Random Process

Lecture 6

October 16, 21, 23 and 28, 2025

Outline

Random (or Stochastic) Process

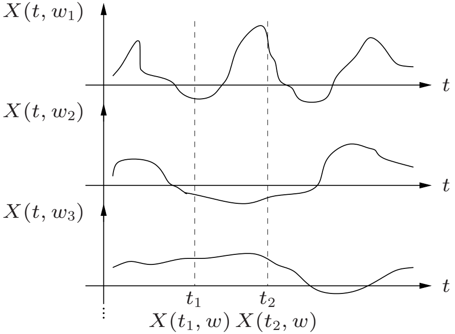
- A random process (RP) (or stochastic process) is an infinite indexed collection of random variables $\{X(t) : t \in \mathcal{T}\}$, defined over a common probability space $(\Omega, \mathcal{F}, P)$. $\mathcal{T}$ is the index set.
- The index parameter t is typically time, but can also be a spatial dimension (for images, videos, ...).
- Random processes are used to model random experiments that evolve in time:
  - Received sequence/waveform at the output of a communication channel
  - Packet arrival times at a node in a communication network
  - Thermal noise in a resistor
  - Daily price of a stock
  - Winnings or losses of a gambler

Questions Involving Random Processes

- Dependencies of the random variables of the process
  - How do future received values depend on past received values?
  - How do future prices of a stock depend on its past values?
- Long term averages
  - What is the proportion of time a queue is empty?
  - What is the average noise power at the output of a circuit?
- Extreme or boundary events
  - What is the probability that a link in a communication network is congested?
  - What is the probability that the maximum power in a power distribution line is exceeded?
  - What is the probability that a gambler will lose all his captial?
- Estimation/detection of a signal from a noisy waveform

Two Ways to View a Random Process

- A random process can be viewed as a function $X(t, \omega)$ of two variables, time $t \in \mathcal{T}$ and the outcome of the underlying random experiment $\omega \in \Omega$
  - For fixed t, $X(t, \omega)$ is a random variable over Ω
  - For fixed ω, $X(t, \omega)$ is a deterministic function of t, called a sample function

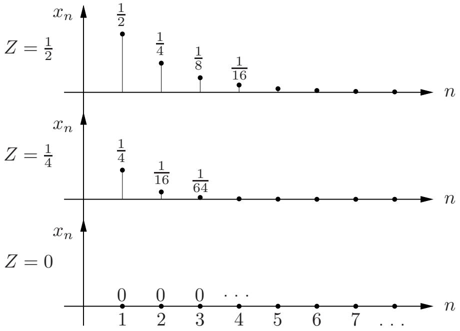


Discrete Time Random Process

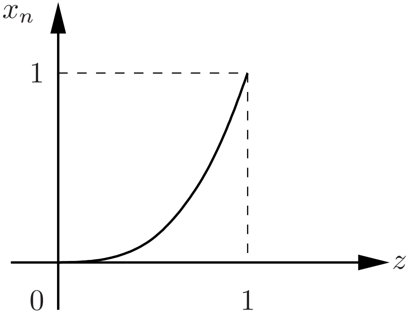
- A random process is said to be discrete time if index set $\mathcal{T}$ is a countably infinite set, e.g.,
  - $\mathbb{N} = \{0, 1, 2, \cdots\}$ = set of nonnegaive integers
  - $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ = set of integers
- In this case the process is typically denoted by X_n, for $n \in \mathbb{N}$ (or $\mathbb{Z}$), a countably infinite set, and is simply an infinite sequence of random variables.
- A sample function for a discrete time process is called a sample sequence or sample path
- In a discrete-time process X_ns can be discrete, continuous, or mixed random variables.

Example

- Let $Z \sim \text{Uniform}[0, 1]$, and define the discrete time process $X_n = Z^n$ for $n \geq 1$
- Sample paths:



- First-order pdf of the process: For each n, $X_n = Z^n$ is an RV; the sequence of pdfs of X_n is called the first-order pdf of the process.



- Since X_n is a differentiable function of the continuous RV Z, we can find its pdf as

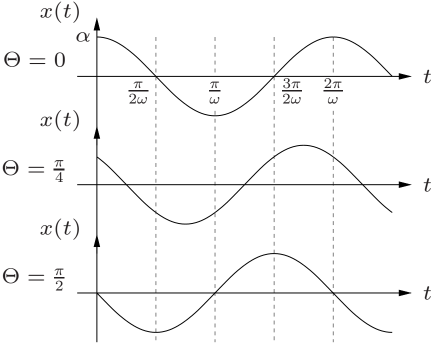
$$f_{X_n}(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{\frac{1}{n}-1} \quad 0 \le x \le 1$$

Continuous Time Random Process

- A random process is continuous time if the index set $\mathcal{T}$ is a continuous set.
- Example: Sinusoidal Signal with Random Phase

$$X(t) = \alpha \cos(\omega t + \Theta)\,, \quad -\infty < t < \infty \text{ or } t \geq 0$$

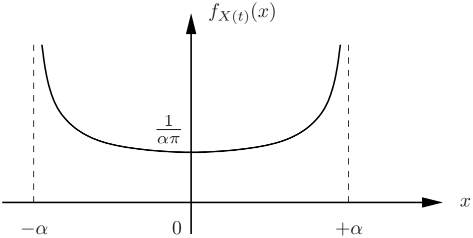
where $\Theta \sim \text{Uniform}[0, 2\pi]$, and α and ω are constants.
- Sample functions:



- The first-order pdf of the process is the pdf of $X(t) = \alpha \cos(\omega t + \Theta)$. It turns out to be

$$f_{X(t)}(x) = \frac{1}{\alpha\pi\sqrt{1-(x/\alpha)^2}} \quad -\alpha < x < \alpha$$

Note that the pdf is independent of t. The graph of the pdf is shown below.



Specifying a Random Process

- In the above examples we specified the random process by describing the set of sample functions (sequences, paths) and explicitly providing a probability measure over the set of events (subsets of sample functions).
- This way of specifying a random process has very limited applicability, and is suited only for very simple processes.
- A random process is typically specified (directly or indirectly) by specifying all its n-th order cdfs (pdfs, pmfs), i.e., the joint cdf (pdf, pmf) of the samples

$$X(t_1), X(t_2), \cdots, X(t_n)$$

  for every order n and for every set of n points $t_1, t_2, \cdots, t_n \in \mathcal{T}$
- The following examples of important random processes will be specified (directly or indirectly) in this manner.

Important Classes of Random Processes

- IID process: $\{X_n : n \in \mathbb{N}\}$ is an IID process if the RVs X_n are i.i.d. (independent and identically distributed).
  Examples:
  - Bernoulli process: $X_1, X_2, \cdots, X_n, \cdots$, i.i.d. $\sim \text{Bern}(p)$
  - Discrete-time white Gaussian noise (WGN): $X_1, X_2, \cdots, X_n, \cdots$, i.i.d. $\sim \mathcal{N}(0, \sigma^2)$
- Here we specified the n-th order pmfs (pdfs) of the processes by specifying the first-order pmf (pdf) and stating that the RVs are independent
- It would be quite difficult to provide the specifications for an IID process by specifying the probability measure over the subsets of the sample space.

The Random Walk Process

- Let $Z_1, Z_2, \cdots, Z_n, \cdots$ be i.i.d., where

$$Z_n = \begin{cases} +1 \text{ (heads)} & \text{with probability } \frac{1}{2} \\ -1 \text{ (tails)} & \text{with probability } \frac{1}{2} \end{cases}$$

- The random walk process is defined by

$$X_0 = 0, \quad X_n = \sum_{i=1}^{n} Z_i\,, \quad n \geq 1$$

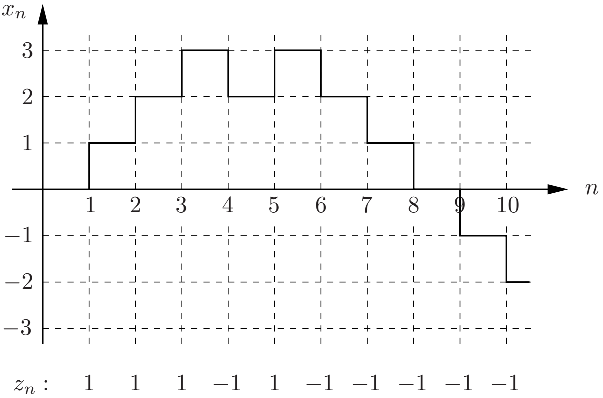
- Again this process is specified by (indirectly) specifying all n-th order pmfs
- Sample path: The sample path for a random walk is a sequence of integers, e.g.,

$$0, +1, 0, -1, -2, -3, -4, \cdots$$

or

$$0, +1, +2, +3, +4, +3, +4, +3, +4, \cdots$$

- Example:



- First-order pmf: It is easy to see that $P(X_n = k) = 0$ if $k < -n$ or $k > n$.

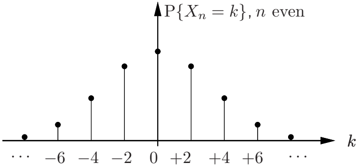
- $P(X_n = k) = ?$. Suppose in n steps, there are r $+1$'s (steps to "right"), then the number of -1's (steps to "left") are $n - r$. So

$$k = r - (n - r) = 2r - n \;\Rightarrow\; r = \frac{n+k}{2}$$

  Since r, k and n are integers, $n + k$ must be an even number. So $P(X_n = k) = 0$ if $n + k$ is odd.

- If n is even, then $k = 2r - n$ must be even. If n is odd, then k must be odd.
- Thus, for $n + k$ even,

$$\begin{aligned} P(X_n = k) &= P\Big(\frac{n+k}{2} \text{ heads in } n \text{ independent coin tosses}\Big) \\ &= \binom{n}{\frac{n+k}{2}} 2^{-n} \text{ for } -n \le k \le n \end{aligned}$$



Markov Processes

- A discrete-time random process X_n is said to be a Markov process if the process future and past are conditionally independent given its present value
- Mathematically this can be rephrased in several ways. For example, if the RVs $\{X_n : n \geq 1\}$ are discrete, then the process is Markov iff the conditional pmfs

$$P_{X_{n+1}|\boldsymbol{X}^n}(x_{n+1}|x_n, \boldsymbol{x}^{n-1}) = P_{X_{n+1}|X_n}(x_{n+1}|x_n)$$

  for every n, where $\boldsymbol{X}^n = \{X_i : 1 \leq i \leq n\}$.
- If the RVs $\{X_n : n \geq 1\}$ are continuous, then the process is Markov iff the pdfs

$$f_{X_{n+1}|\boldsymbol{X}^n}(x_{n+1}|x_n, \boldsymbol{x}^{n-1}) = f_{X_{n+1}|X_n}(x_{n+1}|x_n)$$

  for every n.
- IID processes are Markov

- A continuous-time process $X(t)$ is said to be Markov if $X(t_{k+1})$ and $(X(t_1), X(t_2), \cdots, X(t_{k-1}))$ are conditionally independent given $X(t_k)$ for every $0 \leq t_1 < t_2 < \cdots < t_k < t_{k+1}$ and every $k \geq 2$. Equivalently, iff

$$f_{X(t_{k+1})|X(t_1),X(t_2),\cdots,X(t_k)}(x_{k+1}|x_k, \boldsymbol{x}^{k-1}) = f_{X(t_{k+1})|X(t_k)}(x_{k+1}|x_k)$$

- The random walk process is Markov. To see this consider $(X_n = \sum_{i=1}^{n} Z_i = X_{n-1} + Z_n)$

$$\begin{aligned} P(X_{n+1} = x_{n+1}|\boldsymbol{X}^n = \boldsymbol{x}^n) &= P(X_n + Z_{n+1} = x_{n+1}|\boldsymbol{X}^n = \boldsymbol{x}^n) \\ &= P(X_n + Z_{n+1} = x_{n+1}|X_n = x_n) \\ &= P(X_{n+1} = x_{n+1}|X_n = x_n) \end{aligned}$$

Independent Increment Processes

- A discrete-time random process $\{X_n : n \geq 0\}$ is said to be independent increment if the increment random variables

$$X_{n_1},\ X_{n_2} - X_{n_1},\ \cdots, X_{n_k} - X_{n_{k-1}}$$

are independent for all sequences of indices such that $0 \leq n_1 < n_2 < \cdots < n_k$.

- Example: Random walk is an independent increment process . To see this consider $(X_n = \sum_{i=1}^{n} Z_i)$

$$X_{n_1} = \sum_{i=1}^{n_1} Z_i\ , \quad X_{n_2} - X_{n_1} = \sum_{i=n_1+1}^{n_2} Z_i\ , \quad \cdots$$

$$X_{n_k} - X_{n_{k-1}} = \sum_{i=n_{k-1}+1}^{n_k} Z_i$$

They are independent because they are functions of independent random variables/vectors.

- A continuous-time random process $\{X(t) : t \geq 0\}$ is said to be independent increment if the increment random variables

$$X(t_1),\ X(t_2) - X(t_1),\ \cdots, X(t_k) - X(t_{k-1})$$

are independent for all sequences of time instances such that $0 \leq t_1 < t_2 < \cdots < t_k$.

- The independent increment property makes it easy to find the n-th order pmfs (pdfs) of an independent increment process (such as random walk process) from knowledge only of the first-order pmf/pdf
- Example: Find $P(X_5 = 3, X_{10} = 6, X_{20} = 10)$ for random walk $\{X_n\}$
Solution: We use the independent increment property as follows (recall $X_n = \sum_{i=1}^{n} Z_i$)

$$P(X_5 = 3, X_{10} = 6, X_{20} = 10) = P(X_5 = 3, X_{10} - X_5 = 3, X_{20} - X_{10} = 4)$$

$$= P(X_5 = 3)\, P(X_5 = 3)\, P(X_{10} = 4) = \binom{5}{4} 2^{-5} \times \binom{5}{4} 2^{-5} \times \binom{10}{7} 2^{-10}$$

$$= 3000 \times 2^{-20} \quad \text{(use expression from slide 15)}$$

Notice that $X_{10} - X_5 = \sum_{i=6}^{10} Z_i \sim \sum_{i=1}^{5} Z_i$.

- In general if a process is independent increment, then it is also Markov. To see this let $\{X_n\}$ be an independent increment process. Define the column vector

$$\boldsymbol{\Delta X}^n = [X_1,\ X_2 - X_1,\ \cdots\ , X_n - X_{n-1}]^\top$$

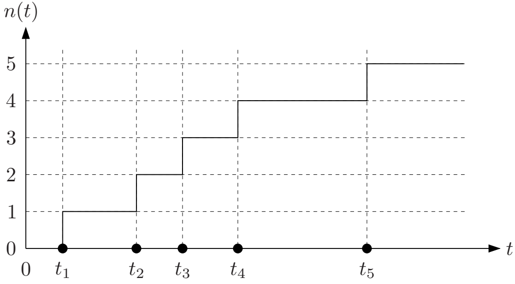
Then the conditional pmf (assuming X_n's are discrete RVs, else use conditional pdf)

$$\begin{aligned} P_{X_{n+1}|\boldsymbol{X}^n}(x_{n+1}|\boldsymbol{x}^n) &= P(X_{n+1} = x_{n+1}|\boldsymbol{X}^n = x^n) \\ &= P(X_{n+1} - X_n = x_{n+1} - x_n|\boldsymbol{\Delta X}^n = \boldsymbol{\Delta x}^n, X_n = x_n) \\ &= P(X_{n+1} - X_n = x_{n+1} - x_n|X_n = x_n) \\ &= P(X_{n+1} = x_{n+1}|X_n = x_n) \end{aligned}$$

- The converse is not necessarily true, e.g., IID processes are Markov but not independent increment.

Counting Processes and Poisson Process

A continuous-time random process $N(t)$, $t \geq 0$, is said to be a counting process if $N(0) = 0$ and $N(t) = n$, $n \in \{0, 1, 2, \cdots\}$, is the number of events in the time interval 0 to t (hence $N(t_2) \geq N(t_1)$ for every $t_2 > t_1 \geq 0$).



$t_1, t_2, \cdots$ are the arrival times or the wait times of the events.
$t_1, t_2 - t_1, \cdots$ are the interarrival times of the events

- The events may be:
  - Photon arrivals at an optical detector
  - Packet arrivals at a router
  - Student arrivals at a class
- The Poisson process is a counting process in which the events are "independent of each other" (independent increment process)
- More precisely, $N(t)$ is a Poisson process with rate (intensity) $\lambda > 0$ if:
  - $N(0) = 0$
  - $N(t)$ is independent increment
  - $N(t_2) - N(t_1) \sim \text{Poisson}(\lambda(t_2 - t_1))$ for all $t_2 > t_1 \geq 0$
- Thus,

$$
\begin{aligned}
P(N(t_2) - N(t_1) = k) &= \frac{[\lambda(t_2 - t_1)]^k}{k!} e^{-\lambda(t_2 - t_1)}, \quad k = 0, 1, \cdots \\
&= \frac{[\lambda(t_2 - t_1)]^k}{k!} e^{-\lambda(t_2 - t_1)} u(k)
\end{aligned}
$$

To find the 2nd order pmf, we use the independent increment property:

$$
\begin{aligned}
P(N(t_1) = n_1, N(t_2) = n_2) &= P(N(t_1) = n_1, N(t_2) - N(t_1) = n_2 - n_1) \\
&= P(N(t_1) = n_1) P(N(t_2) - N(t_1) = n_2 - n_1) \\
&= \frac{[\lambda t_1]^{n_1}}{n_1!} e^{-\lambda t_1} u(n_1) \frac{[\lambda(t_2 - t_1)]^{n_2 - n_1}}{(n_2 - n_1)!} e^{-\lambda(t_2 - t_1)} u(n_2 - n_1) \\
&= \frac{\lambda^{n_2} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1}}{n_1!(n_2 - n_1)!} e^{-\lambda t_2} u(n_1) u(n_2 - n_1)
\end{aligned}
$$

This generalizes to k-th order pmf:

$$
\begin{aligned}
&P(N(t_1) = n_1, N(t_2) = n_2, \cdots, N(t_k) = n_k) \\
&= P(N(t_1) = n_1, N(t_2) - N(t_1) = n_2 - n_1, \cdots, N(t_k) - N(t_{k-1}) = n_k - n_{k-1}) \\
&= P(N(t_1) = n_1) P(N(t_2) - N(t_1) = n_2 - n_1) \cdots P(N(t_k) - N(t_{k-1}) = n_k - n_{k-1}) \\
&= \cdots \quad \cdots
\end{aligned}
$$

Mean and Autocorrelation Functions

- For a random vector $\boldsymbol{X}$ the first and second order moments are
  - mean $\boldsymbol{\mu} = E\{\boldsymbol{X}\}$
  - correlation matrix $\boldsymbol{R}_X = E\{\boldsymbol{X}\boldsymbol{X}^\top\}$
- For a random process $\{X(t)\}$ the first and second order moments are
  - mean function $\mu_X(t) = E\{X(t)\}$ for $t \in \mathcal{T}$
  - autocorrelation function $R_X(t_1, t_2\} = E\{X(t_1)X(t_2)\}$ for $t_1, t_2 \in \mathcal{T}$
- autocovariance function of a random process is defined as

$$C_X(t_1, t_2) = E\{(X(t_1) - E\{X(t_1)\})(X(t_2) - E\{X(t_2)\})\}$$

The autocovariance function can be expressed using the mean and autocorrelation functions as

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

Examples

- IID process $\{X_n\}$:

$$\mu_X(n) = E\{X_1\} = \int x f_{X_1}(x)\, dx$$

$$R_X(n_1, n_2) = E\{X_{n_1} X_{n_2}\} = \begin{cases} E\{X_1^2\} & \text{if } n_1 = n_2 \\ E\{X_{n_1}\}E\{X_{n_2}\} = \mu_X^2(n) & \text{if } n_1 \neq n_2 \end{cases}$$

- Random phase signal process: $X(t) = \alpha \cos(\omega t + \Theta)$

$$\mu_X(t) = E\{\alpha \cos(\omega t + \Theta)\} = \int_0^{2\pi} \frac{\alpha}{2\pi} \cos(\omega t + \theta)\, d\theta = 0$$

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{\alpha^2 \underbrace{\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)}_{0.5\cos(\omega(t_1 - t_2)) + 0.5\cos(\omega(t_1 + t_2) + 2\Theta)}\}$$

$$= \frac{\alpha^2}{2} \cos(\omega(t_1 - t_2)) + \int_0^{2\pi} \frac{\alpha^2}{4\pi} \cos(\omega(t_1 + t_2) + 2\theta)\, d\theta$$

$$= \frac{\alpha^2}{2} \cos(\omega(t_1 - t_2))$$

- Random Walk $\{X_n\}$: $X_n = \sum_{i=1}^{n} Z_i$

$$\mu_X(n) = E\{\sum_{i=1}^{n} Z_i\} = \sum_{i=1}^{n} E\{Z_i\} = \sum_{i=1}^{n} 0 = 0$$

$$\begin{aligned}
R_X(n_1, n_2) &= E\{X_{n_1} X_{n_2}\} \\
&\overset{\text{if } n_2 \geq n_1}{=} E\{X_{n_1}(X_{n_2} - X_{n_1} + X_{n_1})\} \\
&= E\{X_{n_1}\}E\{(X_{n_2} - X_{n_1})\} + E\{X_{n_1}^2\} \\
&= 0 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} E\{Z_i Z_j\} \\
&= \sum_{i=1}^{n_1} \underbrace{E\{Z_i^2\}}_{=1} + \sum_{i=1}^{n_1} \sum_{j=1, j \neq i}^{n_1} E\{Z_i\}E\{Z_j\} = n_1 + 0 = n_1
\end{aligned}$$

Considering both cases, $n_2 \geq n_1$ and $n_2 < n_1$, we have

$$R_X(n_1, n_2) = \min(n_1, n_2) = \begin{cases} n_1 & \text{if } n_1 \leq n_2 \\ n_2 & \text{if } n_1 > n_2 \end{cases}$$

- If $X \sim \text{Poisson}(\lambda)$, then $E\{X\} = \lambda$, $E\{X^2\} = \lambda(\lambda + 1)$
- Poisson Counting Process $N(t)$ with rate λ:

$$
\begin{aligned}
\mu_N(t) &= E\{N(t)\} = \lambda t \\
R_N(t_1, t_2) &= E\{N(t_1)N(t_2)\} \\
&\overset{\text{if } t_2 \geq t_1}{=} E\{N(t_1)(N(t_2) - N(t_1) + N(t_1))\} \\
&= E\{N(t_1)\}E\{(N(t_2) - N(t_1))\} + E\{N^2(t_1)\} \\
&= \lambda t_1 \times \lambda(t_2 - t_1) + \lambda t_1(\lambda t_1 + 1) \\
&= \lambda^2 t_1 t_2 + \lambda t_1
\end{aligned}
$$

Considering both cases, $t_2 \geq t_1$ and $t_2 < t_1$, we have

$$R_N(t_1, t_2) = \lambda \min(t_1, t_2) + \lambda^2 t_1 t_2$$