Probability & Random Processes for Engineers

J. Ravichandran

PROBABILITY AND RANDOM PROCESSES FOR ENGINEERS

Dr. J. Ravichandran
Professor
Department of Mathematics
Amrita Vishwa Vidyapeetham
Coimbatore, India

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Preface

Probability and Random Processes is one of the most important courses being offered in engineering colleges. Particularly, more attention is paid by those who work with signals, random walks, and Markov chains. For aspiring engineering students, whether at graduate level or at postgraduate level, when they take up a project work or research related to signals, image processing, etc., knowledge of random processes is very useful. Looking at the existing texts on Probability and Random Processes, there are no texts that are well-structured ones and as a result it is difficult to understand the basics not to mention about the higher level concepts.

Therefore, in order to help teachers for the best of their teaching and students for the best of their understanding on the subject Probability and Random Processes, first of all I have made attempts to put the contents in well-structured chapters. It is well known that for understanding the concepts of *random processes* from the point of view of teachers and students lies with the understanding of the concepts of *probability and statistics* since the concepts of random processes are built upon the concepts of probability and statistics. Hence, one full chapter is fully dedicated for the topics on probability and statistics.

Probability and Random Processes for Engineers caters to the needs of the engineering students both at graduate and postgraduate levels. The text contains nine chapters that are well organized and presented in an order as the contents progress from one topic in one chapter to another topic in the proceeding chapter. In addition, there are appendixes that help in knowing some of the derivations for the results used in the text. Clearly, the book is user-friendly, as it explains the concepts with suitable examples and graphical representations before solving problems. I am of the opinion that this book will be of much value to the faculty in developing or fine-tuning a good syllabus on *Probability and Random Processes* and also in teaching the subject.

This book will have a tinge of the author's expertise. The author has been teaching this subject for many years at both undergraduate and postgraduate levels and, therefore, this book has been written taking into account the needs of teaching faculty and students. Where appropriate, examples with graphical representations that are engineering in nature are given to illustrate the concepts. A number of problems have been solved and exercise problems are given with answers. Putting it in simple terms, this book is written in such a way that it will stimulate the interest of students in learning of this subject and also in preparing for their examinations.

As an author I always expect from both students and faculty their critical evaluations and suggestions by which the book can be further improved with more fitting scope in future. It will be my pleasure to acknowledge with thanks these criticisms and suggestions.

I take this opportunity to express my sincere thanks to beloved Amma, Her Holiness Mata Amritanandamayi Devi, our Chancellor, Amrita Vishwa Vidyapeetham by whose blessings I initiated this project. I wish to express my gratitude to Br. Abhayamrita Chaitanya, The Pro-Chancellor, Amrita Vishwa Vidyapeetham for his inspiration and support. I am very thankful to Dr. P. Venkat Rangan, The Vice Chancellor, Amrita Vishwa Vidyapeetham, for his support and encouragement. My heartfelt thanks are due to Dr. M. P. Chandrashekaran, former Dean (School of Engineering), Dr. Sasangan Ramanathan, present Dean (School of Engineering) and Dr. S. Krishnamoorty, The Registrar for extending support.

My special thanks are due to all my colleagues in the Department of Mathematics for their cooperation, in particular, to the Chairperson, Dr. K. Somasundaram and Dr. G. Prema, Professor whose continued support helped me to accomplish this goal.

My heartfelt thanks are due to my family members, friends and well-wishers without whose support and encouragement, this book would not have been possible.

Finally, I wish all students and faculty of engineering community for a wonderful learning experience on Probability and Random Processes.

Dr. J. Ravichandran

List of Acronyms

1. Probability (P)
2. Probability mass function (PMF)
3. Probability density function (PDF)
4. Cumulative distribution function (CDF)
5. Joint probability mass function (JPMF)
6. Joint probability density function (JPDF)
7. Expectation (E)
8. Moment generating function (MGF)
9. Variance (V or VAR))
10. Standard deviation (SD)
11. Covariance (C or Cov or Covar)
12. Central limit theorem (CLT)
13. Wide sense stationary (WSS)
14. Strict sense stationary (SSS)
15. Jointly wide sense stationary (JWSS)
16. Jointly strict sense stationary (JSSS)
17. Power spectral density (PSD)
18. Transition probability matrix (TPM)

CHAPTER 1

AN OVERVIEW OF RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

1.0 INTRODUCTION

Probability, random variables and probability distributions play a major role in modeling of random processes in which the outputs observed over a period of time are quite random in nature. As a matter of fact, one can easily understand the concept of random process, if the basics of probability, random variables and distributions are properly understood. As probability goes in tune with random variables and probability distributions, in this chapter we concentrate more on defining the concepts such as random variables and different types of distributions. In fact, in this chapter we present only the essentials that are required for understanding the concept of random process. Similar to that of random variables, in random process also, each outcome is associated with a probability of its happening. Such probabilities may be according to some probability distribution as well. For example, an outcome of a random process may take a form according to the outcomes of tossing a coin, throwing a dice, etc. Or the outcomes of a random process may be according to a uniform distribution, normal distribution, etc. Also, in this chapter, the most required concepts such as expectation, covariance, correlation and multivariable distribution function are considered.

1.1 PROBABILITY

Probability is defined as a measure of degree of uncertainty, that is, a measure of happening or not happening of an event in a trial of a random experiment. Probability can be determined using three different approaches:

1.1.1 Classical Approach

If a trial results in 'n' exhaustive and equally likely events and 'm' of them are favorable to the happening of an event A, then the probability P of happening of the event A, denoted by $P(A)$ is given by

$$P = P(A) = \frac{\text{Favorable number of events}}{\text{Total number of events}} = \frac{m}{n} \tag{1.1}$$

Suppose an event A can occur in 'a' ways and cannot occur in 'b' ways, then the probability that event 'A' can occur is given as

$$P = \frac{m}{n} = \frac{a}{a+b},$$

and the probability that it cannot occur is

$$Q = \frac{n-m}{n} = \frac{b}{a+b} = 1 - P.$$

Obviously, P and Q are non-negative and cannot exceed unity, that is $P + Q = 1$ and $0 \leq P, Q \leq 1$.

1.1.2 Statistical Approach

If a trial is repeated a number of times under essentially homogeneous and identical conditions, meaning that same coin is thrown by same person, then the probability P of happening of the favorable event A, denoted by $P(A)$ is given by

$$P = P(A) = \lim_{n\to\infty} \frac{m}{n} \tag{1.2}$$

where, m is the number of times the favorable event A appears and n is the total number of trials to be conducted.

1.1.3 Axiomatic Approach

Given a sample space, say S, the probability is a function which assigns a non-negative real number in $(0, 1)$ to every event, say A, denoted by $P(A)$ and is called the probability of the event A.

Axioms of Probability

The function $P(A)$ is said to be the probability function defined on a sample space S of exhaustive events if the following axioms hold good.

(i) For each $A \in S$, $P(A)$ is defined, is real and $0 \leq P(A) \leq 1$
(ii) $P(S) = 1$
(iii) If $A_1, A_2, \cdots\cdots, A_n$ are n mutually exclusive (disjoint) events in S, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

1.1.4 Some Important Results

(i) *Additive law:* If A and B are any two events (subsets of sample space) and are not mutually exclusive, then we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{1.3}$$

If the events A and B are mutually exclusive, then the additive law becomes

$$P(A \cup B) = P(A) + P(B) \tag{1.4}$$

(ii) *Conditional probability and multiplicative law:* For two events A and B, we have

$$\begin{aligned} P(A \cap B) &= P(A)\, P(B/A), \quad P(A) > 0 \\ &= P(B)\, P(A/B), \quad P(B) > 0 \end{aligned} \tag{1.5}$$

where $P(A/B)$ represents the conditional probability of occurrence of event A when the event B has already occurred.

If events A and B are independent then

$$P(A \cap B) = P(A)\, P(B) \tag{1.6}$$

If A and B are nonmutually exclusive and independent then we have

$$P(A \cup B) = 1 - P(\overline{A})\; P(\overline{B}) \tag{1.7}$$

1.1.5 Bayes Theorem

If $E_1, E_2, \cdots\cdots, E_n$ are n mutually exclusive events (disjoint events) with $P(E_i) \neq 0$, $i = 1, 2, \cdots\cdots, n$, then for any event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{P(A)}, \qquad i = 1, \cdots\cdots, n \tag{1.8}$$

where

$$P(A) = \sum_{i=1}^{n} P(E_i)\, P(A/E_i). \tag{1.9}$$

1.2 ONE-DIMENSIONAL RANDOM VARIABLE

Random variable is associated with the outcomes of random experiment. A random variable is a variable, say X, that assumes a real number, say x, for each and every outcome of a random experiment. Clearly, in random experiments, the outcomes are associated with their probabilities of happening.

If S is the sample space containing all the n outcomes $\{\xi_1, \xi_2, \cdots\cdots, \xi_n\}$ of an experiment and X is a random variable defined as a function, say $X(\xi)$, on S, then for every outcome ξ_i, $(i = 1, 2, \cdots\cdots n)$, that is, in S, the random variable X will assign a real value x_i as shown below.

Outcome S	ξ_1	ξ_2	$\dots$	ξ_i	$\dots$	ξ_n
Random Variable X	$X(\xi_1)=x_1$	$X(\xi_2)=x_2$	$\dots$	$X(\xi_i)=x_i$	$\dots$	$X(\xi_n)=x_n$

Example 1.1

For example, if two coins are tossed once (or one coin is tossed twice) and if X is a random variable representing number of heads turning up, then we have the possible outcomes and the related random variable as

Outcome S	$\xi_1=HH$	$\xi_2=HT$	$\xi_3=TH$	$\xi_4=TT$
Random Variable X	$X(HH)=2$	$X(HT)=1$	$X(TH)=1$	$X(TT)=0$

Example 1.2

Similarly, if we observe temperature in a place set to be $18\pm2°C$, then we may get the following temperature values at different time points:

Outcome (S): Temperature values at time point	$\xi_1=t_1$	$\xi_2=t_2$	$\xi_3=t_3$	......
Random Variable X	$X(t_1)=18.00$	$X(t_2)=18.03$	$X(t_3)=19.10$	......

From these examples, one can understand that in the first case (Example 1.1) the number of heads obtained can be either 0 or 1 or 2, whereas in the second case (Example 1.2) the temperature at a point of time may be any value within $18\pm2°C$.

1.2.1 Discrete Random Variable

If a random variable assigns only a specific value to each and every outcome of an experiment then such a random variable is called a discrete random variable. In other words, if sample space contains only a finite or countably infinite number of values then the corresponding random variable is called a discrete random variable. Refer Example 1.1 and also see more examples given below:

(i) Number of people arrive at a cinema: 0, 1, 2, ...

(ii) Number of defects per unit of a product: 0, 1, 2, ...

(iii) Readings given in a scale: 0, 0.5, 1.0, 1.5, 2.0, ...

1.2.1.1 Discrete Probability Distribution: Probability Mass Function (PMF)

A discrete random variable obviously assumes a value to each and every outcome of the related random experiment with a probability. For example, out of four outcomes of tossing two coins, HH, (i.e., number of two heads $X = x_3 = 2$) happens once, H and T happen together (i.e., number of one head $X = x_2 = 1$) twice and head does not happen together (i.e., number of heads $X = x_1 = 0$) once. Therefore, the corresponding probabilities (rather, we call them probability masses) are given below:

$X = x$	$x_1 = 0$	$x_2 = 1$	$x_3 = 2$
$P(X = x)$	1/4	1/2	1/4

Here $X = x$ exhausts all possible values 0, 1, 2 and hence the probabilities add to 1. The probabilities shown are, in fact, the weights assigned to each and every value assigned by the random variable. Hence, we have

$$P(X = x_1 = 0) = P(X = 0) = 1/4$$
$$P(X = x_2 = 1) = P(X = 1) = 1/2$$
$$P(X = x_3 = 2) = P(X = 2) = 1/4$$

The probability function $P(X = x)$ of the numerical values of the random variable X, is known as the probability mass function (PMF). A graphical representation of the probability mass function is given in Figure 1.1.

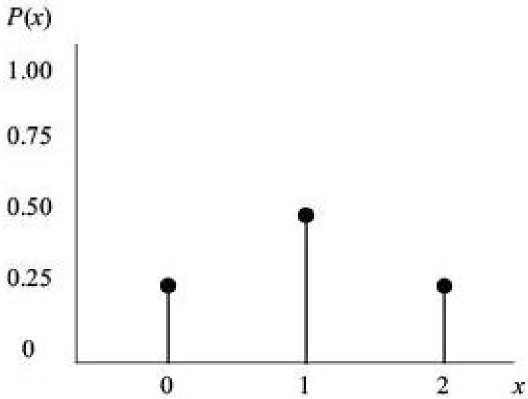


Figure 1.1. Probability mass function $P(X = x)$

Definition

The function $P(X=x)$ of the numerical values of the discrete random variable X is said to be probability mass function (PMF) if it satisfies the following properties:

(i) $0 \leq P(X=x) \leq 1$

(ii) $\sum\limits_{x=-\infty}^{\infty} P(X=x) = 1$

(iii) $P(\bigcup\limits_{i=1}^{n} E_i) = \bigcup\limits_{i=1}^{n} P(E_i)$, if $E_1, E_2, \cdots\cdots, E_n$ are mutually exclusive (disjoint) events.

1.2.2 Continuous Random Variable

If a random variable can assign any value in the given interval covering outcomes of a random experiment, then such a random variable is known as a continuous random variable. In other words, if a random variable can take values on a continuous scale, it is called a continuous random variable. Refer to Example 1.2. Also see the following examples:

(i) Diameter value of a bolt: (2.0 to 2.5) cm

(ii) Circumference of a well: (25 to 32) feet

(iii) Length of a screw: (12.0 to 12.5) mm

(iv) Temperature set in a machine: (16 to 20)°C

One may be of the opinion that since a particular thermometer measures the temperature as 18°C, 19°C, 20°C, 21°C and 22°C, the random variable may be a discrete one. However, there exist measuring equipment (thermometers) that can measure all possible values such as 19.0001°C, 19.0002°C, and so on. This means that the random variable can assign all possible values in the given interval (16 to 20)°C.

1.2.2.1 Continuous Probability Distribution: Probability Density Function (PDF)

A continuous random variable can have probability only for a range assigned by it and as a result it has a zero probability for assuming exactly any of its values. For example, consider a random variable whose values are the heights of all students in a college over 21 years of age. Between any two values, say 165.5 and 166.5 cm, or even 165.99 and 166.01 cm, there are an infinite number of heights, one of which is 166 cm. Therefore, the probability of selecting a person whose height is exactly 166 cm is assigned to be zero. However, we can compute the probability of selecting a person whose height is at least 165 cm but not more than 166 cm, and so on. Therefore, one can deal with an interval rather than a point value of the random variable.

1.2.3 Cumulative Distribution Function (CDF)

If X is a random variable then its cumulative distribution function (CDF) denoted by $F(x)$, also denoted by $F_X(x)$, is given as

(i) $F(x) = P(X \leq x) = \sum_{x_i = -\infty}^{x} P(X = x_i)$ if X is discrete random variable

(ii) $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$ if X is continuous random variable

Properties

(i) $P(a \leq X \leq b) = F(b) - F(a)$ if X is continuous
(ii) $F(b) \geq F(a)$ for all $b \geq a$
(iii) $F(-\infty) = 0$ and $F(\infty) = 1$

It is important to note that, if X is a discrete random variable then the difference $F(b) - F(a)$ gives $P(a < X \leq b)$ and not $P(a \leq X \leq b)$. Therefore, if X is a discrete random variable then, $P(a \leq X \leq b) = F(b) - F$ (the immediate value less than a).

The cumulative distribution function $F(x)$ and the probability density function $f(x)$ are related as

$$F(x) = P\{X \in (-\infty, x)\} = \int_{-\infty}^{x} f(x)dx$$

Differentiating both sides yields

$$F'(x) = \frac{d\,F(x)}{dx} = f(x).$$

That is, the density function is the derivative of the cumulative distribution function.

1.3 EXPECTATION (AVERAGE OR MEAN)

1.3.1 Definition and Important Properties of Expectation

Definition

If X is a random variable then the expectation of X, denoted as $E(X)$ or simply μ_x, is given as

$$E(X) = \mu_x = \sum_{x=-\infty}^{\infty} x\,P(X = x) \quad \text{if } X \text{ is discrete random variable}$$

$$E(X) = \mu_x = \int_{-\infty}^{\infty} x\,f(x)\,dx \quad \text{if } X \text{ is continuous random variable}$$

Since $V(X) \geq 0$, we have

$$E(X^2) \geq \{E(X)\}^2 \tag{1.13}$$

Let $Z = XY$, then

$$E(Z^2) \geq \{E(Z)\}^2$$

$$\Rightarrow \quad E(X^2Y^2) \geq \{E(XY)\}^2 \tag{1.14}$$

If X and Y are not independent, we have

$$E(X^2Y^2) = E(X^2)E(Y^2/X)$$

$$\Rightarrow \quad \{E(XY)\}^2 \leq E(X^2)E(Y^2/X)$$

If X and Y are independent, we have

$$\{E(XY)\}^2 \leq E(X^2)E(Y^2) \tag{1.15}$$

which is known as *Cauchy-Schwarz inequality*.

(iv) ***Standard deviation:*** It is important to note that variance of a random variable gives only a squared average value. Hence, square root is taken over the variance to get a meaningful deviation of each observation from its own mean. That is, the standard deviation of the random variable X, denoted by σ_x is the square root of the variance and is given by

$$SD(X) = \sigma_x = \sqrt{V(X)} \tag{1.16}$$

1.3.2 Moments and Moment Generating Function

Raw moments

It may be noted that $E(X^r)$, $r = 1, 2, 3, 4, \cdots\cdots$ are known as raw moments (or moments about origin) of order r. For example, the mean $E(X)$ is the first order moment and is obtained with $r = 1$ and $E(X^2)$ is the second order is obtained with $r = 2$ and so on. The r^{th} order raw moments $E(X^r)$, $r = 1, 2, 3, 4, \cdots\cdots$ can be obtained as

$$\mu_r' = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = E(X^r)$$

where $M_X(t) = E(e^{tx})$ is known as moment generating function (MGF) and is given as

$$\begin{aligned} M_X(t) &= \sum_{x=-\infty}^{\infty} e^{tx} P(X = x), \quad \text{if } X \text{ is discrete} \\ M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, \quad \text{if } X \text{ is continuous} \end{aligned} \tag{1.17}$$

Central moments

$\mu_r = E[(X - E(X)]^r, \quad r = 1, 2, 3, 4, \cdots\cdots$ are known as the r^{th} order central moments (or moments about mean) as the deviations are taken from the mean. Clearly, the first order moment is given as $\mu_1 = E[(X - E(X)] = 0$ and the *second order moment* is given as $\mu_2 = E[(X - E(X)]^2$, which is the variance of X.

1.3.3 Characteristic Function

Similar to that of moment generating function, raw moments can also be generated by another function called characteristic function, denoted by $\Phi_X(t)$. That is, if X is a random variable, then its characteristic function is defined as

$$\begin{aligned} \Phi_X(t) &= \sum_{x=-\infty}^{\infty} e^{itx} P(X = x), \quad \text{if } X \text{ is discrete} \\ \Phi_X(t) &= \int_{-\infty}^{\infty} e^{itx} f(x)\,dx, \quad \text{if } X \text{ is continuous} \end{aligned} \tag{1.18}$$

where the imaginary number $i = \sqrt{-1}$.

Now the r^{th} order raw moments can be obtained as

$$\mu_r' = (-i)^r \left.\frac{d^r \Phi_X(t)}{dt^r}\right|_{t=0} = E(X^r)$$

1.4 SPECIAL DISTRIBUTION FUNCTIONS

1.4.1 Binomial Random Variable and Its Distribution

Let us suppose that an experiment with n independent trials, each of which results in a success with probability p, is to be performed. If X represents the number of successes that occurs in the n trials, then X is said to be a binomial random variable if its probability mass function is given by

$$P(X = x) = {}^nC_x p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots\cdots, n \tag{1.19}$$

$$\text{Or} \quad P(X = x) = {}^nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \cdots\cdots, n; \quad q = 1 - p$$

In other words, a random variable X that follows Binomial distribution with parameters n and p is usually denoted by: $X \sim B(n, p)$

Since a $B(n, p)$ random variable X represents the number of successes in n independent trials, each of which results in a success with probability p, we can represent it as follows:

$$X = \sum_{i=1}^{n} X_i$$

$$\text{where} \quad X_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ trial is success} \\ 0 & \text{if } i^{\text{th}} \text{ trial is failure} \end{cases} \tag{1.20}$$

Here each X_i is also known as a *Bernoulli random variable*. In other words, a trial with only two outcomes (a success and a failure) is known as a Bernoulli trial. Obviously, an experiment comprising Bernoulli trials is known as a *Bernoulli experiment*. Therefore, the probability of getting x successes out of n Bernoulli trials represents the binomial probability. An example for binomial distribution with $n = 10$ and $p = 0.5$ is shown in Figure 1.3.

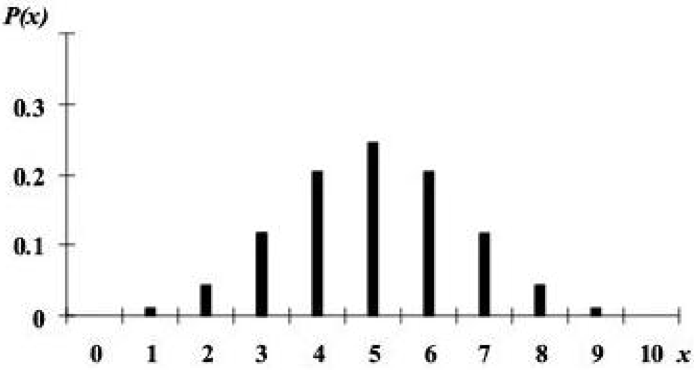


Figure 1.3. Binomial distribution with parameters $n = 10$ and $p = 0.5$

1.4.1.1 Derivation of Mean and Variance using Moment Generating Function

By definition, we know that

$$M_X(t) = E(e^{tx}) = \sum_{x=-\infty}^{\infty} e^{tx} P(X = x)$$

$$= \sum_{x=0}^{n} e^{tx}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x\, (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n$$

$$\text{Now, Mean} = \mu_1' = E(X) = \left.\frac{dM_X(t)}{dt}\right|_{t=0}$$

$$\Rightarrow \quad \frac{dM_X(t)}{dt} = \frac{d}{dt}\left[(pe^t + q)^n\right]$$

$$= n\left(pe^t + q\right)^{n-1} pe^t$$

$$\Rightarrow \quad \left.\frac{dM_X(t)}{dt}\right|_{t=0} = n\,(p+q)^{n-1}\,p$$

$$= np \quad (\because \quad p+q=1)$$

We know that $V(X) = E(X^2) - \{E(X)\}^2 = \mu_2' - \left(\mu_1'\right)^2$

Consider $$\mu_2' = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \frac{d}{dt}\left[npe^t\left(pe^t+q\right)^{n-1}\right]\Big|_{t=0}$$

$$= np[e^t\,(n-1)\left(pe^t+q\right)^{n-2} pe^t + \left(pe^t+q\right)^{n-1} e^t]\Big|_{t=0}$$

$$= np\,[(n-1)\,(p+q)^{n-2}\,p+1]$$

$$= np\,[(n-1)\,p+1]$$

$$\Rightarrow \quad V(X) = np\,[(n-1)\,p+1] - (np)^2 = npq.$$

1.4.2 Poisson Random Variable and Its Distribution

It is established that Poisson distribution is a limiting case of binomial distribution under certain conditions that

(i) n, the number of trials is large, say $n \geq 30$, and

(ii) p, the probability of success is small, say $p \leq 0.05$.

Similar to binomial distribution, in many real life situations one is often interested in measuring the number of incidences happening in a particular point of time, in a particular location, in a particular inspection, etc. Under these circumstances, there are cases where nil or one or more incidents might be happening. Also, since one can always count the number of favorable incidents out of infinite observations the case is approximated as Poisson distribution.

Accordingly, a random variable X that represents the number of incidents with one of the values $0, 1, 2, \cdots\cdots$ is said to be a Poisson random variable with parameter $\lambda > 0$, if its probability mass function is given by

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \cdots\cdots, \tag{1.21}$$

where $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$, is a commonly used constant in mathematics and is approximately equal to 2.7183.

A random variable X that follows Poisson distribution with parameter λ is usually denoted by: $X \sim P(\lambda)$. An example for Poisson distribution with $\lambda = 1.5$ is shown in Figure 1.4.

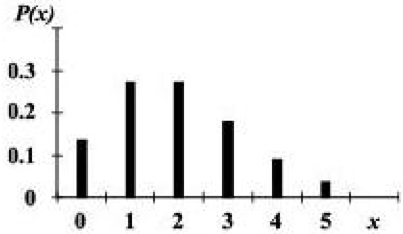


Figure 1.4. Poisson distribution with parameter $\lambda = 1.5$

1.4.2.1 Derivation of Mean and Variance using Moment Generating Function

By definition, moment generating function is given as

$$M_X(t) = E(e^{tx}) = \sum_{x=-\infty}^{\infty} e^{tx} P(X = x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \left[1 + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \ldots\ldots\right]$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

$$\therefore \quad \text{Mean} = \mu_1' = E(X) = \left.\frac{d M_X(t)}{dt}\right|_{t=0}$$

$$\Rightarrow \quad \frac{d M_X(t)}{dt} = \frac{d}{dt}\left[e^{\lambda(e^t - 1)}\right]$$

$$= e^{\lambda(e^t - 1)} (\lambda e^t)$$

$$\Rightarrow \quad \left.\frac{d M_X(t)}{dt}\right|_{t=0} = \lambda$$

Also we know that variance is given by

$$V(X) = E(X^2) - \{E(X)\}^2 = \mu_2' - (\mu_1')^2$$

Consider

$$\begin{aligned}
\mu_2' &= \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} \\
&= \frac{d}{dt}\left.\left[\lambda\, e^{\lambda(e^t-1)+t}\right]\right|_{t=0} \\
&= \left.\left[\left(\lambda\, e^{\lambda(e^t-1)+t}\right)(\lambda\, e^t + 1)\right]\right|_{t=0} \\
&= [\lambda\,(\lambda+1)] \\
&= \lambda^2 + \lambda
\end{aligned}$$

$$\therefore \quad V(X) = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

It is important to note that in case of Poisson distribution, the mean and variance are same.

1.4.3 Uniform Random Variable and Its Distribution

A continuous random variable X is said to be uniformly distributed over the interval (a, b), $a < b$, if its probability density function is given by

$$f(x) = \frac{1}{b-a}, \quad a < x < b \tag{1.22}$$

As shown in Figure 1.5, the random variable X is uniformly distributed over (a, b), meaning that it puts all its mass on that interval and any point in this interval is equally likely to occur. By virtue of its appearance as in Figure 1.5, the uniform distribution is also called a "rectangular distribution".

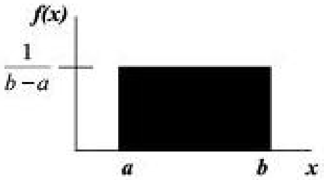


Figure 1.5. Uniform density function distributed in (a, b)

Contents

The cumulative distribution function of the uniform random variable X is given, for $a < x < b$, by

$$P(X \leq x) = \int_a^x \frac{1}{b-a}\, dx = \frac{x-a}{b-a}$$

Notationally, a uniform random variable X taking values in the interval a and b is usually denoted by: $X \sim U(a, b)$.

1.4.3.1 Derivation of Mean and Variance using Moment Generating Function

Consider the r^{th} moment

$$\mu_r' = E(x^r) = \int_a^b x^r \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b = \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)}$$

Now, letting $r = 1$, we have

$$\text{Mean } = \mu_1' = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

Similarly, if we let $r = 2$, we have

$$\mu_2' = \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{(b^2 + ab + a^2)}{3}$$

Therefore, variance is given by

$$V(X) = E(X^2) - \{E(X)\}^2 = \mu_2' - (\mu_1')^2$$

$$= \frac{(b^2 + ab + a^2)}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

1.4.4 Normal Random Variable and Its Distribution

The normal distribution is also popularly called a Gaussian distribution. Normal distribution is the most adaptive distribution to any data on hand. In fact, in many real life situations, knowingly or unknowingly, it is often assumed that the set of observations on hand are assumed to normally distribute. This is due to the reason that many phenomena that occur in nature, industry, and research appear to have the features of normality. Physical measurements in areas such as temperature studies, meteorological experiments, rainfall studies and product dimensions in manufacturing industries are often conveniently explained with a normal distribution. This is particularly true because, when the size of the sample increases, almost all distributions can be approximated to a normal distribution. Of course, the data analyses with normality assumption are always handy.

A random variable X is said to be normally distributed with parameters μ and σ if its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), \quad -\infty < x < \infty \tag{1.23}$$

In fact, in case of normal distribution, we have μ as mean and σ^2 as variance (or σ as standard deviation)

The normal density function is a bell-shaped curve that is symmetric about mean μ (refer Figure 1.6).

A random variable X that follows normal distribution with mean μ and variance σ^2 is notationally given by: $X \sim N(\mu, \sigma^2)$.

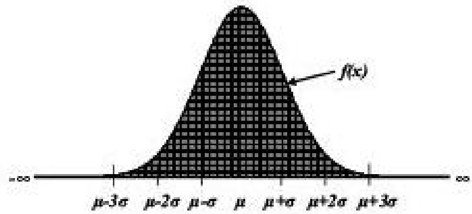


Figure 1.6. Normal density function with parameters (μ and σ)

1.4.4.1 Properties of Normal Distribution

(i) The normal curve is bell-shaped and symmetric about mean μ.

(ii) Mean, median and mode of the normal distribution coincide.

(iii) As x increases numerically, $f(x)$ increases and then decreases rapidly. The maximum ordinate is occurring at the point $x = \mu$, and is given by $\dfrac{1}{\sqrt{2\pi}\sigma}$.

(iv) Skewness is = 0 (being symmetric) and kurtosis is = 3 (nominal peakness)

(v) Area property: probabilities (percentages) of area coverage are
$P(\mu - \sigma < X < \mu + \sigma) = 0.6826$ (68.26%)

$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$ (95.44%) and $P(\mu - 1.96\sigma < X < \mu + 1.96\sigma) = 0.95$ (95%)

$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$ (99.73%) and $P(\mu - 2.58\sigma < X < \mu + 2.58\sigma) = 0.99$ (99%)

1.4.4.2 Standard Normal Density and Distribution

If X is a normal random variable with mean μ and variance σ^2, then for any constants a and b, $aX + b$ is normally distributed with mean $a\mu + b$ and variance $a^2\,\sigma^2$. It follows from this fact that if X is a normal random variable with mean μ and variance σ^2, then the variable given by

$$Z = \frac{X - \mu}{\sigma}$$

is normal with mean 0 and variance 1. Such a random variable Z is said to have a standard normal distribution.

A random variable Z that follows standard normal distribution with mean 0 and variance 1 is usually denoted by: $Z \sim N(0,\ 1)$. Accordingly, the probability density function of the standard normal variate is given as

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right), \quad -\infty < z < \infty, \tag{1.24}$$

If we let $\varphi(z)$ as the distribution function of a standard normal random variable then we have

$$\varphi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz, \quad -\infty < z < \infty,$$

The result that $Z = \dfrac{X - \mu}{\sigma}$ has a standard normal distribution when X is normal with mean μ and variance σ^2 is quite useful because it allows us to evaluate all probabilities concerning X in terms of φ. For example, the distribution function of X can be expressed as

$$\begin{aligned} F(x) = P(X \le x) &= P\left(\frac{X - \mu}{\sigma} \le \frac{x - \mu}{\sigma}\right) \\ &= P(Z \le z) = \varphi(z) \end{aligned}$$

The value of $\varphi(z)$ can be determined either by looking it up in a table or by writing a computer program to approximate it. For $\alpha \in (0,\ 1)$, let z_α be such that $P(Z > z_\alpha) = 1 - \varphi\ (z_\alpha) = \alpha$. That is, a standard normal variate will exceed z_α with probability α, refer Figure 1.7. The value of z_α can be obtained from a table of the values of $\varphi(z)$ (Refer Table given in Appendix C). For example, consider the following:

$$\varphi(1.64) = 0.950 \Rightarrow z_{0.05} = 1.64$$
$$\varphi(1.96) = 0.975 \Rightarrow z_{0.025} = 1.96$$
$$\varphi(2.33) = 0.990 \Rightarrow z_{0.01} = 2.33$$
$$\varphi(2.58) = 0.995 \Rightarrow z_{0.005} = 2.58$$

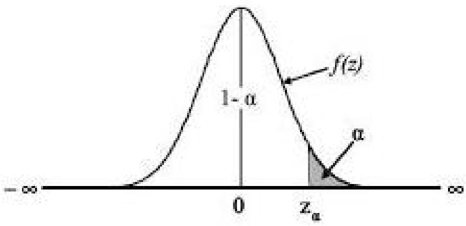


Figure 1.7. Standard normal distribution: $P(Z > z_\alpha) = 1 - \varphi\ (z_\alpha) = \alpha$

1.4.4.3 Derivation of Mean and Variance Using Moment Generating Function

By definition the moment generating function is given by

$$M_X(t) = E\left(e^{tX}\right)$$
$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$
$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Letting $z = (x - \mu)/\sigma \Rightarrow \quad x = \sigma z + \mu$ and $dx = \sigma dz$, we have

$$M_X(t) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{t(z\sigma+\mu)}\, e^{-\frac{1}{2}z^2}\ \sigma dz$$
$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z^2 - 2t\sigma z\right)}\, dz$$

$$= \sigma^2 + \mu^2$$

$$\therefore \quad V(X) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

1.5 CHEBYSHEV'S THEOREM AND CENTRAL LIMIT THEOREM

1.5.1 Chebyshev's Theorem

Chebyshev's theorem is useful in finding the bounds on probability when the distribution of interest is not known but the mean and variance of the distribution are known.

If X is any random variable (whether discrete or continuous) with arbitrary mean $E(X) = \mu$ and arbitrary variance $V(X) = \sigma^2$, then

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2} \quad \text{(Upper bound for probability)}$$

$$\text{or} \quad [P(|X - \mu|) \leq k] \geq 1 - \frac{\sigma^2}{k^2} \quad \text{(Lower bound for probability)} \tag{1.25}$$

where k is a non-zero constant. That is $k > 0$. If we replace k by $k\sigma$, the above inequalities can also be written as

$$[P(|X - \mu|) \geq k\sigma] \leq \frac{1}{k^2}$$

$$\text{or} \quad [P(|X - \mu|) \leq k\sigma] \geq 1 - \frac{1}{k^2}$$

1.5.2 Central Limit Theorem

If $X_1, X_2, \cdots\cdots, X_i, \cdots\cdots X_n$ are statistically independent and identically distributed *(iid)* random variables such that $E(X_i) = \mu$ and $V(X_i) = \sigma^2$, then

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \quad N(0, 1) \text{ as } n \to \infty \tag{1.26}$$

where $\overline{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$. It may be noted that $E(\overline{X}) = \mu$ and $V(\overline{X}) = \dfrac{\sigma^2}{n}$.

1.6 TWO-DIMENSIONAL RANDOM VARIABLES

If X_1 and X_2 are two discrete random variables then their joint probability mass function is denoted by $P(X_1 = x_1, X_2 = x_2)$ or $P_{X_1 X_2}(x_1, x_2)$ or simply $P(x_1, x_2)$ and if they are continuous random variables, then their joint probability density function is denoted by $f_{X_1 X_2}(x_1, x_2)$ or simply $f(x_1, x_2)$. Clearly, the cumulative distribution function is given by

$$F(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2) = \sum_{x=-\infty}^{x_1} \sum_{y=-\infty}^{x_2} P(X_1 = x, X_2 = y)$$

if X_1 and X_2 are discrete and

$$F(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(x, y)\, dx\, dy$$

if X_1 and X_2 are continuous

Expectation of the product of X_1 and X_2 can be obtained as

$$E(X_1 X_2) = \sum_{x_1=-\infty}^{\infty} \sum_{x_2=-\infty}^{\infty} x_1 x_2 P(X_1 = x_1, X_2 = x_2) \quad \text{if } X_1 \text{ and } X_2 \text{ are discrete and}$$

$$E(X_1 X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2)\, dx_1\, dx_2 \quad \text{if } X_1 \text{ and } X_2 \text{ are continuous}$$

1.6.1 Covariance and Correlation

If X_1 and X_2 are two random variables then the covariance between X_1 and X_2 is given by

$$\begin{aligned} C(X_1, X_2) &= E\{[X_1 - E(X_1)][X_2 - E(X_2)]\} \\ &= E(X_1 X_2) - E(X_1)E(X_2) \end{aligned} \tag{1.27}$$

Putting $X_1 = X_2$, we have

$$C(X_1, X_1) = E(X_1 X_1) - E(X_1)E(X_1) = E(X_1^2) - \{E(X_1)\}^2 = V(X_1)$$

or

$$C(X_2, X_2) = E(X_2 X_2) - E(X_2)E(X_2) = E(X_2^2) - \{E(X_2)\}^2 = V(X_2)$$

Now, the correlation between X_1 and X_2 is given by

$$\rho(X_1, X_2) = \rho_{X_1 X_2} = \frac{C(X_1, X_2)}{\sqrt{C(X_1, X_1)}\sqrt{C(X_2, X_2)}} = \frac{\sigma_{X_1 X_2}}{\sigma_{X_1} \sigma_{X_2}} \tag{1.28}$$

or simply $\rho(X_1, X_2) = \rho_{12} = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2}$

Note:

Two random variables X_1 and X_2 are said to be independent if $P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)P(X_2 = x_2)$ for discrete case and $f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$ for continuous case. Also two random variables X_1 and X_2 are said to be uncorrelated if $C(X_1 X_2) = E(X_1 X_2) - E(X_1)E(X_2) = 0$.

1.7 TRANSFORMATION OF ONE OR TWO RANDOM VARIABLES

1.7.1 Discrete Case

Given a discrete random variable X with probability mass function $P(X = x)$ and another discrete random variable Y such that $Y = g(X)$ and there is one-to-one transformation between the values of X and Y so that the equation of $y = g(x)$ can be uniquely solved for x in terms of y, say $x = w(y)$. Then the probability mass function of the random variable Y is given as

$$P(Y = y) = P[w(y)]$$

Similarly, let X and Y be two discrete random variables with joint probability mass function $P(X = x,\ Y = y)$. Let $U = g(X,\ Y)$ and $V = h(X,\ Y)$ define a one-to-one transformation between points $(x,\ y)$ and $(u,\ v)$ so that the equations $u = g(x,\ y)$ and $v = h(x,\ y)$ may be solved uniquely for x and y in terms of u and v, say $x = w_1(u,\ v)$ and $y = w_2(u,\ v)$, then the joint probability mass function of U and V is given as

$$P(U = u,\ V = v) = P[w_1(u,\ v),\ w_2(u, v)] \tag{1.29}$$

1.7.2 Continuous Case

Given a continuous random variable X with probability density function $f(x)$ and another random variable Y such that $Y = g(X)$ and there is one-to-one correspondence between the values of X and Y so that the equation $y = g(x)$ can be uniquely solved for x in terms of y, say $x = w(y)$. Then the probability density function of the random variable Y is given as

$$h(y) = f[w(y)]\,|J| \tag{1.30}$$

where $J = w'(y)$ and is called the Jacobian of the transformation.

Similarly, let us suppose that X and Y are two continuous random variables with joint probability density function $f_{XY}\ (x,y)$. Let $U = g(X,\ Y)$ and $V = h(X,\ Y)$ be the transformations having one-to-one inverse transformations $x = w_1(u,\ v)$ and $y = w_2(u,\ v)$, then the joint probability density function of the random variables U and V is given as

$$f_{UV}(u,\ v) = f[w_1(u,\ v),\ w_2(u,\ v)]\ |J| \tag{1.31}$$

where J is the Jacobian of the transformation given as

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

or

$$f_{UV}(u, v) = f[w_1(u, v), w_2(u, v)]/|J| \tag{1.32}$$

where the Jacobian of the transformation is given by

$$J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

1.8 MULTIVARIATE NORMAL DISTRIBUTION

1.8.1 *d*-Variables Case

Let $X_1, X_2, \cdots\cdots, X_i, \cdots\cdots X_d$ be d normal random variables each with mean $E(X_i) = \mu_i$, variance $V(X_i) = \sigma_i^2$, $i = 1, 2, \cdots\cdots, d$ and covariance $\text{Cov}(X_i, X_j) = \sigma_{ij}$, $i = 1, 2, \cdots\cdots, d$; $j = 1, 2, \cdots\cdots, d$, $i \neq j$, then the d-dimensional, random variable, say X, mean vector, say μ, and variance covariance matrix, say Σ, can be given as

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_i \\ \vdots \\ X_d \end{pmatrix}, \quad \mu = \begin{pmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_i) \\ \vdots \\ E(X_d) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_i \\ \vdots \\ \mu_d \end{pmatrix}$$

$$\text{and} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1j} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2i} & \cdots & \sigma_{2d} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{i1} & \sigma_{i2} & \cdots & \sigma_i^2 & \cdots & \sigma_{id} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \cdots & \sigma_{dj} & \cdots & \sigma_d^2 \end{pmatrix}$$

It may be noted that $V(X_i) = \sigma_i^2 = E[X_i - E(X_i)]^2, \quad i = 1, 2, \cdots\cdots, d$ and

$$\text{Cov}(X_i, X_j) = \sigma_{ij} = E\{[X_i - E(X_i)][X_j - E(X_j)]\}, \quad i = 1, 2, \cdots\cdots, d;$$
$$j = 1, 2, \cdots\cdots, d, \quad i \neq j.$$

Since the correlation coefficient $\rho_{ij} = \dfrac{Cov\,(X_i, X_j)}{\sqrt{V(X_i)}\,\sqrt{V(X_j)}} = \dfrac{\sigma_{ij}}{\sigma_i\,\sigma_j}$, we can write $\sigma_{ij} = \rho_{ij}\,\sigma_i\,\sigma_j$, $i = 1, 2, \cdots\cdots, d$; $\quad j = 1, 2, \cdots\cdots, d$, $i \neq j$. Now, the joint probability density function can be obtained as

$$f(x_1, x_2, \cdots, x_i, \cdots, x_d) = \frac{e^{-\frac{1}{2}(X-\mu)^T(X-\mu)}}{(2\pi)^{d/2}\,|\Sigma|^{1/2}}, \quad -\infty < x_i < \infty, \forall\, i \qquad (1.33)$$

where $\quad X - \mu = \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_i - \mu_i \\ \vdots \\ x_d - \mu_d \end{pmatrix}.$

1.8.2 *d*-Independent Variables Case

If $X_1, X_2, \cdots\cdots, X_i, \cdots\cdots X_d$ are statistically independent normal random variables, then they are uncorrelated as well, and hence we have $\mathrm{Cov}(X_i, X_j) = \sigma_{ij} = 0$, $\forall\, i,\ j$. Therefore, the variance covariance matrix becomes,

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_i^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \sigma_d^2 \end{pmatrix}$$

$$\Rightarrow \quad |\Sigma| = \prod_{i=1}^{d} \sigma_i^2 \quad \text{and} \quad |\Sigma|^{1/2} = \prod_{i=1}^{d} \sigma_i$$

Also,

$$\Sigma^{-1} = \begin{pmatrix} 1/\sigma_1^2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1/\sigma_2^2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/\sigma_i^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1/\sigma_d^2 \end{pmatrix}$$

$$\Rightarrow \quad (X-\mu)^T\Sigma^{-1}(X-\mu) = (x_1-\mu_1\ x_2-\mu_2,\ \dots\ ,x_i-\mu_i,\ \dots\ ,x_d-\mu_d)$$

$$\begin{pmatrix} 1/\sigma_1^2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1/\sigma_2^2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/\sigma_i^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1/\sigma_d^2 \end{pmatrix} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \\ \vdots \\ x_i-\mu_i \\ \vdots \\ x_d-\mu_d \end{pmatrix}$$

$$= (x_1-\mu_1)^2\frac{1}{\sigma_1^2} + (x_2-\mu_2)^2\frac{1}{\sigma_2^2} + \cdots\cdots + (x_i-\mu_i)^2\frac{1}{\sigma_i^2} + \cdots\cdots + (x_d-\mu_d)^2\frac{1}{\sigma_d^2}$$

$$= \sum_{i=1}^{d}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2$$

$$\therefore \quad f(x_1, x_2, \cdots, x_i, \cdots, x_d) = \frac{e^{-\frac{1}{2}\sum\limits_{i=1}^{d}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2}}{(2\pi)^{d/2}\prod\limits_{i=1}^{d}\sigma_i}, \quad -\infty < x_i < \infty,\ \forall\, i \tag{1.34}$$

1.8.3 *d-i.i.d.* Variables Case

If $X_1,\ X_2,\ \cdots\cdots,\ X_i, \cdots\cdots, X_d$ are statistically *independent and identically distributed* (iid) normal random variables, then we have $|\Sigma| = \prod_{i=1}^{d}\sigma^2 = \sigma^{2d}$ and $|\Sigma|^{1/2} = \sigma^d$. Hence, the joint probability density function becomes

$$\therefore \quad f(x_1, x_2, \cdots, x_i, \cdots, x_d) = \frac{e^{-\frac{1}{2}\sum\limits_{i=1}^{d}\left(\frac{x_i-\mu_i}{\sigma}\right)^2}}{(2\pi)^{d/2}\sigma^d}, \quad -\infty < x_i < \infty,\ \forall\, i \tag{1.35}$$

1.8.4 Two-Variable Case: Bivariate Normal Distribution

If X_1 and X_2 are two normal random variables, then we have

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$$

CHAPTER 2

INTRODUCTION TO RANDOM PROCESSES

2.0 INTRODUCTION

In this chapter, the concept of random process is explained in such a way that it is easy to understand. The concepts of random variable and random function are discussed. Many examples are presented to illustrate the nature of various random processes. Since a random process depends on both time and state space, the random process is properly interpreted and classified into different categories based on the combination of time index and state space. Further, the statistical averages of random processes are presented since the outcomes of a random process are probabilistic in nature. Several problems are worked out and exercise problems are given.

2.1 RANDOM VARIABLE AND RANDOM FUNCTION

It is known that a *random variable* is a function, say $X(e)$, that assigns a real value to each and every outcome e of a random experiment. Here, $e = (e_1, e_2, ...)$ is known as sample space or state space. Whereas a *random function* is a function $X(t,e)$ that is chosen randomly from a family of functions $\{X(t,e_i)\}$, $i = 1, 2, \cdots\cdots$ where t is usually known as a time parameter. This implies that for every e_i, $i = 1, 2, \cdots\cdots$ or for the combination of e_i's, we get $X(t,e)$ as a function of the time parameter t.

In order to understand the features of the random variable, random function and hence to define *random process,* let us first have a look at few illustrative examples presented below.

ILLUSTRATIVE EXAMPLE 2.1

Consider a random experiment in which a fair coin is tossed. We know that the outcomes are either a Head (H) or a Tail (T). For a given $e = (e_1, e_2) = (H, T)$ that is known as *sample space* or *state space*, if $X(e)$ is a *random variable* assigning a value '0' when the outcome is T and '1' when the outcome is H, then we can represent the same as given in Table 2.1.

ILLUSTRATIVE EXAMPLE 2.5

In this example, four of the many possible ways that the temperature values could have happened randomly over a continuous time interval $(0,\ t)$ are obtained and plotted in a graph as shown in Figure 2.6. The temperature is set to vary between 18°C and 22°C. Here the random functions can be estimated to have functional expressions (like sine curve, cosine curve, polynomial curve, linear curve, etc.) as given in the preceding illustrative examples. It may be noted that both time and temperature (outcome) are continuous. As discussed in the Illustrative Example 2.4, while we have as many the random functions $X(t,\ \xi_i)$, $i = 1.\ 2,\ \cdots\cdots$ over a period of time, at particular points of time t_i, $i = 1,\ 2,\ \cdots\cdots$, we have the *random variables* $X(t_i,\ \xi)$. In Figure 2.6, at time t_i the intersecting points of the vertical line and the curves show the values of the random variable $X(t_i,\ \xi)$.

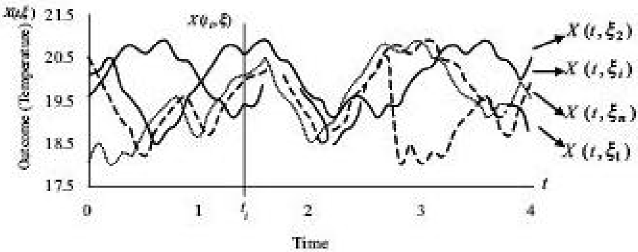


Figure 2.6. Graphical representation of the temperature values against time t

From all the illustrative examples we observe that while $X(t,\ \xi_i)$, $i = 1,\ 2,\cdots\cdots$ are the *random functions* over the time period, say $(0,\ t)$,$X(t_i,\ \xi)$ are the *random variables* at the time points t_i, $i = 1,\ 2,\ \cdots\cdots$ in $(0,\ t)$.

2.2 RANDOM PROCESS

Looking at the illustrative examples, by now we have clearly understood what is a random variable and a random function. Now, we can define random process as follows:

2.2.1 Definition

A random process is defined as the collection of random functions, say $\{X(t,\ \xi)\}$, $\xi = \xi_1,\ \xi_2,\cdots\cdots,\ \xi_i,\cdots\cdots,\ \xi_n\cdots$, together with a probability rule.

The probability rule implies that each random function $X(t,\ \xi_i)$, $i = 1,\ 2,\ \cdots\cdots,n,\cdots\cdots$ is associated with a probability of its happening over a time period.

That is, the random process denoted by $\{X(t,\ \xi)\}$ is the collection of (the uncountably infinite if the state space ξ is continuous or countably infinite if the state space ξ is discrete) random functions $X(t,\ \xi_1)$,$X(t,\ \xi_2),\ldots,\ X(t,\ \xi_i),\ldots,$ $X(t,\ \xi_n),\ldots$ with state space $\xi = \xi_1,\ \xi_2,\cdots\cdots,\ \xi_i,\cdots\cdots,\ \xi_n,\ldots\ldots$ It could be

Table 2.6. Probability distribution of the random process given in Example 2.2

x	$-\sin(1+t)$	$\sin(1+t)$
$P[X(t)=x]$	$1/2$	$1/2$

2.2.2 Interpretation of Random Process

Based on the nature of the state space ξ and time parameter t, we can use different interpretations for the random process $\{X(t, \xi)\}$ as shown in Table 2.7. From Table 2.7, four interpretations can be explicitly given as follows:

Table 2.7. Nature of the function $X(t, \xi)$ based on t and ξ

		State Space (ξ)	
		Fixed	Variable
Time Parameter (t)	Fixed	A value $X(t_i, \xi_i)$	Random variable $X(t_i, \xi)$
	Variable	Single Random function $X(t, \xi_i)$	An ensemble $\{X(t, \xi)\}$

Interpretation 1:

If the state space ξ and time parameter t are fixed at (t_i, ξ_i) for some $i = 1, 2, \cdots\cdots, m, \cdots\cdots$, then the process stops by assigning *a value* say $X(t_i\, \xi_i)$. Consider the Illustrative Example 2.1, in tossing the coin at the fixed time point $t = 4$, for a fixed outcome of a head we have $X(t_4, \xi_1) = 40$ and for a fixed outcome of a tail we have $X(t_4, \xi_2) = -20$ (Refer to Table 2.3).

Interpretation 2:

If the state space ξ is fixed and time parameter t is allowed to vary such as $t_1, t_2, \cdots, t_i, \cdots, t_m, \cdots$ in $(0, t)$, then we have a *single random function* $X(t, \xi_i)$ for some $i = 1, 2, \cdots\cdots, m, \cdots\cdots$. Consider the Illustrative Example 2.2, in which the coin is tossed repeatedly at the time points $t_1, t_2, \cdots, t_i, \cdots, t_{10}$ in $(0, t)$ we get the function $X(t, \xi_1) = -\sin(1+t)$ if the outcome of the toss is a tail and the function $X(t, \xi_2) = \sin(1+t)$ if the outcome results in a head. When these functions are plotted with smooth curves for $t = t_1, t_2, \cdots, t_i, \cdots, t_{10}$ in $(0, t)$ then we get the graphical representation of these two functions as shown in Figure 2.3.

Interpretation 3:

If the state space ξ is allowed to vary such as $\xi = \xi_1, \xi_2, \cdots\cdots, \xi_i, \cdots\cdots, \xi_n, \cdots\cdots$ and time parameter t is fixed at t_i, then we have a random variable $X(t_i, \xi)$ for some $i = 1, 2, \cdots\cdots, m, \cdots\cdots$ in $(0, t)$. Consider the Illustrative Example 2.3, in which if the dice is thrown at a time point t_6 for some $i = 1, 2, \cdots\cdots, m, \cdots\cdots$ in $(0, t)$, we get the *random variable* $X(t_6, \xi_i)$ that assumes the values $X(t_6, \xi_1) = 24$, $X(t_6, \xi_2) = 36$, $X(t_6, \xi_i) = 6$ and $X(t_6, \xi_n) = 18$

(Refer Table 2.5 and Figure 2.4). When these functions are plotted with smooth curves for $t_1, t_2, \cdots, t_i, \cdots, t_{10}$ in $(0, t)$ then we get the graphical representation of these functions as shown in Figure 2.4.

Interpretation 4:

If the state space ξ is allowed to vary such as $\xi = \xi_1, \xi_2, \cdots\cdots, \xi_i, \cdots\cdots, \xi_n, \cdots\cdots$ and time parameter t is allowed to vary such as $t_1, t_2, \cdots, t_i, \cdots, t_m, \cdots$ in $(0, t)$, then we have an ensemble of random functions $\{X(t, \xi)\}$ whose member functions are $X(t, \xi_i)$ for $i = 1, 2, \cdots\cdots, n, \cdots\cdots$ in $(0, t)$. Consider the Illustrative Example 2.4, in which the state space ξ is continuous uniform random variable in $(0, 1)$ and the random functions are observed over a period of time in $(0, t)$ continuously. Similarly, in the Illustrative Example 2.5, the temperature values are continuous between 18°C and 22°C, and is observed over a continuous time interval $(0, t)$. In the given time interval $(0, t)$ the random functions $X(t, \xi_1)$, $X(t, \xi_2)$, $X(t, \xi_i)$ and $X(t, \xi_n)$ are observed and the graphical representation of these functions are shown in Figures 2.5 and 2.6.

2.2.3 Classification of a Random Process

It may be noted that if t and ξ are variables, then we have an ensemble $\{X(t, \xi)\}$ of random functions. Also, t and ξ are being variables they may be either discrete (countably infinite) or continuous (uncountably infinite) or in combination of both. We know that an ensemble of random functions is called random process. This has helped to classify a random process as shown in Table 2.8.

Table 2.8. Classification of the ensemble $\{X(t, \xi)\}$ based on t and ξ

		State Space (ξ)	
		Discrete	Continuous
Time Parameter (t)	Discrete	Discrete random sequence	Continuous random sequence
	Continuous	Discrete random process	Continuous random process

Discrete random sequence: If time parameter t is discrete and state space ξ is also discrete then each member function of the ensemble $\{X(t, \xi)\}$ is called a *discrete random sequence*. Refer to Illustrative Example 2.1 of tossing a coin in which the outcomes ξ are discrete (0 for tail and 1 for head) and Illustrative Example 2.3 of throwing a dice in which the outcomes are discrete (1, 2, 3, 4, 5, 6). In these cases the time parameter t is also discrete as the experiments are conducted at specific time points $t_1, t_2, \cdots, t_i, \cdots, t_m, \cdots$.

Continuous random sequence: If time parameter t is discrete and state space ξ is continuous then each member function of the ensemble $\{X(t, \xi)\}$ is called a *continuous random sequence*. Let us suppose that, temperature is recorded at specific time points $t_1, t_2, \cdots, t_i, \cdots, t_m, \cdots$. Clearly, in this case, the temperature is continuous and the time points are discrete. Also refer to Illustrative Example 2.5 with temperature as discrete.

Discrete random process: If time parameter t is continuous and state space ξ is discrete then each member function of the ensemble $\{X(t, \xi)\}$ is called the discrete random function and the ensemble representing the collection of such functions is called *discrete random process*. Let us suppose that, number of telephone calls (outcomes) received per time unit at a telephone exchange is recorded over a period of time $(0, t)$. In this case, while the state space is discrete (0 call, 1 call, 2 calls, etc.), the time parameter is continuous. Also refer to Illustrative Example 2.2.

Continuous random process: If time parameter t is continuous and state space ξ is also continuous then each member function of the ensemble $\{X(t, \xi)\}$ is called the continuous random function and the ensemble representing the collection of such continuous time functions is called *continuous random process*. For example, let us suppose that temperature is recorded continuously over a period of time $(0, t)$, then in this case both temperature and time are continuous. Also refer to Illustrative Examples 2.4 and 2.5.

2.3 PROBABILITY DISTRIBUTIONS AND STATISTICAL AVERAGES

Given a random process $\{X(t)\}$ observed over a period of time $(0, t)$ and $X(t_1)$, $X(t_2)$ and so on $X(t_m)$ are the random variables of the process $\{X(t)\}$ at different time points, say, $t_1, t_2, \cdots, t_i, \cdots, t_m$, then distribution functions and various statistical averages can be obtained as follows:

2.3.1 Probability Mass Function (PMF) and Probability Density Function (PDF)

The probability mass function (PMF) and the probability density function (PDF) of a random process $\{X(t)\}$ are denoted respectively as $P\{X(t) = x\}$ and $f_X(x,t)$ or $f(x,t)$ or $f_{X(t)}(x)$. It may be noted that all assumptions related to the PMF and PDF of random variable hold good for $P\{X(t) = x\}$ and $f(x,t)$ as well. Now the cumulative distribution function denoted by $F_X(x,t)$ or $F(x,t)$ or $F_{X(t)}(x)$ can be given as

$$\begin{aligned} F_{X(t)}(x) &= P\{X(t) \le x\} \\ &= \sum_{x_i=-\infty}^{x} P(X = x_i) \text{ if the outcome of the process } \{X(t)\} \text{ is discrete} \end{aligned} \tag{2.1}$$

$P\{X_1(t_1)=x_1,\ X_2(t_2)=x_2\}$ and $f_{X_1X_2}(x_1,\ x_2;\ t_1,\ t_2)$. Similarly, the CDF for two variable case is denoted as $F_{X_1(t)X_2(t)}(x_1,\ x_2)$.

2.3.2 Statistical Averages

In the study of random processes, statistical averages play a major role in analyzing the nature and properties of such random processes. Important statistical averages are defined below:

Mean or Expected Value:

Let $\mu_x(t)$ denote the expected value of the random process $\{X(t)\}$. Then we have

$$\mu_x(t)=E\{X(t)\}=\sum_{x=-\infty}^{+\infty} xP\{X(t)=x\} \text{ if the outcome of the process } \{X(t)\} \text{ is discrete}$$
$$=\int_{-\infty}^{\infty} xf(x,t)\,dx \text{ if the outcome of the process } \{X(t)\} \text{ is continuous} \tag{2.4}$$

Autocorrelation:

If $\{X(t)\}$ is a random process and $X(t_1)$ and $X(t_2)$ are the two random variables of the process at two time points t_1 and t_2, then the autocorrelation of the process $\{X(t)\}$ denoted by $R_{xx}(t_1,\ t_2)$ is obtained as the expected value of the product of $X(t_1)$ and $X(t_2)$. That is,

$$R_{xx}(t_1,\ t_2)=E\{X(t_1)X(t_2)\}. \tag{2.5}$$

Autocovariance and Variance:

If $\{X(t)\}$ is a random process and $X(t_1)$ and $X(t_2)$ are the two random variables of the process at two time points t_1 and t_2, then the autocovariance of the process $\{X(t)\}$, denoted by $C_{xx}(t_1,\ t_2)$, is given by

$$C_{xx}(t_1,\ t_2)=E\{X(t_1)X(t_2)\}-E\{X(t_1)\}E\{X(t_2)\}$$
$$=R_{xx}(t_1,\ t_2)-\mu_x(t_1)\ \mu_x(t_2) \tag{2.6}$$

Apparently, when $t_1=t_2=t$, we have

$$C_{xx}(t,\ t)=E\{X(t)X(t)\}-E\{X(t)\}E\{X(t)\}$$
$$=E\left\{X^2(t)\right\}-\{E[X(t)]\}^2 \tag{2.7}$$
$$=V\{X(t)\}=\sigma_x^2(t)$$

$V\{X(t)\}=\sigma_x^2(t)$ is the *variance* of the random process $\{X(t)\}$.

2.3.3 *a*-Dependent Processes

Given a random process $\{X(t)\}$, the random variables $X(t_1)$ and $X(t_2)$ observed at two time points t_1 and t_2 are stochastically dependent (correlated) for any t_1 and t_2. As the time difference increases, that is, as $\tau = |t_2 - t_1| \to \infty$, these random variables become independent. This aspect leads to the following definition:

A random process $\{X(t)\}$ is called $a-$dependent processes if all $X(t)$ values for $t < t_0$ and $t > t_0 + a$ are mutually independent. This implies that the auto-covariance

$$C_{xx}(t_1, t_2) = 0 \text{ for all } |t_1 - t_2| > a. \tag{2.12}$$

2.3.4 White Noise Processes

A random process $\{X(t)\}$ is called *white noise process* if the random variables $X(t_i)$ and $X(t_j)$ of the process observed at two time points t_i and t_j are uncorrelated for every pair t_i and t_j such that $t_i \neq t_j$. That is, the auto-covariance

$$C_{xx}(t_i, t_j) = 0 \text{ for every pair } t_i \text{ and } t_j \text{ such that } t_i \neq t_j \tag{2.13}$$

It may be noted that unless or otherwise it is provided, the mean of the white noise process is always assumed as zero. If $X(t_1)$ and $X(t_2)$ are uncorrelated and independent, then the process $\{X(t)\}$ is called *strictly white noise process*.

Given two random variables $X(t_1)$ and $X(t_2)$ of a white noise process $\{X(t)\}$, the auto-covariance is usually of the form

$$C_{xx}(t_1, t_2) = b(t_1)\delta(t_1 - t_2) \text{ for } b(t) \geq 0 \tag{2.14}$$

SOLVED PROBLEMS

Problem 1. A random process $\{X(t)\}$ has the sample functions of the form $X(t) = Y \sin \omega t$ where ω is a constant and Y is a random variable that is uniformly distributed in (0, 1). Sketch three sample functions for $Y = 0.25,\ 0.5,\ 1$ by fixing $\omega = 2$. Assume $0 \leq t \leq 10$.

SOLUTION:

Since Y is a random variable that is uniformly distributed in (0, 1) we consider three arbitrary values $Y = 0.25,\ 0.5,\ 1$. Now, for $\omega = 2$, we have three sample functions of $\{X(t)\}$ as

$$X(t) = (0.25)\sin 2t$$

$$X(t) = (0.5)\sin 2t$$

$$X(t) = (1)\sin 2t$$

CHAPTER 3

STATIONARITY OF RANDOM PROCESSES

3.0 INTRODUCTION

In the previous chapter the concept of random process is explained. In fact, one can understand from the definition that, a random process is a collection of random variables, each at different points of time. Due to this reason, random processes have all the distributional properties of random variables such as mean, variance, moments and correlation. When dealing with groups of signals or sequences (ensembles) it will be important for us to be able to show whether or not these statistical properties hold good for the entire random process. For this purpose, and to study the nature of a random process, the concept of *stationarity* has been developed. Stationary refers to time invariance of some, or all, statistics of a random process, for example, mean, variance, autocorrelation, m^{th} order distribution, etc. Otherwise, that is, if any of these statistics is not time invariant, then the process is said to be nonstationary. Stationarity of a random process can be classified into two categories: (i) strict sense stationary (SSS) and wide sense stationary (WSS).

3.1 TYPES OF STATIONARITY IN RANDOM PROCESSES

3.1.1 Strict Sense Stationary (SSS) Process

A random process $\{X(t)\}$ is said to be stationary in strict sense, if the distributions of the random variables $X(t_1), X(t_2), \cdots\cdots, X(t_m)$ observed respectively at time points $t_1, t_2, \cdots\cdots, t_m$, *over a period of time interval $(0, t)$ are same. That is, the distributions are time invariant.*

Apparently, for the given time points t_1 and t_2 in the time period $(0, t)$ such that $t_1 \leq t_2$ a random process $\{X(t)\}$ is said to be an SSS process of *order one*, if

$$\begin{aligned} &P\{X(t_1) = x\} = P\{X(t_2) = x\} && \text{in discrete case or} \\ &f_X(x; t_1) = f(x; t_2) && \text{in continuous case.} \end{aligned} \tag{3.1}$$

since $t_1 \leq t_2$ we can write $t_2 = t_1 + \tau$ for some $\tau > 0$, therefore, the above equations can be written as

$$\begin{aligned} &P\{X(t_1) = x\} = P\{X(t_1+\tau) = x\} && \text{in discrete case or} \\ &f_X(x;t_1) = f_X(x;t_1+\tau) && \text{in continuous case and hence} \\ &&& t_2 = t_1 + \tau \text{ for some } \tau > 0, \end{aligned} \tag{3.2}$$

For the given time points t_1 and t_2 in the time period $(0, t)$ such that $t_1 \leq t_2$ and for some $\tau > 0$, a random process $\{X(t)\}$ is said to be SSS process of *order two*, if

$$P\{X(t_1) = x_1, X(t_2) = x_2\} = P\{X(t_1+\tau) = x_1, X(t_2+\tau) = x_2\}$$

in discrete case or

$$f_{XX}(x_1, x_2; t_1, t_2) = f_{XX}(x_1, x_2; t_1+\tau, t_2+\tau) \quad \text{in continuous case.} \tag{3.3}$$

Similarly, for the given time points $t_1, t_2, \cdots\cdots, t_m$ in the time period $(0, t)$ such that $t_1 \leq t_2 \leq \cdots\cdots \leq t_m$ and for some $\tau > 0$, random process $\{X(t)\}$ is said to be SSS process of order m, if

$$\begin{aligned} &P\{X(t_1) = x_1, X(t_2) = x_2, \cdots\cdots, X(t_m) = x_m\} \\ &\qquad = P\{X(t_1+\tau) = x_1,\ X(t_2+\tau) = x_2, \cdots\cdots, X(t_m+\tau) = x_m\} \end{aligned}$$

in discrete case or

$$\begin{aligned} &f(x_1, x_2, \cdots\cdots x_m; t_1, t_2, \cdots\cdots, t_m) \\ &\qquad = f(x_1, x_2, \cdots\cdots x_m; t_1+\tau, t_2+\tau, \cdots\cdots, t_m+\tau) \end{aligned} \tag{3.4}$$

in continuous case.

It is clear that the distribution of SSS process $\{X(t)\}$ is independent of time t and hence it depends only on the time difference $\tau = (t_i + \tau) - t_i$ for $1 \leq i \leq m$.

By virtue of the property of probability distributions, it may be noted that all moments $E\{X^r(t)\}$, $r = 1, 2, \cdots\cdots$ of SSS process are time independent.

3.1.2 Wide Sense Stationary (WSS) Process

A random process $\{X(t)\}$ observed over a time period $(0, t)$ is said to be stationary in wide sense, if its mean is constant and autocorrelation is time invariant. That is,

$$\begin{aligned} &(i) \quad E\{X(t)\} = \mu_x, \quad \text{a constant, and} \\ &(ii) \quad R_{xx}(t_1, t_2) = E\{X(t_1)X(t_2)\} = R_{xx}(\tau) \end{aligned} \tag{3.5}$$

Figure 3.2. ECG recorded at time point t_2

3.1.3 Jointly Strict Sense Stationary (JSSS) Processes

Two random processes $\{X_1(t)\}$ and $\{X_2(t)\}$ are said to be jointly strict sense stationary (i.e., jointly SSS), if the joint distributions are invariant over time. That is,

$$P\{X_1(t_1) = x_1, X_2(t_2) = x_2\} = P\{X_1(t_1+\tau) = x_1, X_2(t_2+\tau) = x_2\}$$

in discrete case or

$$f_{X_1X_2}(x_1, x_2; t_1, t_2) = f_{X_1X_2}(x_1, x_2; t_1+\tau, t_2+\tau) \quad \text{in continuous case.} \tag{3.7}$$

If $\{X_n,\ n \geq 0\}$ is a sequence of identically and independently distributed (iid) random variables, then the sequence $\{X_n,\ n \geq 0\}$ is wide sense stationary if

$$(i) \qquad E(X_n) = 0, \quad \forall\, n$$

$$(ii) \qquad R_n(n, n+s) = \begin{cases} E(X_n X_{n+s}), & \text{for} \quad s \neq 0 \\ E(X_n^2), & \text{for} \quad s = 0 \end{cases} \tag{3.8}$$

That is, mean of the sequence $\{X_n,\ n \geq 0\}$ is constant and the autocorrelation function denoted by $R_n(n,\ n+s)$ depends only on s, the step (time points) difference.

3.1.4 Jointly Wide Sense Stationary (JWSS) Processes

Two random processes $\{X_1(t)\}$ and $\{X_2(t)\}$ are said to be jointly wide sense stationary (i.e., jointly WSS), if each process is individually a wide sense stationary process and the cross-correlation $R_{x_1x_2}(t_1, t_2)$ is time invariance. That is,

(*i*) $E\{X_1(t)\} = \mu_{x_1}$, a constant, and

$$R_{x_1x_1}(t_1, t_2) = E\{X_1(t_1)X_1(t_2)\} = R_{x_1x_1}(\tau) \text{ for } \{X_1(t)\} \text{ to be } WSS$$

(*ii*) $E\{X_2(t)\} = \mu_{x_2}$, a constant, and

$$R_{x_2x_2}(t_1, t_2) = E\{X_2(t_1)X_2(t_2)\} = R_{x_2x_2}(\tau) \text{ for } \{X_2(t)\} \text{ to be } WSS \tag{3.9}$$

(*iii*) $R_{x_1x_2}(t_1, t_2) = E\{X_1(t_1)X_2(t_2)\} = R_{x_1x_2}(\tau)$

3.1.5 Random Processes with Stationary Independent Increments

A random process $\{X(t), t > 0\}$ is said to have independent increments, if whenever $0 < t_1 < t_2 < \cdots\cdots < t_n$

$$X(0),\ X(t_1) - X(0),\ X(t_2) - X(t_1), \cdots\cdots,\ X(t_n) - X(t_{n-1}) \tag{3.10}$$

are independent. Further, if the process $\{X(t), t > 0\}$ has independent increments, and the difference $X(t) - X(s)$ for some $t > s$ has the same distribution as $X(t+\tau) - X(s+\tau)$, then the process $\{X(t), t > 0\}$ is said to have stationary independent increments.

Theorem 3.1: Let $\{X(t)\}$ be a random process with stationary independent increments. If $X(0) = 0$, then $E\{X(t)\} = \mu_1 t$ where $E\{X(1)\} = \mu_1$.

Proof. Let $g(t) = E\{X(t)\} = E\{X(t) - X(0)\}$

$$\Rightarrow \quad g(t+s) = E\{X(t+s) - X(0)\} \quad \text{for any } t \text{ and } s$$

Adding and subtracting $X(s)$, we have

$$\begin{aligned} g(t+s) &= E\{X(t+s) - X(s) + X(s) - X(0)\} \\ &= E\{X(t+s) - X(s)\} + E\{X(s) - X(0)\} \\ &= g(t) + g(s) \end{aligned}$$

It may be noted that the only solution to the equation $g(t+s) = g(t) + g(s)$ can be given by $g(t) = ct$ with c as constant. That is,

$$g(t) = ct \quad \Rightarrow \quad g(t+s) = c(t+s) = ct + cs = g(t) + g(s)$$

Now, clearly since $g(1) = c$, we have $c = g(1) = E\{X(1)\}$ and hence $E\{X(t)\} = g(t) = ct = E\{X(1)\}t = \mu_1 t$ where $\mu_1 = E\{X(1)\}$.

Theorem 3.2: Let $\{X(t)\}$ be a random process with stationary independent increments. If $X(0) = 0$, then $V\{X(t)\} = \sigma_1^2 t$ and $V\{X(t) - X(s)\} = \sigma_1^2 (t-s)$ where $V\{X(1)\} = \sigma_1^2$ and $t > s$.

Proof (i) Let $h(t) = V\{X(t)\} = V\{X(t) - X(0)\}$

$$\Rightarrow \quad h(t+s) = V\{X(t+s) - X(0)\}$$

Adding and subtracting $X(s)$, we have

$$\begin{aligned} h(t+s) &= V\{X(t+s) - X(s) + X(s) - X(0)\} \\ &= V\{X(t+s) - X(s)\} + V\{X(s) - X(0)\} \\ &= h(t) + h(s) \end{aligned}$$

$$\Rightarrow \quad C_{XX}(t,s) = \frac{1}{2}\left(V\{X(t)\} + V\{X(s)\} - V\{X(t) - X(s)\}\right)$$

$$\Rightarrow \quad C_{XX}(t,s) = \frac{1}{2}\left(\sigma_1^2 t + \sigma_1^2 s - \sigma_1^2 (t-s)\right) = \sigma_1^2 s$$

Similarly, if we let, $s > t$ we have

$$\Rightarrow \quad C_{XX}(t,s) = \sigma_1^2 t$$

$$\therefore \quad C_{XX}(t,s) = \begin{cases} \sigma_1^2 s, & \text{if } t > s \\ \sigma_1^2 t, & \text{if } s > t \end{cases} \quad \Rightarrow \quad C_{XX}(t,s) = \sigma_1^2 \min(t,s)$$

3.2 STATIONARITY AND AUTOCORRELATION

As discussed earlier, a random process $\{X(t)\}$ is said to be wide sense stationary, if its autocorrelation is independent of time apart from mean being constant. This means that the process observed at a given time duration may change as the time duration increases or decreases. Here, the time difference matters, not the actual time points. For example, let us suppose that we observe quantum of rainfall for one hour period not minding the exact time when it is observed. Then the amount of rainfall may increase or decrease, depending upon whether it is observed for one hour or less than one hour or more than one hour.

ILLUSTRATIVE EXAMPLE 3.2

In order to illustrate the behavior of mean and autocorrelation of a stationary process, let us consider the process $\{X(t)\}$ such that its member function is given by $X(t) = A\cos(\omega t + \theta)$ where ω is a constant, A is a random variable that has magnitude $+1$ and -1 with equal probabilities, and θ is a random variable that is uniformly distributed in $(0, 2\pi)$. Assume that A and θ are independent. It can be easily proved that $\{X(t)\}$ is a wide sense stationary process because

$$E\{X(t)\} = 0$$

$$R(\tau) = \frac{1}{2}\cos\omega\tau$$

Without loss of generality, if we assume that

$$A = +1, \omega = 2, \theta = \pi/2, t = (0, 10)$$

We can plot both $X(t) = (+1)\cos(2t + \pi/2)$ and $R(\tau) = (0.5)\cos\omega\tau$ as shown in Figure 3.3 and Figure 3.4 respectively. It may be noted that if $X(t) = A\cos(\omega t + \theta)$ is chosen from any time window, there is no change in the pattern of the plot, meaning the process is stationary.

CHAPTER 4

AUTOCORRELATION AND ITS PROPERTIES

4.0 INTRODUCTION

Autocorrelation function of a random process plays a major role in knowing whether the process is stationary. In particular, for a stationary random process, the autocorrelation function is independent of time and hence it becomes dependent only on time difference. Hence, autocorrelation of a stationary process also helps to determine the average of the process as the time difference becomes infinite. Apart from this, the autocorrelation function of a stationary process shows something about how rapidly one can expect a random process to change as a function of time. If the autocorrelation function changes slowly (i.e., decays rapidly) then it is an indication that the corresponding process can be expected to change slowly and vice-versa. Further, if the autocorrelation function has periodic components, then the corresponding process is also expected to have periodic components. Therefore, there is a clear indication that the autocorrelation function contains information about the expected frequency content of the random process.

For example, let us assume that the random process $\{X(t)\}$ represents voltage in waveform across a resistance of unit ohms. Then, the ensemble average of the second order moment of $\{X(t)\}$, that is $E\{X^2(t)\}$ gives the average power delivered to the resistance of unit ohms by $\{X(t)\}$ as shown below. That is, the average power of $\{X(t)\}$ is

$$\frac{\text{Square of voltage}}{\text{Resistance}} = \frac{E\{X^2(t)\}}{1} = E\{X(t)X(t)\} = R_{xx}(t,t) = R_{xx}(0)$$

Hence, it is important to learn in depth about the properties of autocorrelation function of a random process which is the main objective of this chapter.

4.1 AUTOCORRELATION

In Chapter 2, we have defined autocorrelation and in Chapter 3 we have studied the importance of autocorrelation in establishing the stationarity of a random process. Let us recall that if $\{X(t)\}$ is a random process and $X(t_1)$ and $X(t_2)$ are the two

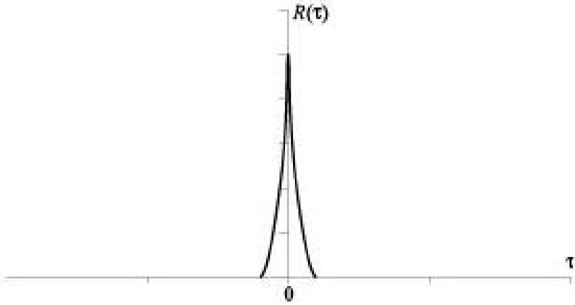


Figure 4.4. Autocorrelation of rapidly changing random process

Figure 4.5. Slowly changing random process

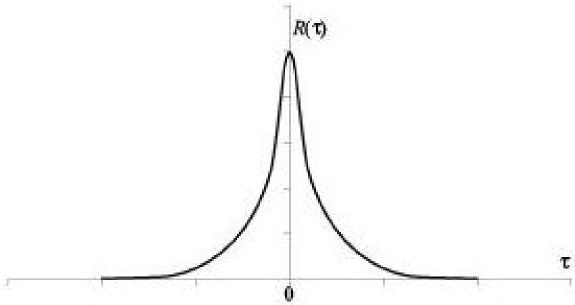


Figure 4.6. Autocorrelation of slowly changing random process

4.2 PROPERTIES OF AUTOCORRELATION

We know that in case of a stationary process $\{X(t)\}$, the autocorrelation is essentially a function of the time difference τ and denoted by $R_{XX}(\tau)$. For a stationary

Note:

If $\{X(t)\}$ is a wide sense stationary process with autocorrelation function $R_{xx}(\tau)$ then

$$R_{xx}(0) \geq 0, \quad R_{x}(\tau) = R_{xx}(-\tau), \quad R_{xx}(0) \geq R_{xx}(\tau)$$

4.3 PROPERTIES OF CROSS-CORRELATION

We know that if $\{X_1(t)\}$ and $\{X_2(t)\}$ are two stationary random processes then the cross-correlation is given by

$$R_{x_1x_2}(t_1, t_2) = E\{X_1(t)X_2(t+\tau)\} = R_{x_1x_2}(\tau)$$

Below are some of the properties of cross-correlation function of two stationary random processes.

Property 4.6: The cross-correlation function $R_{x_1x_2}(\tau)$ of two stationary random processes $\{X_1(t)\}$ and $\{X_2(t)\}$ is an even function. That is, $R_{x_1x_2}(\tau) = R_{x_2x_1}(-\tau)$ or $R_{x_2x_1}(\tau) = R_{x_1x_2}(-\tau)$.

Proof. It is known that given two time points with t and $t+\tau$ the cross-correlation of the stationary random processes $\{X_1(t)\}$ and $\{X_2(t)\}$ is given by

$$\begin{aligned} R_{x_1x_2}(\tau) &= E\{X_1(t)X_2(t+\tau)\} \\ &= E\{X_2(t+\tau)X_1(t)\} = R_{x_2x_1}(-\tau) \end{aligned} \tag{4.15}$$

Similar proof can be given for $R_{x_2x_1}(\tau) = R_{x_1x_2}(-\tau)$.

Property 4.7: If $\{X_1(t)\}$ and $\{X_2(t)\}$ are two stationary random processes with autocorrelation functions $R_{x_1x_1}(\tau)$ and $R_{x_2x_2}(\tau)$ respectively and let $R_{x_1x_2}(\tau)$ be their cross-correlation function, then $|R_{x_1x_2}(\tau)| \leq \sqrt{R_{x_1x_1}(0)\, R_{x_2x_2}(0)}$.

Proof. This can be proved with the help of Schwarz's inequality. That is, if X and Y are two random variables, then $\{E(XY)\}^2 \leq E(X^2)E(Y^2)$.

Now, consider

$$\{E[X_1(t)X_2(t+\tau)]\}^2 \leq E[X_1^2(t)]\, E[X_2^2(t+\tau)]$$

$$\{R_{x_1x_2}(\tau)\}^2 \leq R_{x_1x_1}(0)\, R_{x_2x_2}(0)$$

Since $E[X_1^2(t)] = R_{x_1x_1}(0)$ and $E[X_2^2(t)] = R_{x_2x_2}(0)$

$$\therefore \quad |R_{x_1x_2}(\tau)| \leq \sqrt{R_{x_1x_1}(0)\, R_{x_2x_2}(0)} \tag{4.16}$$

If $X_1(t)$ and $X_2(t+\tau)$ are independent then we have

$$\begin{aligned} R_{x_1x_2}(\tau) &= E[X_1(t)]\,E[X_2(t+\tau)] \\ &= \mu_{x_1}\,\mu_{x_2} \end{aligned} \tag{4.20}$$

Property 4.10: If two stationary random processes $\{X_1(t)\}$ and $\{X_2(t)\}$ are orthogonal, then $R_{x_1x_2}(\tau) = 0$.

Proof. If $R_{x_1x_2}(\tau)$ is the cross-correlation function of the two stationary random processes $\{X_1(t)\}$ and $\{X_2(t)\}$, then

$$R_{x_1x_2}(\tau) = E\{X_1(t)X_2(t+\tau)\}$$

We know that if X and Y are orthogonal random variables then $E(XY) = 0$.

Therefore, if $X_1(t)$ and $X_2(t+\tau)$ are orthogonal, then we have

$$\begin{aligned} & E\{X_1(t)X_2(t+\tau)\} = 0 \\ \Rightarrow\quad & R_{x_1x_2}(\tau) = 0 \end{aligned} \tag{4.21}$$

Note: (See Result A.3.3 in Appendix A for details)

If the random process $\{X(t)\}$ is integrable in mean square sense, then

$$E\left\{\int_a^b X(t)dt\right\}^2 = \int_a^b\int_a^b E\{X(t_1)X(t_2)\}\,dt_1dt_2 = \int_a^b\int_a^b R(t_1,t_2)dt_1dt_2 \tag{4.22}$$

4.4 CORRELATION COEFFICIENT OF STATIONARY RANDOM PROCESS

If $\{X(t)\}$ is a stationary random process and $X(t_1)$ and $X(t_2)$ are the two random variables of the process at two time points t_1 and t_2 with mean $E\{X(t_1)\} = E\{X(t_2)\} = \mu_x$ and autocorrelation function $R_{xx}(t_1,\, t_2)$ then the autocovariance, denoted by $C_{xx}(t_1,\, t_2)$ between $X(t_1)$ and $X(t_2)$ is given as

$$C_{xx}(t_1,\, t_2) = R_{xx}(t_1,\, t_2) - E\{X(t_1)X(t_2)\}$$

Since $\{X(t)\}$ is a stationary random process, as discussed earlier in case of autocorrelation, the autocovariance is also a function of the time difference only, that is, $C_{xx}(t_1,\, t_2) = C_{xx}(\tau)$, where $\tau = |t_1 - t_2|$.

$$\therefore\quad C_{xx}(\tau) = R_{xx}(\tau) - \mu_x^2 \tag{4.23}$$

CHAPTER 5

BINOMIAL AND POISSON PROCESSES

5.0 INTRODUCTION

It is known that random process is associated with probability and probability distributions. That is, every outcome (i.e., member function) of a random process is associated to a probability of its happening. For example, as shown in Illustrative Example 2.2 in Chapter 2, two member functions $X(t) = -\sin(1+t)$ and $X(t) = \sin(1+t)$ of a random process $\{X(t)\}$ can happen as follows:

$$X(t) = \begin{cases} -\sin(1+t) & \text{if tail turns up} \\ \sin(1+t) & \text{if head turns up} \end{cases}$$

Also we know that in tossing a coin, the probabilities are

$$P\{X(t) = -\sin(1+t)\} = P\{X(t) = \sin(1+t)\} = \frac{1}{2}$$

If the experiment of tossing a coin is observed with one trial, this happening can be thought of as according to a *Bernoulli distribution*.

Therefore, random processes can be described by statistical distributions indexed by time parameter depending on the nature of the processes. For example, if number of occurrence of phone calls is observed over a period of time, then the number of occurrences observed over time can be thought of as a *Poisson process*. If a trial is conducted over a period of time and in each trial there are only two outcomes, then the outcomes observed at a time point can be fitted into a *binomial process*.

5.1 BINOMIAL PROCESS

It is known that a random variable X is said to follow *binomial distribution* if its probability mass function is given by

$$P\{X = x\} = {}^{n}C_{x} p^{x} q^{n-x}, \quad x = 0, 1, 2, \cdots\cdots, n \tag{5.1}$$

where n is the number of trials conducted, p is the probability of a success and $q = 1 - p$ is the probability of a failure.

5.2 POISSON PROCESS

It is known that a random variable X is said to follow *Poisson distribution* if its probability mass function is given by

$$P\{X=x\}=\frac{e^{-\lambda}\lambda^{x}}{x!},\quad x=0,\ 1,\ 2,\cdots\cdots \tag{5.3}$$

where the parameter $\lambda>0$ represents the rate of occurrence of events (points).

5.2.1 Poisson Points

The collection of discrete sets of points (time points) from a time domain is called *point process* or *counting process*. Such points are known as *Poisson points*. For example, let us consider the case where we count the number of telephone calls received at random time points, say $t_1,\ t_2,\ t_3,\cdots\cdots$ starting from time point $t_0=0$. In every time interval, say $(0,\ t_i),\quad i=1,\ 2,\cdots\cdots$ we can count the total number of telephone calls received. If there are 10 telephone calls received in the interval $(0,\ t_1)$ and 25 calls in the interval $(0,\ t_2)$ then the number of calls received in the interval $(t_1,\ t_2)=25-10=15$. Here, the points $t_1,\ t_2,\ t_3,\cdots\cdots$ by which the calls are counted are the Poisson points. Clearly, the number of telephone calls received in a given time interval – either it is in $(0,\ t_1)$ or $(0,\ t_2)$ or $(t_1,\ t_2)$ – is a random variable. In general, if the time interval is $(0,\ t)$ then the number of occurrence of phone calls is a random variable and is denoted by $X(t)$ or $n(0,t)$. In case of phone calls received in the time interval $(t_1,\ t_2)$ we have the random variable as $X(t_2)-X(t_1)$ or $n(t_1,\ t_2)$. In the telephone calls, for example, we have $X(t_1)=10$ or $n(0,t_1)=10, X(t_2)=25$ or $n(0,\ t_2)=25$, and $X(t_2)-X(t_1)=25-10=15$ or $n(t_1,\ t_2)=25-10=15$.

In case of Poisson points experiment, an outcome, say ξ, is a set of Poisson points $\{t_i,\ i=1,\ 2,\cdots\cdots\}$ on the time line t, that is t-axis. It may be noted that right from any starting time point $t_0=0$ till the end of each Poisson point, say $t=t_i$, one could see random occurrences of an event (say x occurrences of an event) (Refer to Figure 5.2). Therefore, the probability that there are x occurrences in the time interval $(0,\ t_i)$, that is, from the initial time $t=t_0=0$ up to a Poisson point $t=t_i$ can be obtained as

$$P\{X(t_i)=x\}=\frac{e^{-\lambda t_i}(\lambda t_i)^{x}}{x!},\quad x=0,\ 1,\ 2,\cdots\cdots \tag{5.4}$$

Clearly, $X(t_i)$ is a Poisson random variable and hence $\{X(t)\}$ is a Poisson process. Notationally, x occurrences in the time interval $(0,\ t_i)$ are denoted by $X(t_i)=x$ or $n(0,t_i)=x$. Therefore, the number of occurrences, $X(t_i)=x$ or $n(0,t_i)=x$, in an interval of length $t_i-0=t_i$ follows Poisson distribution with parameter $\lambda t_i>0$ where $\lambda>0$ is the rate of occurrence of events. Obviously, for given two time points t_1 and t_2 in the interval $(0,\ t)$ such that $t_1<t_2$, if the number of occurrences is m up to time t_1 and the number of occurrences is n up to time t_2, then the number

5.2.2 Poisson Process

In general, the random process $\{X(t)\}$ is said to be a Poisson process with parameter $\lambda t > 0$, if the probability mass function of the random variable $X(t)$ is given by

$$P\{X(t)=x\} = \frac{e^{\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \cdots\cdots \tag{5.8}$$

Or, a counting process $\{X(t)\}$ is said to be Poisson process with parameter $\lambda t > 0$ if

(i) $X(t) = 0, \quad$ when $t = 0$

(ii) $\{X(t)\}$ has *independent increments* (that is, if the intervals $(t_1,\ t_2)$ and $(t_2,\ t_3)$ are non-overlapping, then the random variables $n(t_1,t_2) = X(t_2) - X(t_1)$ and $n(t_2,\ t_3) = X(t_3) - X(t_2)$ are independent).

(iii) The number of occurrences in any interval of length t is Poisson with parameter $\lambda t > 0$. That is, for any two time points t_i and t_{i+1} such that $t = t_{i+1} - t_i, \quad i = 0, 1, 2, \ldots\ldots\ldots$, we have

$$P\{X(t_{i+1}) - X(t_i) = x\} = \frac{e^{\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \cdots\cdots$$

For example, if we let $t_i = s$ and $t_{i+1} = t + s$ then

$$P\{X(t+s) - X(s) = x\} = \frac{e^{\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \cdots\cdots$$

5.2.3 Properties of Poisson Points and Process

By now we have understood that Poisson points are specified by the following properties:

Property 5.1: The number of occurrences in an interval $(t_1,\ t_2)$ of length $(t = t_2 - t_1)$ denoted by $n(t_1,\ t_2)$ is a Poisson random variable with parameter $\lambda t > 0$. That is,

$$P\{n(t_1,\ t_2) = x\} = \frac{e^{\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \cdots\cdots \tag{5.9}$$

Property 5.2: If the intervals $(t_1,\ t_2)$ and $(t_2,\ t_3)$ are non-overlapping, then the random variables $n(t_1,\ t_2) = X(t_2) - X(t_1)$ and $n(t_2,\ t_3) = X(t_3) - X(t_2)$ are independent. This is true in case of Poisson process and hence Poisson process is a process with *independent increments*.

Property 5.3: For a specific t, it is known that $\{X(t)\}$ a Poisson random variable with parameter $\lambda t > 0$. Therefore, we have

$$\text{Mean: } E\{X(t)\} = \lambda t$$

$$E\left\{X^2(t)\right\} = \lambda t + (\lambda t)^2$$

$$\text{Variance: } V\{X(t)\} = E\left\{X^2(t)\right\} - \{E[X(t)]\}^2 = \lambda t$$

$$\text{Autocorrelation: } R(t_1,t_2) = E\{X(t_1)X(t_2)\} = \begin{cases} \lambda t_1 + \lambda^2 t_1 t_2, & \text{if} \quad t_1 < t_2 \\ \lambda t_2 + \lambda^2 t_1 t_2, & \text{if} \quad t_1 > t_2 \end{cases}$$

$$\therefore \quad R(t_1,\, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1,\, t_2) \tag{5.10}$$

If $t_1 = t_2 = t$, we have $R_{xx}(t_1,\, t_2) = \lambda t + \lambda^2 t^2 = E\{X^2(t)\}$

$$\text{Autocovariance: } C_{xx}(t_1,\, t_2) = R_{xx}(t_1,\, t_2) - E\{X(t_1)X(t_2)\} = \begin{cases} \lambda t_1, & \text{if} \quad t_1 < t_2 \\ \lambda t_2, & \text{if} \quad t_1 > t_2 \end{cases}$$

$$\therefore \quad C_{xx}(t_1,\, t_2) = \lambda \min(t_1,\, t_2) \tag{5.11}$$

If $t_1 = t_2 = t$, we have $C_{xx}(t_1,\, t_2) = \lambda t$, which is nothing but the variance of the Poisson process $\{X(t)\}$.

5.2.4 Theorems on Poisson Process

Theorem 5.1: If $\{X_1(t)\}$ and $\{X_2(t)\}$ represent two independent Poisson processes with parameters $\lambda_1 t$ and $\lambda_2 t$ respectively, then the process $\{Y(t)\}$, where $Y(t) = X_1(t) + X_2(t)$, is a Poisson process with parameter $(\lambda_1 + \lambda_2)\, t$. (That is, the sum of two independent Poisson processes is also a Poisson process.)

Proof. It is given that $\{X_1(t)\}$ and $\{X_2(t)\}$ are two independent Poisson processes with parameters $\lambda_1 t$ and $\lambda_2 t$ respectively, and $Y(t) = X_1(t) + X_2(t)$ therefore we have

$$P\{X_1(t) = x\} = \frac{e^{-\lambda_1 t}\,(\lambda_1 t)^x}{x!}, \quad x = 0, 1, 2, \cdots\cdots$$

$$P\{X_2(t) = x\} = \frac{e^{-\lambda_2 t}\,(\lambda_2 t)^x}{x!}, \quad x = 0, 1, 2, \cdots\cdots$$

$$\text{Consider } P\{Y(t) = n\} = \sum_{r=0}^{n} P\{X_1(t) = r\}\, P\{X_2(t) = n - r\}$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}\,(\lambda_1 t)^r}{r!}\, \frac{e^{-\lambda_2 t}\,(\lambda_2 t)^{n-r}}{(n-r)!}$$

CHAPTER 6

NORMAL PROCESS (GAUSSIAN PROCESS)

6.0 INTRODUCTION

Due to the nature of normal distribution, all processes can be approximated to *normal process* (also called *Gaussian Process*). In fact, as discussed in the previous chapter, the random processes following any standard statistical distributions are known as *special random processes*. The Gaussian process plays an important role in random process because it is a convenient starting point for many studies related to electrical and computer engineering. Also, in most of the situations, the Gaussian process is useful in modeling the white noise signal observed in practice which can be further interpreted as a filtered white Gaussian noise signal. For a definition of white noise process, readers are referred to Section 2.3.4 in Chapter 2. In this chapter, we study in detail the aspects of normal process. Throughout this book, normal process and Gaussian process are interchangeably used. Some processes depending on stationary normal process are also studied. In addition, in this chapter, the processes such as *random walk process* and *Weiner process* are also considered.

6.1 DESCRIPTION OF NORMAL PROCESS

A random process $\{X(t)\}$ representing the collection of random variables $X(t)$ at time points $t_1, t_2, \cdots, t_i, \cdots t_n$ is called normal process or Gaussian Process, if the random variables $X(t_1), X(t_2), \cdots, X(t_i), \cdots, X(t_n)$ are jointly normal for every $n = 1, 2, \cdots\cdots$ and for any set of time points.

The joint probability density function of n random variables (that is, n^{th} order joint density function) of a Gaussian process is given by

$$f(x_1, x_2, \cdots, x_i, \cdots, x_n; t_1, t_2, \cdots t_i, \cdots, t_n)$$

$$= \frac{e^{-\left(\frac{1}{2|\Sigma|}\right)\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}|\Sigma|_{ij}(x_i-\mu(t_i))^T\left(x_j-\mu(t_j)\right)}}{(2\pi)^{n/2}|\Sigma|^{1/2}}, -\infty < x_i < \infty, \forall i \qquad (6.1)$$

where $\mu(t_i) = E\{X(t_i)\}$, Σ is the n^{th} order square matrix (called *variance-covariance matrix*) with elements $\sigma(t_i, t_j) = C_{xx}(t_i, t_j)$ and $|\Sigma|_{ij}$ is the cofactor of $\sigma(t_i, t_j)$ in $|\Sigma|$. Refer to Section 1.8 of Chapter 1 for derivation of probability density function of n–dimensional normal random variables.

It may be noted that the covariance of two random variables $X(t_i)$ and $X(t_j)$ observed at time points t_i and t_j respectively is given by

$$\sigma(t_i, t_j) = C_{xx}(t_i, t_j) = E\{X(t_i)E(t_j)\} - E\{X(t_i)\}E\{X(t_j)\}$$

$$= E\{X(t_i)E(t_j)\} - \mu(t_i)\mu(t_j)$$

If $i = j$ then we have $\sigma(t_i, t_i) = C_{xx}(t_i, t_i) = V\{X(t_i)\}$ denoted by $\sigma^2(t_i)$ which gives the variance of the random variable $X(t_i)$. If $i \neq j$ then we have $\sigma(t_i, t_j) = C_{xx}(t_i, t_j) = \text{covar}\{X(t_i), X(t_j)\}$. We know that the correlation coefficient between the two random variables $X(t_i)$ and $X(t_j)$ is given by

$$\rho(t_i, t_j) = \frac{C_{xx}(t_i, t_j)}{\sqrt{V\{X(t_i)\}}\sqrt{V\{X(t_j)\}}} = \frac{\sigma(t_i, t_j)}{\sigma(t_i)\sigma(t_j)}$$

$$\Rightarrow \quad \sigma(t_i, t_j) = \rho(t_i, t_j)\sigma(t_i)\,\sigma(t_j) \tag{6.2}$$

It is clear that the normal process involves mean, variance and correlation as parameters that may or may not depend on time. Therefore, if mean and variance of a normal process are constants and correlation coefficient is time invariant, then we can conclude that the process is stationary in strict sense.

6.2 PROBABILITY DENSITY FUNCTION OF NORMAL PROCESS

6.2.1 First Order Probability Density Function of Normal Process

The first order probability density function of normal process $\{X(t)\}$ is given by

$$f(x, t) = \frac{1}{\sqrt{2\pi}\sigma(t)} e^{-\frac{1}{2}\left(\frac{x-\mu(t)}{\sigma(t)}\right)^2}, \quad -\infty < x < \infty \tag{6.3}$$

where $\mu(t)$ is the mean and $\sigma(t)$ is the standard deviation of the normal process.

It is clear that the normal process involves mean and variance as parameters that may or may not depend on time. Therefore, if mean and variance of a normal process are constants then we can conclude that the process is stationary in strict sense.

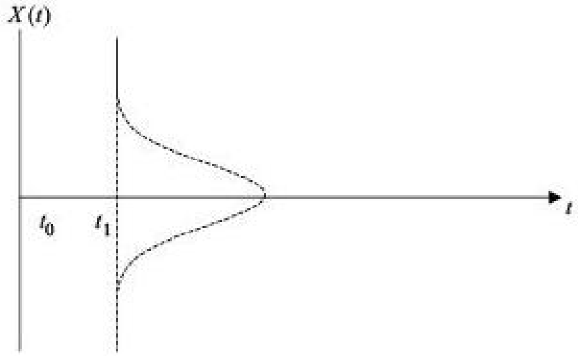


Figure 6.1. The normal distribution of the random process $\{X(t)\}$

Figure 6.1 represents the normal distribution of the random process $\{X(t)\}$ at time point $t = t_1$. This means that at a particular time point, say $t = t_1$, we have a random variable $X(t_1)$ which is normally distributed with some mean, say $\mu(t_1)$ and variance, say $\sigma^2(t_1)$. From any given process, if we observe that $\mu(t_1) = \mu(t_2) = \mu(t_3) = \cdots\cdots = \mu$ and $\sigma^2(t_1) = \sigma^2(t_2) = \sigma^2(t_3) = \cdots\cdots = \sigma^2$, meaning that the mean and variance are constants being independent of time, we say that the process is stationary in strict sense. It may be noted that as many samples of the process $\{X(t)\}$ will be close enough to the mean of the normal distribution.

6.2.2 Second Order Probability Density Function of Normal Process

If $X(t_1)$ and $X(t_2)$ are two random variables observed at time points t_1 and t_2 of normal process $\{X(t)\}$, then the second order probability density function (or the joint probability density function of $X(t_1)$ and $X(t_2)$ is given by

$$f(x_1, x_2; t_1, t_2)$$

$$= \frac{e^{-\frac{1}{2[1-\rho^2(t_1, t_2)]}\left\{\left(\frac{x_1-\mu(t_1)}{\sigma(t_1)}\right)^2 - 2\rho(t_1, t_2)\left(\frac{x_1-\mu(t_1)}{\sigma(t_1)}\right)\left(\frac{x_2-\mu(t_2)}{\sigma(t_2)}\right) + \left(\frac{x_2-\mu(t_2)}{\sigma(t_2)}\right)^2\right\}}}{\sqrt{2\pi}\,\sigma(t_1)\,\sigma(t_2)\sqrt{[1-\rho^2(t_1, t_2)]}}, \tag{6.4}$$

$$-\infty < x_1, x_2 < \infty$$

where $\mu(t_1)$ and $\mu(t_2)$ are the means and $\sigma(t_1)$ and $\sigma(t_2)$ are the standard deviations of the random variables $X(t_1)$ and $X(t_2)$ respectively. Refer to Section 1.8.4 of Chapter 1 for derivation of probability density function of two-dimensional normal random variables.

In this case the normal process involves mean, variance and correlation as parameters that may or may not depend on time. Therefore, if mean and variance of a normal process are constants and correlation coefficient is time invariant, then we can conclude that the process is stationary in strict sense.

If $X(t_1)$ and $X(t_2)$ are independent, then $\rho(t_1, t_2) = 0$, and hence we have

$$f(x_1, x_2; t_1, t_2) = \frac{e^{-\frac{1}{2}\left\{\left(\frac{x_1-\mu(t_1)}{\sigma(t_1)}\right)^2 + \left(\frac{x_2-\mu(t_2)}{\sigma(t_2)}\right)^2\right\}}}{\sqrt{2\pi}\,\sigma(t_1)\,\sigma(t_2)},$$

$$-\infty < x_1, x_2 < \infty \tag{6.5}$$

This is nothing but the product of the two normal densities.

6.2.3 Second Order Stationary Normal Process

It may be noted that in case of second order normal process if mean and variance are constants then only possibility that the process need not be a stationary process is the correlation is not constant (that is, not time invariant). Therefore, under such circumstances, if we can prove that the correlation is time invariant, then the given normal process is stationary.

That is if mean and variance are constants, then the normal process is said to be stationary, if

$$\rho(t_1, t_2) = \rho(t_1 + \tau, t_2 + \tau) \tag{6.6}$$

6.3 STANDARD NORMAL PROCESS (CENTRAL LIMIT THEOREM)

The *central limit theorem* helps to explain the propagation of Gaussian random variable in nature (refer to Section 1.5.2 of Chapter 1). The same can be extended to the case of Gaussian process. If $\{X(t)\}$ is a normal process with mean $\mu(t)$ and standard deviation $\sigma(t)$, then the random process $\{Z(t)\}$, where

$$Z(t) = \frac{X(t) - E\{X(t)\}}{\sqrt{V\{X(t)\}}} = \frac{X(t) - \mu(t)}{\sigma(t)} \tag{6.7}$$

is called standard normal process with mean 0 and variance 1. That is, $E\{Z(t)\} = 0$ and $V\{Z(t)\} = 1$.

6.3.1 Properties of Gaussian (Normal) Process

(i) If a Gaussian process is wide sense stationary, then it is also a strict sense stationary. This is true by the definition of WSS and the parameters involved in Gaussian process.

(ii) If the member functions (random variables) of a Gaussian process are uncorrelated, then they are independent.

(iii) If the input $\{X(t)\}$ of a linear system is Gaussian then the output will also be a Gaussian process. That is, if the random process $\{X(t)\}$ is Gaussian with mean $\mu(t)$ and standard deviation $\sigma(t)$ and if $\{Y(t)\}$ is a random process such that $Y(t) = a + bX(t)$ where a and b are constants, then $\{Y(t)\}$ is also Gaussian with mean $E\{Y(t)\} = E\{a + bX(t)\} = a + b\mu(t)$ and variance $V\{Y(t)\} = V\{a + bX(t)\} = b^2\sigma^2(t)$.

6.4 PROCESSES DEPENDING ON STATIONARY NORMAL PROCESS

6.4.1 Square-Law Detector Process

If $\{X(t)\}$ is a zero mean stationary normal process and if $Y(t) = X^2(t)$ then the process $\{Y(t)\}$ is called a *square-law detector process*.

Some important results:

Let the normal process $\{X(t)\}$ be stationary with mean $E\{X(t)\} = 0$, variance $V\{X(t)\} = \sigma_x^2$ (*say*) and autocorrelation function $R_{xx}(\tau)$. Now,

$$E\{X(t)\} = 0, \quad \Rightarrow \quad V\{X(t)\} = \sigma_x^2 = E\left\{X^2(t)\right\} = R_{xx}(0)$$

$$\therefore \quad E\{Y(t)\} = E\left\{X^2(t)\right\} = R_{xx}(0)$$

Consider $\quad R_{yy}(t_1, t_2) = E\{Y(t_1)Y(t_2)\}$

$$= E\left\{X^2(t_1)X^2(t_2)\right\}$$

$$= E\left\{X^2(t_1)\right\}E\left\{X^2(t_2)\right\} + 2\{E[X(t_1)X(t_2)]\}^2$$

(Refer Eqn. 6.8)

$$\therefore \quad R_{yy}(t_1, t_2) = R_{xx}^2(0) + 2R_{xx}(t_1, t_2)$$

$$\Rightarrow \quad R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau) \quad (\because \quad \{X(t)\} \text{ is stationary})$$

Now, $E\left\{Y^2(t)\right\} = R_{yy}(0) = 3R_{xx}^2(0)$

$$\therefore \quad V\{Y(t)\} = \sigma_y^2 = E\left\{Y^2(t)\right\} - \{E[Y(t)]\}^2 = 3R_{xx}^2(0) - R_{xx}^2(0) = 2R_{xx}^2(0)$$

Also $C_{yy}(\tau) = R_{yy}(\tau) - E\{Y(t_1)\}E\{Y(t_2)\} = R_{xx}^2(0) + 2R_{xx}^2(\tau) - R_{xx}^2(0) = 2R_{xx}^2(\tau)$

Therefore, $\{Y(t)\}$ is a wide sense stationary process with mean and autocorrelation as given below:

$$E\{Y(t)\} = R_{xx}(0) \quad \text{(constant)}$$

$$R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau) \quad \text{(time invariant)} \tag{6.14}$$

6.4.2 Full-Wave Linear Detector Process

If $\{X(t)\}$ is a zero mean stationary normal process and if $Y(t) = |X(t)|$ then the process $\{Y(t)\}$ is called a *full-wave linear detector process*.

Some important results:

Let the normal process $\{X(t)\}$ be stationary with mean $E\{X(t)\} = 0$, variance $V\{X(t)\} = \sigma_x^2$ (*say*) and autocorrelation function $R_{xx}(\tau)$. Now,

$$E\{X(t)\} = 0, \quad \Rightarrow \quad V\{X(t)\} = \sigma_x^2 = E\left\{X^2(t)\right\} = R_{xx}(0)$$

$$\therefore \quad E\{Y(t)\} = E\{|X(t)|\} = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}\sigma_x} e^{-x^2/2\sigma_x^2} dx = \frac{2}{\sqrt{2\pi}\sigma_x} \int_0^{\infty} x e^{-x^2/2\sigma_x^2} dx$$

Let $v = \dfrac{x^2}{2\sigma_x^2} \quad \Rightarrow \quad x\,dx = \sigma_x^2\,dv$

$$\Rightarrow \quad \frac{2}{\sqrt{2\pi}\sigma_x} \int_0^{\infty} x e^{-x^2/2\sigma_x^2} dx = \sqrt{\frac{2}{\pi}}\sigma_x \int_0^{\infty} e^{-v} dv = \sqrt{\frac{2}{\pi}}\sigma_x \qquad \because \quad \int_0^{\infty} e^{-v} dv = 1$$

$$\therefore \quad E\{Y(t)\} = \sqrt{\frac{2}{\pi}}\sigma_x = \sqrt{\frac{2}{\pi}\sigma_x^2} = \sqrt{\frac{2}{\pi}R_{xx}(0)}$$

Consider
$$\begin{aligned} R_{yy}(t_1, t_2) &= E\{Y(t_1)Y(t_2)\} \\ &= E\{|X(t_1)|\,|X(t_2)|\} = E\{|X(t_1)X(t_2)|\} \\ &= \frac{2}{\pi}\sigma_x^2(\cos\alpha + \alpha\sin\alpha) \quad \text{(Refer to Equation (6.11))} \end{aligned}$$

where $\sin\alpha = \rho_{xx}(t_1,\, t_2)$

We know that $\rho(t_1,\, t_2) = \dfrac{\operatorname{Cov}\{X(t_1)X(t_2)\}}{\sqrt{V\{X(t_1)\}}\,\sqrt{V\{X(t_2)\}}} = \dfrac{E\{X(t_1)X(t_2)\}}{\sigma_x^2}$

$\because \quad E\{X(t)\} = 0$

$$\Rightarrow \quad \rho_{xx}(t_1,\, t_2) = \frac{E\{X(t_1)X(t_2)\}}{\sigma_x^2} = \frac{R_{xx}(t_1,\, t_2)}{\sigma_x^2}$$

Since the process $\{X(t)\}$ is stationary, we have

$$\Rightarrow \quad \rho_{xx}(\tau) = \frac{R_{xx}(\tau)}{\sigma_x^2} = \frac{R_{xx}(\tau)}{R_{xx}(0)} = \sin\alpha \quad \text{(Refer to Equation (6.13))}$$

Therefore, $\{Y(t)\}$ is wide sense stationary process with autocorrelation function

$$R_{yy}(\tau) = \frac{2}{\pi}R_{xx}(0)(\cos\alpha + \alpha\sin\alpha)$$

Consider $E\left\{Y^2(t)\right\} = R_{yy}(0) = \dfrac{2}{\pi}R_{xx}(0)\left\{0+\dfrac{\pi}{2}(1)\right\} = R_{xx}(0)$

Because when $\tau = 0$ we have $\sin\alpha = 1 \Rightarrow \alpha = \dfrac{\pi}{2}$

Hence, $V\{Y(t)\} = E\left\{Y^2(t)\right\} - \{E[Y(t)]\}^2 = R_{xx}(0) - \left(\sqrt{\dfrac{2}{\pi}R_{xx}(0)}\right)^2 = \left(1-\dfrac{2}{\pi}\right)R_{xx}(0)$

Therefore, $\{Y(t)\}$ is wide sense stationary process with mean and autocorrelation as given below:

$$E\{Y(t)\} = \sqrt{\frac{2}{\pi}}\sigma_x = \sqrt{\frac{2}{\pi}\sigma_x^2} = \sqrt{\frac{2}{\pi}R_{xx}(0)} \quad \text{(constant)}$$

$$R_{yy}(\tau) = \frac{2}{\pi}R_{xx}(0)(\cos\alpha + \alpha\sin\alpha), \quad \text{(time invariant)}, \tag{6.15}$$

where $\sin\alpha = \rho_{xx}(\tau)$.

6.4.3 Half-Wave Linear Detector Process

If $\{X(t)\}$ is a zero mean stationary normal process and if $Z(t) = \begin{cases} X(t), & \text{if } X(t) \geq 0 \\ 0, & \text{if } X(t) < 0 \end{cases}$ then the process $\{Z(t)\}$ is called a *half-wave linear detector process*. It may be noted that we can also have $Z(t) = \dfrac{1}{2}\{X(t) + |X(t)|\}$.

Some important results:

Let the normal process $\{X(t)\}$ be stationary with mean $E\{X(t)\} = 0$, variance $V\{X(t)\} = \sigma_x^2$ (*say*) and autocorrelation function $R_{xx}(\tau)$. Now,

$$E\{X(t)\} = 0, \quad \Rightarrow \quad V\{X(t)\} = \sigma_x^2 = E\left\{X^2(t)\right\} = R_{xx}(0)$$

Consider $\quad R_{zz}(t_1, t_2) = E\{Z(t_1)\,Z(t_2)\}$

We know that

$$E\{Z(t_1)Z(t_2)\} = E\{E\{Z(t_1)Z(t_2)/X(t_1)X(t_2)\}\}$$

$$Z(t_1)Z(t_2)/X(t_1)X(t_2) = \begin{cases} \dfrac{1}{2}\{X(t_1)X(t_2) + |X(t_1)X(t_2)|\} & \text{if } \; X(t_1)X(t_2) \geq 0 \\ 0 \quad \text{if } \; X(t_1)X(t_2) < 0 \end{cases}$$

Therefore, $\{Z(t)\}$ is wide sense stationary process with mean and autocorrelation as given below:

$$E\{Z(t)\} = \sqrt{\frac{1}{2\pi}R_{xx}(0)} \quad \text{(constant)}$$

$$R_{zz}(\tau) = \frac{1}{4}\left\{R_{xx}(\tau) + \frac{2}{\pi}R_{xx}(0)(\cos\alpha + \alpha\sin\alpha\right\} \quad \text{(time invariant)} \quad (6.16)$$

6.4.4 Hard Limiter Process

If $\{X(t)\}$ is a zero mean stationary normal process and if $Y(t) = \begin{cases} +1, & \text{if } X(t) \geq 0 \\ -1, & \text{if } X(t) < 0 \end{cases}$

then the process $\{Y(t)\}$ is called a *hard limiter process*.

Some important results:

Let the normal process $\{X(t)\}$ be stationary with mean $E\{X(t)\} = 0$, variance $V\{X(t)\} = \sigma_x^2$ (*say*) and autocorrelation function $R_{xx}(\tau)$. Now,

$$E\{X(t)\} = 0, \Rightarrow V\{X(t)\} = \sigma_x^2 = E\left\{X^2(t)\right\} = R_{xx}(0)$$

$$\therefore \quad E\{Y(t)\} = E\{X(t)\} = (+1)\frac{1}{2} + (-1)\frac{1}{2} = 0$$

$$E\left\{Y^2(t)\right\} = E\left\{X^2(t)\right\} = (+1)^2\frac{1}{2} + (-1)^2\frac{1}{2} = 1$$

Hence, $V\{Y(t)\} = E\{Y^2(t)\} - \{E[Y(t)]\}^2 = 1 - 0^2 = 1$

Consider $R_{yy}(t_1, t_2) = E\{Y(t_1)Y(t_2)\}$

Now, $Y(t_1)Y(t_2) = \begin{cases} 1 & \text{if} \quad X(t_1)X(t_2) \geq 0 \\ -1 & \text{if} \quad X(t_1)X(t_2) < 0 \end{cases}$

$$\therefore \quad P\{Y(t_1)Y(t_2) = 1\} = P\{X(t_1)X(t_2) \geq 0\}$$

$$= \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\rho_{xx}(t_1, t_2) \quad \text{(Refer to Equation (6.9))}$$

$$= \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\frac{R_{xx}(t_1, t_2)}{R_{xx}(0)}$$

$$\therefore \quad P\{Y(t_1)Y(t_2) = -1\} = P\{X(t_1)X(t_2) < 0\}$$

$$= \frac{1}{2} - \frac{1}{\pi}\sin^{-1}\rho_{xx}(t_1, t_2) \quad \text{(Refer to Equation (6.10))}$$

$$= \frac{1}{2} - \frac{1}{\pi}\sin^{-1}\frac{R_{xx}(t_1, t_2)}{R_{xx}(0)}$$

$$\begin{aligned}
\therefore \quad R_{yy}(t_1, t_2) &= E\{Y(t_1)Y(t_2)\} \\
&= (+1)\left\{\frac{1}{2} + \frac{1}{\pi}\sin^{-1}\left(\frac{R_{xx}(t_1, t_2)}{R_{xx}(0)}\right)\right\} \\
&\quad + (-1)\left\{\frac{1}{2} - \frac{1}{\pi}\sin^{-1}\left(\frac{R_{xx}(t_1, t_2)}{R_{xx}(0)}\right)\right\} \\
&= \frac{2}{\pi}\sin^{-1}\left(\frac{R_{xx}(t_1, t_2)}{R_{xx}(0)}\right)
\end{aligned}$$

Since the process $\{X(t)\}$ is stationary, we have

$$R_{yy}(\tau) = \frac{2}{\pi}\sin^{-1}\left(\frac{R_{xx}(\tau)}{R_{xx}(0)}\right)$$

Therefore, $\{Y(t)\}$ is wide sense stationary process with mean and autocorrelation as given below:

$$E\{Y(t)\} = 0 \quad \text{(constant)}$$

$$R_{yy}(\tau) = \frac{2}{\pi}\sin^{-1}\left(\frac{R_{xx}(\tau)}{R_{xx}(0)}\right) \quad \text{(time invariant)} \tag{6.17}$$

6.5 GAUSSIAN WHITE-NOISE PROCESS

A random process $\{X(t)\}$ is called *Gaussian white noise process* if and only if it is a stationary Gaussian random process with zero mean and autocorrelation function is of the form given by

$$R_{xx}(t_1, t_2) = b(t_1)\delta(t_1 - t_2) = b_0\delta(\tau) \tag{6.18}$$

That is,

$$R_{xx}(\tau) = b_0\delta(\tau)$$

If $X(t_1), X(t_2), \cdots\cdots, X(t_n)$ is a collection of n independent random variables at time points $t_1, t_2, \cdots\cdots, t_n$ respectively, then the value of the noise $X(t_i)$ at time point t_i says nothing about the value of noise $X(t_j)$ at time point t_j, $t_i \neq t_j$. Which means that $X(t_i)$ and $X(t_j)$ are uncorrelated and hence we have the autocovariance as

$$C_{xx}(t_i, t_j) = 0 \text{ for every pair } t_i \text{ and } t_j \text{ such that } t_i \neq t_j \tag{6.19}$$

However, though Gaussian white noise process is a useful mathematical model it does not conform to any signal that can be observed physically. Further, it may be noted that the average power of white noise is given by

$$E\left\{X^2(t)\right\} = R_{xx}(0) = \infty \tag{6.20}$$

Meaning that the white noise has infinite average power, which is physically not possible. Notwithstanding, it is useful because any Gaussian noise signal observed in a real system can be interpreted as a filtered white Gaussian noise signal with finite power.

6.6 RANDOM WALK PROCESS

If $Y_1, Y_2, \ldots\ldots$ are independently identically distributed random variables such that $P\{Y_n = 1\} = p$ and $P\{Y_n = -1\} = q$, $p+q = 1$ for all n, then the collection of random variables $\{X_n, n \geq 0\}$, where $X_n = \sum\limits_{i=1}^{n} Y_i$, $n = 1, 2, 3, \cdots\cdots$ and $X_0 = 0$, which is a discrete-parameter (or time), discrete-state random process, is known as a *simple random walk*.

6.6.1 More on Random Walk

Let us assume that a process $\{X(t)\}$ moves one step forward of a distance d, if a coin turns head and moves one step backward of a distance d, if the coin turns tail. Let the process $\{X(t)\}$ represent the total distance traveled in the time interval $(0, t)$. Clearly, $X(t) = 0$ initially at time point $t = 0$ and is observed at time intervals each of length T. Then at every time point $t = 1, 2, 3, \cdots\cdots, T$ we have

$$X(t) = \begin{cases} +d, & \text{if head turns} \\ -d, & \text{if tail turns} \end{cases}$$

$$\Rightarrow \quad P\{X(t = +d\} = P\{X(t = -d\} = \frac{1}{2}$$

If the coin is tossed n times, then there could be k heads and $n-k$ tails in the total time of $t = nT$. Therefore, the distance covered by the process is

$$X(nT) = \begin{cases} kd, & \text{ahead for heads} \\ (n-k)d, & \text{backward for tails} \end{cases}$$

That is, after n tosses, the total distance between the origin to the present position of the process is

$$X(nT) = kd - (n-k)d = (2k-n)d$$

The process $\{X(nT)\}$ is known as *random walk* process.

It may be noted that, since $k = 0, 1, 2, 3, \cdots\cdots, n$, we have $X(nT) = -nd$, $(2-n)d$, $(4-n)d$, $\cdots\cdots$, $(-2+n)d$, nd. However, being distance, the quantity $(2k-n)d$ is always taken as positive. Clearly, $X(nT) = (2k-n)d$ is equivalent to getting 'only' k number of heads in n tosses. Then we have

$$P\{X(nT) = (2k-n)d\} = P\{\text{getting only } k \text{ heads in } n \text{ tosses}\} = {}^{n}C_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

Clearly, $E\{X(nT)\} = n\left(\frac{1}{2}(d) + \frac{1}{2}(-d)\right) = 0$

$$E\left\{X^2(nT)\right\} = n\left(\frac{1}{2}(d)^2 + \frac{1}{2}(-d)^2\right) = nd^2$$

$$\therefore \quad V\{X(nT)\} = E\left\{X^2(nT)\right\} - \{E[X(nT)]\}^2 = nd^2$$

Given a binomial distribution with probability mass function

$$P\{X = k\} = {}^nC_k\, p^k\, q^{n-k}, \quad k = 0, 1, 2, \cdots\cdots, n$$

whose mean is np and variance is npq the same can be approximated as a normal distribution with mean $\mu = np$ and variance $\sigma^2 = npq$. That is, the random variable $X \sim N(np, npq)$ which implies

$$ {}^nC_k\, p^k\, q^{n-k} \cong \frac{1}{\sqrt{2\pi npq}} e^{-\frac{1}{2}\left(\frac{k-np}{\sqrt{npq}}\right)^2}$$

But

$$P\{X(nT) = (2k-n)d\} \cong {}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \frac{1}{\sqrt{2\pi}\sqrt{n/4}} e^{-\frac{1}{2}\left(\frac{k-n/2}{\sqrt{n/4}}\right)^2},$$

for $(2k-n)d = -nd,\ (-n+2)d,\ (-4-n)d,\ \cdots\cdots,\ (-2+n)d,\ nd$

6.7 WIENER PROCESS

A random process $\{X(t)\}$ is said to be a Wiener process if

(i) $\{X(t)\}$ has stationary independent increments.
(ii) The increment $X(t_i) - X(t_j), \quad t_i > t_j, \quad \forall\, i, j$ is normally distributed.
(iii) $E\{X(t)\} = 0$.
(iv) $X(0) = 0$.

6.7.1 Random Walk and Wiener Process

Here we show that the Wiener process is a limiting form of random walk. Let us consider the probability distribution of random walk

$$P\{X(nT) = (2k-n)d\} = \frac{1}{\sqrt{2\pi}\sqrt{n/4}} e^{-\frac{1}{2}\left(\frac{k-n/2}{\sqrt{n/4}}\right)^2} = \frac{1}{\sqrt{2\pi}\sqrt{n/4}} e^{-\frac{1}{2}\left(\frac{2k-n}{\sqrt{n}}\right)^2},$$

for $(2k-n)d = -nd,\ (2-n)d,\ (4-n)d,\ \cdots\cdots,\ (-2+n)d,\ nd$

Let $nT = t$, $(2k-n)d = x$ and $d^2 = \omega T$. Now, as $T = 0$ and $n \to \infty$, that is the case of tossing the coin continuously, then we have $\{X(nT)\} = \{X(t)\}$ as a process.

Accordingly, we have $\frac{n}{4} = E\{X^2(nT)\} = nd^2 = n\omega T = \omega t$ and hence we have

$$\frac{(2k-n)}{\sqrt{n}} = \frac{x/d}{\sqrt{t/T}} = \frac{x}{\sqrt{d^2 t/T}} = \frac{x}{\sqrt{\omega t}}$$

$$\therefore \quad P\{x \le X(t) \le x + dx\} = \frac{1}{\sqrt{2\pi}\sqrt{\omega t}} e^{-x^2/2\omega t}$$

Therefore, the probability density function of Wiener process $\{X(t)\}$ is given by

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi}\sqrt{\omega t}} e^{-x^2/2\omega t}, \quad -\infty < x < \infty$$

which is normal with mean 0 and standard deviation $\sqrt{\omega t}$.

6.7.2 Mean, Variance, Autocorrelation and Autocovariance of Wiener Process

For a Wiener process we have

$$E\{X(t)\} = 0 \quad \text{and} \quad V\{X(t)\} = \omega t$$

Since, Wiener process is a process with independent increments, we have letting $t_2 > t_1$

$$X(t_2) - X(t_1) \quad \text{and} \quad X(t_1) - X(t_0) \text{ are independent}$$

Consider $E\{[X(t_2) - X(t_1)]X(t_1)\} = E\{X(t_2) - X(t_1)\}E\{X(t_1)\} = 0$

$\because \quad E\{X(t_1)\} = 0$

$$\Rightarrow E\{X(t_2)X(t_1)\} = E\left\{X^2(t_1)\right\} = \omega t_1$$

$$\Rightarrow R_{xx}(t_2,\ t_1) = \omega t_1$$

Similarly, letting $t_1 > t_2$, we have

$$R_{xx}(t_2,\ t_1) = \omega t_2$$

Therefore, for a Wiener process the autocorrelation is given by

$$R_{xx}(t_2,\ t_1) = \omega \min(t_1,\ t_2)$$

$$\Rightarrow C_{xx}(t_1,\ t_2) = R_{xx}(t_1,\ t_2) - E\{X(t_1)\}E\{X(t_2)\} = R_{xx}(t_1,\ t_2) = \omega \min(t_1,\ t_2).$$

This is true because for a Wiener process

$$E\{X(t_1)\} = E\{X(t_2)\} = 0$$

CHAPTER 7

SPECTRUM ESTIMATION: ERGODICITY

7.0 INTRODUCTION

In random process studies, a central problem in the application of such processes is the estimation of various statistical parameters in terms of real data which are nothing but signals. Similar to that of estimation of parameters in statistical distribution, the parameter estimation in random process is mostly related to find the expected values of some functional forms of the given process. The real challenge in this estimation is the limited availability of data (signals). For example, one may observe only one signal during a time interval out of as many possible signals. Obviously, in this chapter, the problem of estimating the mean and, of course, the variance of a given process is considered. If entire spectrum of signals (i.e., ensemble) is available, *ensemble average* can be obtained which is similar to that of population parameter in statistical studies. However, as discussed above, only one signal (single realization of the process) can be observed during a time interval from which *time average* can be obtained as an estimate of ensemble average. *Ergodicity* is, in fact, related to the estimation of ensemble average using time average. Ergodicity is related to correlation and distribution as well. As a result, one can verify whether a given process is mean ergodic or correlation ergodic or distribution ergodic.

7.1 ENSEMBLE AVERAGE AND TIME AVERAGE

It may be noted that an ensemble of a random process $\{X(t)\}$ contains an infinite number of random functions, say $X(t,\xi_i)$, $i=1,2,3,\cdots\cdots,n\cdots\cdots$. Let us assume that we have a sample of size n of such random functions, that is $X(t,\xi_1)$, $X(t,\xi_2)$, $\cdots\cdots X(t,\xi_i)$, $\cdots\cdots$, $X(t,\xi_n)$. Refer to the examples given in Chapter 2 to know more about the sample functions of a random process. We know that at any time point t, say $t=t_j$, $j=1,2,\cdots\cdots$, the function $X(t_j,\xi)$ becomes a random variable assigning the values

$$X(t_j,\xi_1)=x_1, X(t_j,\xi_2)=x_2,\cdots\cdots X(t_j,\xi_i)=x_i,\cdots\cdots, X(t_j,\xi_n)=x_n.$$

Under this circumstance, if the random variable $X(t_j,\xi)$ is discrete with probability mass function $P[X(t_j,\xi)=x]$ then its expected value (average) can be statistically obtained as simply $X(t) = 1.5\cos(2.5t + 0.05)$. Then, in the interval $(0, t) = (0, T) = (0, 2)$, we have the time average as

$$\overline{X}_T = \frac{1}{2}\int_0^2 1.5\cos(2.5t + 0.05)\,dt$$

$$= \frac{1.5}{2}\left[\frac{\sin(2.5t + 0.05)}{2.5}\right]_0^2 = \frac{1.5}{5}[\sin(5.05) - \sin(0.05)] = -0.2981$$

Therefore, the value -0.2681 can be taken as the time average of the process $X(t) = 1.5\cos(2.5t + 0.05)$. Refer to Figure 7.2.

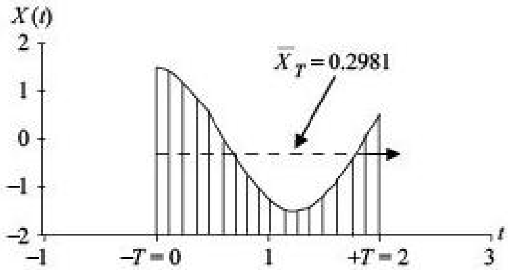


Figure 7.2. Time average of the process $X(t) = 1.5\cos(2.5t + 0.05)$

7.2 DEFINITIONS ON ERGODICITY

7.2.1 Ergodic Process

A stationary random process is said to be *ergodic* if its ensemble average involving the process can be estimated using the time average of one of the sample functions (realizations) of the process. While the subject of ergodicity is complicated one, in most physical applications it is assumed that stationary processes are ergodic.

7.2.2 Mean Ergodic Process

A stationary random process $\{X(t)\}$ defined in the time interval $(-T, T)$ said to be *mean ergodic* if the time average tends to constant ensemble average as $T \to \infty$. That is,

(i) Ensemble average: $E\{X(t)\} = \mu$ (constant).

(ii) Limit of time average: $\lim_{T\to\infty} \overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt = \mu$.

In other words, $E\{X(t)\} = \lim_{T\to\infty} \overline{X}_T = \mu$

7.2.3 Correlation Ergodic Process

A stationary random process $\{X(t)\}$ defined in the time interval $(-T, T)$ said to be *correlation ergodic* if the process $\{Z(t)\}$, where $Z(t) = X(t)X(t+\tau)$ or $Z(t) = X(t+\tau)X(t)$, is mean ergodic. That is,

(i) $E\{Z(t)\} = E\{X(t+\tau)X(t)\} = R(\tau)$ (the autocorrelation function).

(ii) $\lim_{T\to\infty} \overline{Z}_T = \dfrac{1}{2T}\displaystyle\int_{-T}^{T} X(t+\tau)X(t)\,dt = R(\tau)$.

7.2.4 Distribution Ergodic Process

Let $\{X(t)\}$ be a stationary random process and $\{Y(t)\}$ is another stationary random process such that

$$Y(t) = \begin{cases} 1 & \text{if } X(t) \le x \\ 0 & \text{if } X(t) > x \end{cases}$$

where x is some realization of $\{X(t)\}$. Then $\{X(t)\}$ is said to be a distribution ergodic process, if $\{Y(t)\}$ is mean ergodic. That is, the stationary random process $\{X(t)\}$ is said to be distribution ergodic, if

(i) Ensemble average: $E\{Y(t)\} = \eta$ (constant).

(ii) Time average: $\lim_{T\to\infty} Y_T = \dfrac{1}{2T}\displaystyle\int_{-T}^{T} Y(t)\,dt = E\{Y(t)\} = \eta$.

It may be noted that

$$\begin{aligned} E\{Y(t)\} &= (1)P\{X(t) \le x\} + (0)P\{X(t) > x\} \\ &= P\{X(t) \le x\} \\ &= F(x,\ t) \end{aligned}$$

where $F(x,\ t)$ is the cumulative probability distribution of $\{X(t)\}$. Therefore, the stationary random process $\{X(t)\}$ is said to be distribution ergodic, if

$$E\{Y(t)\} = \lim_{T\to\infty} Y_T = \frac{1}{2T}\int_{-T}^{T} Y(t)\,dt \to F(x,t)$$

7.2.5 Estimator of Mean of the Process

If $\{X(t)\}$ is a stationary random process with ensemble average μ and time average μ_T, then μ_T is the *unbiased estimator* of μ.

This is true because, we know that the time average of $\{X(t)\}$ is given by

$$\overline{X}_T = \mu_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$$

$$\Rightarrow E\{\mu_T\} = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\,dt = \mu \qquad \because \quad E\{X(t)\} = \mu$$

7.2.6 Convergence in Probability

If the time average $\mu_T(\xi_i)$ is computed from a single realization $\{X(t,\xi_i)\}$ of the stationary random process $\{X(t)\}$ whose ensemble average is μ, then $\mu_T(\xi_i)$ is said to converge to μ in probability if $P\{|\mu_T(\xi_i) - \mu| \le \varepsilon\} \to 1$ as $T \to \infty$ for some negligible quantity $\varepsilon > 0$.

7.2.7 Convergence in Mean Square Sense

If μ_T is time average of the stationary random process $\{X(t)\}$ whose ensemble mean is μ, then μ_T is said to converge to μ in mean square sense if the variance of μ_T, say σ_T^2, tends to zero as $T \to \infty$. That is, $\mu_T \to \mu$ if $\sigma_T^2 \to 0$ as $T \to \infty$.

7.2.8 Mean Ergodic Theorem

If $\{X(t)\}$ is a stationary random process with a constant ensemble average μ and time average given by $\overline{X}_T = \dfrac{1}{2T}\displaystyle\int_{-T}^{T} X(t)\,dt$, then $\{X(t)\}$ is said to be mean ergodic if $\lim\limits_{T\to\infty} V\{\overline{X}_T\} = 0$.

Proof.
Consider

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$$

$$\Rightarrow E\{\overline{X}_T\} = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\,dt = \mu \qquad \because \quad E\{X(t)\} = \mu$$

By Chebycheff's inequality, we know that

$$P\{|\overline{X}_T - E(\overline{X}_T)| \le \varepsilon\} \ge 1 - \frac{V\{\overline{X}_T\}}{\varepsilon^2}, \qquad \varepsilon > 0$$

$$\Rightarrow \quad P\left\{\left|\lim_{T\to\infty} \overline{X}_T - \mu\right| \le \varepsilon\right\} \ge 1 - \lim_{T\to\infty}\frac{V\{\overline{X}_T\}}{\varepsilon^2}, \qquad \varepsilon > 0$$

CHAPTER 8

POWER SPECTRUM: POWER SPECTRAL DENSITY FUNCTIONS

8.0 INTRODUCTION

In case of stationary random process $\{X(t)\}$ observed in a time interval $(0,t)$, the autocorrelation function, denoted by $R_{xx}(\tau)$, where $\tau = t_1 - t_2$ or $\tau = t_2 - t_1$ with $t_1, t_2 \in (0,t)$, plays an important role in determining the strength of a process that is, particularly, observed in the form of a signal. In fact, the autocorrelation shows how rapidly one can expect the random signal represented by $X(t)$ of a stationary process $\{X(t)\}$ to change as a function of time, t. That is, if $R_{xx}(\tau)$ decays rapidly (deteriorates fast), then it indicates that the signal is expected to change rapidly (fast). In other sense, if $R_{xx}(\tau)$ decays slowly (deteriorates slowly), then it indicates that the signal is expected to change slowly. Further, if the autocorrelation function has periodic components, then such a periodicity will be reflected on the corresponding process as well. Apparently, one can understand the fact that the autocorrelation function $R_{xx}(\tau)$ contains information about the expected frequency content in the signal of the stationary process of interest.

Power spectral density (PSD) describes how the power (or variance or amplitudes) of a time series (a time dependent signal) is distributed with frequency. In other simple terms, the PSD captures the frequency content in a signal. Or otherwise, the PSD refers to the amount of power per unit of frequency as a function of frequency. For example, if a realization $X(t)$ of a stationary random process $\{X(t)\}$ represents a voltage waveform across a one ohm ($1\ \Omega$) resistance, then the ensemble average of the square of $X(t)$ is nothing but the average of power delivered to the $1\ \Omega$ resistance by $X(t)$. That is, $E\{X^2(t)\} = R_{xx}(t,t) = R_{xx}(0)$ gives the average power of $\{X(t)\}$. Therefore, the autocorrelation function at $\tau = 0$ gives the average power of the process $\{X(t)\}$.

It may be noted that in the theory of signals, spectra are associated with Fourier transforms. The Fourier transforms are used to represent a function as a superposition of exponentials for determining signals. In fact, for random signals, the notion of a spectrum has two interpretations: the first one involves transforms of averages and is essentially deterministic and the second one leads to the representation of the process under consideration as superposition of exponentials with random

8.1 POWER SPECTRAL DENSITY FUNCTIONS

8.1.1 Power Spectral Density Function

If $\{X(t)\}$ is a stationary process with autocorrelation function $R_{xx}(\tau)$, then the Fourier transform of $R_{xx}(\tau)$ is called the *power spectral density* (PSD) function of the process $\{X(t)\}$ and is given by

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-i\omega\tau} d\tau \tag{8.1}$$

The frequency content ω is sometimes replaced by $2\pi f$, where f is the frequency variable. In this case, the PSD is a function of f and hence, we have

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{x}(\tau)\, e^{-i2\pi f\tau} d\tau \tag{8.2}$$

Accordingly, if the PSD, $S_{xx}(\omega)$, is known, then the autocorrelation function $R_{xx}(\tau)$ can be obtained as the Fourier inverse transform of $S_{xx}(\omega)$ and is given by

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{+i\tau\omega}\, d\omega \tag{8.3}$$

Or if we let, $\omega = 2\pi f$ then we have

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f) e^{+i2\pi\tau f}\, df \tag{8.4}$$

8.1.2 Cross-power Spectral Density Function

If $\{X(t)\}$ and $\{Y(t)\}$ are two stationary processes with crosscorrelation function $R_{xy}(\tau)$, then the Fourier transform of $R_{xy}(\tau)$ is called the cross-*power spectral density* (cross-PSD) function of the processes $\{X(t)\}$ and $\{Y(t)\}$ and is given by

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-i\omega\tau} d\tau \tag{8.5}$$

With $\omega = 2\pi f$, the cross-PSD becomes a function of f and hence we have

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i2\pi f\tau} d\tau \tag{8.6}$$

Accordingly, if the cross-PSD, $S_{xy}(\omega)$, is known, then the crosscorrelation function $R_{xy}(\tau)$ can be obtained as the Fourier inverse transform of $S_{xy}(\omega)$ and is given by

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)e^{+i\tau\omega}\,d\omega \tag{8.7}$$

Or if we let , $\omega = 2\pi f$ then we have

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(f)\,e^{+i2\pi\tau f}\,df \tag{8.8}$$

8.1.3 Properties of PSD Function

Property 8.1: If $\{X(t)\}$ is a stationary random process with autocorrelation function $R_{xx}(\tau)$, then the value of the PSD function at zero frequency (that is, $\omega = 0$) is equal to the total area under the graph of the autocorrelation function $R_{xx}(\tau)$. That is,

$$S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau)\,d\tau \tag{8.9}$$

Proof. We know that $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\,e^{-i\omega\tau}d\tau$

When $\omega = 0$, we have $S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau)\,d\tau$

Property 8.2: If $\{X(t)\}$ is a stationary random process with autocorrelation function $R_{xx}(\tau)$, then the mean square value (that is, the *second order moment* or the *power*) of the process is equal to the total area under the graph of the PSD function. That is,

$$E\left\{X^2(t)\right\} = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\tau \tag{8.10}$$

Proof. We know that the second order moment of the process $\{X(t)\}$ is given by

$$E\left\{X^2(t)\right\} = R_{xx}(t,t) = R_{xx}(0)$$

$$\Rightarrow \quad R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega$$

8.2 WIENER-KHINCHIN THEOREM

If $\{X_T(t)\}$ is a truncated random process of the original real stationary random process $\{X(t)\}$ such that

$$X_T(t) = \begin{cases} X(t), & |t| \le T \\ 0, & \text{otherwise} \end{cases}$$

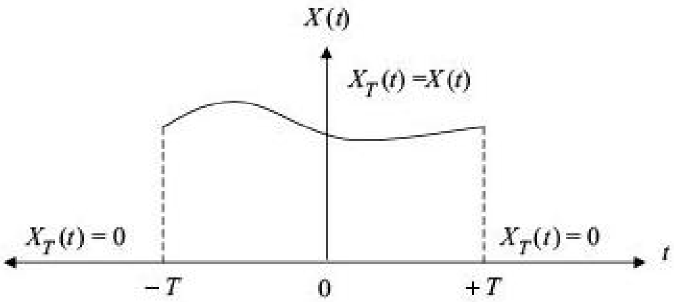
and if $X_T(\omega)$ is the Fourier transform of $\{X_T(t)\}$, then

$$\lim_{T\to\infty}\left\{\frac{1}{2T}E\left\{|X_T(\omega)|^2\right\}\right\} = S_{xx}(\omega)$$

where $S_{xx}(\omega)$ is the power spectral density function of $\{X(t)\}$.

Proof. It is given that $X_T(t) = \begin{cases} X(t), & |t| \le T \\ 0, & \text{otherwise} \end{cases}$

This is shown in the following figure.



Since $X_T(\omega)$ is the Fourier transform of $\{X_T(t)\}$, we have

$$X_T(\omega) = \int_{-\infty}^{\infty} X_T(t)\, e^{-i\omega t} dt = \int_{-T}^{+T} X(t)\, e^{-i\omega t} dt$$

Since the process $\{X(t)\}$ is real and hence $X_T(\omega) = X_T(-\omega)$ as it is even function, we have

$$|X_T(\omega)|^2 = X_T(\omega) X_T(-\omega)$$

$$= \int_{-\infty}^{+\infty} g(\tau)\,d\tau$$

$$\Rightarrow \qquad \lim_{T\to\infty} \frac{1}{2T} E\left\{|X_T(\omega)|^2\right\} = \int_{-\infty}^{+\infty} R(\tau)\,e^{-i\omega\tau}\,d\tau = S_{xx}(\omega)$$

8.3 SYSTEMS WITH STOCHASTIC (RANDOM) INPUTS

Let $\{X(t)\}$ be a random process with sample functions $X(t, \xi_i)$, $i = 1, 2, \ldots\ldots$. If we can obtain the functions $Y(t, \xi_i)$, $i = 1, 2, \ldots\ldots$, which are the sample functions of another process $\{Y(t)\}$, corresponding to $X(t, \xi_i)$, $i = 1, 2, \ldots\ldots$ then we can express $\{Y(t)\}$ as a function of $\{X(t)\}$ as below:

$$\text{Input} \to X(t) = \left\{ \begin{array}{lll} X(t,\xi_1) \to & f\{X(t,\xi_1)\} = & Y(t,\xi_1) \\ X(t,\xi_2) \to & f\{X(t,\xi_2)\} = & Y(t,\xi_2) \\ \vdots & \vdots & \vdots \\ X(t,\xi_i) \to & f\{X(t,\xi_i)\} = & Y(t,\xi_i) \\ \vdots & \vdots & \vdots \end{array} \right\} = Y(t) \to \text{output}$$

Therefore, in simple terms the relation between $\{X(t)\}$ and $\{Y(t)\}$ can be as shown as

$$Y(t) = f\{X(t)\} \tag{8.14}$$

Here f stands for an appropriate functional operator. Now, the process $\{Y(t)\}$ formed so is considered the output of a system whose input is $\{X(t)\}$. It may be noted that the sample functions $X(t, \xi_i)$, $i = 1, 2, \ldots\ldots$ are random in nature and hence the sample functions $Y(t, \xi_i)$, $i = 1, 2, \ldots\ldots$ too are quite random. Such systems are called the *systems with random inputs*. If the relationship given in (8.14) is linear then we have the *systems with linear inputs* otherwise we have *systems with non-linear inputs*. $Y(t) = cX(t)$, where c is constant, is an example for linear case whereas $Y(t) = X^2(t)$ is an example for non-linear case.

If a linear system with input process $\{X(t)\}$ and output process $\{Y(t)\}$ is given then by which we mean that

$$\begin{aligned} Y(t) &= f\{X(t)\} \\ &= f\{a_1X_1(t) + a_2X_2(t) + \ldots\ldots\} \\ &= a_1 f\{X_1(t)\} + a_2 f\{X_2(t)\} + \ldots\ldots \end{aligned}$$

8.3.1 Fundamental Results on Linear Systems

For any linear system $Y(t) = f\{X(t)\}$,

CHAPTER 9

MARKOV PROCESS AND MARKOV CHAIN

9.0 INTRODUCTION

Let us assume that a man starts from one of the k locations, say $L_{0,1}, L_{0,2}, \cdots, L_{0,k}$, at time point t_0 and moves to one of the locations, say $L_{1,1}, L_{1,2}, \cdots, L_{1,k}$, or remains at the same location at time point t_1 and from there he further moves to one of the next locations, say $L_{2,1}, L_{2,2}, \cdots, L_{2,k}$, or remains at the same location at time point t_2 and so on and reaches one of the locations, say $L_{n-1,1}, L_{n-2,2}, \cdots, L_{n-1,k}$, or remains at the same location at time point t_{n-1} from where he moves finally to one of the locations $L_{n,1}, L_{n,2}, \cdots, L_{n,k}$ or remains at the same location at time point t_n. This implies that in the given interval of time, say $(0, t)$, at every point of time $t_0, t_1, \cdots\cdots t_{n-1}, t_n \in (0, t)$ the person has the choice of moving to one of the $k-1$ predefined states (locations) or remain at the same location. This means that at every point of time he has k options (locations) to choose. If we represent $X(t, \xi)$ as the person is in state (location) ξ at time point t, then one such a random move, $X(t, \xi_1)$ of $X(t, \xi)$, is shown in Figure 9.1. Now, if being in one of the states at time point t_n depends only on the state (location) where he was at time point t_{n-1}, then we say that the man's process of making random moves towards a final state is a *Markov process*. Otherwise, it is not a Markov process.

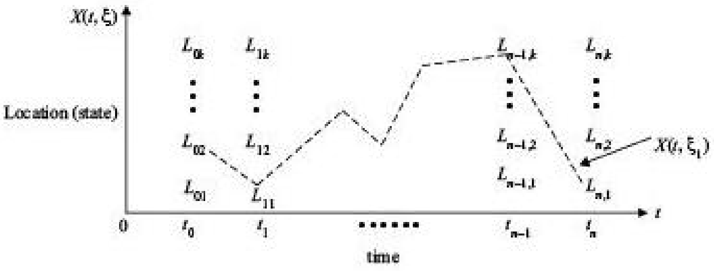


Figure 9.1. Random move of a process at different time points

Let us see another example. In this example, a person involves in a game being played at time points $t_0, t_1, \cdots\cdots t_{n-1}, t_n \in (0, t)$. At every point of time, he gets Rs. 100 if he wins and he gives Rs. 50 if he loses. It is clear that the amount he owns (present state) at any point of time depends on what the amount he had (previous state) in the immediate past time point. Suppose he has Rs. 1000 at a point of time, then the amount he is going to have after the next game will be either Rs. 1100 or Rs. 950. Alternatively, the amount he had in the previous game was Rs. 900 and he won to have Rs. 1000 or he had Rs. 1050 and lost Rs. 50 to have Rs. 1000. In this example also, the process of man having an amount of money after a game was played (present state of the process) at a particular time point depends on how much money the man had after the game (past state of the process) in the immediate past time. Therefore, this process can be termed a Markov process.

Consider the next example where a programmer writes code for an algorithm. Every time point he completes the code and the program is run and its ability is checked to see whether the program works well (state of the process). Next corrections/improvements are done accordingly and the program is run at this time point. Here, if the state that the program works well at a particular time point depends on the state that how the program worked in the previous run in the immediate past time then the process can be termed a Markov process otherwise it cannot be a Markov process.

It may be noted that in all these examples the happening of the state of the process at any point of time is quite random and is observed over a period of time. Therefore, by nature, Markov process is a random process as it is time dependent. Or otherwise, a random process becomes a Markov process under the condition that the state of the process at any point of time depends only on the state of the process at immediate past time.

9.1 CONCEPTS AND DEFINITIONS

9.1.1 Markov Process

A random process $\{X(t)\}$ is said to be a *Markov process*, if given the time points $t_0, t_1, \cdots\cdots t_{n-1}, t_n \in (0, t)$ and the respective states (outcomes) $\xi_0, \xi_1, \cdots\cdots \xi_{n-1}, \xi_n \in \xi$, the state ξ_n of the process at time point t_n depends only on the state ξ_{n-1} of the process at the immediate past time point t_{n-1} but not on the states $\xi_0, \xi_1, \cdots\cdots \xi_{n-2}$ observed respectively at time points $t_0, t_1, \cdots\cdots t_{n-2}$. This implies that

$$P\{X(t_n) = \xi_n / X(t_0) = \xi_0, X(t_1) = \xi_1, \cdots, X(t_{n-2}) = \xi_{n-2}, X(t_{n-1}) = \xi_{n-1}\}$$
$$= P\{X(t_n) = \xi_n / X(t_{n-1}) = \xi_{n-1}\} \tag{9.1}$$

In terms of cumulative probability, we can also write this as

$$P\{X(t_n) \le \xi_n / X(t_0) = \xi_0, X(t_1) = \xi_1, \cdots, X(t_{n-2}) = \xi_{n-2}, X(t_{n-1}) = \xi_{n-1}\}$$
$$= P\{X(t_n) \le \xi_n / X(t_{n-1}) = \xi_{n-1}\} \tag{9.2}$$

It may be noted that a Markov process is indexed by *state space*, ξ , and *time parameter*, t.

9.1.2 Markovian Property

The property that the state of a process at any point of time depends only on the state of the process at immediate past time is called *Markovian property*.

Or otherwise, the property that the future behavior (state) of a random process depends only on the present behavior (state), but not on the past, is called Markovian property.

9.1.3 Markov Chain

A Markov process (or a random process with Markovian property) is said to be a *Markov chain* if the state space ξ is discrete irrespective of whether the time parameter t is discrete or continuous.

In this chapter, we consider only the discrete time points. If the time parameter t is assumed discrete in a Markov chain, it is always represented as *step*, say n, where steps (time points) are denoted by $n = 1, 2, 3, \cdots\cdots\cdots$. Similarly, the discrete states of the state space ξ are represented by small cases a, b, c, i, j, k, etc. Accordingly, (9.1) can be written as

$$P\{X_n = j/X_0 = a, X_1 = b, \cdots, X_{n-2} = c, X_{n-1} = i\} = P\{X_n = j/X_{n-1} = i\} \quad (9.3)$$

Here, $P\{X_n = j/X_{n-1} = i\}$ means the probability that the process that was in state i in $(n-1)^{th}$ step moved to state j in n^{th} step.

9.1.4 Transition Probabilities

The probability that the process that was in state i in $(n-1)^{th}$ step moved to state j in n^{th} step denoted by $P\{X_n = j/X_{n-1} = i\}$, $n = 1, 2, 3, \cdots\cdots\cdots$; $i, j = 1, 2, 3, \cdots\cdots\cdots, k$, is, in fact, the probability of transition occurred in the process in one step. Or otherwise, the *one-step transition probability* is the probability that the state j is reached from the state i in one step.

Here, the step size of 'one' is obtained by $n-(n-1) = 1$. We call $P\{X_n = j/X_{n-1} = i\}$, $n = 1, 2, 3, \cdots\cdots\cdots$; $i, j = 1, 2, 3, \cdots\cdots\cdots, k$, as one-step transition probabilities. In general, given $m \leq n$, if the process that was in state i in m^{th} step moved to state j in n^{th} step, then we say that the transition occurred in $n-m$ number of steps. The related transition probabilities can be given as $P\{X_n = j/X_m = i\}$, $m, n = 1, 2, 3, \cdots\cdots\cdots$; $i, j = 1, 2, 3, \cdots\cdots\cdots, k$. It is customary to denote the transition probabilities of various steps as

One step transition probabilities:

$$P_{ij}^{(1)} = P\{X_n = j/X_{n-1} = i\}, n = 1, 2, 3, \cdots\cdots\cdots; i, j = 1, 2, 3, \cdots\cdots\cdots, k$$

Two-step transition probabilities:

$$P_{ij}^{(2)} = P\{X_n = j/X_{n-2} = i\}, n = 2, 3, \cdots\cdots\cdots; i, j = 1, 2, 3, \cdots\cdots\cdots, k$$

n-step transition probabilities:

$$P_{ij}^{(n)} = P\{X_n = j/X_0 = i\}, \quad n = 1, 2, 3, \cdots\cdots\cdots; \quad i, j = 1, 2, 3, \cdots\cdots\cdots, k$$

Or

$$P_{ij}^{(n)} = P\{X_{m+n} = j/X_m = i\}, \quad m, n = 1, 2, 3, \cdots\cdots\cdots; \quad i, j = 1, 2, 3, \cdots\cdots\cdots, k$$

9.1.5 Homogeneous Markov Chain

A Markov chain is said to be *homogeneous Markov chain* in time if the transition probabilities depend only on the difference of steps but not on the actual steps. For example, if rainfall is observed for one hour duration, then it does not matter it is observed from 10 to 11 a.m or 8 to 9 p.m. This implies that the difference of one hour ($11 - 10 = 1$ or $9 - 8 = 1$) matters but not the actual timings (10 to 11 a.m or 8 to 9 p.m.). Under the homogeneity condition the Markov chain is said to have *stationary transition probabilities* and hence we have

$$P_{ij}^{(1)} = P\{X_n = j/X_{n-1} = i\} = P\{X_m = j/X_{m-1} = i\}, \quad m, n = 1, 2, 3, \cdots\cdots\cdots; \quad i, j = 1, 2, 3, \cdots\cdots\cdots, k$$

$$P_{ij}^{(n)} = P\{X_n = j/X_0 = i\} = P\{X_{m+n} = j/X_m = i\}, \quad m, n = 1, 2, 3, \cdots\cdots\cdots; \quad i, j = 1, 2, 3, \cdots\cdots\cdots, k$$

9.1.6 Transition Probability Matrix (TPM)

Given that the Markov chain is homogeneous, if there are k number of states, then the one-step transition probabilities can be obtained for $i = 1, 2, 3, \cdots\cdots\cdots, k$ and $j = 1, 2, 3, \cdots\cdots\cdots, k$. These probabilities can be further arranged in a matrix form known as *transition probability matrix* (TPM). As a result, the n-step transition probability matrix, denoted by $P^{(n)}$ can be given as

$$P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ k \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & j & \cdots & k \end{array} \\ \begin{bmatrix} P_{11}^{(n)} & P_{12}^{(n)} & \cdots & P_{1j}^{(n)} & \cdots & P_{1k}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} & \cdots & P_{2i}^{(n)} & \cdots & P_{2k}^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{i1}^{(n)} & P_{i2}^{(n)} & \cdots & P_{ij}^{(n)} & \cdots & P_{ik}^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{k1}^{(n)} & P_{k2}^{(n)} & \cdots & P_{kj}^{(n)} & \cdots & P_{kk}^{(n)} \end{bmatrix} \end{array}$$

The TPM must satisfy the following conditions:

(i) $0 \leq P_{ij}^{(n)} \leq 1, \quad \forall\, i, j$

(ii) $\sum_{j=1}^{k} P_{ij}^{(n)} = 1, \quad \forall\, i$ (that is, the probabilities of each row should add to one.)

It may be noted that the state j is said to be not accessible (or not reachable) from state i in n steps, if $P_{ij}^{(n)} = 0$.

By letting $n = 1$, the one-step transition probability matrix can be obtained as

$$P^{(1)} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ k \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & j & \cdots & k \end{array} \\ \begin{bmatrix} P_{11}^{(1)} & P_{12}^{(1)} & \cdots & P_{1j}^{(1)} & \cdots & P_{1k}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} & \cdots & P_{2i}^{(1)} & \cdots & P_{2k}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{i1}^{(1)} & P_{i2}^{(1)} & \cdots & P_{ij}^{(1)} & \cdots & P_{ik}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{k1}^{(1)} & P_{k2}^{(1)} & \cdots & P_{kj}^{(1)} & \cdots & P_{kk}^{(1)} \end{bmatrix} \end{array}$$

Here

(i) $0 \leq P_{ij}^{(1)} \leq 1, \quad \forall\, i, j$

(ii) $\sum\limits_{j=1}^{k} P_{ij}^{(1)} = 1, \quad \forall\, i$

If $P_{ij}^{(1)} = 0$, then state j is not accessible from state i in one step.

Similarly, for $n = 2, 3, \cdots\cdots$ we get two-step transition probability matrix $P^{(2)}$ and three-step transition probability matrix $P^{(3)}$ and so on.

Note:

(i) A matrix is said to be *stochastic matrix* if each row of probabilities adds to one.

(ii) A stochastic matrix say P, is said to be a *regular matrix*, if all the elements of $P^{(n)}$, for some n, are greater than zero.

(iii) A homogeneous Markov chain is said to be regular if its transition probability matrix is regular.

9.2 TRANSITION DIAGRAM

Transition diagram is the representation of the transitions among states of a Markov chain in the form of a network in which each state is considered a node and the transitions among the states are shown by arrows.

9.3 PROBABILITY DISTRIBUTION

The distribution of probability of the Markov chain being in various states at a particular time point (step) is known as the probability distribution of the Markov chain at that point of time (step). For example, given the Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ and if there are k states, $i = 1, 2, 3, \cdots\cdots\cdots, k$, then the probability distribution for $i = 1, 2, 3, \cdots\cdots\cdots, k$ states at n^{th} step can be given as

$$\pi_i^{(n)} = P\{X_n = i\}, \quad i = 1, 2, 3, \cdots\cdots\cdots k$$

such that

$$\sum_{i=1}^{k} \pi_i^{(n)} = 1, \ \forall\, n = 1, 2, \cdots\cdots.$$

Here, $\pi_i^{(n)}$ represents the probability that the Markov chain is in the state i at n^{th} step. This can be represented in the form of a row vector as given below:

$$\pi^{(n)} = [\pi_1^{(n)}, \pi_2^{(n)}, \cdots, \pi_i^{(n)}, \cdots, \pi_k^{(n)}] \tag{9.4}$$

9.3.1 Initial Probability Distribution

In general, before making the first transition, the Markov chain, initially, stays in one of the states. The distribution of probability among the states initially before making the first transition is known an *initial probability distribution*. It is the distribution of probability of the Markov chain being in any of the states at the beginning. The initial probability distribution is given as (Also refer to Figure 9.3)

$$\pi_i^{(0)} = P\{X_0 = i\}, \quad i = 1, 2, 3, \cdots\cdots\cdots k$$

or

$$\pi^{(0)} = [\pi_1^{(0)}, \pi_2^{(0)}, \cdots, \pi_i^{(0)}, \cdots, \pi_k^{(0)}] \tag{9.5}$$

9.3.2 Probability Distribution at n^{th} Step

Given the initial probability distribution $\pi_i^{(0)} = P\{X_0 = i\}, \quad i = 1, 2, 3, \cdots\cdots\cdots k$ and one-step transition probabilities $P_{ij}^{(1)}, \quad i = 1, 2, 3, \cdots\cdots\cdots k, \ j = 1, 2, 3, \cdots\cdots\cdots k$, the probability distribution $\pi_j^{(n)} = P\{X_n = j\}, \quad j = 1, 2, 3, \cdots\cdots\cdots k$ after n^{th} step can be found as follows.

$$\pi_j^{(n)} = P\{X_n = j\} = \sum_{i=1}^{k} P(X_0 = i) P(X_n = j / X_0 = i), \quad j = 1, 2, 3, \cdots\cdots\cdots k$$

$$= \sum_{i=1}^{k} \pi_i^{(0)} P_{ij}^{(n)} \quad j = 1, 2, 3, \cdots\cdots\cdots k$$

where $P_{ij}^{(n)}, \quad i = 1, 2, 3, \cdots\cdots\cdots k, \ j = 1, 2, 3, \cdots\cdots\cdots k$ are the n-step transition probabilities. (Also refer to Figure 9.3.)

9.4 CHAPMAN-KOLMOGOROV THEOREM ON n-STEP TRANSITION PROBABILITY MATRIX

Theorem 9.1: If $P^{(1)}$ is the one-step transition probability matrix of a homogeneous Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$, then the n-step transition probability matrix $P^{(n)}$ can be obtained as the n^{th} power of one-step transition matrix $P^{(1)}$

$$P^{(n)} = \left\{P^{(1)}\right\}^n \tag{9.8}$$

Or otherwise, the $(i, j)^{th}$ element $P_{ij}^{(n)}$ of n-step transition probability matrix $P^{(n)}$ is equal to the $(i, j)^{th}$ element of n^{th} power of one-step transition matrix $P_{ij}^{(1)}$, that is the $(i, j)^{th}$ element of $\left\{P^{(1)}\right\}^n$.

Proof. Consider a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ with k states, $i = 1, 2, 3, \cdots\cdots\cdots, k$. Let us assume that the chain is initially in state i and makes transition to state j in one-step. Then the one-step transition probabilities can be obtained as

$$P_{ij}^{(1)} = P\{X_1 = j/X_0 = i\} \quad \text{for} \quad i = 1, 2, 3, \cdots\cdots\cdots, k; j = 1, 2, 3, \cdots\cdots\cdots, k$$

Similarly, the two-step transition probabilities can be obtained as

$$P_{ij}^{(2)} = P\{X_2 = j/X_0 = i\} \quad \text{for} \quad i = 1, 2, 3, \cdots\cdots\cdots, k; j = 1, 2, 3, \cdots\cdots\cdots, k$$

Here, state j can be reached from state i in two steps, meaning that the chain starts initially from state i and moves to an intermediate state, say k, in the first step and then from k, it moves further to state j in the second step. Therefore, we have

$$\begin{aligned} P_{ij}^{(2)} &= P\{X_2 = j/X_0 = i\} \\ &= P\{X_2 = j/X_1 = k\} P\{X_1 = k/X_0 = i\} \\ &= P\{X_1 = k/X_0 = i\} P\{X_2 = j/X_1 = k\} \\ &= P_{ik}^{(1)} P_{kj}^{(1)} \end{aligned}$$

It may be noted that the intermediate state k can be $1, 2, 3, \cdots\cdots\cdots$ which implies

$$P_{ij}^{(2)} = P_{i1}^{(1)} P_{1j}^{(1)} \text{ or } P_{ij}^{(2)} = P_{i2}^{(1)} P_{2j}^{(1)} \text{ or } P_{ij}^{(2)} = P_{i3}^{(1)} P_{3j}^{(1)} \text{ and so on.}$$

That is,

$$\begin{aligned} P_{ij}^{(2)} &= P_{i1}^{(1)} P_{1j}^{(1)} + P_{i2}^{(1)} P_{2j}^{(1)} + P_{i3}^{(1)} P_{3j}^{(1)} + \cdots\cdots \\ &= \sum_{k=1}^{\infty} P_{ik}^{(1)} P_{kj}^{(1)} \end{aligned}$$

the n-step transition probability matrix $P^{(n)}$, as the $(i, j)^{th}$ element of the n^{th} power of one-step transition matrix $P^{(1)}$, that is the $(i, j)^{th}$ element of $\left\{P^{(1)}\right\}^n$. Hence,

$$P^{(n)} = \left\{P^{(1)}\right\}^n$$

9.4.1 Important Results when One-Step TPM is of Order 2 × 2

Let $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ be a Markov chain with state space $\{1, 2\}$ with the one-step transition probability matrix $p^{(1)} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$, $\quad 0 \leq p, q \leq 1$ then

$$P^{(n)} = \frac{1}{p+q}\left\{\begin{bmatrix} q & p \\ q & p \end{bmatrix} + (1-p-q)^n \begin{bmatrix} p & -p \\ -q & q \end{bmatrix}\right\} \tag{9.9}$$

$$\lim_{n\to\infty} P^{(n)} = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} \tag{9.10}$$

where $P^{(n)}$ is the n-step transition probability matrix.

Proof. Given the matrix $p^{(1)}$, we know by matrix analysis, the characteristic equation of $p^{(1)}$ can be given as

$$\begin{aligned} C(\lambda) &= \left|\lambda I - P^{(1)}\right| = 0 \\ &= \left|\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}\right| = 0 \\ &= \begin{vmatrix} \lambda-(1-p) & -p \\ -q & \lambda-(1-q) \end{vmatrix} = 0 \\ &= (\lambda-1)(\lambda-1+p+q) = 0 \end{aligned}$$

This gives the eigenvalues of $p^{(1)}$ as $\lambda_1 = 1$ and $\lambda_2 = 1-p-q = 0$
Using the spectral decomposition method, $p^{(n)}$ can now be written as

$$P^{(n)} = \left\{P^{(1)}\right\}^n = \lambda_1^n E_1 + \lambda_2^n E_2$$

where $\quad E_1 = \dfrac{1}{\lambda_1-\lambda_2}[P^{(1)} - \lambda_2 I] = \dfrac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix}$

and

$$E_2 = \frac{1}{\lambda_2-\lambda_1}[P^{(1)} - \lambda_1 I] = \frac{1}{p+q}\begin{bmatrix} p & -p \\ -q & q \end{bmatrix}$$

$$\therefore \quad P^{(n)} = (1)^n \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} + (1-p-q)^n \frac{1}{p+q}\begin{bmatrix} p & -p \\ -q & q \end{bmatrix}$$

$$\therefore \quad P^{(n)} = \frac{1}{p+q}\left\{\begin{bmatrix} q & p \\ q & p \end{bmatrix} + (1-p-q)^n \begin{bmatrix} p & -p \\ -q & q \end{bmatrix}\right\}$$

Since $|1-p-q| \leq 1$, we have $\lim_{n\to\infty}(1-p-q)^n = 0$

$$\therefore \quad \lim_{n\to\infty} P^{(n)} = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix}$$

9.5 STEADY-STATE (STATIONARY) PROBABILITY DISTRIBUTION

If a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ is homogeneous and regular, then every sequence of state probabilities, say $\left\{\pi_i^{(n)}\right\}$, $i = 1, 2, \cdots, k$, approaches to a unique fixed probability, say π_i, as the number of steps $n \to \infty$, is called the *steady-state probability* (or *stationary probability*) of the Markov chain. If π represents the row vector of the distribution of such unique probabilities, then we have

$$\begin{aligned} \pi = \lim_{n\to\infty}\left\{\pi^{(n)}\right\} &= \left[\lim_{n\to\infty}\pi_1^{(n)}, \lim_{n\to\infty}\pi_2^{(n)}, \cdots, \lim_{n\to\infty}\pi_i^{(n)}, \cdots, \lim_{n\to\infty}\pi_k^{(n)}\right] \\ &= [\pi_1, \pi_2, \cdots, \pi_i, \cdots, \pi_k] \end{aligned}$$

which is called the *steady-state probability distribution* of the Markov chain. This means that

$$\lim_{n\to\infty}\left\{\pi_i^{(n)}\right\} = \pi_i, \quad i = 1, 2, \cdots, k \tag{9.11}$$

Here, while π_i is called the *limiting probability* of the sequence $\left\{\pi_i^{(n)}\right\}$, $i = 1, 2, \cdots, k$ of state probabilities, π is called the *limiting distribution* of the sequence $\left\{\pi^{(n)}\right\}$, $n = 1, 2, 3, \cdots\cdots$ of distributions.

Note:

The steady-state probabilities π_i, $i = 1, 2, \cdots, k$ of the steady-state probability distribution π can be obtained by solving the equations

$$\pi P^{(1)} = \pi \quad \text{and} \quad \sum_{i=1}^{k}\pi_i = 1 \tag{9.12}$$

The equations in (9.12) can also be written as

$$\Rightarrow [\pi_1, \pi_2, \cdots \pi_i \cdots, \pi_k] \begin{bmatrix} P_{11}^{(1)} & P_{12}^{(1)} & \dots & P_{1j}^{(1)} & \dots & P_{1k}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} & \dots & P_{2i}^{(1)} & \dots & P_{2k}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{i1}^{(1)} & P_{i2}^{(1)} & \dots & P_{ij}^{(1)} & \dots & P_{ik}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{k1}^{(1)} & P_{k2}^{(1)} & \dots & P_{kj}^{(1)} & \dots & P_{kk}^{(1)} \end{bmatrix} = [\pi_1, \pi_2, \cdots \pi_j \cdots, \pi_k] \tag{9.13}$$

$$\pi_1 + \pi_2 + \cdots + \pi_i + \cdots + \pi_k = 1$$

9.6 IRREDUCIBLE MARKOV CHAIN

Given a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$, if every state can be accessible from every other state, then such a Markov chain is said to be *irreducible*. Otherwise it is *reducible*. It is clear that if there are k states, then in case of irreducible Markov chain we have the transition probabilities $P_{ij}^{(n)} > 0$ for some n and for all $i = 1, 2, 3, \cdots\cdots\cdots, k; \ j = 1, 2, 3, \cdots\cdots\cdots, k$.

9.7 CLASSIFICATION OF STATES OF MARKOV CHAIN

9.7.1 Accessible State

Given a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$, with k states, a state j ($j = 1, 2, 3, \cdots\cdots\cdots, k$) is said to be an *accessible state* after n steps ($n = 1, 2, 3, \cdots\cdots\cdots$), if it can be reached from any other state i ($i = 1, 2, 3, \cdots\cdots\cdots, k$). Or otherwise, a state j is said to be an *accessible state* after n steps, if n-step transition probability that it can be reached from any other state i is greater than zero. That is, $P_{ij}^{(n)} > 0, \ j \neq i$.

9.7.2 Communicating States

Given a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$, with k states, states i ($i = 1, 2, 3, \cdots\cdots\cdots, k$) and j ($j = 1, 2, 3, \cdots\cdots\cdots, k$) are said to be *communicating states* at step n if they are accessible from each other. In this case, we have the n-step transition probabilities as $P_{ij}^{(n)} > 0, \ i \neq j$ and $P_{ji}^{(n)} > 0, \ j \neq i$.

9.7.3 Absorbing State

Given a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$, with k states, a state i ($i = 1, 2, 3, \cdots\cdots\cdots, k$) is said to be an *absorbing state* at step n if no other state, say j ($j = 1, 2, 3, \cdots\cdots\cdots, k$), is accessible from it at step n. That is, for an absorbing state i we have $P_{ii}^{(n)} = 1, \ n = 1, 2, 3, \cdots\cdots\cdots$.

9.7.4 Persistent or Recurrent or Return State

Given a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$, with k states, a state i ($i = 1, 2, 3, \cdots\cdots\cdots, k$) is said to be a *persistent* or *recurrent* or *return state*, if the return of the chain to state i having started from the same state i for the *first time* is certain. In this case, the probability that the chain returns to state i for the *first time* having started from the same state i after n steps, that is, after making n transitions, is denoted by $f_{ii}^{(n)}$, $n = 1, 2, 3, \cdots\cdots\cdots$ where

$$f_{ii}^{(m)} = \sum_{\substack{j=1 \\ j \neq i}}^{k} P_{ij}^{(1)} f_{ji}^{(m-1)}, \quad m = 2, 3, \cdots\cdots, \quad i = 1, 2, \cdots\cdots, k$$

Similarly, the probability that the chain goes to state j for the *first time* having started from the state i after n steps, that is after making n transitions, is denoted by $f_{ij}^{(n)}$, $n = 1, 2, 3, \cdots\cdots\cdots$ where

$$f_{ij}^{(m)} = \sum_{\substack{s=1 \\ s \neq j}}^{k} P_{is}^{(1)} f_{sj}^{(m-1)}, \quad m = 2, 3, \cdots\cdots, \quad i = 1, 2, \cdots\cdots, k; \quad j = 1, 2, \cdots\cdots, k,$$

clearly, $f_{ij}^{(1)} = P_{ij}^{(1)}, \quad f_{ij}^{(0)} = 0, \quad \forall\, i, j.$

Therefore, a state i is *persistent* or *recurrent* or *return state* if $\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$ for $i = 1, 2, 3, \cdots\cdots\cdots, k$.

9.7.5 Transient State

Given a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$, with k states, a state i ($i = 1, 2, 3, \cdots\cdots\cdots, k$) is said to be a *transient state* (or non-recurrent state) if $\sum\limits_{n=1}^{\infty} f_{ii}^{(n)} < 1$ for $i = 1, 2, 3, \cdots\cdots\cdots, k$. That is the return of the chain for the first time to state i after n steps having started from the same state i is uncertain.

9.7.6 Mean Time to First Return of a State (Mean Recurrent Time)

Let N_{ii} be the random variable representing the number of steps for a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ to return to state i for the *first time* having started from the same state i. Clearly, N is a discrete random variable whose probability distribution is given below:

$N_{ii} = n$	1	2	3	$\cdots\cdots$
$P\{N_{ii} = n\}$	$f_{ii}^{(1)}$	$f_{ii}^{(2)}$	$f_{ii}^{(3)}$	$\cdots\cdots$

Therefore, having started from a state i, the average number of transitions (steps) made by the chain before returning to the same state i for the *first time*, say $\mu_{ii} = E\{N_{ii}\}$, is known as the *mean recurrent time of state i* and is calculated by

$$\mu_{ii} = E\{N_{ii}\} = \sum_{n=1}^{\infty} n f_{ii}^{(n)} \tag{9.14}$$

9.7.7 Non-null Persistent and Null Persistent States

A state i ($i = 1, 2, 3, \cdots\cdots\cdots, k$) is said to be non-null persistent state, if the mean recurrent time is finite, that is $\mu_{ii} < \infty$.

A state i ($i = 1, 2, 3, \cdots\cdots\cdots, k$) is said to *be null persistent state*, if the mean recurrent time is infinite, that is $\mu_{ii} = \infty$.

9.7.8 Periodicity of a State

Given a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$, with k states, the *greatest common divisor* (GCD) of all n such that $P_{ii}^{(n)} > 0$ is known as the period of the return state i ($i = 1, 2, 3, \cdots\cdots\cdots, k$). Let $d(i)$ be the period of the return state i, then we have

$$d(i) = GCD\left\{n : P_{ii}^{(n)} > 0\right\} \tag{9.15}$$

Here, state i is said to be *periodic* with period $d(i)$ if $d(i) > 1$ and is said to be *aperiodic* if $d(i) = 1$.

9.7.9 Ergodic State

Given a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$, with k states, a state i ($i = 1, 2, 3, \cdots\cdots\cdots, k$) is said to be an *ergodic state* if it is aperiodic and non-null persistent.

Note:

The mean time to the first return for a recurrent state is related to the steady-state probability. In this regard, let us define a sequence of steps $t_1, t_2, t_3, \cdots, t_k, \cdots$, where t_k represents the time between the $(k-1)^{th}$ and k^{th} returns to the state i. That is, if it is assumed that $X_s = i$ for some time constant (step) s which is sufficiently large so that the chain has reached steady state. The chain then returns to state i at steps, $s + t_1$, $s + t_1 + t_2$, $s + t_1 + t_2 + t_3$ and so on. Accordingly, over some period of time (steps), where the chain visits state i exactly n times, the fraction of time the process spends in state i is equal to $n / \sum_{j=1}^{n} t_i$. Now, as $n \to \infty$, (meaning that the chain is ergodic), this fraction becomes the steady-state probability π_i, that the process is in state i. Further, $\sum_{j=1}^{n} t_i$ converges to the mean recurrent time μ_{ii}.

APPENDIX A

SOME IMPORTANT RESULTS RELATED TO RANDOM PROCESSES

A.1 CONTINUITY RELATED TO RANDOM PROCESSES

Result A.1.1:

Cauchy Criterion: A random process $\{X(t)\}$ is said to be continuous in mean square sense if

$$\lim_{\tau \to 0} E\left\{[X(t+\tau)-X(t)]^2\right\}=0$$

(or) In real analysis, a function $f(\tau)$ of some parameter τ converges to a finite value if

$$\lim_{\tau_1-\tau_2 \to 0} \{f(\tau_1)-f(\tau_2)\}=0$$

Result A.1.2:

A random process $\{X(t)\}$ is said to be continuous if its autocorrelation function $R_{xx}(t_1, t_2)$ is continuous.

Proof. Let $t_1 = t$ and $t_2 = t+\tau$

$$\begin{aligned} \text{Consider} \quad E\left\{[X(t+\tau)-X(t)]^2\right\} &= E\left\{X^2(t+\tau)\right\}+E\left\{X^2(t)\right\} \\ &\quad - 2E\{X(t+\tau)X(t)\} \\ &= R_{xx}(t+\tau,t+\tau)+R_{xx}(t,t)-2R_{xx}(t+\tau,t) \end{aligned}$$

If the autocorrelation $R_{xx}(t_1, t_2)$ of the random process $\{X(t)\}$ is continuous, then we have

$$\lim_{\tau \to 0} E\left\{[X(t+\tau)-X(t)]^2\right\} = \lim_{\tau \to 0}\{R_{xx}(t+\tau,t+\tau)+R_{xx}(t,t)-2R_{xx}(t+\tau,t)\}=0$$

APPENDIX B

SERIES AND TRIGONOMETRIC FORMULAS

Important formulas used elsewhere in the text are given below:

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$

2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\cdots$

3. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\cdots$

4. $2(1-x)^{-3} = (1)(2) + (2)(3)x + (3)(4)x^2 + (4)(5)x^3 + \cdots\cdots$

5. $\sin(-A) = -\sin A$, $\cos(-A) = \cos A$

6. $\sin(A \pm n\pi) = (-1)^n \sin A$, $\cos(A \pm n\pi) = (-1)^n \cos A$

7. $\sin(n\pi \pm A) = \pm(-1)^n \sin A$, $\cos(n\pi \pm A) = (-1)^n \cos A$

8. $\sin \pi = 0$, $\cos \pi = -1$

9. $\sin^2 A = \frac{1 - \cos 2A}{2}$, $\cos^2 A = \frac{1 + \cos 2A}{2}$

10. $\sin(a \pm b) = \sin A \cos B \pm \cos A \sin B$

11. $\cos(a \pm b) = \cos A \cos B \mp \sin A \sin B$

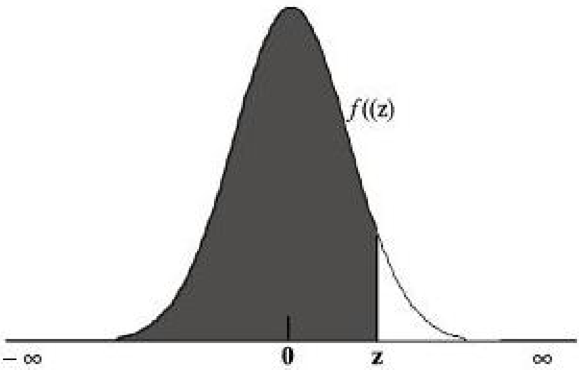
12. $\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

13. $\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$

14. $\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

APPENDIX C

STANDARD NORMAL TABLE



Shaded area shows the cumulative probability $\varphi(z) = P(Z \leq z) = \int_{-\infty}^{z} f(z)dz$. Where $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, $\quad -\infty < z < \infty$ is the standard normal density function of the standard normal variable $Z = \frac{X-\mu}{\sigma}$ whose mean is 0 and standard deviation is 1. Here X is a normal random variable whose mean is μ and standard deviation is σ. The table gives cumulative probabilities for the z-values ranging from -3.99 to $+3.99$.

Therefore, from the table, for a given z one can find the required cumulative probability $\varphi(z) = P(Z \leq z)$ and also for the given cumulative probability $\varphi(z) = P(Z \leq z)$, one can find the value of z. For example, if the cumulative probability is 0.9750, then $z = +1.96$, that is $\varphi(+1.96) = P(Z \leq +1.96) = 0.9750$. Similarly, if the cumulative probability is 0.0250, then $z = +1.96$, that is, $\varphi(-1.96) = P(Z \leq -1.96) = 0.0250$. Otherwise, if $z = +2.58$, then the cumulative probability is 0.9950 and if $z = -2.58$, then the cumulative probability is 0.0050.
Therefore, (area property)

$$
\begin{aligned}
P(-1.96 \leq Z \leq +1.96) &= \varphi(+1.96) - \varphi(-1.96) \\
&= 0.9750 - 0.0250 = 0.95 = 95\% \\
P(-2.58 \leq Z \leq +2.58) &= \varphi(+2.58) - \varphi(-2.58) \\
&= 0.9950 - 0.0050 = 0.99 = 99\%
\end{aligned}
$$

ANSWERS TO EXERCISE PROBLEMS

Chapter 1

1. 0.72
2. 0.929258
3. (i) 0.56, (ii) 0.9642
4. (a) $\frac{1}{72}$, (b) $\frac{1}{18}$,

   (c)

$X = x$	1	2	3	4	5	6	7	8
$F(x)$	$\frac{1}{72}$	$\frac{4}{72}$	$\frac{9}{72}$	$\frac{16}{72}$	$\frac{27}{72}$	$\frac{40}{72}$	$\frac{55}{72}$	$\frac{72}{72} = 1$

5. $\frac{8}{9}$
6. (i) $k = \frac{1}{1 - e^{-a^2}}$, (ii) $\left(\frac{1 - e^{-4}}{1 - e^{-a^2}}\right)^5$
7. 1.5 inch^3
8. Mean $= \frac{\theta}{2}$, Variance$= \frac{\theta^2}{12}$
9. Mean $= 1$, Variance$= 1$
10. 0.04
11. 0.9394

where t_1 and t_2 are two time points in the time period $(0, t)$ such that $0 < t_1 < t_2 < t$ and $\tau = t_2 - t_1$

Note:

(i) If $t_1 = t$ and $t_2 = t + \tau$ for some $\tau > 0$, then,

$$R_{xx}(t_1, t_2) = R_{xx}(t, t+\tau) = E\{X(t)X(t+\tau)\} = R_{xx}(\tau)$$

And in particular, $R_{xx}(0) = E\{X^2(t)\}$, which is called the *average power* of the process.

(ii) Since τ is the distance between t and $t+\tau$, the function $R(\tau)$ can be written in the symmetrical form as follows:

$$R_{xx}(\tau) = E\left\{X\left(t - \frac{\tau}{2}\right)X\left(t + \frac{\tau}{2}\right)\right\} \tag{3.6}$$

From the definitions of SSS and WSS processes, it is clear that SSS implies WSS but the converse is not necessarily true. Therefore, a random process $\{X(t)\}$ is said to be stationary (SSS or WSS) if the autocorrelation function is time invariant. That is, $R_{xx}(t_1, t_2) = R_{xx}(\tau)$. Also, in case of a SSS process, all the statistical properties are independent of time. Due to the time invariance property, in case of SSS process we have $E\{X(t)\}$, $E\{X^2(t)\}$, and hence $V\{X(t)\}$ as constants, free from time (time invariance).

ILLUSTRATIVE EXAMPLE 3.1

As an example of stationarity, let us consider that a person undergoes ECG (Electrocardiogram) test. Under the normal conditions, that is if the person's heart is in good condition, then the ECG recorded, say at time point t_1 appears as shown in Figure 3.1. Since the person is in good health, if ECG is taken at another time point t_2, it would appear in the same fashion. Therefore, we can say the distributional pattern of ECG is same at different points of time.

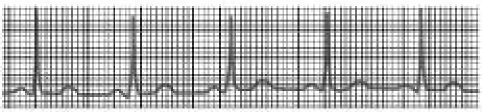


Figure 3.1. ECG recorded at time point t_1

If the ECG recorded at time point t_2 is as shown in Figure 3.2 then it is clear that the distributions of ECG in Figures 3.1 and 3.2 are different and hence we can conclude that the process is not stationary as it has changed over time.

SOLVED PROBLEMS

Problem 1. A random process $\{Y_n\}$ is defined by $Y_n = 3X_n + 1$, where $\{X_n\}$ is a Bernoulli process that assumes 1 with probability $\frac{2}{3}$ and 0 with probability $\frac{1}{3}$. Find the mean and variance of $\{Y_n\}$.

SOLUTION:

Since $\{X_n\}$ is a Bernoulli process, we have

$$X_n = \begin{cases} 1 & \text{with probability} \quad 2/3 \\ 0 & \text{with probability} \quad 1/3 \end{cases}$$

$$\therefore \quad E\{X_n\} = 1(2/3) + 0(1/3) = 2/3$$

$$E\left\{X_n^2\right\} = (1)^2(2/3) + (0)^2(1/3) = 2/3$$

$$\therefore \quad V\{X_n\} = E\left\{X_n^2\right\} - \{E[X_n]\}^2 = 2/3 - (2/3)^2 = (2/3(1 - 2/3) = 2/9$$

Now consider $E\{Y_n\} = E\{3X_n + 1\} = 3E\{X_n\} + 1 = (3)\frac{2}{3} + 1 = 3.$

$$V\{Y_n\} = V\{3X_n + 1\} = 9V\{X_n\} = (9)\left(\frac{2}{9}\right) = 2$$

Problem 2. Let $\{X_n,\ n \geq 1\}$ denote the presence or absence of a pulse at the n^{th} time instance in a digital communication system or digital data processing system. If $x = 1$ represents the presence of a pulse with probability p and $x = 0$ represents the absence of a pulse with probability $q = 1 - p$, then $\{X_n,\ n \geq 1\}$ is a Bernoulli process $\{X_n,\ n \geq 1\}$ with probabilities defined below

$$P\{X_x = x\} = \begin{cases} p & \text{if } x = 1 \\ q = 1 - p & \text{if } x = 0 \end{cases}$$

Show that $\{X_n,\ n \geq 1\}$ is a strict sense stationary process. Or otherwise show that a Bernoulli process $\{X_n,\ n \geq 1\}$ is a strict sense stationary process.

SOLUTION:

In order to prove that $\{X_n,\ n \geq 1\}$ is strict sense stationary, it is enough to show that the probability distributions of $\{X_n,\ n \geq 1\}$ of different orders are same.

Consider the first order probability distribution of X_n as

$\{X_n = x\}$	1	0
$P\{X_n = x\}$	p	q

The random process $\{X(t)\}$ is called *binomial process* if $X(t)$ represents the number of successes, say x, observed by the time t in a sequence of *Bernoulli trials*. A Bernoulli trial can be represented by

$$X(t) = \begin{cases} 0 & \text{if failure is observed at time } t \\ 1 & \text{if success is observed at time } t \end{cases}$$

Then we have

$$P\{X(t) = x\} = \begin{cases} p & \text{if } x \text{ observed at time } t \text{ is success} \\ q = 1 - p & \text{if } x \text{ observed at time } t \text{ is failure} \end{cases}$$

Clearly, $E\{X(t)\}) = p$ and $V\{X(t)\}) = p(1-p) = pq$

Therefore, if x successes are observed out of n trials conducted by the time t, then the probability of getting x successes out of n trials by time t is given by

$$P\{X(t) = x\} = {}^nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \cdots\cdots, n$$

Since the trials are conducted discretely over a period of time, we can also denote this probability as

$$P\{X_k = x\} = {}^nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \cdots\cdots, n \quad \textit{and} \quad k = 1, 2, \cdots\cdots \tag{5.2}$$

for showing the probability of x successes out of n trials conducted at k^{th} step. That is, $X(t),\ t > 0$ is represented by $X_k, \quad k = 1, 2, \cdots\cdots$.

Let us suppose that we observe a sequence of random variables assuming value $+1$ with probability p and value -1 with probability $q = 1 - p$ then a natural example is the sequence of Bernoulli trials, say $X_1, X_2, X_3, \cdots\cdots, X_n, \cdots\cdots$, each with probability of success equal to p (similar to the probability p of getting $+1$) and with probability of failure equal to $q = 1 - p$ (similar to the probability $q = 1 - p$ of getting -1). Here the partial sum, in fact, $S_n = X_1 + X_2 + X_3 + \cdots\cdots + X_n$, $n \geq 0$ with $S_0 = 0$ follows binomial process. This can be thought of as a *random walk* of a particle that takes a unit step up and down randomly with $S_n = X_1 + X_2 + X_3 + \cdots\cdots + X_n$ representing the position after n^{th} step. Refer to Figure 5.1 for one of the realizations of $S_n = X_1 + X_2 + X_3 + \cdots\cdots + X_n$.

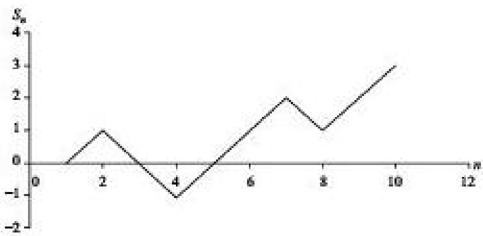


Figure 5.1. One of the realizations of $S_n = X_1 + X_2 + X_3 + \cdots\cdots + X_n$

$$= \frac{2\omega}{T}\left[\tau - \frac{\tau^2}{2T} - \frac{\tau^2}{2\tau_0} + \frac{\tau^3}{3\tau_0 T}\right]_0^{\tau_0}$$

$$= \frac{2\omega}{T}\left(\tau_0 - \frac{\tau_0^2}{2T} - \frac{\tau_0^2}{2\tau_0} + \frac{\tau_0^3}{3\tau_0 T}\right) = \frac{\omega\tau_0}{T}\left(1 - \frac{\tau_0}{3T}\right) \quad (2)$$

It may be noted that as $T \to \infty$ only (2) holds good since in this case $T > \tau_0$.

Therefore, $\lim_{T\to\infty}\{V(\overline{X}_T)\} = \lim_{T\to\infty}\left\{\frac{\omega\tau_0}{T}\left(1 - \frac{\tau_0}{3T}\right)\right\} = 0$

Hence, the process $\{X(t)\}$ is mean ergodic.

Problem 5. A binary transmission process $\{X(t)\}$ has zero mean and autocorrelation function $R(\tau) = 1 - \frac{|\tau|}{T}$. Find the mean and variance of the time average of the process $\{X(t)\}$ over the interval $(0, T)$ and verify whether the process is mean ergodic.

SOLUTION:

We know that the time average $\overline{X}_T$ of the stationary random process $\{X(t)\}$ in the interval $(0, T)$ is given by $\overline{X}_T = \frac{1}{T}\int_0^T X(t)\,dt$.

Therefore, the mean of the time average denoted by $E\{\overline{X}_T\}$ is obtained as

$$E(\overline{X}_T) = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\,dt = E\{X(t)\} = 0$$

In the interval $(0, T)$ the variance of the time average, denoted by $V\{\overline{X}_T\}$, is given by

$$V(\overline{X}_T) = \frac{1}{T}\int_{-T}^{T} C(\tau)\left(1 - \frac{|\tau|}{T}\right)d\tau$$

$$= \frac{1}{T}\int_{-T}^{T} R(\tau)\left(1 - \frac{|\tau|}{T}\right)d\tau$$

Problem 3. If a random process $\{X(t)\}$ is sinusoid with a random frequency $X(t) = \cos(2\pi At)$ where A is random variable uniformly distributed over some interval $(0, a_0)$. Then obtain the mean and variance of the process $\{X(t)\}$.

SOLUTION:

It is given that A is a random variable uniformly distributed over some interval $(0, a_0)$, therefore we have the probability density function of A as

$$f(a) = \begin{cases} \dfrac{1}{a_0}, & 0 < a < a_0 \\ 0, & \text{otherwise} \end{cases}$$

Now, the mean of the random process $\{X(t)\}$ can be obtained as

$$E\{X(t)\} = \int_0^{a_0} \cos(2\pi at) f(a)\,da$$

$$= \frac{1}{a_0}\left[\frac{\sin(2\pi at)}{2\pi t}\right]_0^{a_0} = \frac{\sin(2\pi a_0 t)}{2\pi a_0 t} = \sin c(2a_0 t)$$

which is called the '*cardinal sine function*' or simply '*sinc function*'.

$$\text{Consider} \quad E\left\{X^2(t)\right\} = \int_0^{a_0} \cos^2(2\pi at) f(a)\,da$$

$$= \frac{1}{a_0}\int_0^{a_0} \cos^2(2\pi at)\,da$$

$$= \frac{1}{a_0}\int_0^{a_0}\left\{\frac{1+\cos(4\pi at)}{2}\right\}da$$

$$= \frac{1}{2a_0}\left\{\int_0^{a_0} da + \int_0^{a_0}\cos(4\pi at)\,da\right\}$$

$$= \frac{1}{2a_0}\left\{a_0 + \left[\frac{\sin 4\pi at}{4\pi t}\right]_0^{a_0}\right\}$$

$$= \frac{1}{2a_0}\left\{a_0 + \left[\frac{\sin 4\pi a_0 t}{4\pi t}\right]\right\} = \frac{1}{2}(1 + \sin c(4a_0 t))$$

Hence, the correlation coefficient between $X(t_1)$ and $X(t_2)$, denoted by $\rho_{xx}(\tau)$, can be given as

$$\rho_{xx}(\tau) = \frac{C_{xx}(\tau)}{\sqrt{V\{X(t_1)\}}\sqrt{V\{X(t_2)\}}} = \frac{C_{xx}(\tau)}{\sqrt{C_{xx}(0)}\sqrt{C_{xx}(0)}} = \frac{C_{xx}(\tau)}{C_{xx}(0)} \qquad (4.24)$$

Since $V\{X(t)\} = E\{X(t)\}^2 - \{E[X(t)]\}^2 = R_{xx}(0) - \mu_x^2 = C_{xx}(0)$

Therefore, if we let $t_1 = t$ and $t_2 = t + \tau$ then $C_{xx}(\tau)$ represents the autocovariance and $\rho_{xx}(\tau)$ represents the correlation coefficient between the random variables $X(t)$ and$X(t+\tau)$ of the stationary random process $\{X(t)\}$.

Correlation Time

If the random process $\{X(t)\}$ is an $a-$independent, then covariance $C_{xx}(\tau) = 0$ for $|\tau| > a$. Here, the constant a is called the *correlation time*, say t_c, of the process $\{X(t)\}$. The correlation time for an arbitrary process is also defined as the ratio

$$t_c = \frac{1}{C(0)}\int_{-a}^{a} C(\tau)\,d\tau$$

It may be noted that, in general, the autocovariance function $C_{xx}(\tau) \neq 0$ is for every τ. However, as time difference increases τ, the random variables of process become uncorrelated, hence we have $C(\tau) \to 0$ as $\tau \to \infty$. Also, as discussed in the properties of autocorrelation function, we have $R(\tau) \to \mu^2$ as $\tau \to \infty$.

Theorem 4.1: If $\{X(t)\}$ is a wide sense stationary random process and S is a random variable such that $S = \int_{-T}^{T} X(t)\,dt$, then the variance $\sigma_S^2 = \int_{-T}^{T}\int_{-T}^{T} C(t_1, t_2)dt_1dt_2$

$= \int_{-2T}^{2T} C(\tau)(2T - |\tau|)\,d\tau$ for some $T > o$ where $\tau = t_1 - t_2$ or $\tau = t_1 - t_2$.

Proof. We know that variance of S is given by $V(S) = \sigma_S^2 = E(S^2) - \{E(S)\}^2$

$$\Rightarrow \quad \sigma_S^2 = E\left\{\int_{-T}^{T} X(t)\,dt\right\}^2 - \left\{E\int_{-T}^{T} X(t)\,dt\right\}^2$$

$$= E\left\{\int_{-T}^{T} X(t_1)\,dt_1 \int_{-T}^{T} X(t_2)\,dt_2\right\} - \left\{E\left(\int_{-T}^{T} X(t_1)\,dt\right) E\left(\int_{-T}^{T} X(t_2)\,dt_2\right)\right\}$$

Correlation:

If $\{X(t)\}$ is a random process and $X(t_1)$ and $X(t_2)$ are the two random variables of the process at two time points t_1 and t_2, then the correlation between $X(t_1)$ and $X(t_2)$, denoted by $\rho_{xx}(t_1, t_2)$, is given by

$$\rho_{xx}(t_1, t_2) = \frac{C_{xx}(t_1, t_2)}{\sqrt{V\{X(t_1)\}}\sqrt{V\{X(t_2)\}}} = \frac{R_{xx}(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)}{\{\sigma_x(t_1)\}\ \{\sigma_x(t_2)\}} \tag{2.8}$$

Crosscorrelation and crosscovariance:

If $\{X_1(t)\}$ and $\{X_2(t)\}$ are two random processes observed over a period of time $(0, t)$, then cross correlation between the random variable $X_1(t_1)$ of the process $\{X_1(t)\}$ observed at the time point t_1, and the random variable $X_2(t_2)$ of the process $\{X_2(t)\}$ observed at the time point t_2, denoted by $R_{x_1x_2}(t_1, t_2)$, is given by

$$R_{x_1x_2}(t_1, t_2) = E\{X_1(t_1)X_2(t_2)\} \tag{2.9}$$

And cross-covariance denoted by $C_{x_1x_2}(t_1, t_2)$ is given by

$$\begin{aligned} C_{x_1x_2}(t_1, t_2) &= E\{X_1(t_1)X_2(t_2)\} - E\{X_1(t_1)\}E\{X_2(t_2)\} \\ &= R_{x_1x_2}(t_1, t_2) - \mu_{x_1}(t_1)\ \mu_{x_2}(t_2) \end{aligned} \tag{2.10}$$

In this case, the correlation between the random variables $X_1(t_1)$ and $X_2(t_2)$, denoted as $\rho_{x_1x_2}(t_1, t_2)$, is given by

$$\begin{aligned} \rho_{x_1\,x_2}(t_1, t_2) &= \frac{C_{x_1x_2}(t_1, t_2)}{\sqrt{V\{X_1(t_1)\}}\sqrt{V\{X_2(t_2)\}}} \\ &= \frac{R_{x_1x_2}(t_1, t_2) - \mu_{x_1}(t_1)\mu_{x_2}(t_2)}{\{\sigma_{x_1}(t_1)\ \}\{\sigma_{x_2}(t_2)\}} \end{aligned} \tag{2.11}$$

It may be noted that without loss of generality, and of course for clarity, it is assumed that $t = t_1$ in $\{X_1(t)\}$ and $t = t_2$ in $\{X_2(t)\}$.

Note:

In discrete case, we have random sequence denoted by $\{X_n\}$ instead of $\{X(t)\}$, since the time t is discrete the same is represented in terms of steps, say $n = 0, 1, 2, \cdots\cdots$.

Problem 9. Let $\{X(t)\}$ be a random process with $X(t) = Y\,|\cos(2\pi ft)|$, $t \geq 0$ where ω is a constant and be a rectified cosine signal having a random amplitude Y with exponential probability density function given by

$$f_Y(y) = \begin{cases} \dfrac{1}{10}e^{-y/10}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Then obtain the probability density function of $\{X(t)\}$.

SOLUTION:

Consider the cumulative probability distribution function of $\{X(t)\}$

$$F_{X(t)}(x) = P\{X(t) \leq x\} = P\{Y\,|\cos(2\pi ft)| \leq x\}$$

$$= P\{Y \leq x/\,|\cos(2\pi ft)|\}$$

$$= \int_0^{x/|\cos(2\pi ft)|} f_Y(y)dy$$

$$= \int_0^{x/|\cos(2\pi ft)|} \frac{1}{10}e^{-y/10}dy$$

$$= 1 - e^{-x/10|\cos 2\pi ft|}$$

$$\therefore F_{X(t)}(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x/10|\cos 2\pi ft|}, & x \geq 0 \end{cases}$$

We know that the probability density function of the process $\{X(t)\}$ can be given as

$$f_{X(t)}(x) = F'_{X(t)}(x) = \begin{cases} \dfrac{1}{10\,|\cos 2\pi ft|}e^{-x/10|\cos 2\pi ft|}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Problem 10. Let $\{X(t)\}$ be a random process with $X(t) = A\cos(\omega t + \theta)$, $t \geq 0$ where ω is a constant and A and θ are two independent random variables and θ is uniformly distributed in the interval $(-\pi, \pi)$. Then determine the mean, variance and autocorrelation function of $\{X(t)\}$. Also obtain the covariance of the process $\{X(t)\}$.

seen that each random function of the random process $\{X(t,\ \xi)\}$ is indexed by the time parameter t and the state space ξ. Here, the state space represents the state (outcome) of the random function at time t. The collection of the random functions is also called an *ensemble*.

Further, it may be noted that at each time point t_i, $i = 1, 2, \cdots\cdots, m, \cdots\cdots$ in the interval $(0,\ t)$, we have a random variable denoted by $X(t_i,\ \xi)$ whose realizations are $X(t_i,\ \xi_1),\ X(t_i,\ \xi_2), \ldots, X(t_i,\ \xi_i), \ldots, X(t_i,\ \xi_n)\cdots\cdots$. Therefore, *a random process can also be defined as the collection of random variables* $\{X(t,\ \xi)\}$ *at time points* $t = t_1, t_2, \cdots, t_i, \cdots, t_m, \cdots$ in $(0,\ t)$ *together with a probability rule indexed by time parameter t and the state space* ξ.

Such collection of random variables is uncountably infinite if the time parameter t is continuous or countably infinite if the time parameter t is discrete.

In an ensemble, since the happening of each member function $X(t,\ \xi_i)$, $i = 1, 2, \cdots\cdots, n, \cdots\cdots$, depends on the happening of the corresponding experimental outcome ξ_i according to the known probability rule, the random process is usually denoted by $\{X(t)\}$ or simply $X(t)$. Note that in case of random variable $X(e)$, we denote the same as simply X, omitting e. Apparently, when we denote a random process by $X(t)$, we mean that it is a random process observed in the time interval $(0,\ t)$.

For example, recall the Illustrative Example 2.2 in which the member functions $X(t,\ \xi_1)$ and $X(t,\ \xi_2)$ of the ensemble are given as follows:

$$X(t,\ \xi) = \begin{cases} X(t,\ \xi_1) = -\sin(1+t) & \text{if tail turns up (i.e.,} \ \xi = e_1 = T) \\ X(t, \xi_2) = \sin(1+t) & \text{if head turns up (i.e.,} \ \xi = e_2 = H) \end{cases}$$

Since we know that for a fair coin, probabilities are equal for the happening of a tail or a head, according to the probability law we have

$$P[X(t,\ \xi_1) = -\sin(1+t)] = \frac{1}{2} \quad \text{and} \quad P[X(t,\ \xi_2) = \sin(1+t)] = \frac{1}{2}$$

which can simply be given as

$$P[X(t) = -\sin(1+t)] = \frac{1}{2} \quad \text{and} \quad P[X(t) = \sin(1+t)] = \frac{1}{2}$$

This probability distribution can be represented in tabular form as shown in Table 2.6. However, it may be noted that we have a countably infinite number of random variables $X(t_i,\ \xi)$, $i = 1, 2, \cdots\cdots, m, \cdots\cdots$ at time points in $(0,\ t)$ and when the outcomes of these random variables at these time points are connected a smooth curve of the random function is formed. Refer to Figure 2.3 or the curves of other examples given earlier.

Or with, $\omega = 2\pi f$, we have

$$E\left\{X^2(t)\right\} = R_{xx}(0) = \int_{-\infty}^{\infty} S_{xx}(f)\,df$$

Property 8.3: The PSD function $S_{xx}(\omega)$ of a stationary process $\{X(t)\}$ with autocorrelation function $R_{xx}(\tau)$ is an even function. That is, $S_{xx}(\omega) = S_{xx}(-\omega)$. Also, $S_{xy}(\omega) = S_{yx}(-\omega)$.

Proof. We know that $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\,e^{-i\omega\tau}d\tau$

Now, consider $S_{xx}(-\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\,e^{+i\omega\tau}d\tau$

Let $\tau = -v$

$$\Rightarrow \quad S_{xx}(-\omega) = \int_{-\infty}^{\infty} R_{xx}(-v)\,e^{-i\omega v}dv$$

$$= \int_{-\infty}^{\infty} R_{xx}(v)\,e^{-i\omega v}dv = S_{xx}(\omega) \quad \because \quad R(v) = R(-v)$$

Property 8.4: The PSD function $S_{xx}(\omega)$ and autocorrelation function $R_{xx}(\tau)$ of a stationary process $\{X(t)\}$ form a Fourier cosine transform pair.

Proof. We know that

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\,e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau)\,(\cos\omega\tau - i\sin\omega\tau)d\tau$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau)\,\cos\omega\tau\,d\tau - \int_{-\infty}^{\infty} R_{xx}(\tau) i\sin\omega\tau\,d\tau$$

$$= 2\int_{0}^{\infty} R_{xx}(\tau)\,\cos\omega\tau\,d\tau \qquad \because \int_{-\infty}^{\infty} R_{xx}(\tau) i\sin\omega\tau\,d\tau = 0$$

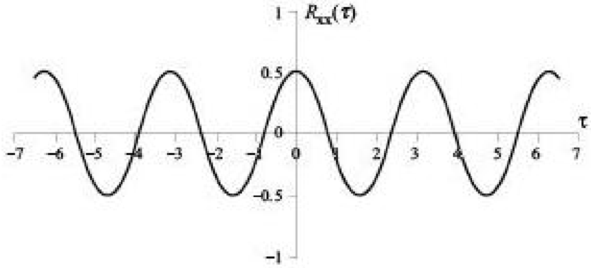


Figure 4.2. Graphical representation of $R_{XX}(\tau) = \frac{1}{2}\cos 2\tau$

The autocorrelation function of a sinusoidal wave with random phase is another sinusoid at the same frequency in the τ – domain. It may be visually verified from Figure 4.2 that autocorrelation function is an even function.

Frequency domain representation of autocorrelation

It may be noted that if the autocorrelation function $R_{XX}(\tau)$ of the random process $\{X(t)\}$ drops (decays) quickly (for example, refer to Figure 4.4), then the samples of the process (signal) are less correlated which, in turn, means that the signal has lot of changes over time (for example, refer to Figure 4.3). Such a signal has high frequency components. If the autocorrelation function $R_{XX}(\tau)$ drops slowly (for example, refer to Figure 4.6), then the signal samples are highly correlated and such a signal has less high frequency components (for example, refer to Figure 4.5). Obviously, the autocorrelation function $R_{XX}(\tau)$ is directly related to the frequency domain representation of the random process. Note that in Figures 4.4 and 4.6, the autocorrelation $R_{XX}(\tau)$ is maximum when $\tau = 0$.

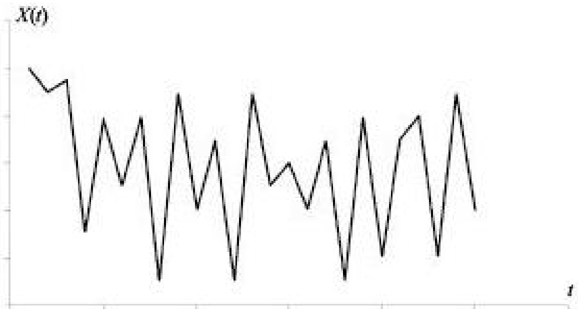


Figure 4.3. Rapidly changing random process

we have

$$E\{Y(t)\} = E\{f[X(t)]\} = f\{E[X(t)]\}$$

where E represents expectation.

$$\Rightarrow \quad \mu_y(t) = f\{\mu_x(t)\}$$

If $h(t)$ is an impulse response function, then we have

$$\mu_y(t) = E\{Y(t)\} = \int_{-\infty}^{\infty} E\{X(t-a)\}\, h(a)da = \mu_x(t)h(t)$$

Similarly, we have

$$R_{xy}(t_1,t_2) = \int_{-\infty}^{\infty} R_{xx}(t_1, t_2 - a)h(a)da$$

$$R_{yy}(t_1,t_2) = \int_{-\infty}^{\infty} R_{xy}(t_1 - a, t_2)h(a)da$$

That is,

$$R_{xx}(t_1,t_2) \underset{h(t_2)}{\rightarrow} R_{xy}(t_1,t_2) \underset{h(t_1)}{\rightarrow} R_{yy}(t_1,t_2)$$

SOLVED PROBLEMS

Problem 1. Given that the random process $\{X(t)\}$ is a wide sense stationary process whose autocorrelation function traps an area of 6.25 square units in the first quadrant. Find the value of the power spectral density function at zero frequency.

SOLUTION:
Let $R_{xx}(\tau)$ be the autocorrelation function of a stationary random process $\{X(t)\}$. It is given that

$$\int_{0}^{\infty} R_{xx}(\tau)\, d\tau = 6.25 \quad \text{Sq. units}$$

We know that the power spectral density function at ω frequency is given by

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-i\omega\tau} d\tau$$

Index

That is, given two values a and b such that $(a < b)$, the probability can be computed as the probability that the random variable X lies between a and b and is denoted as $P(a < X < b)$ or $P(a \le X \le b)$. In this regard, we need a function of the numerical values of the random variable X which could be integrated over the range (a, b) to get the required probability. Such a function is notationally given as $f(x)$ and called probability density function (PDF).

Consider the following probability for a more intuitive interpretation of the density function

$$P(c - \varepsilon/2 \le X \le c + \varepsilon/2) = \int_{c-\varepsilon/2}^{c+\varepsilon/2} f(x)\,dx \cong \varepsilon f(x)$$

where ε is small. This probability is depicted by the shaded area in Figure 1.2.

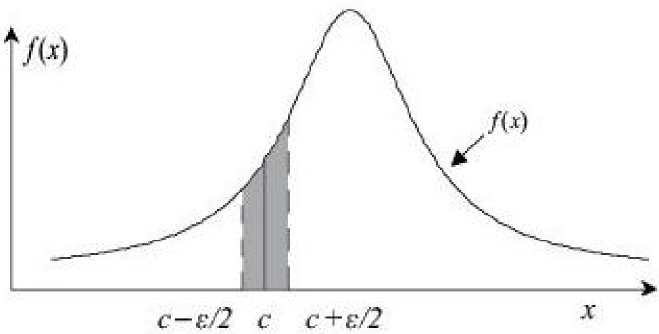


Figure 1.2. Probability density function $f(x)$

Definition

The function $f(x)$, also denoted by $f_X(x)$, of the numerical values of the continuous random variable X is said to be probability density function (PDF) if it satisfies the following properties:

(i) $f(x) \ge 0$ for all $x \in R$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) $P(a \le X \le b) = \int_{a}^{b} f(x)dx$

Properties

(i) If X is a random variable (whether discrete or continuous), and $Y = aX + b$, where a and b are real costants, then

$$\begin{aligned} E(Y) &= E(aX + b) = aE(X) + E(b) \\ &= aE(X) + b \end{aligned} \tag{1.10}$$

It may be noted that $E(b) = b$ implies that the expected value of a constant is the same constant only.

(ii) If X is a random variable and $h(X)$ is a function of X, then

$$\begin{aligned} E[h(X)] &= \sum_{x=-\infty}^{\infty} h(x)\ P(X = x) \quad \text{if } X \text{ is a discrete random variable} \\ E[h(X)] &= \int_{-\infty}^{\infty} h(x)\ f(x)\,dx \quad \text{if } X \text{ is a continuous random variable} \end{aligned} \tag{1.11}$$

(iii) ***Variance:*** If X is a random variable (whether discrete or continuous), then the variance of X, denoted by $V(X)$ or σ_x^2 is given as

$$\begin{aligned} V(X) = \sigma_x^2 &= E\{X - E(X)\}^2 \\ &= E\left\{X^2 - 2XE(X) + [E(X)]^2\right\} \\ &= E(X^2) - 2E[XE(X)] + E\left\{[E(X)]^2\right\} \end{aligned}$$

Since the expected value $E(X)$ is constant, we have

$$E[2XE(X)] = 2E(X)E(X) = 2\{E(X)\}^2$$

Therefore,

$$V(X) = \sigma_x^2 = E(X^2) - \{E(X)\}^2 \tag{1.12}$$

It may be noted that, the variance $V(X)$ is nothing but the average (mean or expectation) of the squared differences of each observation from its own mean value and is always greater than or equal to zero, that is, $V(X) \geq 0$.

If X is a random variable (whether discrete or continuous) and if a sample of n observations is drawn whose mean is $E(X)$, then the variance of X can be defined as

$$V(X) = \frac{1}{n} \sum_{i=1}^{n} [x_i - E(X)]^2$$

Making the exponent a perfect square by adding and subtracting $\sigma^2 t^2$, we have

$$M_X(t) = \frac{e^{\mu t + (\sigma^2 t^2)/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2}\, dz$$

Letting $u = z - \sigma t \quad \Rightarrow \quad du = dz$, then

$$M_X(t) = \frac{e^{\mu t + (\sigma^2 t^2)/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du$$

$$= \frac{e^{\mu t + (\sigma^2 t^2)/2}}{\sqrt{2\pi}} 2\int_{0}^{\infty} e^{-\frac{1}{2}u^2}\, du \quad \text{(by even function property)}$$

Let $y = \frac{1}{2}u^2$ then $dy = udu \Rightarrow du = \frac{dy}{\sqrt{2y}}$, and now we have

$$M_X(t) = \frac{e^{\mu t + (\sigma^2 t^2)/2}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y} y^{\frac{1}{2}-1}\, dy$$

$$= e^{\mu t + (\sigma^2 t^2)/2} \qquad \therefore \int_{0}^{\infty} e^{-y} y^{\frac{1}{2}-1}\, dy = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\therefore \quad \text{Mean} = \mu_1' = E(X) = \left.\frac{dM_X(t)}{dt}\right|_{t=0}$$

$$\Rightarrow \quad \frac{dM_X(t)}{dt} = M_X'(t) = \frac{d}{dt}\left[e^{\mu t + (\sigma^2 t^2)/2}\right]_{t=0}$$

$$= \left[\left(e^{\mu t + (\sigma^2 t^2)/2}\right)\left(\mu + \sigma^2 t\right)\right]_{t=0} = \mu$$

We know that $V(X) = E(X^2) - \{E(X)\}^2 = \mu_2' - (\mu_1')^2$

Consider,

$$\mu'_2 = \left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0}$$

$$= \frac{d}{dt} M_X'(t) = \frac{d}{dt}\left[\left(e^{\mu t + (\sigma^2 t^2)/2}\right)\left(\mu + \sigma^2 t\right)\right]_{t=0}$$

$$= \left[\left(e^{\mu t + (\sigma^2 t^2)/2}\right)\left(\sigma^2\right) + \left(\mu + \sigma^2 t\right)\left(e^{\mu t + (\sigma^2 t^2)/2}\right)\left(\mu + \sigma^2 t\right)\right]_{t=0}$$

Clearly, $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, |\Sigma| = \left(1-\rho_{12}^2\right)\sigma_1^2\sigma_2^2,$

$|\Sigma|^{1/2} = \sqrt{\left(1-\rho_{12}^2\right)}\,\sigma_1\sigma_2$ and

$$\Sigma^{-1} = \frac{1}{\left(1-\rho_{12}^2\right)\sigma_1^2\sigma_2^2}\begin{pmatrix} \sigma_2^2 & -\rho_{12}\sigma_1\sigma_2 \\ -\rho_{12}\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$$

where ρ_{12} is the correlation coefficient between X_1 and X_2. Hence,

$$f(x_1, x_2) = \frac{e^{-\frac{1}{2}(X-\mu)^T\Sigma^{-1}(X-\mu)}}{(2\pi)\,|\Sigma|^{1/2}}$$

$$= \frac{\exp\left\{\left(-\frac{1}{2}\right)\begin{pmatrix} x_1-\mu_1 & x_2-\mu_2 \end{pmatrix}\left\{\frac{1}{\left(1-\rho_{12}^2\right)\sigma_1^2\sigma_2^2}\begin{pmatrix} \sigma_2^2 & -\rho_{12}\sigma_1\sigma_2 \\ -\rho_{12}\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}\right\}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}\right\}}{2\pi\sqrt{\left(1-\rho_{12}^2\right)}\sigma_1\sigma_2}$$

$$= \frac{\exp\left\{\frac{-1}{2\left(1-\rho_{12}^2\right)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}}{2\pi\,\sigma_1\sigma_2\sqrt{\left(1-\rho_{12}^2\right)}} \tag{1.36}$$

If X_1 and X_2 are independent and hence uncorrelated then $\rho_{12} = 0$ and hence we have

$$f(x_1, x_2) = \frac{\exp\left\{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}}{2\pi\,\sigma_1\sigma_2}$$

$$f(x_1, x_2) = \left\{\frac{1}{\sqrt{2\pi}\,\sigma_1}e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}\right\}\left\{\frac{1}{\sqrt{2\pi}\,\sigma_2}e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}\right\},$$
$$-\infty < x_1, x_2 < \infty \tag{1.37}$$

which is the product of the density functions of two independent normal random variables. The graphical representation of the bivariate (two-dimensional) normal probability density function is shown in Figure 1.7.

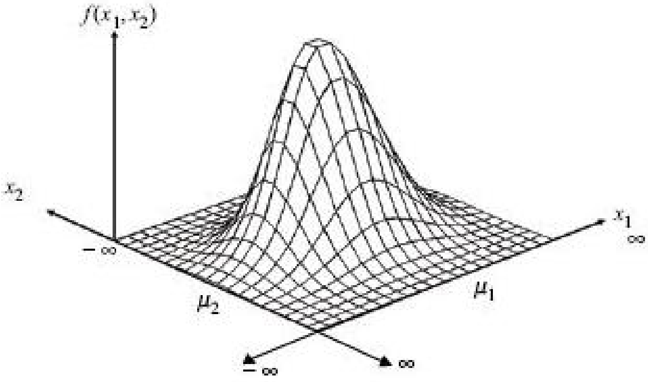


Figure 1.8. Bivariate normal density with parameters μ_1, μ_2, σ_1, σ_2

SOLVED PROBLEMS

Problem 1. A and B are two events such that $P(A \cup B) = 3/4$, $P(A \cap B) = 1/4$ and $P(A^c) = 2/3$. Find $P(A^c/B)$.

SOLUTION:

It is given that

$$P(A \cup B) = \frac{3}{4}, \quad P(A \cap B) = \frac{1}{4} \quad \text{and} \quad P(A^c) = \frac{2}{3}$$

$$\Rightarrow \quad P(A) = 1 - P(A^c) \quad \Rightarrow \quad 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore \quad P(B) = P(A \cup B) - P(A) + P(A \cap B) = \frac{3}{4} - \frac{1}{3} + \frac{1}{4} = \frac{2}{3}$$

We know that

$$P(A^c/B) = 1 - P(A/B)$$

$$= 1 - \frac{P(A \cap B)}{P(B)} = 1 - \frac{1/4}{2/3} = \frac{5}{8}$$

Problem 2. Machine A was put into use 15 years ago and the probability that it may work for the next 10 years is 0.2. Machine B was put into use eight years ago and that it may work for the next 10 years is 0.9. The machines being independent, what is the probability that these two machines can work for the next 10 years?

SOLUTION:

Probability that M/C A will work for next 10 years: $P(A) = 0.2$
Probability that M/C B will work for next 10 years: $P(B) = 0.9$
Since the machines are independent, the probability that M/C A and M/C B will work for the next 10 years can be obtained as

$$P(A \cap B) = P(A) \times P(B) = (0.2)(0.9) = 0.18$$

Problem 3. A, B, and C in order hit a target. The first one to hit the target wins. If A starts, find their respective chances of winning.

SOLUTION:

It is known that

$$P(A \text{ wins}) = P(A) = \frac{1}{2} \quad \text{and } P(A \text{ loses}) = P(\overline{A}) = \frac{1}{2}$$

$$P(B \text{ wins}) = P(B) = \frac{1}{2} \quad \text{and } P(B \text{ loses}) = P(\overline{B}) = \frac{1}{2}$$

$$P(C \text{ wins}) = P(C) = \frac{1}{2} \quad \text{and } P(C \text{ loses}) = P(\overline{C}) = \frac{1}{2}$$

Now we can have the sequence

A, if A wins
$\overline{A}B$, if A loses and B wins
$\overline{A}\,\overline{B}C$, if A and B lose and C wins
$\overline{A}\,\overline{B}\,\overline{C}A$, if A, B, C lose and A wins and so on

$$\therefore \quad P(A \text{ wins}) = P(A) + P(\overline{A}\,\overline{B}\,\overline{C}A) + P(\overline{A}\,\overline{B}\,\overline{C}\,\overline{A}\,\overline{B}\,\overline{C}A) + \cdots\cdots$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^7 + \cdots\cdots$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \cdots\cdots\right]$$

$$= \left(\frac{1}{2}\right)\left(1 - \frac{1}{8}\right)^{-1} = \frac{4}{7}$$

$$P(B \text{ wins}) = P(\overline{A}B) + P(\overline{A}\,\overline{B}\,\overline{C}\,\overline{A}B) + P(\overline{A}\,\overline{B}\,\overline{C}\,\overline{A}\,\overline{B}\,\overline{C}\,\overline{A}B) + \cdots\cdots$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^8 + \cdots\cdots$$

$$= \left(\frac{1}{2}\right)^2 \left[1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \cdots\cdots\right]$$

$$= \left(\frac{1}{2}\right)^2 \left(1 - \frac{1}{8}\right)^{-1} = \frac{2}{7}$$

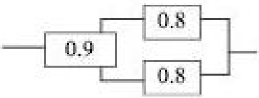
$$P(C \text{ wins}) = P(\overline{A}\,\overline{B}C) + P(\overline{A}\,\overline{B}\,\overline{C}\,\overline{A}\,\overline{B}C) + P(\overline{A}\,\overline{B}\,\overline{C}\,\overline{A}\,\overline{B}\,\overline{C}\,\overline{A}\,\overline{B}C) + \cdots\cdots$$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \cdots\cdots$$

$$= \left(\frac{1}{2}\right)^3 \left[1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \cdots\cdots\right]$$

$$= \left(\frac{1}{2}\right)^3 \left(1 - \frac{1}{8}\right)^{-1} = \frac{1}{7}$$

Problem 4. A system with three components with the probabilities that they work is given below. Calculate the probability that the system will work.



Solution:

The system S has two independent subsystems S_1 and S_2 and S will work if both S_1 and S_2 work. That is,

$$P(S) = P(S_1 \cap S_2) = P(S_1)P(S_2)$$

S_1 contains one component with probability $= 0.90$

That is, $P(S_1) = 0.90$

S_2 contains two components (C_1 and C_2) each with probability 0.80

$$\Rightarrow \quad P(C_1) = 0.80 \quad \text{and} \quad P(C_2) = 0.80$$

$$\Rightarrow \quad P(\overline{C_1}) = 1 - P(C_1) = 1 - 0.80 = 0.2 \quad \text{and}$$

$$P(\overline{C_2}) = 1 - P(C_2) = 1 - 0.80 = 0.2$$

Subsystem S_2 will work if either (C_1 or C_2) works.

That is,

$$P(S_2) = P(C_1 \cup C_2) = 1 - P\overline{(C_1 \cap C_2)}$$

$$= 1 - P(\overline{C_1})P(\overline{C_2}) = 1 - (0.2)(0.2) = 0.96$$

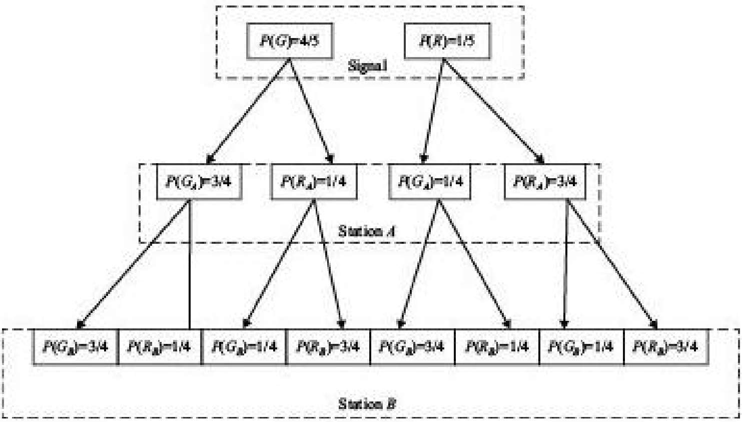
Therefore, probability that the system will work is

$$P(S) = P(S_1 \cap S_2) = P(S_1)P(S_2) = (0.90)(0.96) = 0.864$$

Problem 5. A signal which can be green with probability $4/5$ or red with probability $1/5$ is received by Station A and then transmitted to Station B. The probability of each station receiving the signals correctly is $3/4$. If the signal received at Station B is green, then find (i) the probability that the original signal was green and (ii) the probability that the original signal was red. Also if the signal received at Station B is red, then find (iii) the probability that the original signal was green and (iv) the probability that the original signal was red.

SOLUTION:

Let us present the problem situation using a flow diagram as shown below:



Here, G be the event that original signal is green, then $P(G) = 4/5$

R be the event that original signal is red, then $P(R) = 4/5$

G_A be the event of receiving green signal at Station A

R_A be the event of receiving red signal at Station A

G_B be the event of receiving green signal at Station B

R_B be the event of receiving red signal at Station B

Let E be the event that a signal at Station B is received, then E can be either green, say E_G or red, say E_R. Therefore, we have

$$\begin{aligned}
P(E_G) &= P(GG_AG_B) + P(GR_AG_B) + P(RG_AG_B) + P(RR_AG_B) \\
&= P(G)P(G_A)P(G_B) + P(G)P(R_A)P(G_B) + P(R)P(G_A)P(G_B) \\
&\quad + P(R)P(R_A)P(G_B) \\
&= \left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{5}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) \\
&\quad + \left(\frac{1}{5}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{46}{80}
\end{aligned}$$

(i) Now, if the signal received at Station B is green, then the probability that the original signal was green can be obtained using Bayes formula as follows:

$$\begin{aligned}
P(G/E_G) &= \frac{P(G)P(E_G/G)}{P(E_G)} = \frac{P(G)P(G_A)P(G_B) + P(G)P(R_A)P(G_B)}{P(E_G)} \\
&= \frac{\left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)}{P(E_G)} = \frac{40/80}{46/80} = \frac{20}{23}
\end{aligned}$$

(ii) It may be noted that

$$P(R/E_G) = 1 - P(G/E_G) = 1 - \frac{20}{23} = \frac{3}{23}$$

(iii) Similarly,

$$\begin{aligned}
P(E_R) &= P(GG_AR_B) + P(GR_AR_B) + P(RG_AR_B) + P(RR_AR_B) \\
&= P(G)P(G_A)P(R_B) + P(G)P(R_A)P(R_B) + P(R)P(G_A)P(R_B) \\
&\quad + P(R)P(R_A)P(R_B) \\
&= \left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{5}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) \\
&\quad + \left(\frac{1}{5}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = \frac{34}{80}
\end{aligned}$$

$$P(G/E_R) = \frac{P(G)P(E_R/G)}{P(E_R)} = \frac{P(G)P(G_A)P(R_B) + P(G)P(R_A)P(R_B)}{P(E_R)}$$

$$= \frac{\left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)+\left(\frac{4}{5}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)}{P(E_R)} = \frac{24/80}{34/80} = \frac{24}{34} = \frac{12}{17}$$

(iv) $\therefore \quad P(R/E_R) = 1 - P(G/E_R) = 1 - \frac{12}{17} = \frac{5}{17}$

Problem 6. An assembly consists of three mechanical components. Suppose that the probabilities that the first, second, and third components meet specifications are 0.95, 0.98, and 0.99. Assume that the components are independent. Determine the probability mass function of the number of components in the assembly that meet specifications.

SOLUTION:

$$\begin{aligned}
\text{Probability that first component meets specification} &= 0.95 \\
\text{Probability that second component meets specification} &= 0.98 \\
\text{Probability that third component meets specification} &= 0.99
\end{aligned}$$

Out of three components, let X be the number of components that meets specifications. Therefore, we have $X = 0, 1, 2, 3$.

Now,

$$\begin{aligned}
P(X=0) &= P\text{ (No component meets the specification)} \\
&= (1-0.95)(1-0.98)(1-0.99) \\
&= 10^{-5} = 0.00001 \\
P(X=1) &= P\text{ (One component meets the specification)} \\
&= (0.95)(1-0.98)(1-0.99) + (1-0.95)(0.98) \\
&\quad (1-0.99) + (1-0.95)(1-0.98)(0.99) \\
&= 0.00167 \\
P(X=2) &= P\text{ (Two components meet the specification)} \\
&= (0.95)(0.98)(1-0.99) + (0.95)(1-0.98)(0.99) \\
&\quad + (1-0.95)(0.98)(0.99) \\
&= 0.07663 \\
P(X=3) &= P\text{ (Three components meet the specification)} \\
&= (0.95)(0.98)(0.99) \\
&= 0.92169
\end{aligned}$$

Therefore, the probability distribution is given as

$X=x$	0	1	2	3
$P(X=x)$	0.00001	0.00167	0.07663	0.92469

Problem 7. A car agency sells a certain brand of foreign car either equipped with power steering or not equipped with power steering. The probability distribution of number of cars with power steering sold among the next 4 cars is given as

$$P(X=x)=\left({}^{4}C_{x}\right)/16, \quad x=0,\ 1,\ 2,\ 3,\ 4$$

Find the cumulative distribution function of the random variable X. Using cumulative probability approach verify that $P(X=2)=3/8$.

Solution:

It is given that $P(X=x)=\dfrac{{}^{n}C_{x}}{16}, \quad x=0,\ 1,\ 2,\ 3,\ 4$

Probability distribution is

$$P(X=0)=\frac{{}^{4}C_{0}}{16}=\frac{1}{16},\ P(X=1)=\frac{{}^{4}C_{1}}{16}=\frac{4}{16},\ P(X=2)=\frac{{}^{4}C_{2}}{16}=\frac{6}{16},$$

$$P(X=3)=\frac{{}^{4}C_{3}}{16}=\frac{4}{16},\quad P(X=4)=\frac{{}^{4}C_{4}}{16}=\frac{1}{16}$$

Cumulative distribution function:

x	$F(x)=P(X\leq x)$
$x<0$	0
$0\leq x<1$ $(x=0)$	$0+\frac{1}{16}=\frac{1}{16}$
$1\leq x<2$ $(x=1)$	$\frac{1}{16}+\frac{4}{16}=\frac{5}{16}$
$2\leq x<3$ $(x=2)$	$\frac{5}{16}+\frac{6}{16}=\frac{11}{16}$
$3\leq x<4$ $(x=3)$	$\frac{11}{16}+\frac{4}{16}=\frac{15}{16}$
$x\geq 4$ $(x=4)$	$\frac{15}{16}+\frac{1}{16}=\frac{16}{16}=1$

$$\text{Now, } P(X=2) = P(X \le 2) - P(X \le 1)$$

$$= \frac{11}{16} - \frac{5}{16} = \frac{3}{8} \quad \text{(Verified)}$$

Problem 8. Let X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 mm. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function given below:

$$f(x) = \begin{cases} 20e^{-20(x-12.5)}, & x \ge 12.5 \\ 0, & \text{Otherwise} \end{cases}$$

If a part with a diameter larger than 12.6 mm is scrapped, (i) what proportion of parts is scrapped? and (ii) what proportion of parts is not scrapped?

SOLUTION:

(i) A part is scrapped if $X \ge 12.6$ then

$$P(X \ge 12.6) = \int_{12.6}^{\infty} f(x)dx$$

$$\Rightarrow \quad \int_{12.6}^{\infty} 20e^{-20(x-12.5)}dx = 20\left|\frac{-e^{-20(x-12.5)}}{-20}\right|_{12.6}^{\infty} = 0.135$$

It may be noted that we can obtain this probability value using the relationship

$$P(X \ge 12.6) = 1 - P(12.5 \le X \le 12.6)$$

(ii) A part is not scrapped if $X < 12.6$

$$P(X < 12.6) = 1 - P(X \ge 12.6) = 1 - 0.135 = 0.865$$

Problem 9. If X is a random variable, then find k so that the function $f(x) = \begin{cases} 0, & \textit{for } x \le 0 \\ kxe^{-4x^2} & \textit{for } x > 0 \end{cases}$ can serve as the probability density function of X.

SOLUTION:

It is given that

$$f(x) = kxe^{-4x^2}, \quad 0 \le x \le \infty$$

We know that

$$\int_{-\infty}^{\infty} f(x)dx = 1 \quad \Rightarrow \quad \int_{0}^{\infty} kxe^{-4x^2}dx = 1$$

Let $4x^2 = y \quad \Rightarrow \quad 8x\,dx = dy \quad \Rightarrow \quad xdx = \frac{1}{8}dy$

$$\Rightarrow \quad \frac{k}{8}\int_{0}^{\infty} e^{-y}dy = 1 \quad \Rightarrow \quad \frac{k}{8}\left[\frac{e^{-y}}{-1}\right]_0^{\infty} = 1$$

$$\Rightarrow \quad \frac{k}{8}(1) = 1 \quad \Rightarrow \quad k = 8$$

Problem 10. With equal probability, the observations 5, 10, 8, 2 and 7, show the number of defective units found during five inspections in a laboratory. Find (a) the first four central moments and (b) the moments about origin (raw moments).

SOLUTION:

(a) *Central moments:* In order to find the first four central moments, we first obtain the computations for $\mu_r = E[X - E(X)]^r, \quad r = 1, 2, 3, 4$ as given in the following table with

$$E(X) = \frac{1}{5}\sum_{i=1}^{5} x_i = \frac{32}{5} = 6.4$$

	X	$X-6.4$	$(X-6.4)^2$	$(X-6.4)^3$	$(X-6.4)^4$
	5	−1.4	1.96	−2.744	3.8416
	10	3.6	12.96	46.656	167.9616
	8	1.6	2.56	4.096	6.5536
	2	−4.4	19.36	−85.184	374.8096
	7	0.6	0.36	0.216	0.1296
Total	**32**	**0**	**37.2**	**−36.96**	**553.296**

We know that the central moments can be obtained as:

$$\mu_r = E[X - E(X)]^r, \quad r = 1, 2, 3, 4,$$

that is, $\mu_1 = E[X - E(X)]^1 = 0, \quad \mu_2 = E[X - E(X)]^2 = \frac{37.2}{5} = 7.44$

$$\mu_3 = E[X - E(X)]^3 = \frac{-36.96}{5} = -7.392,$$

$$\mu_4 = E[X - E(X)]^4 = \frac{553.296}{5} = 110.66$$

(b) *Raw moments:* In order to obtain the raw moments we consider the following table:

	X	X^2	X^3	X^4
	5	25	125	625
	10	100	1000	10000
	8	64	512	4096
	2	4	8	16
	7	49	343	2401
Total	**32**	**242**	**1988**	**17138**

We know that the raw moments can be obtained as:

$$\mu_r = E(X^r), \quad r = 1, 2, 3, 4, \quad \text{that is}$$

$$\mu_1' = E(X) = \frac{1}{5}\sum_{i=1}^{5} x_i = \frac{1}{5}(32) = 6.4$$

$$\mu_2' = E(X^2) = \frac{1}{5}\sum_{i=1}^{5} x_i^2 = \frac{1}{5}(242) = 48.4$$

$$\mu_3' = E(X^3) = \frac{1}{5}\sum_{i=1}^{5} x_i^3 = \frac{1}{5}(1988) = 397.6$$

$$\mu_4' = E(X^4) = \frac{1}{5}\sum_{i=1}^{5} x_i^4 = \frac{1}{5}(17138) = 3427.6$$

Problem 11. A man draws 3 balls from an urn containing 5 white and 7 black balls. He gets Rs. 10 for each white ball and Rs. 5 for black ball. Find his expectation.

SOLUTION:

Out of three balls drawn, the following combinations are possible:

(i) 3 white balls, $(3W)$
(ii) 2 white balls and 1 black ball, $(2W, 1B)$
(iii) 1 white ball and 2 black balls $(1W, 2B)$ and
(iv) 3 black balls $(3B)$

Let X be the amount from each draw, then we have

Balls drawn	**Amount (in Rs) from each draw (X)**
$3W$	$3 \times 10 = 30$
$2W, 1B$	$2 \times 10 + 1 \times 5 = 25$
$1W, 2B$	$1 \times 10 + 2 \times 5 = 20$
$3B$	$3 \times 5 = 15$

Therefore, possible values of X are: 15, 20, 25 and 30

Now,

$$P(X = 15) = P(3B) = \frac{({}^5C_0)\,({}^7C_3)}{{}^{12}C_3} = \frac{7}{44}$$

$$P(X = 20) = P(1W, 2B) = \frac{({}^5C_1)\,({}^7C_2)}{{}^{12}C_3} = \frac{21}{44}$$

$$P(X = 25) = P(2W, 1B) = \frac{({}^5C_2)\,({}^7C_1)}{{}^{12}C_3} = \frac{14}{44}$$

$$P(X = 30) = P(3W) = \frac{({}^5C_3)\,({}^7C_0)}{{}^{12}C_3} = \frac{2}{44}$$

Therefore, the probability distribution is

$X = x$	15	20	25	30
$P(X = x)$	$\frac{7}{44}$	$\frac{21}{44}$	$\frac{14}{44}$	$\frac{2}{44}$

$$\therefore \quad E(X) = \sum_x xP(X = x) = \sum_{x=15}^{30} xP(X = x)$$

$$= (15)\left(\frac{7}{44}\right) + (20)\left(\frac{21}{44}\right) + (25)\left(\frac{14}{44}\right) + (30)\left(\frac{2}{44}\right)$$

$$= \text{Rs. } 21.25$$

Problem 12. Let X be a random variable with probability density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find (i) mean (ii) variance and (iii) standard deviation of X. Also obtain the expected value and variance of $g(X) = 4X + 3$.

SOLUTION:

(i) Since X is a continuous random variable, by definition, we know that

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-1}^{2} x\left(\frac{x^2}{3}\right) dx = \left|\frac{x^4}{12}\right|_{-1}^{2} = \frac{16}{12} - \frac{1}{12} = \frac{15}{12}$$

(ii) For finding variance of X, by definition, we know that

$$V(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$$

By definition we have

$$E\left(X^2\right) = E[g(X)] = \int_{-\infty}^{\infty} g(x)\ f(x)\, dx$$

$$= \int_{-1}^{2} x^2\left(\frac{x^2}{3}\right) dx = \left|\frac{x^5}{15}\right|_{-1}^{2} = \frac{32}{15} - \frac{-1}{15} = \frac{33}{15}$$

Therefore,

$$V(X) = \frac{33}{15} - \left(\frac{15}{12}\right)^2 = \frac{33}{15} - \frac{225}{144} = 0.6375$$

(iii) For finding standard deviation, by definition, we know that

$$SD(X) = \sqrt{V(X)} = \sqrt{0.6375} = 0.7984$$

Now we find the mean and variance of $g(X) = 4X + 3$ as follows:

$$E[g(X)] = E(4X + 3) = 4E(X) + 3 = 4(15/12) + 3 = 8$$

$$V[g(X)] = V(4X + 3) = 16V(X) = 16(0.6375) = 10.2$$

Problem 13. The fraction X of male runners and the fraction Y of female runners who compete in marathon races is described by the joint density function

$$f(x,y)=\begin{cases} 8xy, & 0<x<1, \quad 0<y<x \\ 0, & \text{otherwise} \end{cases}$$

Find the covariance of X and Y.

SOLUTION:

In order to find the covariance, first we have to find the marginal probability density functions for X and Y as follows. By definition we know that

$$f(x)=\int_0^x 8xy\,dy = 8x\left[\frac{y^2}{2}\right]_0^x = 4x^3, \quad 0<x<1$$

Similarly,

$$f(y)=\int_y^1 8xy\,dx = 8y\left[\frac{x^2}{2}\right]_y^1 = 4y(1-y^2), \quad 0<y<1$$

Now from the marginal density functions given above, we can compute the expected values of X and Y as follows:

$$E(X)=\int_0^1 xf(x)dx=\int_0^1 4x^4dx = 4\left[\frac{x^5}{5}\right]_0^1=\frac{4}{5}$$

$$E(Y)=\int_0^1 yf(y)dy=\int_0^1 4y^2(1-y^2)dy = 4\left[\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1=\frac{8}{15}$$

Also using joint density function of X and Y, we can find $E(XY)$ as follows:

$$E(XY)=\int_0^1\int_0^x xyf(x,y)\,dydx$$

$$=\int_0^1\int_0^x xy(8xy)\,dydx=\int_0^1 8x^2\left[\int_0^x y^2dy\right]dx$$

$$=\int_0^1 8x^2\left[\frac{y^3}{3}\right]_0^x dx=\frac{8}{3}\int_0^1 x^5dx$$

$$= \frac{8}{3}\left[\frac{x^6}{6}\right]_0^1 = \frac{4}{9}$$

Then, the covariance of X and Y can be obtained using the definition

$$\text{Cov}(X, Y) = \sigma_{xy} = E(XY) - E(X)E(Y) = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}$$

Problem 14. The variables X and Y have the joint probability function

$$f(x, y) = \begin{cases} \frac{1}{3}(x+y), & 0 < x < 1, \quad 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the correlation between X and Y.

SOLUTION:

The marginal probability density functions of X and Y are given by

$$f(x) = \int_0^2 f(x, y)dy = \int_0^2 \frac{1}{3}(x+y)dy = \frac{2}{3}(1+x), \quad 0 < x < 1$$

$$f(y) = \int_0^1 f(x, y)dx = \int_0^1 \frac{1}{3}(x+y)dx = \frac{1}{3}\left(\frac{1}{2}+y\right), \quad 0 < y < 2$$

The correlation between X and Y is given as

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - [E(X)]^2}\sqrt{E(Y^2) - [E(Y)]^2}}$$

Consider

$$E(X) = \int_0^1 xf(x)dx = \int_0^1 x\left(\frac{2}{3}(1+x)\right)dx = \frac{5}{9}$$

$$E(X^2) = \int_0^1 x^2 f(x)dx = \int_0^1 x^2\left(\frac{2}{3}(1+x)\right)dx = \frac{7}{18}$$

$$E(Y) = \int_0^2 yf(y)dy = \int_0^2 y\left(\frac{1}{3}\left(\frac{1}{2}+y\right)\right)dy = \frac{11}{9}$$

$$E(Y^2) = \int_0^2 y^2 f(y)dy = \int_0^2 y^2\left(\frac{1}{3}\left(\frac{1}{2}+y\right)\right)dy = \frac{16}{9}$$

$$E(XY) = \int_0^1\int_0^2 xyf(x,y)dydx = \int_0^1\int_0^2 xy\left(\frac{1}{3}(x+y)\right)dy\,dx = \frac{2}{3}$$

$$\therefore \quad \rho_{XY} = \frac{\frac{2}{3}-\left(\frac{5}{9}\right)\left(\frac{11}{9}\right)}{\sqrt{\frac{7}{18}-\left(\frac{5}{9}\right)^2}\sqrt{\frac{16}{9}-\left(\frac{11}{9}\right)^2}} = \frac{-\frac{1}{81}}{\sqrt{\frac{13}{162}}\sqrt{\frac{23}{81}}} = -\sqrt{\frac{2}{299}}$$

Problem 15. Suppose that 10 cones are selected for weight test. From the past records 2 out of the 10 cones on the lot are expected to be below standards for weight, what is the probability that at least 2 cones will be found not meeting weight standards?

Solution:

It is given that the probability that the cones are below standards, say

$$p = \frac{2}{10} = 0.2$$
$$q = 1 - p = 0.8$$

Let X be the number of cones not meeting standards, then the probability that out of 10, at least two cones will not meet weight standards

$$P(X \geq 2) = \sum_{x=2}^{10} {}^{10}C_x p^x q^{10-x} = \sum_{x=2}^{10} {}^{10}C_x (0.2)^x (0.8)^{10-x}$$
$$= 1 - \sum_{x=0}^{1} {}^{10}C_x (0.2)^x (0.8)^{10-x}$$
$$= 1 - \left[{}^{10}C_0 (0.2)^0 (0.8)^{10} + {}^{10}C_1 (0.2)^1 (0.8)^{10-1}\right]$$
$$= 1 - [0.10737 + 0.26844]$$
$$= 0.6242$$

Problem 16. A manufacturer of electric bulbs knows that 5% of his products are defective. If he sells bulbs in boxes of 100 and guarantees that no more than 10 bulbs will be defective, what is the probability that a box will fail to meet the guaranteed quality?

SOLUTION:

It is given that the probability that a bulb is defective, say $p = 0.05$

Number of bulbs in one box, say $n = 100$

Since n is large and $p \leq 0.05$, we can use Poisson approximation with

$$\lambda = np = (100)(0.05) = 5$$

Therefore, a box will fail to meet the guaranteed quality if the number of defective bulbs, say X, exceeds 10. Then the required probability is

$$\begin{aligned}
P(X \geq 11) &= 1 - P(X \leq 10) \\
&= 1 - \sum_{x=0}^{10} \frac{e^{-\lambda}\lambda^x}{x!} = 1 - \sum_{x=0}^{\infty} \frac{e^{-5}5^x}{x!} \\
&= 1 - \left\{ e^{-5}\left(1 + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!} + \frac{5^5}{5!} + \frac{5^6}{6!} + \frac{5^7}{7!} + \frac{5^8}{8!} + \frac{5^9}{9!} + \frac{5^{10}}{10!}\right)\right\} \\
&= 1 - \left\{ e^{-5}\begin{pmatrix} 1 + 5 + 12.5 + 20.83 + 26.04 + 26.04 + 21.70 \\ +15.5 + 9.68 + 5.38 + 2.69 \end{pmatrix}\right\} \\
&= 1 - e^{-5}(146.36) = 0.0137
\end{aligned}$$

Problem 17. In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is 0.005 and accidents are independent of each other. (i) What is the probability that in any given period of 400 days there will be an accident on one day? (ii) What is the probability that there are at most three days with an accident?

SOLUTION:

Let the probability of an accident on any given day be $p = 0.005$

It is given that the number of days is $n = 400$

Since n is large and p is small, we can approximate this to a Poisson distribution as mean $\lambda = np = (400)(0.005) = 2$

Now, if we let X as the random variable that represents number of accidents, then X follows a Poisson distribution with mean $\lambda = 2$. Therefore,

$$P(X = x) = \frac{e^{-2}2^x}{x!}, \quad x = 0,\ 1,\ 2, \ldots\ldots$$

(i) Now the probability that there is one accident on a day is given by

$$P(X=1)=\frac{e^{-2}2^{1}}{1!}\quad=0.271$$

(ii) The probability that there are at most three days with an accident is given by

$$P(X\le 3)=\sum_{x=0}^{3}\frac{e^{-2}2^{x}}{x!}$$

$$=P(X=0)+P(X=1)+P(X=2)+P(X=3)$$

$$=\frac{e^{-2}2^{0}}{0!}+\frac{e^{-2}2^{1}}{1!}+\frac{e^{-2}2^{2}}{2!}+\frac{e^{-2}2^{3}}{3!}\quad=0.857$$

Problem 18. The process of drilling holes in printed circuit boards produces diameters with standard deviation 0.01 mm. How many diameters must be measured so that the probability is at least 8/9 that the average of the measured diameters is within 0.005 mm of the process mean diameter?

SOLUTION:

It is given that mean $E(X)=\mu \Rightarrow E(\bar{x})=\mu$

And standard deviation $\sigma=0.01$ mm

Then variance $V(X)=\sigma^2 \Rightarrow V(\bar{x})=\frac{\sigma^2}{n}=\frac{(0.01)^2}{n}$

$$\Rightarrow \frac{\sigma}{\sqrt{n}}=\frac{0.01}{\sqrt{n}}$$

We need to find the sample size n such that $P\left[\left|\bar{X}-\mu\right|\le 0.005\right]\ge\frac{8}{9}$

By Chebyshev's theorem, we know that

$$\left[P\left(\left|\bar{X}-\mu\right|\right)\le k\frac{\sigma}{\sqrt{n}}\right]\ge 1-\frac{1}{k^2}$$

$$\Rightarrow\quad \left[P\left(\left|\bar{X}-\mu\right|\right)\le k\left(\frac{0.01}{\sqrt{n}}\right)\right]\ge 1-\frac{1}{k^2}$$

Let $1-\frac{1}{k^2}=\frac{8}{9}\quad\Rightarrow\quad\varepsilon=3$

$$\therefore\quad\left[P\left(\left|\bar{X}-\mu\right|\right)\le 3\left(\frac{0.01}{\sqrt{n}}\right)\right]\ge\frac{8}{9}$$

Letting $3\left(\frac{0.01}{\sqrt{n}}\right)=0.005\quad\Rightarrow\quad n=36$

Therefore, 36 diameters must be measured that will ensure that the average of the measured diameters is within 0.005 mm of the process mean diameter with probability at least 8/9.

Problem 19. The thickness of photo resist applied to wafers in semiconductor manufacturing at a particular location on the wafer is uniformly distributed between 0.205 and 0.215 micrometers. What thickness is exceeded by 10% of the wafers?

SOLUTION:

Let X be the thickness of the photo resist. Then X is uniform in the interval (0.205, 0.215)

$$f(x) = \frac{1}{b-a} = \frac{1}{0.215-0.205}, \quad 0.205 < x < 0.215$$

$$= \frac{1}{0.01}, \quad 0.205 < x < 0.215$$

Let a be the thickness such that 10% of the wafers exceed this thickness

That is,

$$P(X \geq a) = 0.10$$

$$\Rightarrow \quad \int_a^{0.215} f(x)dx = \int_a^{0.215} \frac{1}{0.01} dx = 0.10$$

$$\Rightarrow \quad \frac{0.215-a}{0.01} = 0.10$$

$$\Rightarrow \quad a = 0.214$$

Problem 20. Let X be a normal variable with mean μ and standard deviation σ. If Z is the standard normal variable such that $Z = -0.8$ when $X = 26$ and $Z = 2$ when $X = 40$, then find μ and σ. Also find $P(X > 45)$ and $P(|X-30|) > 5)$.

SOLUTION:

Given a normal random variable X with mean μ and standard deviation σ the standard normal variate is given by $Z = \dfrac{X-\mu}{\sigma}$

It is given that $\quad Z = \dfrac{26-\mu}{\sigma} = -0.8 \Rightarrow -0.8\sigma + \mu = 26$

and $$Z = \frac{40-\mu}{\sigma} = 2 \Rightarrow 2\sigma + \mu = 40$$

Solving the above two equations, we get $\mu = 30$ and $\sigma = 5$

Now, consider

$$\begin{aligned} P(X > 45) &= P\left(\frac{X-30}{5} > \frac{45-30}{5}\right) \\ &= P(Z > 3) \\ &= 0.00135 \quad \text{(Refer Appendix C)} \end{aligned}$$

Similarly,

$$\begin{aligned} P(|X-30| > 5) &= 1 - P(|X-30| \le 5) \\ &= 1 - P(-5 \le X - 30 \le 5) \\ &= 1 - P(25 \le X \le 35) \\ &= 1 - P\left(\frac{25-30}{5} \le \frac{X-30}{5} \le \frac{35-30}{5}\right) \\ &= 1 - P(-1 \le Z \le 1) \\ &= 1 - [P(Z \le 1) - P(Z \le -1)] \\ &= 1 - [\varphi(1) - \varphi(-1)] \\ &= 1 - (0.8413 - 0.1581) = 0.3168 \quad \text{(Refer Appendix C)} \end{aligned}$$

Problem 21. If X is uniform in the interval $(0, 1)$ then find the probability density function of the random variable $Y = \sqrt{X}$.

Solution:

Since X is uniform in the interval $(0, 1)$, we have

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Now, $y = g(x) = \sqrt{x} \Rightarrow x = w(y) = y^2$

$$J = w'(y) = \frac{dx}{dy} = 2y$$

We know that the probability density function of Y say $h(y)$ can be given as

$$h(y) = f[w(y)]\,|J| = f(y^2)(2y) = 2y$$

When $\quad 0 < x < 1, \quad 0 < y^2 < 1 \quad \Rightarrow \quad 0 < y < 1$

$$\Rightarrow \quad h(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Problem 22. The joint probability density function of random variables X and Y is given by

$$f(x, y) = \begin{cases} e^{-(x+y)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Then find whether X and Y are independent. Also find the probability density function of the random variable $U = (X+Y)/2$.

SOLUTION:

Given
$$f(x, y) = \begin{cases} e^{-(x+y)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Now,
$$f(x) = \int_0^{\infty} e^{-(x+y)} dy = e^{-x}, \quad x > 0$$

$$f(y) = \int_0^{\infty} e^{-(x+y)} dx = e^{-y}, \quad y > 0$$

$$\because \quad f(x)f(y) = (e^{-x})(e^{-y}) = e^{-(x+y)} = f(x, y)$$

Therefore, X and Y are independent.

Now, consider $U = \dfrac{X+Y}{2}$ and let $V = Y$

$$\Rightarrow \quad x = 2u - v = w_1(u, v) \text{ and } y = v \quad \Rightarrow \quad w_2(u, v)$$

$$\Rightarrow \quad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

$$\therefore \quad f(u, v) = f[w_1(u, v),\, w_2(u, v)] = e^{-[(2u-v)+v]}(2) = 2e^{-2u}$$

Since $x > 0 \Rightarrow \quad 2u - v > 0 \Rightarrow 2u > v \Rightarrow u > v/2$

Also, $y > 0 \Rightarrow \quad v > 0$

$$\therefore \quad f(u, v) = \begin{cases} 2e^{-2u}, & 0 < v < 2u, \quad v/2 < u < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad f(u) = \int_0^{2u} 2e^{-2u} du = 4ue^{-2u}, \quad u > 0$$

$$\therefore \quad f(u) = \begin{cases} 4ue^{-2u}, & u > 0 \\ 0, & \text{otherwise} \end{cases}$$

Problem 23. If $X = (X_1, X_2)$ is a bivariate normal random vector with mean vector $(0,\ 0)'$ and covariance matrix $\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ then obtain the means and variances of X_1 and X_2 and also the correlation between X_1 and X_2. Hence, obtain the joint probability density function of X_1 and X_2.

Solution:

It is given that $\mu = (\mu_1,\ \mu_2)' = (0,\ 0)'$

$$\Rightarrow \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} E(X_1) \\ E(X_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow E(X_1) = 0 \text{ and } E(X_2) = 0$$

Also it is given that $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

$$\Rightarrow V(X_1) = \sigma_1^2 = 5, \quad V(X_2) = \sigma_2^2 = 5$$

and $C(X_1, X_2) = \sigma_{12} = 4$

But correlation coefficient $\rho_{12} = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2} = \dfrac{4}{\sqrt{5}\sqrt{5}} = \dfrac{4}{5} = 0.8$

Therefore, the joint probability density function of X_1 and X_2 becomes

$$f(x_1,x_2) = \frac{\exp\left\{\frac{-1}{2\left(1-\rho_{12}^2\right)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}}{2\pi\,\sigma_1\sigma_2\sqrt{\left(1-\rho_{12}^2\right)}}$$

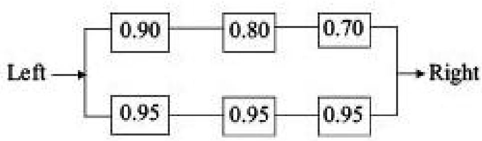
$$= \frac{\exp\left\{\frac{-1}{2\left(1-0.8^2\right)}\left[\left(\frac{x_1-0}{\sqrt{5}}\right)^2 - 2(0.8)\left(\frac{x_1-0}{\sqrt{5}}\right)\left(\frac{x_2-0}{\sqrt{5}}\right)+\left(\frac{x_2-0}{\sqrt{5}}\right)^2\right]\right\}}{2\pi\,\sqrt{5}\sqrt{5}\sqrt{\left(1-0.8^2\right)}}$$

$$= \frac{1}{6\pi}e^{(-1/3.6)\left(x_1^2-1.6x_1x_2+x_2^2\right)}, \quad -\infty < x_1, x_2 < \infty$$

EXERCISE PROBLEMS

1. Two newspapers A and B are published in a city and a survey of readers indicates that 20% read A, 16% read B, and 8% read both A and B. For a person chosen at random, find the probability that he reads none of the papers.

2. The following circuit operates if and only if there is a path of functional devices from left to right. The probability that each functions is as shown in figure. Assume that the probability that a device is functional does not depend on whether or not the other devices are functional. What is the probability that the circuit operates?



3. In a binary communication channel the probability of receiving a '1' from a transmitted '1' is 0.90 and the probability of receiving a '0' from a transmitted '0' is 0.95. If the probability of transmitting a '1' is 0.60 then obtain the probability that (i) a '1' is received and (i) a '1' is received from a transmitted '1'.

4. A discrete random variable X has the following probability distribution

$X = x$	1	2	3	4	5	6	7	8
$P(X = x)$	a	$3a$	$5a$	$7a$	$11a$	$13a$	$15a$	$17a$

(a) Find the value of a, (b) Find $P(X < 3)$ and (c) find the cumulative probability distribution of X.

5. If the cumulative distribution function of a random variable X is given by

$$F(X) = \begin{cases} 0, & x < 0 \\ x^2/2, & 0 \le x < 4 \\ 1, & x \ge 4 \end{cases} \quad \text{then find} \quad P(X > 1/X < 3).$$

6. A batch of small caliber ammunition is accepted as satisfactory if none of a sample of five shots falls more than 2 feet from the centre of the target at a given range. If R, the distance from the centre of the target to a given impact has the probability density function

$$f(r) = \begin{cases} 2kre^{-r^2}, & 0 \le r \le a \\ 0, & \text{otherwise} \end{cases}$$

where a and k are constants, then find the value of k and find the probability that a given batch will be accepted.

7. A box is to be constructed so that its height is 5 inches and its base is X inches by X inches, where X is a random variable described by the probability density function

$$f(x) = \begin{cases} 6x(1-x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the expected volume of the box.

8. Obtain the moment generating function of the random variable X whose probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\theta}, & 0 \le x \le \theta \\ 0, & \text{otherwise} \end{cases}$$

Also obtain the mean and variance of X.

9. Find the characteristic function of the random variable X whose probability density function is given as

$$f(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

and hence find the mean and variance of X.

10. In 1 out of 6 cases, material for bulletproof vests fails to meet puncture standards. If 405 specimens are tested, what does Chebyshev's theorem tell us about the probability of getting at most 30 or more than 105 cases that do not meet puncture standards?

11. The lifetime of a certain brand of electric bulb may be considered as a random variable with mean 1200 hours and standard deviation 250 hours. Find the probability, using central limit theorem, that the average lifetime of 60 bulbs exceeds 1250 hours.

12. Let X and Y be two random variables with the joint probability density function given as

$$f(x,y) = \begin{cases} x(1+3y^2)/4, & 0 < x < 2, \quad 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) Verify whether X and Y are independent

(ii) Find $E(X/Y)$ and $E(Y/X)$

(ii) Show that $E(XY) = E(X)E(Y)$ and

(iii) Evaluate $P\left(\frac{1}{4} < X < \frac{1}{2} \middle| Y = \frac{1}{3}\right)$

13. If X is a standard normal variate and $Y = aX + b$ then find the probability density function of Y.

14. If X and Y are two independent random variables each normally distributed with mean 0 and variance σ^2, then find the density functions of the random variables $R = \sqrt{X^2 + Y^2}$ and $\theta = \tan(Y/X)$.

15. If $X = (X_1, X_2)$ is a bivariate normal random vector with mean vector $(0, 0)'$ and covariance matrix $\begin{pmatrix} 2 & 0.5 \\ 0.5 & 2 \end{pmatrix}$ then obtain the means and variances of X_1 and X_2 and also the correlation between X_1 and X_2. Hence, obtain the joint probability density function of X_1 and X_2.

Table 2.1. Outcomes of tossing a coin once

Outcome(e)	$e_1 = T$	$e_2 = H$
$X(e)$	0	1

If this coin is tossed continuously at time points, say $t_1, t_2, \cdots, t_i, \cdots, t_m, \cdots\cdots$ in a given interval of time $(0,\ t)$, with a fair coin, in every toss at a point of time, we may expect either H or T. Accordingly, the outcomes may occur in different combinations as shown in Figure 2.1.

Let us assume that the coin is tossed ten times at different points of time $t_1,\ t_2,\ \cdots,\ t_i,\ \cdots,\ t_{10}$ in $(0,\ t)$. If we let $X(t,\ e)$, where t is a *time parameter*, as the *random variable* assigning 0 for tail and 1 for head for the outcomes of e at time point t, then we have ten such random variables given as $X(t_1,\ e), X(t_2,\ e), \cdots, X(t_i,\ e), \cdots, X(t_{10},\ e)$. That is, at time point t_1, the value of the random variable $X(t_1, e)$ is either 0 or 1. Hence, from the first column of Table 2.2 we have $X(t_1,\ e_1) = 0$ or $X(t_1, e_2) = 1$, from the second column the values of the random variable $X(t_2, e)$ at time point t_2 are $X(t_2,\ e_1) = 0$, $X(t_2,\ e_2) = 1$ and so on.

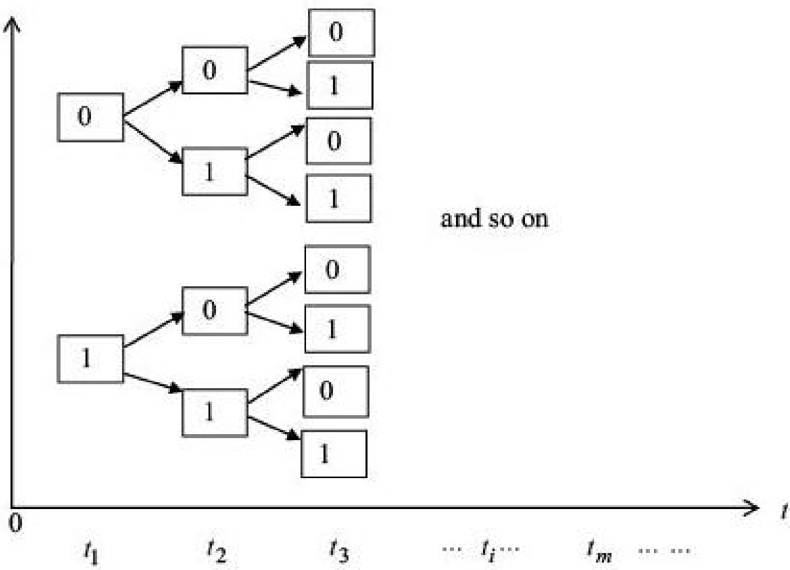


Figure 2.1. Possible combinations of outcomes of tossing a coin at different time points

Table 2.2. Possible outcomes from ten tosses of a coin

Outcome	Time									
	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
ξ_1	1	0	0	1	0	1	0	0	1	0
ξ_2	0	1	1	0	1	0	0	1	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ξ_i	0	0	1	0	0	1	1	0	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ξ_n	1	1	0	0	0	0	1	0	0	0

For illustration purpose, let us assume that the values shown in the rows of Table 2.2 are the n possible outcomes, say $\xi = \xi_1, \xi_2, \cdots, \xi_i, \cdots, \xi_n$, when the coin is tossed ten times, that is at time points $t_1, t_2, \cdots, t_i, \cdots, t_{10}$ in $(0, t)$. For example, one of the possible outcomes (there are 2^{10} outcomes of different combinations in total) from ten tosses (first row of Table 2.2) of the coin at time points $t_1, t_2, \cdots, t_i, \cdots, t_{10}$ could be, say $\xi_1 = 1, 0, 0, 1, 0, 1, 0, 0, 1, 0$ respectively. Naturally, the events of $\xi_1 \in (e_1, e_2)$ assume the values from $(0, 1)$. It may be noted that in every time point the possible outcomes are either $e_1 = 0$ or $e_2 = 1$, whereas the outcomes for 10 tosses may happen in combination of '0's and '1's. Similarly, we can obtain ξ_i, $i = 2, 3, \cdots\cdots, n$. In general, the events of $\xi \in (e_1, e_2)$ assume values from $(0, 1)$.

Let us assume that for every toss you will get rupees ten multiplied by the time t at which it is tossed and for every toss you will lose rupees five multiplied by the time t at which it is tossed. That is, if the time point is t_1 then the multiplying factor is 1, if the time point is t_2, then the multiplying factor is 2 and so on. Under this condition, if we define the random function $X(t, \xi)$ as the amount gained at time point t, then we will have the values of random functions $X(t, \xi_i)$, $i = 1, 2, 3, \cdots\cdots, n$ as given in Table 2.3.

Table 2.3. Possible outcomes for gain in rupees when a coin is tossed

Outcome	Time									
	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
ξ_1	10	−10	−15	40	−25	60	−35	−40	90	−50
ξ_2	−5	20	30	−20	50	−30	−35	80	−45	100
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ξ_i	−5	−10	30	−20	50	60	70	80	90	100
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ξ_n	10	20	−15	−20	−25	−30	70	−40	−45	−50

For example, from the first row of Table 2.3, we have $X(t_1, \xi_1) = 10$, $X(t_2, \xi_1) = -10$, $X(t_3, \xi_1) = -15$ and so on $X(t_{10}, \xi_1) = -50$. Therefore, in this example, based on the outcomes ξ at the time points $t_1 = 1, \cdots\cdots, t = t_{10} = 10$ in $(0, t)$, we can represent the random function $X(t, \xi)$ as:

$$X(t, \xi) = \begin{cases} -5t & \text{if tail turns up (i.e., } \xi = e_1 = T) \\ 10t & \text{if head turns up (i.e., } \xi = e_2 = H) \end{cases}$$

If we plot this function connecting the points with a smooth curve (at this stage never mind the time points are discrete) then we have the graph as shown in Figure 2.2. Note that each curve is occurring in a random fashion. Also, in this case, it may be noted that both the outcome ξ and the time parameter t change randomly simultaneously. In addition, the changes in the outcomes depend on the changes in the time points. However, it may be noted that at a particular time point t_i for some $i = 1, 2, \cdots\cdots, m, \cdots\cdots$ we have the random variable $X(t_i, \xi)$ with different outcomes. For example, In Figure 2.2, at time t_4 the intersecting points of the vertical line and the curves show the values (outcomes) of the random variable $X(t_4, \xi)$.

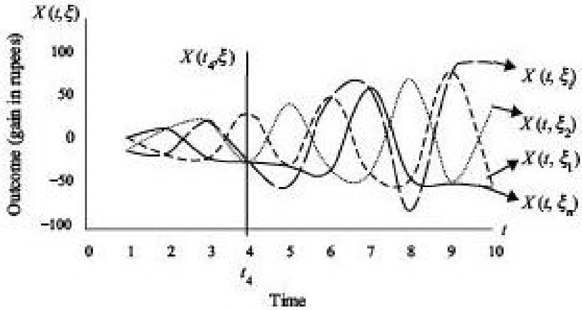


Figure 2.2. Graphical representation of data in Table 2.3

ILLUSTRATIVE EXAMPLE 2.2

In this example, in the time interval $(0, t)$, let us define the random function $X(t, \xi)$ as follows:

$$X(t, \xi) = \begin{cases} -\sin(1+t) & \text{if tail turns up (i.e., } \xi = e_1 = T) \\ \sin(1+t) & \text{if head turns up (i.e., } \xi = e_2 = H) \end{cases}$$

Here, it may be noted that the experimental outcomes are fixed. That is, either ξ_1 or ξ_2 happens based on which the function changes over a given period of time. Hence, we get the random function as $X(t, \xi_1) = -\sin(1+t)$ when tail turns up $(e_1 = T)$ or the function $X(t, \xi_2) = \sin(1+t)$ when head turns up $(e_2 = H)$. If these functions are plotted against the time points, we obtain the smooth curves as shown in Figure 2.3.

In Figure 2.3, at time t_1 the intersecting points of the vertical line and the curves show the values of the random variable $X(t_1, \xi)$.

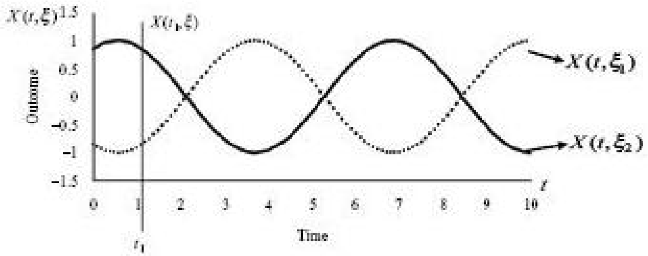


Figure 2.3. Graphical representation of random functions in Illustrative Example 2.2

ILLUSTRATIVE EXAMPLE 2.3

Now, let us consider another example in which a six faced dice is thrown. We know that if we define the outcomes as $e = (e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6) = (1,\ 2,\ 3,\ 4,\ 5,\ 6)$ then the random variable $X(e)$ assigns the values 1, 2, 3, 4, 5 or 6. Let us assume that the dice is thrown ten times continuously in a row at time points $t_1,\ t_2, \cdots,\ t_i,\ \cdots, t_{10}$ in the interval $(0,\ t)$. Let $X(t,\ \xi)$ be the random function representing the outcome ξ occurring over a period of time t, then for illustration purpose, the values of the n random functions $X(t, \xi_i)$, $i = 1,\ 2,\ 3,\ \cdots\cdots, n$ based on the possible outcomes $\xi = \xi_1, \xi_2, \cdots, \xi_i, \cdots, \xi_n$ are presented in Table 2.4.

Table 2.4. Possible outcomes from ten throws of a dice

Outcome	Time									
	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
ξ_1	6	3	6	6	2	4	2	2	5	1
ξ_2	2	5	1	3	3	6	2	1	2	4
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
ξ_i	1	5	3	4	4	1	5	1	4	3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
ξ_n	4	2	1	5	2	3	2	6	1	2

Let us assume that you win an amount equivalent to the face of the dice that turned up multiplied by the time t at which it is thrown. If we let the random function $X(t,\ \xi)$ as the amount won at time point t, the values of the n random functions $X(t,\ \xi_i), i = 1, 2, 3, \cdots\cdots, n$ based on the possible outcomes $\xi = \xi_1,\ \xi_2, \cdots, \xi_i,\ \cdots, \xi_n$ are presented in Table 2.5.

Table 2.5. Possible outcomes for gain in rupees when a dice is thrown

Outcome	Time									
	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
ξ_1	6	6	18	24	10	24	14	16	45	10
ξ_2	2	10	3	12	15	36	14	8	18	40
⋮	⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
ξ_i	1	10	9	16	20	6	35	8	36	30
⋮	⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
ξ_n	4	4	3	20	10	18	14	48	9	20

Looking at Table 2.5, in this example of throwing a dice ten times, for time points $t_1 = 1, \cdots\cdots, t = t_{10} = 10$ in $(0, t)$, we can represent the random function $X(t, \xi)$ as a function of t and ξ as follows:

$$X(t, \xi) = at$$

where $a = 1, 2, 3, 4, 5$ *or* 6. If we plot this function connecting the points (though discrete) with a smooth curve then we have the graph as shown in Figure 2.4. Note that each curve is occurring in a random fashion.

Also, in this case, it may be noted that both the outcome ξ and the time parameter t change randomly simultaneously. In addition, the changes in the outcomes depend on the changes in the time points.

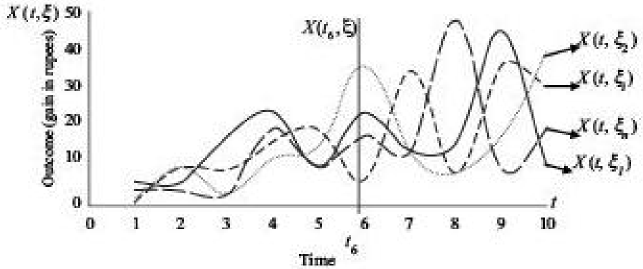


Figure 2.4. Graphical representation of data in Table 2.5

In Figure 2.4, at time t_6 the intersecting points of the vertical line and the curves show the values of the random variable $X(t_6, \xi)$.

ILLUSTRATIVE EXAMPLE 2.4

In this example, we consider a random function $X(t,\ \xi)$ of t and ξ as $X(t,\xi) = A\cos(\omega t + \xi)$ where $t > 0$ is the time parameter, ξ is a uniformly distributed random variable in the interval (0,1), and A and ω are known constants. Without loss of generality, for the given values of $A = 1.5$ and $\omega = 2.5$, if we consider four randomly chosen values (say experimental outcomes) of ξ, (say, $\xi_1 = 0.05$, $\xi_2 = 0.4$, $\xi_i = 0.75$, $\xi_n = 0.95$) from a set of n possible values of $\xi = \xi_1, \xi_2, \cdots\cdots, \xi_i, \cdots\cdots, \xi_n$ in the interval (0,1) then we have four different random functions as

(i) $X(t,\ \xi_1) = 1.5\cos(2.5t + 0.05)$
(ii) $X(t,\ \xi_2) = 1.5\cos(2.5t + 0.40)$
(iii) $X(t,\ \xi_i) = 1.5\cos(2.5t + 0.75)$
(iv) $X(t,\ \xi_n) = 1.5\cos(2.5t + 0.95)$

Further, without loss of generality we can write these functions in the following fashion as well:

(i) $X(t,\ \xi) = 1.5\cos(2.5t + 0.05)$ when $\xi = \xi_1 = 0.05$
(ii) $X(t,\ \xi) = 1.5\cos(2.5t + 0.40)$ when $\xi = \xi_2 = 0.40$
(iii) $X(t,\ \xi) = 1.5\cos(2.5t + 0.75)$ when $\xi = \xi_i = 0.75$
(iv) $X(t,\ \xi) = 1.5\cos(2.5t + 0.95)$ when $\xi = \xi_n = 0.95$

It may be noted that for the fixed outcomes of ξ, the changes in the value of the function depends on the changes in time parameter. If these functions are plotted against t, then we can have its graphical representation as shown in Figure 2.5.

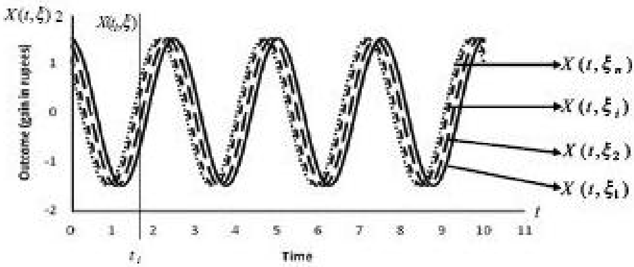


Figure 2.5. Graphical representation of the random function $X(t,\ \xi)$ in Example 2.4

Here, if we look at the values of the function at a particular time point of the time parameter, say t_i, then the function $X(t,\ \xi)$ becomes a random variable as $X(t_i,\ \xi_i)$ at time point t_i with outcomes $\xi_1 = 0.05$, $\xi_2 = 0.4$, $\xi_i = 0.75$, $\xi_n = 0.95$, that is, the points of the four curves as they pass through the time point t_i. Also, it could be seen that at time point t_i, the outcome ξ is uniformly distributed in the interval (0, 1). In Figure 2.5, at time t_i the intersecting points of the vertical line and the curves show the values of the random variable $X(t_i, \xi)$.

$$F_{X(t)}(x) = P\{X(t) \le x\}$$

$$= \int_{\infty}^{x} f(x,t)\, dx \text{ if the outcome of the process } \{X(t)\} \text{ is continuous.}$$

Here, $P\{X(t) = x\}$ and $f(x,t)$ are respectively called the *first order* PMF and *first order* PDF of the random process $\{X(t)\}$. The *second order* PMF (joint PMF) and *second order* PDF (joint PDF) of the random process $\{X(t)\}$ are respectively denoted as $P\{X(t_1) = x_1, X(t_2) = x_2\}$ and $f_{XX}(x_1,\, x_2;\, t_1,\, t_2)$ or $f(x_1,\, x_2;\, t_1,\, t_2)$ or $f_{X(t_1)X(t_2)}(x_1,\, x_2)$. Now, the *second order* CDFs for discrete and continuous cases denoted by $F_{XX}(x_1,\, x_2;\, t_1,\, t_2)$ or $F(x_1,\, x_2;\, t_1,\, t_2)$ or $F_{X(t_1)X(t_2)}(x_1,x_2)$ can be obtained as

$$F_{X(t_1)X(t_2)}(x_1,x_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

$$= \sum_{x_i=-\infty}^{x_1} \sum_{x_j=-\infty}^{x_2} P\{X(t_1) = x_i, X(t_2) = x_j\} \tag{2.2}$$

$$F_{X(t_1)X(t_2)}(x_1,x_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(x_1, x_2; t_1, t_2)\, dx_2 dx_1$$

Similarly, the m^{th} *order* PMF and m^{th} *order* PDF the random process $\{X(t)\}$ are respectively given as $P\{X(t_1) = x_1,\, X(t_2) = x_2,\, \cdots\cdots,\, X(t_m) = x_m\}$ and $f(x_1, x_2, \cdots\cdots, x_m;\, t_1, t_2, \cdots\cdots,\, t_m)$ and hence the m^{th} *order* CDFs for discrete and continuous cases can be obtained as

$$P\{X(t_1) \le x_1, X(t_2) \le x_2, \cdots\cdots, X(t_m) \le x_m\}$$

$$= \sum_{x_i=-\infty}^{x_1} \sum_{x_j=-\infty}^{x_2} \cdots \sum_{x_l=-\infty}^{x_m} P\{X(t_1) = x_i, X(t_2) = x_j, \cdots\cdots X(t_m) = x_l\} \tag{2.3}$$

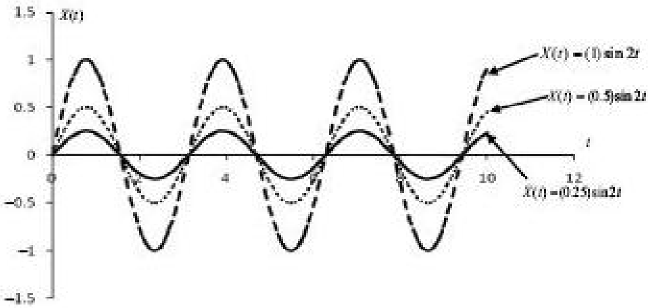
$$P\{X(t_1) \le x_1, X(t_2) \le x_2, \cdots\cdots X(t_m) \le x_m\}$$

$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_m} f(x_1, x_2, \cdots\cdots, x_m; t_1, t_2, \cdots\cdots,\, t_m)\, dx_m \cdots\cdots dx_2 dx_1$$

Note:

If $\{X_1(t)\}$ and $\{X_2(t)\}$ are two random processes observed over a period of time $(0,\, t)$, and $X_1(t_1)$ is a random variable of the process $\{X_1(t)\}$ at the time point t_1, and $X_2(t_2)$ is a random variable of the process $\{X_2(t)\}$ at the time point t_2, then their joint PMF and joint PDF are respectively denoted by

Now, the sample functions can be graphically shown as below:



Problem 2. If $\{X(t)\}$is a random process then obtain the probability mass function of the sample function $X(t)$at time point $t = 5$ in cases of (i) repeated tossing of a coin (ii) repeated rolling of a die.

SOLUTION:

(i) Let X (5) be the outcome of the coin tossed at time point 5, and let

$$X(5) = \begin{cases} 1, & \text{if head turns at time point 5} \\ 0, & \text{if tail turns at time point 5} \end{cases}$$

Therefore, the probability mass function becomes

$$P\{X(5) = x\} = \begin{cases} \dfrac{1}{2}, & x = 0,\ 1 \\ 0, & \text{otherwise} \end{cases}$$

(ii) Now, let X (5) be the outcome of the die rolled at time point 5, then we have

$$X(5) = 1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6$$

Therefore, the probability mass function becomes

$$P\{X(5) = x\} = \begin{cases} \dfrac{1}{6}, & x = 0,\ 1,\ 3,\ 4,\ 5,\ 6 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \quad V\{X(t)\} = E\left\{X^2(t)\right\} - \{E[X(t)]\}^2$$

$$= \frac{1}{2}\{1 + \text{sin}c\,(4a_0t)\} - (\text{sin}c\,(2a_0t))^2$$

Problem 4. If $\{Z(t)\}$ is a random process defined *by* $Z(t) = Xt + Y$ where X and Y are a pair of random variables with means μ_x and μ_y, variances σ_x^2 and σ_y^2 respectively, and correlation coefficient ρ_{xy}. Find (i) mean (ii) variance (iii) autocorrelation and (iv) autocovariance of $\{Z(t)\}$.

SOLUTION:

It is given that

$$E(X) = \mu_x,\ E(Y) = \mu_y,\ V(X) = \sigma_x^2 \text{ and } V(Y) = \sigma_y^2$$

The correlation coefficient between X and Y is ρ_{xy}
Also, note that

$$V(X) = E(X^2) - \{E(X)\}^2 \Rightarrow \sigma_x^2 = E(X^2) - \mu_x^2$$

$$\Rightarrow \quad E(X^2) = \sigma_x^2 + \mu_x^2$$

$$V(Y) = E(Y^2) - \{E(Y)\}^2 \Rightarrow \sigma_y^2 = E(Y^2) - \mu_y^2,$$

$$\Rightarrow \quad E(Y^2) = \sigma_y^2 + \mu_y^2$$

$$C(X,\ Y) = E(XY) - E(X)E(Y) \Rightarrow E(XY) = C(X,Y) + E(X)E(Y)$$

$$E(XY) = C(X,Y) + \mu_x\mu_y$$

(i) mean of $\{Z(t)\}$
That is, $E\{Z(t)\} = E(Xt+Y) = tE(X) + E(Y) = t\,\mu_x + \mu_y$

(ii) variance of $\{Z(t)\}$

$$V\{Z(t)\} = V(Xt+Y) = V(tX) + V(Y) + 2t\ \text{Cov}(X,\ Y)$$

$$= t^2V(X) + V(Y) + 2t\rho_{xy}\sigma_x\sigma_y \qquad \because \quad \rho_{xy} = \frac{\text{Cov}(X,\ Y)}{\sigma_x\,\sigma_y}$$

$$= t^2\sigma_x^2 + \sigma_y^2 + 2t\rho_{xy}\sigma_x\sigma_y$$

(iii) Autocorrelation of $\{Z(t)\}$

$$R_{zz}(t_1,\ t_2) = E\{Z(t_1)Z(t_2)\} = E\{(Xt_1+Y)(Xt_2+Y)\}$$

$$= E\left\{(X^2t_1t_2 + Xt_1Y + YXt_2 + Y^2)\right\}$$

$$= t_1t_2E(X^2) + t_1E(XY) + t_2E(YX) + E(Y^2)$$

$$= t_1t_2(\sigma_x^2 + \mu_x^2) + (t_1 + t_2)\left\{\text{Cov}(X,Y) + \mu_x\mu_y\right\} + (\sigma_y^2 + \mu_y^2)$$

$$= t_1t_2(\sigma_x^2 + \mu_x^2) + (t_1 + t_2)(\rho_{xy}\sigma_x\sigma_y + \mu_X\mu_Y) + (\sigma_y^2 + \mu_y^2)$$

(iv) Autocovariance of $\{Z(t)\}$

$$C_{zz}(t_1,t_2) = R_{zz}(t_1,t_2)$$

$$= t_1t_2(\sigma_x^2 + \mu_x^2) + (t_1 + t_2)(\rho_{xy}\sigma_x\sigma_y + \mu_x\mu_y) + (\sigma_y^2 + \mu_y^2)$$

$$- (t_1\mu_x + \mu_y)(t_2\mu_x + \mu_y)$$

$$= t_1t_2\sigma_x^2 + (t_1 + t_2)\rho_{xy}\sigma_x\sigma_y + \sigma_y^2$$

Note:

(i) Variance of $\{Z(t)\}$ may also be obtained by letting $t_1 = t_2 = t$ in $C_{zz}(t_1,t_2)$ given in (iv).

(ii) If X and Y are independent random variables, then we have $E(XY) = E(X)\,E(Y)$ and hence $\rho_{xy} = \dfrac{\text{Cov}(X,\,Y)}{\sigma_x\,\sigma_x} = 0$, therefore we have the results

$$V\{Z(t)\} = t^2\sigma_x^2 + \sigma_x^2$$

$$R_{zz}(t_1,t_2) = t_1t_2(\sigma_x^2 + \mu_x^2) + (t_1 + t_2)\mu_x\mu_y + (\sigma_y^2 + \mu_y^2)$$

$$C_{zz}(t_1,t_2) = t_1t_2\sigma_x^2 + \sigma_y^2$$

Problem 5. Suppose that $\{X(t)\}$ is a random process with $\mu(t) = 3$ and $C(t_1,t_2) = 4e^{-0.2|t_1-t_2|}$. Find (i) $P[X(5) \le 2]$ and (ii) $P[|X(8) - X(5)| \le 1]$ using central limit theorem.

SOLUTION:

It is given that $E\{X(t)\} = \mu(t) = 3$ and $V\{X(t)\} = C(t,t) = 4e^{-0.2|0|} = 4$

(i) Consider $P[X(5) \le 2] = P\left(\dfrac{X(5) - E\{X(5)\}}{\sqrt{V\{X(5)\}}} < \dfrac{2 - E\{X(5)\}}{\sqrt{V\{X(5)\}}}\right)$

$$= P\left(Z < \frac{2 - \mu(5)}{\sqrt{C(5,5)}}\right) = P\left(Z < \frac{2-3}{\sqrt{4}}\right)$$

$$= P(Z < -0.5) = 0.309 \text{ (Refer to Appendix C)}$$

(ii) Consider

$$P\{|X(8)-X(5)|\le 1\}$$
$$= P\left(\left|\frac{X(8)-X(5)-E\{X(8)-X(5)\}}{\sqrt{V\{X(8)-X(5)\}}}\right| < \frac{1-E\{X(8)-X(5)\}}{\sqrt{V\{X(8)-X(5)\}}}\right)$$

But we know that

$$E\{X(8)-X(5)\} = E\{X(8)\} - E\{X(5)\} = 3-3=0$$
$$V\{X(8)-X(5)\} = V\{X(8)\} + V\{X(5)\} - 2C(8,5)$$
$$= 4+4-2\left\{4e^{-0.2|8-5|}\right\} = 3.608$$

$$\therefore \quad P\{|X(8)-X(5)|\le 1\} = P\left(|Z| < \frac{1-E\{X(8)-X(5)\}}{\sqrt{V\{X(8)-X(5)\}}}\right)$$
$$= P\left(|Z| \le \frac{1-0}{\sqrt{3.608}}\right)$$
$$= P(|Z|\le 0.526)$$
$$= P(-0.526 \le Z \le 0.526)$$
$$= 0.40 \text{ (Refer to Appendix C)}$$

Problem 6. If $\{X(t)\}$ is a random process with $\mu(t) = 8$ autocorrelation $R(t_1,t_2) = 64+10e^{-2|t_1-t_2|}$ then find (i) mean, (ii) variance and (iii) covariance of the random variables $Z = X(6)$ and $W = X(9)$.

SOLUTION:

It is given that $E\{X(t)\} = \mu(t) = 8$ and $R(t_1,t_2) = 64+10e^{-2|t_1-t_2|}$

(i) *Mean*

$$\text{Consider } E(Z) = E\{X(6)\} = \mu(6) = 8$$
$$\text{Consider } E(W) = E\{X(9)\} = \mu(9) = 8$$

(since mean is given as constant)

(ii) *Variance*

$$\text{Consider } E(Z^2) = E\{X(6)X(6)\} = R(6,\ 6) = 64 + 10 = 74$$

$$\text{Similarly, } E(W^2) = E\{X(9)X(9)\} = R(9,\ 9) = 64 + 10 = 74$$

$$\text{Therefore,} \quad V(Z) = E(Z^2) - \{E(Z)\}^2 = 74 - 8^2 = 10$$

$$\text{Similarly,} \quad V(W) = E(W^2) - \{E(W)\}^2 = 74 - 8^2 = 10$$

(iii) *Covariance*

$$\text{Cov}\,(Z, W) = E(ZW) - E(Z)E(W)$$

$$\text{Consider } E(ZW) = E\{X(6)X(9)\} = R(6,9) = 64 + 10e^{-6} = 64.0248$$

$$\Rightarrow \quad \text{Cov}\,(Z, W) = 64.0248 - (8)(8) = 0.0248$$

Problem 7. Let $\{X(t)\}$ be a random process with $X(t) = Y\cos\omega t, \quad t \geq 0$ where ω is a constant Y is a uniform random variable in the interval $(0,\ 1)$. Then determine the probability density functions of $\{X(t)\}$ at *(i)* $t = 0$, *(ii)* $t = \pi/4\omega$, *(iii)* $t = \pi/2\omega$ and *(iv)* $\quad t = \pi/\omega$.

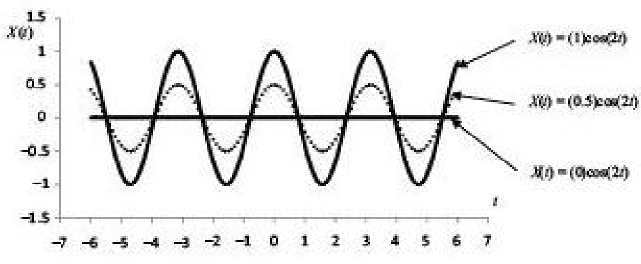
SOLUTION:

Given $X(t) = Y\cos\omega t$

Since Y is a uniform random variable in the interval $(0,\ 1)$ the probability density function (PDF) of Y is given as

$$f(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

The following figure depicts three samples of $X(t) = Y\cos(2t)$ for $Y = 0, Y = 0.5$ and $Y = 1.0$ for a fixed value of $\omega = 2$.



We know that if the PDF of the random variable Y is known and the random variable $X = g(Y) \Rightarrow y = w(x)$ then the PDF of the random variable X can be obtained using the transformation

$$f(x) = f[w(x)]\,|J| \quad \text{Refer Equation (1.30)}$$

where $|J| = w'(x) = \left|\dfrac{dy}{dx}\right|$.

Unlike Equation 1.30, note that in this case the new variable is X and the old variable is Y.
When, $t = 0$ we have $X(0) = Y\cos 0 = Y$
Therefore, the probability density function $f(x)$ of $\{X(t)\}$ at $t = 0$ becomes

$$f(x) = f[w(x)]\left|\frac{dy}{dx}\right| = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) When, $t = \pi/4\omega$, we have $X(\pi/4\omega) = Y\cos(\omega\pi/4\omega) = Y\cos\pi/4 = \dfrac{1}{\sqrt{2}}Y$

Therefore, the probability density function $f(x)$ of $\{X(t)\}$ at $t = \pi/4\omega$ becomes

$$f(x) = f[w(x)]\left|\frac{dy}{dx}\right| = \begin{cases} \sqrt{2}, & 0 < x < 1/\sqrt{2} \\ 0, & \text{otherwise} \end{cases}$$

(ii) When, $t = \pi/2\omega$ we have $X(\pi/2\omega) = Y\cos(\omega\pi/2\omega) = Y\cos\pi/2 = 0$

In this case, we have $X(\pi/2\omega) = 0$ irrespective of the value of Y. Therefore, here we have the probability mass function (PMF) at $X(\pi/2\omega) = 0$. As follows:

$$P\{X(t) = 0\} = \begin{cases} 1, & t = \pi/2\omega \\ 0, & \text{otherwise} \end{cases}$$

(iii) When, $t = \pi/\omega$ we have $X(\pi/\omega) = Y\cos(w\pi/\omega) = Y\cos\pi = -Y$

Therefore, the probability density function $f(x)$ of $\{X(t)\}$ at $t = \pi/4\omega$ becomes

$$f(x) = f[w(x)]\left|\frac{dy}{dx}\right| = \begin{cases} 1, & -1 < x < 0 \\ 0, & \text{otherwise} \end{cases}$$

Problem 8. In an experiment of tossing a fair coin, the random process $\{X(t)\}$ is defined as

$$X(t) = \begin{cases} \sin \pi t, & \text{if head turns up} \\ 2t, & \text{if tail turns up} \end{cases}$$

(i) Find $E\{X(t)\}$ at $t = 1/4$. and (ii) Find the probability distribution function $F(x, t)$ at $t = 1/4$.

SOLUTION:

We know that in the experiment of tossing a fair coin,

$$P(\text{head}) = P(\text{tail}) = \frac{1}{2}$$

(i) Now,

$$P(\text{head}) = \frac{1}{2} \Rightarrow P\{X(t) = \sin \pi t\} = \frac{1}{2}$$

$$P(\text{tail}) = \frac{1}{2} \Rightarrow \quad P\{X(t) = 2t\} = \frac{1}{2}$$

$$E\{X(t)\} = \sum_x xP\{X(t) = x)\}$$

$$= (\sin \pi t)P\{X(t) = \sin \pi t\} + (2t)P\{X(t) = 2t\}$$

$$= (\sin \pi t)\frac{1}{2} + (2t)\frac{1}{2} = \frac{1}{2}(\sin \pi t) + t$$

$$\therefore \quad E\{X(1/4)\} = \frac{1}{2}(\sin \pi/4) + 1/4 = \frac{1}{2\sqrt{2}} + \frac{1}{4} = 0.6036$$

(ii) When $t = 1/4$, we have

$$P\{X(t) = \sin \pi t\} = \frac{1}{2} \Rightarrow P\{X(1/4) = \sin \pi/4\} = \frac{1}{2}$$

$$\Rightarrow P\left\{X(1/4) = 1/\sqrt{2}\right\} = \frac{1}{2}$$

$$P\{X(t) = 2t\} = \frac{1}{2} \Rightarrow P\{X(1/4) = 2/4\} = \frac{1}{2}$$

$$\Rightarrow P\{X(1/4) = 1/2\} = \frac{1}{2}$$

$$\therefore \quad F(x, t) = \begin{cases} 0, & \text{if} \quad x < 1/2 \\ \frac{1}{2}, & \text{if} \quad 1/2 \le x < 1/\sqrt{2} \\ 1, & \text{if} \quad x \ge 1/\sqrt{2} \end{cases}$$

SOLUTION:

Given $X(t) = A\cos(\omega t + \theta)$

Since θ is a uniform random variable distribution in the interval $(-\pi, \pi)$, we have the probability density function of θ as

$$f(\theta) = \frac{1}{2\pi}, \qquad -\pi \le \theta \le \pi$$

We know that mean of $\{X(t)\}$ is given by

$E\{X(t)\} = E(A)E[\cos(\omega t + \theta)]$ (since A and θ are independent)

$$E[\cos(\omega t + \theta)] = \int_{-\pi}^{\pi} \cos(\omega t + \theta)\frac{1}{2\pi}d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)d\theta$$

$$= \frac{1}{2\pi}[\sin(\omega t + \theta)]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[\sin(\omega t + \pi) - \sin(\omega t - \pi)] = 0$$

$$\therefore \quad E\{X(t)\} = E(A)(0) = 0$$

Now, $E\{X^2(t)\}$ is given by

$E\{X^2(t)\} = E[A^2\cos^2(\omega t + \theta)] = E(A^2)E[\cos^2(\omega t + \theta)]$ (since A and θ are independent)

Consider $$E[\cos^2(\omega t + \theta)] = E\left\{\frac{1 + \cos 2(\omega t + \theta)}{2}\right\}$$

$$= \frac{1}{2} + \frac{1}{2}E[\cos 2(\omega t + \theta)]$$

$$= \frac{1}{2} + \frac{1}{2}\int_{-\pi}^{\pi} \cos 2(\omega t + \theta)\frac{1}{2\pi}d\theta$$

$$= \frac{1}{2} + \frac{1}{4\pi}\int_{-\pi}^{\pi} \cos 2(\omega t + \theta)d\theta$$

$$= \frac{1}{2} + \frac{1}{4\pi}\left[\frac{\sin(2\omega t + 2\theta)}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} + \frac{1}{4\pi}[\sin(2\omega t + 2\pi) - \sin(2\omega t - 2\pi)]$$

$$= \frac{1}{2}$$

$$\therefore \quad E\left\{X^2(t)\right\} = E(A^2)E[\cos^2(\omega t + \theta)] = E(A^2)\left(\frac{1}{2}\right)$$

Variance of $\{X(t)\}$ is given by

$$V\{X(t)\} = E\left\{X^2(t)\right\} - \{E[X(t)]\}^2 = \frac{1}{2}E(A^2) - (0)^2 = \frac{1}{2}E(A^2)$$

Autocorrelation of $\{X(t)\}$is given by

$$R_{xx}(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{A\cos(\omega t_1 + \theta)A\cos(\omega t_2 + \theta)\}$$

$$= E(A^2)E\{\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)\}$$

$$= \frac{1}{2}E(A^2)E\{\cos\omega(t_1 - t_2) + \cos[\omega(t_1 + t_2) + 2\theta]\}$$

$$= \frac{1}{2}E(A^2)\{E[\cos\omega(t_1 - t_2)] + E(\cos[\omega(t_1 + t_2) + 2\theta])\}$$

$$= \frac{1}{2}E(A^2)\left\{\int_{-\pi}^{\pi}\cos\omega(t_1 - t_2)\ f(\theta)d\theta + \int_{-\pi}^{\pi}\cos[\omega(t_1 + t_2) + 2\theta]\ f(\theta)d\theta\right\}$$

$$= \frac{1}{2}E(A^2)\left\{\int_{-\pi}^{\pi}\cos\omega(t_1 - t_2)\ \frac{1}{2\pi}d\theta + \int_{-\pi}^{\pi}\cos[\omega(t_1 + t_2) + 2\theta]\ \frac{1}{2\pi}d\theta\right\}$$

$$= \frac{1}{2}E(A^2)\left\{\cos\omega(t_1 - t_2) - \frac{1}{2\pi}\left|\sin[\omega(t_1 + t_2) + 2\theta]\right|_{-\pi}^{\pi}\right\}$$

$$= \frac{1}{2}E(A^2)\cos\omega(t_1 - t_2) = \frac{1}{2}E(A^2)\cos\omega\tau$$

Note that
$|\sin[\omega(t_1+t_2)+2\theta]|_{-\pi}^{\pi} = \sin[\omega(t_1+t_2)+2\pi] - \sin[\omega(t_1+t_2) \;\; -2\pi] = 0$

$$\therefore \quad R_{xx}(t_1,t_2) = \frac{1}{2}E(A^2)\cos\omega\tau$$

We know that covariance is given by

$$C_{xx}(t_1,t_2) = R_{xx}(t_1,t_2) - E\{X(t_1)\}E\{X(t_2)\} = \frac{1}{2}E(A^2)\cos\omega\tau - (0)(0)$$

$$= \frac{1}{2}E(A^2)\cos\omega\tau$$

Problem 11. A random process $\{X(t)\}$ has the sample functions of the form $X(t) = A\cos(\omega t + \theta)$ where ω is a constant, A is a random variable that has magnitude $+1$ and -1 with equal probabilities, and θ is a random variable that is uniformly distributed in $(0,\ 2\pi)$. Assume that A and θ are independent. Find the probability density functions of $X(t)$ when $A = \pm 1$. Also plot the probability density functions when $A = +1,\ t = 1,\ \omega = 2,\ \theta = (0,\ 2\pi)$ and $A = +1,\ t = 1,\ \omega = 2,\ \theta = (0,\ 2\pi)$.

SOLUTION:

For $A = \pm 1$, we have $X(t) = \begin{cases} +\cos(\omega t+\theta), & A = +1 \\ -\cos(\omega t+\theta), & A = -1 \end{cases}$

We know that

$$f_{X(t)}(x) = f[w(x)]\left|\frac{d\theta}{dx}\right|$$

Now $\quad x = A\cos(\omega t+\theta) \quad \Rightarrow \quad \theta = \cos^{-1}(x/A) - \omega t$

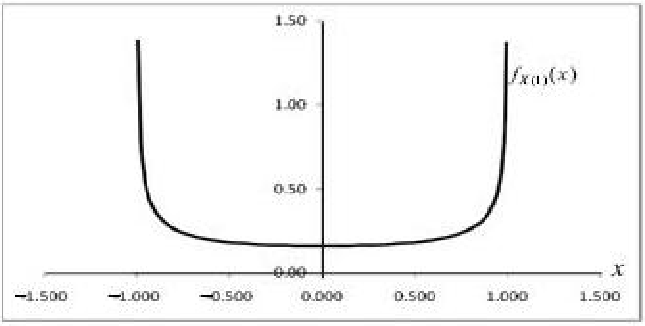
$$\Rightarrow \quad \frac{d\theta}{dx} = \frac{-1}{\sqrt{1-(x/A)^2}} \quad \Rightarrow \left|\frac{d\theta}{dx}\right| = \left|\frac{-1}{\sqrt{1-(x/A)^2}}\right| = \frac{1}{\sqrt{1-(x/A)^2}}$$

For $\quad A = \pm 1$, we have $\left|\dfrac{d\theta}{dx}\right| = \dfrac{1}{\sqrt{1-(x)^2}}$

However, when $A = 1$, the limit becomes $+\cos\omega t \le x \le +\cos(\omega t + 2\pi)$
$A = -1$, the limit becomes $-\cos(\omega t+2\pi) \le x \le -\cos\omega t$

$$\therefore \quad f_{X(t)}(x) = \begin{cases} \dfrac{1}{2\pi}\dfrac{1}{\sqrt{1-x^2}}, & \cos\omega t \le x \le \cos(\omega t+2\pi) \quad \text{or} \\ & -\cos(\omega t+2\pi) \le x \le -\cos\omega t \\ 0, & \text{otherwise} \end{cases}$$

Therefore, the plot of probability density function when $A = +1$, $t = 1$, $\omega = 2$, $\theta = (0, 2\pi)$ or $A = -1$, $t = 1$, $\omega = 2$, $\theta = (0, 2\pi)$ is shown below. It is nothing but the probability density function $f_{X(1)}(x)$ of the random variable, $X(1)$.



EXERCISE PROBLEMS

1. A random process $\{X(t)\}$ has the sample functions of the form $X(t) = Y \cos \omega t$ where ω is a constant and Y is a random variable that is uniformly distributed in $(0, 1)$. Sketch three sample functions for $Y = 0.25, 0.5, 1$ by fixing $\omega = 2$ without loss of generality. Assume $0 \leq t \leq 10$.
2. Let the receiver carrier signal of an AM radio be a random process $\{X(t)\}$ has the received carrier signal $X(t) = A\cos(2\pi f t + \theta)$ where f is the carrier frequency with a random phase θ which is a uniform random variable in the interval $(0, 2\pi)$ then what is the expected value of the process $\{X(t)\}$.
3. If a random process $\{X(t)\}$ is sinusoid with a random frequency $X(t) = A\sin(\omega_0 t)$ where A is random variable uniformly distributed over some interval $(0, 1)$. Then obtain the mean and variance of the process.
4. In an experiment of throwing a fair six-faced dice, the random process $\{X(t)\}$ is defined as

$$X(t) = \begin{cases} \sin \pi t, & \text{if odd number shows up} \\ 2t, & \text{if even number shows up} \end{cases}$$

   (i) Find $E\{X(t)\}$ at $t = 0.25$ and (ii) Find the probability distribution function $F(x, t)$ at $t = 0.25$.
5. Let $\{X(t)\}$ be a random process with $X(t) = \cos(\omega t + \theta)$, $t \geq 0$ where ω is a constant and θ is a random variable uniformly distributed in the interval$(-\pi, \pi)$. Then (i) show that the first and second order moments of $\{X(t)\}$ are independent of time and (ii) if θ is constant will the ensemble mean of $\{X(t)\}$ be independent of time?

6. Let $\{X(t)\}$ be random process such that $X(t) = \sin(\omega t + \theta)$ is a sinusoidal signal with random phase θ which is a uniform random variable in the interval $(-\pi, \pi)$. If both time t and the radial frequency ω are constants, then find the probability density function of the random variable $X(t)$. Also comment on the dependence of probability density function of $X(t)$.
7. Consider the continuous random process $\{X(t)\}$ such that $X(t) = A\cos(\omega t + \theta)$, where A is a random variable that has a uniform density in the range $(-1, 1)$. Find the mean of $\{X(t)\}$.
8. The random process $\{X(t)\}$ is defined as $X(t) = 2e^{-At}\sin(\omega t + B)u(t)$, where $u(t)$ is the unit step function and the random variables A and B are independent where A is uniformly distributed in $(0, 2)$ and B is uniformly distributed in $(-\pi, \pi)$. Find the autocovariance of the random process.
9. A random process $\{X(t)\}$ has the sample functions of the form $X(t) = A\cos(\omega t + \theta)$ where ω is a constant, A is a random variable that has magnitude $+1$ and -1 with equal probabilities, and θ is a random variable that is uniformly distributed in $(0, 2\pi)$. Assume that A and θ are independent. Find mean and variance of the random process $\{X(t)\}$.
10. Consider a random process $\{X(t)\}$ such that $X(t) = U\cos t + V\sin t$ where U and V are independent random variables each of which assumes the values -2 and -1 with probabilities $1/4$ and $3/4$ respectively. Obtain $E\{X(t)\}$ and $V\{X(t)\}$.

It may be noted that the only solution to the equation $h(t+s) = h(t) + h(s)$ can be given by $h(t) = kt$ with k as constant. That is,

$$h(t) = kt \quad \Rightarrow \quad h(t+s) = c(t+s) = ct + cs = h(t) + h(s)$$

Now, clearly since $h(1) = k$, we have $k = h(1) = V\{X(1)\}$ and hence $V\{X(t)\} = h(t) = kt = V\{X(1)\}t = \sigma_1^2 t$ where $\sigma_1^2 = V\{X(1)\}$.

(ii) With $t > s$, we have

$$\begin{aligned} V\{X(t)\} &= V\{X(t) - X(s) + X(s) - X(0)\} \\ &= V\{X(t) - X(s)\} + V\{X(s) - X(0)\} \\ &= V\{X(t) - X(s)\} + V\{X(s)\} \end{aligned}$$

$$\Rightarrow \quad V\{X(t) - X(s)\} = V\{X(t)\} - V\{X(s)\} = \sigma_1^2 t - \sigma_1^2 s = \sigma_1^2 (t-s)$$

Note:
From the results of the Theorems 3.1 and 3.2, it may be noted that, if we assume that $X(0) = 0$, then we have

$$E\{X(t)\} = E\{X(1) - X(0)\} = E\{X(1)\} = \mu_1 t$$

$$V\{X(t)\} = V\{X(1) - X(0)\} = V\{X(1)\} = \sigma_1^2 t$$

Clearly, we understand that the processes with stationary independent increments are non-stationary as mean and variance are time dependent.

Theorem 3.3: Let $\{X(t)\}$ be a random process with stationary independent increments. If $X(0) = 0, V\{X(t)\} = \sigma_1^2 t$ and $V\{X(s)\} = \sigma_1^2 s$ for some t and s, then $C\{X(t), X(s)\} = C_{xx}(t,s) = \sigma_1^2 \min(t, s)$ where $V\{X(1)\} = \sigma_1^2$.

Proof. Let $t > s$

By definition, we know that

$$\begin{aligned} V\{X(t) - X(s)\} &= E\{[X(t) - X(s)] - E[X(t) - X(s)]\}^2 \\ &= E\{[X(t) - E[X(t)] - [X(s) - E[X(s)]\}^2 \\ &= E\{[X(t) - E[X(t)]]\}^2 - 2E\{[X(t) - E[X(t)]]\,[X(s) - E[X(s)]]\} \\ &\quad + E\{X(s) - E[X(s)]\}^2 \\ &= V\{X(t)\} - 2C_{xx}(t,s) + V\{X(s)\} \end{aligned}$$

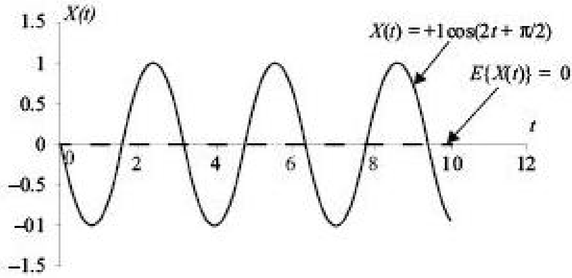


Figure 3.3. Plot of $X(t) = (+1)\cos(2t + \pi/2)$ and $E\{X(t)\} = 0$

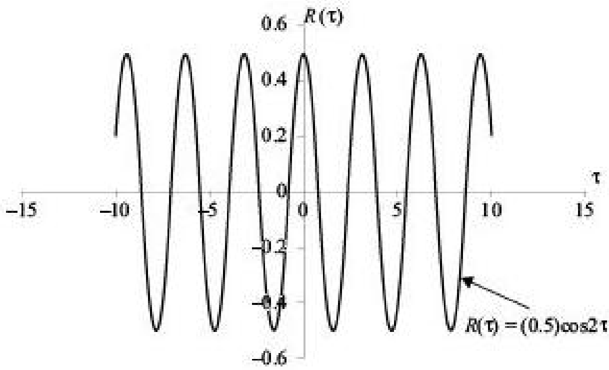


Figure 3.4. Plot of $R(\tau) = (0.5)\cos 2\tau$

SOLVED PROBLEMS

Problem 1. If $\{X(t)\}$ is a random process with $X(t) = Y\cos t + Z\sin t$ for all t where Y and Z are independent random variables, each of which assumes the values -2 and 1 with probabilities $1/3$ and $2/3$ respectively. Prove that $\{X(t)\}$ is a stationary process in wide sense but not stationary in strict sense.

SOLUTION:

Since Y and Z are discrete random variables, the probability distribution of random variable Y can be represented as

$Y = y$	-2	1
$P(Y = y)$	1/3	2/3

and the probability distribution of random variable Z can be given as

$Z=z$	-2	1
$P(Z=z)$	1/3	2/3

Since Y and Z are independent random variables, we have the joint probability distribution as

		$Y=y$		$P(Y=y)$
		-2	1	
$Z=z$	-2	1/9	2/9	1/3
	1	2/9	4/9	2/3
$P(Z=z)$		1/3	2/3	

Consider

$$E(Y)=E(Z)=(-2)\frac{1}{3}+(1)\frac{2}{3}=0$$

$$E(Y^2)=E(Z^2)=(-2)^2\frac{1}{3}+(1)^2\frac{2}{3}=2$$

$$\therefore \quad V(Y)=E(Y^2)-\{E(Y)\}^2=2-0=2 \quad \text{and}$$

$$V(Z)=E(Z^2)-\{E(Z)\}^2=2-0=2$$

Since Y and Z are independent random variables, we have

$$E(YZ)=\sum_{y=-2,1}\sum_{z=-2,1} yzP(Y=y,Z=z)$$

$$=\sum_{y=-2,1}\sum_{z=-2,1} yzP(Y=y)P(Z=z)$$

$$E(YZ)=(-2)(-2)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)+(-2)(1)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)+(1)(-2)$$

$$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)+(1)(1)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)=0$$

Consider $E\{X(t)\}=E(Y\cos t+Z\sin t)=\cos tE(Y)+\sin tE(Z)=0$ (a constant)

$$R(t_1,t_2)=E\{X(t_1)X(t_2)\}$$

$$=E\{(Y\cos t_1+Z\sin t_1)(Y\cos t_2+Z\sin t_2)\}$$

$$= \cos t_1 \cos t_2 E(Y^2) + (\cos t_1 \sin t_2 + \sin t_1 \cos t_2) E(YZ)$$

$$+ \sin t_1 \sin t_2 E(Z^2)$$

$$= 2(\cos t_1 \cos t_2 + \sin t_1 \sin t_2) + (\cos t_1 \sin t_2 + \sin t_1 \cos t_2) E(YZ)$$

$$= 2\cos(t_1 - t_2) + \sin(t_2 + t_1) E(YZ)$$

$$= 2\cos(t_1 - t_2) = 2\cos(t_2 - t_1) = 2\cos\tau$$

Since $E\{X(t)\} = 0$ is constant and autocorrelation function $R(t_1, t_2) = 2\cos\tau$ is a function of the time difference, the given random process $\{X(t)\}$ is a WSS process.

Consider

$$E\left\{X^2(t)\right\} = E(Y\cos t + Z\sin t)^2$$

$$= E\left\{Y^2\cos^2 t + 2YZ\cos t\sin t + Z^2\sin^2 t\right\}$$

$$= \cos^2 tE(Y^2) + 2\cos t\sin tE(YZ) + \sin^2 tE(Z^2) = 2$$

$$\therefore \quad V\{X(t)\} = E\left\{X^2(t)\right\} - \{E[X(t)\}^2 = 2 - 0 = 2$$

Or

Consider

$V\{X(t)\} = V(Y\cos t + Z\sin t) = \cos^2 tV(Y) + \sin^2 tV(Z) = 2$ (a constant)

Now, consider

$$E\left\{X^3(t)\right\} = E(Y\cos t + Z\sin t)^3$$

$$= E\left\{Y^3\cos^3 t + Y^2Z\cos^2 t\sin t + YZ^2\cos t\sin^2 t + Z^3\sin^3 t\right\}$$

$$= \cos^3 tE(Y^3) + \cos^2 t\sin tE(Y^2Z) + \cos t\sin^2 tE(YZ^2)$$

$$+ \sin^3 tE(Z^3)$$

But

$$E(Y^3) = E(Z^3) = (-2)^3\frac{1}{3} + (1)^3\frac{2}{3} = -2$$

$$E(Y^2Z) = E(Y^2)E(Z) = 0 \text{ and } E(YZ^2) = E(Y)E(Z^2) = 0$$

(since Y and Z are independent)

$$\therefore \quad E\left\{X^3(t)\right\} = -2(\cos^3 t + \sin^3 t)$$

Which is not time invariant as it depends on the time t. But by definition, for a random process to be stationary in strict sense, all the moments must be independent of time. Therefore, the process $\{X(t)\}$ is not an SSS process.

Problem 2. If $\{X_n,\ n \geq 0\}$is a sequence of identically and independently distributed (iid) random variables, each with mean 0 and variance 1, then show that the sequence $\{X_n,\ n \geq 0\}$ is wide sense stationary.

SOLUTION:

It is given that $E(X_n) = 0, \quad \forall n$

The autocorrelation function is given by

$$R_n(n, n+s) = \begin{cases} E(X_n X_{n+s}), & \text{for} \quad s \neq 0 \\ E(X_n^2), & \text{for} \quad s = 0 \end{cases}$$

Since $X_n^{'}$s are independent and, $E(X_n) = 0, \quad \forall n$ we have

$$R_n(n, n+s) = \begin{cases} E(X_n)E(X_{n+s}) = 0, & \text{for} \quad s \neq 0 \\ E(X_n^2) = V(X_n) = 1, & \text{for} \quad s = 0 \end{cases}$$

Clearly the autocorrelation function $R_n(n, n+s)$ depends on s, the step difference only. Since mean of the sequence $\{X_n,\ n \geq 0\}$ is constant and the autocorrelation function $R_n(n,\ n+s)$ depends only on the step difference but not on the steps, we conclude that $\{X_n,\ n \geq 0\}$ is a wide sense stationary.

Problem 3. A random process $\{X(t)\}$ has the probability distribution

$$P\{X(t) = x)\} = \begin{cases} \dfrac{(at)^{x-1}}{(1+at)^{x+1}}, & x = 1, 2, 3.... \\ \dfrac{at}{1+at}, & x = 0 \end{cases}.$$

Show that the process is evolutionary (that is, non-stationary).

SOLUTION:

The probability distribution of $\{X(t)\}$ can be represented as follows:

$X(t) = x$	0	1	2	3	....
$P\{X(t) = x\}$	$\dfrac{at}{1+at}$	$\dfrac{1}{(1+at)^2}$	$\dfrac{at}{(1+at)^3}$	$\dfrac{(at)^2}{(1+at)^4}$	....

We know that by definition of expectation

$$E\{X(t)\} = \sum_{x=0}^{\infty} xP\{X(t)=x\}$$

$$= 0 + \frac{1}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3\,(at)^2}{(1+at)^3} + \cdots\cdots$$

$$= \frac{1}{(1+at)^2}\left\{1+2\left(\frac{at}{1+at}\right)+3\left(\frac{at}{1+at}\right)^2+\cdots\cdots\right\}$$

$$= \frac{1}{(1+at)^2}\left\{1-\left(\frac{at}{1+at}\right)\right\}^{-2} = 1$$

$$E\left\{X^2(t)\right\} = \sum_{x=0}^{\infty} x^2 P\{X(t)=x\}$$

$$= \sum_{x=0}^{\infty} x(x+1)P\{X(t)=x\} - \sum_{x=0}^{\infty} xP\{X(t)=x\}$$

Now, consider

$$\sum_{x=0}^{\infty} x(x+1)P\{X(t)=x\} = 0 + (1)(2)\frac{1}{(1+at)^2} + (2)(3)\frac{at}{(1+at)^3}$$

$$+ (3)(4)\frac{(at)^2}{(1+at)^3} + \cdots\cdots$$

$$= \frac{1}{(1+at)^2}\left\{(1)(2)+(2)(3)\left(\frac{at}{1+at}\right)+(3)(4)\left(\frac{at}{1+at}\right)^2+\cdots\cdots\right\}$$

$$= \frac{1}{(1+at)^2}(2)\left\{1-\left(\frac{at}{1+at}\right)\right\}^{-3} = 2(1+at)$$

$$\therefore \quad E\left\{X^2(t)\right\} = 2(1+at) - 1 = 1 + 2at$$

We know that by definition of variance

$$V\{X(t)\} = E\left\{X^2(t)\right\} - \{E[X(t)]\}^2 = 1 + 2at - (1)^2 = 2at$$

Though $E\{X(t)\}$ is constant, $V\{X(t)\}$ is not time invariance as it depends on t. Therefore, the given random process $\{X(t)\}$ is not stationary and is evolutionary.

Problem 4. If $\{X(t)\}$ is a wide sense stationary process with autocorrelation function $R(\tau) = 4e^{-2|\tau|}$ then find the second moment of the random variable $Z = X(t+\tau) - X(t)$.

SOLUTION:
We know that the second moment of the random variable $Z = X(t+\tau) - X(t)$ is given by

$$E(Z^2) = E\left\{[X(t+\tau) - X(t)]^2\right\}$$

Now consider

$$E\left\{[X(t+\tau) - X(t)]^2\right\} = E\left\{X^2(t+\tau) + X^2(t) - 2X(t+\tau)X(t)\right\}$$

Since the given random process $\{X(t)\}$ is a WSS process we know that

$$E\left\{X^2(t+\tau)\right\} = E\left\{X^2(t)\right\} = R(0)$$

and

$$E\{X(t+\tau)X(t)\} = R(\tau)$$

Therefore, we have

$$\begin{aligned} E\left\{[X(t+\tau) - X(t)]^2\right\} &= R(0) + R(0) - 2R(\tau) \\ &= 4 + 4 - 2\left(4e^{-2|\tau|}\right) = 8\left(1 - e^{-2|\tau|}\right) \end{aligned}$$

Problem 5. Consider a random process $\{X(t)\}$ such that $X(t) = A\cos(\omega t + \theta)$ where A and ω are constants and θ is a uniform random variable distributed in the interval $(-\pi, \pi)$. Check whether the process $\{X(t)\}$ is a stationary process in wide sense.

SOLUTION:
Given $X(t) = A\cos(\omega t + \theta)$
Since θ is a uniform random variable distributed in the interval $(-\pi, \pi)$, we have the PDF of θ as

$$f(\theta) = \frac{1}{2\pi}, \qquad -\pi \le \theta \le \pi$$

Consider

$$E\{X(t)\} = E[A\cos(\omega t+\theta)] = \int_{-\pi}^{\pi} A\cos(\omega t+\theta) f(\theta)d\theta$$

$$= \int_{-\pi}^{\pi} A\cos(\omega t+\theta)\frac{1}{2\pi}d\theta$$

$$= \frac{A}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t+\theta)d\theta$$

$$= \frac{A}{2\pi}[\sin(\omega t+\theta)]_{-\pi}^{\pi}$$

$$= \frac{A}{2\pi}[\sin(\omega t+\pi)-\sin(\omega t-\pi)] = 0$$

Consider

$$R_{xx}(t_1,t_2) = E\{X(t_1)X(t_2)\} = E\{A\cos(\omega t_1+\theta)A\cos(\omega t_2+\theta)\}$$

$$= A^2E\{\cos(\omega t_1+\theta)\cos(\omega t_2+\theta)\}$$

$$= \frac{A^2}{2}E\{\cos\omega(t_1-t_2)+\cos[\omega(t_1+t_2)+2\theta]\}$$

$$= \frac{A^2}{2}\{E\{\cos\omega(t_1-t_2)\}+E\{\cos[\omega(t_1+t_2)+2\theta]\}\}$$

$$= \frac{A^2}{2}\left\{\int_{-\pi}^{\pi}\cos\omega(t_1-t_2)f(\theta)d\theta+\int_{-\pi}^{\pi}\cos[\omega(t_1+t_2)+2\theta]f(\theta)d\theta\right\}$$

$$= \frac{A^2}{2}\left\{\int_{-\pi}^{\pi}\cos\omega(t_1-t_2)\frac{1}{2\pi}d\theta+\int_{-\pi}^{\pi}\cos[\omega(t_1+t_2)+2\theta]\frac{1}{2\pi}d\theta\right\}$$

$$= \frac{A^2}{2}\cos\omega(t_1-t_2)-\frac{A^2}{8\pi}|\sin[\omega(t_1+t_2)+2\theta]|_{-\pi}^{\pi}$$

$$= \frac{A^2}{2}\cos\omega(t_1-t_2) = \frac{A^2}{2}\cos\omega\tau$$

Note that

$$|\sin[\omega(t_1+t_2)+2\theta]|_{-\pi}^{\pi} = \sin[\omega(t_1+t_2)+2\pi]$$
$$-\sin[\omega(t_1+t_2)-2\pi] = 0$$

$$\therefore \quad R_{xx}(t_1,t_2) = \frac{A^2}{2}\cos\omega\tau = R(\tau)$$

Since mean of the random process $\{X(t)\}$ is constant and autocorrelation function is invariant of time, $R(t_1,t_2) = R(\tau)$, the process $\{X(t)\}$ is stationary in wide sense.

Problem 6. If $R(\tau)$ is the autocorrelation function of a wide sense stationary process $\{X(t)\}$ with zero mean, then using Chebyshev's inequality show that

$$P\{|X(t+\tau)-X(t)| \geq \varepsilon\} \leq 2\{R(0)-R(\tau)\}/\varepsilon^2 \quad \text{for some } \varepsilon > 0.$$

SOLUTION:
If X is a random variable, then we know that by Chebyshev's theorem,

$$P\{|X-E(X)| \geq \varepsilon\} \leq V(X)/\varepsilon^2$$

for some $\varepsilon > 0$. Accordingly, we have

$$P\{|[X(t+\tau)-X(t)]-E[X(t+\tau)-X(t)]| \geq \varepsilon\} \leq V[X(t+\tau)-X(t)]/\varepsilon^2$$

Consider $V[X(t+\tau)-X(t)] = V\{X(t+\tau)\}+V\{X(t)\}-2Cov(t+\tau,\, t)$

Since $\{X(t)\}$ is a wide sense stationary process with zero mean, we have

$$V[X(t+\tau)-X(t)] = E\left\{X^2(t+\tau)\right\}+E\left\{X^2(t)\right\}-2R(\tau)$$

$$V[X(t+\tau)-X(t)] = R(0)+R(0)-2R(\tau) = 2[R(0)-R(\tau)]$$

$$\therefore \quad P\{|X(t+\tau)-X(t)| \geq \varepsilon\} \leq 2[R(0)-R(\tau)]/\varepsilon^2$$

Problem 7. If $\{X(t)\}$ is a stationary random process with mean μ and autocorrelation function $R_{xx}(\tau)$ and if S a random variable such that $S = \int_a^b X(t)\,dt$ then find (i) mean and (ii) variance of S.

SOLUTION:

(i) ***Mean:***
If S a random variable, then its mean is nothing but the expected value, that is, $E(S)$.
Consider $E(S) = E\left\{\int_a^b X(t)\,dt\right\} = \int_a^b \mu\,dt = (b-a)\mu \quad \because \; E\{X(t)\} = \mu$

(ii) ***Variance:***
Variance of the random variable S is given by $V(S) = E(S^2) - \{E(S)\}^2$
Consider

$$E(S^2) = E\left\{\int_a^b X(t)\,dt\right\}^2 = E\left\{\left(\int_a^b X(t_1)\,dt_1\right)\left(\int_a^b X(t_2)\,dt_2\right)\right\}$$

$$= E\left\{\int_a^b\int_a^b X(t_1)X(t_2)dt_1dt_2\right\}$$

(Refer Appendix A: Result A.3.3)

$$= \int_a^b\int_a^b E\{X(t_1)X(t_2)\}\,dt_1dt_2$$

$$= \int_a^b\int_a^b R_{xx}(t_1,t_2)dt_1dt_2$$

$$= \int_{2a}^{2b} R_{xx}(\tau)[(b-a)-|\tau|]d\tau$$

(Refer Appendix A: Result A.4.1)

$$V(S) = \int_a^b R_{xx}(\tau)[(b-a)-|\tau|]d\tau - [(b-a)\mu]^2$$

Problem 8. If $\{Z(t)\}$ is a random process defined *by* $Z(t) = Xt + Y$ where X and Y are a pair of random variables with means μ_x and μ_y, variances σ_x^2 and σ_y^2 respectively, and correlation coefficient ρ_{xy}. Find (i) mean (ii) variance (iii) autocorrelation and (iv) autocovariance of $\{Z(t)\}$ under the assumption that: *Case (i):* X and Y are not independent and *Case (ii):* X and Y are independent. Verify whether the process $\{Z(t)\}$ is stationary.

SOLUTION:

Case (i): When X and Y are not independent.
It is given that

$$E(X) = \mu_X,\ E(Y) = \mu_Y,\ V(X) = \sigma_X^2 \text{ and } V(Y) = \sigma_Y^2$$

The correlation coefficient between X and Y is ρ_{xy}

(i) Mean of $\{Z(t)\}$

That is, $E\{Z(t)\} = E(Xt+Y) = tE(X)+E(Y) = t\mu_X + \mu_Y$

(ii) Variance of $\{Z(t)\}$

$$V\{Z(t)\} = V(Xt+Y) = V(tX)+V(Y)+2t\,\text{Cov}(X,Y)$$

$$= t^2V(X)+V(Y)+2\rho_{XY}\sigma_X\sigma_Y \quad \because \quad \rho_{XY} = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y}$$

$$= t^2\sigma_X^2+\sigma_Y^2+2\rho_{XY}\sigma_X\sigma_Y$$

(iii) Autocorrelation of $\{Z(t)\}$

$$R_{zz}(t_1,t_2) = E\{Z(t_1)Z(t_2)\} = E\{(Xt_1+Y)(Xt_2+Y)\}$$

$$= E\left\{(X^2t_1t_2+Xt_1Y+YXt_2+Y^2)\right\}$$

$$= t_1t_2E(X^2)+t_1E(XY)+t_2E(YX)+E(Y^2)$$

$$= t_1t_2(\sigma_X^2+\mu_X^2)+(t_1+t_2)\{\text{Cov}(X,Y)+\mu_X\mu_Y\}$$
$$+(\sigma_Y^2+\mu_Y^2)$$

$$= t_1t_2(\sigma_X^2+\mu_X^2)+(t_1+t_2)(\rho_{XY}\sigma_X\sigma_Y+\mu_X\mu_Y)$$
$$+(\sigma_Y^2+\mu_Y^2)$$

(iv) Autocovariance of $\{Z(t)\}$

$$C_{zz}(t_1,t_2) = R_{zz}(t_1,t_2) - E\{Z(t_1)\}E\{Z(t_2)\}$$

$$= t_1t_2(\sigma_X^2+\mu_X^2)+(t_1+t_2)(\rho_{XY}\sigma_X\sigma_Y+\mu_X\mu_Y)$$
$$+(\sigma_Y^2+\mu_Y^2)-(t_1\mu_X+\mu_Y)(t_2\mu_X+\mu_Y)$$

$$= t_1t_2\sigma_X^2+(t_1+t_2)\rho_{XY}\sigma_X\sigma_Y+\sigma_Y^2$$

Note: Variance of $\{Z(t)\}$ may also be obtained by letting $t_1 = t_2 = t$ in $C_{zz}(t_1,t_2)$ given in (iv).

Case (ii): When X and Y are independent.

If X and Y are independent random variables, then we have $E(XY) = E(X)V(Y)$ and $\rho_{XY} = \dfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = 0$, therefore we have the results

$$V\{Z(t)\} = t^2\sigma_X^2 + \sigma_Y^2$$

$$R_{zz}(t_1,t_2) = t_1t_2(\sigma_X^2+\mu_X^2) + (t_1+t_2)\mu_X\mu_Y + (\sigma_Y^2+\mu_Y^2)$$

$$C_{zz}(t_1,t_2) = t_1t_2\sigma_X^2 + \sigma_Y^2$$

Since mean, variance, autocorrelation and covariance are all not time invariant, the random process $\{Z(t)\}$ is not a stationary process.

Problem 9. Consider the random variable Y with characteristic function $\phi(\omega) = E\{e^{i\omega Y}\} = E\{\cos\omega Y + i\sin\omega Y\}$ and a random process $\{X(t)\}$ defined by $X(t) = \cos(at+Y)$. Show that $\{X(t)\}$ is a stationary process in wide sense if $\phi(1) = \phi(2) = 0$.

Solution:

Given $\phi(1) = 0 \Rightarrow E\{\cos Y + i\sin Y\} = 0 \Rightarrow E(\cos Y) = E(\sin Y) = 0$

Given $\phi(2) = 0 \Rightarrow E\{\cos 2Y + i\sin 2Y\} = 0 \Rightarrow E(\cos 2Y) = E(\sin 2Y) = 0$

Consider

$$\begin{aligned} E\{X(t)\} &= E\{\cos(at+Y)\} \\ &= E\{\cos at\cos Y - \sin at\sin Y\} \\ &= \cos atE(\cos Y) - \sin atE(\sin Y) = 0 \end{aligned}$$

$$\begin{aligned} R_{xx}(t_1,t_2) &= E\{X(t_1)X(t_2)\} = E\{\cos(at_1+Y)\cos(at_2+Y)\} \\ &= E\left\{\frac{\cos[a(t_1+t_2)+2Y] + \cos[a(t_2-t_1)]}{2}\right\} \end{aligned}$$

Consider

$$\begin{aligned} \frac{1}{2}E\{\cos[a(t_1+t_2)+2Y]\} &= \frac{1}{2}E\{\cos a(t_1+t_2)\cos 2Y - \sin a(t_1+t_2)\sin 2Y\} \\ &= \frac{1}{2}\{\cos a(t_1+t_2)E(\cos 2Y) - \sin a(t_1+t_2)E(\sin 2Y)\} \end{aligned}$$

Given $\phi(2) = 0 \Rightarrow E\{\cos 2Y + i\sin 2Y\} = 0 \Rightarrow E(\cos 2Y) = E(\sin 2Y) = 0$

$$\Rightarrow \quad \frac{1}{2}\{\cos a(t_1+t_2)E(\cos 2Y) - \sin a(t_1+t_2)E(\sin 2Y)\} = 0$$

Consider

$$\frac{1}{2}E\{\cos[a(t_2-t_1)]\} = \frac{1}{2}\cos[a(t_2-t_1)]$$
$$\therefore \quad R(t_1,t_2) = \frac{1}{2}\cos[a(t_2-t_1)] = \frac{1}{2}\cos[a(\tau)] \quad [\because \quad \tau = t_2 - t_1]$$

Since mean of the random process $\{X(t)\}$ is constant and autocorrelation is time invariant we conclude that the process is a stationary process in wide sense.

Problem 10. Two random processes $\{X(t)\}$ and $\{Y(t)\}$ are defined by

$$X(t) = A\cos\omega t + B\sin\omega t$$
$$Y(t) = B\cos\omega t - A\sin\omega t$$

Then show that $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary, if A and B are uncorrelated random variables with zero means and equal variances and ω is a constant.

SOLUTION:

$$E(A) = E(B) = 0$$
$$V(A) = V(B) = \sigma^2 \ (say) \Rightarrow E(A^2) = E(B^2) = 0$$

Since A and B are uncorrelated random variables, we have $E(AB) = 0$

Consider

$$E\{X(t)\} = E\{A\cos\omega t + B\sin\omega t\}$$
$$= \cos\omega t E(A) + \sin\omega t E(B) = 0$$

Consider

$$R_{xx}\{t_1, t_2\} = E\{(A\cos\omega t_1 + B\sin\omega t_1)(A\cos\omega t_2 + B\sin\omega t_2)\}$$
$$= \cos\omega t_1 \cos\omega t_2 E(A^2) + \sin\omega t_1 \sin\omega t_2 E(B^2)$$
$$+\{\cos\omega t_1 \sin\omega t_2 + \cos\omega t_1 \cos\omega t_2\} E(AB)$$
$$= \{\cos\omega t_1 \cos\omega t_2 + \sin\omega t_1 \sin\omega t_2\}\,\sigma^2$$

$$= \cos(\omega t_1 - \omega t_2)\sigma^2 = \sigma^2 \cos\omega(t_1 - t_2)$$

$$= \sigma^2 \cos\omega\tau \quad [\text{with} \quad \tau = t_1 - t_2]$$

On similar lines, we can show that

$$R_{yy}(t_1, t_2) = \sigma^2 \cos\omega\tau$$

Since $E\{X(t)\} = 0$ is constant and $R_{xx}(t_1, t_2)$ is time invariant, the random processes $\{X(t)\}$ and $\{Y(t)\}$ are individually wide sense processes.

Now consider

$$R_{xy}\{t_1, t_2\} = E\{(A\cos\omega t_1 + B\sin\omega t_1)(B\cos\omega t_2 - A\sin\omega t_2)\}$$

$$= \sin\omega t_1 \cos\omega t_2 E(B^2) - \cos\omega t_1 \sin\omega t_2 E(a^2)$$

$$+ \{\cos\omega t_1 \cos\omega t_2 - \sin\omega t_1 \sin\omega t_2\} E(AB)$$

$$= \{\sin\omega t_1 \cos\omega t_2 - \cos\omega t_1 \sin\omega t_2\}\sigma^2$$

$$= \sin(\omega t_1 - \omega t_2)\sigma^2 = \sigma^2 \sin\omega(t_1 - t_2)$$

$$= \sigma^2 \cos\omega\tau \quad [\textit{with} \quad \tau = t_1 - t_2]$$

Here, $R_{xy}(t_1, t_2)$ is time invariant.

Therefore, since the random processes $\{X(t)\}$ and $\{Y(t)\}$ are individually wide sense processes and $R_{xx}(t_1, t_2)$ is time invariant, by definition, we conclude that $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide sense stationary processes.

EXERCISE PROBLEMS

1. The random process $\{X(t)\}$ is defined as $X(t) = 2e^{-At}\sin(\omega t + B)u(t)$, where $u(t)$ is the unit step function and the random variables A and B are independent where A is uniformly distributed in $(0, 2)$ and B is uniformly distributed in $(-\pi, \pi)$. Verify whether the process is wide sense stationary.
2. Let $\{X(t)\}$ be random process such that $X(t) = \sin(\omega t + \theta)$ is a sinusoidal signal with random phase θ which is a uniform random variable in the interval $(-\pi, \pi)$. If both time t and the radial frequency ω are constants, then find the probability density function of the random variable $X(t)$ at $t = t_0$. Also comment on the dependence of the probability density function of $X(t)$ and on the stationarity of the process $\{X(t)\}$.
3. Consider the random process $\{X(t)\}$ with $X(t) = A(t)\cos(2\pi + \theta)$, where the amplitude $A(t)$ is a zero-mean wide sense stationary process with autocorrelation function $R_A(\tau) = e^{-|\tau|/2}$, the phase θ is a uniform random variable in the interval $(0, 2\pi)$, and $A(t)$ and θ are independent. Is $\{X(t)\}$ a wide sense stationary process? Justify you answer.

4. A random process $\{X(t)\}$ has the sample functions of the form $X(t) = A\cos(\omega t+\theta)$ where ω is a constant, A is a random variable that has magnitude $+1$ and -1 with equal probabilities, and θ is a random variable that is uniformly distributed in $(0, 2\pi)$. Assume that A and θ are independent.
   (i) Find mean and variance of the random process $\{X(t)\}$.
   (ii) Is $\{X(t)\}$ first order strict sense stationary? Give reason for your answer.
   (iii) Find the autocorrelation function of $\{X(t)\}$.
   (iv) Is $\{X(t)\}$ wide-sense stationary? Give reasons for your answer.
   (v) Plot the sample functions of $\{X(t)\}$ when $A = \pm 1$, $t = 1$, $\omega = 2$, $\theta = 2\pi$.
5. Consider a random process $\{Y(t)\}$ such that $Y(t) = X(t)\cos(\omega t+\theta)$, where ω is a constant, $\{X(t)\}$ is a wide sense stationary random process, θ is a uniform random variable in the interval $(-\pi, \pi)$ and is independent of $X(t)$. Show that $\{Y(t)\}$ is also a wide sense stationary process.
6. If $\{X(t)\}$ is a random process with $X(t) = A\cos(\omega t+\theta)$ where ω is a constant, A is a random variable that has magnitude $+1$ and -1 with equal probabilities, and θ is a random variable that is uniformly distributed in $(0, 2\pi)$. Assume that A and θ are independent. The autocorrelation of the process $\{X(t)\}$ is given as $R(\tau) = \frac{A^2}{2}\cos\omega\tau$. Plot the sample function and autocorrelation when $A = +1$, $t = (0, 10)$, $\omega = 2$, $\theta = \pi$.
7. If $\{X(t)\}$ is a stationary random process and $\{Y(t)\}$ is another random process such that $Y(t) = X(t+a)$ where a is an arbitrary constant, then verify whether the process $\{Y(t)\}$ is stationary.
8. Let $\{X(t)\}$ and $\{Y(t)\}$ be two independent wide sense stationary processes with expected values μ_x and μ_y and autocorrelation functions $R_{xx}(\tau)$ and $R_{yy}(\tau)$ respectively. Let $W(t) = X(t)Y(t)$, then find μ_w and $R_{ww}(t, t+\tau)$. Also, verify whether $\{X(t)\}$ and $\{W(t)\}$ are jointly wide sense stationary.
9. Let $\{X(t)\}$ be a wide sense stationary random process. Verify whether the processes $\{Y(t)\}$ and $\{Z(t)\}$ defined below are wide sense stationary. Also, determine whether $\{Z(t)\}$ and each of the other two processes are jointly wide sense stationary.
   (i) $Y(t) = X(t+a)$
   (ii) $Z(t) = X(at)$
10. Consider the random process $\{X(t)\}$ such that $X(t) = \cos(\omega t+\theta)$ where θ is a uniform random variable in the interval $(-\pi, \pi)$. Show that first and second order moments of $\{X(t)\}$ are independent of time. Also find variance of $\{X(t)\}$.

random variables of the process at two time points t_1 and t_2, then the autocorrelation of the process $\{X(t)\}$ denoted by $R_{xx}(t_1, t_2)$ is obtained as the expected value of the product of $X(t_1)$ and $X(t_2)$. That is,

$$R_{xx}(t_1, t_2) = E\{X(t_1)X(t_2)\} \tag{4.1}$$

Also, if $\{X_1(t)\}$ and $\{X_2(t)\}$ are two random processes observed over a period of time $(0, t)$ and $X_1(t_1)$ and $X_2(t_2)$ are the two random variables respectively of the process $\{X_1(t)\}$ at the time point t_1 and $X_2(t_2)$ at the time point t_2 then (4.1) is given as

$$R_{x_1x_2}(t_1, t_2) = E\{X_1(t_1)X_2(t_2)\} \tag{4.2}$$

If $\{X(t)\}$ is a stationary process, then we know that the autocorrelations given in (4.1) and (4.2) are time invariant and hence are the functions of the time difference, say $|t_1 - t_2| = \tau$, only for some $\tau > 0$. This implies

$$R_{xx}(t_1, t_2) = E\{X(t_1)X(t_2)\} = R(t_2 - t_1) = R_{xx}(\tau) \tag{4.3}$$

Therefore, for stationary process $\{X(t)\}$ with $t_1 = t$ and $t_2 = t + \tau$, we have

$$R_{xx}(t_1, t_2) = E\{X(t)X(t+\tau)\} = R(t + \tau - t) = R_{xx}(\tau) \tag{4.4}$$

And for two stationary processes $\{X_1(t)\}$ and $\{X_2(t)\}$, we have

$$R_{x_1x_2}(t_1, t_2) = E\{X_1(t)X_2(t+\tau)\} = R_{x_1x_2}(\tau) \tag{4.5}$$

Clearly, when $\tau = 0$, from (4.3), we have

$$R_{xx}(0) = E\{X(t)X(t)\} = E\left\{X^2(t)\right\} \tag{4.6}$$

As discussed in previous chapter, $E\left\{X^2(t)\right\}$ this is known as the *average power* of the process $\{X(t)\}$.

From (4.4), we have

$$R_{x_1x_2}(0) = E\{X_1(t)X_2(t)\} \tag{4.7}$$

ILLUSTRATIVE EXAMPLE 4.1

Consider a random process (sinusoidal with random phase) $\{X(t)\}$ where $X(t) = a\sin(\omega t + \theta)$. Here a and ω are constants and the random variable θ is uniform in the interval $(0, 2\pi)$. Then the autocorrelation function of the process can be obtained as

$$R_{xx}(t, t+\tau) = E\{X(t)X(t+\tau)\} = E\{a\sin(\omega t+\theta)\, a\sin[\omega(t+\tau)+\theta]\}$$

$$= \frac{a^2}{2}E\{\cos\omega t - \cos[\omega(t+\tau)+\theta]\}$$

$$\Rightarrow \quad R_{xx}(\tau) = \frac{a^2}{2}E(\cos\omega\tau) - \frac{a^2}{2}E\{\cos[\omega(2t+\tau)+2\theta]\}$$

Since θ is uniform in the interval $(0, 2\pi)$, we have

$$f(\theta) = \frac{1}{2\pi}, \quad 0 \le \theta \le 2\pi$$

$$\therefore \quad R_{xx}(\tau) = \frac{a^2}{2}\int_0^{2\pi} \cos\omega\tau \frac{1}{2\pi}\, d\theta - \frac{a^2}{2}\int_o^{2\pi} \cos[\omega(2t+\tau)+2\theta]\frac{1}{2\pi}d\theta$$

$$= \frac{a^2}{2}\cos\omega\tau - \frac{a^2}{2}\int_o^{2\pi}\cos[\omega(2t+\tau)+2\theta]\frac{1}{2\pi}d\theta$$

$$= \frac{a^2}{2}\cos\omega\tau$$

Since the second term integrates to zero.

Now without loss of generality, let us assume $a = 1$ and $\omega = 2$. Since θ is uniform in the interval $(0, 2\pi)$ let us assume $\theta = 1$. Then the plots of the process $X(t)$ and its autocorrelation function $R_{xx}(\tau)$ can be given as shown in Figures 4.1 and 4.2. Note that both $X(t)$ and $R_{xx}(\tau)$ are periodic.

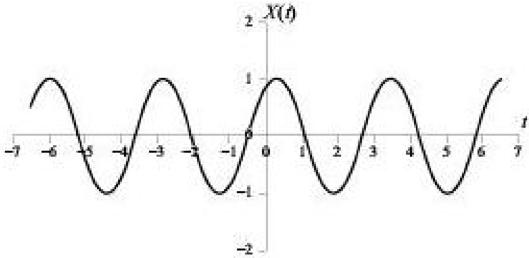


Figure 4.1. Graphical representation of $X(t) = (1)\sin(2t+1)$

process, this autocorrelation function holds some important properties which are given below:

Property 4.1: Autocorrelation function $R_{XX}(\tau)$ of a stationary random process $\{X(t)\}$ is an even function. That is, $R_{XX}(\tau) = R_{XX}(-\tau)$.

Proof. It is known that given two time points t and $t+\tau$, then the autocorrelation of the stationary random process $\{X(t)\}$ is given by

$$\begin{aligned} R_{XX}(\tau) &= E\{X(t)X(t+\tau)\} \\ &= E\{X(t+\tau)X(t)\} = R_{XX}(-\tau) \end{aligned} \tag{4.8}$$

Property 4.2: Autocorrelation function $R_{XX}(\tau)$ of a stationary random process $\{X(t)\}$ is maximum at $\tau = 0$. That is $|R_{XX}(\tau)| \leq R_{XX}(0)$.

Proof. This can be proved with the help of *Cauchy-Schwarz inequality* (Refer to Equation 1.15 of Chapter 1). That is if X and Y are two random variables, then

$$\{E(XY)\}^2 \leq E(X^2)E(Y^2) \tag{4.9}$$

Now, consider

$$\{E[X(t)X(t+\tau)]\}^2 \leq E[X^2(t)]E[X^2(t+\tau)]$$

$$\{R_{XX}(\tau)\}^2 \leq \left\{E[X^2(t)]\right\}^2$$

Since $\{X(t)\}$ is a stationary process, it has a constant mean, implying that $E[X(t)] = E[X(t+\tau)]$

$$\therefore \quad \{R_{XX}(\tau)\}^2 \leq \{R_X(0)\}^2 \quad \because \quad E\left\{X^2(t)\right\} = R_{XX}(0)$$

$$\Rightarrow \quad |R_{XX}(\tau)| \leq R_{XX}(0) \tag{4.10}$$

Property 4.3: If $R_{XX}(\tau)$ is the autocorrelation function of a stationary random process $\{X(t)\}$, then the mean of the process, say $E[X(t)] = \mu_X$, can be obtained as

$$\mu_X = \sqrt{\lim_{\tau \to \infty} R_{XX}(\tau)}$$

Proof. We know that autocorrelation function $R_{XX}(\tau)$ of a stationary random process $\{X(t)\}$ is given by

$$R_{XX}(\tau) = E\{X(t)X(t+\tau)\}$$

It may be noted that as $\tau \to \infty$, $X(t)$ and $X(t+\tau)$ become independent and therefore, we have

$$\lim_{\tau \to \infty} R_{xx}(\tau) = E\{X(t)\}\, E\{X(t+\tau)\}$$

Since $\{X(t)\}$ is a stationary process, it has a constant mean, implying that $E[X(t)] = E[X(t+\tau)]$ and hence we have

$$\lim_{\tau \to \infty} R_{xx}(\tau) = \{E[X(t)]\}^2 = \mu_x^2$$

$$\Rightarrow \quad \mu_x = \sqrt{\lim_{\tau \to \infty} R_{xx}(\tau)} \tag{4.11}$$

Property 4.4: The autocorrelation function $R_{xx}(\tau)$ of a stationary random process $\{X(t)\}$ is periodic with period h, for some $h \neq 0$, if $R_{xx}(h) = R_{xx}(0)$, or otherwise $R_{xx}(\tau + h) = R_{xx}(\tau)$.

Proof. If we consider $\{E[X(t+\tau+h) - X(t+\tau)]X(t)\}^2$

Then according to Schwarz's inequality,

$$\{E[X(t+\tau+h) - X(t+\tau)]X(t)\}^2 \leq E[X(t+\tau+h) - X(t+\tau)]^2\, E[X^2(t)]$$

On simplification we have

$$\{R_{xx}(\tau+h) - R_{xx}(\tau)\}^2 \leq 2[R(0) - R_{xx}(h)]R_{xx}(0) \tag{4.12}$$

If we let $R_{xx}(h) = R_{xx}(0)$, then the right-hand side of (4.12) becomes zero and obviously, the left-hand size of (4.12) also becomes zero for every τ which yields the result that

$$R_{xx}(\tau + h) = R_{xx}(\tau) \tag{4.13}$$

Property 4.5: If the autocorrelation function $R_{xx}(\tau)$ of a stationary random process $\{X(t)\}$ is continuous at $\tau = 0$, then it is continuous for all τ.

Proof. If the autocorrelation function $R_{xx}(\tau)$ is continuous at $\tau = 0$, then for any $h \neq 0$, it follows that $R_{xx}(h) \to R_{xx}(0)$.

This implies that from (4.13), we have

$$R_{xx}(\tau + h) - R_{xx}(\tau) \to 0 \quad \text{for every } \tau \text{ as } h \to 0. \tag{4.14}$$

Property 4.8: If $\{X_1(t)\}$ and $\{X_2(t)\}$ are two stationary random processes, then $|R_{x_1x_2}(\tau)| \leq \frac{1}{2}\{R_{x_1x_1}(0) + R_{x_2x_2}(0)\}$.

Proof. Let $\{X_1(t)\}$ and $\{X_2(t)\}$ be the two stationary random processes with autocorrelation functions $R_{x_1x_1}(\tau)$ and $R_{x_2x_2}(\tau)$ respectively and let $R_{x_1x_2}(\tau)$ be their cross-correlation function. Now consider

$$E\{X_1(t) - X_2(t+\tau)\}^2 = E[X_1^2(t)] + E[X_2^2(t+\tau)] - 2X_1(t)X_2(t+\tau)$$
$$= R_{x_1x_1}(0) + R_{x_2x_2}(0) - 2R_{x_1x_2}(\tau)$$
$$\Rightarrow \quad 2R_{x_1x_2}(\tau) = R_{x_1x_1}(0) + R_{x_2x_2}(0) - E\{X_1(t) - X_2(t+\tau\}^2$$

Eliminating the third term $-E\{X_1(t) - X_2(t+\tau\}^2$ from right-hand side, we have

$$2R_{x_1x_2}(\tau) \leq R_{x_1x_1}(0) + R_{x_2x_2}(0)$$
$$\Rightarrow \quad R_{x_1x_2}(\tau) \leq \frac{1}{2}\{R_{x_1x_1}(0) + R_{x_2x_2}(0)\} \tag{4.17}$$

Now, consider

$$E\{X_1(t) + X_2(t+\tau)\}^2 = E[X_1^2(t)] + E[X_2^2(t+\tau)] + 2X_1(t)X_2(t+\tau)$$
$$= R_{x_1x_1}(0) + R_{x_2x_2}(0) + 2R_{x_1x_2}(\tau)$$
$$\Rightarrow \quad -2R_{x_1x_2}(\tau) = R_{x_1x_1}(0) + R_{x_2x_2}(0) - E\{X_1(t) - X_2(t+\tau\}^2$$

Eliminating the third term $-E\{X_1(t) - X_2(t+\tau\}^2$ from right-hand side, we have

$$-2R_{x_1x_2}(\tau) \leq R_{x_1x_1}(0) + R_{x_2x_2}(0)$$
$$\Rightarrow \quad -R_{x_1x_2}(\tau) \leq \frac{1}{2}\{R_{x_1x_1}(0) + R_{x_2x_2}(0)\} \tag{4.18}$$

From (4.17) and (4.18), we have

$$|R_{x_1x_2}(\tau)| \leq \frac{1}{2}\{R_{x_1x_1}(0) + R_{x_2x_2}(0)\} \tag{4.19}$$

Property 4.9: If $\{X_1(t)\}$ and $\{X_2(t)\}$ are two independent stationary random processes with mean values $E[X_1(t)] = \mu_{x_1}$ and $E[X_2(t)] = \mu_{x_2}$, then $R_{x_1x_2}(\tau) = \mu_{x_1}\,\mu_{x_2}$.

Proof. If $R_{x_1x_2}(\tau)$ is the cross-correlation function of the two stationary random processes $\{X_1(t)\}$ and $\{X_2(t)\}$, then

$$R_{x_1x_2}(\tau) = E\{X_1(t)X_2(t+\tau)\}$$

$$= \left\{ \int_{-T}^{T} \int_{-T}^{T} E\{X(t_1)X(t_2)\} dt_1 dt_2 \right\} - \left\{ \left(\int_{-T}^{T} \int_{-T}^{T} E\{X(t_1)\} E\{X(t_2)\} dt_1 dt_2 \right) \right\}$$

$$= \left\{ \int_{-T}^{T} \int_{-T}^{T} E\{X(t_1)X(t_2)\} - E\{X(t_1)\} E\{X(t_2)\} dt_1 dt_2 \right\}$$

$$= \int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2) dt_1 dt_2$$

Using the Result A.4.1 from Appendix A, we have

$$\sigma_S^2 = \int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2) dt_1 dt_2 = \int_{-2T}^{2T} C(\tau)(2T - |\tau|)\, d\tau \tag{4.25}$$

Note

(i) For any arbitrary *white noise process* (Refer to Section 2.3.4 of Chapter 2 for definition) the autocovariance is given as $C(t_1, t_2) = b(t_1)\delta(t_1 - t_2)$ for $b(t) \geq 0$. Therefore, in case of stationary white noise process, we have $C(\tau) = b\delta(\tau)$ with b as constant. In fact, if we can express $C(\tau) = b\delta(\tau)$, where b is constant, then we have the variance as

$$\sigma_S^2 = b \int_{-2T}^{2T} \delta(\tau)(2T - |\tau|)\, d\tau$$

$$= b \int_{-2T}^{2T} \delta(\tau)(2T) d\tau - b \int_{-2T}^{2T} \delta(\tau)\, |\tau|\, d\tau = 2Tb \tag{4.26}$$

(ii) If the random process $\{X(t)\}$ is an $a-$*independent* (Refer to Section 2.3.3 of Chapter 2 for definition) and a is considerably small compared to T, that is, $a << T$, then

$$\sigma_S^2 = \int_{-2T}^{2T} C(\tau)(2T - |\tau|)\, d\tau$$

$$= \int_{-2T}^{2T} C(\tau)(2T) d\tau - \int_{-2T}^{2T} C(\tau)|\tau| d\tau \approx 2T \int_{-a}^{a} C(\tau)\, d\tau$$

$$\Rightarrow \quad \int_{-a}^{a} C(\tau)(2T)d\tau - \int_{-a}^{a} C(\tau)|\tau|d\tau \approx 2T\int_{-a}^{a} C(\tau)d\tau \qquad (4.27)$$

This is true because $\tau \to 0$ in the interval$(-a, a)$.

Looking at (4.26) and (4.27), an $a-$independent process with $a << T$ can be replaced by white noise with

$$b = \int_{-a}^{a} C(\tau)d\tau \qquad (4.28)$$

SOLVED PROBLEMS

Problem 1. Can the function $R_{xx}(\tau) = \dfrac{1+\tau^4}{1+\tau^6}$ serve as a valid autocorrelation function for a continuous time real valued wide sense stationary process $X(t)$? Justify.

Solution:

If $\{X(t)\}$ is a wide sense stationary process with autocorrelation function $R_{xx}(\tau)$ then

$$R_{xx}(0) \geq 0, \quad R_x(\tau) = R_{xx}(-\tau), \quad R_{xx}(0) \geq |R_{xx}(\tau)|$$

Consider

$$R_{xx}(\tau) = \frac{1+\tau^4}{1+\tau^6}$$

$$R(0) = \frac{1+0^4}{1+0^6} = 1 \geq 0$$

$$R(-\tau) = \frac{1+(-\tau)^4}{1+(-\tau)^6} = \frac{1+\tau^4}{1+\tau^6} = R(\tau)$$

$$R(0) = \frac{1+0^4}{1+0^6} = 1, \quad |R(\tau)| = \left|\frac{1+\tau^4}{1+\tau^6}\right| \leq 1, \quad \because \tau^4 \geq \tau^6,$$

$$\Rightarrow \quad R(0) \geq |R(\tau)|$$

Therefore, $R_{xx}(\tau) = \dfrac{1+\tau^4}{1+\tau^6}$ is a valid autocorrelation function.

Problem 2. Let $\{X(t)\}$ be a random process and $X(t_1)$ and $X(t_2)$ are the two random variables of the process at two time points t_1 and t_2 with autocorrelation function $R_{xx}(t_1, t_2)$. If $\{Y(t)\}$ is another random process such that $Y(t) = X(t_1) + X(t_2)$ with autocorrelation function $R_{yy}(t_1, t_2)$, then show that

$$R_{yy}(t, t) = R_{xx}(t_1, t_1) + R_{xx}(t_2, t_2) + 2R_{xx}(t_1, t_2)$$

SOLUTION:
Consider

$$E\{Y(t)\}^2 = E\{X(t_1) + X(t_2)\}^2$$

$$= E\left\{X^2(t_1) + X^2(t_2) + 2X(t_1)X(t_2)\right\}$$

$$= E[X^2(t_1)] + E[X^2(t_2)] + 2E[X(t_1)X(t_2)]$$

$$= E[X(t_1)X(t_1)] + E[X(t_2)X(t_2)] + 2E[X(t_1)X(t_2)]$$

$$R_{yy}(t, t) = R_{xx}(t_1, t_1) + R_{xx}(t_2, t_2) + 2R_{xx}(t_1, t_2)$$

It may be noted that if $\{X(t)\}$ and $\{Y(t)\}$ are stationary processes, then we have

$$R_{yy}(0) = R_{xx}(0) + R_{xx}(0) + 2R_{xx}(\tau) = 2[R_{xx}(0) + R_{xx}(\tau)]$$

$$\Rightarrow \quad E[Y^2(t)] = 2\left\{E[X^2(t) + R_{xx}(\tau)\right\}$$

This implies that for the determination of $E[Y^2(t)]$, the *average power* of the output process $\{Y(t)\}$, the *average power* of the input process $E[X^2(t)$ alone is not sufficient, but the knowledge of the autocorrelation function $R_{xx}(\tau)$ is also required.

Problem 3. If $\{X(t)\}$ is a wide sense stationary process with autocorrelation function $R(\tau) = 4e^{-2|\tau|}$ then find

$$E\left\{[X(t+\tau) - X(t)]^2\right\}$$

Which is the second order moment of $[X(t+\tau) - X(t)]$

SOLUTION:
Consider

$$E\left\{[X(t+\tau) - X(t)]^2\right\} = E\left\{X^2(t+\tau) + X^2(t) - 2X(t+\tau)X(t)\right\}$$

$$= E\left\{X^2(t+\tau)\right\} + E\left\{X^2(t)\right\} - 2E\{X(t+\tau)X(t)\})$$

Since the given random process $\{X(t)\}$ is a WSS process, we know that

$$E\left\{X^2(t+\tau)\right\} = E\left\{X^2(t)\right\} = R(0)$$

and

$$E\{X(t+\tau)X(t)\} = R(\tau)$$

Therefore, we have

$$E\left\{[X(t+\tau) - X(t)]^2\right\} = R(0) + R(0) - 2R(\tau)$$

$$= 4 + 4 - 2\left(4e^{-2|\tau|}\right) = 8\left(1 - e^{-2|\tau|}\right)$$

Problem 4. If $\{X(t)\}$ is a wide sense stationary process with autocorrelation function $R_{xx}(\tau)$ and if $\{Y(t)\}$ is another wide sense stationary random process such that $Y(t) = X(t+a) - X(t-a))$ where a is constant, then show that

$$R_{yy}(\tau) = 2R_{xx}(\tau) - R_{xx}(\tau + 2a) - R_{xx}(\tau - 2a)$$

Solution:

We know that since $\{X(t)\}$ is a wide sense stationary process, and $\{Y(t)\}$ is another wide sense stationary random process such that $Y(t) = X(t+a) - X(t-a)$, the autocorrelation function $R_{yy}(\tau)$ can be given as

$$R_{yy}(\tau) = E\{Y(t)Y(t+\tau)\}$$

$$R_{yy}(\tau) = E\{[X(t+a) - X(t-a)]\,[X(t+\tau+a) - X(t+\tau-a)]\}$$

$$= E[X(t+a)X(t+\tau+a)] - E[X(t+a) - X(t+\tau-a)]$$

$$- E[X(t-a)X(t+\tau+a)] + E[X(t-a)X(t+\tau-a)]$$

$$= R_{xx}(\tau) - R_{xx}(\tau+2a) - R_{xx}(\tau-2a) + R_{xx}(\tau)$$

$$= 2R_{xx}(\tau) - R_{xx}(\tau+2a) - R_{xx}(\tau-2a)$$

Problem 5. Given that $\{X(t)\}$ and $\{Y(t)\}$ are two independent and stationary random processes. If $\{Z(t)\}$ is another process such that $Z(t) = aX(t)Y(t)$, then find $R_{zz}(t, t+\tau)$.

SOLUTION:
We know that since $\{X(t)\}$ and $\{Y(t)\}$ are independent stationary processes, and $Z(t) = aX(t)Y(t)$, the autocorrelation function $R_{zz}(t, t+\tau)$ can be given as

$$\begin{aligned} R_{zz}(\tau, t+\tau) = R_{zz}(\tau) = E\{Z(t)\,Z(t+\tau)\} \\ = E\{[aX(t)Y(t)]\,[aX(t+\tau)Y(t+\tau)]\} \\ = a^2 E\{[X(t)X(t+\tau)]\,[Y(t)Y(t+\tau)]\} \end{aligned}$$

Since $\{X(t)\}$ and $\{Y(t)\}$ are independent, we have

$$\begin{aligned} R_{zz}(\tau) &= a^2 E\{X(t)X(t+\tau)\}E\{Y(t)Y(t+\tau)\} \\ &= a^2 R_{xx}(\tau) R_{yy}(\tau) \end{aligned}$$

Problem 6. If there are two stationary random processes $\{X(t)\}$ and $\{Y(t)\}$ such that $Z(t) = X(t) + Y(t)$ then find $R_{x+y}(\tau)$.

SOLUTION:
We know that since $\{X(t)\}$ and $\{Y(t)\}$ are stationary processes, and $Z(t) = X(t) + Y(t)$, the autocorrelation function $R_{x+y}(\tau)$ can be given as

$$\begin{aligned} R_{x+y}(\tau) &= R_{zz}(\tau) = E\{Z(t)\,Z(t+\tau)\} \\ &= E\{[X(t)+Y(t)]\,[X(t+\tau)+Y(t+\tau)]\} \\ &= E\{[X(t)X(t+\tau) + X(t)Y(t+\tau) \\ &\quad + Y(t)X(t+\tau) + Y(t)Y(t+\tau)]\} \\ &= E\{X(t)X(t+\tau)\} + E\{X(t)Y(t+\tau)\} + E\{Y(t)X(t+\tau)\} \\ &\quad + E\{Y(t)Y(t+\tau)\} \\ &= R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau) \\ &= R_{xx}(\tau) + 2R_{xy}(\tau) + R_{yy}(\tau) \end{aligned}$$

Problem 7. A random process $\{X(t)\}$ is defined as $X(t) = A\sin(\omega t + \theta)$, where A and ω are constants and θ is uniformly distributed between $-\pi$ and π. Then (i) find the autocorrelation of the random process $\{X(t)\}$, (ii) find mean and autocorrelation of the random process $\{Y(t)\}$ where $Y(t) = X^2(t)$.

SOLUTION:
It is given $X(t) = A\sin(\omega t + \theta)$
Since θ is uniformly distributed between $-\pi$ and π, we have its PDF as

$$f(\theta) = \frac{1}{2\pi}, \quad -\pi \le \theta \le \pi$$

(i) The autocorrelation of the random process $\{X(t)\}$ is defined as

$$R_{xx}(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{A\sin(\omega t_1 + \theta)A\sin(\omega t_2 + \theta)\}$$

$$= A^2 E\{\sin(\omega t_1 + \theta)\sin(\omega t_2 + \theta)\}$$

$$= A^2 E\left\{\frac{\cos\omega(t_1 - t_2) - \cos[\omega(t_1 + t_2) + 2\theta]}{2}\right\}$$

$$= \frac{A^2}{2}\left\{\int_{-\pi}^{\pi} \cos\omega(t_1 - t_2) f(\theta) d\theta - \int_{-\pi}^{\pi} \cos[\omega(t_1 + t_2) + 2\theta] f(\theta) d\theta\right\}$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi} \cos\omega(t_1 - t_2)\frac{1}{2\pi} d\theta - \frac{A^2}{2}\int_{-\pi}^{\pi} \cos[\omega(t_1 + t_2) + 2\theta]\frac{1}{2\pi} d\theta$$

$$= \frac{A^2}{2}\cos\omega(t_1 - t_2) - \frac{A^2}{8\pi}|\sin[\omega(t_1 + t_2) + 2\theta]|_{-\pi}^{\pi}$$

$$= \frac{A^2}{2}\cos\omega(t_1 - t_2) = \frac{A^2}{2}\cos\omega\tau$$

Since $|\sin[\omega(t_1 + t_2) + 2\theta]|_{-\pi}^{\pi} = \sin[\omega(t_1 + t_2) + 2\pi] - \sin[\omega(t_1 + t_2) - 2\pi] = 0$ as $\sin(n\pi \pm A) = (-1)^n \sin A$

$$\therefore \quad R_{xx}(t_1, t_2) = \frac{A^2}{2}\cos\omega\tau = R_{xx}(\tau)$$

(ii) Mean of the random process $\{Y(t)\}$ where $Y(t) = X^2(t)$ is given by

$$E\{Y(t)\} = E\left\{X^2(t)\right\} = R_{xx}(0)$$

Autocorrelation of the random process $\{Y(t)\}$ where $Y(t) = X^2(t)$ is given by

$$\begin{aligned} R_{yy}(t_1, t_2) &= E\{Y(t_1)Y(t_2)\} = E\left\{X^2(t_1)X^2(t_2)\right\} \\ &= E\left\{X^2(t_1)\right\}E\left\{X^2(t_2)\right\} + \{E[X(t_1)X(t_2)]\}^2 \\ &\qquad \because \quad E\left(X^2Y^2\right) = E(X^2)E(Y^2) + \{E(XY)\}^2 \\ &= \{R_{xx}(0)\}^2 + \{R_{xx}(t_1,t_2)\}^2 \\ &= R^2_{xx}(0) + R^2_{xx}(t_1,t_2) \end{aligned}$$

Problem 8. Let $\{X(t)\}$ be a stationary random process with autocorrelation function $R_{xx}(\tau)$. If $S = \int_0^{10} X(t)\,dt$, then show that

$E\{S^2\} = \int_{-10}^{10} (10 - |\tau|)\, R_{xx}(\tau)\, d\tau$. Also if $E\{X(t)\} = 8$ and autocorrelation function,

$R_{xx}(\tau) = 64 + 10e^{-2|\tau|}$, then find the mean and variance of S.

SOLUTION:

Mean of S is given by $E(S) = E\left\{\int_0^{10} X(t)dt\right\} = \int_0^{10} E\{X(t)\}\,dt = \int_0^{10} 8dt = 80$

$$\begin{aligned} \text{Consider } E(S^2) &= E\left\{\int_0^{10} X(t)dt\right\}^2 \\ &= E\left\{\int_0^{10} X(t)dt \int_0^{10} X(t)dt\right\} \\ &= \int_0^{10}\int_0^{10} E\{X(t)X(t)\}\, dtdt \end{aligned}$$

For our convenience and without loss of generality, we can also write as follows:

$$E(S^2) = E\left\{\int_0^{10} X(t)dt\right\}^2 = \int_0^{10}\int_0^{10} E\{X(t_1)X(t_2)\}\, dt_1dt_2$$

$$= \int_0^{10}\int_0^{10} R(t_1,t_2)dt_1dt_2$$

$$= \int_{-10}^{10} (10-|\tau|)R_{xx}(\tau)d\tau$$

(Using the Result A.4.1 in Appendix A)

$$\therefore E\left\{S^2\right\} = \int_{-10}^{10} (10-|\tau|)R_{xx}(\tau)d\tau$$

We know that variance of S is given by $V(S) = E(S^2) - \{E(S)\}^2$
Consider

$$E(S^2) = \int_{-10}^{10} (10-|\tau|)R_{xx}(\tau)\,d\tau = \int_{-10}^{10} (10-|\tau)|)\left(64+10e^{-2|\tau|}\right)d\tau$$

$$= 2\int_0^{10} (10-\tau)\left(64+10e^{-2|\tau|}\right)d\tau$$

$$= 2\int_0^{10} \left(640-64\tau+100e^{-2\tau}-10\tau\, e^{-2\tau}\right)d\tau$$

$$= 6495$$

$$\therefore \quad V(S) = 6495 - 80^2 = 95$$

Problem 9. A stationary zero mean random process $\{X(t)\}$ has the autocorrelation function $R_{xx}(\tau) = 10e^{-0.1\tau^2}$. Find the mean and variance of $S = \frac{1}{5}\int_0^5 X(t)\,dt$.

Solution:
Given $\{X(t)\}$ is a stationary zero mean random process

Autocorrelation function $R_{xx}(\tau) = 10\,e^{-0.1\tau^2}$

Consider

$$E(S) = E\left\{\frac{1}{5}\int_0^5 X(t)\,dt\right\} = \frac{1}{5}\int_0^5 E\{X(t)\}\,dt = 0 \quad \because \quad E\{X(t)\} = 0$$

Consider

$$E(S^2) = E\left\{\frac{1}{5}\int_0^5 X(t)\,dt\right\}^2 = \frac{1}{5}\int_0^5\int_0^5 E\{X(t_1)X(t_2)\}\,dt_1dt_2$$

$$= \frac{1}{5}\int_0^5\int_0^5 R_{xx}(t_1,t_2)dt_1dt_2$$

$$= \frac{1}{5}\int_{-5}^{5} [5-|\tau|]\,R_{xx}(\tau)d\tau$$

Since $\int_0^T\int_0^T R_{xx}(t_1,t_2)dt_1dt_2 = \int_{-T}^{T}(T-|\tau|)R(\tau)d\tau$ (Refer Result A.4.1 in Appendix A), we have

$$E(S^2) = \frac{1}{5}\int_{-5}^{5}(5-|\tau|)(10e^{-0.1\tau^2})d\tau$$

$$= 2\int_{-5}^{5}(5-|\tau|)e^{-0.1\tau^2}d\tau$$

$$= 4\int_0^5 (5-\tau)e^{-0.1\tau^2}d\tau$$

$$= 20\int_0^5 e^{-0.1\tau^2}d\tau - 4\int_0^5 \tau e^{-0.1\tau^2}d\tau$$

$$= I_1 - I_2$$

It may be noted that the integral part $I_1 = 20\int_0^5 e^{-0.1\tau^2}d\tau$ can be evaluated using any numerical integration methods.

Now, consider, the integral part $I_2 = 4\int_0^5 \tau e^{-0.1\tau^2}d\tau$

Let $u = \frac{\tau^2}{10} \Rightarrow du = \frac{\tau}{5}d\tau \quad \Rightarrow \tau d\tau = 5du$

$$\Rightarrow \quad I_2 = 4\int_0^{2.5} e^{-u}du = 4(0.9179) = 3.6716$$

$$\therefore \quad E(S^2) = I_1 - I_2 = 20\int_0^5 e^{-0.1\tau^2}d\tau - 3.6716$$

We know that $V(S) = E(S^2) - \{E(S)\}^2$

$$= 20\int_0^5 e^{-0.1\tau^2}\,d\tau - 3.6716 \quad \because E(S) = 0$$

Problem 10. A stationary random process $\{X(t)\}$ has an autocorrelation function $R_{xx}(\tau) = \dfrac{25\tau^2+36}{6.25\tau^2+4}$, then find mean and variance of the process.

SOLUTION:

We know that if $\{X(t)\}$ is a stationary random process with autocorrelation function $R_{xx}(\tau)$, then the mean of the process, say $E[X(t)] = \mu_x$ can be obtained as

$$\mu_x = \sqrt{\lim_{\tau\to\infty} R_{xx}(\tau)}$$

$$\mu_x = \sqrt{\lim_{\tau\to\infty}\frac{25\tau^2+36}{6.25\tau^2+4}} = \sqrt{\lim_{\tau\to\infty}\frac{25\tau^2(1+36/25\tau^2)}{6.25\tau^2(1+4/6.25\tau^2)}}$$

$$\mu_x = \sqrt{\lim_{\tau\to\infty}\frac{4(1+36/25\tau^2)}{(1+4/6.25\tau^2)}} = \sqrt{4} = 2$$

It is known that variance of the stationary process $\{X(t)\}$ can be obtained as

$$V(\{X(t)\} = E\{X(t)\}^2 - \{E[X(t)]\}^2$$

$$= R(0) - \{\mu_x\}^2$$

$$= \frac{25(0)+36}{6.25(0)+4} - 2^2 = 5$$

Problem 11. Let $\{X(t)\}$ and $\{Y(t)\}$ be two random stationary random process such that $X(t) = 3\cos(\omega t+\theta)$ and $Y(t) = 2\cos(\omega t+\theta-\pi/2)$ where θ is a random variable uniformly distributed in $(0, 2\pi)$. Then prove that

$$\sqrt{R_{xx}(0)R_{yy}(0)} \geq |R_{xy}(\tau)|$$

SOLUTION:
Consider

$$R_{xx}(\tau) = E\{X(t+\tau)X(t)\}$$
$$= \{3\cos\omega(t+\tau)+(\theta)\}\{3\cos(\omega t+\theta)\}$$
$$= 9E\left\{\frac{\cos[\omega(t+\tau)+\omega(t+2\theta)]+\cos\omega\tau}{2}\right\}$$
$$= \frac{9}{2}E\{\cos[\omega(t+\tau)+\omega(t+2\theta)]\}+\frac{9}{2}E\{\cos\omega\tau\}$$

Since θ is a random variable uniformly distributed in $(0, 2\pi)$, its probability density function is given by

$$f(\theta) = \frac{1}{2\pi}, \qquad 0 \le \theta \le 2\pi$$

Consider

$$E\{\cos[\omega(t+\tau)+\omega t+2\theta)]\} = \int_0^{2\pi}\cos[\omega(t+\tau)+\omega t+2\theta)]\frac{1}{2\pi}d\theta$$
$$= \frac{1}{2\pi}\left\{\frac{\sin(2\omega t+\omega\tau+2\theta)}{2}\right\}_0^{2\pi}$$
$$= \frac{1}{4\pi}\{\sin(2\omega t+\omega\tau+4\pi)-\sin(2\omega t+\omega\tau)\}$$
$$= \frac{1}{4\pi}\{\sin(2\omega t+\omega\tau)-\sin(2\omega t+\omega\tau)\} = 0$$

Consider

$$\frac{9}{2}E\{\cos\omega t\} = \frac{9}{2}\int_0^{2\pi}\cos\omega\tau\frac{1}{2\pi}d\theta$$
$$= \frac{9}{2}\cos\omega\tau$$
$$\therefore \quad R_{xx}(\tau) = \frac{9}{2}\cos\omega\tau \quad \Rightarrow \quad R_{xx}(0) = \frac{9}{2}$$

Now consider

$$R_{yy}(\tau) = E\{Y(t+\tau)Y(t)\}$$

$$= \{2\cos[\omega(t+\tau)+\theta-\pi/2]\}\ \{2\cos(\omega t+\theta-\pi/2)\}$$

$$= 4E\left\{\frac{\cos[2\omega t+\omega t+2\theta-\pi]+\cos\omega\tau}{2}\right\}$$

$$= 2E\{\cos[2\omega t+\omega t+2\theta-\pi]\}+2E\{\cos\omega\tau\}$$

Consider

$$E\{\cos[2\omega t+\omega t+2\theta-\pi]\} = \int_0^{2\pi}\cos[2\omega t+\omega t+2\theta-\pi]\,\frac{1}{2\pi}d\theta$$

$$= \frac{1}{2\pi}\left\{\frac{\sin(2\omega t+\omega\tau+2\theta-\pi)}{2}\right\}_0^{2\pi}$$

$$= \frac{1}{4\pi}\{\sin(2\omega t+\omega\tau+3\pi)-\sin(2\omega t+\omega\tau-\pi)\}$$

$$= \frac{1}{4\pi}\{\sin(2\omega t+\omega\tau)-\sin(2\omega t+\omega\tau)\}=0$$

Consider

$$2E\{\cos\omega t\} = 2\int_0^{2\pi}\cos\omega\tau\frac{1}{2\pi}d\theta$$

$$= 2\cos\omega\tau$$

$$\therefore\quad R_{yy}(\tau) = 2\cos\omega\tau \quad\Rightarrow\quad R_{yy}(0)=2$$

Now, consider

$$R_{xy}(\tau) = E\{X(t+\tau)Y(t)\}$$

$$= \{3\cos[\omega(t+\tau)+\theta]\}\ \{2\cos(\omega t+\theta-\pi/2)\}$$

$$= 6E\left\{\frac{\cos(2\omega t+\omega t+2\theta-\pi/2)+\cos(\omega\tau+\pi/2)}{2}\right\}$$

$$= 3E\{\cos[2\omega t+\omega t+2\theta-\pi/2]\}+3E\{\sin\omega\tau\}$$

$$= 3\sin\omega\tau$$

$$\therefore\quad \sqrt{R_{xx}(0)R_{yy}(0)} = \sqrt{\frac{9}{2}(2)} = 3$$

But $R_{xy}(\tau) = 3\sin\omega\tau \le 3, \qquad \because \quad -1 \le \sin\omega\tau \le 1$

$$\therefore \quad \sqrt{R_{xx}(0)R_{yy}(0)} \ge R_{xy}(\tau)$$

Since,
$\tau = t_1 - t_2$ or $\tau = t_2 - t_1$ we have $\sqrt{R_{xx}(0)R_{yy}(0)} \ge |R_{xy}(\tau)|$.

EXERCISE PROBLEMS

1. Which of the following functions are valid autocorrelation functions for the respective wide sense stationary processes?

   (i) $R(\tau) = e^{-|\tau|}$

   (ii) $R(\tau) = e^{-\tau}\cos\tau$

   (iii) $R(\tau) = e^{-\tau^2}$

   (iv) $R(\tau) = e^{-\tau^2}\sin\tau$

2. If $\{X(t)\}$ is a wide sense stationary process with autocorrelation function $R(\tau) = Ae^{-\alpha|\tau|}$ then find the second order moment of $[X(8) - X(5)]$.
3. Find the mean, mean-square value (second order moment or average power) and variance of the stationary random process $\{X(t)\}$ whose autocorrelation function is given by

   (i) $R(\tau) = e^{-\tau^2/2}$ (ii) $R(\tau) = 2 + 4e^{2|\tau|}$, (iii) $R(\tau) = 25 + \dfrac{4}{1+6\tau^2}$ and (iv) $R(\tau) = \dfrac{4\tau^2+6}{\tau^2+1}$
4. If $\{X(t)\}$ is a stationary random process with autocorrelation function $R_{xx}(\tau) = 10e^{-0.1|\tau|}$ and if S is a random variable is such that $S = \frac{1}{5}\int_0^5 X(t)\,dt$ then find (i) mean and (ii) variance of S.
5. The autocorrelation function of a stationary process $\{X(t)\}$ is given by $R(\tau) = 9 + 2e^{-|\tau|}$. Find the mean value of the random variable $Y = \int_0^2 X(t)\,dt$ and the variance of $\{X(t)\}$.
6. If $R(\tau) = e^{-|\tau|}$ is the autocorrelation function of a wide sense stationary process $\{X(t)\}$ then using Chebyshev's inequality obtain $P\{|X(10) - X(8)| \ge 2\}$.
7. If $\{X(t)\}$ is a random process with $X(t) = A\sin(\omega t + \theta)$, where A and ω are constants and θ is random variable uniformly distributed over $(-\pi, \pi)$, then find the autocorrelation of $\{Y(t)\}$ where $Y(t) = X^2(t)$.

8. If $\{X(t)\}$ is a random process such that the sample function is given by $X(t) = Y \sin \omega t$, where ω is constant and Y is random variable uniformly distributed over $(0, 1)$, then find mean autocorrelation and autocovariance of $\{X(t)\}$.

9. If $\{X(t)\}$ is a wide sense stationary random process with mean μ_x and autocorrelation function $R_{xx}(\tau)$ and $\{Y(t)\}$ is another random process such that $Y(t) = \{X(t+\tau) - X(t)\}/\tau$ then find mean and autocorrelation of $\{Y(t)\}$. Also verify whether $\{Y(t)\}$ is wide sense stationary.

10. Let us suppose that we are interested in the random process $\{X(t)\}$ but due to possible existence of noise, its sample function $X(t)$ is observed only in the form of $Y(t) = X(t) + N(t)$ where $Y(t)$ can be viewed as a sample function of the process $\{Y(t)\}$ and $N(t)$ is a sample function of a noise process$\{N(t)\}$. If $\{X(t)\}$ and $\{N(t)\}$ are independent wide sense stationary processes with means $E\{X(t)\} = \mu_x$ and $E\{N(t)\} = \mu_n = 0$ and the autocorrelation functions $R_{xx}(\tau)$ and $R_{nn}(\tau)$ respectively, then obtain the autocorrelation function of $\{Y(t)\}$. Also obtain the cross correlation functions of $\{Y(t)\}$ and $\{X(t)\}$ and $\{Y(t)\}$ and $\{N(t)\}$.

of occurrences in the time interval $(t_1,\, t_2)$ of length $(t_2 - t_1)$ denoted as $n\,(t_1,\, t_2)$ (Refer Figure 5.3) can be obtained as

$$n\,(t_1,\, t_2) = \{X(t_2) - X(t_1)\} = n - m = x\ (say) \tag{5.5}$$

It may be noted that while $X(t_1) = m$ or $n\,(0,\, t_1) = m$ represents there are m occurrences in the time interval $(0,\, t_1)$, $X(t_2) = n$ or $n(0,\, t_2) = n$ represents there are n occurrences in the time interval $(0,\, t_2)$ and so on. And hence we have

$$P\,\{n(t_1,\, t_2) = x\} = P\,\{X(t_2) - X(t_1) = x\} = \frac{e^{\lambda(t_2 - t_1)}[\lambda\,(t_2 - t_1)]^x}{x!}, \quad x = 0,\, 1,\, 2, \cdots\cdots \tag{5.6}$$

If we let $t_1 = s$ and $t_2 = t + s$ then

$$P\{X(t+s) - X(s) = x\} = \frac{e^{\lambda t}(\lambda\, t)^x}{x!}, \quad x = 0,\, 1,\, 2, \cdots\cdots \tag{5.7}$$

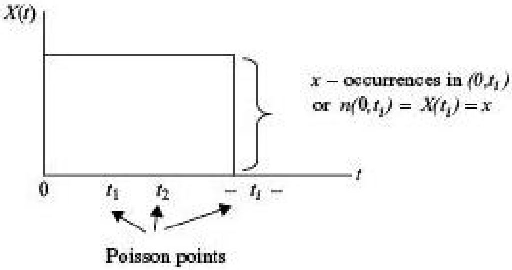


Figure 5.2. Poisson points and occurrences

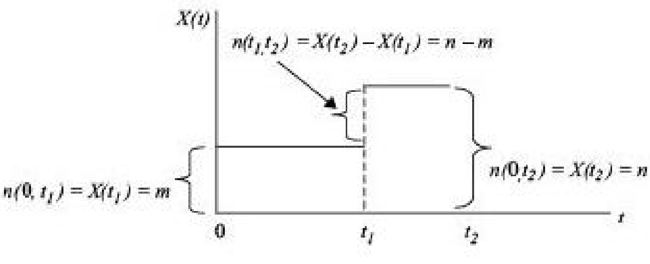


Figure 5.3. Number of occurrences in the time interval (t_1, t_2)

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{n} \frac{n!}{n!} \frac{(\lambda_1 t)^r}{r!} \frac{(\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{r=0}^{n} {}^nC_r (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t} \{(\lambda_1+\lambda_2)t\}^n}{n!}$$

Which implies $Y(t) = X_1(t) + X_2(t)$ is a Poisson process with parameter $(\lambda_1 + \lambda_2)\, t$. Therefore, sum of two independent Poisson processes is also a Poisson process.

Alternative proof:

Consider $Y(t) = X_1(t) + X_2(t)$

$$E\{Y(t)\} = E\{X_1(t) + X_2(t)\} = E\{X_1(t)\} + E\{X_2(t)\} = \lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2)\, t$$

$$V\{Y(t)\} = V\{X_1(t) + X_2(t)\} = V\{X_1(t)\} + V\{X_2(t)\} = \lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2)\, t$$

Since mean and variance are equal, we conclude that the sum of two independent Poisson processes is also a Poisson process with parameter $(\lambda_1 + \lambda_2)\, t$.

Theorem 5.2: If $\{X_1(t)\}$ and $\{X_2(t)\}$ represent two independent Poisson processes with parameters $\lambda_1 t$ and $\lambda_2 t$ respectively, then the process $\{Y(t)\}$, where $Y(t) = X_1(t) - X_2(t)$, is not a Poisson process. (That is, the difference of two independent Poisson processes is not a Poisson process.)

Proof. It is given that $\{X_1(t)\}$ and $\{X_2(t)\}$ are two independent Poisson processes with parameters λ_1 and λ_2 respectively, and $Y(t) = X_1(t) - X_2(t)$ therefore we have

$$P\{X_1(t) = x\} = \frac{e^{-\lambda_1 t} (\lambda_1 t)^x}{x!}, \quad x = 0, 1, 2 \cdots\cdots$$

$$P\{X_2(t) = x\} = \frac{e^{-\lambda_2 t} (\lambda_2 t)^x}{x!}, \quad x = 0, 1, 2, \cdots\cdots$$

$$\text{Consider } P\{Y(t) = n\} = \sum_{r=0}^{n} P\{X_1(t) = n+r\} P\{X_2(t) = r\}$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t} (\lambda_1 t)^{n+r}}{(n+r)!} \frac{e^{-\lambda_2 t} (\lambda_2 t)^r}{r!}$$

$$= e^{-(\lambda_1+\lambda_2)t} \left(\frac{\lambda_1}{\lambda_2}\right)^{n/2} \sum_{r=0}^{n} \frac{\left(t\sqrt{\lambda_1 \lambda_2}\right)^{n+2r}}{r!\,(n+r)!}$$

This is not in the form of a probability mass function of Poisson distribution which implies $Y(t) = X_1(t) - X_2(t)$ is not a Poisson process. Therefore, the difference of two independent Poisson processes is not a Poisson process.

Alternative proof:

Consider $Y(t) = X_1(t) - X_2(t)$

$$E\{Y(t)\} = E\{X_1(t) - X_2(t)\} = E\{X_1(t)\} - E\{X_2(t)\} = \lambda_1 t - \lambda_2 t = (\lambda_1 - \lambda_2)\, t$$

$$V\{Y(t)\} = V\{X_1(t) - X_2(t)\} = V\{X_1(t)\} + V\{X_2(t)\} = \lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2)\, t$$

Since mean and variance are not equal, we conclude that the difference of two independent Poisson processes is not a Poisson process.

Theorem 5.3: If $\{X_1(t)\}$ and $\{X_2(t)\}$ represent two independent Poisson processes with parameters $\lambda_1 t$ and $\lambda_2 t$ respectively, then $P[X_1(t) = x/\{X_1(t) + X_2(t) = n\}]$ is binomial with parameters n and p where $p = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$. That is,

$$P[X_1(t) = x/\{X_1(t) + X_2(t) = n\}] = {}^nC_x p^x q^{n-x}$$

where $p = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$ and $q = 1 - p = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$.

Proof.

Consider $P[X_1(t) = x/\{X_1(t) + X_2(t) = n\}] = \dfrac{P\{(X_1(t) = x) \cap (X_1(t) + X_2(t) = n)\}}{P\{X_1(t) + X_2(t) = n\}}$

$$= \frac{P\{(X_1(t) = x) \cap (X_2(t) = n - x)\}}{P\{X_1(t) + X_2(t) = n\}}$$

$$= \frac{P\{X_1(t) = x\}\, P\{X_2(t) = n - x\}}{P\{X_1(t) + X_2(t) = n\}}$$

(Since $\{X_1(t)\}$ and $\{X_2(t)\}$ are independent)

$$= \frac{\left\{e^{-\lambda_1 t} (\lambda_1 t)^x / x!\right\} \left\{e^{-\lambda_2 t} (\lambda_2 t)^{n-x} / (n-x)!\right\}}{e^{-(\lambda_1 + \lambda_2)t} \{(\lambda_1 + \lambda_2)\, t\}^n / n!}$$

$$= \frac{n!}{x!\,(n-x)!} \frac{(\lambda_1 t)^x (\lambda_2 t)^{n-x}}{\{(\lambda_1 + \lambda_2)\, t\}^n}$$

$$= {}^nC_x \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-x}$$

$$= {}^nC_x p^x q^{n-x}$$

where $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$ and $q = 1-p = \dfrac{\lambda_2}{\lambda_1+\lambda_2}$.

Therefore, $P[X_1(t) = x/\{X_1(t)+X_2(t) = n\}]$ is binomial with parameters n and p

where $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

Theorem 5.4: If $\{X(t)\}$ is a Poisson process with parameter λt then $P[X(t_1) = x/\{X(t_2) = n\}]$ is binomial with parameters n and $p = \dfrac{t_1}{t_2}$. That is, the conditional probability of a subset of two Poisson events is, in fact, binomial.

Proof. Let t_1 and t_2 be two time points and let $X(t_1)$ and $X(t_2)$ be two random variables at these time points forming a subset of the Poisson process $\{X(t)\}$. Let $t_1 < t_2$, and consider

$$P[X(t_1) = x/\{X(t_2) = n\}] = \frac{P\{X(t_1) = x, \quad n(t_1, t_2) = n-x\}}{P\{X(t_2) = n\}}$$

$$= \frac{P\{X(t_1) = x\}P\{n(t_1,t_2) = n-x\}}{P\{X(t_2) = n\}}$$

(Since $\{X(t)\}$ and $n(t_1, t_2) = \{X(t_2) - X(t_1)\}$ are independent)

$$= \frac{\left\{e^{-\lambda t_1}(\lambda t_1)^x/x!\right\}\left\{e^{-\lambda(t_2-t_1)}\{\lambda(t_2-t_1)\}^{n-x}/(n-x)!\right\}}{e^{-\lambda t_2}(\lambda t_2)^n/n!}$$

$$= {}^nC_x\left(\frac{t_1}{t_2}\right)^x\left(1-\frac{t_1}{t_2}\right)^{n-x}$$

$$= {}^nC_x p^x q^{n-x}$$

where $p = \dfrac{t_1}{t_2}$ and $q = 1-p$

Therefore, $P[X(t_1) = x/\{X(t_2) = n\}]$ is binomial with parameters n and $p = \dfrac{t_1}{t_2}$. That is, the conditional probability of a subset of two Poisson events is binomial.

Theorem 5.5: Let$\{X(t)\}$ be a Poisson process with parameter λt and let us suppose that each occurrence gets tagged independently with probability p. Let $\{Y(t)\}$ be the total number of tagged events and let $\{Z(t)\}$ be the total number of untagged events in the interval $(0, t)$, then $\{Y(t)\}$ is a Poisson process with parameter λpt and $\{Z(t)\}$ is a Poisson process with parameter λqt where $q = 1-p$.

Proof. Let E_x be the event that "x occurrences are tagged out of n occurrences". Then we have

$$P(E_x)=P\{x \text{ tagged occurrences}/n \text{ occurrences in } (0,t)\}P\{n \text{ occurrences in } (0,t)\}$$

$$= P\{x \text{ tagged and } (n-x) \text{ untagged occurrences out of } n \text{ occurrences}\}$$

$$P\{X(t)=n\}$$

$$= \{{}^nC_x p^x q^{n-x}\}\left\{\frac{e^{\lambda t}(\lambda t)^n}{n!}\right\}, \quad x = 0,\ 1,\ 2, \cdots\cdots n$$

It may be noted that the event $\{Y(t) = x\}$ represents the mutually exclusive union of the events E_x, E_{x+1}, $E_{x+2}, \cdots\cdots$ meaning that there should be a minimum of x occurrences out of which all are tagged, that is the minimum value of n is x.

$$\therefore \quad P\{Y(t)=x\} = \sum_{n=x}^{\infty} E_n = \sum_{n=x}^{\infty} {}^nC_x p^x q^{n-x}\frac{e^{\lambda t}(\lambda t)^n}{n!}, \quad x = 0, 1, 2, \cdots\cdots$$

$$= e^{\lambda t}\sum_{n=x}^{\infty}\frac{n!}{x!\,(n-x)!}p^x q^{n-x}\frac{(\lambda t)^n}{n!}$$

$$= e^{\lambda t}\sum_{n=x}^{\infty}\frac{(\lambda t)^n}{x!\,(n-x)!}p^x q^{n-x}$$

$$= e^{\lambda t}\frac{(\lambda pt)^x}{x!}\sum_{r=0}^{\infty}\frac{(\lambda qt)^r}{r!} = e^{-\lambda t}\frac{(\lambda pt)^x}{x!}e^{\lambda qt}$$

$$= \frac{e^{-\lambda(1-q)t}(\lambda pt)^x}{x!} = \frac{e^{-\lambda pt}(\lambda pt)^x}{x!}, \quad x = 0,\ 1,\ 2, \cdots\cdots$$

Which is a Poisson process with parameter λpt. Similarly, we can prove that $\{Z(t)\}$ is a Poisson process with parameter λqt where $q = 1 - p$.

Theorem 5.6: The time X (waiting time or service time) between the occurrences of events in a Poisson process with parameter λx is an exponential.

Or, if there is an arrival at time point t_0 and the next arrival is at time point t_1 then the time between these two Poisson points, t_0 and t_1 given by $X = t_1 - t_0$ follows exponential distribution with probability density function $f(x) = \lambda e^{-\lambda x}$, $x > 0$.

Proof. We know that the probability of getting k occurrences in the interval (t_0, t_1) of length say $x = t_1 - t_0$ is Poisson with parameter $\lambda x > 0$ and is given by

$$P\{n(t_0,\ t_1) = k\} = \frac{e^{-\lambda x}(\lambda x)^k}{k!}, \quad k = 0,\ 1,\ 2, \cdots\cdots; \quad x = t_1 - t_0$$

Therefore, the probability of getting no occurrences in the interval (t_0, t_1) of length $x = t_1 - t_0$ can be obtained as follows by letting $k = 0$

$$P\{n(t_0, t_1) = 0\} = e^{-\lambda x}, \quad x = t_1 - t_0$$

Let the first occurrence happen only beyond time point t_1 then we have $X > t_1 - t_0 = x$, then clearly $X > x$. This implies that there are no occurrences in the interval (t_0, t_1) (Refer to Figure 5.4). Hence,

$$\begin{aligned} P(X > x) &= P\{\text{no occurrences in the interval } (t_0, t_1)\} \\ &= P\{n(t_0, t_1) = 0\} = e^{-\lambda x}, \quad x = t_1 - t_0 \end{aligned}$$

Now, consider

$$F(x) = P(X \leq x) = 1 - P\{X > x\} = 1 - e^{-\lambda x}$$

$$\Rightarrow \quad f(x) = F'(x) = \lambda e^{\lambda x}, \quad x > 0$$

where $x = t_1 - t_0$.

Therefore, the time between arrivals, represented by the random variable X follows exponential distribution with parameter λ.

Note: Notationally, if we let $t_1 - t_0 = t$ then we have $f(t) = F'(t) = \lambda e^{\lambda t}, \quad t > 0$ where $t = t_1 - t_0$.

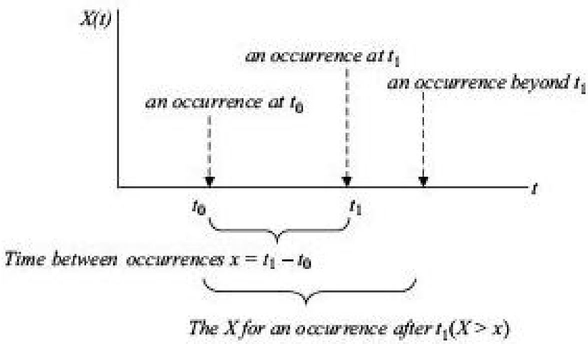


Figure 5.4. Poisson points and time between occurrences

It may be noted that this first order distribution is same for $X_{n+\tau}$ for some $\tau > 0$ also. That is the first order distribution is time invariant.

Let us consider the second order joint probability distribution of the process $\{X_n, n \geq 1\}$ for some $n = r$ and $n = s$. Then we have the second order joint probability distribution $P\{X_r = r, X_s = s\}$ of X_r and X_s as

		X_s	
		1	0
X_r	1	p^2	pq
	0	pq	q^2

It may be noted that this first order joint probability distribution is same for $X_{r+\tau}$ and $X_{s+\tau}$ also. That is, the second order distribution is time invariant.

Similarly, consider the third order distribution $P\{X_r = r, X_s = s, X_t = t\}$ of the process $\{X_n, n \geq 1\}$ for some $n = r$, $n = s$ and $n = t$ as follows:

X_r	X_s	X_t	$P\{X_r = r, X_s = s, X_t = t\}$
0	0	0	q^3
0	0	1	pq^2
0	1	0	pq^2
1	0	0	pq^2
0	1	1	p^2q
1	0	1	p^2q
1	1	0	p^2q
1	1	1	p^3

We can show that the third order joint probability distribution of X_r, X_s and X_t and the third order joint probability distribution of $X_{r+\tau}$, $X_{s+\tau}$ and $X_{t+\tau}$ are same. That is, the third order distribution is time invariant.

Continuing this way, we can prove that the distributions of all orders are time invariant. Therefore, we conclude that the process $\{X_n, n \geq 1\}$ is stationary in strict sense.

Problem 3. At a service counter customers arrive according to Poisson process with mean rate of 3 per minute. Find the probabilities that during a time interval of 2 minutes, (i) exactly 4 customers arrive and (ii) more than 4 customers arrive.

SOLUTION:

Let $\{X(t)\}$ be Poisson process with parameter λt. Then the probability of x arrivals at time t (or x arrivals in the time interval $(0, t)$) is given by

$$P\{X(t) = x\} = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \cdots\cdots$$

It is given that mean arrival rate $\lambda = 3$ per minute, that is, in every time interval of $(t_i, t_{i+1}) = 1$ min, $i = 0, 1, 2, \cdots\cdots$ with $t_0 = 0$, the average arrival rate is $\lambda = 3$. This implies that in the time interval $(0, t_1) = (0, 1)$ there are $\lambda t_1 = (3)(1) = 3$ arrivals on the average and in the time interval $(0, t_2) = (0, 2)$ there are $\lambda t_2 = (3)(2) = 6$ arrivals on the average and in the time interval $(0, t_3) = (0, 3)$ there are $\lambda t_3 = (3)(3) = 9$ arrivals on the average and so on (Refer to Figure 5.5).

Therefore, the probability that x customers arrive in the interval of $(0, 2)$ minutes is

$$P\{X(2) = x\} = P\{n(0, 2) = x\} = \frac{e^{-3(2)}[(3)(2)]^x}{x!} = \frac{e^{-6}6^x}{x!},$$
$$x = 0, 1, 2, \cdots\cdots$$

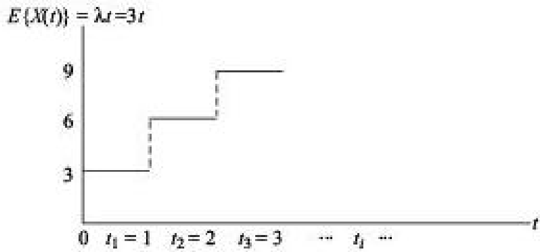


Figure 5.5. Average number of arrivals in the time interval $(0, t_i)$, $i = 1, 2, \cdots\cdots$

(i) Hence, the probability that exactly 4 customers arrive during a time interval of $(0, 2)$ minutes is given as

$$P\{X(2) = 4\} = \frac{e^{-6}6^4}{4!} = 0.1339$$

(ii) The probability that more than 4 customers arrive during a time interval of $(0, 2)$ minutes is given as

$$\begin{aligned} P\{X(2) \geq 5\} &= 1 - P\{X(2) \leq 4\} \\ &= 1 - \sum_{x=0}^{4} \frac{e^{-6}6^x}{x!} \\ &= 1 - e^{-6}\left(1 + \frac{6}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!}\right) = 0.7174 \end{aligned}$$

Problem 4. Suppose that customers arrive at a counter from town A at the rate of 1 per minute and from town B at the rate of 2 per minute according to two independent Poisson processes. Find the probability that the interval between two successive arrivals is more than 1 minute.

SOLUTION:

It is given that $\{X_1(t)\}$ is Poisson process with parameter $\lambda_1 t = (1)(1) = 1$ and $\{X_2(t)\}$ is an independent Poisson process with parameter $\lambda_2 t = (2)(1) = 2$ since $t = 1$ minute. Therefore, the average arrival, say λ, of customers at the counter is also Poisson with parameter

$$\lambda = E\{X_1(t) + X_2(t)\} = E\{X_1(t)\} + E\{X_2(t)\}$$
$$= \lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2)\, t = (1+2)(1) = 3$$

It may be noted that the interval between two successive Poisson arrivals follows an exponential distribution with parameter λ. If X is an exponential random variable representing the interval between two successive arrivals, then the required probability is

$$P\{X > 1\} = \int_1^\infty \lambda e^{-\lambda x} dx = \int_1^\infty 3e^{-3x} dx = 3\left[\frac{e^{-3x}}{-3}\right]_1^\infty = 3\left(0 - \frac{e^{-3}}{-3}\right)$$
$$= e^{-3} = 0.0498$$

Problem 5. If $\{X(t)\}$ is a Poisson process with parameter λt then show that $P[X(t_1) = x/\{X(t_2) = n\}]$ is binomial with parameters n and $p = \dfrac{t_1}{t_2}$. Hence, obtain $P\{X(2) = 2/X(6) = 6\}$.

SOLUTION:

Let t_1 and t_2 be two time points and let $X(t_1)$ and $X(t_2)$ be two random variables at these time points forming a subset of the Poisson process $\{X(t)\}$. Let $t_1 < t_2$, and consider

$$P[X(t_1) = x/\{X(t_2) = n\}] = \frac{P\{X(t_1) = x, \quad n(t_1, t_2) = n - x\}}{P\{X(t_2) = n\}}$$
$$= \frac{P\{X(t_1) = x\}\, P\{n(t_1, t_2) = n - x\}}{P\{X(t_2) = n\}}$$

(Since $\{X(t)\}$ and $n(t_1, t_2) = \{X(t_2) - X(t_1)\}$ are independent)

$$= \frac{\left\{e^{-\lambda t_1}(\lambda t_1)^x / x!\right\}\left\{e^{-\lambda(t_2 - t_1)}\{\lambda(t_2 - t_1)\}^{n-x} / (n-x)!\right\}}{e^{-\lambda t_2}(\lambda t_2)^n / n!}$$

$$= {}^nC_x \left(\frac{t_1}{t_2}\right)^x \left(1 - \frac{t_1}{t_2}\right)^{n-x}$$

$$= {}^nC_x p^x q^{n-x}$$

where $p = \frac{t_1}{t_2}$ and $q = 1 - p$

Therefore, $P[X(t_1) = x/\{X(t_2) = n\}]$ is binomial with parameters n and $p = \frac{t_1}{t_2}$.

Consider $P\{X(2) = 2/X(6) = 6\}$

$$\Rightarrow \quad x = 2,\ n = 6,\ t_1 = 2,\ t_2 = 6$$

$$\Rightarrow \quad p = \frac{t_1}{t_2} = \frac{2}{6} = \frac{1}{3}, \quad \Rightarrow q = 1 - p = \frac{2}{3}$$

$$\therefore \quad P\{X(2) = 2/X(6) = 6\} = {}^6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{6-2}$$

$$= \frac{(6)(5)}{(1)(2)} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{6-2} = (15)\frac{2^4}{3^6}$$

$$= \frac{(15)(16)}{729} = 0.3292$$

Problem 6. If $\{X(t)\}$ is a Poisson process such that $E\{X(9)\} = 6$ then (a) find the mean and variance of $X(8)$, (b) find $P\{X(4) \leq 5/X(2) = 3\}$ and (c) $P\{X(4) \leq 5/X(2) \leq 3\}$.

SOLUTION:

(a) It is given that $E\{X(9)\} = 6$

Since $\{X(t)\}$ is the Poisson process with parameter λt, we know that

$$E\{X(t)\} = \lambda t \quad \Rightarrow \quad E\{X(9)\} = \lambda(9) = 6$$

$$\Rightarrow \quad \lambda = \frac{6}{9} = \frac{2}{3}$$

Since $\{X(t)\}$ is the Poisson process with parameter λt, we know that

$$E\{X(t)\} = \lambda t \quad \Rightarrow \quad E\{X(8)\} = \lambda(8) = \frac{2}{3}(8) = \frac{16}{3}$$

Similarly,

$$V\{X(t)\} = \lambda t \quad \Rightarrow \quad V\{X(8)\} = \lambda(8) = \frac{2}{3}(8) = \frac{16}{3}$$

(b) We know that $P\{X(4) \le 5/X(2) = 3\} = \dfrac{P\{X(2) = 3, X(4) \le 5\}}{P\{X(2) = 3\}}$

Now consider $P\{X(4) \le 5/X(2) = 3\}$ which implies that less than or equal to 5 occurrences have occurred in the interval (0, 4) given that a maximum of 3 occurrences have occurred in the interval (0, 2). This follows that there have to be utmost 2 occurrences only in the interval (2, 4). Therefore, the required probability becomes

$$P\{X(4) \le 5/X(2) = 3\} = \frac{P\{X(2) = 3\}\, P\{n(2, 4) \le 2\}}{P\{X(2) = 3\}}$$

$$= P\{n(2, 4) \le 2\}$$

where $P\{n(t_1, t_2) \le x\}$ is described as follows:
The number of Poisson points occurred in the interval (t_1, t_2) of length $t = t_2 - t_1$ say $n(t_1, t_2)$, is a Poisson random variable with parameter λt. That is,

$$P\{n(t_1, t_2) = x\} = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, 3, \cdots\cdots$$

$$\Rightarrow \quad P\{n(2, 4) = x\} = \frac{e^{-2\lambda}(2\lambda)^x}{x!}, \quad x = 0, 1, 2, 3, \cdots\cdots$$

$$\therefore \quad P\{n(2, 4) \le 2\} = \sum_{x=0}^{2} \frac{e^{-2\lambda}(2\lambda)^x}{x!}$$

$$= \sum_{x=0}^{2} \frac{e^{-4/3}(4/3)^x}{x!}$$

$$= \frac{e^{-4/3}(4/3)0}{0!} + \frac{e^{-4/3}(4/3)^1}{1!} + \frac{e^{-4/3}(4/3)^2}{2!}$$

$$= e^{-4/3}\left\{\frac{(4/3)^0}{0!} + \frac{(4/3)^1}{1!} + \frac{(4/3)^2}{2!}\right\}$$

$$= (0.2636)\left\{1 + \frac{4}{3} + \frac{16}{18}\right\} = 0.8494$$

(c) We know that

$$P\{X(4) \le 5/X(2) \le 3\} = \frac{P\{X(2) \le 3,\, X(4) \le 5\}}{P\{X(2) \le 3\}}$$

$$= \frac{\sum_{k=0}^{3} \{P\{n(0,\,2) = k\}\, P\{n(2,\,4) \le 5-k\}\}}{\sum_{k=0}^{3} P\{n(0,\,2) = k\}}$$

$$= \frac{\sum_{k=0}^{3} \left\{P\{n(0,\,2) = k\} \sum_{r=0}^{5-k} P\{n(2,\,4) = r\}\right\}}{\sum_{k=0}^{3} P\{n(0,\,2) = k\}}$$

$$= \frac{\sum_{k=0}^{3} \left\{\left\{e^{-4/3}(4/3)^k/k!\right\} \sum_{r=0}^{5-k} \left\{e^{-4/3}(4/3)^r/r\right\}\right\}}{\sum_{k=0}^{3} P\{n(0,\,2) = k\}}$$

$$= \frac{1}{\sum_{k=0}^{3} \left[e^{-4/3}(4/3)^k/k!\right]} \left\{ \begin{array}{l} \left[e^{-4/3}(4/3)^0/0!\right] \sum_{r=0}^{5} e^{-4/3}(4/3)^r/r \\ + \left[e^{-4/3}(4/3)^1/1!\right] \sum_{r=0}^{4} e^{-4/3}(4/3)^r/r \\ + \left[e^{-4/3}(4/3)^2/2!\right] \sum_{r=0}^{3} e^{-4/3}(4/3)^r/r \\ + \left[e^{-4/3}(4/3)^3/3!\right] \sum_{r=0}^{2} e^{-4/3}(4/3)^r/r \end{array} \right\}$$

$$= \frac{1}{0.9535} \left\{ \begin{array}{l} (0.2636)\,(0.9975) + (0.3515)\,(0.9882) \\ +(0.2343)\,(0.9535) + (0.1041\,(0.8494) \end{array} \right\}$$

$$= \frac{0.9221}{0.9535} = 0.9671$$

Problem 7. Let $\{X(t)\}$ be the Poisson process with parameter λt such that $X(t) = 1$ if the number of occurrences (Poisson points) is even in the interval $(0, t)$ and $X(t) = -1$ if the number of occurrences is odd (this process is known as *semi random process* in *telegraphic signal* studies) obtain the mean and autocorrelation of the process.

SOLUTION:

Since $\{X(t)\}$ is the Poisson process with parameter λt we have

$$P\{X(t) = x\} = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0,\, 1,\, 2,\, 3, \cdots\cdots$$

Also we know that the number of occurrences denoted by $n(t_1, t_2)$ in the interval (t_1, t_2) of length $t = t_2 - t_1 > 0$ is Poisson with probability mass function

$$P\{n(t_1, t_2) = x\} = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, 3, \cdots\cdots$$

Now, we have

$$\begin{aligned} P\{X(t) = 1\} &= P\{n(0, t) = 0\} + P\{n(0, t) = 2\} + P\{n(0, t) = 4\} + \cdots\cdots \\ &= \frac{e^{-\lambda t}(\lambda t)^0}{0!} + \frac{e^{-\lambda t}(\lambda t)^2}{2!} + \frac{e^{-\lambda t}(\lambda t)^4}{4!} + \cdots\cdots \\ &= e^{-\lambda t}\left\{1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\cdots\right\} = e^{-\lambda t}\cosh\lambda t \end{aligned}$$

Similarly,

$$\begin{aligned} P\{X(t) = -1\} &= P\{n(0, t) = 1\} + P\{n(0, t) = 3\} + P\{n(0, t) = 5\} + \cdots\cdots \\ &= \frac{e^{-\lambda t}(\lambda t)^1}{1!} + \frac{e^{-\lambda t}(\lambda t)^3}{3!} + \frac{e^{-\lambda t}(\lambda t)^5}{5!} + \cdots\cdots \\ &= e^{-\lambda t}\left\{\lambda t + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^5}{5!} + \cdots\cdots\right\} = e^{-\lambda t}\sinh\lambda t \end{aligned}$$

Therefore, the mean $E\{X(t)\}$ of the process $\{X(t)\}$ can be obtained as

$$\begin{aligned} E\{X(t)\} &= \sum_{x=1,\,-1} xP\{X(t) = x\} \\ &= (1)P\{X(t) = 1\} + (-1)P\{X(t) = -1\} \\ &= (1)e^{-\lambda t}\cosh\lambda t + (-1)e^{-\lambda t}\sinh\lambda t \\ &= e^{-\lambda t}(\cosh\lambda t - \sinh\lambda t) \\ &= \left(e^{-\lambda t}\right)\left(e^{-\lambda t}\right) = e^{-2\lambda t} \end{aligned}$$

Therefore, the autocorrection $R_{xx}(t_1, t_2)$of the process $\{X(t)\}$ can be obtained as follows:

$$R_{xx}(t_1, t_2) = \sum_{x_1=1,\,-1}\sum_{x_2=1,\,-1} x_1x_2P\{X(t_1) = x_1, X(t_2) = x_2\}$$

If we let $t = t_1 - t_2 > 0$ and $X(t_2) = 1$ then for the even number of Poisson points in the interval (t_1, t_2), we have $X(t_2) = 1$. This gives

$$P\{X(t_1) = 1/X(t_2) = 1\} = P\{n(t_1, t_2) \text{ is even}\} = e^{-\lambda t}\cosh\lambda t$$

Now multiplying both sides by $P\{X(t_2)=1\}$, we have

$$P\{X(t_1)=1/X(t_2)=1\}P\{X(t_2)=1\}=\left(e^{-\lambda t}\cosh\lambda t\right)P\{X(t_2)=1\}$$

$$P\{X(t_1)=1,\,X(t_2)=1\}=\left(e^{-\lambda t}\cosh\lambda t\right)\,\left(e^{-\lambda t_2}\cosh\lambda t_2\right)$$

Consider

$$P\{X(t_1)=1/X(t_2)=-1\}=P\{n\,(t_1,\,t_2)\text{ is odd}\}=e^{-\lambda t}\sinh\lambda\,t$$

Now multiplying both sides by $P\{X(t_2)=-1\}$ and simplifying we have

$$P\{X(t_1)=1,\,X(t_2)=-1\}=\left(e^{-\lambda t}\sinh\lambda t\right)\,\left(e^{-\lambda t_2}\sinh\lambda t_2\right)$$

Consider

$$P\{X(t_1)=-1/X(t_2)=1\}=P\{n\,(t_1,\,t_2)\text{ is odd}\}=e^{-\lambda t}\sinh\lambda\,t$$

Now multiplying both sides by $P\{X(t_2)=1\}$ and simplifying we have

$$P\{X(t_1)=-1,\,X(t_2)=1\}=\left(e^{-\lambda t}\sinh\lambda t\right)\,\left(e^{-\lambda t_2}\cosh\lambda t_2\right)$$

Consider

$$P\{X(t_1)=-1/X(t_2)=-1\}=P\{n\,(t_1,\,t_2)\text{ is even}\}=e^{-\lambda t}\cosh\lambda\,t$$

Now multiplying both sides by $P\{X(t_2)=-1\}$ and simplifying we have

$$P\{X(t_1)=-1,\,X(t_2)=-1\}=\left(e^{-\lambda t}\cosh\lambda t\right)\,\left(e^{-\lambda t_2}\sinh\lambda t_2\right)$$

Therefore, $R_{xx}(t_1,\,t_2)$ becomes

$$R_{xx}(t_1,t_2)=(1)(1)P\{X(t_1)=1,X(t_2)=1\}+(1)(-1)P\{X(t_1)=1,X(t_2)=-1\}$$
$$+(-1)(1)P\{X(t_1)=-1,X(t_2)=1\}+(-1)(-1)P\{X(t_1)=-1,X(t_2)=-1\}$$
$$=\left(e^{-\lambda t}\cosh\lambda t\right)\,\left(e^{-\lambda t_2}\cosh\lambda t_2\right)-\left(e^{-\lambda t}\sinh\lambda t\right)\,\left(e^{-\lambda t_2}\sinh\lambda t_2\right)$$
$$-\left(e^{-\lambda t}\sinh\lambda\,t\right)\,\left(e^{-\lambda t_2}\cosh\lambda t_2\right)+\left(e^{-\lambda t}\cosh\lambda t\right)\,\left(e^{-\lambda t_2}\sinh\lambda t_2\right)$$

Combining appropriate terms, we have

$$R_{xx}(t_1,\,t_2)=e^{-\lambda t}e^{-\lambda t_2}\,(\cosh\lambda t\cosh\lambda t_2+\cosh\lambda t\sinh\lambda t_2)$$
$$-\,e^{-\lambda t}e^{-\lambda t_2}\,(\sinh\lambda t\sinh\lambda t_2+\sinh\lambda t\cosh\lambda t_2)$$

$$= e^{-\lambda(t+t_2)} \cosh\lambda t\,(\cosh\lambda t_2 + \sinh\lambda t_2)$$

$$- e^{-\lambda(t+t_2)} \sinh\lambda t\,(\sinh\lambda t_2 + \cosh\lambda t_2)$$

$$= e^{-\lambda(t+t_2)} e^{\lambda t_2}(\cosh\lambda t - \sinh\lambda t)$$

$$= e^{-\lambda(t+t_2)} e^{\lambda t_2}\, e^{-\lambda t} = e^{-2\lambda t}$$

Letting $t = t_1 - t_2$, we have

$$R_{xx}(t_1,\, t_2) = e^{-2\lambda(t_1 - t_2)}$$

Similarly, if we let $t = t_2 - t_1 > 0$ and proceed in the similar way, we get

$$R_{xx}(t_1,\, t_2) = e^{-2\lambda(t_2 - t_1)}$$

Combining, we finally get

$$R_{xx}(t_1,\, t_2) = e^{-2\lambda\,|t_2 - t_1|}$$

Problem 8. Let $\{X(t)\}$ be a Poisson process with parameter λt and let us suppose that each occurrence gets tagged independently with probability $p = \frac{2}{3}$. If the average rate of occurrence is 3 per minute then obtain the probability that exactly 4 occurrences are tagged in the time interval (0, 2).

SOLUTION:

If we let $\{Y(t)\}$ as the number of tagged occurrences then we know that

$$P\{Y(t) = x\} = \frac{e^{-\lambda pt}(\lambda pt)^x}{x!}$$

It is given that $\lambda = 3,\ p = \frac{2}{3},\quad t = 2$

Therefore, the probability that exactly 4 occurrences are tagged in the time interval (0, 2) can be obtained as

$$P\{Y(2) = 4\} = \frac{e^{-(3)\left(\frac{2}{3}\right)(2)}\left[(3)\left(\frac{2}{3}\right)(2)\right]^4}{4!} = \frac{e^{-4}4^4}{4!} = 0.1954$$

Problem 9. If arrival of customers at a counter is in accordance with a Poisson process with a mean arrival rate of 2 per minute, then find the probability that the interval between 2 consecutive arrivals is (i) more than 1 minute, (ii) between 1 and 2 minutes and (iii) less than or equal to 4 minutes.

SOLUTION:

Let $\{X(t)\}$ be a Poisson process with parameter λt where t is the time between arrivals. If we let X as the random variable representing the time between arrivals then it follows exponential distribution whose probability density function is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

It is given that the average arrival rate is according to Poisson and is $\lambda = 2$

$$\therefore \quad f(x) = 2e^{-2x}, \quad x > 0$$

(i) The probability that the time interval between two consecutive arrivals is more than 1 minute is obtained as

$$P\{X > 1\} = \int_1^\infty \lambda e^{-\lambda x} dx = \int_1^\infty 2e^{-2x} dx = 2\left[\frac{e^{-2x}}{-2}\right]_1^\infty$$

$$= 2\left(0 - \frac{e^{-2}}{-2}\right) = e^{-2} = 0.1353$$

(ii) The probability that the time interval between two consecutive arrivals is between 1 and 2 minutes

$$P\{1 < X < 2\} = \int_1^2 \lambda e^{-\lambda x} dx = \int_1^2 2e^{-2x} dx = 2\left[\frac{e^{-2x}}{-2}\right]_1^2$$

$$= \left(-e^{-4} + e^{-2}\right) = 0.1170$$

(iii) The probability that the time interval between two consecutive arrivals is less than or equal to 4 minutes

$$P\{X \leq 4\} = \int_0^4 \lambda e^{-\lambda x} dx = \int_0^4 2e^{-2x} dx = 2\left[\frac{e^{-2x}}{-2}\right]_0^4$$

$$= \left(-e^{-8} + 1\right) = 0.9997$$

Problem 10. Consider a random telegraph signal process $\{X(t)\}$ in which the sample function is $X(t) = 0$ or $X(t) = 1$. It is supposed that the process starts at time $t = 0$ in the zero state $X(t) = 0$ and then it remains there for a time interval equal to T_1 at which point it switches to the state $X(t) = 1$ and remains there for a time interval equal to T_2 then switches state again and so on. Find the first order probability mass function of the process and hence find the mean of the process.

SOLUTION:

Since the given telegraph signal process $\{X(t)\}$ is binary valued, any sample will be a Bernoulli random variable. That is, if $X_k = X(t_k)$ is a Bernoulli variable, then it is required to find the probabilities of $X_k = X(t_k) = 0$ and $X_k = X(t_k) = 1$.

Let us suppose that there are exactly n switches in the time interval $(0, t_k)$. Then $S_n = T_1 + T_2 + \cdots\cdots + T_n$ is the random variable representing the time taken for n switches. We know that

$$P(n \text{ switches in } (0, t_k) = P\{X(t_k) = n\} = \frac{e^{-\lambda t_k}(\lambda t_k)^n}{n!}, \quad n = 0, 1, 2, \cdots\cdots$$

This is Poisson with parameter λt_k.

Therefore, the number of switches in the time interval $(0, t_k)$ follows a Poisson distribution. Since the sample member $X(t_k) = 0$ of the random process will be equal to 0 if the number of switches is even, we have

$$P\{X(t_k) = 0\} = \sum_{n \text{ is even}} P(n \text{ switches in } (0, t_k) = \sum_{n \text{ is even}} \frac{e^{-\lambda t_k}(\lambda t_k)^n}{n!}$$

$$= e^{-\lambda t_k}\cosh(\lambda t_k) = \frac{1}{2}\left(1 + e^{-2\lambda t_k}\right)$$

Similarly, the sample member $X(t_k) = 1$ of the random process will be equal to 1 if the number of switches is odd, we have

$$P\{X(t_k) = 1\} = \sum_{n \text{ is odd}} P(n \text{ switches in } (0, t_k) = \sum_{n \text{ is odd}} \frac{e^{-\lambda t_k}(\lambda t_k)^n}{n!}$$

$$= e^{-\lambda t_k}\sinh(\lambda t_k) = \frac{1}{2}\left(1 - e^{-2\lambda t_k}\right)$$

Therefore, the probability mass function of the telegraphic signal process can be described by a Bernoulli distribution given by

$X(t_k) = n$	0	1
$P\{X(t_k) = n\}$	$\frac{1}{2}\left(1 + e^{-2\lambda t_k}\right)$	$\frac{1}{2}\left(1 - e^{-2\lambda t_k}\right)$

Therefore, the mean of the process can be obtained as

$$E\{X(t_k)\} = (0)P\{X(t_k) = 0\} + (1)P\{X(t_k) = 1\} = \frac{1}{2}\left(1 - e^{-\lambda t_k}\right)$$

It may be noted that for large t_k many switches will likely to occur and in this case it is equally likely that the process will take on the values 0 or 1 with equal probabilities.

EXERCISE PROBLEMS

1. A random process $\{Y_n\}$ is defined by $Y_n = aX_n + b$, where $\{X_n\}$ is a Bernoulli process that assumes 1 or 0 with equal probabilities. Find the mean and variance of $\{Y_n\}$.

2. If $\{X_1(t)\}$ is a Poisson process with rate of occurrence $\lambda_1 = 2$ and $\{X_2(t)\}$ is another independent Poisson process with rate of occurrence $\lambda_2 = 3$. Then obtain (i) the probability mass function of the random process $\{Y(t)\}$ where $Y(t) = X_1(t) + X_2(t)$, (ii) Find $P\{Y(2) = 5\}$ and (iii) the mean and variance of $\{Y(t)\}$ and also (iv) obtain the parameters under these processes when $t = 2$.

3. Suppose that customers are arriving at a ticket counter according to a Poisson process with a mean rate of 2 per minute. Then, in an interval of 5 minutes, find the probabilities that the number of customers arriving is (i) exactly 3, (ii) greater than 3 and (iii) less than 3.

4. Patients arrive at the doctor's clinic according to a Poisson process with rate parameter $\lambda = 1/10$ minutes. The doctor will not attend a patient until at least three patients are in the waiting room. Then

   (i) find the expected waiting time until the first patient is admitted to see the doctor; and

   (ii) what is the probability that no patient is admitted to see the doctor in the first one hour.

5. Let T_n denote the time taken for the occurrence of the n^{th} event of a Poisson process with rate parameter λ. Let us suppose that one event has occurred in the time interval $(0, t)$. Then obtain the conditional distribution of arrival time T_1 over $(0, t)$.

6. Let T_n denote the time taken for the occurrence of the n^{th} event of a Poisson process with rate parameter λ. Let us suppose that one event has occurred in the time $(0, 10)$ interval. Then obtain $P\{T_1 \leq 4/X(10) \leq 1\}$.

7. It is given that $\{X_1(t)\}$ and $\{X_2(t)\}$ represent two independent Poisson processes and $X_1(2)$ and $X_2(2)$ are random variables observed from these processes at $t = 2$ with parameters 6 and 8 respectively. Then obtain $P\{Y(2) = 1\}$, where $Y(2) = X_1(2) + X_2(2)$.

8. Suppose the arrival of calls at a switch board is modeled as a Poisson process with the rate of calls per minute being $\lambda = 0.1$. Then

   (i) What is the probability that the number of calls arriving in a 10 minutes interval is less than 3?

   (ii) What is the probability that one call arrives during the first 10 minutes interval and two calls arrive during the second 10 minutes interval?

9. If $\{X(t)\}$ is a Poisson process such that $E\{X(8)\} = 6$ then (a) find the mean and variance of $X(7)$, (b) find $P\{X(3) \leq 3/X(1) \leq 1\}$.
10. Let $\{X(t)\}$ be a Poisson process with parameter λt and let us suppose that each occurrence gets tagged independently with probability $p = \frac{3}{4}$. If the average rate of occurrence is 4 per minute then obtain the probability that exactly 3 occurrences are tagged in the time interval $(0, 3)$.

(iv) If $X(t)$ and $Y(t)$ are two random variables of the normal processes $\{X(t)\}$ and $\{Y(t)\}$ with zero means, then we have

$$E\left\{X^2(t)Y^2(t)\right\} = E[X^2(t)]E[Y^2(t)] + 2\{E[X(t)Y(t)]\}^2 \tag{6.8}$$

(v) If $X(t)$ and $Y(t)$ are two random variables of the normal processes $\{X(t)\}$ and $\{Y(t)\}$ with zero means, variances σ_x^2 and σ_y^2, and correlation coefficient $\rho_{xy}(t_1,t_2)$, then we have

$$P\{X(t_1)Y(t_2) = positive\} = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\rho_{xy}(t_1,t_2) \tag{6.9}$$

which gives the probability that $X(t)$ and $Y(t)$ are of same signs and,

$$P\{X(t_1)Y(t_2) = negative\} = \frac{1}{2} - \frac{1}{\pi}\sin^{-1}\rho_{xy}(t_1,t_2) \tag{6.10}$$

which gives the probability that $X(t)$ and $Y(t)$ are of different signs.

(vi) If $X(t)$ and $Y(t)$ are two random variables of the normal processes $\{X(t)\}$ and $\{Y(t)\}$ with zero means, variances σ_x^2 and σ_y^2, and correlation coefficient, $\rho_{xy}(t_1,t_2)$ then we have

$$E\{|X(t_1Y(t_2))|\} = \frac{2}{\pi}\sigma_x\,\sigma_y(\cos\alpha + \alpha\sin\alpha) \tag{6.11}$$

where

$$\sin\alpha = \rho_{xy}(t_1,t_2) = \frac{R_{xy}(t_1,t_2)}{\sigma_x\,\sigma_y} \tag{6.12}$$

Note:

If $X(t_1)$ and $X(t_2)$ are two random variables of the same normal process observed at time points t_1 and t_2, with zero means, variances σ_x^2 and σ_y^2, and correlation coefficient, $\rho_{xy}(t_1,t_2)$, then we have

$$\rho_{xx}(t_1,t_2) = \frac{R_{xx}(t_1,t_2)}{\sigma_x^2} = \frac{R_{xx}(t_1,t_2)}{R_{xx}(0)} = \sin\alpha$$

Since normal process is stationary process, we have

$$\rho_{xx}(\tau) = \frac{R_{xx}(\tau)}{\sigma_x^2} = \frac{R_{xx}(\tau)}{R_{xx}(0)} = \sin\alpha \tag{6.13}$$

Following are the different possibilities and their respective probabilities

$X(t_1)$	+	+	–	–
$X(t_2)$	+	–	+	–
$X(t_1)X(t_2)$	+	–	–	+

$$\therefore \quad P\{X(t_1)X(t_2) = +\} = \frac{1}{2}, \quad P\{X(t_1)X(t_2) = -\} = \frac{1}{2}$$

$$\Rightarrow \quad E\{X(t_1)Y(t_2)/X(t_1)X(t_2)\} = \frac{1}{2}\{X(t_1)X(t_2) + |X(t_1)X(t_2)|\}\left(\frac{1}{2}\right) + (0)\left(\frac{1}{2}\right)$$

$$= \frac{1}{4}\{X(t_1)X(t_2) + |X(t_1)X(t_2)|\}$$

$$\therefore \quad E\{Z(t_1)Z(t_2)\} = \frac{1}{4}\{E\{X(t_1)X(t_2)\} + E\{|X(t_1)X(t_2)|\}\}$$

$$\Rightarrow \quad R_{zz}(t_1, t_2) = \frac{1}{4}\{R_{xx}(t_1, t_2) + R_{yy}(t_1, t_2)\}$$

where $R_{yy}(t_1 t_2)$ is the autocorrelation of the full-wave linear detector process $Y(t) = |X(t)|$.

Since the processes $\{X(t)\}$ and $\{Y(t)\}$ are stationary, we have

$$\therefore \quad R_{zz}(\tau) = \frac{1}{4}\left\{R_{xx}(\tau) + \frac{2}{\pi}R_{xx}(0)(\cos\alpha + \alpha\sin\alpha\right\}, \quad \sin\alpha = \frac{R_{xx}(\tau)}{R_{xx}(0)}$$

$$\tau = 0 \Rightarrow \sin\alpha = 1 \text{ but } \cos^2\alpha = 1 - \sin^2\alpha \Rightarrow \cos\alpha = 0 \Rightarrow \alpha = \frac{\pi}{2}$$

$$\therefore \quad R_{zz}(0) = \frac{1}{4}\left\{R_{xx}(0) + \frac{2}{\pi}R_{xx}(0)\frac{\pi}{2}\right\} = \frac{1}{2}R_{xx}(0)$$

$$\Rightarrow \quad E\left\{Z^2(t)\right\} = R_{zz}(0) = \frac{1}{2}R_{xx}(0)$$

Consider $\quad E\{Z(t)\} = \dfrac{1}{2}\{E\{X(t)\} + E\{|X(t)|\}\}$

$$= \frac{1}{2}\{0 + E\{Y(t)\}\} = \frac{1}{2}\sqrt{\frac{2}{\pi}R_{xx}(0)}$$

$$= \sqrt{\frac{1}{2\pi}R_{xx}(0)} \quad \because \quad Y(t) = |X(t)|$$

$$\therefore \quad V\{Z(t)\} = E\left\{Z^2(t)\right\} - \{E[Z(t)]\}^2$$

$$= \frac{1}{2}R_{xx}(0) - \left(\sqrt{\frac{1}{2\pi}R_{xx}(0)}\right)^2 = \frac{1}{2}\left(1 - \frac{1}{\pi}\right)R_{xx}(0)$$

Alternative proof for $E\{Z(t)\}$, $E\{Z^2(t)\}$ and $V\{Z(t)\}$

$$E\{Z(t)\} = E\{X(t)\} = \int_0^\infty x \frac{1}{\sqrt{2\pi}\sigma_x} e^{-x^2/2\sigma_x^2} dx = \frac{1}{\sqrt{2\pi}\sigma_x} \int_0^\infty x e^{-x^2/2\sigma_x^2} dx$$

Let $v = \dfrac{x^2}{2\sigma_x^2} \Rightarrow xdx = \sigma_x^2 dv$

$$\Rightarrow \quad \frac{2}{\sqrt{2\pi}\sigma_x} \int_0^\infty x e^{-x^2/2\sigma_x^2} dx = \frac{1}{\sqrt{2\pi}} \sigma_x \int_0^\infty e^{-v} dv = \frac{1}{\sqrt{2\pi}} \sigma_x \quad \int_0^\infty e^{-v} dv = 1$$

$$\therefore \quad E\{Z(t)\} = \frac{1}{\sqrt{2\pi}} \sigma_x = \sqrt{\frac{1}{2\pi}\sigma_x^2} = \sqrt{\frac{1}{2\pi} R_{xx}(0)}$$

$$\therefore \quad E\left\{Z^2(t)\right\} = E\left\{X^2(t)\right\} = \int_0^\infty x^2 \frac{1}{\sqrt{2\pi}\sigma_x} e^{-x^2/2\sigma_x^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} \int_0^\infty x^2 e^{-x^2/2\sigma_x^2} dx$$

Let $v = \dfrac{x^2}{2\sigma_x^2} \quad \Rightarrow \quad xdx = \sigma_x^2 dv$

$$\Rightarrow \quad \frac{2}{\sqrt{2\pi}\sigma_x} \int_0^\infty x^2 e^{-x^2/2\sigma_x^2} dx = \frac{1}{\sqrt{2\pi}\sigma_x} \int_0^\infty (2v\sigma_x^2) e^{-v} \left(\frac{\sigma_x^2}{\sqrt{2\sigma_x^2 v}} \right) dv$$

$$= \frac{\sigma_x^2}{\sqrt{\pi}} \int_0^\infty \sqrt{v} e^{-v} dv = \frac{1}{2}\sigma_x^2$$

$$\because \quad \int_0^\infty v^{\frac{3}{2}-1} e^{-v} dv = \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$$

$$\therefore \quad E\left\{Z^2(t)\right\} = \frac{1}{2}\sigma_x^2 = \frac{1}{2} R_{xx}(0)$$

Hence,

$$V\{Z(t)\} = E\left\{Z^2(t)\right\} - \{E[Z(t)]\}^2$$

$$= \frac{1}{2} R_{xx}(0) - \left(\sqrt{\frac{1}{2\pi} R_{xx}(0)} \right)^2 = \frac{1}{2}\left(1 - \frac{1}{\pi}\right) R_{xx}(0)$$

SOLVED PROBLEMS

Problem 1. Given a normal process with mean 0, autocorrelation function $R(\tau) = 4e^{-3|\tau|}$, where $\tau = t_1 - t_2$, and the random variables $Y = X(t+1)$ and $W = X(t-1)$ then find

(a) (i) $E(YW)$, (ii) $E\{(Y+W)^2\}$ and (iii) the correlation coefficient between Y and W.

(b) (i) Find probability density function $f(y)$, (ii) cumulative probability $P(Y < 1)$ and (iii) the joint probability density function $f(y, w)$.

(a) SOLUTION:

(i) Consider

$$E(YW) = E[X(t+1)X(t-1)]$$
$$= R(t+1,\ t-1)$$
$$= R(2) = 4e^{-3(2)} = 4e^{-6} = 0.0099$$

(ii) Consider

$$E\left\{(Y+W)^2\right\} = E\left\{X^2(t+1) + X^2(t-1) + 2X(t+1)X(t-1)\right\}$$
$$E\left\{(Y+W)^2\right\} = E\left\{X^2(t+1)\right\} + E\left\{X^2(t-1)\right\} + 2E\{X(t+1)X(t-1)\}$$
$$E\left\{(Y+W)^2\right\} = R(0) + R(0) + 2R(2)$$
$$= 2[R(0) + R(2)]$$
$$= 2\left\{4 + 4e^{-6}\right\} = 8.0198$$

(iii) Consider

$$\rho_{ZW} = \frac{Cov(Y, W))}{\sqrt{V(Y)}\sqrt{V(W)}}$$
$$= \frac{E(YW) - E(Y)E(W)}{\sqrt{E(Y^2) - [E(Y)]^2}\sqrt{E(W^2) - [E(W)]^2}}$$
$$= \frac{E(YW)}{\sqrt{E(Y^2)}\sqrt{E(W^2)}} \qquad (\text{Since } E(Y) = E(W) = 0)$$
$$= \frac{E(YW)}{\sqrt{R(0)}\sqrt{R(0)}} = \frac{E(YW)}{R(0)} = \frac{0.0099}{4} = 0.00248$$

(b) Solution:

(i) It is given that mean $E(Y) = \mu_y = 0$
We know that standard deviation $\sigma_y = \sqrt{R(0)} = \sqrt{4} = 2$
This implies that the random variable Y is normal with mean 0 and standard deviation 2.
Therefore, the probability density function of Y is given as

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}$$
$$= \frac{1}{2\sqrt{2\pi}} \exp\left(\frac{-y^2}{8}\right) \quad \text{for} \quad -\infty < y < \infty$$

(ii) Consider

$$P(Y < 1) = P(X(t+1) < 1)$$
$$= P\left(\frac{X(t+1) - E\{X(t+1)\}}{\sqrt{V\{X(t+1)\}}} < \frac{1 - E\{X(t+1)\}}{\sqrt{E\{X^2(t+1))\}}}\right)$$
$$= P\left(Z < \frac{1 - E\{X(t+1)\}}{\sqrt{E\{X^2(t+1))\}}}\right)$$

where Z is standard normal variable with mean 0 and standard deviation 1. Also

$$V\{X(t+1)\} = E\left\{X^2(t+1)\right\} = R(0) = 4$$

Therefore, we have

$$P(Z < z) = P\left(Z < \frac{1 - E\{X(t+1)\}}{\sqrt{E\{X^2(t+1))\}}}\right) = P\left(Z < \frac{1}{2}\right) = 0.6915$$

(iii) We have shown that Y is standard normal variable with mean 0 and standard deviation 2. Similarly, we can show that W is also standard normal variable with mean 0 and standard deviation 2. Since it is not known that Z and W are uncorrelated random variables, the correlation, r, between these random variables can be obtained as follows:

$$r = \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1)}\sqrt{C(t_2, t_2)}} = \frac{C(\tau)}{C(0)} = \frac{R(\tau)}{R(0)} = \frac{4e^{-3|\tau|}}{4} = e^{-3|\tau|}$$

Since $Y = X(t+1)$ and $W = X(t-1)$, we have $\tau = 2$

$$\therefore \quad r = e^{-3|\tau|} = e^{-6} = 0.00248 \Rightarrow \sqrt{1-r^2} = 0.999 \approx 1$$

Also we have $\sigma^2 = R(0) = 4$

Now we know that the joint density function $f(y, w)$ can be obtained as (refer to Section 1.8.4 of Chapter 1)

$$\begin{aligned} f(y, w) &= \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2\sigma^2(1-r^2)}\left(y^2 - 2ryw + w^2\right)\right\} \\ &= \frac{1}{2\pi(4)(1)} \exp\left\{-\frac{1}{2(4)(1)}\left(y^2 - 2(0.00248)yw + w^2\right)\right\} \\ &= \frac{1}{8\pi} \exp\left\{-\frac{1}{8}\left(y^2 - (0.005)yw + w^2\right)\right\}, \\ & \quad -\infty < y, w < +\infty \end{aligned}$$

Problem 2. Suppose that $\{X(t)\}$ is a random process with $\mu(t) = 3$ and $C(\tau) = 4e^{-0.2|\tau|}$, where $\tau = t_1 - t_2$. Find (i) $P[X(5) \le 2]$ and (ii) $P[|X(8) - X(5)| \le 1]$ using central limit theorem.

SOLUTION:

It is given that $E\{X(t)\} = \mu(t) = 3$ and $V\{X(t)\} = C(0) = 4e^{-0.2|0|} = 4$

(i) Consider

$$\begin{aligned} P[X(5) \le 2] &= P\left(\frac{X(5) - E\{X(5)\}}{\sqrt{V\{X(5)\}}} < \frac{2 - E\{X(5)\}}{\sqrt{V\{X(5)\}}}\right) \\ &= P\left(Z < \frac{2-\mu(5)}{\sqrt{C(0)}}\right) = P\left(Z < \frac{2-3}{\sqrt{4}}\right) \\ &= P(Z < -0.5) = 0.309 \end{aligned}$$

(ii) Consider

$$P\{|X(8) - X(5)| \le 1\} = P\left(\left|\frac{X(8) - X(5) - E\{X(8) - X(5)\}}{\sqrt{V\{X(8) - X(5)\}}}\right| < \frac{1 - E\{X(8) - X(5)\}}{\sqrt{V\{X(8) - X(5)\}}}\right)$$

But we know that

$$E\{X(8) - X(5)\} = E\{X(8)\} - E\{X(5)\} = 3 - 3 = 0$$

$$V\{X(8)-X(5)\} = V\{X(8)\} + V\{X(5)\} - 2C(8, 5)$$

$$= 4 + 4 - 2\left\{4e^{-0.2|8-5|}\right\} = 3.608$$

$$\therefore \quad P\{|X(8)-X(5)| \le 1\} = P\left(|Z| < \frac{1 - E\{X(8)-X(5)\}}{\sqrt{V\{X(8)-X(5)\}}}\right)$$

$$= P\left(|Z| \le \frac{1-0}{\sqrt{3.608}}\right) = P(|Z| \le 0.526)$$

$$= P(-0.526 \le Z \le 0.526)$$

$$= 0.40$$

Problem 3. If $\{X(t)\}$ is a Gaussian process with mean $\mu(t) = 10$ and autocovariance $C(t_1, t_2) = 16e^{-0.2|t_1 - t_2|}$, then find (i) $P[X(10) \le 8]$ and (ii) $P[|X(10) - X(6)| \le 4]$ using central limit theorem.

SOLUTION:

It is given that $\{X(t)\}$ is a Gaussian process with $E\{X(t)\} = \mu(t) = 10$ and $V\{X(t)\} = C(0) = 16e^{-0.2|0|} = 16$

(i) Consider

$$P[X(10) \le 8] = P\left(\frac{X(10) - E\{X(10)\}}{\sqrt{V\{X(10)\}}} < \frac{8 - E\{X(10)\}}{\sqrt{V\{X(10)\}}}\right)$$

$$= P\left(Z < \frac{8 - \mu(10)}{\sqrt{C(0)}}\right) = P\left(Z < \frac{8-10}{\sqrt{16}}\right)$$

$$= P(Z < -0.5) = 0.309$$

(ii) Consider

$$P\{|X(10) - X(6)| \le 4\} = P\left(\left|\frac{X(10) - X(6) - E\{X(10) - X(6)\}}{\sqrt{V\{X(10) - X(6)\}}}\right| < \frac{4 - E\{X(10) - X(6)\}}{\sqrt{V\{X(10) - X(6)\}}}\right)$$

But we know that

$$E\{X(10) - X(6)\} = E\{X(10)\} - E\{X(6)\} = 10 - 10 = 0$$

$$V\{X(10)-X(6)\} = V\{X(10)\}+V\{X(6)\}-2C(10,6)$$
$$= 16+16-2\left\{16e^{-|10-6|}\right\} = 31.4139$$

$$P\{|X(10)-X(6)| \leq 4\} = P\left(|Z| < \frac{4-E\{X(10)-X(6)\}}{\sqrt{V\{X(10)-X(6)\}}}\right)$$

$$= P\left(|Z| \leq \frac{4-0}{\sqrt{31.4139}}\right) = P(|Z| \leq 0.7137)$$
$$= P(-0.7137 \leq Z \leq 0.7137)$$
$$= 0.48$$

Problem 4. If $\{X(t)\}$ is a zero mean stationary Gaussian process with $Y(t) = X^2(t)$ then show that $C_{yy}(\tau) = 2C_{xx}^2(\tau)$.

SOLUTION:

It is given that $Y(t) = X^2(t)$ and $E\{X(t)\} = 0$

We know that $C_{yy}(t_1,\, t_2) = R_{yy}(t_1,\, t_2) - E\{Y(t_1)\}E\{Y(t_2)\}$

$$= E\{Y(t_1)Y(t_2)\} - E\{Y(t_1)\}E\{Y(t_2)\}$$
$$= E\left\{X^2(t_1)X^2(t_2)\right\} - E\left\{X^2(t_1)\right\}E\left\{X^2(t_2)\right\}$$

But $E\left\{X^2(t_1)X^2(t_2)\right\} = E\left\{X^2(t_1)\right\}E\left\{X^2(t_2)\right\} + 2\left\{E[X(t_1)X(t_2)]\right\}^2$

$$\therefore \quad C_{yy}(t_1,\, t_2) = E\left\{X^2(t_1)\right\}E\left\{X^2(t_2)\right\} + 2\left\{E[X(t_1)X(t_2)]\right\}^2$$
$$- E\left\{X^2(t_1)\right\}E\left\{X^2(t_2)\right\}$$
$$= 2\left\{E[X(t_1)X(t_2)]\right\}^2 = 2\left\{C_{xx}(t_1,\, t_2)\right\}^2$$

$$\therefore \quad C_{yy}(t_1,\, t_2) = 2\left\{C_{xx}(t_1,\, t_2)\right\}^2$$

Since $\{X(t)\}$ is a stationary process, we have

$$C_{yy}(\tau) = 2\left\{C_{xx}(\tau)\right\}^2$$

Problem 5. If $\{X(t)\}$ is a zero mean stationary Gaussian process with mean $\mu(t) = 0$ and autocorrelation function $R_{xx}(\tau) = 4e^{-3|\tau|}$ then find a system $g(x)$ such that the first order density $f(y;t)$ of the resulting output $Y(t) = g\{X(t)\}$ is uniform in the interval $(6, 9)$.

SOLUTION:

It is given that $E\{X(t)\} = \mu(t) = 0$ and $R_{xx}(\tau) = 4e^{-3|\tau|}$

$$\Rightarrow \quad \sigma^2(t) = R_{xx}(0) = 4e^{-3|0|} = 4$$

Therefore, $\{X(t)\}$ is normal with mean 0 and variance 4 and hence the first order probability density function becomes

$$f(x,\ t) = \frac{1}{2\sqrt{2\pi}}e^{-\frac{x^2}{8}}, \quad -\infty < x < \infty$$

Since $Y(t)$ is uniform in the interval (6, 9), we have

$$f(y,\ t) = \frac{1}{3}, \quad 6 < y < 9$$

It may be noted that x and y are the realizations of $X(t)$ and $Y(t)$ respectively.

We know that the probability density function $f(y,\ t)$ of $Y(t)$ can also be expressed as

$$f(y;t) = \frac{f(g(y);t)}{|J|}$$

Where $x = g(y)$ and $J = g'(y)$

$$\Rightarrow \ g'(x) = \frac{f(g(y);t)}{f(y;t)} = (3)\,f(g(y);t) = \frac{3}{2\sqrt{2\pi}}e^{-x^2/8}$$

$$\Rightarrow \ g(x) = \frac{3}{2}\int \frac{1}{\sqrt{2\pi}}e^{-x^2/8} + C \tag{1}$$

Alternative proof:

It is given that $Y(t) = g\{X(t)\} \ \Rightarrow \ X(t) = g^{-1}\{Y(t)\}$

Consider the cumulative distribution function

$$F(y;t) = P\{Y(t) \le y\} = P\{g[X(t)] \le y\} = P\left\{X(t) \le g^{-1}(y)\right\}$$

$$= P\{X(t) \le x\} = F(x;\ t)$$

We know that $F(y;t) = \int\limits_6^y f(y,\ t)dy = \frac{1}{3}(y-6)$

$$\therefore \quad \frac{1}{3}\{g(x) - 6\} = F(x;t) \ \Rightarrow \ g(x) = 6 + 3F(x;t)$$

$$= 6 + \frac{3}{2}\int\limits_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-x^2/8}dx \tag{2}$$

From (1) and (2), we have $C = 6$

We can also write this expression as

$$g(x) = 6 + \frac{3}{2}P\{X(t) \leq x\} = 6 + \frac{3}{2}P\left\{Z(t) \leq \frac{x}{2}\right\}$$

where $Z(t)$ is the random variable of standard normal process $\{Z(t)\}$.

Problem 6. It is given that $\{X(t)\}$ is a random process such that $X(t) = Y\cos\omega t + W\sin\omega t$, where ω is constant and Y and W are two independent normal random variables with $E(Y) = E(W) = 0$ and $E(Y^2) = E(W^2)$, then prove that $\{X(t)\}$ is a stationary process of order 2.

SOLUTION:

It is given that $E(Y) = E(W) = 0$ and $E(Y^2) = E(W^2) \Rightarrow \quad V(Y) = E(W)$

Let $V(Y) = E(W) = \sigma^2$

Since $X(t)$ is a linear combination of two independent random variables Y and W, we know that $X(t)$ is also a normal random variable with mean and variance given as

$$E\{X(t)\} = E\{Y\cos\omega t + W\sin\omega t\} = \cos\omega t E(Y) + \sin\omega t E(W) = 0$$

$$\because \quad E(Y) = E(W) = 0$$

$$V\{X(t)\} = V\{Y\cos\omega t + W\sin\omega t\} = \cos^2\omega t V(Y) + \sin^2\omega t V(W) = \sigma^2$$

Therefore, $X(t)$ follows a normal distribution with mean 0 and variance σ^2. If we consider two random variables $X(t_1)$ and $X(t_2)$ then each of these random variables is normal with mean 0 and variance σ^2 and their joint probability density can be given as (refer to Section 1.8.4 of Chapter 1)

$$f(x_1, x_2; t_1, t_2) = \frac{e^{-\frac{1}{2[1-\rho^2(t_1, t_2)]\sigma^2}\left\{x_1^2 - 2\rho(t_1, t_2)x_1x_2 + x_2^2\right\}}}{\sqrt{2\pi}\,\sigma^2\sqrt{[1-\rho^2(t_1, t_2)]}},$$

$$-\infty < x_1, x_2 < \infty$$

where $\rho(t_1, t_2)$ is the correlation coefficient between $X(t_1)$ and $X(t_2)$ which is given by

$$\rho(t_1, t_2) = \frac{C_{xx}(t_1, t_2)}{\sqrt{V\{X(t_1)\}}\sqrt{V\{X(t_2)\}}} = \frac{E\{X(t_1)X(t_2)\}}{\sigma^2}$$

Sine mean and variance of the random process $\{X(t)\}$ are constants, the joint density probability function $f(x_1, x_2; t_1, t_2)$ of $X(t_1)$ and $X(t_2)$ depends only on $\rho(t_1, t_2)$ as it is a function of the time points t_1 and t_2. We know that the random process $\{X(t)\}$ is stationary in second order if

$$f(x_1, x_2;\ t_1, t_2) = f(x_1, x_2;\ t_1 + \tau,\ t_2 + \tau)$$

Which implies that the second order probability density function is time invariant. Clearly, it is true if $\rho(t_1,\ t_2) = \rho(t_1 + \tau,\ t_2 + \tau) = \rho(\tau)$, where $\tau = t_1 - t_2$ or $\tau = t_2 - t_1$. Therefore, in order to show that the random process $\{X(t)\}$ is stationary in second order, if is sufficient to show that the correlation coefficient between $X(t_1)$ and $X(t_2)$ is time invariant. That is,

$$\rho(t_1,\ t_2) = \rho(t_1 + \tau,\ t_2 + \tau)$$

Now, consider

$$\rho(t_1,\ t_2) = \frac{1}{\sigma^2} E\left\{(Y\cos\omega t_1 + W\sin\omega t_1)(Y\cos\omega t_2 + W\sin\omega t_2)\right\}$$

$$= \frac{1}{\sigma^2}\left\{\begin{array}{l} \cos\omega t_1 \cos\omega t_2 E(Y^2) + \sin\omega t_1 \sin\omega t_2 E(W^2) \\ \quad + (\cos\omega t_1 \sin\omega t_2 + \sin\omega t_1 \cos\omega t_2)E(YW) \end{array}\right\}$$

$$= \cos\omega t_1 \cos\omega t_2 + \sin\omega t_1 \sin\omega t_2$$

$$\left(\because \quad E(Y^2) = E(W^2) = \sigma^2 \quad \text{and} \quad E(YW) = 0\right)$$

$$= \cos(\omega t_1 - \omega t_2) = \cos\omega(t_1 - t_2) = \cos\omega\tau$$

$$\therefore \quad \rho(t_1,\ t_2) = \rho(\tau) = \cos\omega\tau$$

Similarly, the correlation coefficient between $X(t_1 + \tau)$ and $X(t_2 + \tau)$ can be obtained as

$$\rho(t_1 + \tau,\ t_2 + \tau) = \cos\omega[(t_1 + \tau) - (t_2 + \tau) = \cos\omega(t_1 - t_2)$$

$$= \cos\omega\tau = \rho(\tau)$$

$$\therefore \quad \rho(t_1,\ t_2) = \rho(t_1 + \tau,\ t_2 + \tau) = \cos\omega\tau = \rho(\tau)$$

This implies that $f(x_1, x_2;\ t_1, t_2) = f(x_1, x_2;\ t_1 + \tau,\ t_2 + \tau)$, is the second order joint probability density function of $X(t_1)$ and $X(t_2)$ a nd that of $X(t_1 + \tau)$ and $X(t_2 + \tau)$ are same. Therefore, the random process $\{X(t)\}$ is a strict sense stationary process of order 2.

Problem 7. Let $\{X(t)\}$ be a zero mean Gaussian random process with autocorrelation function $R_{xx}(\tau) = 4e^{-2|\tau|}$. Find the joint probability density function of $Y = X(t)$ and $W = X(t + \tau)$ as $\tau \to \infty$.

SOLUTION:
Since $\{X(t)\}$ is a zero mean Gaussian random process, we have
Given $E\{X(t)\} = 0$ and $E\{X(t+\tau)\} = 0$

$$R_{xx}(\tau) = 4e^{-2|\tau|}$$

We know that $E\{X^2(t)\} = R_{xx}(0) = 4$

$$\therefore \quad V\{X(t)\} = E\left\{X^2(t)\right\} - \{E[X(t)]\}^2 = 4 - 0^2 = 4$$

$$E(Y) = E\{X(t)\} = 0 \quad \text{and} \quad E(W) = E\{X(t+\tau)\} = 0$$

$$V(Y) = V(W) = V\{X(t)\} = 4$$

Consider

$$r = r_{yw} = \frac{C_{yw}(t_1, t_2)}{\sqrt{C_{yy}(t_1, t_1)}\sqrt{C_{ww}(t_2, t_2)}} = \frac{C(\tau)}{C(0)} = \frac{R(\tau)}{R(0)} = \frac{4e^{-2|\tau|}}{4} = e^{-2|\tau|}$$

Now we know that the joint density function $f(y, w)$ can be obtained as

$$f(y, w) = \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2\sigma^2(1-r^2)}\left(y^2 - 2ryw + w^2\right)\right\}$$

$$= \frac{1}{2\pi(4)\sqrt{1-e^{-4|\tau|}}} \exp\left\{-\frac{1}{2(4)(1-e^{-4|\tau|})}\left(y^2 - 2(e^{-2|\tau|})yw + w^2\right)\right\}$$

$$= \frac{1}{8\pi\sqrt{1-e^{-4|\tau|}}} \exp\left\{-\frac{1}{8(1-e^{-4|\tau|})}\left(y^2 - 2(e^{-2|\tau|})yw + w^2\right)\right\}$$

When $\tau \to \infty$, we have $r = e^{-2|\infty|} = 0$

$$\therefore \quad f(y, w) = \left(\frac{1}{\sqrt{2\pi}(2)} e^{-\frac{1}{8}y^2}\right)\left(\frac{1}{\sqrt{2\pi}(2)} e^{-\frac{1}{8}w^2}\right)$$

This shows that $Y = X(t)$ and $W = X(t+\tau)$ are two independent normal random variables as $\tau \to \infty$.

Problem 8. Find the mean and variance of the simple random walk given by $\{X_n, n \geq 0\}$ where $X_n = \sum_{i=1}^{n} Y_i$, $n = 1, 2, 3, \cdots\cdots$, $X_0 = 0$ and $Y_1, Y_2, \ldots\ldots$ are independently identically distributed random variables with $P\{Y_n = 1\} = p$ and $P\{Y_n = -1\} = q$, $p + q = 1$ for all n.

SOLUTION:
It is given that

$$X_n = \sum_{i=1}^{n} Y_i,\ n = 1, 2, 3, \cdots\cdots$$

$$\Rightarrow X_n = X_{n-1} + Y_n, \quad n = 1, 2, \cdots\cdots$$

Here, $X_0 = 0$ and $Y_1, Y_2, \ldots\ldots$ are independently identically distributed random variables with

$P\{Y_n = 1\} = p$ and $P\{Y_n = -1\} = q$, $p + q = 1$ for all n.

Now, from $X_n = X_{n-1} + Y_n, \quad n = 1, 2, \cdots\cdots$ we have

$$X_1 = X_0 + Y_1 \quad = Y_1$$

$$X_2 = X_1 + Y_2 \quad = Y_1 + Y_2$$

And so on

$$X_n = Y_1 + Y_2 + \cdots\cdots + Y_n$$

Now,

$$E(X_n) = E(Y_1 + Y_2 + \cdots\cdots + Y_n) = nE(Y_i),\ i = 1 \text{ or } 2 \text{ or } 3 \cdots\cdots$$

Consider

$$E(Y_i) = \{(1)(p) + (-1)q\} = (p - q) = (2p - 1)$$

$$\therefore \quad E\{X_n\} = nE(Y_i) = n(2p - 1)$$

Consider

$$E(Y_i^2) = (1)^2(p) + (-1)^2 q = p + q = 1$$

$$\therefore \quad V(Y_i) = E(Y_i^2) - \{E(Y_i)\}^2 = 1 - (2p - 1)^2 = 4pq$$

$$\Rightarrow \quad V(X_n) = V(Y_1 + Y_2 + \cdots\cdots + Y_n) = nV(Y_i) = n(4pq)$$

$$= 4npq = 4np(1 - p)$$

If $p = q = \dfrac{1}{2}$, the mean and variance of $\{X_n,\ n \geq 0\}$ become

$$E\{X_n\} = n\left(2\frac{1}{2} - 1\right) = 0$$

$$V(X_n) = 4n\frac{1}{2}\left(1-\frac{1}{2}\right) = n$$

Problem 9. Let $\{X(t)\}$ be a Gaussian white noise process and $\{Y(t)\}$ is another process such that $Y(t) = \int_0^t X(\alpha)d\alpha$ then

(i) Find the autocorrelation function.
(ii) Show that $\{Y(t)\}$ is a Wiener process.

SOLUTION:

(i) We know that the autocorrelation of a Gaussian white noise process is given by

$$R_{xx}(t_1, t_2) = b(t_1)\delta(t_1 - t_2) = b_0\delta(\tau)$$

Referring to Result A.3.4 in Appendix A, we have

$$R_{yy}(t, s) = \int_0^t \int_0^s R_{xx}(\alpha, \beta) d\beta d\alpha$$

$$= \int_0^t \int_0^s b_0 \delta(\alpha - \beta) d\beta d\alpha$$

$$= b_0 \int_0^s u(t-\beta)d\beta \quad \text{or} \quad b_0 \int_0^t u(s-\alpha)d\alpha$$

where $u(x-y)$ is a unit step function defined by

$$u(x-y) = \begin{cases} 1 & \text{if} \quad x > y \\ 0 & \text{if} \quad x < y \end{cases}$$

$$\therefore \quad R_{yy}(t, s) = b_0 \int_0^{\min(t,\, s)} d\beta = b_0 \min(t, s)$$

(ii) By definition we know that the autocorrelation function of the Wiener process is same as the one obtained in (i). Also $Y(0) = 0$ and since $\{X(t)\}$ is a Gaussian white noise process we have $E\{Y(t)\} = 0$, and hence we conclude that $\{Y(t)\}$ is a Wiener process.

Problem 10. Let $\{X(t)\}$ be a Wiener process with parameter b_0 and $\{Y(t)\}$ is another process such that $Y(t) = \int_0^t X(\alpha)d\alpha$, then find the mean and variance of $\{Y(t)\}$.

SOLUTION:
Since $\{X(t)\}$ is a Wiener process, we have $E\{X(t)\} = 0$

$$\Rightarrow \quad E\{Y(t)\} = E\left\{\int_0^t X(\alpha)d\alpha\right\} = \int_0^t E\{X(\alpha)\}\,d\alpha = 0$$

$$\Rightarrow \quad E\left\{Y^2(t)\right\} = E\int_0^t\int_0^t \{X(\alpha)X(\beta)\}\,d\alpha d\beta$$

$$= \int_0^t\int_0^t E\{X(\alpha)X(\beta)\}\,d\alpha d\beta$$

$$= \int_0^t\int_0^t R_{xx}(\alpha, \beta)d\beta d\alpha$$

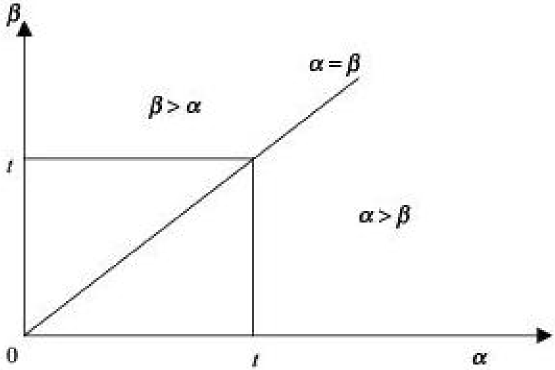
We know that for a Wiener process

$$R_{xx}(\alpha, \beta) = b_0 \min(\alpha, \beta)$$

$$\therefore\ V\{Y(t)\} = E\left\{Y^2(t)\right\} - \{E[Y(t)]\}^2 = \int_0^t\int_0^t b_0 \min(\alpha, \beta)d\alpha d\beta$$

$$= b_0\int_0^t\left\{\int_0^\beta \alpha d\alpha\right\}d\beta + b_0\int_0^t\left\{\int_0^\alpha \beta d\beta\right\}d\alpha \text{ (Refer to the Figure below)}$$

$$= \frac{b_0 t^3}{3}$$



EXERCISE PROBLEMS

1. If $\{X(t)\}$ is a Gaussian process with mean 0, autocorrelation function $R_{xx}(\tau) = 2^{-|\tau|}$, where $\tau = t_1 - t_2$, then obtain $P\{|X(t)| \le 0.5\}$.
2. If $\{X(t)\}$ is a random process whose sample path is given by $X(t) = W \sin(\pi t) + Y$, where Y is a positive random variable with mean μ and variance σ^2 and W is the standard normal random variable independent of Y. Obtain mean and variance of $\{X(t)\}$. Comment on the stationarity of the process $\{X(t)\}$.
3. If $\{X(t)\}$ is a Gaussian process with mean 0 and autocorrelation function $R(\tau) = 4e^{-|\tau|}$, where $\tau = t_1 - t_2$, then find $P(W > 2)$ where the random variable $W = X(t-1)$.
4. Given a normal process $\{X(t)\}$ with mean 0, autocorrelation function $R_{xx}(\tau) = 2^{-|\tau|}$, where $\tau = t_1 - t_2$, then what is the joint probability density function of the random variables $Y = X(t)$ and $W = X(t+1)$.
5. Suppose $\{X(t)\}$ is a Gaussian random process with mean $E\{X(t)\} = 0$ and autocorrelation function $R_{xx}(\tau) = e^{-|\tau|}$. If A is a random variable such that $A = \int_0^1 X(t)dt$. Then determine expectation and variance of A.
6. If $\{X(t)\}$ is a zero mean stationary Gaussian process with $R_{xx}(\tau) = \cos(\tau)$ find the mean, variance and autocorrelation of the square law detector process of $\{X(t)\}$.
7. If $\{X(t)\}$ is a zero mean stationary Gaussian process with autocorrelation function $R_{xx}(\tau) = \cos(\tau)$ and if $\{Z(t)\}$ is a half-wave linear detector process then obtain mean and variance of $\{Z(t)\}$.
8. If $\{X(t)\}$ is a zero mean stationary Gaussian process with autocorrelation function $R(\tau) = 4e^{-3|\tau|}$ and if $\{Y(t)\}$ is a hard limiter process then obtain mean, variance and autocorrelation of $\{Y(t)\}$ when the time points are t and $t+2$.

9. Find the autocorrelation of the simple random walk given by $\{X_n, n \geq 0\}$ where $X_n = \sum_{i=1}^{n} Y_i$, $n = 1, 2, 3, \cdots\cdots$, $X_0 = 0$ and Y_1, $Y_2, \ldots\ldots$ are independently identically distributed random variables with $P\{Y_n = 1\} = p$ and $P\{Y_n = -1\} = q$, $p + q = 1$ for all n.

10. Let $\{X(t)\}$ be a Gaussian random process with mean $E\{X(t)\} = 0$ and auto-correlation function $R_{xx}(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T}, & |\tau| \leq T \\ 0, & \text{otherwise} \end{cases}$. Let $\{X(t_i),\ i = 1, 2, \cdots\cdots, n\}$ be a sequence of n samples of the process taken at the time points $t_i = i\dfrac{T}{2}$, $\quad i = 1, 2, \cdots\cdots, n$, then find the mean and variance of the sample mean given by $\bar{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X(t_i)$

$$E\left[X(t_j, \xi)\right] = \sum_{i=1}^{n} x_i P\left[X\left(t_j, \xi_i\right) = x_i\right] \quad \text{for} \quad j = 1, 2, \cdots\cdots \tag{7.1}$$

For example, refer to Figure 7.1 taken from Chapter 2 for illustration. In this figure, if we assume that at time point t_6 the process $X(t, \xi)$ will assume the values

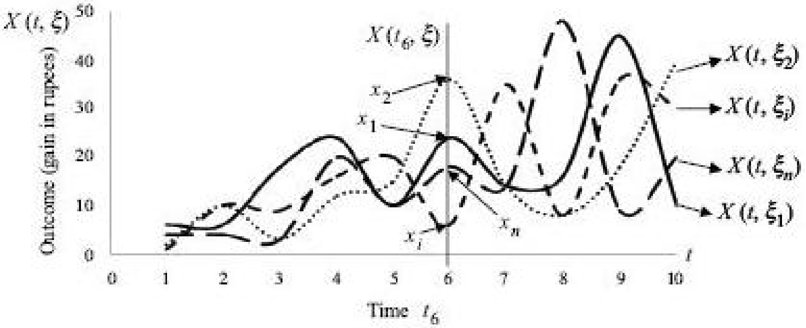


Figure 7.1. Observations (x_i's) at time point t_6, that is $X(t_6, \xi) = x_i \quad i = 1, 2, \cdots\cdots n$

$$X(t_6, \xi_1) = x_1, X(t_6, \xi_2) = x_2, \cdots\cdots X(t_6, \xi_i) = x_i, \cdots\cdots, X(t_6, \xi_n) = x_n$$

With equal chance, then we have

$$E[X(t_6, \xi)] = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \because \quad P[X(t_6, \xi_i) = x_i] = \frac{1}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} X(t_6, \xi_i)$$

In general, notationally we can write (7.1) as

$$E[X(t)] = \sum_{i=1}^{n} x_i P[X(t) = x_i] \tag{7.2}$$

If the random variable $X(t_j, \xi)$ is continuous with probability density function $f(x, t_j)$ such that the realizations $x \in (-\infty, \infty)$, then the expected value can be obtained as

$$E\left[X(t_j)\right] = \int_{-\infty}^{\infty} x f(x, t_j) dx \tag{7.3}$$

Since this is true for all values of $t = t_j,\ j = 1, 2, \cdots\cdots$, in general, we can write (7.3) as

$$E[X(t)] = \int_{-\infty}^{\infty} x f(x, t) dx \tag{7.4}$$

It may be noted that $E[X(t)]$ obtained so may be a function depending on t. However, for a fixed t, $f(x, t)$is the function of x only, and hence we have

$$E[X(t)] = \int_{-\infty}^{\infty} x f(x) dx \tag{7.5}$$

And in this case $E[X(t)]$ will be a constant.

This average, given either in (7.2) or (7.4) can then be used as a representative of the average of the ensemble itself. In the terminology of random processes, this average is known as the *ensemble average* of the random process $\{X(t)\}$ as it makes use of the values from each and every random function (signal) of the ensemble.

For better understanding, consider the Example 2.4 given in Chapter 2 in which the random process is given as $X(t,\ \xi) = A\cos(\omega t + \xi)$ where $t > 0$ is the time parameter, ξ is a uniformly distributed random variable in the interval $(0, 1)$, and A and ω are known constants. Without loss of generality, for the given values of $A = 1.5$ and $\omega = 2.5$ the function now becomes $X(t, \xi) = 1.5\cos(2.5t + \xi)$. Now, for a fixed value of time point t, say $t = 2$, we have the random variable $X(2) = 1.5\cos(5 + \xi)$ which is a function of ξ. Therefore, according to (7.3), the ensemble average can be obtained as

$$E[X(2)] = \int_{-\infty}^{\infty} 1.5\cos(5 + \xi) f(\xi) d\xi$$

It is given that ξ is a uniformly distributed random variable in the interval $(0, 1)$ and hence we have

$$f(\xi) = \begin{cases} 1, & 0 \le \xi \le 1 \\ 0, & \ldots\text{otherwise} \end{cases}$$

This implies

$$E[X(2)] = \int_{0}^{1} 1.5\cos(5 + \xi) d\xi = 1.5[\sin(6) - \sin(5)] = 1.5x0.6795 = 1.0193$$

However, if t is kept as it is, we have

$$E[X(t)] = \int_{-\infty}^{\infty} 1.5\cos(2.5t + \xi) f(\xi, t) d\xi$$

and $$f(\xi, t) = \begin{cases} 1, & 0 \le \xi \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$E[X(t)] = \int_0^1 1.5\cos(2.5t + \xi)\,d\xi = 1.5\,[\sin(2.5t + 1) - \sin(2.5t)]$$

which remains as the function of t.

It may be noted that if the random variable $X(t_j, \xi)$ is discrete, we may require all the realizations of ξ or if the random variable $X(t_j, \xi)$ is continuous we may require the domain in which all the realizations of ξ are defined to get the ensemble average. That is, in order to know the ensemble average we must know all the member functions of the random process $\{X(t)\}$. However, in practice, one cannot get all the member functions of the ensemble in the truncated time interval during which the process is observed.

For example, when a signal is recorded, we can get only one form of the signal from the ensemble that may contain infinite number of forms of the signal, i.e., the spectrum. Therefore, an attempt is made to find an average for the random process $\{X(t)\}$ from the available lone signal that is assumed to have occurred in a two-sided truncated time interval $(-T, T)$. Since, we have a member function of the process during this time interval, say the truncated process $\{X_T(t)\}$, integrating this function gives the area under the curve of this function. Then dividing this area by the length $T - (-T) = 2T$, we get approximately the height of the member function and this is taken as an average of the process and is known as the *time average*, denoted by $\overline{X}_T$ of the random process $\{X(t)\}$ and is given as follows:

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X_T(t)\,dt$$

Since, the functions $X_T(t)$ and $X(t)$ are same over a period of time, in general, we present the time average simply as

$$X_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt \tag{7.6}$$

However, if the process is observed in the time interval $(0, T)$, then we have

$$X_T = \frac{1}{T}\int_{0}^{T} X(t)\,dt \tag{7.7}$$

Again, consider the Example 2.4 of Chapter 2, in which one of the member functions of the random process $\{X(t)\}$ is given by $X(t, \xi_1) = 1.5\cos(2.5t + 0.05)$ or

If $\lim_{T\to\infty} V\{\overline{X}_T\} = 0$, we have $P\left\{\left|\lim_{T\to\infty} \overline{X}_T - \mu\right| \le \varepsilon\right\} = 1$

$$\Rightarrow \quad \lim_{T\to\infty} \overline{X}_T = \mu = E\{X(t)\}$$

Hence, the proof.

Note:

The sufficient condition that a stationary random process $\{X(t)\}$ with autocorrelation function $C(\tau)$ is said to be mean ergodic is (Refer to Problem 3 for proof)

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} C(\tau)\left(1 - \frac{|\tau|}{2T}\right) d\tau = 0$$

Or

$$\lim_{T\to\infty} \frac{1}{T} \int_{0}^{2T} C(\tau)\left(1 - \frac{\tau}{2T}\right) d\tau = 0$$

SOLVED PROBLEMS

Problem 1. If $\{X_1(t)\}$ and $\{X_2(t)\}$ are two mean ergodic processes with means μ_1 and μ_2 respectively and $\{X(t)\}$ is another random process such that $X(t) = X_1(t) + AX_2(t)$ where A is a random variable independent of $X_2(t)$. The random variable A assumes 0 and 1 with equal probabilities. Then show that the process $\{X(t)\}$ is not mean ergodic.

SOLUTION:

Consider
$$\begin{aligned} E\{X(t)\} &= E\{X_1(t) + AX_2(t)\} \\ &= E\{X_1(t)\} + E\{AX_2(t)\} \\ &= E\{X_1(t)\} + E(A)\{X_2(t)\} \end{aligned}$$

But $E(A) = (0)\frac{1}{2} + (1)\frac{1}{2} = \frac{1}{2}$

$$\therefore \quad E\{X(t)\} = E\{X_1(t)\} + \frac{1}{2}\{X_2(t)\}$$

Let $\xi = \{\xi_1, \xi_2\} = \{0,\ 1\}$ be the set of all possible outcomes of the random variable A. It is possible that $A(\xi) = 0$ for a particular ξ. Then we have

$$E\{X(t)\} = E\{X_1(t)\} \quad \Rightarrow \quad \mu_t = \mu_1 \quad \text{(constant)}$$

where $E\{X(t)\} = \mu_t$

$$E\{X(t)\} = E\{X_1(t)\} \quad \Rightarrow \quad \mu_T \to \mu_1 \quad \text{as} \quad T \to \infty$$

Similarly, it is possible that $A(\xi) = 1$ for another other ξ. Then we have

$$E\{X(t)\} = E\{X_1(t)\} + \{X_2(t)\} \quad \Rightarrow \quad \mu_t = \mu_1 + \mu_2 \quad \text{(constant)}$$
$$\therefore \quad \mu_T \to \mu_1 + \mu_2 \quad \text{as} \quad T \to \infty$$

Though $E\{X(t)\}$ is constant in both the cases, the values are different, therefore, the process $\{X(t)\}$ is not mean ergodic.

Problem 2. Show that the random process $\{X(t)\}$ with constant mean is mean ergodic, if

$$\lim_{T\to 0}\left\{\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)\,dt_1dt_2\right\} = 0$$

SOLUTION:

We know that according to mean ergodic theorem, the stationary random process $\{X(t)\}$ with a constant ensemble average μ and time average given by $\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$ is mean ergodic if $\lim_{T\to\infty} V\{\overline{X}_T\} = 0$

Consider $\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$

$$\Rightarrow \quad E(\overline{X}_T) = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\,dt = E\{X(t)\}$$

Now consider

$$\overline{X}_T^2 = \left\{\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right\}^2 = \left(\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right)\left(\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right)$$

Since the process $\{X(t)\}$ is stationary, given two time points t_1 and t_2 the time averages at these time points are same. Therefore, we can write

$$\overline{X}_T^2 = \left(\frac{1}{2T}\int_{-T}^{T} X(t_1)\,dt_1\right)\left(\frac{1}{2T}\int_{-T}^{T} X(t_2)\,dt_2\right)$$

$$= \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} X(t_1)X(t_2)\,dt_1 dt_2$$

$$\Rightarrow \quad E\left(\overline{X}_T^2\right) = \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} E\{X(t_1)X(t_2)\}\,dt_1 dt_2$$

$$= \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} R(t_1, t_2)\,dt_1 dt_2$$

We know that

$$V(X_T) = E(X_T^2) - \{E(X_T\}^2$$

$$= \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} R(t_1, t_2)\,dt_1 dt_2 - \left\{E\left(\frac{1}{2T}\int_{-T}^{T} X(t)dt\right)\right\}^2$$

$$= \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} R(t_1, t_2)\,dt_1 dt_2 - E\left(\frac{1}{2T}\int_{-T}^{T} X(t_1)dt_1\right)$$

$$E\left(\frac{1}{2T}\int_{-T}^{T} X(t_2)dt_2\right)$$

$$= \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} R(t_1, t_2)\,dt_1 dt_2 - \left(\frac{1}{2T}\int_{-T}^{T} E\{X(t_1)\}\,dt_1\right)$$

$$\left(\frac{1}{2T}\int_{-T}^{T} E\{X(t_2)\}\,dt_2\right)$$

$$= \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} R(t_1, t_2)\,dt_1 dt_2 - \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} E\{X(t_2)\}E\{X(t_1)\}\,dt_1 dt_2$$

$$= \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} [R(t_1, t_2) - E\{X(t_1)\}E\{X(t_2)\}]\,dt_1 dt_2$$

$$= \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2) dt_1 dt_2$$

$$\therefore \quad \lim_{T \to \infty} \{V(\overline{X}_T\} = 0 \quad \Rightarrow \quad \lim_{T \to \infty} \left\{ \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2) dt_1 dt_2 \right\} = 0$$

Therefore, the random process $\{X(t)\}$ with constant mean is mean ergodic, if

$$\lim_{T \to 0} \left\{ \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2)\, dt_1 dt_2 \right\} = 0$$

Problem 3. If $\overline{X}_T$ is the time average of a stationary random process $\{X(t)\}$ over $(-T, T)$ then prove that

$$V\left(\overline{X}_T\right) = \frac{1}{T} \int_{0}^{2T} C(\tau) \left(1 - \frac{\tau}{2T}\right) d\tau$$

And hence prove that the sufficient condition for the mean-ergodicity of the process $\{X(t)\}$ is

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{2T} C(\tau) \left(1 - \frac{\tau}{2T}\right) d\tau = 0$$

which in turn implies that $\int_{-\infty}^{\infty} |C(\tau)|\, d\tau < \infty$.

SOLUTION:

It is known that $V\left(\overline{X}_T\right)$ can be expressed as (See Problem 2)

$$V\left(\overline{X}_T\right) = \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2)\, dt_1 dt_2$$

Applying the Jacobean of transformation as described in Result A.4.1 in Appendix A to the double integral, we have

$$\int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2)\, dt_1 dt_2 = \int_{-2T}^{2T} C(\tau)\left(2T - |\tau|\right) d\tau$$

$$\therefore \quad V\left(\overline{X}_T\right) = \frac{1}{4T^2} \int_{-2T}^{2T} C(\tau)(2T - |\tau|)d\tau$$

$$= \frac{1}{2T} \int_{-2T}^{2T} C(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau$$

$$= \frac{1}{T} \int_{0}^{2T} C(\tau)\left(1 - \frac{\tau}{2T}\right)d\tau$$

(Since the integrand is an even function)

We know that the random process $\{X(t)\}$ is said to be mean ergodic if $\lim_{T\to\infty} V\left(\overline{X}_T\right) = 0$. This implies that $\frac{1}{T}\int_0^{2T} C(\tau)\left(1 - \frac{\tau}{2T}\right)d\tau = 0$. The sufficient condition for the mean-ergodicity of the process $\{X(t)\}$ is

$$\lim_{T\to\infty} \frac{1}{T} \int_{0}^{2T} C(\tau)\left(1 - \frac{\tau}{2T}\right)d\tau = 0$$

Obviously, when $\lim_{T\to\infty} V\left(\overline{X}_T\right) = 0$, we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} C(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau = 0$$

We know that τ varies from $-2T$ to $+2T$ and hence $|\tau| \le 2T$. Therefore, we have $1 - \frac{|\tau|}{2T} \le 1$ and which follows that

$$\frac{1}{2T} \int_{-2T}^{2T} C(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau \le \frac{1}{2T} \int_{-2T}^{2T} |C(\tau)|d\tau$$

Here, $|C(\tau)|$is considered to ensure that the inequality is valid. Now,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} C(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau = 0 \text{ is true only if } \lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} |C(\tau)|d\tau = 0$$

This leads to the conclusion that $\int_{-\infty}^{\infty} |C(\tau)|d\tau < \infty$ (finite). Therefore, the sufficient condition for mean-ergodicity of the stationary process $\{X(t)\}$ can also be given as $\int_{-\infty}^{\infty} |C(\tau)|d\tau < \infty$.

Problem 4. If $\{X(t)\}$ is a wide sense stationary process with constant mean and autocovariance function

$$C(\tau) = \begin{cases} \omega\left(1 - \frac{\tau}{\tau_0}\right), & \text{for} \quad 0 \le \tau \le \tau_0 \\ 0, & \text{otherwise} \end{cases}$$

where ω is a constant, then find the variance of the time average of $\{X(t)\}$ over the interval $(0, T)$. Also examine if the process $\{X(t)\}$ is mean ergodic.

SOLUTION:

We know that the time average $\overline{X}_T$ of the stationary random process $\{X(t)\}$ in the interval $(-T, T)$ is given by $\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$ and its variance is given by $V(\overline{X}_T) = \frac{1}{2T}\int_{-2T}^{2T} C(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau$.

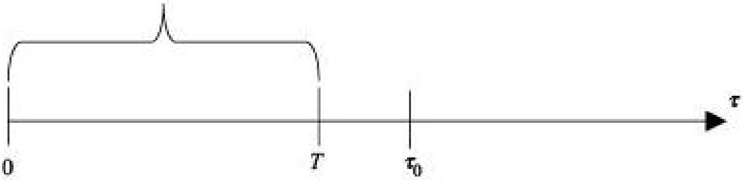
Accordingly, we can write the time average and variance of time average in the interval $(0, T)$ as

$$\overline{X}_T = \frac{1}{T}\int_0^T X(t)\,dt$$

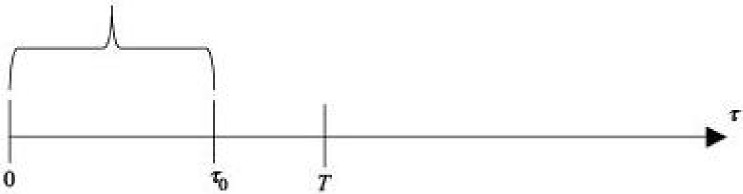
$$V(\overline{X}_T) = \frac{1}{T}\int_{-T}^{T} C(\tau)\left(1 - \frac{|\tau|}{T}\right)d\tau$$

$$= \frac{2}{T}\int_0^T C(\tau)\left(1 - \frac{\tau}{T}\right)d\tau$$

It may be noted that given the value τ_0 and the interval $(0, T)$, it is possible to have either Case (i) $\tau_0 > T$ or Case (ii) $\tau_0 < T$ (Refer the Figure below).



Case (*i*) : $T < \tau_0 \Rightarrow 0 < \tau < T$



Case (*ii*) : $T > \tau_0 \Rightarrow 0 < \tau < \tau_0$

Case (i): $\tau_0 > T$

Therefore, if $0 < T < \tau_0$, we have

$$V(\overline{X}_T) = \frac{2}{T}\int_0^T \omega\left(1-\frac{\tau}{\tau_0}\right)\left(1-\frac{\tau}{T}\right)d\tau$$

$$= \frac{2\omega}{T}\int_0^T \left(1-\frac{\tau}{T}-\frac{\tau}{\tau_0}+\frac{\tau^2}{\tau_0 T}\right)d\tau$$

$$= \frac{2\omega}{T}\left[\tau-\frac{\tau^2}{2T}-\frac{\tau^2}{2\tau_0}+\frac{\tau^3}{3\tau_0 T}\right]_0^T$$

$$= \frac{2\omega}{T}\left(T-\frac{T}{2}-\frac{T^2}{2\tau_0}+\frac{T^2}{3\tau_0}\right) = \omega\left(1-\frac{T}{3\tau_0}\right) \tag{1}$$

Case (ii): $\tau_0 < T$

Therefore, if $0 < \tau_0 < T$, we have

$$V(\overline{X}_T) = \frac{2}{T}\int_0^{\tau_0} \omega\left(1-\frac{\tau}{\tau_0}\right)\left(1-\frac{\tau}{T}\right)d\tau$$

$$= \frac{2\omega}{T}\int_0^{\tau_0} \left(1-\frac{\tau}{T}-\frac{\tau}{\tau_0}+\frac{\tau^2}{\tau_0 T}\right)d\tau$$

Since $C(\tau) = R(\tau) - E\{X(t+\tau)\}E\{X(t)\} = R(\tau) - (0)(0) = R(\tau)$

$$V(\overline{X}_T) = \frac{1}{T}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)\left(1-\frac{|\tau|}{T}\right)d\tau$$

$$= \frac{1}{T}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)^2 d\tau$$

$$= \frac{2}{T}\int_{0}^{T}\left(1-\frac{\tau}{T}\right)^2 d\tau$$

$$= \frac{2}{T}\left[\frac{(1-\tau/T)^3}{3(-1/T)}\right]_0^T = \frac{2}{3}$$

Therefore, $\lim_{T\to\infty}\{V(\overline{X}_T)\} = \lim_{T\to\infty}\left(\frac{2}{3}\right) = \frac{2}{3} \neq 0$

Since $\lim_{T\to\infty}\{V(\overline{X}_T)\} \neq 0$ the process is not mean ergodic.

Problem 6. If $\{X(t)\}$ is a wide sense stationary process such that $X(t) = 10\cos(100t+\theta)$ where θ is a random variable uniformly distributed over $(-\pi, \pi)$ then show that the process $\{X(t)\}$ is correlation ergodic.

SOLUTION:

Since θ is a random variable uniformly distributed over $(-\pi, \pi)$, we have the probability density function of θ as

$$f(\theta) = \frac{1}{2\pi}, \quad -\pi \le \theta \le \pi$$

We know that autocorrelation function of the stationary random process $\{X(t)\}$ is given by

$$R(\tau) = E\{X(t+\tau)X(t)\}$$

$$= E\{10\cos[100(t+\tau)+\theta]10\cos[100t+\theta\}$$

$$= 100E\{\cos[100(t+\tau)+\theta]\cos[100t+\theta\}$$

$$= 50E\{\cos(200t+100\tau+2\theta)+\cos 100\tau\}$$

$$= 50E\{\cos(200t+100\tau+2\theta)\} + 50E\{\cos 100\tau\}$$

$$= 50\int_{-\pi}^{\pi}\cos(200t+100\tau+2\theta)f(\theta)d\theta + 50\int_{-\pi}^{\pi}\cos 100\tau\, f(\theta)d\theta$$

$$= \frac{25}{\pi}\int_{-\pi}^{\pi}\cos(200t+100\tau+2\theta)d\theta + \frac{25}{\pi}\int_{-\pi}^{\pi}\cos 100\tau d\theta$$

$$= \frac{25}{\pi}(0) + 50\cos 100\tau \qquad \text{(See Appendix B)}$$

$$= 50\cos 100\tau$$

Now, consider $Z_T = X(t+\tau)X(t)$

$$\Rightarrow E(Z_T) = E\{X(t+\tau)X(t)\} = R(\tau) = 50\cos 100\tau$$

We know that the time average of the process $\{Z(t)\}$ in the interval $(-T, T)$ is given by

$$Z_T = \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)\,dt$$

$$= \frac{1}{2T}\int_{-T}^{T} 100\cos[100(t+\tau)+\theta]\cos[100t+\theta]dt$$

$$= \frac{50}{T}\int_{-\pi}^{\pi}\frac{\cos(200t+100\tau+2\theta)+\cos 100\tau}{2}dt$$

$$= \frac{25}{T}\int_{-\pi}^{\pi}[\cos(200t+100\tau+2\theta)]\,dt + \frac{25}{T}\int_{-T}^{T}\cos 100\tau\,dt$$

$$= \frac{25}{T}(0) + 50\cos 100\tau = 50\cos 100\tau$$

$$= 50\cos 100\tau$$

$$\therefore \quad \lim_{T\to\infty} Z_T = \lim_{T\to\infty}\{50\cos 100\tau\} = 50\cos 100\tau = R(\tau)$$

Since $\lim_{T\to\infty}\overline{Z}_T = R(\tau)$, the process is correlation ergodic.

Problem 7. If $\{X(t)\}$ is a zero mean wide sense stationary process with $R_{xx}(\tau) = e^{-2|\tau|}$, show that $\{X(t)\}$ is mean ergodic.

Solution:

It is given that $E\{X(t)\} = \mu = 0$ (that is, the ensemble average is a constant).

Now consider the time average of the process $\{X(t)\}$ in the interval $(-T, T)$ given by

$$\bar{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt \quad \Rightarrow \quad E(\bar{X}_T) = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\,dt = \mu = 0$$

We know that $V(\bar{X}_T) = E(\bar{X}_T^2) - \{E(\bar{X}_T)\}^2$

Consider $\bar{X}_T^2 = \frac{1}{2T}\int_{-T}^{T} X^2(t)\,dt = \left(\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right)\left(\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right)$

Since the process $\{X(t)\}$ is stationary, given two time points t_1 and t_2 the time averages at these time points are same. Therefore, we can write

$$\begin{aligned}
\bar{X}_T^2 &= \left(\frac{1}{2T}\int_{-T}^{T} X(t_1)\,dt_1\right)\left(\frac{1}{2T}\int_{-T}^{T} X(t_2)\,dt_2\right) \\
&= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} X(t_1)X(t_2)\,dt_1 dt_2 \\
\Rightarrow \quad E(\bar{X}_T^2) &= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X(t_2)\}\,dt_1 dt_2 \\
&= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} R(t_1, t_2)\,dt_1 dt_2 \\
&= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1, t_2)\,dt_1 dt_2 \qquad \because \quad E\{X(t)\} = 0
\end{aligned}$$

Applying the Jacobean of transformation as described in Result A.4.1 in Appendix A to the double integral and simplifying, we have

$$\begin{aligned}
V(\bar{X}_T) &= \frac{1}{T}\int_{0}^{2T} C(\tau)\left(1 - \frac{\tau}{2T}\right)d\tau \\
&= \frac{1}{T}\int_{0}^{2T} R(\tau)\left(1 - \frac{\tau}{2T}\right)d\tau \quad \because E\{X(t)\} = 0,\ C(\tau) = R(\tau) \\
&= \frac{1}{T}\int_{0}^{2T} e^{-2|\tau|}\left(1 - \frac{\tau}{2T}\right)d\tau
\end{aligned}$$

$$= \frac{1}{T}\int_0^{2T} e^{-2\tau}\left(1-\frac{\tau}{2T}\right)d\tau$$

$$= \frac{1}{T}\int_0^{2T} e^{-2\tau}d\tau - \frac{1}{2T^2}\int_0^{2T} e^{-2\tau}\tau d\tau$$

$$= \frac{1}{T}\left[\frac{e^{-2\tau}}{-2}\right]_0^{2T} - \frac{1}{2T^2}\left[\tau\frac{e^{-2\tau}}{-2}-(1)\frac{e^{-2\tau}}{4}\right]_0^{2T}$$

$$= \frac{1}{T}\left[\frac{e^{-4T}}{-2}-\frac{1}{-2}\right]-\frac{1}{2T^2}$$

$$\left[\left(2T\frac{e^{-4T}}{-2}-(1)\frac{e^{-4T}}{4}\right)-\left((0)-\frac{1}{4}\right)\right]$$

$$= \frac{1}{2T}\left[1+\frac{e^{-4T}}{4T}-\frac{1}{4T}\right]$$

$$\therefore \quad \lim_{T\to\infty} V(\overline{X}_T) = \lim_{T\to\infty}\left\{\frac{1}{2T}\left[1-\frac{1-e^{-4T}}{4T}\right]\right\}=0$$

Since $\lim_{T\to\infty} V(\overline{X}_T)=0$, the process $\{X(t)\}$ is mean ergodic.

Problem 8. A random process $\{X(t)\}$ has the sample functions of the form $X(t) = A\cos(\omega t+\theta)$ where ω is a constant, A is a random variable that has magnitude $+1$ and -1 with equal probabilities, and θ is a random variable that is uniformly distributed between 0 and 2π. Assume that A and θ are independent. Is $\{X(t)\}$ a mean ergodic process?

Solution:

In order to show that a random process $\{X(t)\}$ defined in the time interval $(-T,T)$ is *mean ergodic* we have to show that

(i) Ensemble average: $E\{X(t)\}=\mu$ (constant).

(ii) Time average: $\lim_{T\to\infty}\overline{X}_T = \frac{1}{2T}\int_{-T}^{T}X(t)\,dt=\mu$.

Since A and θ are independent we have

$$E\{X(t)\}=E\{A\cos(\omega t+\theta)\}=E(A)E\{\cos(\omega t+\theta)\}$$

But $E(A)=(+1)\frac{1}{2}+(-1)\frac{1}{2}=0$

Also $E[\cos(\omega t+\theta)] = \int_0^{2\pi} \cos(\omega t+\theta) f(\theta) d\theta$

Since θ is a random variable that is uniformly distributed between 0 and 2π, we have

$$f(\theta) = \frac{1}{2\pi}, \quad 0 \le \theta \le 2\pi$$

$$\therefore \quad E[\cos(\omega t+\theta)] = \int_0^{2\pi} \cos(\omega t+\theta) \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2\pi} [\sin(\omega t+\theta)]_0^{2\pi}$$

$$= \frac{1}{2\pi} [\sin(\omega t+2\pi) - \sin \omega t] = 0$$

$$\therefore \quad E\{X(t)\} = E(A) E[\cos(\omega t+\theta)] = 0$$

Consider the time average

$$\overline{X_T} = \frac{1}{2T} \int_{-T}^{T} A\cos(\omega t+\theta) dt = \frac{A}{2T} \left[\frac{\sin(\omega t+\theta)}{\omega} \right]_{-T}^{T}$$

$$= \frac{A}{2T\omega} [\sin(\theta+\omega T) - \sin(\theta - \omega T)]$$

$$= \frac{A}{2T\omega} [2\cos\theta \sin \omega T]$$

$$= A\cos\theta \left(\frac{\sin \omega T}{\omega T} \right)$$

$$\lim_{T\to\infty} \overline{X_T} = A\cos\theta \lim_{T\to\infty} \left(\frac{\sin \omega T}{\omega T} \right) = (A\cos\theta)(0) = 0$$

Since, $E\{X(t)\} = \lim_{T\to\infty} \overline{X_T} = \mu = 0$, we conclude that the given process is mean ergodic.

Problem 9. Consider the sinusoid with random phase $X(t) = a\sin(\omega t+\theta)$ where a is constant and θ is a random variable uniformly distributed over $(0, 2\pi)$. Show that the process $\{X(t)\}$ is correlation ergodic.

SOLUTION:

In order to show that a stationary random process $\{X(t)\}$ defined in the time interval $(-T, T)$ is *correlation ergodic* we have to show that the process $\{Z(t)\}$ is mean ergodic, where $Z(t) = X(t)X(t+\tau)$ or $Z(t) = X(t+\tau)X(t)$. That is,

(i) $E\{Z(t)\} = E\{X(t+\tau)X(t)\} = R(\tau)$ (the autocorrelation function).

(ii) $\lim_{T\to\infty} \overline{Z}_T = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)\,dt = R(\tau)$.

Consider

$$\begin{aligned}
E\{Z(t)\} &= E\{X(t+\tau)X(t)\} \\
&= E\{a\sin[\omega(t+\tau)+\theta]\,a\sin(\omega t+\theta)\} \\
&= a^2E\{\sin[\omega t+\omega\tau+\theta]\sin(\omega t+\theta)\} \\
&= a^2E\left\{\frac{\cos\omega\tau-\cos(2\omega t+\omega\tau+2\theta)}{2}\right\} \\
&= \frac{a^2}{2}E(\cos\omega\tau)-\frac{a^2}{2}E\{\cos(2\omega t+\omega\tau+2\theta)\} \\
&= \frac{a^2}{2}\int_0^{2\pi}\cos\omega\tau\frac{1}{2\pi}d\theta-\frac{a^2}{2}\int_0^{2\pi}\cos(2\omega t+\omega\tau+2\theta)\frac{1}{2\pi}d\theta \\
&= \frac{a^2}{2}\cos\omega\tau-\frac{a^2}{4\pi}\int_0^{2\pi}\cos(2\omega t+\omega\tau+2\theta)d\theta
\end{aligned}$$

Consider

$$\begin{aligned}
\frac{a^2}{4\pi}\int_0^{2\pi}\cos(2\omega t+\omega\tau+2\theta)d\theta &= \frac{a^2}{8\pi}\left[\sin(2\omega t+\omega\tau+2\theta)\right]_0^{2\pi} \\
&= \frac{a^2}{8\pi}\left[\sin(2\omega t+\omega\tau+4\pi)-\sin(2\omega t+\omega\tau)\right]=0
\end{aligned}$$

$$\therefore \quad E\{Z(t)\} = R(\tau) = \frac{a^2}{2}\cos\omega\tau$$

Now, consider

$$\overline{Z}_T = \frac{1}{2T}\int_{-T}^{T}X(t+\tau)X(t)\,dt = \frac{1}{2T}\int_{-T}^{T}a\sin[\omega(t+\tau)+\theta]a\sin(\omega t+\theta)dt$$

$$= \frac{a^2}{2T}\int_{-T}^{T} \sin[\omega(t+\tau)+\theta]\sin(\omega t+\theta)dt$$

$$= \frac{a^2}{2T}\int_{-T}^{T} \frac{\cos\omega\tau - \cos(2\omega t+\omega\tau+2\theta)}{2}dt$$

$$= \frac{a^2}{4T}\int_{-T}^{T} \cos\omega\tau dt - \frac{a^2}{4T}\int_{-T}^{T}\cos(2\omega t+\omega\tau+2\theta)\,dt$$

$$= \frac{a^2}{2}\cos\omega\tau - \frac{a^2}{4T}\int_{-T}^{T}\cos(2\omega t+\omega\tau+2\theta)\,dt$$

Consider

$$\frac{a^2}{4T}\int_{-T}^{T}\cos(2\omega t+\omega\tau+2\theta)\,dt = \frac{a^2}{8\omega T}\left[\sin(2\omega t+\omega\tau+2\theta)\right]_{-T}^{T}$$

$$= \frac{a^2}{8\omega T}\left[\sin(\omega\tau+2\theta+2\omega T)-\sin(\omega\tau+2\theta-2\omega T)\right]$$

$$= \frac{a^2}{8\omega T}\left[2\cos(\omega\tau+2\theta)\sin(2\omega T)\right]$$

$$= \frac{a^2}{4\omega T}\left[\cos(\omega\tau+2\theta)\sin(2\omega T)\right]$$

$$\therefore \quad \lim_{T\to\infty}\overline{Z}_T = \lim_{T\to\infty}\left\{\frac{a^2}{2}\cos\omega\tau - \frac{a^2}{4\omega T}\left[\cos(\omega\tau+2\theta)\sin(2\omega T)\right]\right\}$$

$$= \frac{a^2}{2}\cos\omega\tau = R(\tau)$$

Since $E\{Z(t)\} = \lim_{T\to\infty}\overline{Z}_T = R(\tau)$ the process $\{X(t)\}$ is correlation ergodic.

Problem 10. If $\{X(t)\}$ is a wide sense stationary process such that $X(t) = 4\cos(50t+\theta)$ where θ is a random variable uniformly distributed over $(-\pi,\pi)$ then show that the process $\{X(t)\}$ is correlation ergodic.

Solution:

Since θ is a random variable uniformly distributed over $(-\pi,\pi)$, we have the probability density function of θ as

$$f(\theta) = \frac{1}{2\pi}, \quad -\pi \le \theta \le \pi$$

We know that autocorrelation function of the stationary random process $\{X(t)\}$ is given by

$$\begin{aligned}
R(\tau) &= E\{X(t+\tau)X(t)\} \\
&= E\{4\cos[50(t+\tau)+\theta]4\cos[50t+\theta]\} \\
&= 16E\{\cos[50(t+\tau)+\theta]\cos[50t+\theta]\} \\
&= 8E\{\cos(100t+50\tau+2\theta)+\cos 50\tau\} \\
&= 8E\{\cos(100t+50\tau+2\theta)\}+8E\{\cos 50\tau\} \\
&= 8\int_{-\pi}^{\pi}\cos(100t+50\tau+2\theta)f(\theta)d\theta+8\int_{-\pi}^{\pi}\cos 50\tau\, f(\theta)d\theta \\
&= \frac{4}{\pi}\int_{-\pi}^{\pi}\cos(100t+50\tau+2\theta)d\theta+\frac{4}{\pi}\int_{-\pi}^{\pi}\cos 50\tau d\theta \\
&= \frac{4}{\pi}(0)+8\cos 50\tau \quad \text{(See Appendix B)} \\
&= 8\cos 50\tau
\end{aligned}$$

Now, consider $\qquad Z_T = X(t+\tau)X(t)$

$$\Rightarrow \quad E(Z_T) = E\{X(t+\tau)X(t)\} = R(\tau) = 8\cos 50\tau$$

We know that the time average of the process $\{Z(t)\}$ in the interval $(-T, T)$ is given by

$$\begin{aligned}
\overline{Z}_T &= \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)\,dt \\
&= \frac{1}{2T}\int_{-T}^{T} 4\cos[100(t+\tau)+\theta]4\cos[100t+\theta]\,dt \\
&= \frac{8}{T}\int_{-\pi}^{\pi}\frac{\cos(100t+50\tau+2\theta)+\cos 50\tau}{2}\,dt \\
&= \frac{4}{T}\int_{-\pi}^{\pi}[\cos(100t+50\tau+2\theta)]\,dt+\frac{4}{T}\int_{-T}^{T}\cos 50\tau\,dt
\end{aligned}$$

$$= \frac{4}{T}(0) + 8\cos 50\tau = 8\cos 50\tau$$

$$\therefore \quad \lim_{T\to\infty} \overline{Z}_T = \lim_{T\to\infty} \{8\cos 50\tau\} = 8\cos 50\tau = R(\tau)$$

Since $\lim_{T\to\infty} Z_T = R(\tau)$, the process is correlation ergodic.

EXERCISE PROBLEMS

1. If $\{X(t)\}$ is a random process such that $X(t) = A$ where A is a random variable with mean μ_A then show that the process $\{X(t)\}$ is not mean ergodic.
2. If $\{X(t)\}$ is a zero mean wide sense stationary process with $R_{xx}(\tau) = 4e^{-|\tau|}$, show that $\{X(t)\}$ is mean ergodic.
3. Consider the sinusoid with random phase $X(t) = a\sin(\omega t + \theta)$ where a is constant and θ is a random variable uniformly distributed over $(0, 2\pi)$. Show that the process $\{X(t)\}$ is mean ergodic.
4. If $\{X(t)\}$ is a wide sense stationary process such that $X(t) = \cos(t + \theta)$ where θ is a random variable uniformly distributed over $(-\pi, \pi)$ then show that the process $\{X(t)\}$ is correlation ergodic.
5. Show that the stationary random process $\{X(t)\}$ whose autocovariance function is given by $C(\tau) = qe^{-\alpha\tau}$, where q and α are constants, is mean ergodic.
6. If $\{X(t)\}$ is a stationary random process such that $X(t) = \eta + W(t)$ where $W(t)$ is a white noise process with autocorrelation function $R_{ww}(\tau) = q\delta(\tau)$, where q is constant and δ is the unit impulse function, then show that $\{X(t)\}$ is mean ergodic.
7. If $\{X(t)\}$ is a stationary process with $X(t) = A$, where A is a random variable, having an arbitrary probability density function. Check whether the process $\{X(t)\}$ is mean ergodic.
8. If $\{X(t)\}$ is a stationary random telegraph signal with mean 0 and autocorrelation function $R(\tau) = e^{-2\lambda\tau}$, where λ is constant, then find the mean and variance of the time average of $\{X(t)\}$. Also verify whether $\{X(t)\}$ is mean ergodic.
9. If $\{X(t)\}$ is a stationary Gaussian process with zero mean and autocorrelation function $R(\tau) = 10e^{-|\tau|}$ then show that the process $\{X(t)\}$ is mean ergodic.
10. If $\{X(t)\}$ is a stationary random process with autocorrelation function $R(\tau)$ then show that the process $\{X(t)\}$ is correlation ergodic if and only if

$$\lim_{T\to\infty} \frac{1}{2T}\left\{\int_0^{2T} \phi(\tau)\left(1 - \frac{\tau}{2T}\right) d\tau\right\} \to \{R(\tau)\}^2$$

where

$$\phi(\tau) = E\{X(t_1+\tau)X(t_1)X(t_2+\tau)X(t_2)\}.$$

coefficients. In this chapter, we consider the frequency domain techniques captured by Fourier transforms from the perspective of deterministic signals and systems.

For an illustration, consider the signal $X(t)$, a single realization of the stationary random process $\{X(t)\}$ with magnified amplitudes as given in Figure 8.1. The distribution of frequencies of amplitudes is shown in the histogram given in Figure 8.2. The histogram is fitted with a smooth normal (Gaussian) curve for illustration purpose. This curve shows the spectral density (distributional pattern) of the amplitudes of signal observed over a period of time.

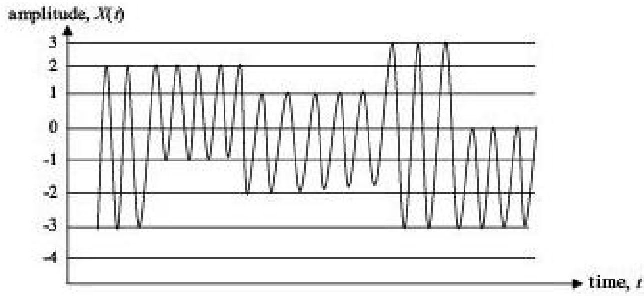


Figure 8.1. Magnified amplitudes of a single realization of the random process $\{X(t)\}$

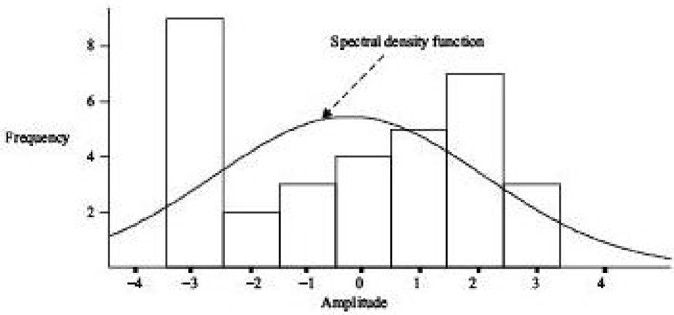


Figure 8.2. Frequency distribution of amplitudes and the fitted normal curve as spectral density function

However, in practice, one cannot manually count the frequencies for each amplitude as signal may vary rapidly. This job is, in fact, done by the Fourier transform of autocorrelation of the signal under study since autocorrelation captures the changes in the signal over time. As discussed earlier, since autocorrelation is used, this transformation will result in *power spectral density function* of the stationary random process.

$$\therefore \quad S_{xx}(\omega) = \int_{0}^{\infty} 2R_{xx}(\tau)\cos\omega\tau d\tau \tag{8.11}$$

which is a Fourier cosine transform of $2R_{xx}(\tau)$.
Now consider

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{i\tau\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)(\cos\tau\omega + i\sin\tau\omega)\,d\omega$$

$$= \int_{-\infty}^{\infty} S_{xx}(\omega)\cos\tau\omega\,d\omega + \int_{-\infty}^{\infty} S_{xx}(\omega)i\sin\tau\omega\,d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} S_{xx}(\omega)\cos\tau\omega d\omega \qquad \because \int_{-\infty}^{\infty} S_{xx}(\omega)i\sin\tau\omega d\omega = 0$$

$$\therefore \quad R_{xx}(\tau) = \int_{0}^{\infty} \frac{1}{\pi}S_{xx}(\omega)\cos\tau\omega\,d\omega \tag{8.12}$$

which is a Fourier inverse cosine transform of $\frac{1}{\pi}S_{xx}(\omega)$.

Property 8.5: The PSD function of the output process $\{Y(t)\}$ corresponding to the input process $\{X(t)\}$ in the system that has an impulse response $h(t) = e^{-\beta t}U(t)$, where $U(t)$ is the unit step function, is given as

$$S_{yy}(\omega) = |H(\omega)|^2\,S_{xx}(\omega) \tag{8.13}$$

where $H(\omega)$ is the Fourier transform of $h(t)$.

Property 8.6: The PSD function $S_{xx}(\omega)$ of a stationary process $\{X(t)\}$ (whether real or complex) with autocorrelation function $R_{xx}(\tau)$ is a real and non-negative function. That is, $S_{xx}^*(\omega) = S_{xx}(\omega)$. Also, $S_{xy}^*(\omega) = S_{yx}(\omega)$, where S^* is the complex conjugate.

Proof. Consider

$$R_{xx}(\tau) = E\{X(t)X^*(t-\tau)\} \text{ where } X^* \text{ is the complex conjugate.}$$

$$\therefore \quad R_{xx}^*(\tau) = E\{X(t-\tau)X^*(t)\} = R_{xx}(-\tau)$$

Similarly,

$$R_{xx}(-\tau) = E\{X(t)X^*(t+\tau)\}$$
$$R^*_{xx}(-\tau) = E\{X(t+\tau)X^*(t)\} = R_{xx}(\tau)$$

Now consider

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$S^*_{xx}(\omega) = \int_{-\infty}^{\infty} R^*_{xx}(\tau)\, e^{i\omega\tau}\, d\tau = \int_{-\infty}^{\infty} R_{xx}(-\tau)\, e^{i\omega\tau}\, d\tau$$

Let $v = -\tau$

$$\Rightarrow \quad S^*_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(v)\, e^{-i\omega v} dv = S_{xx}(\omega)$$

Similarly, we can show that $S^*_{xy}(\omega) = S_{xy}(\omega)$

Therefore, $S_{xx}(\omega)$ is a real function.

Let us assume that $S_{xx}(\omega_0) < 0$ at $\omega = \omega_0$

Then for some small $\varepsilon > 0$, we have $S_{xx}(\omega) < 0$ at $\omega_0 - \frac{\varepsilon}{2} < \omega < \omega_0 + \dfrac{\varepsilon}{2}$

Now, consider a system function of a *narrow band filter*

$$H(\omega) = \begin{cases} 1, & \omega_0 - \dfrac{\varepsilon}{2} < \omega < \omega_0 + \dfrac{\varepsilon}{2} \\ 0, & \text{otherwise} \end{cases}$$

Let the PSDs of the input and output processes in the system are connected by the relation $S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$, where $|H(\omega)|$ is the Fourier transform of unit impulse function, say $h(t)$.

$$\Rightarrow S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) = \begin{cases} S_{xx}(\omega), & \omega_0 - \dfrac{\varepsilon}{2} < \omega < \omega_0 + \dfrac{\varepsilon}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow E\left\{Y^2(t)\right\} = R_{yy}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\omega)\, d\omega = \frac{1}{2\pi}\int_{\omega_0-\varepsilon/2}^{\omega_0+\varepsilon/2} S_{yy}(\omega) d\omega = \frac{\varepsilon}{2\pi} S_{xx}(\omega_0)$$

This is true because $S_{xx}(\omega)$ is constant as the band is narrow and is equal to $S_{xx}(\omega_0)$.

Since $E\{Y^2(t)\} \geq 0 \quad \Rightarrow \quad S_{xx}(\omega_0) \geq 0$

This is contrary to the assumption made that $S_{xx}(\omega_0) < 0$. Therefore, $S_{xx}(\omega)$ is non-negative function.

Since $\{X(t)\}$ is stationary, for a given two time points $t_1, t_2 \in (-T, +T)$, we have

$$|X_T(\omega)|^2 = \int_{-T}^{+T} X(t_1)e^{-i\omega t_1}dt_1 \int_{-T}^{+T} X(t_2)e^{i\omega t_2}dt_2$$

$$= \int_{-T}^{+T}\int_{-T}^{+T} X(t_1)X(t_2)e^{-i\omega(t_1-t_2)}dt_1dt_2$$

$$\Rightarrow \quad E\left\{|X_T(\omega)|^2\right\} = \int_{-T}^{+T}\int_{-T}^{+T} E\{X(t_1)X(t_2)\}\, e^{-i\omega(t_1-t_2)}dt_1dt_2$$

$$= \int_{-T}^{+T}\int_{-T}^{+T} R(t_1, t_2)\, e^{-i\omega(t_1-t_2)}dt_1dt_2$$

Since $\{X(t)\}$ is stationary, $R(t_1, t_2)$ is a function of $\tau = t_1 - t_2$ or $\tau = t_2 - t_1$, therefore,

$$E\left\{|X_T(\omega)|^2\right\} = \int_{-T}^{+T}\int_{-T}^{+T} R(t_1 - t_2)\, e^{-i\omega(t_1-t_2)}dt_1dt_2$$

$$= \int_{-T}^{+T}\int_{-T}^{+T} g(t_1 - t_2)\, dt_1dt_2$$

where $g(t_1 - t_2) = R(t_1 - t_2)\, e^{-i\omega(t_1-t_2)} \quad \Rightarrow \quad g(\tau) = R(\tau)\, e^{-i\omega\tau}$

Transforming double integral to single integral as shown in Result A.4.1 in Appendix A, we have

$$E\left\{|X_T(\omega)|^2\right\} = \int_{-2T}^{+2T} g(\tau)\,(2T - |\,\tau\,|)\,d\tau$$

where $g(\tau) = R(\tau)e^{-i\omega\tau}$.

$$\Rightarrow \quad \frac{1}{2T}E\left\{|X_T(\omega)|^2\right\} = \int_{-2T}^{+2T} g(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau$$

$$\therefore \quad \lim_{T\to\infty}\frac{1}{2T}E\left\{|X_T(\omega)|^2\right\} = \lim_{T\to\infty}\int_{-2T}^{+2T} g(\tau)\,d\tau - \lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{+2T} g(\tau)\,|\tau|\,d\tau$$

Therefore, the power spectral density function at 0 frequency is given by

$$S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau)\,d\tau = 2\int_{0}^{\infty} R_{xx}(\tau)\,d\tau \qquad \because R_{xx}(\tau) \quad \text{is an even function}$$

$$= 2(6.25) = 12.5 \quad \text{Sq. units}$$

Problem 2. If the power spectral density of a stationary random process $\{X(t)\}$ is given by

$$S(\omega) = \begin{cases} \frac{b}{a}(a - |\omega|)\,, & |\omega| \le a \\ 0, & |\omega| > a \end{cases}$$

then show that the autocorrelation function is given in the form $R_{xx}(\tau) = \frac{ab}{2\pi}\left(\frac{\sin a\tau/2}{a\tau/2}\right)^2$.

Solution:

It is given that

$$S(\omega) = \begin{cases} \frac{b}{a}(a - |\omega|)\,, & |\omega| \le a \\ 0, & |\omega| > a \end{cases}$$

We know that the autocorrelation function of a stationary random process $\{X(t)\}$ is given by

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,e^{i\tau\omega}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a} \frac{b}{a}(a - |\omega|)\,e^{i\tau\omega}\,d\omega$$

$$= \frac{b}{2\pi a}\int_{-a}^{a} (a - |\omega|)\,(\cos\tau\omega + i\sin\omega\tau)\,d\tau$$

$$= \frac{b}{2\pi a}\int_{-a}^{a} (a - |\omega|)\cos\tau\omega d\omega + \frac{b}{2\pi a}\int_{-a}^{a} (a - |\omega|)\,i\sin\tau\omega d\omega$$

$$= \frac{b}{2\pi a}\int_{-a}^{a} (a - |\omega|)\cos\tau\omega d\omega - 0 \qquad \because (a - |\omega|)\sin\omega\tau \text{ is an odd function}$$

$$= \frac{b}{\pi a}\int_{0}^{a} (a - \omega)\cos\tau\omega d\omega \qquad \because (a - |\omega|\,)\cos\tau\omega \text{ is an even function}$$

$$= \frac{b}{\pi a}\left[(a-\omega)\frac{\sin\tau\omega}{\tau} - \frac{\cos\tau\omega}{\tau^2}\right]_0^a$$

$$= \frac{b}{\pi a}\left[-\frac{\cos a\tau}{\tau^2} + \frac{1}{\tau^2}\right]$$

$$= \frac{b}{\pi a}\left[\frac{1-\cos a\tau}{\tau^2}\right]$$

$$= \frac{b}{\pi a}\frac{2\sin^2 a\tau/2}{\tau^2}$$

$$= \frac{b}{\pi a}\frac{\sin^2 a\tau/2}{\tau^2/2} = \frac{ab}{2\pi}\left(\frac{\sin a\tau/2}{a\tau/2}\right)^2$$

Problem 3. Find the power spectral density function of the random process whose autocorrelation function is given by $R(\tau) = e^{-a\tau^2}$.

SOLUTION:
It is given that

$$R_{xx}(\tau) = e^{-a\tau^2}$$

We know that the power spectral density function of a stationary random process $\{X(t)\}$ is given by

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau^2} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-(a\tau^2+i\omega\tau)} d\tau$$

Add and subtract $(i\omega/2a)^2$ to make the exponent a perfect square. This gives

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} e^{-a\left\{\tau^2+2(i\omega/2a)\tau+(i\omega/2a)^2-(i\omega/2a)^2\right\}} d\tau$$

$$= e^{-\omega^2/4a}\int_{-\infty}^{\infty} e^{-a\{\tau+i\omega/2a\}^2} d\tau$$

$$= 2e^{-\omega^2/4a}\int_{0}^{\infty} e^{-a\{\tau+i\omega/2a\}^2} d\tau$$

Let $u = \sqrt{a}\left(\tau + \frac{i\omega}{2a}\right) \quad \Rightarrow \quad d\tau = \frac{du}{\sqrt{a}}$

$$\therefore \quad S_{xx}(\omega) = \frac{e^{-\omega^2/4a}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \frac{2e^{-\omega^2/4a}}{\sqrt{a}} \int_{0}^{\infty} e^{-u^2} du$$

Now let $v = u^2 \quad \Rightarrow \quad dv = 2udu = 2\sqrt{v}du \quad \Rightarrow \quad du = \frac{1}{2}v^{-1/2}dv$

$$\therefore \quad S_{xx}(\omega) = \frac{e^{-\omega^2/4a}}{\sqrt{a}} \int_{0}^{\infty} e^{-v} v^{\frac{1}{2}-1} dv$$

$$= \frac{e^{-\omega^2/4a}}{\sqrt{a}} \Gamma(1/2) = \frac{e^{-\omega^2/4a}}{\sqrt{a}} \sqrt{\pi}$$

$$\therefore \quad S_{xx}(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

Problem 4. If $\{X(t)\}$ is a stationary process with autocorrelation function $R_{xx}(\tau)$ and if $\{Y(t)\}$ is another stationary random process such that $Y(t) = X(t+a) - X(t-a)$, where a is constant, then show that

(i) $R_{yy}(\tau) = 2R_{xx}(\tau) - R_{xx}(\tau + 2a) - R_{xx}(\tau - 2a)$

(ii) Prove that $S_{yy}(\omega) = 4\sin^2 a\omega\, S_{xx}(\omega)$, where $S_{yy}(\omega)$ is the power spectral density function of $\{Y(t)\}$ and $S_{xx}(\omega)$ is the power spectral density function of $\{X(t)\}$.

SOLUTION:

(i) We know that since $\{X(t)\}$is a wide sense stationary process, and $\{Y(t)\}$ is another wide sense stationary random process such that $Y(t) = X(t+a) - X(t-a)$, the autocorrelation function $R_{yy}(\tau)$ can be given as

$$R_{yy}(\tau) = E\{Y(t)Y(t+\tau)\}$$

$$R_{yy}(\tau) = E\{[X(t+a) - X(t-a)]\,[X(t+\tau+a) - X(t+\tau-a)]\}$$

$$= E[X(t+a)X(t+\tau+a)] - E[X(t+a)X(t+\tau-a)] - E[X(t-a)X(t+\tau+a)] + E[X(t-a)X(t+\tau-a)]$$

$$= R_{xx}(\tau) - R_{xx}(\tau+2a) - R_{xx}(\tau-2a) + R_{xx}(\tau)$$

$$= 2R_{xx}(\tau) - R_{xx}(\tau+2a) - R_{xx}(\tau-2a)$$

(ii) Now taking Fourier transforms on both sides, we have

$$\int_{-\infty}^{\infty} R_{yy}(\tau)e^{-i\omega\tau}d\tau = 2\int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i\omega\tau}d\tau - \int_{-\infty}^{\infty} R_{xx}(\tau+2a)e^{-i\omega\tau}d\tau$$
$$-\int_{-\infty}^{\infty} R_{xx}(\tau-2a)e^{-i\omega\tau}d\tau$$

$$\Rightarrow \quad S_{yy}(\omega) = 2S_{xx}(\omega) - \int_{-\infty}^{\infty} R_{xx}(\tau+2a)e^{-i\omega\tau}d\tau - \int_{-\infty}^{\infty} R_{xx}(\tau-2a)e^{-i\omega\tau}d\tau$$

Now let $\tau+2a = u \quad \Rightarrow \quad \tau = u-2a \quad \Rightarrow \quad d\tau = du$
Similarly, let $\tau-2a = v \quad \Rightarrow \quad \tau = v+2a \quad \Rightarrow \quad d\tau = dv$
Consider

$$\int_{-\infty}^{\infty} R_{xx}(\tau+2a)e^{-i\omega\tau}d\tau = \int_{-\infty}^{\infty} R_{xx}(u)e^{-i\omega(u-2a)}du$$
$$= e^{2i\omega a}\int_{-\infty}^{\infty} R_{xx}(u)e^{-i\omega u}du = e^{2i\omega a}S_{xx}(\omega)$$

$$\int_{-\infty}^{\infty} R_{xx}(\tau-2a)e^{-i\omega\tau}d\tau = \int_{-\infty}^{\infty} R_{xx}(v)e^{-i\omega(u+2a)}du$$
$$= e^{-2i\omega a}\int_{-\infty}^{\infty} R_{xx}(u)e^{-i\omega u}du = e^{-2i\omega a}S_{xx}(\omega)$$

$$\therefore \quad S_{yy}(\omega) = 2S_{xx}(\omega) - e^{2i\omega a}S_{xx}(\omega - e^{-2i\omega a}S_{xx}(\omega)$$
$$= 2S_{xx}(\omega) - \left(e^{2i\omega a} + e^{-2i\omega a}\right)S_{xx}(\omega)$$
$$= 2S_{xx}(\omega) - 2\cos 2a\omega\, S_{xx}(\omega)$$
$$= 2(1-\cos 2a\omega)S_{xx}(\omega)$$
$$= 2(2\sin^2 a\omega)S_{xx}(\omega)$$
$$= 4\sin^2 a\omega\, S_{xx}(\omega)$$

Problem 5. Determine the autocorrelation function of a stationary random process $\{X(t)\}$ whose power spectral density function is given by

$$S_{xx}(\omega) = \begin{cases} 1, & |\omega| < a \\ 0, & \text{otherwise} \end{cases}$$

Also show that the member functions $X(t)$ and $X\left(t+\frac{\pi}{a}\right)$ of the process $\{X(t)\}$ are uncorrelated.

SOLUTION:
Let $R_{xx}(\tau)$ be the autocorrelation function of a stationary random process $\{X(t)\}$. Then we know that

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\tau\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a} (1)\, e^{i\tau\omega}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{i\tau}}{i\tau}\right]_{-a}^{a}$$

$$= \frac{1}{2\pi}\frac{e^{i\tau a} - e^{-i\tau a}}{i\tau}$$

$$= \frac{1}{2\pi}\frac{2i\sin a\tau}{i\tau} = \frac{\sin a\tau}{\pi\tau}$$

Consider the autocorrelation

$$E\left\{X(t)X\left(t+\frac{\pi}{a}\right)\right\} = R_{xx}\left(\frac{\pi}{a}\right) = \frac{\sin a\frac{\pi}{a}}{\pi\frac{\pi}{a}} = \frac{a}{\pi^2}\sin\pi = 0$$

Again consider the autocovariance

$$C_{xx}\left\{X(t)X\left(t+\frac{\pi}{a}\right)\right\} = E\left\{X(t)X\left(t+\frac{\pi}{a}\right)\right\} - E\{X(t)\}\,E\left\{X\left(t+\frac{\pi}{a}\right)\right\}$$

$$= R_{xx}\left(\frac{\pi}{a}\right) - (0)(0) = 0$$

Since covariance of $X(t)$ and $X\left(t+\dfrac{\pi}{a}\right)$ is zero, the member functions $X(t)$ and $X\left(t+\dfrac{\pi}{a}\right)$ of the process $\{X(t)\}$ are uncorrelated.

Problem 6. If $\{X(t)\}$ and $\{Y(t)\}$ are two stationary processes with the cross-power spectral density function given by

$$S_{xy}(\omega) = \begin{cases} a + (ib\omega/\alpha), & |\omega| \le \alpha \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha > 0$, a and b are constants then obtain the crosscorrelation function.

SOLUTION:
If $\{X(t)\}$ and $\{Y(t)\}$ are two stationary processes with crosscorrelation function $R_{xy}(\tau)$, then crosscorrelation function $R_{xy}(\tau)$ can be obtained as the Fourier inverse transform of $S_{xy}(\omega)$ and is given by

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)e^{+i\tau\omega}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\left(a+\frac{ib\omega}{\alpha}\right)e^{+i\tau\omega}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} a\,e^{+i\tau\omega}\,d\omega + \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\frac{ib\omega}{\alpha}e^{+i\tau\omega}\,d\omega$$

$$= \frac{a}{2\pi}\int_{-\alpha}^{\alpha} e^{+i\tau\omega}\,d\omega + \frac{ib}{2\pi\alpha}\int_{-\alpha}^{\alpha}\omega\,e^{+i\tau\omega}\,d\omega$$

$$= \frac{a}{2\pi}\left[\frac{e^{+i\tau\omega}}{i\tau}\right]_{-\alpha}^{\alpha} + \frac{ib}{2\pi\alpha}\left[\omega\frac{e^{+i\tau\omega}}{i\tau} - (1)\frac{e^{+i\tau\omega}}{(i\tau)^2}\right]_{-\alpha}^{\alpha}$$

$$= \frac{a}{2\pi}\left[\frac{e^{+i\tau\alpha}}{i\tau} - \frac{e^{-i\tau\alpha}}{i\tau}\right] + \frac{ib}{2\pi\alpha}\left[\left(\alpha\frac{e^{+i\tau\alpha}}{i\tau} - \frac{e^{+i\tau\alpha}}{(i\tau)^2}\right) - \left(-\alpha\frac{e^{-i\tau\alpha}}{i\tau} - \frac{e^{-i\tau\alpha}}{(i\tau)^2}\right)\right]$$

$$= \frac{a}{2\pi}\left[\frac{e^{+i\tau\alpha} - e^{-i\tau\alpha}}{i\tau}\right] + \frac{ib}{2\pi\alpha}\left[\left(\alpha\frac{e^{+i\tau\alpha} + e^{-i\tau\alpha}}{i\tau}\right) - \left(\frac{e^{+i\tau\alpha} - e^{-i\tau\alpha}}{(i\tau)^2}\right)\right]$$

$$= \frac{a}{2\pi}\left[\frac{2i\sin\alpha\tau}{i\tau}\right] + \frac{ib}{2\pi\alpha}\left[\left(\alpha\frac{2\cos\alpha\tau}{i\tau}\right) - \left(\frac{2i\sin\alpha\tau}{(i\tau)^2}\right)\right]$$

$$= \frac{1}{\pi\tau^2}\left(a\tau\sin\alpha\tau + b\tau\cos\alpha\tau - \frac{b}{\alpha}\sin\alpha\tau\right)$$

$$= \frac{1}{\pi\tau^2}\left[a\tau - \frac{b}{\alpha})\sin\alpha\tau + b\tau\cos\alpha\tau\right]$$

Problem 7. Obtain the power spectral density function of the output process $\{Y(t)\}$ corresponding to the input process $\{X(t)\}$ in the system that has an impulse response $h(t) = e^{-\beta t}U(t)$.

SOLUTION:
Let $S_{xx}(\omega)$ and $S_{yy}(\omega)$ be the power spectral density functions of the processes $\{X(t)\}$ and $\{Y(t)\}$ respectively. Then we know that by Property 5 of PSD

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

where $H(\omega)$ is the Fourier transform of the impulse response $h(t)$.

Consider

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = \int_{0}^{\infty} e^{-\beta t}e^{-i\omega t}dt$$

$$= \int_{0}^{\infty} e^{-(\beta+i\omega)t}dt = \left[\frac{e^{-(\beta+i\omega)t}}{-(\beta+i\omega)}\right]_0^{\infty}$$

$$= \frac{1}{(\beta+i\omega)}$$

$$\Rightarrow |H(\omega)|^2 = H(\omega)H^*(\omega) = \left(\frac{1}{(\beta+i\omega)}\right)\left(\frac{1}{(\beta-i\omega)}\right) = \frac{1}{\beta^2+\omega^2}$$

$$\therefore \quad S_{yy}(\omega) = \frac{1}{\beta^2+\omega^2}S_{xx}(\omega)$$

Problem 8. If $\{X(t)\}$ is a stationary random process with power spectral density function $S_{xx}(\omega) = \frac{1}{(1+\omega^2)^2}$, then find the autocorrelation function of $\{X(t)\}$ and average power.

SOLUTION:
It is given that

$$S_{xx}(\omega) = \frac{1}{(1+\omega^2)^2}$$

We know that the autocorrelation function of a stationary random process $\{X(t)\}$ is given by

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{i\tau\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2}e^{i\tau\omega}d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} (\cos \tau\omega + i \sin \tau\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} \cos \tau\omega d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} i \sin \tau\omega\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} \cos \tau\omega d\omega + 0 \qquad \because \quad \frac{1}{(1+\omega^2)^2} i \sin \tau\omega \quad \textit{is odd}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} \cos \tau\omega d\omega$$

By using complex integration (Refer to Result A.5.1 in Appendix A), we have

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} \cos \tau\omega d\omega = \frac{(1+\tau)e^{-\tau}}{4}$$

Average power of the process $\{X(t)\}$ is given by

$$E\left\{X^2(t)\right\} = R_{xx}(0) = \frac{1}{4} = 0.25$$

Problem 9. If $\{X(t)\}$ is a stationary random process such that $X(t) = a\cos(bt + \theta)$, where a and b are constants and θ is a random variable uniformly distributed in $(0, 2\pi)$, then find the power spectral density function of $\{X(t)\}$.

SOLUTION:
We know that the power spectral density function of a stationary random process $\{X(t)\}$ with autocorrelation function $R_{xx}(\tau)$ is given by

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau$$

Consider

$$R_{xx}(\tau) = E\{X(t+\tau)X(t)\}$$
$$= E\{[a \cos b(t+\tau) + \theta)][a \cos(bt+\theta)]\}$$

$$= a^2 E\left\{\frac{\cos[b(t+\tau)+bt+2\theta)]+\cos b\tau}{2}\right\}$$

$$= \frac{a^2}{2}E\{\cos[b(t+\tau)+bt+2\theta)]\}+\frac{a^2}{2}E\{\cos b\tau\}$$

Since θ is a random variable uniformly distributed in $(0, 2\pi)$, its probability density function is given by

$$f(\theta) = \frac{1}{2\pi}, \qquad 0 \le \theta \le 2\pi$$

Consider

$$E\{\cos[b(t+\tau)+bt+2\theta)]\} = \int_0^{2\pi} \cos[b(t+\tau)+bt+2\theta)]\frac{1}{2\pi}d\theta$$

$$= \frac{1}{2\pi}\left\{\frac{\sin(2bt+b\tau+2\theta)}{2}\right\}_0^{2\pi}$$

$$= \frac{1}{4\pi}\{\sin(2bt+b\tau+4\pi)-\sin(2bt+b\tau)\}$$

$$= \frac{1}{4\pi}\{\sin(2bt+b\tau)-\sin(2bt+b\tau)\} = 0$$

Consider

$$\frac{a^2}{2}E\{\cos b\tau\} = \frac{a^2}{2}\int_0^{2\pi}\cos b\tau\frac{1}{2\pi}d\theta$$

$$= \frac{a^2}{2}\cos b\tau$$

$$R_{xx}(\tau) = \frac{a^2}{2}\cos b\tau$$

$$\therefore \quad S_{xx}(\omega) = \int_{-\infty}^{\infty}\frac{a^2}{2}\cos b\tau\, e^{-i\omega\tau}d\tau$$

$$= \text{Fourier transform of } \frac{a^2}{2}\cos b\tau$$

Consider inverse Fourier transform

$$F^{-1}\left\{\pi\frac{a^2}{2}[\delta(\omega-b)+\delta(\omega+b)]\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\frac{a^2}{2}[\delta(\omega-b)+\delta(\omega+b)]e^{i\tau\omega}d\omega$$

$$=\frac{a^2}{4}\left(e^{i\tau b}+e^{-i\tau b}\right)=\frac{a^2}{2}\cos b\tau$$

This is true because $\int_{-\infty}^{\infty}\phi(x)\delta(x-c)dx=\phi(c)$

$$\therefore\quad F\left\{\frac{a^2}{2}\cos b\tau\right\}=\frac{\pi a^2}{2}[\delta(\omega-b)+\delta(\omega+b)]$$

$$=\frac{a^2\pi}{2}\{\delta(\omega-b)+\delta(\omega+b)\}$$

(Refer to Result A.6.1 in Appendix A for more details of this result.)

Problem 10. Find the power spectral density function of the random process whose autocorrelation function is given by $R(\tau)=e^{-a\tau^2}\cos b\tau$ where a and b are constants.

SOLUTION:
It is given that

$$R(\tau)=e^{-a\tau^2}\cos b\tau$$

We know that the power spectral density function of a stationary random process $\{X(t)\}$ is given by

$$S_{XX}(\omega)=\int_{-\infty}^{\infty}R_{XX}(\tau)e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty}e^{-a\tau^2}\cos b\tau e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty}e^{-a\tau^2}\left(\frac{e^{ib\tau}+e^{-ib\tau}}{2}\right)e^{-i\omega\tau}d\tau$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}e^{-a\tau^2}\left(e^{-i(\omega-b)\tau}+e^{-i(\omega+b)\tau}\right)d\tau$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left\{e^{-\left[a\tau^2+i(\omega-b)\tau\right]}+e^{-\left[a\tau^2+i(\omega+b)\tau\right]}\right\}d\tau$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}e^{-\left[a\tau^2+i(\omega-b)\tau\right]}d\tau+\frac{1}{2}\int_{-\infty}^{\infty}e^{-\left[a\tau^2+i(\omega+b)\tau\right]}d\tau$$

Consider

$$\frac{1}{2}\int_{-\infty}^{\infty}e^{-\left[a\tau^2+i(\omega-b)\tau\right]}d\tau$$

Add and subtract $[i(\omega-b)/2a]^2$to make the exponent a perfect square. This gives

$$\frac{1}{2}\int_{-\infty}^{\infty}e^{-\left\{a\tau^2+i(\omega-b)\tau)^2\right\}}d\tau=\int_{-\infty}^{\infty}e^{-a\left\{\tau^2+2[i(\omega-b)/2a]\tau+[i(\omega-b)/2a]^2-[i(\omega-b)/2a]^2\right\}}d\tau$$

$$=\frac{e^{-(\omega-b)^2/4a}}{2}\int_{-\infty}^{\infty}e^{-a\{\tau+i(\omega-b)/2a\}^2}d\tau$$

Let $u=\sqrt{a}\left(\tau+\frac{i(\omega-b)}{2a}\right) \quad\Rightarrow\quad d\tau=\frac{du}{\sqrt{a}}$

$$\Rightarrow\quad \frac{1}{2}\int_{-\infty}^{\infty}e^{-\left\{a\tau^2+i(\omega-b)\tau)^2\right\}}d\tau=\frac{e^{-(\omega-b)^2/4a}}{2\sqrt{a}}\int_{-\infty}^{\infty}e^{-u^2}du$$

$$=\frac{e^{-(\omega-b)^2/4a}}{\sqrt{a}}\int_{0}^{\infty}e^{-u^2}du$$

Now let $v=u^2 \quad\Rightarrow\quad dv=2udu=2\sqrt{v}du \quad\Rightarrow\quad du=\frac{dv}{2}v^{-1/2}$

$$\Rightarrow\quad \frac{e^{-(\omega-b)^2/4a}}{\sqrt{a}}\int_{0}^{\infty}e^{-u^2}du=\frac{e^{-(\omega-b)^2/4a}}{2\sqrt{a}}\int_{0}^{\infty}e^{-v}v^{\frac{1}{2}-1}dv$$

$$=\frac{e^{-(\omega-b)^2/4a}}{2\sqrt{a}}\Gamma(1/2)$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-(\omega-b)^2/4a}$$

Similarly, we can show that

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-\left[a\tau^2+i(\omega+b)\tau\right]}d\tau = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-(\omega+b)^2/4a}$$

$$\therefore\quad S_{xx}(\omega) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\left\{e^{-(\omega-b)^2/4a}+e^{-(\omega+b)^2/4a}\right\}$$

Problem 11. If $\{X(t)\}$ is a random process such that $X(t)=Y(t)Z(t)$, where $Y(t)$ and $Z(t)$ are independent wide sense stationary processes. Then show that

(i) $R_{xx}(\tau)=R_{yy}(\tau)R_{zz}(\tau)$

(ii) $S_{xx}(\omega)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} S_{yy}(a)S_{zz}(\omega-a)\,da$

SOLUTION:

(i) We know that since $\{Y(t)\}$ and $\{Z(t)\}$ are independent stationary processes, and $X(t)=Y(t)Z(t)$, the autocorrelation function $R_{xx}(t,t+\tau)$ can be given as

$$\begin{aligned} R_{xx}(\tau,t+\tau)=R_{xx}(\tau) &= E\{X(t)X(t+\tau)\} \\ &= E\{[Y(t)Z(t)]\,[Y(t+\tau)Z(t+\tau)]\} \\ &= E\{[Y(t)Y(t+\tau)]\,[Z(t)Z(t+\tau)]\} \end{aligned}$$

Since $\{Y(t)\}$ and $\{Z(t)\}$ are independent, we have

$$\begin{aligned} R_{xx}(\tau) &= E\{Y(t)Y(t+\tau)\}E\{Z(t)Z(t+\tau)\} \\ &= R_{yy}(\tau)R_{zz}(\tau) \end{aligned}$$

(ii) We know that the power spectral density function of $\{X(t)\}$ given by $S_{xx}(\omega)$ is the Fourier transformation of $R_{xx}(\tau)$. That is,

$$S_{xx}(\omega)=F\left[R_{xx}(\tau)\right]=F\left[R_{yy}(\tau)R_{zz}(\tau)\right]$$

Consider the Fourier inverse transform

$$F^{-1}\left\{\int_{-\infty}^{\infty} S_{yy}(a)S_{zz}(\omega-a)\,da\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{S_{yy}(a)S_{zz}(\omega-a)da\}\,e^{j\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(a)S_{zz}(\omega-a)\,e^{j\omega\tau}da\,d\omega$$

Letting $a=y$ and $\omega-a=z$, we have by transformation of variables

$$da\,d\omega = \begin{vmatrix} \dfrac{\partial a}{\partial y} & \dfrac{\partial a}{\partial z} \\ \dfrac{\partial \omega}{\partial y} & \dfrac{\partial \omega}{\partial z} \end{vmatrix} dydz = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = dydz$$

$$\therefore \quad F^{-1}\left\{ \int_{-\infty}^{\infty} S_{yy}(a) S_{zz}(\omega - a)\, da \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(y) S_{zz}(z)\, e^{i(y+z)\tau} dydz$$

$$= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(y) e^{iy\tau} dy \right)$$

$$\left(\int_{-\infty}^{\infty} S_{zz}(z)\, e^{iz\tau} dz \right)$$

$$= F^{-1}\{S_{yy}(\omega)\} \left(2\pi F^{-1}\{S_{zz}(\omega)\} \right),$$

$$\textit{by letting} \quad y, z = \omega$$

$$= 2\pi R_{yy}(\tau) R_{zz}(\tau)$$

$$\therefore \quad F\{R_{yy}(\tau) R_{zz}(\tau)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(a) S_{zz}(\omega - a)\, da$$

$$\Rightarrow S_{xx}(\omega) = F\{R_{yy}(\tau) R_{zz}(\tau)\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(a) S_{zz}(\omega - a)\, da$$

Problem 12. If $\{X(t)\}$ is a zero-mean stationary Gaussian random process with power spectral density function $S_{xx}(\omega)$ then obtain the power spectral density function of the square law detector process $\{Y(t)\}$ where $Y(t) = X^2(t)$.

SOLUTION:

It is given that the process $\{X(t)\}$ is stationary Gaussian random process with mean $E\{X(t)\} = 0$, variance $V\{X(t)\} = \sigma_x^2$ (say) and autocorrelation function $R_{xx}(\tau)$. Now,

$$E\{X(t)\} = 0, \quad \Rightarrow V\{X(t)\} = \sigma_x^2 = E\left\{X^2(t)\right\} = R_{xx}(0)$$

$$\therefore \quad E\{Y(t)\} = E\left\{X^2(t)\right\} = R_{xx}(0)$$

Consider

$$R_{yy}(t_1,t_2) = E\{Y(t_1)Y(t_2)\}$$
$$= E\left\{X^2(t_1)X^2(t_2)\right\}$$
$$= E\left\{X^2(t_1)\right\}E\left\{X^2(t_2)\right\} + 2\{E[X(t_1)X(t_2)]\}^2$$

(Refer Eqn. 6.8 in Chapter 6)

$$\therefore \quad R_{yy}(t_1,t_2) = R_{xx}^2(0) + 2R_{xx}^2(t_1,t_2)$$
$$\Rightarrow \quad R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau) \quad (\because \quad \{X(t)\} \text{ is stationary})$$

The power spectral density function of the process $\{Y(t)\}$ is given by

$$S_{yy}(\omega) = \int_{-\infty}^{\infty} R_{yy}(\tau)\, e^{-i\omega\tau} d\tau$$
$$= \int_{-\infty}^{\infty} \left\{R_{xx}^2(0) + 2R_{xx}^2(\tau)\right\} e^{-i\omega\tau} d\tau$$
$$= \int_{-\infty}^{\infty} R_{xx}^2(0) e^{-i\omega\tau} d\tau + 2\int_{-\infty}^{\infty} R_{xx}^2(\tau) e^{-i\omega\tau} d\tau$$
$$= 2\pi R_{xx}^2(0)\delta(\omega) + 2F\{R_{xx}(\tau)R_{xx}(\tau)\}$$

where $\delta(\omega)$ is the unit impulse function. This is true because we know that F^{-1} $\{2\pi\alpha^2\delta(\omega)\} = \alpha^2$.

Now, consider

$$F^{-1}\{S_{xx}(\omega) * S_{xx}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{S_{xx}(a)S_{xx}(\omega - a)da\}\, e^{i\omega\tau} d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(a)S_{xx}(\omega - a)e^{i\omega\tau} da\, d\omega$$

Here '*' stands for convolution.

Letting $a = u$ and $\omega - a = v$, we have by transformation of variables

$$da\,d\omega = \begin{vmatrix} \dfrac{\partial a}{\partial u} & \dfrac{\partial a}{\partial v} \\ \dfrac{\partial \omega}{\partial u} & \dfrac{\partial \omega}{\partial v} \end{vmatrix} du\,dv = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = du\,dv$$

$$F^{-1}\{S_{xx}(\omega) * S_{xx}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(u)S_{xx}(v)e^{i(u+v)\tau}du\,dv$$

$$= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(u)e^{iy\tau}du\right)$$

$$\left(\int_{-\infty}^{\infty} S_{xx}(v)\,e^{iz\tau}dv\right)$$

$$= F^{-1}\{S_{yy}(\omega)\}\left(2\pi F^{-1}\{S_{zz}(\omega)\}\right),$$

$$\text{by letting} \quad y, z = \omega$$

$$= 2\pi R_{xx}(\tau)R_{xx}(\tau)$$

$$\therefore \quad F\{R_{xx}(\tau)R_{xx}(\tau)\} = \frac{1}{2\pi}S_{xx}(\omega) * S_{xx}(\omega)$$

$$\therefore \quad S_{yy}(\omega) = 2\pi R_{xx}^2(0)\delta(\omega) + \frac{1}{\pi}S_{xx}(\omega) * S_{xx}(\omega)$$

EXERCISE PROBLEMS

1. Given that the random process $\{X(t)\}$ is a wide sense stationary process whose power spectral density function traps an area of 12.5 square units in the first quadrant. Find the power of the random process.

2. Find the power spectral density function of the random process whose autocorrelation function is given by $R(\tau) = e^{-\tau^2}$.

3. Determine the autocorrelation function of a stationary random process $\{X(t)\}$ whose power spectral density function is given by

$$S_{xx}(\omega) = \begin{cases} k, & |\omega| < a \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

4. A stationary random process $\{X(t)\}$ is known to have an autocorrelation function of the form

$$R_{xx}(\tau) = \begin{cases} 1-|\tau|, & -1 < \tau < 1 \\ 0, & \text{otherwise} \end{cases}$$

Show that the power spectral density function is given in the form $S_{xx}(\omega) = \left(\dfrac{\sin \omega/2}{\omega/2}\right)^2$.

5. If the power spectral density of a stationary random process $\{X(t)\}$ is given by

$$S(\omega) = \begin{cases} 1+\omega^2, & |\omega| \le 1 \\ 0, & |\omega| > 1 \end{cases}$$

then obtain the autocorrelation function.

6. If the autocorrelation function of a stationary process $\{X(t)\}$ is given by $R(\tau) = a^2 e^{-2b|\tau|}$, where a and b are constants, then obtain the power spectral density function of the process $\{X(t)\}$.

7. If the autocorrelation function of a stationary process $\{X(t)\}$ is given by $R(\tau) = a e^{-a|\tau|}$, where a is constant, then obtain the power spectral density function of the process $\{X(t)\}$. Obtain the power spectral density when $a = 10$.

8. If $\{X(t)\}$ and $\{Y(t)\}$ are two stationary processes with the cross-power spectral density function given by

$$S_{xy}(\omega) = \begin{cases} a+ib\omega, & |\omega| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha > 0$, a and b are constants then obtain the crosscorrelation function.

9. Given that a process $\{X(t)\}$ has the autocorrelation function $R_{xx}(\tau) = A\, e^{-a|\tau|}$ $\cos b\tau$ where $A > 0$, $a > 0$ and b are real constants, then find the power spectral density function of $\{X(t)\}$.

10. If $\{X(t)\}$ is a stationary process with power spectral density function given by

$$S_{xy}(\omega) = \begin{cases} \omega^2, & |\omega| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

then obtain the autocorrelation function.

Let us suppose that the one-step transition probability matrix with three states 1, 2 and 3 is given as follows:

$$P^{(1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.3 & 0.7 & 0.0 \\ 0.6 & 0.0 & 0.4 \\ 0.5 & 0.5 & 0.0 \end{bmatrix} \end{array}$$

Based on the one-step transition probability matrix, the transitions from one state to one or more available states can be depicted using a transition diagram as shown in Figure 9.2. In this figure, the possible transitions among three states 1, 2 and 3 are shown.

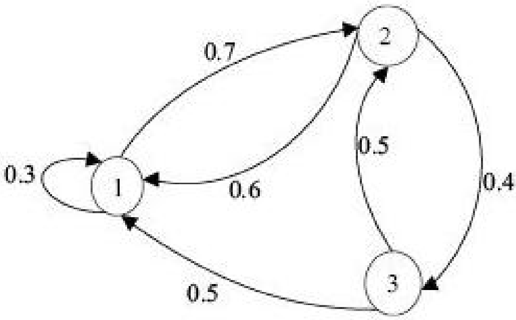


Figure 9.2. Transition diagram showing transitions among three states 1, 2 and 3 and the corresponding transition probabilities

In Figure 9.2, state 1 is accessible from states 2, 3 and its own, state 2 is accessible from 1 and state 3 whereas state 3 is accessible only from state 2 all in one step. This can be guessed from looking at the columns of the transition matrix. This implies that $P_{11}^{(1)} = 0.3$, $P_{21}^{(1)} = 0.6$, $P_{31}^{(1)} = 0.5$; $P_{12}^{(1)} = 0.7$, $P_{22}^{(1)} = 0.0$, $P_{32}^{(1)} = 0.5$; $P_{13}^{(1)} = 0.0$, $P_{23}^{(1)} = 0.4$, $P_{33}^{(1)} = 0.0$.

Higher order transition probability matrices

The first cycle in the transition diagram gives one-step transition probabilities $P_{ij}^{(1)}$, second cycle gives two-step transition probabilities $P_{ij}^{(2)}$ and so on and n^{th} cycle gives the n-step transition probabilities $P_{ij}^{(n)}$. For example, higher order transition probabilities and matrices can be obtained as follows.

Two-step transition probabilities (i.e., reachability of states in two steps):

$$P_{11}^{(2)} = (0.3)(0.3) + (0.7)(0.6) = 0.51, P_{12}^{(2)} = (0.3)(0.7) = 0.21,$$
$$P_{13}^{(2)} = (0.7)(0.4) = 0.28$$

$$P_{21}^{(2)} = (0.6)(0.3) + (0.4)(0.5) = 0.38, P_{22}^{(2)} = (0.6)(0.7) + (0.4)(0.5) = 0.62,$$
$$P_{23}^{(2)} = 0$$
$$P_{31}^{(2)} = (0.5)(0.3) + (0.5)(0.6) = 0.45, P_{32}^{(2)} = (0.5)(0.7) = 0.35, P_{33}^{(2)} = 0.20$$

Therefore, the two-step transition probability matrix becomes

$$P^{(2)} = \begin{bmatrix} 0.51 & 0.21 & 0.28 \\ 0.38 & 0.62 & 0.0 \\ 0.45 & 0.35 & 0.20 \end{bmatrix}$$

Three-step transition probabilities (i.e., reachability of states in three steps):

$$P_{11}^{(3)} = (0.3)(0.3)(0.3) + (0.3)(0.7)(0.6) + (0.7)(0.6)(0.3)$$
$$+ (0.7)(0.4)(0.5) = 0.419,$$

$$P_{12}^{(3)} = (0.7)(0.4)(0.5) + (0.7)(0.6)(0.7) + (0.3)(0.3)(0.7) = 0.497,$$

$$P_{13}^{(3)} = (0.3)(0.7)(0.4) = 0.084,$$

$$P_{21}^{(3)} = (0.4)(0.5)(0.3) + (0.6)(0.7)(0.6) + (0.4)(0.5)(0.6)$$
$$+ (0.6)(0.3)(0.3) = 0.486,$$

$$P_{22}^{(3)} = (0.6)(0.3)(0.7) + (0.4)(0.5)(0.7) = 0.266,$$

$$P_{23}^{(3)} = (0.4)(0.5)(0.4) + (0.6)(0.7)(0.4) = 0.248,$$

$$P_{31}^{(3)} = (0.5)(0.4)(0.5) + (0.5)(0.3)(0.3) + (0.5)(0.6)(0.3)$$
$$+ (0.5)(0.7)(0.6) = 0.445,$$

$$P_{32}^{(3)} = (0.5)(0.4)(0.5) + (0.5)(0.3)(0.7) + (0.5)(0.6)(0.7) = 0.415,$$

$$P_{33}^{(3)} = (0.5)(0.7)(0.4) + (0.5)(0.3)(0.7) + (0.5)(0.6)(0.7) = 0.140$$

Therefore, the three-step transition probability matrix becomes

$$P^{(3)} = \begin{bmatrix} 0.419 & 0.497 & 0.084 \\ 0.486 & 0.266 & 0.248 \\ 0.445 & 0.415 & 0.140 \end{bmatrix}$$

Similarly, we can find all higher order transition probabilities and transition probability matrices. It may be noted that in every transition probability matrix of different order, each row total equals to one.

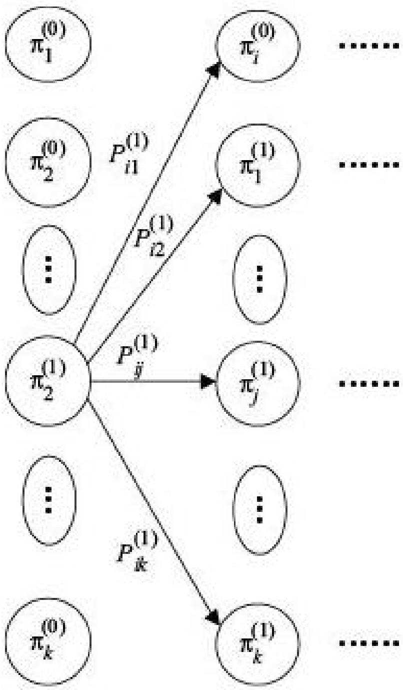


Figure 9.3. State probabilities ($\pi_i^{(n)}, i = 1,2,3,\cdots\cdots k, n = 0,1,2,3,\cdots\cdots$) and transition probabilities ($P_{ij}^{(n)}, n = 1,2,3,\cdots\cdots\cdots; i,j = 1,2,3,\cdots\cdots\cdots, k$)

Further, given the one-step transition probability matrix $P^{(1)}$ and if $\pi^{(0)} = [\pi_1^{(0)}, \pi_2^{(0)}, \cdots, \pi_i^{(0)}, \cdots, \pi_k^{(0)}]$ is the initial state probability distribution, then the state probability distribution after one step $\pi^{(1)}$, two steps $\pi^{(2)}$ and so on n steps $\pi^{(n)}$ and can be obtained as

$$\pi^{(n)} = \pi^{(0)} P^{(n)}, \quad n = 1, 2, \cdots\cdots \tag{9.6}$$

where $P^{(n)}$ is the n-step transition probability matrix. The state probability distributions can also be obtained by using the relationship

$$\pi^{(n)} = \pi^{(n-1)} P^{(1)}, \quad n = 1, 2, \cdots\cdots \tag{9.7}$$

Note:

A Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ is said to be completely specified, if initial probability distribution and transition probability matrix are known.

Expanding $\sum_{k=1}^{\infty} P_{ik}^{(1)} P_{kj}^{(1)}$, we get the $(i, j)^{th}$ element $P_{ij}^{(2)}$, $i = 1, 2, 3, \cdots\cdots\cdots, k$; $j = 1, 2, 3, \cdots\cdots\cdots, k$, of two-step transition probability matrix $P^{(2)}$ as the $(i, j)^{th}$ element of the square of the one-step transition probability matrix $P^{(1)}$, that is, the $(i, j)^{th}$ element of $\left\{P^{(1)}\right\}^2$. Hence,

$$P^{(2)} = \left\{P^{(1)}\right\}^2$$

Now, consider three-step transition probabilities

$$P_{ij}^{(3)} = P\{X_3 = j/X_0 = i\} \text{ for } i = 1, 2, 3, \cdots\cdots\cdots, k; j = 1, 2, 3, \cdots\cdots\cdots, k$$

Here, state j can be reached from state i in three steps. Let us assume that the chain starts initially from state i and moves to an intermediate state, say k, in two steps and then from k, it moves further to state j in the third step. Therefore, we have

$$\begin{aligned}
P_{ij}^{(3)} &= P\{X_3 = j/X_0 = i\} \\
&= P\{X_3 = j/X_2 = k\} P\{X_2 = k/X_0 = i\} \\
&= P\{X_0 = k/X_2 = i\} P\{X_3 = j/X_2 = k\} \\
&= P_{ik}^{(2)} P_{kj}^{(1)}
\end{aligned}$$

Since the intermediate state k can be $1, 2, 3, \cdots\cdots\cdots$ we have

$$P_{ij}^{(3)} = P_{i1}^{(2)} P_{1j}^{(1)} \text{ or } P_{ij}^{(3)} = P_{i2}^{(2)} P_{2j}^{(1)} \text{ or } P_{ij}^{(3)} = P_{i3}^{(2)} P_{3j}^{(1)} \text{ and so on.}$$

That is,

$$\begin{aligned}
P_{ij}^{(3)} &= P_{i1}^{(2)} P_{1j}^{(1)} + P_{i2}^{(2)} P_{2j}^{(1)} + P_{i3}^{(2)} P_{3j}^{(1)} + \cdots\cdots \\
&= \sum_{k=1}^{\infty} P_{ik}^{(2)} P_{kj}^{(1)}
\end{aligned}$$

Expanding $\sum_{k=1}^{\infty} P_{ik}^{(2)} P_{kj}^{(1)}$, we get the $(i, j)^{th}$ element $P_{ij}^{(3)}$, $i = 1, 2, 3, \cdots\cdots\cdots, k$; $j = 1, 2, 3, \cdots\cdots\cdots, k$, of the three-step transition probability matrix $P^{(3)}$ as the $(i, j)^{th}$ element of the cubic power of the one-step transition probability matrix $P^{(1)}$, that is the $(i, j)^{th}$ element of $\left\{P^{(1)}\right\}^3$. Hence,

$$P^{(3)} = \left\{P^{(1)}\right\}^3$$

Continuing in this way, we obtain $P_{ij}^{(n)} = \sum_{k=1}^{\infty} P_{ik}^{(n-1)} P_{kj}^{(1)}$, and by expanding the sum we get the $(i, j)^{th}$ element $P_{ij}^{(n)}$, $i = 1, 2, 3, \cdots\cdots\cdots, k$, $j = 1, 2, 3, \cdots\cdots\cdots, k$, of

This results in the fact that for an irreducible, aperiodic, recurrent Markov chain, the steady-state distribution is unique and can be given as

$$\pi_i = \frac{1}{\mu_{ii}} \tag{9.16}$$

Clearly, if $\mu_{ii} < \infty$, that is non-null persistent, we have $\pi_i > 0$ and if $\mu_{ii} = \infty$, that is null persistent, we have $\pi_i = 0$ and vice-versa.

SOLVED PROBLEMS

Problem 1. In tossing of a fair coin, let $\{X_n\}$, $n = 1, 2, 3, \cdots$ denote the outcome of n^{th} toss, then show that the process of getting total number of heads in the first n trials is a Markov process. Or using a suitable example show that *Bernoulli process* is a Markov process.

SOLUTION:

Let us suppose that we observe a sequence of random variables $\{X_n\}$, $n=1, 2, 3, \cdots$ with probability $1/2$ when head turns (say 1) and with probability $1/2$ when tail turns (say 0). That is,

$$X_n = \begin{cases} 0 & \text{if tail turns at } n^{th} \text{ toss} \\ 1 & \text{if head turns at } n^{th} \text{ toss} \end{cases}$$

This is similar to the sequence of Bernoulli trials, say $X_1, X_2, X_3, \cdots\cdots, X_n, \cdots\cdots$, each with probability of success equal to p and with probability of failure equal to $q = 1 - p$.

Let $S_n = X_1 + X_2 + X_3 + \cdots\cdots + X_n$ be the state (sum) of the process outcomes representing the total of heads in the first n trials. Then the possible values of S_n are $0, 1, 2, \cdots\cdots, n$. Let us suppose that $S_n = x$, $x = 0, 1, 2, 3, \cdots\cdots, n$. Clearly, the state of the process after $n+1$ trials is given by $S_{n+1} = S_n + X_{n+1}$ and it can assume any two possible values, namely, $x+1$ if $(n+1)^{th}$ trial turns as head and x if $(n+1)^{th}$ trial turns as tail. Hence, we have

$$P(S_{n+1} = x+1/S_n = x) = \frac{1}{2}$$
$$P(S_{n+1} = x/S_n = x) = \frac{1}{2}$$

Therefore, the outcome at $(n+1)^{th}$ trial is affected only by the outcome of the n^{th} trial but not by the outcome up to $(n-1)^{th}$ trial. Hence, it is concluded that in tossing of a fair coin, the process of getting total number of heads in the first n trials is a Markov process. Therefore, Bernoulli process is a Markov process.

Problem 2. Show that Poisson process is a Markov process.

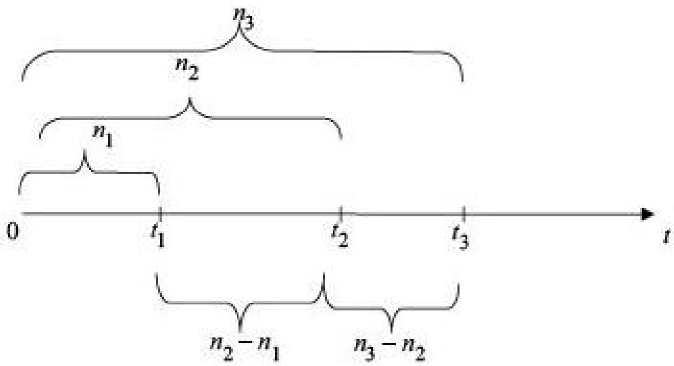
SOLUTION:

Let $\{X(t)\}$ be a Poisson process defined in the interval $(0, t)$. Then $\{X(t) = n\}$ or $n(0, t) = n$ represents that there are n Poisson points in the time interval $(0, t)$. That is,

$$P\{X(t) = n\} = \frac{e^{\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \cdots$$

or

$$P\{n(0,t) = n\} = \frac{e^{\lambda(t-0)}\{\lambda(t-0)\}^n}{n!} = \frac{e^{\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \cdots$$



If $\{X(t_1) = n_1\}$, $\{X(t_2) = n_2\}$ and $\{X(t_3) = n_3\}$ represent respectively the occurrence of n_1, n_2 and n_3 number of Poisson points in the time intervals $(0, t_1)$, $(0, t_2)$ and $(0, t_3)$, $(0 < t_1 < t_2 < t_3 < t)$, (refer to Figure), then in order to show that $\{X(t)\}$ is a Markov process, it is sufficient to show that

$$P\{X(t_3) = n_3/X(t_1) = n_1, X(t_2) = n_2\} = P\{X(t_3) = n_3/X(t_2) = n_2\}$$

This means that the number of Poisson points occurred at time point t_3 depends only on the number of Poisson points occurred at the most recent time point t_2. Now,

$$P\{X(t_3) = n_3/X(t_1) = n_1, X(t_2) = n_2\} = \frac{P\{X(t_1) = n_1, X(t_2) = n_2, X(t_3) = n_3\}}{P\{X(t_1) = n_1, X(t_2) = n_2\}}$$

$$= \frac{P\{n(0,t_1) = n_1\}P\{n(t_1,t_2) = n_2 - n_1\}\{n(t_2,t_3) = n_3 - n_2\}}{P\{n(0,t_1) = n_1\}P\{n(t_1,t_2) = n_2 - n_1\}}$$

$$= P\{n(t_2, t_3) = n_3 - n_2\}$$

$$= \frac{e^{\lambda(t_3 - t_2)}\{\lambda(t_3 - t_2)\}^{n_3 - n_2}}{(n_3 - n_2)!}, \quad n = 0, 1, 2, \cdots$$

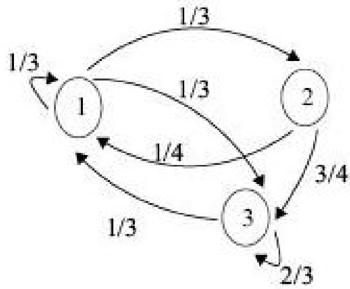
But

$$P\{n(t_2,t_3) = n_3 - n_2\} = P\{n(0,t_3) = n_3 / n(0,t_2) = n_2\} = P\{X(t_3) = n_3 / X(t_2) = n_2\}$$

$$\Rightarrow \quad P\{X(t_3) = n_3 / X(t_1) = n_1, X(t_2) = n_2\} = P\{X(t_3) = n_3 / X(t_2) = n_2\}$$

Therefore, Poisson process is a Markov process.

Problem 3. Obtain the one-step transition probability matrix for a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ with the following state transition diagram.



Solution:
The state space of the given Markov chain has three states 1, 2 and 3. Therefore, from the transition diagram, the one-step transition probability matrix can be given as follows:

$$P^{(1)} = \begin{pmatrix} P_{11}^{(1)} & P_{12}^{(1)} & P_{13}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} & P_{23}^{(1)} \\ P_{31}^{(1)} & P_{32}^{(1)} & P_{33}^{(1)} \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 0 & 3/4 \\ 1/3 & 0 & 2/3 \end{pmatrix}$$

Problem 4. Let $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ be a Markov chain with state space $\{1, 2\}$ with the following transition probability matrix

$$\begin{bmatrix} 1 & 0 \\ 0.25 & 0.75 \end{bmatrix}$$

Show that (i) state 1 is recurrent and (ii) state 2 is transient.

SOLUTION:

In order to show that state 1 is recurrent we have to show that $\sum_{n=1}^{\infty} f_{11}^{(n)} = 1$.

Consider

$$\sum_{n=1}^{\infty} f_{11}^{(n)} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} + \cdots\cdots$$

Here, $f_{ij}^{(1)} = P_{ij}^{(1)}, \quad \forall\, i, j$

From the given transition probability matrix, we have

$$f_{11}^{(1)} = P_{11}^{(1)} = 1, \quad f_{12}^{(1)} = P_{12}^{(1)} = 0, \quad f_{21}^{(1)} = P_{21}^{(1)} = 0.25, \quad f_{22}^{(1)} = 0.75$$

$$\Rightarrow \quad f_{11}^{(2)} = \sum_{j=2} P_{1k}^{(1)} f_{k1}^{(1)} = P_{12}^{(1)} f_{21}^{(1)} = (0)\,(0.25) = 0$$

$$f_{22}^{(2)} = \sum_{j=1} P_{2k}^{(1)} f_{k2}^{(1)} = P_{21}^{(1)} f_{12}^{(1)} = (0.25)\,(0) = 0$$

Now,

$$f_{12}^{(2)} = \sum_{j=1} P_{1k}^{(1)} f_{k2}^{(1)} = P_{11}^{(1)} f_{12}^{(1)} = (1)\,(0) = 0$$

$$f_{21}^{(2)} = \sum_{j=2} P_{2k}^{(1)} f_{k1}^{(1)} = P_{22}^{(1)} f_{21}^{(1)} = (0.75)\,(0.25) = 0.1875$$

Now, consider

$$f_{11}^{(3)} = \sum_{j=2} P_{1k}^{(1)} f_{k1}^{(2)} = P_{12}^{(1)} f_{21}^{(2)} = (0)\,(0.1875) = 0$$

$$f_{22}^{(3)} = \sum_{j=1} P_{1k}^{(1)} f_{k1}^{(2)} = P_{21}^{(1)} f_{12}^{(2)} = (0.25)\,(0) = 0$$

Note that $f_{11}^{(n)} = 0$ and $f_{22}^{(n)} = 0$ for $n \geq 2$. Therefore, we have

$$\sum_{n=1}^{\infty} f_{11}^{(n)} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} + \cdots\cdots = 1 + 0 + 0 + \cdots\cdots = 1$$

which implies that state 1 is recurrent, and

$$\sum_{n=1}^{\infty} f_{22}^{(n)} = f_{22}^{(1)} + f_{22}^{(2)} + f_{22}^{(3)} + \cdots\cdots = 0.75 + 0 + 0 + \cdots\cdots = 0.75 < 1$$

Which implies that state 2 is transient.

Problem 5. Let $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ be a Markov chain with state space $\{1, 2, 3, 4\}$ and initial probability distribution $P\{X_0 = i\} = \frac{1}{4}$, $\quad i = 1, 2, 3, 4$. The one-step transition probability matrix is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(i) Find $P(X_1 = 2/X_0 = 1)$, $P(X_2 = 2/X_1 = 1)$, $P(X_2 = 1/X_1 = 2)$ and $P(X_2 = 2/X_0 = 2)$

(ii) Show that $P_{22}^{(2)} = \sum\limits_{k=1}^{4} P_{2k}^{(1)} P_{k2}^{(1)}$

(iii) Find $P(X_2 = 2, X_1 = 3/X_0 = 2)$

(iv) Find $P(X_2 = 2, X_1 = 3, X_0 = 2)$

(v) Find $P(X_3 = 4, X_2 = 2, X_1 = 3, X_0 = 2)$

Solution:
It is given that $\pi_i^{(0)} = P\{X_0 = i\} = \frac{1}{4}$, $\quad i = 1, 2, 3, 4$

$$\Rightarrow \quad \pi^{(0)} = \left(\pi_1^{(0)}, \pi_2^{(0)}, \pi_3^{(0)}, \pi_4^{(0)}\right) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

The one-step transition probability matrix $P^{(1)}$ is given as

$$P^{(1)} = \begin{pmatrix} P_{11}^{(1)} & P_{12}^{(1)} & P_{13}^{(1)} & P_{14}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} & P_{23}^{(1)} & P_{24}^{(1)} \\ P_{31}^{(1)} & P_{32}^{(1)} & P_{33}^{(1)} & P_{34}^{(1)} \\ P_{41}^{(1)} & P_{42}^{(1)} & P_{43}^{(1)} & P_{44}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $P_{ij}^{(1)} = P\{X_n = j/X_{n-1} = i\}$, $\quad i, j = 1, 2, 3, 4, \quad n = 1, 2, 3, \cdots\cdots$
According to Chapman-Kolmogorov theorem, the two-step transition probability matrix $P^{(2)}$ can be obtained as

$$P^{(2)} = \begin{pmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} & P_{14}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} & P_{24}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} & P_{34}^{(2)} \\ P_{41}^{(2)} & P_{42}^{(2)} & P_{43}^{(2)} & P_{44}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $P_{ij}^{(2)} = P\{X_n = j/X_{n-2} = i\}, \quad i, j = 1, 2, 3, 4, \quad n = 2, 3, \cdots\cdots$

$$\Rightarrow P^{(2)} = \begin{pmatrix} 0.30 & 0.00 & 0.70 & 0.00 \\ 0.00 & 0.51 & 0.00 & 0.49 \\ 0.09 & 0.00 & 0.91 & 0.00 \\ 0 & 0.30 & 1.00 & 0.70 \end{pmatrix}$$

(i) Consider

$$P(X_1 = 2/X_0 = 1) = P_{12}^{(1)} = 1$$

$$P(X_2 = 2/X_1 = 1) = P_{12}^{(1)} = 1$$

$$P(X_2 = 1/X_1 = 2) = P_{21}^{(1)} = 0.3$$

$$P(X_2 = 2/X_0 = 2) = P_{22}^{(2)} = 0.51$$

(ii) Consider $\sum_{k=1}^{4} P_{2k}^{(1)} P_{k2}^{(1)} = P_{21}^{(1)} P_{12}^{(1)} + P_{22}^{(1)} P_{22}^{(1)} + P_{23}^{(1)} P_{32}^{(1)} + P_{24}^{(1)} P_{42}^{(1)}$

From one-step transition probability matrix $P^{(1)}$, we have

$$\sum_{k=1}^{4} P_{2k}^{(1)} P_{k2}^{(1)} = (0.3)(1) + (0)(0) + (0.7)(0.3) + (0)(0) = 0.51 \qquad (1)$$

But from the two-step transition probability matrix $P^{(2)}$, we have

$$P_{22}^{(2)} = 0.51 \qquad (2)$$

From (1) and (2), we have $P_{22}^{(2)} = \sum_{k=1}^{4} P_{2k}^{(1)} P_{k2}^{(1)}$

(iii) Consider $P(X_2 = 2, X_1 = 3/X_0 = 2)$

This implies that the Markov chain was initially in state 2, and then moved to state 3 in one step and from state 3 it moved to state 2 in another one step. Therefore, we have

$$\begin{aligned} P(X_2 = 2, X_1 = 3/X_0 = 2) &= P(X_1 = 3/X_0 = 2)\, P(X_2 = 2/X_1 = 3) \\ &= P_{23}^{(1)}\, P_{32}^{(1)} \\ &= (0.7)(0.3) = 0.21 \end{aligned}$$

(iv) Consider $P(X_2=2, X_1=3, X_0=2)$. Using the formula $P(A \cap B) = P(A/B)\,P(B)$, we have

$$P(X_2=2, X_1=3, X_0=2) = P(X_2=2, X_1=3/X_0=2)\,P(X_0=2)$$

$$= (0.21)\,\pi_2^{(0)}$$

$$= (0.21)\left(\frac{1}{4}\right) = 0.0525$$

(v) Consider $P(X_3=4, X_2=2, X_1=3, X_0=2)$. Using the formula $P(A \cap B) = P(A/B)\,P(B)$, we have

$$P(X_3=4, X_2=2, X_1=3, X_0=2)$$
$$= P(X_3=4/X_2=2, X_1=3, X_0=2)\,P(X_2=2, X_1=3, X_0=2)$$

By Markovian property, we know that

$$P(X_3=4/X_2=2, X_1=3, X_0=2) = P(X_3=4/X_2=2)$$

$$\therefore \quad P(X_3=4, X_2=2, X_1=3, X_0=2) = P(X_3=4/X_2=2)$$
$$P(X_2=2, X_1=3, X_0=2)$$

$$= P_{24}^{(1)}\,(0.0525)$$

$$= (0)\,(0.0525) = 0$$

Problem 6. Let $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ be a Markov chain with three states 1, 2 and 3 with initial probability distribution $\Rightarrow \pi^{(0)} = (0.7,\ 0.2,\ 0.1)$. If the one-step transition probability matrix is given by

$$\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

(i) Find $P(X_2=3)$

(ii) Find $P(X_3=2, X_2=3, X_1=3, X_0=2)$

SOLUTION:

It is given that $\quad \pi^{(0)} = \left(\pi_1^{(0)}, \pi_2^{(0)}, \pi_3^{(0)}\right) = (0.7,\ 0.2,\ 0.1)$

$\Rightarrow \quad P\{X_0=1\} = \pi_1^{(0)} = 0.7,\ P\{X_0=2\} = \pi_2^{(0)} = 0.2,\ P\{X_0=3\} = \pi_3^{(0)} = 0.1$

The one-step transition probability matrix $P^{(1)}$ is given as

$$P^{(1)} = \begin{pmatrix} P_{11}^{(1)} & P_{12}^{(1)} & P_{13}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} & P_{23}^{(1)} \\ P_{31}^{(1)} & P_{32}^{(1)} & P_{33}^{(1)} \end{pmatrix} = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

where $P_{ij}^{(1)} = P\{X_n = j/X_{n-1} = i\}, \quad i,j = 1,2,3, \quad n = 1,2,3,\cdots\cdots$
According to Chapman-Kolmogorov theorem, the two-step transition probability matrix $P^{(2)}$ can be obtained as

$$P^{(2)} = \begin{pmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{pmatrix} = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

where $P_{ij}^{(2)} = P\{X_n = j/X_{n-2} = i\}, \quad i,j = 1,2,3, \quad n = 2,3,\cdots\cdots$

(i) Consider

$$P(X_2 = 3) = \pi_3^{(2)} = \sum_{k=1}^{3} \pi_k^{(0)} P_{k3}^{(2)}$$
$$= \pi_1^{(0)} P_{13}^{(2)} + \pi_2^{(0)} P_{23}^{(2)} + \pi_3^{(0)} P_{33}^{(2)}$$
$$= (0.7)(0.26) + (0.2)(0.34) + (0.1)(0.29) = 0.279$$

(ii) Consider, by applying Markovian property,

$$P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$$
$$= P(X_3 = 2/X_2 = 3, X_1 = 3, X_0 = 2)P(X_2 = 3, X_1 = 3, X_0 = 2)$$
$$= P(X_3 = 2/X_2 = 3)P(X_2 = 3,/X_1 = 3, X_0 = 2)P(X_1 = 3, X_0 = 2)$$
$$= P(X_3 = 2/X_2 = 3)P(X_2 = 3,/X_1 = 3)P(X_1 = 3/X_0 = 2)P(X_0 = 2)$$
$$= P_{32}^{(1)} P_{33}^{(1)} P_{23}^{(1)} \pi_2^{(0)}$$
$$= (0.4)(0.3)(0.2)(0.2) = 0.0048$$

Problem 7. A raining process is regarded as a two state Markov chain, $\{X_n\}$, $n = 1,2,3,\cdots\cdots\cdots$. If it rains, it is considered to be in state 1, and if it does not rain the chain is in state 2. The transition probability matrix of the chain is given as $\begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$ with initial probabilities of states 1 and 2 are given as 0.4 and 0.6 respectively. (i) Find the probability that it will rain for three days from today assuming that it is raining today. (ii) Find the probability that it will rain after three days. (iii) Find the probability that it will not rain after three days.

SOLUTION:

It is given that the Markov chain starts on day one (today). Then the probability distribution that there is rain or no rain today gives the initial probability distribution as

$$\pi^{(0)} = \left(\pi_1^{(0)}, \pi_2^{(0)}\right) = (0.4,\ 0.6) \quad \Rightarrow \quad P\{X_0 = 1\} = \pi_1^{(0)} = 0.4,$$
$$P\{X_0 = 2\} = \pi_2^{(0)} = 0.6$$

where $X_0 = 1$ stands for there is rain today and $X_0 = 2$ for no rain today.
The one-step transition probability matrix $P^{(1)}$ is given as

$$P^{(1)} = \begin{pmatrix} P_{11}^{(1)} & P_{12}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$$

where $P_{ij}^{(1)} = P\{X_n = j/X_{n-1} = i\}, \quad i, j = 1, 2 \quad n = 1, 2, 3, \cdots\cdots$

(i) The probability that it will rain for three days from today assuming that it is raining today can be obtained using the probabilities of transitions given by

P(rain for 3 days from today)

$$= P(\text{rain next to next day, rain next day/rain today})$$
$$P(\text{rain today})$$

$$\begin{aligned} P(X_2=1, X_1=1, X_0=1) &= P(X_2 = 1, X_1 = 1/X_0 = 1)P(X_0 = 1) \\ &= P(X_2 = 1/X_1 = 1)P(X_1 = 1/X_0 = 1)P(X_0 = 1) \\ &= P_{11}^{(1)} P_{11}^{(1)} \pi_1^{(0)} \\ &= (0.6)(0.6)(0.4) = 0.144 \end{aligned}$$

(ii) Since $X_0 = 1$, $X_1 = 1$, $X_2 = 1$, $X_3 = 1$ represent respectively rain on first day, second day, third day and fourth day, the probability that there will be rain after three days (that is fourth day) can be obtained as

$$\begin{aligned} P(X_3 = 1) = \pi_1^{(3)} &= \sum_{k=1}^{2} \pi_k^{(0)} P_{k3}^{(3)} \\ &= \pi_1^{(0)} P_{11}^{(3)} + \pi_2^{(0)} P_{21}^{(3)} \end{aligned}$$

Consider

$$P^{(2)} = \begin{pmatrix} P_{11}^{(2)} & P_{12}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{pmatrix}$$

$$P^{(3)} = \begin{pmatrix} P_{11}^{(3)} & P_{12}^{(3)} \\ P_{21}^{(3)} & P_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{pmatrix} \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{pmatrix}$$

$$\therefore \quad P(X_3 = 1) = (0.4)(0.376) + (0.6)(0.312) = 0.3376$$

(iii) Since $X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 2$ represent respectively no rain on first day, second day, third day and fourth day, the probability that there will be no rain after three days (that is fourth day) can be obtained as

$$P(X_3 = 2) = \pi_2^{(3)} = \sum_{k=1}^{2} \pi_k^{(0)} P_{k2}^{(3)}$$

$$= \pi_1^{(0)} P_{12}^{(3)} + \pi_2^{(0)} P_{22}^{(3)}$$

$$\therefore \quad P(X_3 = 2) = (0.4)(0.624) + (0.6)(0.688) = 0.6624$$

Note:
Alternatively, we can obtain $P(X_3 = 1) = \pi_1^{(3)}$ and $P(X_3 = 2) = \pi_2^{(3)}$ using the relationship,

$$\pi^{(3)} = \left(\pi_1^{(3)}, \ \pi_2^{(3)} \right) = \pi_1^{(0)} P^{(3)}$$

$$= \left(0.4 \ \ 0.6 \right) \begin{pmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{pmatrix}$$

$$\Rightarrow P(X_3 = 1) = \pi_1^{(3)} = (0.4)(0.376) + (0.6)(0.312) = 0.3376$$

$$P(X_3 = 2) = \pi_2^{(3)} = (0.4)(0.624) + (0.6)(0.688) = 0.6624$$

Problem 8. If the one-step transition probability matrix is given by $P^{(1)} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$ then obtain (i) the n-step TPM $P^{(n)}$, (ii) obtain $\lim_{n \to \infty} P^{(n)}$ and (iii) steady-state probability distribution if the initial state probability distribution is given as $\pi^{(0)} = \left[\, 0.5 \ \ 0.5 \, \right]$.

SOLUTION:

(i) We know that if the one-step transition probability matrix is of the form $P^{(1)} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$, $\quad 0 \le p, q \le 1$ then the n-step TPM can be given as (Refer to (9.9))

$$P^{(n)} = \frac{1}{p+q}\left\{\begin{bmatrix} q & p \\ q & p \end{bmatrix} + (1-p-q)^n \begin{bmatrix} p & -p \\ -q & q \end{bmatrix}\right\}$$

We have $P^{(1)} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \Rightarrow \quad p = 0.1, \quad q = 0.2$

$$\therefore \quad P^{(n)} = \frac{1}{0.1+0.2}\left\{\begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix} + (1-0.1-0.2)^n \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}\right\}$$

$$= \frac{1}{0.3}\begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix} + \frac{(0.7)^n}{0.3}\begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2+(0.7)^n}{3} & \dfrac{1-(0.7)^n}{3} \\ \dfrac{2-2(0.7)^n}{3} & \dfrac{1+2(0.7)^n}{3} \end{bmatrix}$$

(ii) $\lim\limits_{n\to\infty} P^{(n)} = \lim\limits_{n\to\infty} \begin{bmatrix} \dfrac{2+(0.7)^n}{3} & \dfrac{1-(0.7)^n}{3} \\ \dfrac{2-2(0.7)^n}{3} & \dfrac{1+2(0.7)^n}{3} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{3} & \dfrac{1}{3} \\ \dfrac{2}{3} & \dfrac{1}{3} \end{bmatrix}, \quad \because \lim\limits_{n\to\infty} (0.7)^n = 0$

(iii) We know that steady-state probability distribution $\pi^{(n)}$ after n steps, if the initial state probability distribution is $\pi^{(0)}$, can be given as (refer (9.6))

$$\pi^{(n)} = \pi^{(0)} P^{(n)}$$

$$= \begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix} \begin{bmatrix} \dfrac{2+(0.7)^n}{3} & \dfrac{1-(0.7)^n}{3} \\ \dfrac{2-2(0.7)^n}{3} & \dfrac{1+2(0.7)^n}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2+(0.7)^n}{6} + \dfrac{2-2(0.7)^n}{6} & \dfrac{1-(0.7)^n}{6} + \dfrac{1+2(0.7)^n}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2}{3} & \dfrac{1}{3} \end{bmatrix}$$

$$\Rightarrow \quad P(X_n = 1) = \frac{2}{3}, \quad P(X_n = 2) = \frac{1}{3}$$

Therefore, the steady-state probability distribution is given as

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$
$$\Rightarrow \quad P(X_\infty = 1) = \frac{2}{3}, \quad P(X_\infty = 2) = \frac{1}{3}$$

Problem 9. A man either drives a car or goes by train to office each day. He never goes two days in a row by train. If he drives one day then he is just as likely to drive again the next day as he is to travel by train. Now suppose that on the first day of the week, the man tosses a fair die and then drives to work if and only if a 6 appears. Under this circumstance, (i) verify whether the process of going to office is Markovian, (ii) obtain initial probability distribution and the one-step transition probability matrix, (iii) what is the probability that he will go by train on the third day and (iv) obtain the probability that he will drive to work in the long run.

SOLUTION:

(i) Since the decision process of either going by car or train on a particular day depends on the mode of transport used just on the previous day, we conclude that the process is Markovian.

(ii) Since the state space is discrete, the man's travel pattern is clearly a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ with two states 1 and 2, where 1 stands for travel by train and 2 stands for drive by car. Since the first day of the week is the starting point the initial state distribution (the probability of going by train or car on first day) can be given as

$$\begin{aligned} P\{X_0 = 1\} &= \pi_1^{(0)} = P\{\text{Going by train}\} \\ &= P\{\text{Getting no six in the toss of a die}\} = \frac{5}{6} \\ P\{X_0 = 2\} &= \pi_2^{(0)} = P\{\text{Going by car}\} \\ &= P\{\text{Getting a six in the toss of a die}\} = \frac{1}{6} \end{aligned}$$
$$\Rightarrow \quad \pi^{(0)} = \left(\pi_1^{(0)}, \pi_2^{(0)}\right) = \left(\frac{5}{6}, \frac{1}{6}\right)$$

The one-step transition probability matrix becomes

$$P^{(1)} = (n-1)^{th}\text{ day}\left|\begin{array}{c} \\ \text{train(1)} \\ \text{car(2)} \end{array}\right. \begin{array}{c} n^{th}\text{day} \\ \begin{array}{cc} \text{train(1)} & \text{car(2)} \end{array} \\ \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} \end{array} = \begin{pmatrix} P_{11}^{(1)} & P_{12}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} \end{pmatrix}$$

$$\Rightarrow \quad P^{(1)} = \begin{pmatrix} P_{11}^{(1)} & P_{12}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix}$$

(iii) Here, if the starting day is the first day of the week, then $n = 1$ means the second day (after one step), $n = 2$ means the third day (after two steps), and so on. That is, the second day is reached after one step, third day is reached in two steps, and so on. Therefore, on n^{th} day (that is after $n-1$ steps), we have

$$X_{n-1} = \begin{cases} 1 & \text{If the man goes by train} \\ 2 & \text{If the man goes by car} \end{cases}$$

Hence, the probability that the man will go by train or car on the third day can be computed by finding the state probabilities $P\{X_2 = 1\} = \pi_1^{(2)}$ and $P\{X_2 = 2\} = \pi_2^{(2)}$ respectively as follows:

$$\pi^{(2)} = \left(\pi_1^{(2)},\ \pi_2^{(2)}\right) = \pi^{(0)}P^{(2)}$$

Consider

$$P^{(2)} = \begin{pmatrix} P_{11}^{(2)} & P_{12}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix}$$

$$\therefore \quad \pi^{(2)} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix} = \begin{pmatrix} \frac{11}{24} & \frac{13}{24} \end{pmatrix}$$

Therefore, the probability that the person will travel by train on third day is

$$P\{X_2 = 1\} = \pi_1^{(2)} = \frac{11}{24}$$

(iv) The probability that the man will drive car in the long run can be obtained by finding the steady-state probability distribution called the limiting distribution $\pi = (\pi_1,\ \pi_2)$, where π_1 is the probability that the man will travel by train in the long run and π_2 is the probability that the man will travel by car in the long run. We know that if the initial state probability distribution $\pi^{(0)}$ and one-step transition probability matrix $P^{(1)}$ are known, then the steady-state probability distribution, can be obtained by using the equations

$$\pi P^{(1)} = \pi \quad \text{and} \quad \sum_{i=1}^{k} \pi_i = 1$$

$$\Rightarrow \quad (\pi_1 \quad \pi_2)\begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix} = (\pi_1 \quad \pi_2) \quad \text{and} \quad \pi_1 + \pi_2 = 1$$

$$\Rightarrow \quad (0)\pi_1 + (0.5)\pi_2 = \pi_1, (1)\pi_1 + (0.5)\pi_2 = \pi_2, \pi_1 + \pi_2 = 1$$

$$\Rightarrow \quad \pi_1 = \frac{1}{3} \quad \text{and} \quad \pi_2 = \frac{2}{3}$$

Therefore, the probability that the man will travel by car in the long run is $\frac{2}{3}$.

Problem 10. Three persons A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states.

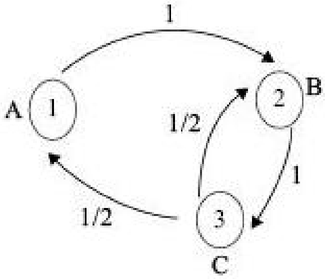
SOLUTION:

Since the state space is discrete, and the state of ball being with A or B or C depends on who was having the ball in the immediate past, the pattern of receiving the ball is clearly a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ with three states 1, 2 and 3 where 1 stands for the ball being with A, 2 stands for the ball being with B and 3 stands for the ball being with C. Therefore, the process is Markovian.

The one-step transition probability matrix becomes

$$\begin{array}{cc} & n^{th}\, step \\ & \begin{array}{ccc} A(1) & B(2) & C(3) \end{array} \\ (n-1)^{th}\, step \quad \begin{array}{c} A(1) \\ B(2) \\ C(3) \end{array} & \begin{pmatrix} P_{11}^{(1)} & P_{12}^{(1)} & P_{13}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} & P_{23}^{(1)} \\ P_{31}^{(1)} & P_{32}^{(1)} & P_{33}^{(1)} \end{pmatrix} \end{array} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

The state transition diagram becomes



Classification of states

(i) Irreducibility

From the state transition diagram, we get

$$P_{21}^{(2)} > 0, \quad P_{31}^{(1)} > 0, \quad P_{12}^{(1)} > 0, \quad P_{32}^{(1)} > 0, \quad P_{13}^{(2)} > 0, \quad P_{23}^{(1)} > 0$$

This implies state 1 is accessible from state 2 (in two steps) and from state 3 (in one step). The state 2 is accessible from both the states 1 and 3 (in one step). The state 3 is accessible from state 1 (in two steps) and from state 2 (in one step). Since every state is accessible from every other state in some step, the chain is *irreducible*.

(ii) Periodicity

From the state transition diagram, we get

$$\begin{aligned}
& P_{11}^{(n)} > 0 \quad \text{for} \quad n = 3, 5, 6 \quad etc.. \\
\Rightarrow \quad & d(1) = GCD\{3, 5, 6, \cdots\} = 1 \\
& P_{22}^{(n)} > 0 \quad \text{for} \quad n = 2, 3, 4, 5, 6 \quad etc.. \\
\Rightarrow \quad & d(2) = GCD\{2, 3, 4, 5, 6, \cdots\} = 1 \\
& P_{33}^{(n)} > 0 \quad \text{for} \quad n = 2, 3, 4, 5, 6 \quad etc.. \\
\Rightarrow \quad & d(3) = GCD\{2, 3, 4, 5, 6, \cdots\} = 1
\end{aligned}$$

This implies that all the three states are *aperiodic*.

(iii) Null-persistent or non-null persistent states

Since the Markov chain is irreducible and aperiodic, the steady-state distribution of the states can be obtained to determine whether the chain is null persistent or non-null persistent. We know that the steady-state probabilities can be obtained using the equations

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \quad \text{and} \quad \pi_1+\pi_2+\pi_3=1$$

$$\Rightarrow \quad 0.5\pi_3 = \pi_1, \quad \pi_1+0.5\pi_3 = \pi_2, \quad \pi_2 = \pi_3$$

$$\Rightarrow \quad 0.5\pi_3+\pi_3+\pi_3 = 1 \quad \Rightarrow 2.5\pi_3 = 1 \quad \Rightarrow \pi_3 = 0.4$$

$$\therefore \quad \pi_1 = 0.2, \quad \pi_2 = 0.4, \quad \pi_3 = 0.4$$

Now using the result given in (9.16), the mean recurrent times for the three states can be obtained as

$$\pi_1 = \frac{1}{\mu_1} \quad \Rightarrow 0.2 = \frac{1}{\mu_1} \quad \Rightarrow \mu_1 = 5 < \infty$$

$$\pi_2 = \frac{1}{\mu_2} \quad \Rightarrow 0.4 = \frac{1}{\mu_2} \quad \Rightarrow \mu_2 = 2.5 < \infty$$

$$\pi_3 = \frac{1}{\mu_3} \quad \Rightarrow 0.4 = \frac{1}{\mu_3} \quad \Rightarrow \mu_3 = 2.5 < \infty$$

This shows that all the three states are non-null persistent.

(iv) Ergodicity

Since all the three states are aperiodic and non-null persistent, the given Markov chain is ergodic.

Problem 11. A fair dice is tossed repeatedly. If X_n denotes the maximum of the numbers occurring in the first n tosses, then find the transition probability matrix of the Markov chain. Also find the two-step transition probability matrix and hence find the probability that the maximum of the numbers occurring in the first two tosses, that is $P\{X_2 = 6\}$.

SOLUTION:

We know that in case of the process of tossing a fair dice, the state space (possible outcomes) is given as $\{1, 2, 3, 4, 5, 6\}$. It may be noted that in the initial toss one of these six numbers can happen and it will be the maximum number. Therefore, we have the initial state probability distribution as

$$\pi^{(0)} = \left(\pi_1^{(0)}, \pi_2^{(0)}, \pi_3^{(0)}, \pi_4^{(0)}, \pi_5^{(0)}, \pi_6^{(0)}\right) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

That is, $\pi_i^{(0)} = P\{X_0 = i\} = \frac{1}{6}, \quad i = 1, 2, 3, 4, 5, 6$

Let $X_n = i$ denote the maximum of the numbers occurring in the first n tosses and let $X_{n+1} = j$ be the maximum of the numbers occurring after $(n+1)^{th}$ toss. Then different possibilities are obtained as

If $i = 1$, then $j = 1, 2, 3, 4, 5$ and 6

$$\Rightarrow P\{X_{n+1} = j/X_n = 1\} = \frac{1}{6}, \quad j = 1,2,3,4,5,6$$

If $i = 2$, then $j = 2, 3, 4, 5$ and 6

$$\Rightarrow P\{X_{n+1} = 2/X_n = 2\} = \frac{2}{6} \quad \text{and} \quad P\{X_{n+1} = j/X_n = 2\} = \frac{1}{6}, \quad j = 3,4,5,6$$

If $i = 3$, then $j = 3, 4, 5$ and 6

$$\Rightarrow P\{X_{n+1} = 3/X_n = 3\} = \frac{3}{6} \quad \text{and} \quad P\{X_{n+1} = j/X_n = 3\} = \frac{1}{6}, \quad j = 4,5,6$$

If $i = 4$, then $j = 4, 5$ and 6

$$\Rightarrow P\{X_{n+1} = 4/X_n = 4\} = \frac{4}{6} \quad \text{and} \quad P\{X_{n+1} = j/X_n = 4\} = \frac{1}{6}, \quad j = 5,6$$

If $i = 5$, then $j = 5$ and 6

$$\Rightarrow P\{X_{n+1} = 5/X_n = 5\} = \frac{5}{6}, \quad P\{X_{n+1} = 6/X_n = 5\} = \frac{1}{6}$$

If $i = 6$, then $j = 6$

$$\Rightarrow P\{X_{n+1} = 6/X_n = 6\} = 1 = \frac{6}{6}$$

Therefore, the one-step transition probability matrix becomes

$$P^{(1)} = \text{max after 'n' tosses} \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \text{max after 'n+1' tosses} \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 6/6 \end{pmatrix} \end{array}$$

The two-step transition probability matrix can be obtained using Chapman-Kolmogorov theorem as

$$P^{(2)} = \left\{P^{(1)}\right\}^2 = \begin{pmatrix} 1/36 & 3/36 & 5/36 & 7/36 & 9/36 & 11/36 \\ 0 & 4/36 & 5/36 & 7/36 & 9/36 & 11/36 \\ 0 & 0 & 9/36 & 7/36 & 9/36 & 11/36 \\ 0 & 0 & 0 & 16/36 & 9/36 & 11/36 \\ 0 & 0 & 0 & 0 & 25/36 & 11/36 \\ 0 & 0 & 0 & 0 & 0 & 36/36 \end{pmatrix}$$

Therefore, the probability that the maximum of the numbers occurring in the first two tosses can be obtained as

$$P\{X_2 = 6\} = \sum_{i=1}^{6} \pi_i^{(0)} P_{i6}^{(2)}.$$

$$P\{X_2 = 6\} = \pi_1^{(0)} P_{16}^{(2)} + \pi_2^{(0)} P_{26}^{(2)} + \pi_3^{(0)} P_{36}^{(2)} + \pi_4^{(0)} P_{46}^{(2)} + \pi_5^{(0)} P_{56}^{(2)} + \pi_6^{(0)} P_{66}^{(2)}$$

$$P\{X_2 = 6\} = \left(\frac{1}{6}\right)\left(\frac{11}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{11}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{11}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{11}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{11}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{36}{36}\right)$$

$$P\{X_2 = 6\} = \left(\frac{1}{6}\right)\left(\frac{1}{36}\right)(11 + 11 + 11 + 11 + 11 + 36) = \frac{91}{216} = 0.4123$$

EXERCISE PROBLEMS

1. Let $\{X_n\}$ be a Markov chain with state space $\{1, 2, 3\}$ with initial probability distribution $\pi^{(0)} = (1/4, 1/2, 1/4)$. If the one-step transition probability matrix is given by

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

   Then compute the probabilities
   (i) $P(X_2 = 1)$
   (ii) $P(X_0 = 1, X_1 = 2, X_2 = 2)$
   (iii) $P(X_2 = 2, X_1 = 2/X_0 = 1)$ and
   (iv) $P_{12}^{(2)}$

2. Let $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ be a Markov chain with two state space $\{1, 2\}$ with the following transition probability matrix

$$\begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

Show that (i) state 1 is recurrent and (ii) state 2 is transient.

3. From the following the transition probability matrix of a Markov chain $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$ with state space $\{1, 2, 3,\}$, (i) obtain the state transition diagram (ii) obtain the two-step transition matrix.

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 0 & 3/4 \\ 1/3 & 0 & 2/3 \end{pmatrix}$$

4. A raining process is considered a two state Markov chain, $\{X_n\}$, $n = 1, 2, 3, \cdots\cdots\cdots$. If it rains, it is considered to be in state 1, and if it does not rain the chain is in state 2. The transition probability matrix of the chain is given as $\begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$ with initial probabilities of states 1 and 2 are given as 0.4 and 0.6 respectively. Find (i) the probability that it will rain after third day and (ii) find the probability that it will rain in the long run.

5. Let $\{X_n\}$ be a Markov chain with state space $\{1, 2, 3\}$ with one-step transition probability matrix is given by

$$P^{(1)} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(i) Draw the transition diagram (ii) Show that state 1 is periodic with period 2.

6. An air-conditioner is in one of the three states: off (state 1), low (state 2) or high (state 3). If it is in off position, the probability that it will be turned to low is 1/3. If it is in low position, then it will be turned either to off or high with equal probabilities 1/4. If it is in high position, then the probability that it will be turned to low is 1/3 or to off is 1/6. (i) Draw the transition diagram, (ii) obtain the transition probability matrix and (iii) obtain the steady-state probabilities.

7. Obtain the transition diagram and classify the states of the Markov chain with the transition probability matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$

8. The transition probability matrix of a homogeneous Markov chain with states 1, 2 and 3 is given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Find (a) the transition diagram, (b) the steady-state probability distribution and (c) classify the states.

9. A professor has three pet questions, one of which occurs in every test he gives. The students know his habit well. He never gives a question twice in a row. If he had given question one last time, he tosses a coin and gives question two if head comes up. If he had given question two last time, he tosses two coins and switches to question three, if both heads come up. If he had given question three last time, he tosses three coins and switches to question one if all three heads come up. (i) Show that the process is Markovian. (ii) Write the transition probability matrix of the corresponding Markov chain. (iii) Obtain the probabilities of the pet questions being given in the long run.

10. A man is at an integral part of x-axis between the origin and the point 3. He takes unit steps to the right with probability 1/3 or to the left with probability 2/3 unless he is at the origin. If he is at the origin he takes a step to the right to reach the point 1 or if he is at the point 3 he takes a step to the left to reach the point 2. (i) Obtain the transition probability matrix, (ii) what is the probability that he is at the point 1 after 2 walk if the initial probability vector is $(1/4,\ 1/4,\ 1/4,\ 1/4)$ and (iii) what is the probability that he is at position 1 after a long run.

Therefore, $\{X(t)\}$ is continuous in mean square sense.

Now, let us consider

$$\begin{aligned} R_{xx}(t+\tau_1, t+\tau_2) - R_{xx}(t,t) &= E\{[X(t+\tau_1)-X(t)][X(t+\tau_2)-X(t)]\} \\ &\quad + E\{[X(t+\tau_1)-X(t)]X(t)\} \\ &\quad + E\{[X(t+\tau_2)-X(t)]X(t)\} \end{aligned}$$

Using Cauchy-Schwarz inequality that $[E(XY)]^2 \leq E(X^2)E(Y^2)$, we have

$$\begin{aligned} R_{xx}(t+\tau_1, t+\tau_2) \ - \ R_{xx}(t,t) &\leq \left\{E[X(t+\tau_1)-X(t)]^2 E[X(t+\tau_2)-X(t)]^2\right\}^{1/2} \\ &+ \ \left\{E[X(t+\tau_1)-X(t)]^2 E[X(t)]^2\right\}^{1/2} \\ &+ \ \left\{E[X(t+\tau_2)-X(t)]^2 E[X(t)]^2\right\}^{1/2} \end{aligned}$$

Therefore, if $\{X(t)\}$ is continuous in mean square sense, we have

$$\begin{aligned} \lim_{\tau_1,\tau_2\to 0}\{R_{xx}(t+\tau_1, t+\tau_2) - R_{xx}(t,t)\} &= 0 \\ \Rightarrow \quad \lim_{\tau_1,\tau_2\to 0} R_{xx}(t+\tau_1, t+\tau_2) &= R_{xx}(t,t) \end{aligned}$$

Which implies $R_{xx}(t_1, t_2)$ is continuous. Hence the proof.

Result A.1.3:

If $\{X(t)\}$ is a stationary process, then it is continuous in mean square sense if and only if its autocorrelation function $R_{xx}(\tau)$ is continuous at $\tau = 0$.

Proof. If $\{X(t)\}$ is a stationary process, we have

$$\begin{aligned} E\left\{[X(t+\tau)-X(t)]^2\right\} &= E\left\{X^2(t+\tau)\right\} + E\left\{X^2(t)\right\} - 2E\{X(t+\tau)X(t)\} \\ &= R_{xx}(t+\tau, t+\tau) + R_{xx}(t,t) - 2R_{xx}(t+\tau, t) \\ &= 2R_{xx}(0) - 2R_{xx}(\tau) \end{aligned}$$

Therefore, if $R_{xx}(\tau)$ is continuous at $\tau = 0$, then

$$\lim_{\tau\to 0}\{R_{xx}(\tau) - R_{xx}(0)\} = 0$$

$$\therefore \quad \lim_{\tau\to 0} E\left\{[X(t+\tau)-X(t)]^2\right\} = 0$$

Which implies that $\{X(t)\}$ is continuous in mean square sense.

Similarly, it can be shown that if $\{X(t)\}$ is continuous in mean square sense, then $R_{xx}(\tau)$ is continuous at $\tau = 0$. Hence the proof.

Result A.1.4:
If $\{X(t)\}$ is continuous in mean square sense, then its mean is continuous. That is,

$$\lim_{\tau\to 0} \mu_x(t+\tau) = \mu_x(t)$$

Proof. We know that

$$\begin{aligned} V\{X(t+\tau)-X(t)\} &= E\left\{[X(t+\tau)-X(t)]^2\right\} - \{E[X(t+\tau)-X(t)]\}^2 \geq 0 \\ &\Rightarrow \quad E\left\{[X(t+\tau)-X(t)]^2\right\} \geq \{E[X(t+\tau)-X(t)]\}^2 \\ &= \{[\mu_x(t+\tau)-\mu_x(t)]\}^2 \end{aligned}$$

If $\{X(t)\}$ is continuous in mean square sense then

$$\begin{aligned} \lim_{\tau\to 0} E\left\{[X(t+\tau)-X(t)]^2\right\} &= 0 \\ \Rightarrow \quad \lim_{\tau\to 0} \{[\mu_x(t+\tau)-\mu_x(t)]\}^2 &= 0 \\ \therefore \quad \lim_{\varepsilon\to 0} \mu_x(t+\tau) &= \mu_x(t) \end{aligned}$$

A.2 DERIVATIVES RELATED TO RANDOM PROCESSES

Result A.2.1:
A random process $\{X(t)\}$ is said to have a derivative denoted as $X'(t)$ if

$$\lim_{\tau\to 0} \frac{X(t+\tau)-X(t)}{\tau} = X'(t)$$

Result A.2.2:
A random process $\{X(t)\}$ is said to have a derivative denoted as $X'(t)$ in mean square sense if

$$\lim_{\tau\to 0} E\left\{\left[\frac{X(t+\tau)-X(t)}{\tau} - X'(t)\right]\right\}^2 = 0$$

Result A.2.3:
A random process $\{X(t)\}$ with autocorrelation function $R_{xx}(t_1, t_2)$ has a derivative in mean square sense if $\dfrac{\partial^2 R_{xx}(t_1, t_2)}{\partial t_1 \partial t_2}$ exists at $t = t_1 = t_2$.

Proof. Let $Y(t,\tau) = \dfrac{X(t+\tau)-X(t)}{\tau}$ and with an assumption that $t = t_1 = t_2$, let $Y(t_1,\tau_1) = \dfrac{X(t_1+\tau_1)-X(t_1)}{\tau_1}$ and $Y(t_2,\tau_2) = \dfrac{X(t_2+\tau_2)-X(t_2)}{\tau_2}$

By Cauchy criterion, the mean square derivative of $\{X(t)\}$ exists, if

$$\lim_{\tau_1,\tau_2\to 0} E\left\{[Y(t_2,\tau_2)-Y(t_1,\tau_1)]^2\right\} = 0$$

Consider

$$\begin{aligned} E\left\{[Y(t_2,\tau_2)-Y(t_1,\tau_1)]^2\right\} &= E\left\{Y^2(t_2,\tau_2)\right\} + E\left\{Y^2(t_1,\tau_1)\right\} \\ &\quad - 2E\left\{Y(t_2,\tau_2)Y(t_1,\tau_1)\right\} \end{aligned} \tag{A1}$$

Now, consider

$$\begin{aligned} E\{Y(t_2,\tau_2)Y(t_1,\tau_1)\} &= E\left\{\left(\frac{X(t_2+\tau_2)-X(t_2)}{\tau_2}\right)\left(\frac{X(t_1+\tau_1)-X(t_1)}{\tau_1}\right)\right\} \\ &= \frac{1}{\tau_1\tau_2} E\left\{[X(t_2+\tau_2)-X(t_2)]\,[X(t_1+\tau_1)-X(t_1)]\right\} \end{aligned}$$

Expanding we get

$$\begin{aligned} E\{Y(t_2,\tau_2)Y(t_1,\tau_1)\} &= \frac{1}{\tau_1\tau_2}\{R_{xx}(t_2+\tau_2,t_1+\tau_1) - R_{xx}(t_2+\tau_2,t_1) \\ &\quad - R_{xx}(t_2,t_1+\tau_1) + R_{xx}(t_2,t_1)\} \\ \lim_{\tau_1\to 0} E\{Y(t_2,\tau_2)Y(t_1,\tau_1)\} &= \frac{1}{\tau_2}\lim_{\tau_1\to 0}\left\{\frac{R_{xx}(t_2+\tau_2,t_1+\tau_1) - R_{xx}(t_2+\tau_2,t_1)}{\tau_1}\right. \\ &\quad \left. - \frac{R_{xx}(t_2,t_1+\tau_1) - R_{xx}(t_2,t_1)}{\tau_1}\right\} \\ \lim_{\tau_2\to 0}\left\{\lim_{\tau_1\to 0} E\{Y(t_2,\tau_2)Y(t_1,\tau_1)\}\right\} &= \lim_{\tau_2\to 0}\frac{1}{\tau_2}\left\{\frac{\partial R_{xx}(t_2+\tau_2,t_1)}{\partial t_1} - \frac{\partial R_{xx}(t_2,t_1)}{\partial t_1}\right\} \\ &= \frac{\partial^2 R(t_2,t_1)}{\partial t_1 \partial t_2} \end{aligned}$$

But $\dfrac{\partial^2 R(t_2,t_1)}{\partial t_1\partial t_2} = \dfrac{\partial^2 R(t_1,t_2)}{\partial t_1\partial t_2}$ since $R_{xx}(t_1,t_2)$ is an even function.

$$\therefore \quad \lim_{\tau_1\to 0,\tau_2\to 0} E\{Y(t_2,\tau_2)Y(t_1,\tau_1\} = \frac{\partial^2 R(t_2,t_1)}{\partial t_1\partial t_2} \tag{A2}$$

This is true provided $\dfrac{\partial^2 R(t_1,t_2)}{\partial t_1 \partial t_2}$ exists.

Setting $\tau_1 = \tau_2$ with $t = t_1 = t_2$, in (A2), we have

$$\lim_{\tau_1 \to 0} E\left\{Y^2(t_1,\tau_1)\right\} = \lim_{\tau_2 \to 0} E\left\{Y^2(t_2,\tau_2)\right\} = \frac{\partial^2 R(t_1,t_2)}{\partial t_1 \partial t_2} \tag{A3}$$

Substituting (A2) and A(3) in (A1), we have

$$\lim_{\tau_1,\tau_2 \to 0} E\left\{[Y(t_2,\tau_2) - Y(t_1,\tau_1)]^2\right\} = \frac{\partial^2 R(t_1,t_2)}{\partial t_1 \partial t_2} + \frac{\partial^2 R(t_1,t_2)}{\partial t_1 \partial t_2} - 2\frac{\partial^2 R(t_1,t_2)}{\partial t_1 \partial t_2} = 0$$

Therefore, we conclude that a random process $\{X(t)\}$ with autocorrelation function $R_{xx}(t_1,t_2)$ has a derivative in mean square sense if $\dfrac{\partial^2 R_{xx}(t_1,t_2)}{\partial t_1 \partial t_2}$ exists at $t = t_1 = t_2$. Further if $\{X(t)\}$ is a stationary random process, then this is equivalent to the existence of $\dfrac{\partial^2 R_{xx}(\tau)}{\partial t_1 \partial t_2}$ at $\tau = 0$. It may be noted that $\tau = |t_1 - t_2|$.

A.3 INTEGRALS RELATED TO RANDOM PROCESSES

Result A.3.1:

A random process $\{X(t)\}$ is said to be integrable if

$$S = \int_a^b X(t)\, dt = \lim_{\Delta t_i \to 0} \sum_i X(t_i)\Delta t_i$$

where $a < t_0 < t_1 < t_2 < \cdots < t_i < \cdots t_n < \cdots < b$ and $\Delta t_i = t_{i+1} - t_i$.

Result A.3.2:

The random process $\{X(t)\}$ is integrable in the mean square sense if the following integral

$$\int_a^b \int_a^b R_{xx}(t_i,t_j)\, dt_i dt_j$$

exists for any t_i and t_j.

Proof. We know that the mean square integral of the random process $\{X(t)\}$ is given by

$$S = \int_a^b X(t)\, dt = \lim_{\Delta t_i \to 0} \sum_i X(t_i)\Delta t_i$$

Let

$$S = \int_a^b X(t)\, dt = \lim_{\Delta t_i \to 0} \sum_i X(t_i)\Delta t_i = \lim_{\Delta t_j \to 0} \sum_j X(t_j)\Delta t_j \tag{A4}$$

where $a < t_0 < t_1 < t_2 < \cdots < t_i < \cdots t_n < \cdots < b$, $\Delta t_i = t_{i+1} - t_i$ and $\Delta t_j = t_{j+1} - t_j$.

According to Cauchy criterion, the mean square integral of $\{X(t)\}$ exists if

$$E\left\{\left|\sum_i X(t_i)\Delta t_i - \sum_j X(t_j)\Delta t_j\right|^2\right\} = 0 \quad \text{as} \quad \Delta t_i, \Delta t_j \to 0$$

Now, consider $E\left\{\left|\sum_i X(t_i)\Delta t_i - \sum_j X(t_j)\Delta t_j\right|^2\right\}$

$$= E\left(\sum_i X(t_i)\Delta t_i\right)^2 + E\left(\sum_j X(t_j)\Delta t_j\right)^2 - 2E\left\{\sum_i X(t_i)\Delta t_i \sum_j X(t_j)\Delta t_j\right\} \tag{A5}$$

$$= E\left(\sum_i X(t_i)\Delta t_i \sum_i X(t_i)\Delta t_i\right) + E\left(\sum_j X(t_j)\Delta t_j \sum_j X(t_j)\Delta t_j\right)$$

$$-2E\left\{\sum_i \sum_j X(t_i)X(t_j)\Delta t_i \Delta t_j\right\}$$

$$\therefore \quad \lim_{\Delta t_i, \Delta t_j \to 0} E\left\{\left|\sum_i X(t_i)\Delta t_i - \sum_j X(t_j)\Delta t_j\right|^2\right\}$$

$$= E\left(\lim_{\Delta t_i \to 0} \sum_i X(t_i)\Delta t_i \lim_{\Delta t_i \to 0} \sum_i X(t_i)\Delta t_i\right) + E\left(\lim_{\Delta \tau_j \to 0} \sum_j X(t_j)\Delta t_j \lim_{\Delta t_j \to 0} \sum_j X(t_j)\Delta t_j\right)$$

$$-2E\left\{\lim_{\Delta t_i \to 0} \lim_{\Delta \tau_j \to 0} \sum_i \sum_j X(t_i)X(t_j)\Delta t_i \Delta t_j\right\}$$

By (A4), we have

$$\therefore \quad \lim_{\Delta t_i, \Delta t_j \to 0} E\left\{\left|\sum_i X(t_i)\Delta t_i - \sum_j X(t_j)\Delta t_j\right|^2\right\}$$

$$= E\left(\lim_{\Delta t_i \to 0} \lim_{\Delta t_j \to 0} \sum_i \sum_j X(t_i)\Delta t_i X(t_j)\Delta t_j\right) + E\left(\lim_{\Delta t_i \to 0} \lim_{\Delta t_j \to 0} \sum_i \sum_j X(t_i)\Delta t_i X(t_j)\Delta t_j\right)$$

$$- 2E\left\{\lim_{\Delta t_i \to 0} \lim_{\Delta \tau_j \to 0} \sum_i \sum_j X(t_i)X(t_j)\Delta t_i \Delta t_j\right\}$$

$$= \lim_{\Delta t_i \to 0} \lim_{\Delta t_j \to 0} \sum_i \sum_j E\left\{X(t_i)\Delta t_i X(t_j)\Delta t_j\right\} + \lim_{\Delta t_i \to 0} \lim_{\Delta t_j \to 0} \sum_i \sum_j E\left\{X(t_i)\Delta t_i X(t_j)\Delta t_j\right\}$$

$$- 2 \lim_{\Delta t_i \to 0} \lim_{\Delta t_j \to 0} \sum_i \sum_j E\left\{X(t_i)\Delta t_i X(t_j)\Delta t_j\right\}$$

$$= \lim_{\Delta t_i \to 0, \Delta t_j \to 0} \sum_i \sum_j R_{xx}(t_i, t_j)\Delta t_i \Delta t_j + \lim_{\Delta t_i \to 0, \Delta t_j \to 0} \sum_i \sum_j R_{xx}(t_i, t_j)\Delta t_i \Delta t_j$$

$$- 2 \lim_{\Delta t_i \to 0, \Delta t_j \to 0} \sum_i \sum_j R_{xx}(t_i, t_j)\Delta t_i \Delta t_j$$

$$= \int_a^b \int_a^b R_{xx}(t_i, t_j)\, dt_i dt_j + \int_a^b \int_a^b R_{xx}(t_i, t_j)\, dt_i dt_j \tag{A6}$$

$$- 2 \int_a^b \int_a^b R_{xx}(t_i, t_j)\, dt_i dt_j = 0$$

$$\therefore \quad E\left\{\left|\sum_i X(t_i)\Delta t_i - \sum_j X(t_j)\Delta t_j\right|^2\right\} = 0 \quad \text{as} \quad \Delta t_i, \Delta t_j \to 0$$

This is true if $\int_a^b \int_a^b R_{xx}(t_i, t_j)\, dt_i dt_j < \infty$
Therefore, the random process $\{X(t)\}$ is integrable in the mean square sense if the integral $\int_a^b \int_a^b R_{xx}(t_i, t_j)\, dt_i dt_j$ exists. Letting $t_i = t_1, t_j = t_2$, the integral $\int_a^b \int_a^b R_{xx}(t_i, t_j)$ $dt_i dt_j$ becomes $\int_a^b \int_a^b R_{xx}(t_1, t_2)\; dt_1 dt_2$.

Result A.3.3:
If $\{X(t)\}$ is a random process with autocorrelation function $R_{xx}(t_1, t_2)$ and if S is a random variable such that $S = \int_a^b X(t)\, dt$, then

$$E\left\{\int_a^b X(t) dt\right\}^2 = \int_a^b \int_a^b R_{xx}(t_1, t_2)\; dt_1 dt_2$$

Proof. From (A5) and (A6), we have, as $\Delta t_i, \Delta t_j \to 0$

$$E\left(\sum_i X(t_i)\Delta t_i\right)^2 + E\left(\sum_j X(t_j)\Delta t_j\right)^2 - 2E\left\{\sum_i X(t_i)\Delta t_i \sum_j X(t_j)\Delta t_j\right\} = 0$$

$$\Rightarrow \quad E\left(\lim_{\Delta t_i \to 0} \sum_i X(t_i)\Delta t_i\right)^2 + E\left(\lim_{\Delta t_j \to 0} \sum_j X(t_j)\Delta t_j\right)^2$$

$$- 2E\left\{\lim_{\Delta t_i \to 0} \lim_{\Delta \tau_j \to 0} \sum_i \sum_j X(t_i) X(t_j)\, \Delta t_i \Delta t_j\right\} = 0$$

By (A4), we have

$$2E\left(\lim_{\Delta t_i\to 0}\sum_i X(t_i)\Delta t_i\right)^2 - 2E\left\{\lim_{\Delta t_i\to 0}\lim_{\Delta\tau_j\to 0}\sum_i\sum_j X(t_i)X(t_j)\Delta t_i\Delta t_j\right\} = 0$$

Or

$$2E\left(\lim_{\Delta t_j\to 0}\sum_j X(t_j)\Delta t_j\right)^2 - 2E\left\{\lim_{\Delta t_i\to 0}\lim_{\Delta\tau_j\to 0}\sum_i\sum_j X(t_i)X(t_j)\Delta t_i\Delta t_j\right\} = 0$$

Again by means of (A4), we have

$$2E\left(\int_a^b X(t)dt\right)^2 - 2\int_a^b\int_a^b R_{xx}(t_i,t_j)dt_i dt_j = 0$$

Letting $t_i = t_1, t_j = t_2$, we have

$$2E\left(\int_a^b X(t)dt\right)^2 - 2\int_a^b\int_a^b R_{xx}(t_1,t_2)dt_1 dt_2 = 0$$

$$\Rightarrow \quad E\left\{\int_a^b X(t)\,dt\right\}^2 = \int_a^b\int_a^b R_{xx}(t_1,t_2)\,dt_1\,dt_2 \tag{A7}$$

$$S = \int_a^b X(t)\,dt = \lim_{\Delta t_i\to 0}\sum_i X(t_i)\Delta t_i$$

Therefore, if $\{X(t)\}$ is a random process and S is a random variable such that $S = \int_a^b X(t)\,dt$ then we have

$$E(S^2) = E\left\{\int_a^b X(t)dt\right\}^2 = \int_a^b\int_a^b R_{xx}(t_1,t_2)\,dt_1 dt_2$$

A.4 TRANSFORMATION OF THE INTEGRAL

Result A.4.1:
If $\{X(t)\}$ is a random process with autocovariance function $C(t_1,t_2)$ then

$$\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)dt_1 dt_2 = \int_{-2T}^{2T} C(\tau)\,(2T - |\tau|)\,d\tau$$

where $\tau = t_1 - t_2$ or$\tau = t_2 - t_1$.

Proof. The contour region for $C(t_1,t_2)$ is shown in Figure A.1. In order to convert the double integral into a single integral, let us consider the following new variables defined as

$$t_1 - t_2 = \tau \quad \text{and} \quad t_1 + t_2 = \upsilon$$
$$\Rightarrow \quad t_1 = \frac{\tau + \upsilon}{2} \quad \text{and} \quad t_2 = \frac{\upsilon - \tau}{2}$$

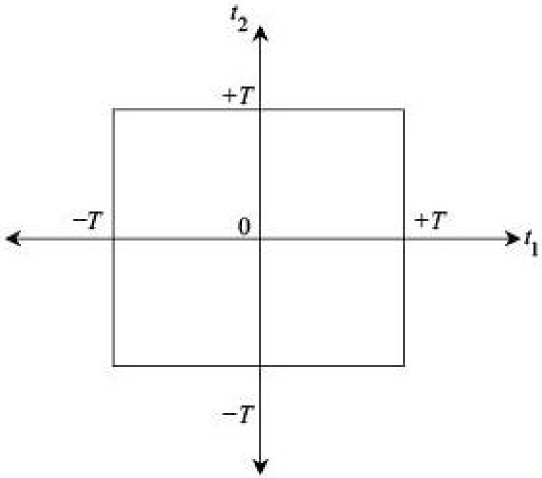


Figure A.1. The contour region for $C(t_1, t_2)$

Now the Jacobean of the transformation becomes

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Since the process is stationary, we can have

$$C(t_1, t_2) = C(t_1 - t_2) = C\left(\frac{\tau + \upsilon}{2} - \frac{\upsilon - \tau}{2}\right) = C(\tau)$$

therefore the function remains as $C(\tau)$ after the transformation. Now we have,

$$\int_{-T}^{T}\int_{-T}^{T} C(t_1, t_2)\, dt_1 dt_2 = \int\int C(\tau)\frac{1}{2} d\,\upsilon\, d\tau = \frac{1}{2}\int\int C(\tau)\; d\,\upsilon\, d\tau$$

Limits for τ and υ are computed as follows:

When $-T \le t_1 \le T \Rightarrow \quad -2T \le \tau + \upsilon \le 2T$ and when $-T \le t_2 \le T \Rightarrow \quad -2T \le \upsilon - \tau \le 2T$ (Refer to Figure A.2).

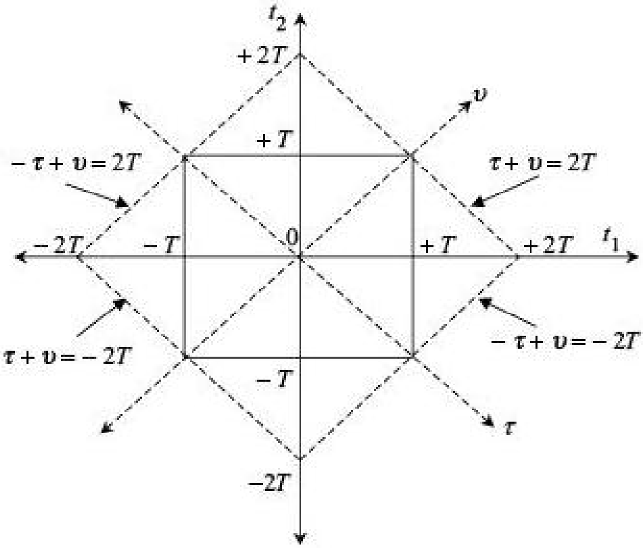


Figure A.2. The contour region for the transferred function of $C(t_1, t_2)$

Therefore,

for $-2T \leq \tau \leq 0$, we have $-2T-\tau \leq \upsilon \leq 2T+\tau$ and

for $0 \leq \tau \leq 2T$, we have $-2T+\tau \leq \upsilon \leq 2T-\tau$

$$
\begin{aligned}
\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)\, dt_1 dt_2 &= \int_{-2T}^{0}\int_{-2T-\tau}^{2T+\tau} C(\tau)\,\frac{1}{2} d\upsilon\, d\tau + \int_{0}^{2T}\int_{-2T+\tau}^{2T-\tau} C(\tau)\,\frac{1}{2} d\upsilon\, d\tau \\
&= \int_{-2T}^{0} C(\tau)\,(2T+\tau)\, d\tau + \int_{0}^{2T} C(\tau)\,(2T-\tau)\, d\tau \\
&= \int_{-2T}^{2T} C(\tau)\,(2T-|\tau|)\, d\tau \\
\Rightarrow dt_1 dt_2 &= (2T-|\tau|)d\tau \\
\therefore \quad \int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)\, dt_1 dt_2 &= \int_{-2T}^{2T} C(\tau)\,(2T-|\tau|)\, d\tau
\end{aligned}
$$

The autocovariance function $C(\tau)$ and the function $2T-|\tau|$ for an arbitrary process are depicted in Figure A.3 for better understanding.

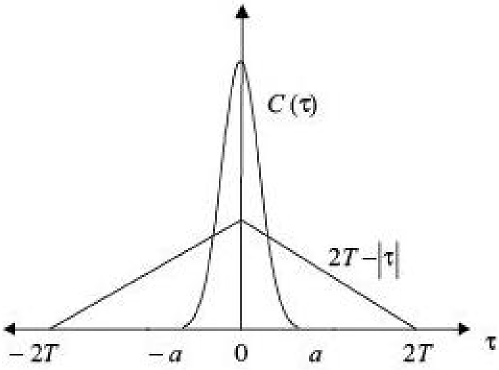


Figure A.3. Plot of $C(\tau)$ and $(2T-|\tau|)$

A.5 EVALUATION OF THE INTEGRAL $\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} \cos\tau\omega \, d\omega$

Result A.5.1:

The integral $\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} \cos\tau\omega\, d\omega$ can be evaluated by the contour integration technique. Consider $\int_C \frac{e^{iaz}}{(1+z^2)^2} dz$, where C is the closed contour consisting of the real axis from $-R$ to $+R$ and the upper half of the circle is $|z| = R$. Refer to the Figure A.4 given below. The only singular of the integrand lying within C is the double pole $z = i$.

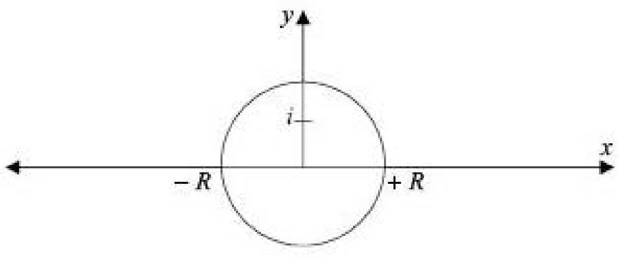


Figure A.4. Closed contour and real axis from $-R$ to $+R$

We know that

$$\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} \cos\tau\omega\, d\omega = \text{real part of } \left\{ \int_{-\infty}^{\infty} \frac{e^{i\tau\omega}}{(1+\omega^2)^2} d\omega \right\}$$

Now consider

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{iaz}}{(1+z^2)^2} dz &= 2\pi i \left(\text{i.e. } f(z) = \frac{e^{iaz}}{(1+z^2)^2} \quad \text{at} \quad z = i \right) \\
&= 2\pi i \lim_{z \to i} \frac{d}{dz} \left\{ (z-i)^2 f(z) \right\} \\
&= 2\pi i \lim_{z \to i} \frac{d}{dz} \left(\frac{e^{iaz}}{(i+z)^2} \right) \\
&= 2\pi i \lim_{z \to i} \left\{ \frac{(i+z^2)^2 e^{iaz} ia - e^{iaz} 2(z+i)}{(i+z)^4} \right\} \\
&= 2\pi i \lim_{z \to i} \left\{ \frac{(i+z) iae^{iaz} - 2e^{iaz}}{(i+z)^3} \right\} \\
&= 2\pi i \left\{ \frac{2i^2 ae^{i^2 a} - 2e^{i^2 a}}{-8i} \right\} = \frac{\pi}{2}(1+a)e^{-a}
\end{aligned}$$

$$\therefore \quad \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} \cos \tau\omega d\omega = \frac{\pi}{2}(1+\tau)e^{-\tau}$$

A.6 POWER SPECTRAL DENSITY FUNCTION OF SINUSOIDAL PROCESS OF THE TYPE $X(t) = A\sin(\omega t + \theta)$ OR $X(t) = A\cos(\omega t + \theta)$

Result A.6.1:

Let $\{X(t)\}$ be a stationary random process such that $X(t) = A\sin(\omega t + \theta)$ where A is the *amplitude*, and $\omega = 2\pi f$ is the *angular frequency* that are assumed constants ($A = a$ and $\omega = \omega_0$) the *phase* θ is a random variable uniformly distributed in $(0, 2\pi)$. Here the amplitude A, and phase θ are assumed as independent to each other. However, since amplitude is random, an arbitrary distribution can be assumed. Now, the power spectral density function of $\{X(t)\}$ can be obtained as follows:

Since each realization of the process $\{X(t)\}$ is a sinusoid at frequency, say ω_0, the expected power in this process should be located at $\omega = \omega_0$ and $\omega = -\omega_0$. For given $-T \le t \le T$, mathematically this truncated sinusoid implies,

$$X_T(t) = a\sin(\omega_0 t + \theta) \operatorname{rect}\left(\frac{t}{2T}\right)$$

where $\operatorname{rect}\left(\frac{t}{2T}\right)$ is a square impulse of unit height and unit width and centered at $t = 0$. Now, the truncated Fourier transform of this truncated sinusoid becomes

$$X_T(\omega) = -iTae^{i\theta} \operatorname{sin}c[2(\omega - \omega_0)t] + iTae^{-i\theta} \operatorname{sin}c[2(\omega + \omega_0)t]$$

where *sinc function* is given by $\operatorname{sinc}(x) = \dfrac{\sin \pi x}{\pi x}$.

$$E\left\{|X_T(\omega|^2\right\} = a^2T^2\left\{\operatorname{sinc}^2[2(\omega-\omega_0)T] + \operatorname{sinc}^2[2(\omega+\omega_0)T]\right\}$$

According to Wiener-Khinchin theorem, the power spectral density function of $\{X(t)\}$ becomes

$$\begin{aligned} S_{xx}(\omega) = \lim_{T\to\infty}\frac{1}{2T}E\left\{|X_T(\omega|^2\right\} &= \lim_{T\to\infty}\frac{1}{2T}E\left(a^2\right)T^2\left\{\operatorname{sinc}^2[2(\omega-\omega_0)T]\right. \\ &\quad \left. + \operatorname{sinc}^2[2(\omega+\omega_0)T]\right\} \\ &= \lim_{T\to\infty}\frac{1}{2}E\left(a^2\right)T\left\{\operatorname{sinc}^2[2(\omega-\omega_0)T]\right. \\ &\quad \left. + \operatorname{sinc}^2[2(\omega+\omega_0)T]\right\} \end{aligned}$$

It may be observed that as $T \to \infty$, the function, say $g(\omega) = T\operatorname{sinc}^2 2\omega T$, becomes increasingly narrower and taller and as a result, we can have the limit as an infinitely tall and infinitely narrow pulse which gives the *delta function* $\delta(\omega)$ *such that* $\int \delta(\omega)d\omega = 1$. As a result, we have

$$\int_{-\infty}^{\infty} g(\omega)\,d\omega = \int_{-\infty}^{\infty} T\operatorname{sinc}^2(2\omega T)\,d\omega$$

Letting $v = 2\omega T$, we have

$$\int_{-\infty}^{\infty} g(\omega)\,d\omega = \frac{1}{2}\int_{-\infty}^{\infty}\operatorname{sinc}^2 v\,dv = \frac{1}{2}$$

This implies $\lim\limits_{T\to\infty} T\operatorname{sinc}^2 2\omega T = \dfrac{1}{2}\delta(\omega)$ and hence the power spectral density function becomes

$$S_{xx}(\omega) = \frac{E\left(A^2\right)}{2}\left\{\frac{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)}{2}\right\}$$

It may be noted that, the average power in a sinusoid process with amplitude $A = a$ is $\dfrac{E\left(A^2\right)}{2} = \dfrac{a^2}{2}$. In fact, this power is evenly split between two frequency points $\omega = \omega_0$ and $\omega = -\omega_0$.

15. $\cos A \sin B = \frac{1}{2}\{\sin(A+B) - \sin(A-B)\}$

16. $\sin 2A = 2\sin A \cos A$

17. $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

18. $1 - \cos 2A = 2\sin^2 A$

19. $1 - \cos A = 2\sin^2 A/2$

20. $\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{2}}, \cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}$

21. $e^{iA} = \cos A + i\sin A$

22. $e^{-iA} = \cos A - i\sin A$

23. $\frac{d\cos x}{dx} = -\sin x, \frac{d\sin x}{dx} = +\cos x$

24. $\int \cos x dx = \sin x, \int \sin x dx = -\cos x$

25. $\int_{-\pi}^{\pi} \cos(A+2x)dx = \left[\frac{\sin(A+2x)}{2}\right]_{-\pi}^{\pi} = \frac{1}{2}[\sin(A+2\pi) - \sin(A-2\pi)] = 0$
[Refer (6)]

26. $\int_{-\pi}^{\pi} e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

27. $\int_{-\pi}^{\pi} e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

28. $\int u dv = uv - \int v du$

29. $(uv)' = u'v + uv'$

30. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Standard Normal Table

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.6	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0002	0.0002
−3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
−3.4	0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
−3.3	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005
−3.2	0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007
−3.1	0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010
−3.0	0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0014
−2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
−2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
−2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
−2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
−2.5	0.0048	0.0050	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
−2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
−2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
−2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
−2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
−2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
−1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
−1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
−1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
−1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
−1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
−1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
−1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
−1.2	0.0985	0.1003	0.1020	0.1038	0.1057	0.1075	0.1093	0.1112	0.1131	0.1151
−1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
−1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
−0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
−0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
−0.7	0.2148	0.2177	0.2207	0.2236	0.2266	0.2297	0.2327	0.2358	0.2389	0.2420
−0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
−0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
−0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
−0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
−0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
−0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5329	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5558	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8079	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9950	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	1.0000
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

12. (i) $f(x) = \frac{x}{2}, \quad 0 < x < 2,\ f(y) = \frac{(1+3y^2)}{2}, \quad 0 < y < 1,\ f(x/y) = f(x)$ (X and Y are independent)

(ii) $\frac{5}{8}$, (iii) $E(X)E(Y) = \left(\frac{4}{3}\right)\left(\frac{5}{8}\right) = \frac{5}{6} = E(XY)$, (iv) $\frac{3}{64}$

13. $$h(y) = \begin{cases} \frac{1}{\sqrt{2\pi}a} e^{-\frac{1}{2}\left(\frac{y-b}{a}\right)^2}, & -\infty < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

14. $$f(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, & 0 < r < \infty \\ 0, & \text{otherwise} \end{cases}, \quad \therefore \quad f(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 < \theta < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

15. $$f(x_1,x_2) = \frac{1}{4\pi\sqrt{0.9375}} e^{(-1/3.75)\left(x_1^2 - 0.50x_1x_2 + x_2^2\right)}, \quad -\infty < x_1, x_2 < \infty$$

Chapter 2

1. Obtain the graph for the following sample functions

$$X(t) = (0.25)\cos 2t$$
$$X(t) = (0.5)\cos 2t$$
$$X(t) = (1)\cos 2t$$

2. $E\{X(t)\} = E[A\cos(2\pi ft + \theta)] = 0$

3. $E\{X(t)\} = \frac{1}{2}\sin(\omega_0 t), V\{X(t)\} = \left\{\frac{1}{3} - \frac{1}{4}\sin(\omega_0 t)\right\}\sin(\omega_0 t)$

4. $$E\{X(1/4)\} = 0.6036, F(x,t) = \begin{cases} 0, & if \quad x < 1/2 \\ \frac{1}{2}, & if \quad 1/2 \le x < 1/\sqrt{2} \\ 1, & if \quad x \ge 1/\sqrt{2} \end{cases}$$

5. (i) $E\{X^2(t)\} = E(A^2)E[\cos^2(\omega t + \theta)] = E(A^2)\left(\frac{1}{2}\right)$

(ii) If θ is constant, we have $E\{X(t)\} = E\{\cos(\omega t + \theta)\} = \cos(\omega t + \theta)$ which is not time independent

6. $f_{X(t_0)}(x) = F'(x) = \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}}, \quad -\sin\omega_0 t_0 \le x \le +\sin\omega_0 t_0$

The probability density function $f_{X(t_0)}(x)$ of $X(t_0)$ is time independent.

7. $E\{X(t)\} = 0$

   $V\{X(t)\} = \dfrac{1}{3}\cos^2(\omega t + \theta)$

8. $C(t_1, t_2) = E\{X(t_1)X(t_2)\} - E\{X(t_1)\}E\{X(t_2)\} = \dfrac{1-e^{-2(t_1+t_2)}}{t_1+t_2}\cos\omega(t_1 - t_2)$

9. $E\{X(t)\} = 0, V\{X(t)\} = \dfrac{1}{2}$

10. $E\{X(t)\} = \left(\dfrac{-5}{4}\right)(\cos t + \sin t), V\{X(t)\} = \dfrac{3}{16}$

Chapter 3

1. $E\{X(t)\} = 0, R(t_1, t_2) = \dfrac{1-e^{-2(t_1+t_2)}}{t_1+t_2}\cos\omega(t_1 - t_2)$, $\{X(t)\}$ is not stationary.

2. $f_{X(t_0)}(x) = F'(x) = \dfrac{1}{2\pi}\dfrac{1}{\sqrt{1-x^2}}, \quad -\sin\omega_0 t_0 \le x \le +\sin\omega_0 t_0$, $\{X(t)\}$is SSS process.

3. $E\{X(t)\} = 0, R_{xx}(t_1, t_2) = \dfrac{1}{2}e^{-|\tau|/2}$, $\{X(t)\}$ is WSS process.

4. (i) $E\{X(t)\} = 0, V\{X(t)\} = \dfrac{1}{2}$,

   (ii) $\{X(t)\}$ is SSS process,

   (iii) $R(t_1, t_2) = \dfrac{A^2}{2}\cos\omega\tau$,

   (iv) $\{X(t)\}$ is WSS process,

   (v) Plot for $X(t) = (+1)\cos(2t + \pi)$ and $X(1) = (-1)\cos(2t + \pi)$ taking $t =$ (0, 10)

5. $E\{Y(t)\} = 0, R_{yy}(t_1, t_2) = \dfrac{R_{xx}(\tau)}{2}\cos\omega\tau$, $\{Y(t)\}$ is WSS process.

6. Plot for $X(t) = \cos(2t + \pi)$ taking $t = (0, 10)$ and for $R(\tau) = (0.5)\cos 2\tau$ taking $\tau = (-10, 10)$

7. $F_{Y(t)}(y) = F_{Y(t+a)}(y)$, $\{Y(t)\}$ is SSS process.

8. $R_{ww}(t, t+\tau) = R_{xx}(\tau)R_{yy}(\tau)$, $\{W(t)\}$ is WSS process and $R_{xw}(t, t+\tau) = R_{xx}(\tau)\mu_y$, $\{X(t)\}$ and $\{W(t)\}$ are JWSS processes.

9. (i) $E\{Y(t)\} = \mu_x\ R_{yy}(\tau) = R_{xx}(\tau)$, $\{Y(t)\}$ is WSS process.
   (ii) $E\{Z(t)\} = \mu_x$, $R_{zz}(t, t+\tau) = R_{xx}(a\tau)$, $\{Z(t)\}$ is a WSS process.
   $R_{xy}(t, t+\tau) = R_{xx}(\tau + a)$, $\{X(t)\}$ and $\{Y(t)\}$ are JWSS processes.
   $R_{xz}(t, t+\tau) = R_{xx}[(a-1)t + a\tau]$, $\{X(t)\}$and $\{Z(t)\}$ are not JWSS processes.

10. $E\{X(t)\} = 0, E\{X^2(t)\} = \frac{1}{2}, V\{X(t)\} = \frac{1}{2}$

Chapter 4

1. (i) and (iii) are valid autocorrelation functions, (ii) and (iv) are not valid autocorrelation functions

2. $2A\left(1 - e^{-3\alpha}\right)$

3. (i) $E\{X(t)\} = 0, E\{X^2(t)\} = 1, V\{X(t)\} = 1,$
   (ii) $E\{X(t)\} = \sqrt{2}, E\{X^2(t)\} = 6, V\{X(t)\} = 4,$
   (iii) $E\{X(t)\} = 5, E\{X^2(t)\} = 29, V\{X(t)\} = 4,$
   (iv) $E\{X(t)\} = 2, E\{X^2(t)\} = 6, V\{X(t)\} = 2$

4. (i) $E(S) = 0$ (ii) $V(S) = 24.8296$

5. (i) $E(Y) = 6$ (ii) $V\{X(t)\} = 2$

6. 0.43235

7. $R_{yy}(\tau) = \frac{A^4}{4}\left(1 + \frac{1}{2}\cos 2\omega\tau\right)$

8. (i) $E\{X(t)\} = \frac{1}{2}\cos\omega t,$
   (ii) $R_{xx}(t_1, t_2) = \frac{1}{3}\cos\omega t_1 \cos\omega t_2,$
   (iii) $C_{xx}(t_1, t_2) = \frac{1}{12}\cos\omega t_1 \cos\omega t_2$

9. $E\{Y(t)\} = 0$, $R_{yy}(\tau) = \frac{1}{\tau^2}\{2R_{xx}(\tau) - R_{xx}(0) - R_{xx}(2\tau)\}$, $\{Y(t)\}$ is WSS process

10. $R_{yy}(\tau) = R_{xx}(\tau) + R_{xn}(\tau) + R_{nx}(\tau) + R_{nn}(\tau), R_{yx}(\tau) = R_{xx}(\tau) + R_{nx}(\tau), R_{yn}(\tau) = R_{xn}(\tau) + R_{nn}(\tau)$

Chapter 5

1. $E\{Y_n\} = \dfrac{a+2b}{2}, V\{Y_n\} = \dfrac{a^2}{4}$

2. (i) $P\{Y(t) = n\} = \dfrac{e^{-5t}(5t)^n}{n!}, \quad n = 0, 1, 2, \cdots\cdots,$

   (ii) $P\{Y(2) = 5\} = 0.0378,$

   (iii) $E\{Y(2)\} = V\{Y(2)\} = 10,$

   (iv) $\lambda_1 t = 4$, $\lambda_2 t = 6$, $(\lambda_1 + \lambda_2)t = 10$

3. (i) $P\{X(5) = 3\} = 0.0076,$

   (ii) $P\{X(5) \geq 4\} = 0.9897,$

   (iii) $P\{X(5) \leq 2\} = 0.0028$

4. (i) $E\{X(t)\} = 30$ min ,

   (ii) $P\{n(0, 60) \leq 2\} = 0.06197$

5. $P\{T_1 \leq t_0/X(t) \leq 1\} = \dfrac{t_0}{t}$

6. $P\{T_1 \leq 4/X(10) \leq 1\} = \dfrac{2}{5}$

7. $P\{Y(2) = 3\} = 0.00038$

8. (i) $P\{n(0, 10) \leq 3\} = 0.9810,$

   (ii) $P\{X(10) = 1/X(2) = 3\} = 0.375$

9. (i) $E\{X(7)\} = V\{X(7)\} = \dfrac{21}{4},$

   (ii) $P\{X(3) \leq 3/X(1) \leq 1\} = 0.3466$

10. $P\{Y(3) = 3\} = 0.0899$

Chapter 6

1. $P\{|X(t)| \leq 0.5\} = 0.3830$

2. $E\{X(t)\} = \mu, V\{X(t)\} = \sin^2 \pi t + \sigma^2$, $\{X(t)\}$ is not WSS process.

3. $P(W > 2) = 0.1587$

4. $f(y,w) = \dfrac{1}{1.732\pi} \exp\left\{-\dfrac{1}{1.5}\left(y^2 - (0.5)yw + w^2\right)\right\}, \quad -\infty < y, w < +\infty$

5. $E(A) = 0, V(A) = 0.8647$

6. $E\{Y(t)\} = 1, V\{Y(t)\} = 2, R_{YY}(\tau) = 1 + 2\cos^2(\tau)$

7. $E\{Z(t)\} = \sqrt{\dfrac{1}{2\pi}}, V\{Z(t)\} = \dfrac{1}{2}\left(1 - \dfrac{1}{\pi}\right)$

8. $E\{Y(t)\} = 0, V\{Y(t)\} = 1, R_{YY}(2) = 0.0016$

9. $R_{XX}(m,n) = \begin{cases} m\left(1 + (n-1)(2p-1)^2\right), & if \quad m < n \\ n\left(1 + (m-1)(2p-1)^2\right), & if \quad n < m \end{cases}$

10. $E(X_n) = 0, V(X_n) = \dfrac{1}{n^2}(2n-1)$

Chapter 7

1. $\mu_t = \mu_A, \mu_T = A(\xi) \neq \mu_A \; as \quad T \to \infty$, $\{X(t)\}$ is not mean ergodic.

2. $\lim\limits_{T\to\infty} V(\overline{X}_T) = \lim\limits_{T\to\infty} \left\{\dfrac{4}{T}\left[1 - \dfrac{1 - e^{-4T}}{2T}\right]\right\} = 0$, $\{X(t)\}$ is mean ergodic.

3. $E\{X(t)\} = 0, E\{X(t)\} = \lim\limits_{T\to\infty} \overline{X}_T = 0$, $\{X(t)\}$ is mean ergodic.

4. $R_{XX}(\tau) = \dfrac{1}{2}\cos\tau$, $E(Z_T) = \dfrac{1}{2}\cos\tau$, $\lim\limits_{T\to\infty} \overline{Z}_T = \dfrac{1}{2}\cos\tau = R_{XX}(\tau)$, $\{X(t)\}$ is correlation ergodic.

5. $\lim_{T\to\infty} \frac{1}{T} \int_0^{2T} c(\tau)\left(1-\frac{\tau}{2T}\right) d\tau = \lim_{T\to\infty} \frac{q}{\alpha T}\left(1-\frac{1-e^{-2\alpha T}}{2\alpha T}\right) = 0$, $\{X(t)\}$ is mean ergodic.

6. $\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} c(\tau)\left(1-\frac{\tau}{2T}\right) d\tau = \lim_{T\to\infty} \frac{q}{2T} = 0$, $\{X(t)\}$ is mean ergodic.

7. $\lim_{T\to\infty} \frac{1}{T} \int_0^{2T} c(\tau)\left(1-\frac{\tau}{2T}\right) d\tau = \lim_{T\to\infty} \sigma_A^2 = \sigma_A^2 \neq 0$, $\{X(t)\}$ is not mean ergodic.

8. $E\left\{\overline{X}_T\right\} = 0$, $V\left\{\overline{X}_T\right\} = \frac{1}{4\lambda T}\left(1-\frac{1-e^{4\lambda T}}{2\lambda T}\right)$, $\lim_{T\to\infty} V(\overline{X}_T) = 0$, $\{X(t)\}$ is mean ergodic.

9. $\lim_{T\to\infty} V(\overline{X}_T) = \lim_{T\to\infty}\left\{\frac{10}{T}\left[1-\frac{1-e^{-4T}}{2T}\right]\right\} = 0$, $\{X(t)\}$ is mean ergodic.

10. $\lim_{T\to\infty} V\left\{\overline{Z}_T\right\} = \lim_{T\to\infty}\left\{\frac{1}{2T}\left\{\int_0^{2T} \phi(\tau)\left(1-\frac{\tau}{2T}\right) d\tau\right\} - \{R(\tau)\}^2\right\}$,

$\therefore \lim_{T\to\infty} V\{\overline{Z}_T\} \to 0$ if and only if $\lim_{T\to\infty} \frac{1}{2T}\left\{\int_0^{2T} \phi(\tau)\left(1-\frac{\tau}{2T}\right) d\tau\right\} \underset{T\to\infty}{\to} \{R(\tau)\}^2$

where $\phi(\tau) = E\{X(t_1+\tau)X(t_1)X(t_2+\tau)X(t_2)\}$

Chapter 8

1. $R_{xx}(0) = \frac{12.5}{\pi}$ Sq. units

2. $S_{xx}(\omega) = \sqrt{\pi}\, e^{-\omega^2/4}$

3. $R_{xxx}(\tau) = \frac{k}{\pi\tau} \sin a\tau$

4. $S_{xx}(\omega) = 2\left[\frac{1-\cos\omega}{\omega^2}\right] = \frac{2\sin^2\omega/2}{\omega^2/2} = \left(\frac{\sin\omega/2}{\omega/2}\right)^2$

5. $R_{xx}(\tau) = \frac{2}{\pi\tau^3}\left(\tau^2\sin\tau + \tau\cos\tau - \sin\tau\right)$

6. $S_{xx}(\omega) = \dfrac{4a^2b}{4b^2+\omega^2}$
7. $S_{xx}(\omega) = \dfrac{2}{1+(\omega/10)^2}$
8. $R_{xy}(\tau) = \dfrac{1}{\pi\tau^2}\left[a\tau - b)\sin\tau + b\tau\cos\tau\right]$
9. $S_{xx}(\omega) = Aa\left[\dfrac{1}{a^2+(\omega+b)^2} + \dfrac{1}{a^2+(\omega-b)^2}\right]$
10. $R_{xx}(\tau) = \dfrac{1}{\pi\tau^3}\left(\tau^2\sin\tau + 2\tau\cos\tau - 2\sin\tau\right)$

Chapter 9

1. (i) $P\{X_2 = 1\} = 0.1962$,
   (ii) $P(X_0 = 1, X_1 = 2, X_2 = 2) = 0.0625$,
   (iii) $P(X_2 = 2, X_1 = 2/X_0 = 1) = 0.25$, (iv) $P_{12}^{(2)} = 0.4375$

2. $\sum_{n=1}^{\infty} f_{11}^{(n)} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} + \cdots\cdots = 1 + 0 + 0 + \cdots\cdots = 1$ (state 1 is recurrent)
   $\sum_{n=1}^{\infty} f_{22}^{(n)} = f_{22}^{(1)} + f_{22}^{(2)} + f_{22}^{(3)} + \cdots\cdots = 0.5 + 0 + 0 + \cdots\cdots = 0.5 < 1$ (state 2 is transient)

3. (ii) $P^{(2)} = \begin{pmatrix} \frac{11}{36} & \frac{4}{36} & \frac{21}{36} \\ \frac{4}{12} & \frac{1}{12} & \frac{7}{12} \\ \frac{3}{9} & \frac{1}{9} & \frac{5}{9} \end{pmatrix}$

4. (i) $P(X_3 = 1) = 0.3376$,
   (ii) $P(X_\infty = 1) = \dfrac{2}{3}$

5. (ii) $d(1) = GCD\left\{n : P_{11}^{(n)} > 0\right\} = GCD\{2, 4, 6.\cdots\cdots\} = 2$, state 1 is periodic with period 2.

6. $\pi_1 = \dfrac{2}{5}, \quad \pi_2 = \dfrac{2}{5}, \quad \pi_3 = \dfrac{1}{5}$

7. States are irreducible, Periodic with period 2, Non-null persistent, Not ergodic.

8. (b) $\pi_1 = \pi_2 = \pi_3 = \dfrac{1}{3}$, (c) States are irreducible, Periodic with period 3, Non-null persistent, Not ergodic.

9. (ii) $P^{(1)} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 1/8 & 7/8 & 0 \end{pmatrix}$,

(iii) $\pi_1 = \dfrac{1}{3}, \pi_2 = \dfrac{2}{5}, \pi_3 = \dfrac{4}{15}$

10. (i) $P^{(1)} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 2/3 & 1/3 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 2/3 & 1/3 \end{pmatrix} \end{array}$,

(ii) $\pi_1^{(2)} = \dfrac{5}{18}$,

(iii) $\pi_1 = \dfrac{4}{15}$