Student Information

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Answer 1

a) For independence we have to check the equation:
$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

$$f_{XY}(x,y) = \begin{cases} (1/\pi) & \text{if } x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

From Part b below we have obtain that:

$$f_X x = \begin{cases} (2/\pi)\sqrt{1-x^2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y y = \begin{cases} (2/\pi)\sqrt{1-y^2} & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

They are NOT independent because, $f_{XY}(x,y) \neq f_X(x) \cdot f_Y(y)$

b) For $-1 \le x \le 1$, we have
$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) = \ dy$
$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1/\pi)\, dy$
$=(2/\pi) \cdot \int_0^{\sqrt{1-x^2}} \ dy$
$= (2/\pi)\sqrt{1-x^2}$
Hence, we have ;

$$f_X x = \begin{cases} (2/\pi)\sqrt{1-x^2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Likewise:

$$f_Y y = \begin{cases} (2/\pi)\sqrt{1-y^2} & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

c) E(x)= $\frac{1}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\, dx\, dy$
$= \int_{-1}^{1} 2x\sqrt{1-x^2}\, dx$
$= 0$

d) $\text{Var(X)} = E(x-\mu)^2$

Var(X)$= \frac{1}{\pi} \cdot \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x-\mu)^2 \, dx \, dy$

Var(X)$= \frac{1}{\pi} \cdot \int_{-1}^{1} \frac{6\mu^2\sqrt{-y^2+1}+2\cdot(\sqrt{-y^2+1})^3}{3} \, dy$

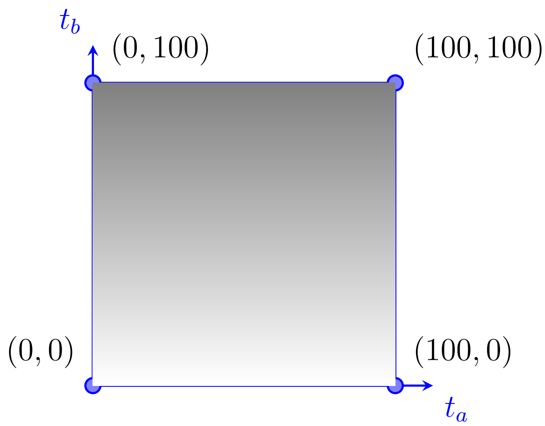
Var(X)$=\frac{1}{\pi} \cdot \left(\frac{\pi\cdot(4\mu^2+1)}{4}\right)$

Var(X)$= \mu^2 + 0.25$ and $\mu = 0$

Thus we have the variance;
Var(X)=0.25

Answer 2

a) $f(t_a, t_b) = \begin{cases} c & 0 \leq t_a, t_b \leq 100 \\ 0 & \text{otherwise} \end{cases}$
where c is constant
First, we sketch a bird's eye view of the joint p.d.f



The support of the distribution is the gray square. The surface is a constant height above the square, so the total volume under it is;
(area of base) $\cdot$ (height of surface) $= (100-0)^2 \cdot c$
We know that the total volume must be 1 (probability has to equal 1), which means that;
c$= \frac{1}{100^2}$

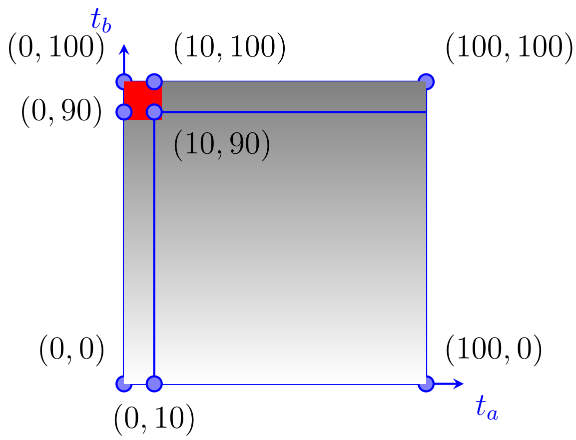
c$= \frac{1}{10000}$

We have obtain that joint p.d.f is
$f(t_a, t_b) = \begin{cases} \frac{1}{10000} & 0 \leq t_a, t_b \leq 100 \\ 0 & \text{otherwise} \end{cases}$

Above We found joint p.d.f
To find the joint CDF for x≥0 and y≥ 0, we need to integrate the joint PDF:
$f_{(t_at_b)}(t_a, t_b) = \int_{-\infty}^{tb} \int_{-\infty}^{ta} f_{(t_at_b)}(u, v)\, du\, dv$
$=\int_0^{tb} \int_0^{ta} f_{(t_at_b)}(u, v)\, du\, dv$
$=\int_0^{min(tb,100)} \int_0^{min(ta,100)} (1/10000)\, du\, dv$

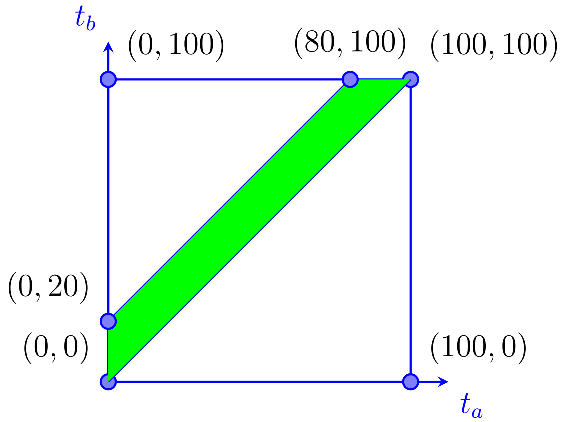
For $0 \leq t_a, t_b \leq 100$ we obtain;
$f_{(t_at_b)}(t_a, t_b) = \int_0^{tb} \int_0^{ta} (1/10000)\, du\, dv$
$\int_0^{tb} (t_a/10000)\, dv$
$= ((t_a \cdot t_b)/10000)$

b) The red shaded area demonstrates A pushes to first 10 seconds and B pushes to last 10 seconds
That is the probability we are looking for and whole gray and red area our sample space



Therefore; P= $\frac{\text{Red Area}}{\text{Gray Area}}$
P= $\frac{10*10}{100*100}$
P= $\frac{1}{100} = 0.01$

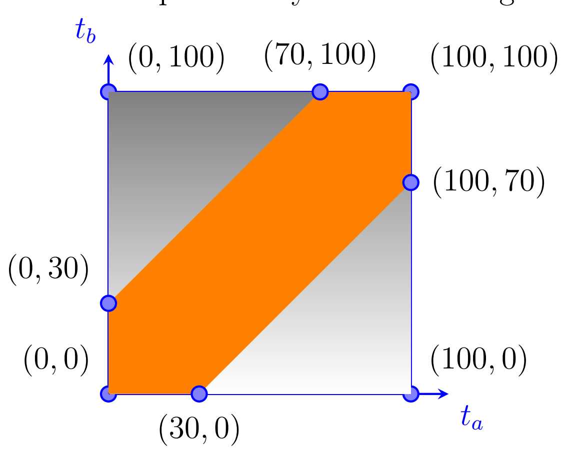
c) The green shaded area demonstrates A pushes the button at most 20 seconds after subject B
That is the probability we are looking for and whole gray and green area our sample space



Hence, we have the probability;
P= $\frac{1800}{100*100}$

P= $\frac{18}{100} = 0.18$

d) The orange shaded area demonstrates subjects fail the test if their elapsed time differ by more than 30 seconds
That is the probability we are looking for and whole gray and orange area our sample space



Thus, we have the probability;
P=$\frac{100*100-70*70}{100*100}$

P=$\frac{5100}{100*100}$

P=$\frac{51}{100} = 0.51$

Answer 3

a) $X_1, X_2, X_3..., X_{n-1}, X_n$ be independent random variables, with X_i having an Exponential(λ_i) distribution. Then the distribution of the min($X_1, X_2, X_3..., X_n$) is exponential ($\lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n$), and the probability which is the minimum is $X_i = \lambda_i/(\lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n)$
P(min($X_1, X_2, X_3, . . ., X_{n-1}, X_n$)) = P($X_1 > t, X_2 > t, X_3 > t, . . ., X_{n-1} > t, X_n > t$)

The conditions are independent, Therefore;
= P($X_1 > t$)$\cdot$ P($X_2 > t$) $\cdot$ P($X_3 > t$) . . . $\cdot$ P($X_{n-1} > t$) $\cdot$ P($X_n > t$)
$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot e^{-\lambda_3 t} \cdot\ .\ .\ .\ e^{-\lambda_{n-1} t} \cdot e^{-\lambda_n t}$
$= e^{-t(\lambda_1+\lambda_2+\lambda_3+.\ .\ .+\lambda_{n-1}+\lambda_n)}$

b) λ stands for the expected time before one of the computers fails. It comes from the minimum of the lifetime of the computer. It is C_{10} and λ =1
We have to calculate cumulative distribution function;
P(min($C_1, C_2, C_3,\ .\ .\ .\ , C_9, C_{10} \leq t$)) = $F(C_1) \cdot F(C_2) \cdot F(C_3)\ .\ .\ .\ \cdot F(C_9) \cdot F(C_{10})$
$= (1 - e^{-\lambda t})^n$; where n=10 and λ =1
$= (1 - e^{-t})^{10}$
And to calculate probability density function we have to differentiate c.d.f
$f(t) = 10 \cdot (1 - e^{-t})^9 * e^{-t}$

The expected time before one of the computer that fails can calculate below
$\int_0^\infty t \cdot f(t)\, dt$
E(T)=$\int_0^\infty t \cdot 10(1 - e^{-t})^9 \cdot e^{-t}\, dt$
E(T)= 2.92896

Answer 4

a)

n denotes sample size
μ denotes sample mean
σ denotes standard deviation
p denotes population proportion
$\mu = n \cdot p$ and $\sigma = \sqrt{npq}$ where q = 1-p

p=0.74, $\mu = 0.74 \cdot 100$= 74 , and $\sigma = \sqrt{100 \cdot 0.74 \cdot 0.26}$=4.38634
We have to calculate P(X>70)
Now we have to calculate Z-score(Z)
Z= $\frac{X-\mu}{\sigma}$
Z= P($Z \geq \frac{70-74}{4.39}$)
Z= P($Z \geq -0.91192$) = P($Z < +0.91192$)

P($Z < +0.91192$)=0.8191

b)

n,p,μ,σ and q explained above
n=100, p=0.1, q=0.9, $\mu = 10$, $\sigma = 3$
We have to calculate P(X$\leq$ 5)
Now we have to calculate Z-score(Z)
Z= $\frac{X-\mu}{\sigma}$
Z= P($Z \leq \frac{5-10}{3}$)
Z= -1.66667
P(X$\leq$ 5) = 1- P(X$>$ 5)
P($Z > +1.66667$) = 0.95221
Therefore;
P(X$\leq$ 5) = 1- 0.95221
P(X$\leq$ 5) = 0.04779