ELEC 7410 Solution: Homework Assignment #4 September 25, 2025

1. (**Problem 30, Chapter 5, Gubner**.) Given $X \sim \frac{1}{2}\delta(x) + \frac{1}{2}I_{(0,1]}(x) = f_X(x)$. We have

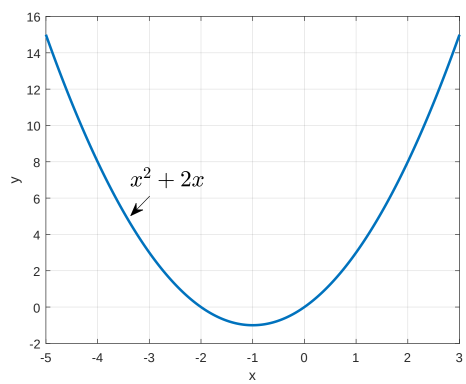
$$E\{e^X\} = \int_{-\infty}^{\infty} e^x f_X(x)\,dx = \frac{1}{2}e^0 + \frac{1}{2}\int_0^1 e^x\,dx = \frac{1}{2} + \frac{1}{2}e^x\Big|_0^1 = \frac{e}{2} = 1.3591\,.$$

$$P\Big(X = 0 \big| X \le \frac{1}{2}\Big) = \frac{P(\{X=0\}\cap\{X \le \frac{1}{2}\})}{P(X \le \frac{1}{2})} = \frac{P(X=0)}{P(X \le \frac{1}{2})} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}\int_0^{\frac{1}{2}} dx}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{4}} = \frac{2}{3}\,.$$

2. (**Problem 36, Chapter 5, Gubner**.) Given $X \sim$ uniform$[-3,1]$, $Y = X(X+2) = X^2 + 2X$. Therefore, $f_X(x) = 1/4$ for $-3 \le x \le 1$, and $= 0$ otherwise.

$$F_Y(y) = P(Y \le y) = P(X^2 + 2X - y \le 0).$$



For a fixed y, $x^2 + 2x - y = 0 \Rightarrow x = -1 \pm \sqrt{1+y}$ which is real for $1 + y \ge 0$. Hence, $F_Y(y) = 0$ for $y < -1$. Now $x^2 + 2x - y \le 0$ iff $(x + 1 + \sqrt{1+y})(x + 1 - \sqrt{1+y}) \le 0$. Hence $-1 - \sqrt{1+y} \le x \le -1 + \sqrt{1+y}$ leads to $x^2 + 2x - y \le 0$, so long as $1 + y \ge 0$. For $-3 \le x \le 1$, $\max(x^2 + 2x) = 3$ and $\min(x^2 + 2x) = -1$. Therefore,

$$F_Y(y) = \begin{cases} 0, & y < -1 \\ 1, & y > 3 \end{cases}\,.$$

For $-1 \le y \le 3$,

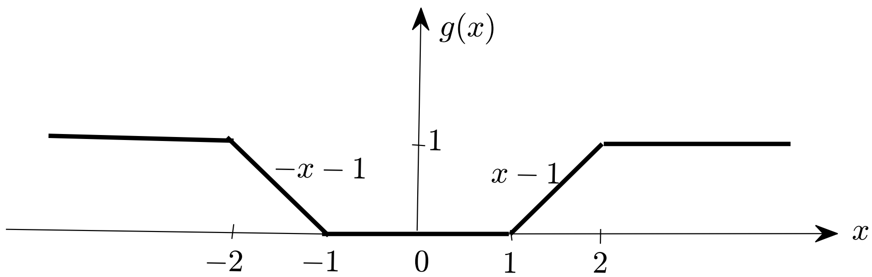
$$\begin{aligned} F_Y(y) =& P(-1 - \sqrt{1+y} \le X \le -1 + \sqrt{1+y}) \\ =& \frac{-1 + \sqrt{1+y} - (-1 - \sqrt{1+y})}{4} = \frac{\sqrt{1+y}}{2}\,. \end{aligned}$$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 0 & y < -1 \\ \frac{1}{4\sqrt{1+y}} & -1 \le y \le 3 \\ 0 & y > 3 \end{cases}\,.$$

3. (**Problem 42, Chapter 5, Gubner**.) Given

$$g(x) = \begin{cases} 0, & |x| < 1 \\ |x| - 1, & 1 \le |x| \le 2 \\ 1, & |x| > 2\,. \end{cases}$$

$Y = g(X)$.



(a) $X \sim$ uniform$[-1, 1] \Rightarrow f_X(x) = \frac{1}{2}$ for $-1 \le x \le 1$, and it is 0 otherwise. For $|x| \le 1$, $y = g(x) = 0$, therefore, $Y = 0$ with probability $P(-1 \le X \le 1) = 1$. Other values of y correspond to a null set. Hence

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ 1, & y \ge 0 \end{cases} \quad \Rightarrow \quad f_Y(y) = \delta(y)$$

(b) $X \sim$ uniform$[-2, 2] \Rightarrow f_X(x) = \frac{1}{4}$ for $-2 \le x \le 2$, and it is 0 otherwise. For $-1 \le x \le 1$, $y = 0$, for $1 < |x| \le 2$, $0 < y \le 1$, and for $|x| > 2$, $y = 1$. Thus,

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ 1, & y > 1 \end{cases}$$

We have

$$\begin{aligned} P(Y = 0) =& P(|X| \le 1) = \frac{2}{4} = \frac{1}{2} \\ P(0 < Y \le y) =& 2P(0 < X - 1 \le y) \text{ if } 0 < y \le 1 \\ =& 2\frac{y}{4} = \frac{y}{2} \text{ for } 0 < y \le 1 \end{aligned}$$

Hence

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{1+y}{2}, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases} \quad \Rightarrow \quad f_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{\delta(y)+1}{2}, & 0 \le y \le 1 \\ 0, & y > 1 \end{cases}$$

(c) $X \sim$ uniform$[-3, 3] \Rightarrow f_X(x) = \frac{1}{6}$ for $-3 \le x \le 3$, and it is 0 otherwise. As before,

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{1}{3}, & y = 0, & (\text{since } P(Y = 0) = P(|X| \le 1) = \frac{2}{6}) \\ \frac{1+y}{3}, & 0 < y \le 1, & (= P(Y = 0) + P(0 < Y \le y) \\ & & \quad = P(Y = 0) + 2P(1 < X \le y + 1)) \\ 1, & y > 1 \end{cases}$$

Therefore,

$$f_Y(y) = \frac{1}{3}\left[u(y) - u(y-1) + \delta(y) + \delta(y-1)\right], \quad -\infty < y < \infty$$

There are jumps in $F_Y(y)$ at $y = 0$ and at $y = 1$.

(d) $X \sim \text{Laplace}(\lambda) \Rightarrow f_X(x) = \frac{\lambda}{2}e^{-\lambda|x|}$. We have

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-\lambda}, & y = 0, \quad \text{(since } P(Y=0) = P(|X| \le 1) = 2 \times \frac{\lambda}{2}\int_0^1 e^{-\lambda x}dx) \\ 1 - e^{-\lambda(y+1)}, & 0 < y < 1, \quad (= P(Y=0) + P(0 < Y \le y) \text{ where } P(0 < Y \le y) \\ & \qquad = 2P(1 < X \le y+1) = e^{-\lambda} - e^{-\lambda(y+1)}) \\ 1, & y \ge 1 \end{cases}$$

Therefore,

$$f_Y(y) = \lambda e^{-\lambda(y+1)}\left[u(y) - u(y-1)\right] + (1-e^{-\lambda})\delta(y) + e^{-2\lambda}\delta(y-1), \quad -\infty < y < \infty$$

There are jumps in $F_Y(y)$ at $y = 0$ and at $y = 1$.

4. (**Problem 7, Chapter 7, Gubner**.) Given

$$F_{XY}(x,y) = \begin{cases} \frac{2}{7}(1 - e^{-2y}), & 2 \le x \le 3,\ y \ge 0 \\ \frac{7 - 2e^{-2y} - 5e^{-3y}}{7}, & x \ge 3,\ y \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

We have

$$F_X(x) = F_{XY}(x,\infty) = \begin{cases} \frac{2}{7}, & 2 \le x \le 3 \\ 1, & x \ge 3 \\ 0, & \text{otherwise} \end{cases}$$

$$F_Y(y) = F_{XY}(\infty, y) = \begin{cases} \frac{7 - 2e^{-2y} - 5e^{-3y}}{7}, & y \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

For X and Y to be independent, must have $F_{XY}(x,y) = F_X(x)F_Y(y)$ for every pair (x,y). This is **not true** here: $F_{XY}(x,y) \neq F_X(x)F_Y(y)$ for $2 \le x \le 3,\ y \ge 0$.

5. (**Problem 16, Chapter 7, Gubner**.) Given $Z = Y - X$. For a fixed y, $y - x \le z$ implies that $x \ge y - z$. We have

$$F_Z(z) = P(Y - X \le z) = \int_{-\infty}^{\infty}\left[\int_{y-z}^{\infty} f_{XY}(x,y)\,dx\right]dy$$

Using Leibniz rule,

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} f_{XY}(y-z, y)\,dy \overset{x=y-z}{=} \int_{-\infty}^{\infty} f_{XY}(x, x+z)\,dx$$

6. (**Problem 30, Chapter 7, Gubner**.) Given $X \sim \exp(\lambda)$, $Y \sim \exp(\mu)$ and $f_{XY}(x,y) = f_X(x)f_Y(y)$. We have

$$P(X \le Y) = \int_{-\infty}^{\infty} P(X \le y | Y = y) f_Y(y)\, dy = \int_{-\infty}^{\infty} P(X \le y) f_Y(y)\, dy$$

where the last equality above follows from independence of X and Y. Now

$$P(X \le y) = \begin{cases} 0 & \text{for } y < 0 \\ \int_0^y \lambda e^{-\lambda x}\, dx = 1 - e^{-\lambda y} & \text{for } y \ge 0 \end{cases}$$

Therefore,

$$P(X \le Y) = \int_0^{\infty} (1 - e^{-\lambda y}) \mu e^{-\mu y}\, dy = 1 - \mu \underbrace{\int_0^{\infty} e^{-(\lambda+\mu)y}\, dy}_{=1/(\lambda+\mu)} = \frac{\lambda}{\lambda + \mu}$$