Notes for chapter 1 of Applied Functional Analysis

Preliminaries

Definition 0.1: Some types of relations

A **relation** R on A can be seen as a subset of $A \times A$

$$xRy \iff (x, y) \in R \subset A \times A$$

R on A is said to be ...

- **Symmetric** iff $xRy \implies yRx$
- **Anti-symmetric** iff $xRy \wedge yRx \implies x = y$
- **Reflexive** iff $\forall x \in A,\ \ xRx$
- **Transitive** iff $xRy,\ \ yRz \implies xRz$

A relation R on A is said to be a **partial ordering** of A iff R is ...

- Transitive
- Reflexive
- Anti-symmetric

A relation R on A is said to be an **equivalence relation** on A iff R is ...

- Transitive
- Reflexive
- Symmetric

Equivalence relations are frequently denoted by $\sim$ or $=$. Given an equivalence relation $\sim$ on A, the **equivalence class** of $x \in A$ is defined as follows

$$[x] = \{y \in A \quad | \quad x \sim y\}$$

The collection of equivalence classes of $\sim$ on A is denoted as ${}^{A}/_{\sim}$ and called **quotient set**.

Definition 0.2: Binary operation

A **binary operation** $*$ on set A is a function

$$\begin{aligned} f : A \times A &\to A \\ (a, b) &\mapsto a * b \end{aligned}$$

A subset $E \subset A$ is said to be *closed under* $*$ iff $a, b \in E \implies a * b \in E$

$*$ is said to be ...

- **Associative** iff $\forall a, b, c \in A$
$$(a * b) * c = a * (b * c)$$

- **Commutative** iff $\forall a, b \in A$
$$a * b = b * a$$

Definition 0.3: Groupoids, Semi-groups, monoids & Groups

Let G be a set and $*$ a binary operation on said set. $\{G, *\}$ is said to be ...

- ... a **groupoid**.
- ... a **semi-group** iff $*$ is associative.
- ... a **monoid** iff it is a semi-group and $*$ has an identity element :
$$\exists e \in G \ \mid \ \forall x \in G, \ x * e = e * x = x$$
- ... a **group** iff it is a monoid and each element admits an inverse element (relative to $*$) :
$$\forall x, \ \exists x^{-1} \in G \ \mid \ x * x^{-1} = e$$

If, additionally, $*$ is commutative, the group is said to be *Abelian.* Notice that in general you can define a group with only left-identity and left-inverse, and right-identity/inverse will follow. For a monoid, you can not stop at defining left-identity.

Definition 0.4: Ring

A **ring** is a set R together with two binary operations, $+$ & $\cdot$, such that

- $(F, +)$ forms an Abelian group with identity element 0
- $(F, \cdot)$ forms a monoid with identity element 1
- Operation $\cdot$ distributes (left & right) over $+$

If additianally $\cdot$ is commutative the ring itself is said to be commutative.

Definition 0.5: Field

A **field** is a set F together with two binary operations, $+$ & $\cdot$, such that

- $(F, +)$ forms an Abelian group with identity element 0
- $(F \setminus \{0\}, \cdot)$ forms an Abelian group with identity element 1
- Operation $\cdot$ distributes (left & right) over $+$

Proposition 0.1: Algebraic properties of the real numbers

i. $\{\mathbb{R}, \cdot, +\}$ is a field

ii. $\leq$ is a total/connex ordering of $\mathbb{R}$

iii. $\mathbb{R}$ is order-complete under $\leq$

iv. $\leq$ is compatible with the field structure of $\mathbb{R}$:

- $x \leq y \implies x + z \leq y + z$
- $x \geq 0 \wedge y \geq 0 \implies x \cdot y \geq 0$

Definition 0.6: Limit of sequence

Let X be a topological space. A sequence $s : \mathbb{N} \to X$ has **limit** l iff

$$\forall M \in \mathcal{F}_l,\ \exists N \in \mathbb{N} \ \mid\ n \geq N \implies x_n \in M$$

Definition 0.7: Limit of a function at a point

Let X, Y be two topological spaces. A function $f : X \to Y$ has **limit** l at the accumulation point x_0 iff for all neighborhoods of l, there exists a neighborhood of x_0 such that the image of the neighborhood of x_0 is contained in the neighborhood of l . Mathematically,

$$\lim_{x \to x_0} f(x) = l \iff \forall N \in \mathcal{F}_l,\ \exists M \in \mathcal{F}_{x_0} \ \mid\ f(M) \subset N$$

If x_0 is not an accumulation point, there exists a neighborhood whose image is a singleton or the empty set, which will trivially satisfy the requirement.

REMARK A sequence converging to some limit can be seen as a special case of 0.7 with $\mathbb{N}$ equipped with the *right order topology.*

Definition 0.8: Continuity

Let X, Y be two topological spaces. A function $f : X \to Y$ is **continuous** at the accumulation point $x_0 \in X$ iff the limit of the function at x_0 is equal to the value of the function at said point. Mathematically,

$$f \text{ is continuous at } x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$$

f is said to be *globally continuous* iff it is continuous at every point in A

Definition 0.9: Derivative

Let $x_0 \in A \subset \mathbb{R}$ an accumulation point. A real number $f'(x_0)$ is said to be the **derivative** of f at x_0 iff

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

If $f'(a)$ exists, the function is *differentiable at a.* If it is differentiable at each point of A, the function is *differentiable on A*

Some reads about the definition at an accumulation point:

- Non-unicity of derivative at accumulation points in higher dimensions
- Linear approximation in some neighborhood