ELEC 7410 Homework Assignment #3 Sept. 9, 2025

(Due: Sept. 16, 2025)

1. (10 points) (**Conditional Probability**) Problem 30, Chapter 3, Gubner.

   30. Let $X \sim \text{Poisson}(\lambda)$, and suppose that given $X = n$, $Y \sim \text{Bernoulli}(1/(n+1))$. Find $\mathsf{P}(X = n|Y = 1)$.

2. (10 points) (**Continuous Random Variables**) Problem 7, Chapter 4, Gubner.

   7. A certain computer is equipped with a hard drive whose lifetime, measured in months, is $X \sim \exp(\lambda)$. The lifetime of the monitor (also measured in months) is $Y \sim \exp(\mu)$. Assume the lifetimes are independent.

   (a) Find the probability that the monitor fails during the first 2 months.

   (b) Find the probability that both the hard drive and the monitor fail during the first year.

   (c) Find the probability that either the hard drive or the monitor fails during the first year.

   Add part (d):

   (d) Find the probability that the hard drive fails after the monitor fails, i.e., find $P(X > Y)$. Find its numerical value if $\mu^{-1} = 48$ months and $\lambda^{-1} = 24$ months.
   [Hint: Integrate $\int\int_{\{(x,y)\,:\,x>y\}} f_{XY}(xy)\,dx\,dy$]

3. (10 points) (**Expectation of a single random variable**) Problem 35, Chapter 4, Gubner.

   35. A small airline makes five flights a day from Chicago to Denver. The number of passengers on each flight is approximated by an exponential random variable with mean 20. A flight makes money if it has more than 25 passengers. Find the probability that at least one flight a day makes money. Assume that the numbers of passengers on different flights are independent.

4. (10 points) (**Transform Methods**) Problem 40, Chapter 4, Gubner.
   [Hint: Show that the moment generating function of Y is identical to that of $X \sim \exp(1)$.]

   40. If $X \sim \text{uniform}(0,1)$, show that $Y = \ln(1/X) \sim \exp(1)$ by finding its moment generating function for $s < 1$.

5. (10 points) (**Probability Bounds**) Problem 67, Chapter 4, Gubner.
[Hint: Continuation of (solved) Example 4.29 on p. 166 in Gubner; reproduced on p. 2.]

*67. Let X be an exponential random variable with parameter $\lambda = 1$. Compute the Markov inequality, the Chebyshev inequality, and the Chernoff bound to obtain bounds on $\mathsf{P}(X \geq a)$ as a function of a. Also compute $\mathsf{P}(X \geq a)$.

(a) For what values of a is the Markov inequality smaller than the Chebyshev inequality?

(b) **MATLAB.** Plot the Markov bound, the Chebyshev bound, the Chernoff bound, and $\mathsf{P}(X \geq a)$ for $0 \leq a \leq 6$ on the same graph. For what range of a is the Markov bound the smallest? the Chebyshev? Now use MATLAB command `semilogy` to draw the same four curves for $6 \leq a \leq 20$. Which bound is the smallest?

6. (10 points) (**Continuous Random Variables**) Problem 14, Chapter 5, Gubner.
[Hint: First find the cdf and the pdf of $Z = \max(X, Y)$.]

14. If X and Y are independent $\exp(\lambda)$ random variables, find $\mathsf{E}[\max(X,Y)]$.

***Example* 4.29.** Let X be a continuous random variable having exponential density with parameter $\lambda = 1$. Compute $\mathsf{P}(X \geq 7)$ and the corresponding Markov, Chebyshev, and Chernoff bounds.

Solution. The exact probability is $\mathsf{P}(X \geq 7) = \int_7^\infty e^{-x}\,dx = e^{-7} = 0.00091$. For the Markov and Chebyshev inequalities, recall that from Example 4.17, $\mathsf{E}[X] = 1/\lambda$ and $\mathsf{E}[X^2] = 2/\lambda^2$. For the Chernoff bound, we need $M_X(s) = \lambda/(\lambda - s)$ for $s < \lambda$, which was derived in Example 4.16. Armed with these formulas, we find that the Markov inequality yields $\mathsf{P}(X \geq 7) \leq \mathsf{E}[X]/7 = 1/7 = 0.143$ and the Chebyshev inequality yields $\mathsf{P}(X \geq 7) \leq \mathsf{E}[X^2]/7^2 = 2/49 = 0.041$. For the Chernoff bound, write

$$\mathsf{P}(X \geq 7) \;\leq\; \min_s e^{-7s}/(1-s),$$

where the minimization is over $0 \leq s < 1$. The derivative of $e^{-7s}/(1-s)$ with respect to s is

$$\frac{e^{-7s}(7s-6)}{(1-s)^2}.$$

Setting this equal to zero requires that $s = 6/7$. Hence, the Chernoff bound is

$$\mathsf{P}(X \geq 7) \;\leq\; \left.\frac{e^{-7s}}{(1-s)}\right|_{s=6/7} \;=\; 7e^{-6} \;=\; 0.017.$$