Exercise 1

Due Date: 8/5

General Instructions

You may **only** use material taught in class or from Vazirani's or Shmoys and Williamson's books. I trust you no to use any other material (either online or offline). You may discuss the exercise with other students (although it is recommended that you try to solve it on your own), as long as you **specify their names**. You still have to write down answers in your own words and **fully prove** your claims. Submission is in pairs. Solutions should be typed and submitted by Moodle.

Questions

1. Recall that in the Traveling Salesperson problem we are given a complete directed weighted graph and our objective is to find a min cost Hamiltonian cycle. Consider the case in which the costs are symmetric (i.e., $c(u, v) = c(v, u)$) and the cost of each edge is either 1 or 2.

   (a) Present a polynomial time algorithm that computes a min cost cycle cover of the nodes of the graph (a collection of cycles such that each node is in exactly a single cycle.) You may use the fact that a min cost prefect matching can be computed in polynomial time. It is okay to have length 2 cycles.

   (b) Use the previous algorithm to get an approximation algorithm that guarantees an approximation ratio of 3/2 for TSP where the cost of each edge is 1 or 2.

   (c) Which property of the cycle cover do you need to get a 4/3-approximation algorithm?

   (d) (Bonus) Present an algorithm that computes a cycle cover with the property mentioned above.

2. Consider the node-weighted Steiner tree problem. G is an undirected graph, each node has a cost $c(v)$ and each edge has a cost $c(e)$. Given a subset of the nodes $R \subseteq V$ we need to find a minimum cost spanning tree that includes all the nodes of R (and potentially some other nodes). You may assume that the cost of all the nodes in R is 0.

   (a) Give an approximation algorithm that guarantees an approximation ratio of 2 for this problem in case the edge costs are metric (i.e., obey the triangle inequality).

   (b) Show that in the general case there exists some c such that there is no $(c \cdot \ln |R|)$-approximation algorithm for this problem unless P=NP.
   Hint: Use one of the problems we saw in class.

   (c) (Bonus) present an $O(\ln R)$ approximation algorithm for the node-weighted Steiner tree problem .

3. Consider a directed graph such that each edge e has a cost $c(e)$ and a length $l(e)$. Given a source node s, target node t and maximum length L our goal is to find the path connecting s and t of minimum cost that its length is at most L. Provide an FPTAS for this problem.

   Hint: Let $c_1, \ldots c_m$ denote the costs of the edges sorted from smallest to largest. You might want to choose between different solutions each uses only edges of cost at most c_j for some $1 \leq j \leq m$.

4. **Do not use any books for solving this question.** A valid k-coloring of a graph is an assignment of colors $1, 2, \cdots, k$ to the vertices of the graph such that no two adjacent vertices receive the same color. A graph is k-colorable if there exists a valid k-coloring of its vertices. The problem of finding a valid k-coloring of a k-colorable graph is NP-complete for $k \geq 3$.

   (a) Prove that graphs with maximum degree Δ are $(\Delta + 1)$-colorable. Also, give a polynomial time algorithm for finding a $(\Delta+1)$ -coloring and a polynomial time algorithm for 2-coloring a bipartite graph.

   (b) Relying on part (a), give a polynomial time algorithm for finding an $O(\sqrt{n})$-coloring of a 3-colorable graph. (Hint: Verify and use the fact that the neighborhood of any vertex in a 3-colorable graph is 2-colorable.)

   (c) Extend the algorithm from part (c) to obtain an $O\left(n^{2/3}\right)$-coloring for a 4-colorable graph in polynomial time.