ELEC 7410 Solution: Homework Assignment #2 Sep. 9, 2025

1. $p = 0.95$, $q = 1 - p = 0.05$. Suppose she buys n (≥ 10) chips. Need at least 10 working chips. Let k denote the number of working chips out of n. Then

$$P_n(k \geq 10) = \sum_{k=10}^{n} \binom{n}{k} p^k q^{n-k} \overset{?}{>} 0.99$$

Solve by trial-and-error. Suppose $n = 10$. Then $P_{10}(k = 9) = (0.95)^{10} = 0.5987 < 0.99$. Let $n = 11$. Then

$$\begin{aligned} P_{11}(k \geq 10) =& \binom{11}{10}(0.95)^{10}(0.05)^1 + \binom{11}{11}(0.95)^{11}(0.05)^0 \\ =& 11 \times (0.95)^{10}(0.05) + (0.95)^{11} = 0.8981 < 0.99 \end{aligned}$$

Let $n = 12$. Then

$$\begin{aligned} P_{12}(k \geq 10) =& \binom{12}{10}(0.95)^{10}(0.05)^2 + \binom{12}{11}(0.95)^{11}(0.05)^1 + \binom{12}{12}(0.95)^{12}(0.05)^0 \\ =& 66 \times (0.95)^{10}(0.05)^2 + 12 \times (0.95)^{11}(0.05)^1 + (0.95)^{12} \\ =& 0.9804 < 0.99 \end{aligned}$$

Now let $n = 13$. Then

$$\begin{aligned} P_{13}(k \geq 10) =& \binom{13}{10}(0.95)^{10}(0.05)^3 + \binom{13}{11}(0.95)^{11}(0.05)^2 + \binom{13}{12}(0.95)^{12}(0.05)^1 \\ &+ \binom{13}{13}(0.95)^{13}(0.05)^0 \\ =& 286 \times (0.95)^{10}(0.05)^3 + 78 \times (0.95)^{11}(0.05)^2 + 13 \times (0.95)^{12}(0.05)^1 + (0.95)^{13} \\ =& 0.9969 > 0.99 \end{aligned}$$

Buy (at least) 13 chips.

2. (**Problem 15 in Gubner, Chapter 2**.) m galaxies with X_i denoting the number of black holes in the ith galaxy. Assume X_i's are independent and $X_i \sim$ Poisson(λ).

   (a) Need to find the probability of the set $\bigcup_{i=1}^{n}\{X_i \geq 2\}$. We have

$$P(\bigcup_{i=1}^{n}\{X_i \geq 2\}) = 1 - P(\bigcap_{i=1}^{n}\{X_i \leq 1\}) \overset{\text{independence}}{=} 1 - \prod_{i=1}^{n}\left(\sum_{k=0}^{1}\frac{\lambda^k e^{-\lambda}}{k!}\right) = 1 - e^{-n\lambda}\big(1 + \lambda\big)^n.$$

   (b) Need to find the probability of the set $\bigcap_{i=1}^{n}\{X_i \geq 1\}$. We have

$$P(\bigcap_{i=1}^{n}\{X_i \geq 1\}) \overset{\text{independence}}{=} \prod_{i=1}^{n}(1 - P(X_i = 0)) = \prod_{i=1}^{n}\big(1 - e^{-\lambda}\big) = \big(1 - e^{-\lambda}\big)^n.$$

(b) Need to find the probability of the set $\bigcap_{i=1}^n \{X_i = 1\}$. We have

$$P(\bigcap_{i=1}^{n}\{X_i = 1\}) \overset{\text{independence}}{=} \prod_{i=1}^{n} P(X_i = 1) = \prod_{i=1}^{n} \lambda e^{-\lambda} = \lambda^n \, e^{-n\lambda}\,.$$

3. (**Problem 18 in Gubner, Chapter 2**.) If $X \sim \text{geometric}_1(p)$, then $P(X = k) = (1-p)p^{k-1}$, $k = 1, 2, \cdots$. We have

$$P(\min(X_1, \cdots, X_n) > \ell) = P(\cap_{k=1}^n \{X_k > \ell\}) \overset{\text{independence}}{=} \prod_{k=1}^{n} P(X_k > \ell)$$

and $P(X_k > \ell) = \sum_{i=\ell+1}^{\infty} P(X_k = i) = (1-p) \sum_{i=\ell+1}^{\infty} p^{i-1} \overset{m=i-\ell-1}{=} (1-p)p^\ell \underbrace{\sum_{m=0}^{\infty} p^m}_{\frac{1}{1-p}} = p^\ell\,.$

Hence $P(\min(X_1, \cdots, X_n) > \ell) = \prod_{k=1}^{n} p^\ell = p^{n\ell}$

Similarly, $P(\max(X_1, \cdots, X_n) \le \ell) = P(\cap_{k=1}^n \{X_k \le \ell\}) \overset{\text{independence}}{=} \prod_{k=1}^{n} P(X_k \le \ell)$

$$= \prod_{k=1}^{n} \Big(1 - \underbrace{P(X_k > \ell)}_{p^\ell}\Big) = (1-p^\ell)^n$$

4. (**Problem 21 in Gubner, Chapter 2**.) Given $X \sim \text{geometric}_1(p)$

   (a) $P(X > n) = p^n$ is proved in Problem 3.

   (b) We have

$$P(\{X > n+k\}|\{X > n\}) = \frac{P(\{X > n+k\} \cap \{X > n\})}{P(\{X > n\})} = \frac{P(\{X > n+k\})}{P(\{X > n\})}$$
$$= \frac{p^{n+k}}{p^n} = p^k\,.$$

5. (**Problem 38 in Gubner, Chapter 2**.) Given $E\{X\} = m$ and $\text{var}(X) = \sigma^2$. Therefore, $E\{X^2\} = \sigma^2 + m^2$. For some constant c, we have

$$g(c) = E\{(X-c)^2\} = E\{X^2 - 2Xc + c^2\} = \sigma^2 + m^2 - 2mc + c^2\,.$$

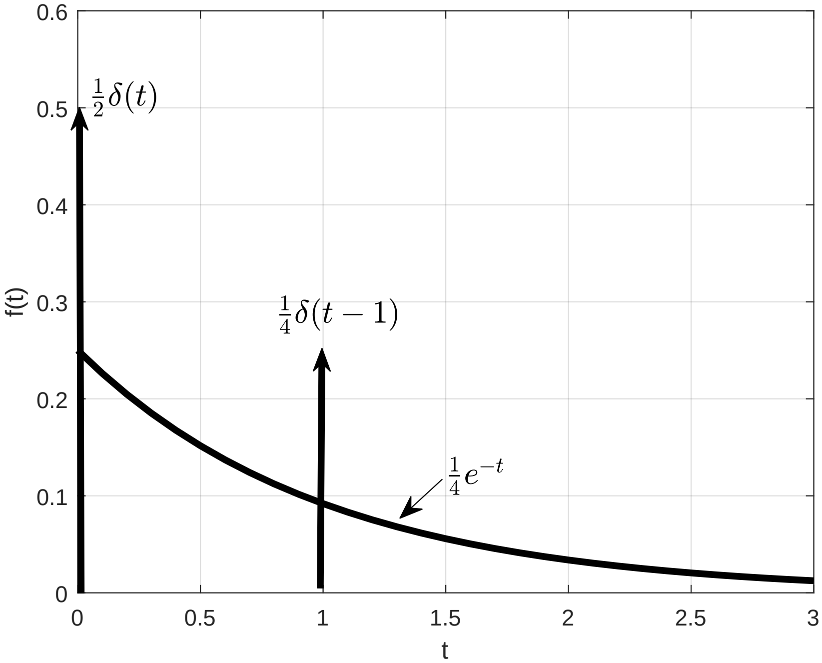
To minimize $g(c)$ with respect to c, we must have

$$0 = \frac{dg(c)}{dc} = -2m + 2c \quad \Rightarrow \quad c = m\,.$$

[Since $\frac{d^2 g(c)}{dc^2} = 2 > 0$, $c = m$ yields a minimum; $\frac{dg(c)}{dc} = 0$ is a necessary condition for both maximum and minimum.]

6. (**Similar to Problem 29 in Gubner, Chapter 5.**) $X \sim f_X(t) = \frac{1}{4}e^{-t}u(t) + \frac{1}{2}\delta(t) + \frac{1}{4}\delta(t-1)$

(a) Sketch



(b) $P(X=0) = \int_{0^-}^{0^+} f_X(t)\,dt = \frac{1}{2}$ and $P(X=1) = \int_{1^-}^{1^+} f_X(t)\,dt = \frac{1}{4}$

(c) $P(0 < X < 1) = \int_{0^+}^{1^-} f_X(t)\,dt = \frac{1}{4}\frac{e^{-t}}{-1}\Big|_0^1 = \frac{1}{4}(1-e^{-1})$ and $P(X > 1) = \int_{1^+}^{\infty} f_X(t)\,dt = \frac{1}{4}\frac{e^{-t}}{-1}\Big|_1^{\infty} = \frac{e^{-1}}{4}$

(d) $P(0 \le X \le 1) = P(X=0)+P(X=1)+P(0<X<1) = \frac{1}{2}+\frac{1}{4}+\frac{1}{4}(1-e^{-1}) = 1-\frac{e^{-1}}{4}$, and $P(X \ge 1) = P(X=1) + P(X>1) = \frac{1}{4} + \frac{e^{-1}}{4}$.

(e) $E\{X\} = \int_{0^-}^{\infty} t f_X(t)\,dt = 0 \cdot P(X=0) + 1 \cdot P(X=1) + \frac{1}{4}\underbrace{\int_0^{\infty} te^{-t}\,dt}_{1} = 0 + \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$