$, $\forall B \in \mathcal{B}_x^o, \ \forall y \in B, \ B \in \mathcal{B}_y^o \quad (\textit{iff } B = \mathring{B})$ ✓
- The set $\mathcal{T} = \{U \subset X \quad | \quad U = \text{int } U\}$ (definition 1.4) forms an open topology (definition 1.2):
  - $\varnothing, X \in \mathcal{T}$: both sets are neighborhoods of all their points. ✓
  - $\bigcup_{U \in \mathcal{T}} U \in \mathcal{T}$: each U is a neighborhood of all its points & the union is a superset of each U. ✓
  - $\bigcap_{U \in \mathcal{T}}^{N} U \in \mathcal{T}$: by induction from the second property of filters. ✓

It only remains to check whether by going forth and forth we return to our starting point:

Proposition 1.2

Let $(X, \mathcal{T}^o)$ be a topological space furnished with an open topology. Then

$$\mathcal{T}^o = \mathcal{T}^{o^{f^o}} \left(:= \{\text{Collection of open sets induced by } \mathcal{F}(\mathcal{B}_x^o)\}\right)$$

Proof

- $\subset$: Let $U \in \mathcal{T}^o$. By definition 1.4, $U \in \mathcal{T}^o \implies U \in \mathcal{B}_x^o \ \forall x \in U$. That is, $U = \text{int } U$. This implies $U \in \mathcal{T}^{o^{f^o}}$
- $\supset$: Let $V \in \mathcal{T}^{o^{f^o}}$. $V = \text{int } V \iff V \in \mathcal{F}_x \ \forall x \in V \iff \forall x \in V \ \exists B_x \in \mathcal{B}_x^o \ | \ x \in B_x \subset V$. We can write V as the union of such sets, which are open in $\mathcal{T}^o$. Thus V is open in $\mathcal{T}^o$.

■

Proposition 1.3

Let $(X, \mathcal{T}^f)$ be a topological space equipped with a filter topology. Then

$$\mathcal{T}^f = \mathcal{T}^{f^{o^f}} \left(:= \text{Mapping } x \mapsto \mathcal{F}_x\right)$$

Proof We show that $\mathcal{B}_x \sim \mathcal{B}_x^{\text{induced}}$ in the sense of $\succ$:

- $\succ$: Let $B \in \mathcal{B}_x$. We use the fact that $\text{int } B = \text{int int } B$: $\text{int } B \in \mathcal{B}_x^o$ & $\text{int } B \subset B$
- $\prec$: Let $B' \in \mathcal{B}_x^o$. $B' \in \mathcal{B}_x^o \implies B' \in \mathcal{T}^{f^o} \implies \text{int } B' = B'$. In particular, B' is a neighborhood of x i.e. $B' \in \mathcal{F}_x \iff \exists B \in \mathcal{B}_x \ | \ B \subset B'$

Thus $\mathcal{B}_x \sim \mathcal{B}_x^{\text{induced}} \iff \mathcal{F}_x = \mathcal{F}_x^{\text{induced}}$ by proposition 1.1. ■

We conclude that both approaches are equivalent. We can set sail on the path of filters without looking back.

Proposition 1.4: Properties of the interior operation

i. int int $A =$ int A

ii. int $(A \cap B) =$ int $A \cap$ int B

iii. int $(A \cup B) \supset$ int $A \cup$ int B

iv. $A \subset B \implies$ int $A \subset$ int B

Proof

iv : $\forall x \in$ int A, $B \supset A \in \mathcal{F}_x$ i.e. neighbourhood A is such that $x \in A \subset B \implies x \in$ int B

i : $\subset$ follows from *iv.* Consider $x \in$ int A. A is a neighbourhood of $x \iff$ int A is a neighbourhood of $x \iff x \in$ int int A

ii : $\subset$ follows from *iv.* Consider $x \in$ int $A \cap$ int B. $A, B \in \mathcal{F}_x \iff A \cap B \in \mathcal{F}_x \iff$ int $(A \cap B) \in \mathcal{F}_x$

iii : $\supset$ follows from *iv.*

■

REMARK The interior of a set corresponds to the biggest open set contained inside of him i.e.

$$\text{int } A = \bigcup_{\substack{U \in \mathcal{T}^o \\ U \subset A}} U$$

Example 2. Equivalent bases in $\mathbb{R}^n$ Define the bases $\mathcal{B}_x$ as the collection of open balls centered at x. They define the *fundamental topology* in $\mathbb{R}^n$. Note that the following bases are equivalent to it

- Open balls with radii $\frac{1}{n}$
- Closed balls centered at x
- Open balls defined by the L^1 norm or any other L^p norm, $p \in \mathbb{N}$

Example 3. Discrete & trivial topologies Let X be an arbitrary set.

- If we use as a base the *singletons* $\{x\}$, every x is mapped to $\mathcal{P}(x)\big($that is, $\mathcal{F}_x = \mathcal{P}(x)\big)$. This topology is dubbed *discrete toplogy.*
- If we use as a base X, every x is mapped to $\{X\}\big($that is, $\mathcal{F}_x = \{X\}\big)$. This topology is dubbed *trivial toplogy.*

These are respectively the *strongest* and *weakest* topologies.

Definition 1.5: Accumulation points, closure of a set, closed set & dense set

Consider a topological space $(X, \mathcal{T}^f)$ and $A \subset X$

- x is an **accumulation point** of A iff $\forall N \in \mathcal{F}_x, (N \cap A) \setminus \{x\} \neq \varnothing$
- **Closure** of A, $\overline{A} := A \cup \hat{A}$, where $\hat{A}$ denotes the set of accumulation points of A
- A set $C \subset X$ is **closed** if it corresponds to its own closure i.e. $C = \overline{C}$

- A set $D \subset X$ is said to be **dense in X** iff $\overline{D} = X$

STATEMENT 2. A set is closed iff its complement is open.

PROOF

$$\begin{cases} x \notin C \\ \text{C closed} \end{cases} \iff \begin{cases} x \notin C \\ x \notin \hat{C} \end{cases} \iff \begin{cases} x \notin C \\ \exists N \in \mathcal{F}_x \quad | \quad N \cap C \setminus \{x\} = \varnothing \end{cases} \iff$$

$$\iff \exists N \in \mathcal{F}_x \quad | \quad N \cap C = \varnothing \iff \exists N \in \mathcal{F}_x \quad | \quad N \subset C' \iff C' \in \mathcal{F}_x \iff \begin{cases} x \in C' \\ \text{C' open} \end{cases}$$

■

Proposition 1.5: Properties / Definition of closed sets

As always, these properties can be taken as a definition and will induce a topology.

- $\varnothing, X$ are closed.
- Finite union of closed sets is closed.
- Arbitrary intersection of closed sets is closed.

2 Topological subspaces & product topologies

Definition 2.1: Subspace topology

A topological space $(X, \mathcal{T})$ induces a topology on its subsets $Y \subset X$

$$\forall x \in Y,\ \ B \in \mathcal{B}_x^Y \iff B = B' \cap Y,\ \ B' \in \mathcal{B}_x$$

Subset Y is said to have some topological property iff $(X, \mathcal{T}\big|_Y)$ possesses said property.

Proposition 2.1

Let $Y \subset X$ a topological subspace & E a subset of Y :

- The closure of E in $\mathcal{T}\big|_Y$ is $\overline{E} \cap Y$
- U is open in $\mathcal{T}\big|_Y \iff U = V \cap Y,\ \ V \in \mathcal{T}$
- F is closed in $\mathcal{T}\big|_Y \iff F = C \cap Y,\ \ C$closed in $\mathcal{T}$

Definition 2.2: Product topology

Consider two topological spaces, X, Y. Their respective topologies induce a topology on $X \times Y$

$$\mathcal{B}_{(x,y)} := \{B \times B' \ \ | \ \ B \in \mathcal{B}_x^X,\ \ B' \in \mathcal{B}_y^Y\}$$

Note that things work differently for infinite cartesian products. In words, the product topology is the **coarsest** topology for which all the projections are continuous.

Proposition 2.2

- $U \in \mathcal{T}_X,\ \ V \in \mathcal{T}_Y \iff U \times V \in \mathcal{T}^*$
- C closed in $\mathcal{T}_X,\ \ V$ closed in $\mathcal{T}_Y \iff U \times F$ closed in $\mathcal{T}^*$

3 Continuity and compactness

Definition 3.1: Continuous function

Let X, Y be two topological spaces and let $f : X \to Y$ be a function. A function is continuous at $x \in X$ iff

$$f(\mathcal{B}_x) \succ \mathcal{B}_{f(x)} \iff f(\mathcal{F}_x) \succ \mathcal{F}_{f(x)}$$

If f is continuous $\forall x \in X$, f is *globally* continuous.

Proposition 3.1: Statements regarding continuity

Let X, Y, Z be topological spaces. Let $f : X \to Y,\ \ g : Y \to Z,\ \ h : X \to V$ be a bunch of functions:

i. f is globally continuous $\iff \forall G$ open in $\mathcal{T}_Y,\ \ f^{-1}(G)$ open in $\mathcal{T}_X$

ii. f is globally continuous $\iff \forall C$ closed in $\mathcal{T}_Y,\ \ f^{-1}(C)$ closed in $\mathcal{T}_X$

iii. f is continuous at x, g is continuous at $f(x) \implies g \circ f$ continuous at x

iv. f is continuous at x, h is continuous at $x \implies (f,\ h)$ continuous at x

$$\begin{aligned}(f,\ h) : X &\to Y \times V \\ x &\mapsto (f(x),\ h(x))\end{aligned}$$

v. Consider the functions $F : A \to B,\ \ G : C \to D$

$$F \text{ is continuous at } x \in A,\ \ G \text{ is continuous at } y \in B \implies F \times G \text{ is continuous at } (x, y)$$

PROOF

- i, $\Rightarrow$: Let U be an open set in the codomain. If there is no $x \in X$ such that $f(x) \in U,\ \ f^{-1}(U) = \varnothing$, which is open by definition. Else, $\exists x \ \mid\ f(x) \in U$. Because U is open, it is a neighborhood of all its points and in particular $f(x)$. By continuity, there exists a neighborhood $N \in \mathcal{F}_x \ \mid\ f(x) \in f(N) \subset U$. We exploit the fact that $N \subset f^{-1}(f(N))$

$$x \in N \subset f^{-1}(U)$$

- i, $\Leftarrow$: Let $N \in \mathcal{F}_{f(x)}$. $\text{int } N$ is a open $f(x) \implies f^{-1}(\text{int } N)$ is open $\implies f^{-1}(\text{int } N)$ is a neighborhood of x such that

$$f(f^{-1}(\text{int } N)) \subset N$$

  We used the fact that the image of the preimage is contained in the original set.

- ii $\Leftrightarrow$ iii : The equivalence follows immediately from $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$ and the complement characterizations of open & closed sets.

- iii : Consider a neighborhood of $f \circ g(x),\ \ L$. By continuity of f, $\exists$ a neighborhood of $g(x),\ \ M \ \mid\ f(M) \subset L$. By continuity of g, $\exists$ a neighborhood of $x,\ \ N \ \mid\ g(N) \subset M$. Thus, $f(g(N)) \subset L\ \&\ f \circ g\,(\mathcal{F}_x) \succ \mathcal{F}_{f \circ g\,(x)}$

- iv : Consider a neighborhood of $(f(x), h(x)) \in Y \times V,\ \ A \times B$. By definition of the product topology, $A\ \&\ B$ are neighborhoods of $f(x),\ h(x)$, respectively. By continuity of $f\ \&\ h$, $\exists M, N \in \mathcal{F}_x \ \mid\ f(M) \subset A,\ \ h(N) \subset B$. Thus $N \cap M \in \mathcal{F}_x\ \&\ (f,h)\,(N \cap M) \subset A \times B$

- v : Consider a neighborhood of $F \times G\,(x, y),\ \ L \times K$. By definition of the product topology, $L \in \mathcal{F}_{F(x)} \subset A,\ \ K \in \mathcal{F}_{G(y)} \subset C$. By continuity of $F\ \&\ G$, $\exists M \in \mathcal{F}_x,\ N \in \mathcal{F}_y \ \mid\ F(M) \subset L,\ \ G(N) \subset K$. By definition of the product topology, this time from the domain's point of view, $M \times N \in \mathcal{F}_{F \times G\,(x,y)}$

■

Definition 3.2: Hausdorff spaces

A topological space $(X, \mathcal{T})$ is said to be **Hausdorff** iff

$$\forall x, y \in X, \ \exists N \in \mathcal{F}_x, \ M \in \mathcal{F}_y \ | \ N \cap M = \varnothing$$

It is a topological property.

Definition 3.3: Compact spaces

A topological space $(X, \mathcal{T})$ is said to be **compact** iff

- X is Hausdorff
- Every open covering has a finite subcovering.

It is a topological property.

An indexed collection of subsets of X, $\mathcal{A} = \{A_i, i \in I\}$, has the **finite intersection property** (FIP) iff any finite subcollection $J \subset I$ has a non-empty intersection i.e. $\bigcap_{i \in J} A \neq \varnothing$

Let $\mathcal{B}$ be a base. x is a **limit point** of $\mathcal{B}$ iff

$$x \in \bigcap_{B \in \mathcal{B}} \overline{B}$$

NB : Nobody else calls this *limit point*.

Proposition 3.2: Characterization of compactness

Let X be a Hausdorff space. The following statements are equivalent.

i. Every open covering has a finite subcovering.

ii. Every base has a limit point.

iii. Every collection of closed sets with the FIP has a non-empty intersection.

Proof

- i $\Rightarrow$ ii : Consider a base $\mathcal{B} \subset \mathcal{P}(X)$. The collection of the closures of the elements of $\mathcal{B}$, is also a base. Assume by contradiction that $\mathcal{B}$ has no limit point i.e.

  $$\bigcap_{\mathcal{B}} \overline{B} = \varnothing \iff \bigcup_{\mathcal{B}} X \setminus \overline{B} = X$$

  In words, the complements form an open covering. By i, there exists a finite subcovering $\mathcal{V} \subset \mathcal{B}$, $|\mathcal{V}| < \infty$ such that

  $$\bigcup_{\mathcal{V}} X \setminus \overline{B} = X \iff \bigcap_{\mathcal{V}} \overline{B} = \varnothing$$

  We have both that $\{\overline{B}, B \in \mathcal{B}\}$ is a base and that there is an empty finite intersection of elements. $\perp$

- ii $\Leftarrow$ i : Consider an open covering $\mathcal{U}$. Assume by contradiction that there is no finite subcovering i.e.

  $$\forall \mathcal{V} \subset \mathcal{U} \ | \ |\mathcal{V}| < \infty, \ \bigcup_{\mathcal{V}} U \neq X \iff \bigcup_{\mathcal{V}} X \setminus U \neq \varnothing$$

  In words, for any finite subcollection, the intersection of the complements is non-empty. The complements thus qualify to form a base. By ii, their intersection is non-empty i.e.

  $$\bigcap_{\mathcal{U}} X \setminus U \neq \varnothing \iff \bigcup_{\mathcal{U}} U \neq X$$

We both have that $\mathcal{U}$ is a covering and that it is not a covering, $\perp$

- iii $\Rightarrow$ ii : A base is a collection of sets with the FIP, meaning the closures of its elements is a collection of closed sets with the FIP.
- iii $\Leftarrow$ i : Consider a collection of closed sets with the FIP, $\mathcal{C} \subset \mathcal{P}(X)$. Assume by contradiction that their intersection is empty

$$\bigcap_{\mathcal{C}} C = \varnothing \iff \bigcup_{\mathcal{C}} X \setminus C = X$$

The collection of the complements is an open covering meaning there is a finite subcovering. Like in the first proof, this contradicts the FIP property.

$\blacksquare$

Proposition 3.3: Statements about compacity

i. Every compact set is closed.

ii. Every closed subset of a compact set is compact.

iii. Cartesian products of compact sets are compact.

iv. Let $f : X \to Y$ be a continuous function with X compact. Then $f(X)$ is also compact.

Proof

i. Consider $K \subset X$ compact. Let x be an accumulation point of K and assume by contradiction that it is not contained in K. Because the space is Hausdorff

$$\forall y \in K, \ \exists N_y \in \mathcal{F}_y, \, M_y \in \mathcal{F}_x \ \ | \ \ \text{int } N_y \cap M_y \subset N_y \cap M_y = \varnothing$$

The collection $\{ \text{ int } N_y, \, y \in K\}$ is an open covering of K. By compactness, we can extract a finite subcovering of K, defined by some finite indexing set I

$$\bigcap_I M_{y_i} \cap K \subset \bigcap_I M_{y_i} \cap \bigcap_I N_{y_i} = \bigcap_I M_{y_i} \cap N_{y_i} = \varnothing$$

We both have that x is an accumulation point and that the neighborhood $\bigcap\limits_I M_{y_i}$ does not intersect K, $\perp$

ii. We use the FIP characterization of compacity : consider a collection of closed sets in $F \subset X$, $\mathcal{C}$ with the FIP. The subset topology dictates that

$$\forall C \in \mathcal{C}, \ \ C = C' \cap F$$

Thus the elements of $\mathcal{C}$ are also closed in X and by compacity of X their intersection is non-empty.

iii. Skipped

iv. Consider an open covering of $f(X), \mathcal{U}$. By continuity, $\{f^{-1}(U), U \in \mathcal{U}\}$ is a collection of open sets. It is also an open covering of X. By compacity of X, there exists a finite indexing set I such that

$$\bigcup_{i \in I} f^{-1}(U_i) = f^{-1}(\bigcup_{i \in I} U_i) = X \iff \bigcup_{i \in I} U_i \supset f(X)$$

i.e. I defines a finite subcovering of $f(X)$

$\blacksquare$

Theorem 3.1: Heine-Borel

A set $E \subset \mathbb{R}$ is compact iff it is closed & bounded.

Proof

- $\Rightarrow$: By proposition 3.3, we only need to prove that compacity in $\mathbb{R}$ implies boundedness. Consider $K \subset \mathbb{R}$ compact. Assume by contradiction that it is not bounded. Without loss of generality assume K has no upper bound. The collection $\{[c, \infty[\cap K,\, c \in \mathbb{N}\}$ forms a base / a collection of closed sets with the FIP. We both have that the intersection of the collection is empty and non-empty, $\perp$

- $\Leftarrow$: It is sufficient to show that sets of the form $[a, b]$ are compact; closed subsets of compact sets are also compact (proposition 3.3) and the subset topology inherited from $[a, b]$ is the same as the subset topology inherited from $\mathbb{R}$. We'll use the limit point characterization of compacity :

  Let $\mathcal{V}$ be a base in $[a, b]$. Thus $\mathcal{B} = \{B = \overline{V},\, B \in \mathcal{V}\}$ is also a base. Because the elements of $\mathcal{B}$ are closed, bounded and non-empty, they contain their supremum. We argue now that

  $$z = \inf_{\mathcal{B}} \max B \in B \quad \forall B \in \mathcal{B}$$

  By definition of the infimum, $\forall \epsilon > 0$

  $$\exists B_\epsilon \in \mathcal{B} \quad | \quad z + \epsilon > B_\epsilon$$

  Since $\mathcal{B}$ is a base

  $$\forall B \in \mathcal{B},\ \exists B' \in \mathcal{B} \quad | \quad B' \subset B \cap B_\epsilon$$

  Since z is the infimum of the set of maximums

  $$z \leq \max B' \leq \max B \cap B_\epsilon \leq \max B_\epsilon < z + \epsilon$$

  Thus for any $B \in \mathcal{B}$, for any $\epsilon > 0$,

  $$]z - \epsilon, z + \epsilon[\cap B \neq \varnothing$$

  This implies that $z \in B$ since B is closed. Thus z is a limit point of $\mathcal{B}$

$\blacksquare$

4 Sequences

Definition 4.1: Sequences & convergence

Let A be an arbitrary set. A **sequence** is any function $s : \mathbb{N} \to A$. A **subsequence** is a sequence derived from deleting some or no elements of a sequence.

Let A be a topological space. A sequence **converges** to $x \in A$, denoted $x_n \to x$, iff for all neighborhoods F of x, F eventually contains the sequence.

Consider again a sequence x_n in a topological space. x is an **accumulation point** of x_n iff all neighborhoods of x contain infinitely many elements of $\{x_n\}$ i.e.

$$\forall F \in \mathcal{F}_x, \ \ \forall N > 0, \ \ \exists n \geq N \ \ | \ \ x_n \in F$$

Example 1. Consider a sequence x_n in a topological space X.

- If the topology on X is the trivial one, the sequence converges to all $x \in X$.
- If the topology on X is the discrete one, the sequence converges to $x \iff \exists N \ \ | \ \ \forall n > N, \ \ x_n = x$

STATEMENT 1. Let X be a Hausdorff topological space. If a sequence converges in x, its limit is unique.

PROOF Assume that x_n converges to two points, $x \neq y$. Then $\exists M \in \mathcal{F}_x, \ \exists N \in \mathcal{F}_y \ \ | \ \ N \cap M = \varnothing$. Eventually, both neighborhoods contain the sequence i.e. their intersection contains the sequence, but their intersection is empty $\perp$ ∎

Definition 4.2: Bases of countable type & first countability

A base is said to be of **countable type** if it is equivalent (in the sense of $\succ$) to a countable base $\mathcal{C} = \{C_i, \ \ i \in \mathbb{N}\}$

REMARK Note that $\mathcal{C} \sim \mathcal{D}$, where $\mathcal{D}$ writes

$$\mathcal{D} = \{D_k, \ \ D_k = C_1 \cap C_2 \cap C_3 \cap ... \cap C_k\}$$

Notice that $\mathcal{D}$ forms a new countable base of nested basis elements i.e. a base of countable type is equivalent to a base of nested basis elements.

A topological space is said to be **first countable** iff every point possesses a base of first countable type.

Definition 4.3: Sequential properties

Consider a topological space X :

- A set G is said to be **sequentially open** iff

$$x \in G, \ \ x_n \to x \implies x_n \text{ is eventually contained in } G$$

- A set E is said to be **sequentially closed** iff

$$\{x_n\} \subset E, \ \ x_n \to x \implies x \in E$$