Linear Phase Low-Pass IIR Digital Differentiators

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***Abstract*—A novel approach to designing approximately linear phase infinite-impulse-response (IIR) digital filters in the passband region is introduced. The proposed approach yields digital IIR filters whose numerators represent linear phase finite-impulse-response (FIR) filters. As an example, low-pass IIR differentiators are introduced. The range and high-frequency suppression of the proposed low-pass differentiators are comparable to those obtained by higher order FIR low-pass differentiators. In addition, the differentiators exhibit almost linear phases in the passband regions.**

***Index Terms*—Al-Alaoui operator, analog filters, bilinear transformation, digital differentiators, finite-impulse-response (FIR) filters, infinite-impulse-response (IIR) filters, linear phase, low-pass digital differentiators.**

I. Introduction

THIS paper describes the design of infinite-impulse response (IIR) low-pass differentiators which exhibit almost linear phases in the passband region. In many applications, differentiation is followed by low-pass filtering [1]–[6]. Differentiation is used to extract information about rapid transients in the signal. Low-pass filters are used to reject noise frequencies higher than the cutoff frequencies of the signal. Low-pass filtering and differentiation can be implemented as a single low-pass differentiator filter [2]–[5] or by using a low-pass filter and a differentiating filter in cascade [1], [6]. The approach of this paper is the second approach. The resulting low-pass differentiators are IIR differentiators. It is shown that fourth-order differentiators compare favorably with the much higher order state of the art finite-impulse-response (FIR) low-pass differentiators of Kumar and Dutta Roy [2]–[4], and Selesnick [5]. The low order of the proposed low-pass differentiators make them suitable for real-time applications. The accuracy of the proposed low-pass differentiators is comparable to that obtained by higher order filters. An additional advantage is that an almost linear phase is also obtained in the passband region.

The contributions of this paper are as follows.

1) The proposed approach of obtaining an IIR filter with a linear phase in the passband region. The approach utilizes the linear phase properties of FIR filters and the steeper roll-off properties of recursive IIR filters to obtain filters with linear phase in the passband region. This approach although used here to obtain low-pass differentiators could have wider applications.
2) Two methods are proposed to obtain novel low-pass IIR differentiators with linear phases in the passband regions. The first method employs cascading a wide-band differentiator with appropriate low-pass filters so that a linear phase in the passband region is obtained. The second method is a constrained optimization method where the numerator of the transfer function of the IIR filter corresponds to a linear phase filter while the coefficients of the denominator are allowed to vary to obtain an optimum solution. The resulting low-pass differentiators compare favorably with the state of the art of higher order, six or seven times higher, FIR low-pass differentiators.
3) Although this paper presents low-pass differentiators, the proposed approach can be generalized to design other types of filters.

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II. Linear Phase IIR Filters

Digital filters with exact linear phase may only be obtained by designing symmetric coefficients FIR filters. IIR filters may be designed to meet the magnitude requirements with much smaller orders than their FIR counterparts, at the expense of obtaining nonlinear phases. However, it is possible to design IIR filters with approximately linear phases in the passbands. Typically, for the same magnitude response specifications, the order of the resulting IIR filter is one sixth the order of corresponding FIR filter [7]. With the additional requirement of an approximately linear phase in the passband, the ratio of the order of the IIR filter to the corresponding FIR filter should increase, however it should remain low enough to be attractive to design IIR filters that have approximately linear phases in the passbands and meet the magnitude specifications. The prevalent approaches to obtain IIR filters with approximately linear phases in the passbands are two-step processes and are briefly mentioned below.

A. Traditional Approach

The traditional approach consists of the following two-step process [8]:

1) an IIR filter is designed to satisfy the magnitude specifications;
2) an all-pass equalizer is designed which is cascaded with the IIR filter that was obtained in the first step.

The equalizer provides the proper phase compensation such that the overall system would have an approximately linear phase in the passband. The equalizer increases considerably the order of the overall IIR filter and thus reduces the computational advantage of the approach.

B. Model Reduction Approaches

The model reduction approaches employ the following two-step processes [9]–[12]:

1) design linear phase FIR filters that meet the magnitude specifications;
2) obtain an IIR filter that satisfies the magnitude specifications and approximates a linear phase in the passband by using model reduction procedures.

The design processes can be quite involved and the reported order reductions to IIR from FIR varies from 12/29 in [9], 12/46 in [11], to 12/60 in [12].

C. Polyphase Approach

Another approach uses a two path polyphase structure to obtain approximately linear phase IIR filters. A modification of the above standard structure replaces the all-pass filter in the lower branch with a bulk delayer of the same order as the all-pass filter in the top branch [13]. The reported IIR filters are of order 14. Although order reduction was not employed, and hence no comparison with corresponding FIR filters were given, it may be inferred from the result in Section II-B that the ratio is no lower than 14/60.

D. Powell and Chau Approach

Another famous approach is the Powell and Chau approach, which implements a linear phase IIR filter as a tandem connection of an arbitrary transfer function $\mathrm{H}(\mathrm{z})$ and a time-reversal version of the same function $\mathrm{H}(\mathrm{z}^{-1})$ [14]–[16]. Maeng and Lee [15], state in their concluding remarks that the fundamental limitation of the approach's "LIFO based implementation is in the group delay…the application of this method could be limited in real-time interactive signal processing."

III. Basic Concept of Linear Phase IIR Low-Pass Differentiators

The basic concept is to obtain an IIR filter that satisfies the magnitude specifications and exhibits a linear phase in the passband region. This could be realized by designing an IIR filter whose transfer function numerator polynomial represents a linear phase FIR filter and whose transfer function denominator polynomial contributes poles that provides the desired steep roll-off for the magnitude of the frequency response while preserving the linear phase in the passband region. The above is possible if the resulting function had poles and zeros that are sufficiently far apart. The proposed approach utilizes the linear phase properties of the FIR filters and the smaller transition band and the steeper roll-off properties of the IIR filters, compared to the same order FIR filter, to obtain low-order low-pass differentiators with an approximately linear phase in the passband region. The resulting IIR filter order should approximate a corresponding FIR filter with a greater order of reduction than had been obtained so far by other methods. The lowest ratio that was reported of the orders of the IIR filters, with linear phases in the passband, to the order of the FIR filters that they were approximating, was one fifth.

At the outset, an IIR filter that approximates an FIR filter should have an order at most equal to one fifth the order of the FIR filter. In this paper, the Selesnick's FIR low-pass differentiators that are being approximated are of order 29 or 30 (types III and IV) [5]. Thus, the IIR filter is chosen to be order 4 or 5.

It was tempting at this stage to apply least-squares (LS) methods to obtain the fourth-order or a fifth-order IIR low-pass differentiators that approximate the Selesnick differentiators, by optimizing both the numerator and denominator coefficients. The time-domain methods of Prony and Shank and the frequency-domain approach utilizing the iterative Fletcher and Powell method were employed [18]. The time-domain methods of Prony and Shank yielded unsatisfactory results for both the amplitude and the phase. The frequency-domain method of Fletcher and Powell yielded unsatisfactory results for the magnitude response while yielding a good approximation for the phase response. The results are omitted for brevity. Better results would have been obtained if the orders of the IIR filters were increased.

In this paper the basic concept is realized by one of two approaches, described below.

A. Cascade Approach

This approach employs the following two-step procedure:

1) obtain a low-order IIR differentiator whose transfer function has a numerator that represents a linear phase FIR filter;
2) cascade the above differentiator with an IIR low-pass filter whose numerator also represents a linear phase IIR filter.

It is shown that the desired low-pass IIR filters may be obtained by applying the bilinear transformation to the class of all-pole analog filters or to analog filters that have zeros only in the stopbands.

B. Constrained Optimization Approach

This approach employs the following two-step procedure:

1) design a low-order FIR filter, the same order as that chosen for the IIR filter, to meet the specifications in the passband (this fixes the coefficients of the numerator);
2) apply optimization procedures to obtain the coefficients of the denominator of the IIR filter (this step is a constrained optimization procedure).

It should be noted that the second approach could be applied to design any type of IIR filters, not only low-pass differentiators.

In the rest of the paper, the Nyquist frequency, which refers to half the sampling frequency, will be normalized to 1. This is equivalent to dividing the radian frequency, in radians per sample, by π. Thus, the frequency axis will be in units of π.

IV. Cascading Approach

A. First Step

Two IIR differentiators are used in the construction of the proposed low-pass differetiators.

1) Differentiator I: This is a first-order, wide-band differentiator, obtained by interpolating the trapezoidal and the rectangular integration rules [18]–[20], and it is also known as the

Al-Alaoui operator [21]. The transfer function of this differentiator is given as follows:

$$H(z) = 0.3638\frac{(1 - z^{-1})}{\left(1 + \frac{z^{-1}}{7}\right)}. \quad (1)$$

It is to be noted that the numerator of (1) represents a length-2 odd symmetric FIR filter. It is also interesting to note that the numerator of (1) is the same, except for a gain factor, as that relevant to the two-point difference differentiator [22].

2) Differentiator II: This is a second-order, low-pass differentiator, obtained by inverting the transfer function of the Simpson integrator. The resulting transfer function is then stabilized and its magnitude is compensated [23], [24]. The transfer function of this differentiator is given as follows:

$$H\ (z) = 0.25586\frac{(1 - z^{-2})}{(1 + 0.5358\ z^{-1} + 0.0718\ z^{-2})}. \quad (2)$$

B. Second Step

To obtain a low-pass differentiator, cascade any of the differentiators of the first step with an appropriate low-pass filter. An appropriate cascade of a differentiator and a low-pass filter should be chosen to effect a differentiation action up to the selected cutoff frequency and a sharp roll-off after the cutoff frequency. It is also desirable to maintain the linear phase as close to the cutoff frequency as possible. To that effect, choose an appropriate all pole analog low-pass filter, and to obtain the digital low-pass filter, apply to it the bilinear transformation given as follows:

$$s = \frac{2(1 - z^{-1})}{T(1 + z^{-1})}. \quad (3)$$

Applying the bilinear transformation to an all pole analog filter of order n will yield a digital low-pass transfer function whose numerator will be, up to a multiplying constant, of the form $(1+z^{-1})^n$, and thus symmetry is preserved in the numerator. In addition, both the numerator and denominator will be of order n. Note that the value of T in (3) is immaterial when prewarping is used [25]. In addition, any analog low-pass filter with zeros only in the stopband, including Chebyshev II and elliptic filters, will have linear phase numerators after they are transformed using the bilinear transform into digital filters. These filters will have even better stopband attenuation.

The transfer function resulting from cascading is expressed in (4), shown at the bottom of the page, where $m = n + 1$ if Differentiator I were employed and $m = n + 2$ if Differentiator II were utilized.

In the following, the results of using a third-order Chebyshev I low-pass filter, with 0.1-dB ripple in the passband, will be reported. However, other all pole filter types, such as Butterworth filters, also gave good results. The numerator polynomial of the resulting digital third-order low-pass filter will be, up to a multiplying constant, as follows:

$$(1 + z^{-1})^3 = 1 + 3z^{-1} + 3z^{-2} + z^{-3}. \quad (5)$$

The choice of third order was adopted because it was the lowest order that gave results that were clearly competitive with the recently introduced Selesnick FIR low-pass differentiators. The rest of this section presents the low-pass IIR differentiators, which are obtained by cascading the IIR differentiators of the first step, with the third-order digital Chebyshev I low-pass filters, with 0.1-dB ripple in the passband, which are obtained by applying the bilinear transformation to the corresponding analog all-pole low-pass filters.

1) Low-Pass Differentiator I: The differentiator of (1) was cascaded with third order Chebyshev I low-pass filters having 0.1-dB ripple in the passband and obtained by using the bilinear transform. The numerator polynomial of the resulting transfer function, up to a multiplying constant, is as follows:

$$(1 - z^{-1})(1 + z^{-1})^3 = 1 + 2z^{-1} + 0z^{-2} - 2z^{-3} - z^{-4}. \quad (6)$$

Equation (6) represents an antisymmetric with an odd length, type III, linear phase FIR filter. The coefficients of the resulting low-pass differentiators are ordered in descending powers of z as in (7) below; note that in this case $m = 4$.

$$\begin{aligned} H(z) &= \frac{B(z)}{A(z)} \\ &= \frac{b(1)+b(2)z^{-1}+b(3)z^{-2}+b(4)z^{-3}+b(5)z^{-4}}{a(1)+a(2)z^{-1}+a(3)z^{-2}+a(4)z^{-3}+a(5)z^{-4}}. \end{aligned} \quad (7)$$

Table I shows the coefficients $a(i)$ and $b(i)$ of the resulting low-pass differentiators which reveal the antisymmetry of the numerator.

Fig. 1 shows the amplitude responses of the resulting fourth-order low-pass differentiators compared with the Selesnick type III FIR low-pass differentiators at the normalized cutoff frequencies 0.35, 0.42, 0.52, and 0.7 of full band, respectively. The amplitude responses of the resulting fourth-order low-pass differentiators compared with the Selesnick type IV FIR low-pass differentiators are similar to those shown in Fig. 1 and are omitted for brevity. Fig. 1 shows clearly that the proposed differentiator magnitudes exhibit shorter transition regions and thus better suppression of high-frequency noise compared with the state-of-the-art FIR filters [2]–[5] represented by the Selesnick low-pass differentiators. Fig. 2 shows the phase responses for four representative cutoff frequency

$$H(z) = \frac{b(1) + b(2)z^{-1} + b(3)z^{-2} + \cdots + b(m)z^{1-m} + b(m+1)z^{-m}}{a(1) + a(2)z^{-1} + a(3)z^{-2} + a(4)z^{-3} + \cdots + a(m)z^{1-m} + a(m+1)z^{-m}}. \quad (4)$$

TABLE I
COEFFICIENTS OF THE PROPOSED LOW-PASS DIFFERENTIATORS I

Wc (rad/s)	0.35	0.42	0.52	0.7
b(1)	0.0386	0.0573	0.0897	0.1649
b(2)	0.0772	0.1147	0.1794	0.3298
b(3)	0.0000	0.0000	0.0000	0.0000
b(4)	-0.0772	-0.1147	-0.1794	-0.3298
b(5)	-0.0386	-0.0573	-0.0897	-0.1649
a(1)	1.0000	1.0000	1.0000	1.0000
a(2)	-0.4398	0.0133	0.6228	1.6240
a(3)	0.4672	0.4366	0.5531	1.1710
a(4)	-0.0403	0.0003	0.0768	0.3223
a(5)	-0.0170	-0.0092	0.0011	0.0265

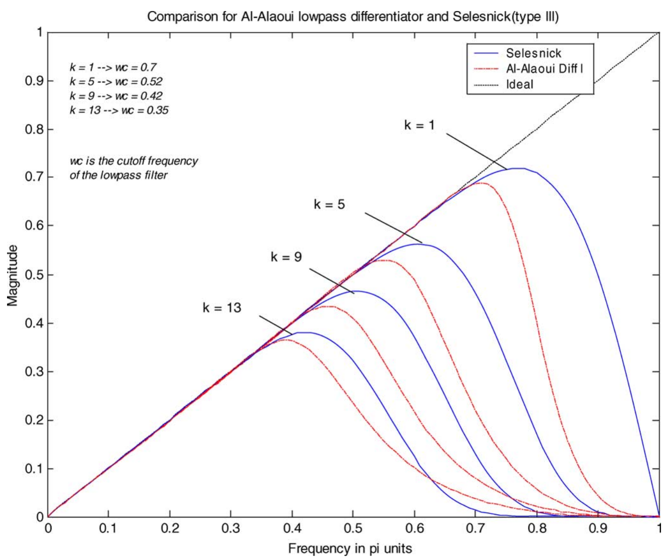

Fig. 1. Comparison of the magnitude responses of the proposed low-pass Differentiator I and the Selesnick low-pass differentiators type III.

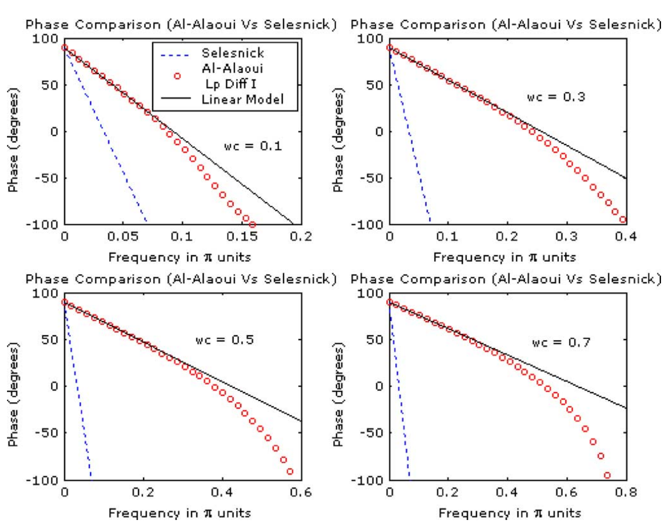

Fig. 2. Comparison of the phases of the proposed low-pass Differentiator I and the Selesnick low-pass differentiators.

which clearly demonstrate that the phases of the proposed differentiators are almost linear up to the cutoff frequencies.

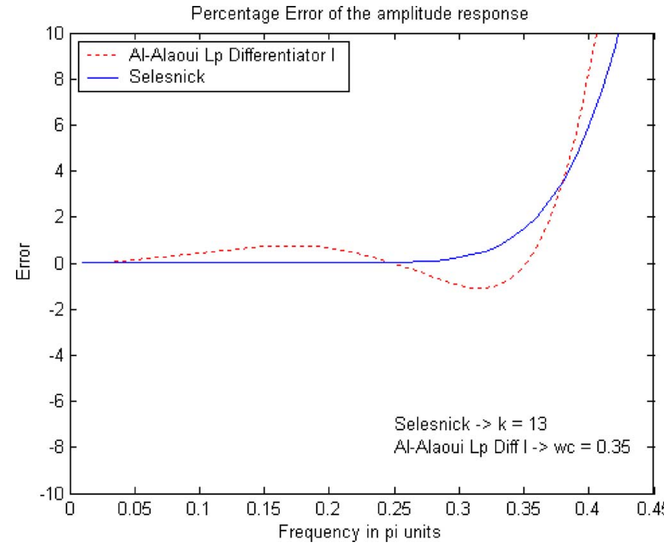

Fig. 3. Comparison of the percentage error of the magnitude response of the proposed low-pass Differentiator I and the Selesnick low-pass differentiators.

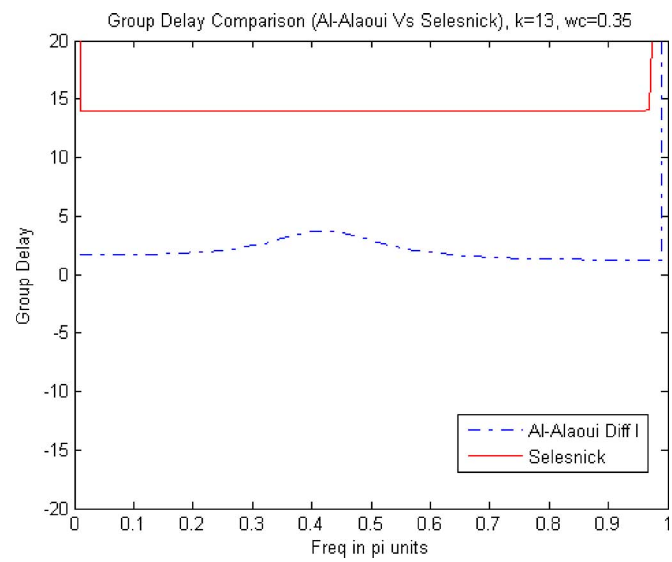

Fig. 4. Comparison of group delays of the Selesnick and the proposed low-pass differentiator for $\omega_c = 0.35\pi$.

Also shown is the much steeper phase response, due to the greater amount of delay, of the Selesnick FIR low-pass linear phase differentiators. Fig. 3 compares the percentage error of the proposed low-pass differentiator with a normalized cutoff frequency of 0.35 with that the FIR differentiator of Selesnick. While the FIR low-pass filters outperform the proposed differentiator, the 1% error is acceptable in many applications.

In addition, the group delay of the proposed IIR low-pass differentiators was found to be almost constant over the passband as evidenced from Fig. 4 for Differentiator I and $\omega_c = 0.35\pi$. Similar results were obtained for other values of passbands and were omitted for brevity.

2) Low-Pass Differentiator II: The differentiator relevant to (2) was cascaded with third order Chebyshev I low-pass filters having 0.1-dB ripple in the passband and obtained by using the

TABLE II
COEFFICIENTS OF THE PROPOSED LOW-PASS DIFFERENTIATORS II

Wc (rad/s)	**0.22**	**0.29**	**0.38**
b(1)	0.0092	0.0178	0.0325
b(2)	0.0277	0.0533	0.0975
b(3)	0.0185	0.0355	0.0650
b(4)	-0.0185	-0.0355	-0.0650
b(5)	-0.0277	-0.0533	-0.0975
b(6)	-0.0092	-0.0178	-0.0325
a(1)	1.0000	1.0000	1.0000
a(2)	-0.9429	-0.4525	0.1501
a(3)	0.3151	0.2623	0.3620
a(4)	0.1809	0.1386	0.1439
a(5)	-0.0691	-0.0427	-0.0150
a(6)	-0.0192	-0.0127	-0.0068

bilinear transform. The numerator polynomial of the resulting transfer function, up to a multiplying constant, is as follows:

$$(1-z^{-2})(1+z^{-1})^3 = 1+3z^{-1}+2z^{-2}-2z^{-3}-3z^{-4}-z^{-5}. \tag{8}$$

Equation (8) represents an antisymmetric with an even length, type IV, linear-phase FIR filter. The coefficients of the resulting low-pass differentiators are ordered in descending powers of z as in (9) below; note that in this case, $m = 5$.

Table II shows the coefficients $a(i)$ and $b(i)$ of the resulting low-pass differentiators, which reveal the antisymmetry of the numerator.

$$\begin{aligned} H(z) &= \frac{B(z)}{A(z)} \\ &= \frac{b(1)+b(2)z^{-1}+b(3)z^{-2}+b(4)z^{-3}+b(5)z^{-4}+b(6)z^{-5}}{a(1)+a(2)z^{-1}+a(3)z^{-2}+a(4)z^{-3}+a(5)z^{-4}+a(6)z^{-5}}. \end{aligned} \tag{9}$$

Fig. 5 shows the amplitude responses of the resulting fifth-order low-pass differentiators compared with the Selesnick type IV FIR low-pass differentiators at the normalized cutoff frequencies 0.22, 0.29, and 0.38. The amplitude responses of the resulting fifth-order low-pass differentiators compared with the Selesnick type III FIR low-pass differentiators are similar to those shown in Fig. 5 and are omitted for brevity.

Fig. 5 shows clearly that the proposed differentiator magnitudes exhibit shorter transition regions and thus better suppression of high-frequency noises compared with the state-of-the-art FIR filters [2]–[5] represented by the Selesnick low-pass differentiators. Fig. 8 shows the phase responses, for four representative cutoff frequencies, which clearly show that the phases of the proposed differentiators are almost linear up to the cutoff frequencies. Also shown is the much steeper linear phase, due to the greater amount of delay, of the Selesnick FIR low-pass differentiators. Fig. 6 compares the percentage error of the proposed low-pass differentiator with a normalized cutoff frequency of 0.35 with that the FIR differentiator of Selesnick.

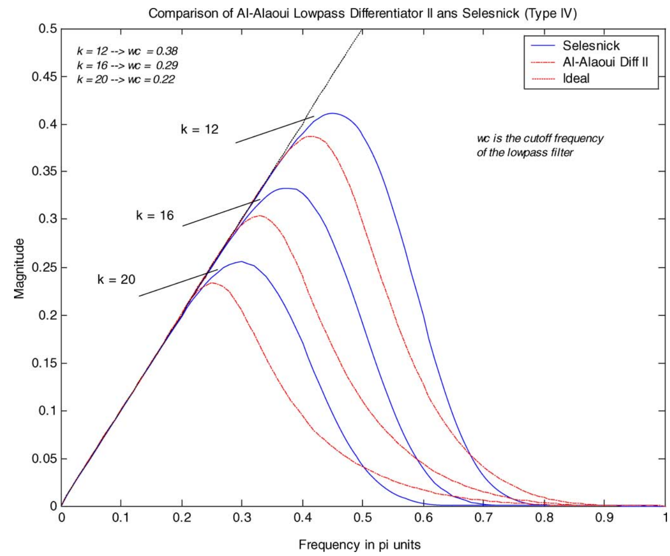

Fig. 5. Comparison of the magnitude responses of the proposed low-pass Differentiator II and the Selesnick low-pass differentiators type IV.

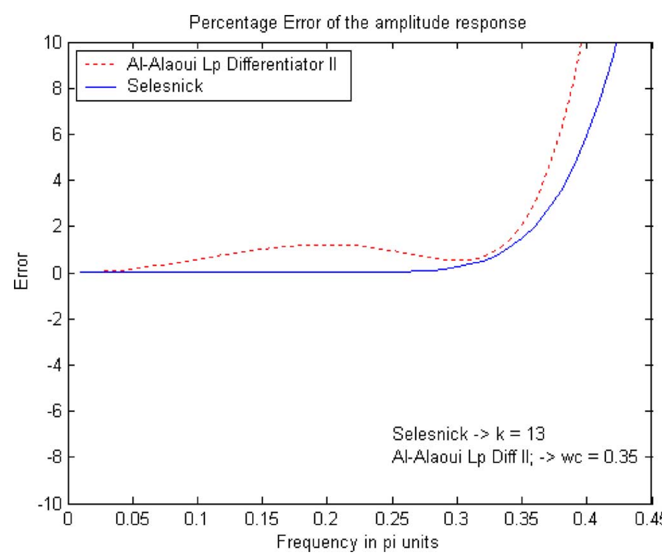

Fig. 6. Comparison of the percentage error of the magnitude response of the proposed low-pass Differentiator II and the Selesnick low-pass differentiators.

While the FIR low-pass filters outperform the proposed differentiator, the 1% error is acceptable in many applications.

C. *Comparison With Parks–McClellan Differentiators*

The responses of order 30 and 31, type III and IV, respectively, of Parks–McClellan linear-phase FIR differentiators were investigated [26], [27]. The results for the magnitudes of the frequency responses, using type III, are shown for the normalized cutoff frequency of 0.7 of full band in Fig. 7(a), and for the normalized cutoff frequency of 0.35 of full band in Fig. 7(b), and reveal that the new differentiators compare favorably with the Parks–McClellan differentiators. The results for the magnitudes of the frequency responses, using type IV and order 31, are almost coincident with those obtained using type III and order 30 and are omitted for brevity. The phases of

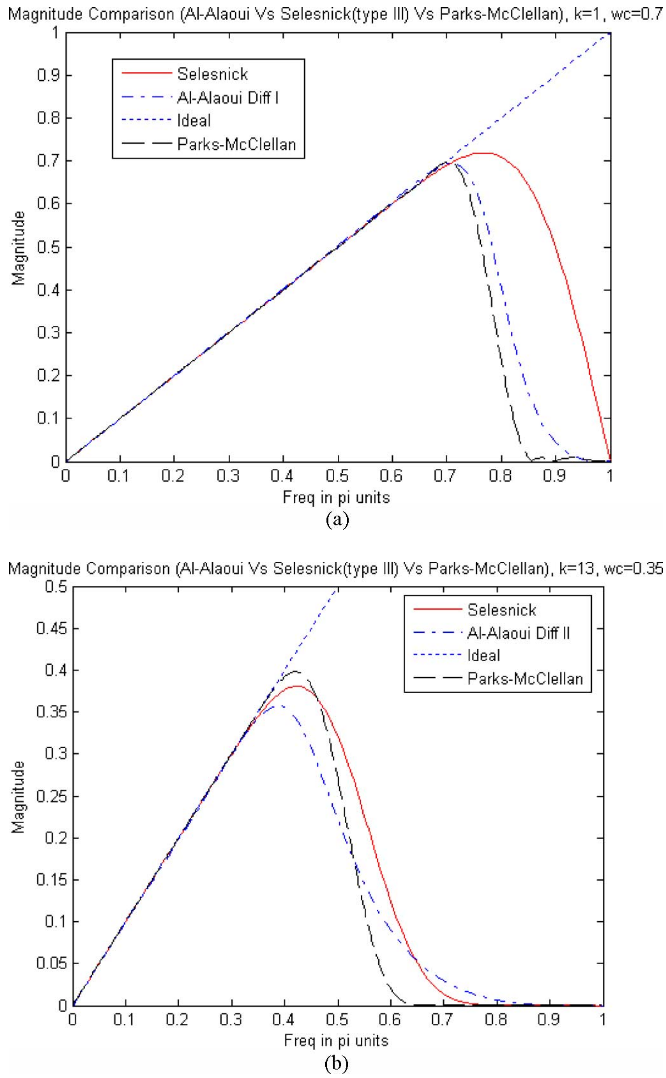

Fig. 7. (a) Comparison of the magnitude responses of the proposed low-pass Differentiator I, with the type III Selesnick and Parks–McClellan low-pass differentiators of order 30 and cutoff frequency of 0.7 of full band. (b) Comparison of the magnitude responses of the proposed low-pass Differentiator I, with the type III Selesnick and Parks–McClellan low-pass differentiators of order 30 and cutoff frequency of 0.35 of full band.

the Parks–McClellan differentiators coincide with the phases of the Selesnick differentiators and are omitted.

V. Constrained Optimization Approach

A. First Step

A choice of one of the following two FIR differentiators, constituting the numerator of the IIR low-pass differentiator, could be employed.

1) FIR Differentiator I: The resulting equation to be optimized is (7). The numerator of (7) is fixed as in (6). Only the gain K and the coefficients of the denominator $a(1)$–$a(5)$ are adjusted as follows:

$$H(z) = \frac{K(1+2z^{-1}-2z^{-3}-z^{-4})}{1+a(1)z^{-1}+a(2)z^{-2}+a(3)z^{-3}+a(4)z^{-4}}.$$

2) FIR Differentiator II: The resulting equation to be optimized is (9). The numerator of (7) is fixed as in (8). Only the gain K and the coefficients of the denominator $a(1)$–$a(6)$ are adjusted as follows:

$$H(z)=\frac{K(1+3z^{-1}+2z^{-2}-3z^{-3}-2z^{-4}-z^{-5})}{1+a(1)z^{-1}+a(2)z^{-2}+a(3)z^{-3}+a(4)z^{-4}+a(5)z^{-5}}.$$

B. Second Step

The coefficients of the denominator polynomials (7) and (9) will be allowed to vary in such a manner as to satisfy the optimization criterion. The approach is thus a constrained least-squares method [28]–[30].

Applying the time-domain approaches of Pade and Prony to the modified (7) and (9), trying to approximate the response of the Selesnick filter, did not yield satisfactory results [17]. The Pade approach yielded good linear phase responses, however, the magnitude responses lost the steep roll-off property that characterizes the new unoptimized low-pass differentiators. On the other hand, the Prony approach yielded a better magnitude response than Pade's; however, the phase response was not sufficiently linear.

Thus, the work was shifted to the frequency-domain approach [17]. In the frequency-domain approach, the magnitude error and the group delay error are included in the optimization criterion. The details of the approach is the same as in [17] and is elaborated further in the Appendix.

In the following we report examples of optimizing low-pass Differentiators I (corresponding to (7), and low-pass Differentiators II, corresponding to the modified (9).

1) Examples of Optimizing Lowpass Differentiator I: The results for the cutoff frequencies $\omega_c = 0.3\pi$, 0.5π, and 0.7π were determined. However, for brevity, only the magnitude, phase, and group delay responses for $\omega_c = 0.5\pi$ are plotted and shown in Fig. 8(a)–(c), respectively. In the figures, the optimized, nonoptimized, and the Selsenick responses are shown. The plots reveal improvements of the phase and group delay responses and a reduction in the magnitude roll-off for the optimized case in comparison with the nonoptimized case.

In the following you may refer to Fig. 8(b). for the case of $\omega_c = 0.5\pi$. At $\omega_c = 0.3\pi$, the value of the ideal linear phase is -12.9°, the initial phase is -31.375°, whereas the optimized phase is -22°, respectively, which corresponds to an improvement of 50.46%.

In addition, at $\omega_c = 0.5\pi$, the value of the ideal linear phase is -15.84°, the initial phase is -47.33°, whereas the optimized phase is -35.35°, respectively, which corresponds to an improvement of 38%.

While at $\omega_c = 0.7\pi$, the value of the ideal linear phase is -10.18°, the initial phase is -66.27°, whereas the optimized phase is -54.98°, respectively, which corresponds an improvement of 20.13%.

The magnitude responses, shown in Fig. 8(a), reveal that the optimized response have a reduced roll-off as compared to the nonoptimized response. Thus, the optimization procedure provides a tradeoff between magnitude responses and phase responses. The results are additionally tabulated in Table III.

2) Examples of Optimizing Low-Pass Differentiator II: The results for the cutoff frequencies $\omega_c = 0.3\pi$, 0.4π, and 0.5π were determined. However, for brevity, only the magnitude,

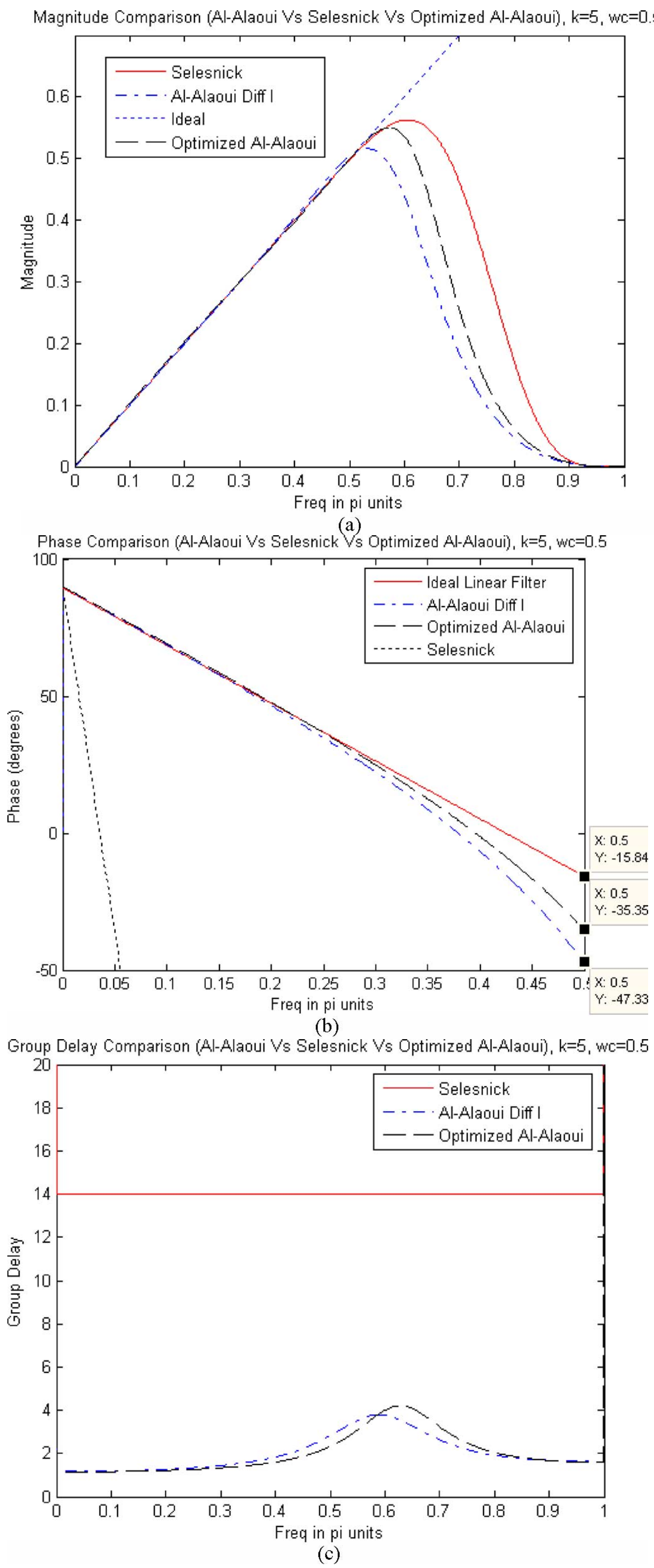

Fig. 8. (a) Comparison of the magnitude responses of the optimized low-pass Differentiator I, low-pass Differentiator I, and the Selesnick low-pass differentiator for $\omega_c = 0.5\pi$. (b) Comparison of the phase responses of the optimized low-pass Differentiator I, low-pass Differentiator I, and the Selesnick low-pass differentiator for $\omega_c = 0.5\pi$. (c) Comparison of the group-delay responses of the optimized low-pass Differentiator I, low-pass Differentiator I, and the Selesnick low-pass differentiator for $\omega_c = 0.5\pi$.

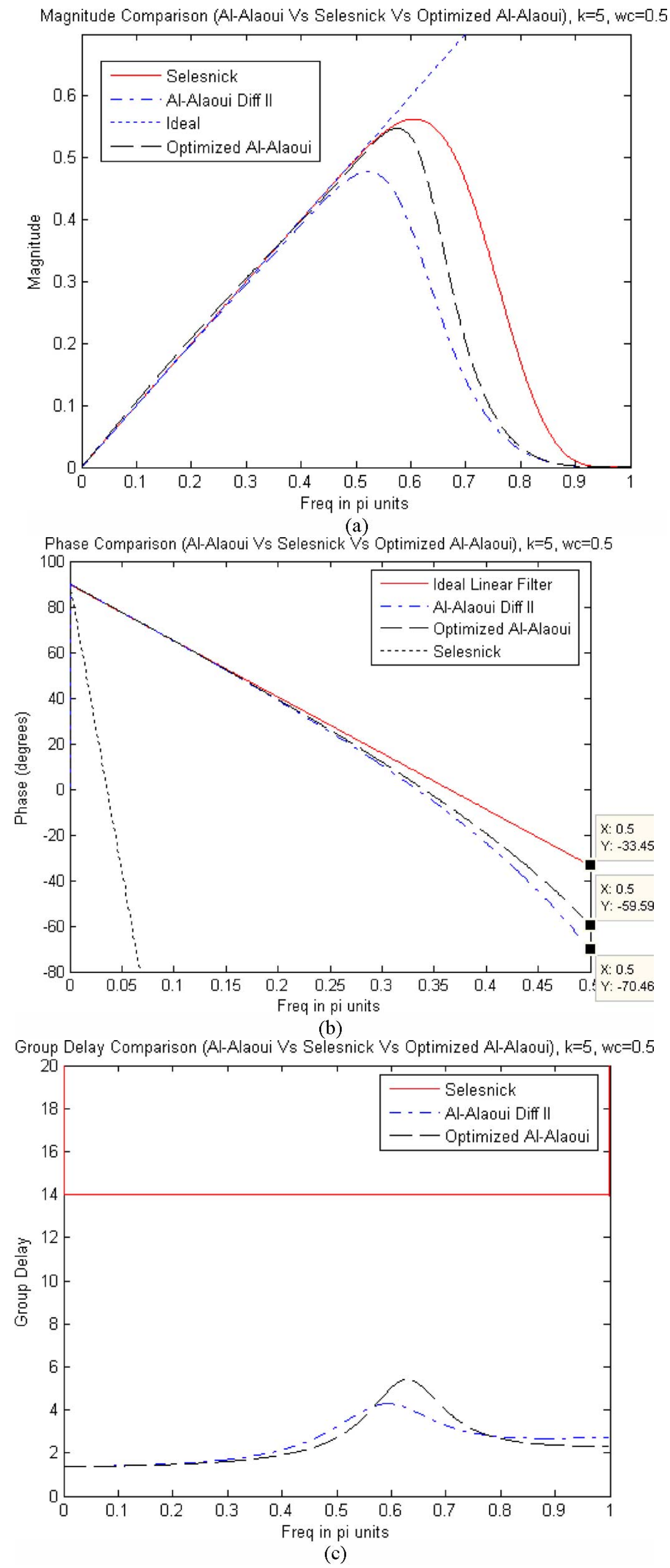

Fig. 9. (a) Comparison of the magnitude responses of the optimized low-pass Differentiator II, low-pass Differentiator II, and the Selesnick low-pass differentiator for $\omega_c = 0.5\pi$. (b) Comparison of the phase responses of the optimized low-pass Differentiator II, low-pass Differentiator II, and the Selesnick low-pass differentiator for $\omega_c = 0.5\pi$. (c) Comparison of the group-delay responses of the optimized low-pass Differentiator II, low-pass Differentiator II, and the Selesnick low-pass differentiator for $\omega_c = 0.5\pi$.

phase, and group-delay responses for $\omega_c = 0.5\pi$ are plotted and shown in Fig. 9(a)–(c), respectively. In the figures, the optimized, nonoptimized, and the Selsenick responses are shown. The plots reveal improvements of the phase and group

TABLE III
COEFFICIENTS OF THE PROPOSED OPTIMIZED LOW-PASS DIFFERENTIATORS

	K	a(1)	a(2)	a(3)	a(4)	a(5)
Dif I – Wc=0.3	0.0953	-0.6744	0.5425	-0.0777	-0.0272	0
Dif I – Wc=0.5	0.2918	0.6524	0.5687	0.0454	-0.0008	0
Dif I – Wc=0.7	0.5324	1.6901	1.2187	0.3095	0.0276	0
Dif II – Wc=0.3	0.0660	-0.3332	0.3041	0.1178	-0.0454	-0.0137
Dif II – Wc=0.4	0.1178	0.3021	0.4396	0.1430	-0.0093	-0.0053
Dif II – Wc=0.5	0.2032	0.9515	0.8383	0.2321	0.0255	-0.0004

delay responses and a reduction in the magnitude roll-off for the optimized case in comparison with the nonoptimized case. In the following you may refer to Fig. 9(b). for the case of $\omega_c = 0.5\pi$.

At $\omega_c = 0.3\pi$, the value of the ideal linear phase is $-24.38°$, the initial phase is $-43.24°$, whereas the optimized phase is $-36.36°$, respectively, which corresponds to an improvement of 36.48%.

In addition, at $\omega_c = 0.4\pi$, the value of the ideal linear phase is $-30.18°$, the initial phase is $-55.97°$ whereas the optimized phase is $-52.53°$, respectively which corresponds to an improvement of 13.34%.

While at $\omega_c = 0.5\pi$, the value of the ideal linear phase is $-33.45°$, the initial phase is $-70.46°$ whereas the optimized phase is $-59.59°$, respectively which corresponds an improvement of 29.37%.

The magnitude responses, shown in Fig. 9(a), reveal that the optimized response have a reduced roll-off as compared with the nonoptimized response. Thus, the optimization procedure provides a tradeoff between magnitude responses and phase responses. The results are additionally tabulated in Table III.

VI. CONCLUSION

Two novel approaches to designing linear phase recursive digital low-pass differentiators were presented. The approaches utilize the linear phase properties of the FIR filters and the steeper magnitude roll-off properties of the IIR filters to obtain IIR low-pass digital differentiators..

The proposed low-pass differentiators were shown to have shorter transition regions, and thus better ability to suppress high frequency noise, for much lower order filters, than the corresponding FIR filters, In addition, the new low-pass differentiators exhibit almost linear phases in their corresponding passband. The examples demonstrate the versatility of the proposed approaches.

The low order of the resulting differentiators makes them suitable for real-time applications. The new low-pass differentiators compare favorably with the state-of-the-art low-pass FIR digital differentiators.

In addition, the approaches could well be employed to design other types of IIR filters that approximate linear phases in the passbands and meet the magnitude specifications at a lower computational cost than the corresponding FIR filters.

APPENDIX
OPTIMIZED LOW-PASS DIFFERENTIATORS

Optimization Criterion: We choose the optimization criterion in such a fashion that the resulting filter will approximate a linear phase filter, or equivalently a constant group delay filter, in the passband region.

The proposed optimization approach is a frequency domain approach.

The transfer function of the low-pass differentiator can be expressed as [17]

$$H(z) = G\prod_{k=1}^{K}\frac{1+\beta_{k1}z^{-1}+\beta_{k2}z^{-2}}{1+\alpha_{k1}z^{-1}+\alpha_{k2}z^{-2}} \tag{10}$$

or as a magnitude and phase response

$$H(\omega) = GA(\omega)e^{j\theta(\omega)} \tag{11}$$

$$A(\omega) = = G\prod_{k=1}^{K}\left|\frac{1+\beta_{k1}z^{-1}+\beta_{k2}z^{-2}}{1+\alpha_{k1}z^{-1}+\alpha_{k2}z^{-2}}\right|_{z=e^{j\omega}}. \tag{12}$$

The group delay is expressed as

$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega} = \tau_g(z)|_{z=e^{j\omega}} = -\left[\frac{d\theta(z)}{dz}\right]_{z=e^{j\omega}}\frac{dz}{d\omega}. \tag{13}$$

It can be shown that

$$\tau_g(z) = \mathrm{Re}\left\{\sum_{k=1}^{K}\left[\frac{\beta_{k1}z+2\beta_{k2}}{z^2+\beta_{k1}z+\beta_{k2}} - \frac{\alpha_{k1}z+2\alpha_{k2}}{z^2+\alpha_{k1}z+\alpha_{k2}}\right].\right. \tag{14}$$

The optimality criterion is to minimize the following error:

$$\varepsilon(\vec{p},G) = (1-\lambda)\sum_{n=1}^{L} w_n\left[GA(\omega_n)-A_d(\omega_n)\right]^2 + \lambda\sum_{n=1}^{L} v_n\left[\tau_g(\omega_n)-\tau_g(\omega_0)-\tau_d(\omega_n)\right]^2. \tag{15}$$

ε is the total weighted least-squares error over all frequencies

$$\omega_1,\omega_2,\ldots,\omega_L \text{ in } 0 \le \omega \le \pi.$$

$\vec{p}$ is the $4K$-dimensional vector of the coefficients $\{\alpha_{k1}\}$, $\{\alpha_{k2}\}$, $\{\beta_{k1}\}$, and $\{\beta_{k2}\}$.

And $\{\lambda\}$, $\{w_n\}$, and $\{v_n\}$ are weighting factors selected by the designer.

The error in magnitude at a frequency ω_n is $GA(\omega_n) - A_d(\omega_n)$, where $A_d(\omega_n)$ is the desired magnitude response, and the delay error is $\tau_g(\omega_n) - \tau_g(\omega_0) - \tau_d(\omega_n)$, where $\tau_g(\omega_0)$ is the filter delay at some nominal center frequency in the passband, and $\tau_d(\omega_n)$ is the desired delay response of the filter relative to $\tau_g(\omega_0)$.

The gain $\hat{G}$ that minimizes ε is given by

$$\hat{G} = \frac{\sum_{n=1}^{L} w_n A(\omega_n)A_d(\omega_n)}{\sum_{n=1}^{L} w_n A^2(\omega_n)}. \tag{16}$$

The Fletcher and Powell method used in the iteration is described as follows.

1) Assume initial set of parameter values $\vec{p} = \vec{p}_0$.
2) Compute $\varepsilon(\vec{p},\hat{G})$.
3) Evaluate $\partial\varepsilon/\partial\alpha_{k1}, \partial\varepsilon/\partial\alpha_{k2}, \partial\varepsilon/\partial\beta_{k1}, \partial\varepsilon/\partial\beta_{k2}$.

4) Use the derivatives information to change the values of the parameters in a direction that leads to the minimum of $\varepsilon(\vec{p},\hat{G}) \rightarrow$ we get a new set of parameters $\vec{p}=\vec{p}_1$.
5) Go to Step 2) and repeat the above steps.

We want to optimize the coefficients of the denominator of the transfer function only, i.e., we work only with the $\{\alpha_{k1}\},\{\alpha_{k2}\}$ coefficients. And since $\mathrm{a}(1),\ldots,\mathrm{a}(5)$ are a linear combination of $\{\alpha_{k1}\},\{\alpha_{k2}\}$, we use the derivatives with respect to the a's in the algorithm instead of the derivatives with respect to the α's.

Examples of Optimizing Lowpass Differentiator I: The selected frequencies are chosen such that

$$\omega_n = 0 + (\mathrm{n}-1)\cdot\Delta\omega \ \text{ where } \ \Delta\omega = \frac{\pi}{1000}.$$

The group delay was selected to be that of the center passband frequency $\omega_c/2$. The initial coefficients are those of the unoptimized filters. We optimize low-pass differentiator I at the following cutoff frequencies with the corresponding weighting factors.

a) $\boldsymbol{\omega_c = 0.3\pi}$:
 $\lambda = 0.05$
 $\mathrm{w}_n = 1$ for $\omega_n = 0,\ldots,\omega_c$ and $\mathrm{w}_n = 0.8$ for $\omega_n = \omega_c + \Delta\omega,\ldots,\pi$
 $v_n = 1$ for $\omega_n = 0,\ldots,\omega\mathrm{c}$ and $v_n = 0$ for $\omega_n = \omega_c + \Delta\omega,\ldots,\pi$
 stop when $\varepsilon < 3$.

b) $\boldsymbol{\omega_c = 0.5\pi}$:
 $\lambda = 0.1$
 $\mathrm{w}_n = 1$ for $\omega_n = 0,\ldots,\omega_c$ and $\mathrm{w}_n = 0.65$ for $\omega_n = \omega_c + \Delta\omega,\ldots,\pi$
 $v_n = 1$ for $\omega_n = 0,\ldots,\omega\mathrm{c}$ and $v_n = 0$ for $\omega_n = \omega_c + \Delta\omega,\ldots,\pi$
 stop when $\varepsilon < 25$.
 The results are shown in Fig. 8(b).

c) $\boldsymbol{\omega_c = 0.7\pi}$:
 $\lambda = 0.2$
 $\mathrm{w}_n = 1$ for $\omega_n = 0,\ldots,\omega_c$ and $\mathrm{w}_n = 0$ for $\omega_n = \omega_c + \Delta\omega,\ldots,\pi$
 $v_n = 1$ for $\omega_n = 0,\ldots,\omega_c$ and $v_n = 0$ for $\omega_n = \omega_c + \Delta\omega,\ldots,\pi$
 stop when $\varepsilon < 40$.
 The results are tabulated also in Table III.

Examples of Optimizing Low-Pass Differentiator II: The selected frequencies are chosen such that:

$$\omega_n = 0 + (\mathrm{n}-1)\cdot\Delta\omega \ \text{ where } \ \Delta\omega = \frac{\pi}{1000}.$$

The group delay was selected to be that of the center passband frequency $\omega_c/2$. The initial coefficients are those of the unoptimized filters.

We used the following weighting factors:
 $\lambda = 0.1$
 $\mathrm{w}_n = 1.2$ for $\omega_n = 0,\ldots,\omega_c$ and $\mathrm{w}_n = 1.2 * 0.8 = 0.96$ for $\omega_n = \omega_c + \Delta\omega,\ldots,1.1\omega_c$
 $v_n = 1$ for $\omega_n = 0,\ldots,\omega\mathrm{c}$ and $v_n = 0$ for $\omega_n = \omega_c + \Delta\omega,\ldots,\pi$.

We optimize low-pass differentiator II at the following cutoff frequencies:

a) $\boldsymbol{\omega_c = 0.3\pi}$ **:** stop when $\varepsilon < 4$;
b) $\boldsymbol{\omega_c = 0.4\pi}$ **:** stop when $\varepsilon < 7$;
c) $\boldsymbol{\omega_c = 0.5\pi}$ **:** stop when $\varepsilon < 10$.

The results are shown in Fig. 9(b) and are also tabulated in Table III.

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