ELEC 7410 Homework Assignment #2 Aug. 28, 2025

(6 problems, Due: Sep. 9, 2025)

Any late homework submission will incur 15% penalty per day, with any fraction of a day counted as full day.

1. (10 points) (**Combinatorics**) A student needs 10 chips of a certain type to build a circuit. It is known that 5% of these chips are defective. How many chips should she buy for there to be a greater than 99% probability of having enough chips for the circuit?
[Hint: Solve iteratively, by trial-and-error. Suppose she buys n ($\geq$ 10) chips. What is the probability of having enough chips when n =10, or 11, or 12, $\cdots$?]

2. (10 points) (**Multiple Discrete Random Variables**) Problem 15, Chapter 2, Gubner.

   15. An astronomer has recently discovered n similar galaxies. For $i = 1,\ldots,n$, let X_i denote the number of black holes in the ith galaxy, and assume the X_i are independent Poisson(λ) random variables.

   (a) Find the probability that at least one of the galaxies contains two or more black holes.

   (b) Find the probability that all n galaxies have at least one black hole.

   (c) Find the probability that all n galaxies have exactly one black hole.

   Your answers should be in terms of n and λ.

   [Hint: Similar to (solved) Example 2.10 in Gubner, p. 72.]

   ***Example* 2.10.** A webpage server can handle r requests per day. Find the probability that the server gets more than r requests at least once in n days. Assume that the number of requests on day i is $X_i \sim \text{Poisson}(\lambda)$ and that $X_1,\ldots,X_n$ are independent.

   Solution. We need to compute

$$\begin{aligned}
\mathsf{P}\left(\bigcup_{i=1}^{n}\{X_i > r\}\right) &= 1 - \mathsf{P}\left(\bigcap_{i=1}^{n}\{X_i \le r\}\right) \\
&= 1 - \prod_{i=1}^{n}\mathsf{P}(X_i \le r) \\
&= 1 - \prod_{i=1}^{n}\left(\sum_{k=0}^{r}\frac{\lambda^k e^{-\lambda}}{k!}\right) \\
&= 1 - \left(\sum_{k=0}^{r}\frac{\lambda^k e^{-\lambda}}{k!}\right)^n.
\end{aligned}$$

3. (10 points) (**Multiple Discrete Random Variables**) Problem 18, Chapter 2, Gubner.

   18. Suppose that $X_1,\ldots,X_n$ are independent, geometric$_1(p)$ random variables. Evaluate $\mathsf{P}(\min(X_1,\ldots,X_n) > \ell)$ and $\mathsf{P}(\max(X_1,\ldots,X_n) \le \ell)$.

[Hint: Read p. 73 in Gubner: see page 3 of this assignment]

4. (10 points) Problem 21, Chapter 2, Gubner.

   21. Let $X \sim \text{geometric}_1(p)$.

   (a) Show that $\mathsf{P}(X > n) = p^n$.

   (b) Compute $\mathsf{P}(\{X > n+k\}|\{X > n\})$. *Hint:* If $A \subset B$, then $A \cap B = A$.

   Remark. Your answer to (b) should not depend on n. For this reason, the geometric random variable is said to have the **memoryless property**. For example, let X model the number of the toss on which the first heads occurs in a sequence of coin tosses. Then given a heads has not occurred up to and including time n, the conditional probability that a heads does not occur in the next k tosses does not depend on n. In other words, given that no heads occurs on tosses $1, \ldots, n$ has no effect on the conditional probability of heads occurring in the future. Future tosses do not remember the past.

   [Hint: Geometric random variable X ~geometric$_1$(p) is defined on p. 74 in Gubner: see page 4 of this assignment]

5. (10 points) (**Expectation**) Problem 38, Chapter 2, Gubner.

   38. Let X be a random variable with mean m and variance σ^2. Find the constant c that best approximates the random variable X in the sense that c minimizes the **mean-squared error** $\mathsf{E}[(X-c)^2]$.

6. (10 points) (**Mixed Random Variables**) Similar to Problem 29, Chapter 5, Gubner.

   29. A random variable X has generalized density

   $$f(t) = \tfrac{1}{4}e^{-t}u(t) + \tfrac{1}{2}\delta(t) + \tfrac{1}{4}\delta(t-1),$$

   where u is the unit step function defined in Section 2.1, and δ is the Dirac delta function defined in Section 5.3.

   (a) Sketch $f(t)$.

   (b) Compute $\mathsf{P}(X = 0)$ and $\mathsf{P}(X = 1)$.

   (c) Compute $\mathsf{P}(0 < X < 1)$ and $\mathsf{P}(X > 1)$.

   (d) Use your above results to compute $\mathsf{P}(0 \le X \le 1)$ and $\mathsf{P}(X \ge 1)$.

   (e) Compute $\mathsf{E}[X]$.

Max and min problems

Calculations similar to those in the preceding example can be used to find probabilities involving the maximum or minimum of several independent random variables.

***Example* 2.11.** For $i = 1, \ldots, n$, let X_i model the yield on the ith production run of an integrated circuit manufacturer. Assume yields on different runs are independent. Find the probability that the highest yield obtained is less than or equal to z, and find the probability that the lowest yield obtained is less than or equal to z.

***Solution*.** We must evaluate

$$\mathsf{P}(\max(X_1, \ldots, X_n) \leq z) \quad \text{and} \quad \mathsf{P}(\min(X_1, \ldots, X_n) \leq z).$$

Observe that $\max(X_1, \ldots, X_n) \leq z$ if and only if all of the X_k are less than or equal to z; i.e.,

$$\boxed{\{\max(X_1, \ldots, X_n) \leq z\} \;=\; \bigcap_{k=1}^{n} \{X_k \leq z\}.}$$

It then follows that

$$\begin{aligned}
\mathsf{P}(\max(X_1, \ldots, X_n) \leq z) &= \mathsf{P}\left(\bigcap_{k=1}^{n} \{X_k \leq z\}\right) \\
&= \prod_{k=1}^{n} \mathsf{P}(X_k \leq z),
\end{aligned}$$

where the second equation follows by independence.

For the min problem, observe that $\min(X_1, \ldots, X_n) \leq z$ if and only if at least one of the X_i is less than or equal to z; i.e.,

$$\boxed{\{\min(X_1, \ldots, X_n) \leq z\} \;=\; \bigcup_{k=1}^{n} \{X_k \leq z\}.}$$

Hence,

$$\begin{aligned}
\mathsf{P}(\min(X_1, \ldots, X_n) \leq z) &= \mathsf{P}\left(\bigcup_{k=1}^{n} \{X_k \leq z\}\right) \\
&= 1 - \mathsf{P}\left(\bigcap_{k=1}^{n} \{X_k > z\}\right) \\
&= 1 - \prod_{k=1}^{n} \mathsf{P}(X_k > z).
\end{aligned}$$

Geometric random variables

For $0 \le p < 1$, we define two kinds of **geometric** random variables.

We write $X \sim \text{geometric}_1(p)$ if

$$\mathsf{P}(X = k) \;=\; (1-p)p^{k-1}, \quad k = 1, 2, \ldots.$$

As the example below shows, this kind of random variable arises when we ask how many times an experiment has to be performed until a certain outcome is observed.

We write $X \sim \text{geometric}_0(p)$ if

$$\mathsf{P}(X = k) \;=\; (1-p)p^{k}, \quad k = 0, 1, \ldots.$$

This kind of random variable arises in Chapter 12 as the number of packets queued up at an idealized router with an infinite buffer. A plot of the $\text{geometric}_0(p)$ pmf is shown in Figure 2.7.

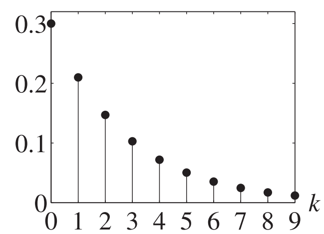

***Figure* 2.7.** The $\text{geometric}_0(p)$ pmf $p_X(k) = (1-p)p^k$ with $p = 0.7$.

By the geometric series formula (Problem 27 in Chapter 1), it is easy to see that the probabilities of both kinds of random variable sum to one (Problem 16).

If we put $q = 1 - p$, then $0 < q \le 1$, and we can write $\mathsf{P}(X = k) = q(1-q)^{k-1}$ in the $\text{geometric}_1(p)$ case and $\mathsf{P}(X = k) = q(1-q)^k$ in the $\text{geometric}_0(p)$ case.

***Example* 2.12.** When a certain computer accesses memory, the desired data is in the cache with probability p. Find the probability that the first cache miss occurs on the kth memory access. Assume presence in the cache of the requested data is independent for each access.

***Solution*.** Let $T = k$ if the first time a cache miss occurs is on the kth memory access. For $i = 1, 2, \ldots$, let $X_i = 1$ if the ith memory request is in the cache, and let $X_i = 0$ otherwise. Then $\mathsf{P}(X_i = 1) = p$ and $\mathsf{P}(X_i = 0) = 1 - p$. The key observation is that the *first* cache miss occurs on the kth access if and only if the first $k-1$ accesses result in cache hits and the kth access results in a cache miss. In terms of events,

$$\{T = k\} \;=\; \{X_1 = 1\} \cap \cdots \cap \{X_{k-1} = 1\} \cap \{X_k = 0\}.$$

Since the X_i are independent, taking probabilities of both sides yields

$$\begin{aligned}
\mathsf{P}(T = k) &= \mathsf{P}(\{X_1 = 1\} \cap \cdots \cap \{X_{k-1} = 1\} \cap \{X_k = 0\}) \\
&= \mathsf{P}(X_1 = 1) \cdots \mathsf{P}(X_{k-1} = 1) \cdot \mathsf{P}(X_k = 0) \\
&= \; p^{k-1}(1-p).
\end{aligned}$$