Multiple Random Variables

Lecture 4

Sept. 16, 18 and 23, 2025

Outline

Conditioning on an RV

- Let $(X, Y) \sim f_{XY}(x, y)$. We have seen that if $f_X(x) > 0$, the conditional pdf of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

- The conditional expectation of $g(X, Y)$ given $X = x$, is defined as

$$E\{g(X, Y) \,|\, X = x\} = \int_{-\infty}^{\infty} g(x, y) f_{Y|X}(y|x)\, dy$$

- Example: If $g(X, Y) = XY$, then $E\{XY \,|\, X = x\} = xE\{Y \,|\, X = x\}$.
- Conditional Expectation as an RV We define the conditional expectation of $g(X, Y)$ given X as the random variable $E\{g(X, Y) \,|\, X\}$, which is a function of the random variable X.
- In particular, $E\{Y|X\}$ is the conditional expectation of Y given X, a random variable that is a function of X.

Iterated Expectation

- In general we can find $E\{g(X,Y)\}$ using iterated expectation as

$$E\{g(X,Y)\} = E_Y\left\{E_X\{g(X,Y) \mid Y\}\right\}$$

where E_X means expectation w.r.t. $f_{X|Y}(x|y)$ and E_Y means expectation w.r.t. $f_Y(y)$. To show this, consider

$$\begin{aligned} E_Y\left\{E_X\{g(X,Y) \mid Y\}\right\} &= \int_{-\infty}^{\infty} E_X\{g(X,Y) \mid Y = y\} f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y)\, dx \right) f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y) f_Y(y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\, dx\, dy = E\{g(X,Y)\} \end{aligned}$$

- This result can be very useful in computing expectation.

- Example: A coin has random bias $P \in [0,1]$ with pdf $f_P(p) = 2(1-p)$. (The coin flips as heads with probability P.) The coin is flipped n times. Let N_h be the number of heads. Find $E\{N_h\}$.
Solution: We have

$$
\begin{aligned}
E\{N_h\} =& E_P\left\{E_{N_h}\{N_h \mid P\}\right\} = E_P\{nP\} \\
=& nE_P\{P\} = n\int_0^1 2(1-p)p\,dp = 2n\left[\frac{p^2}{2} - \frac{p^3}{3}\right]\Big|_0^1 = \frac{n}{3}
\end{aligned}
$$

- Example: Let $E\{X|Y\} = Y^2$ and $Y \sim U[0,1]$. Find $E\{X\}$.
Solution: We cannot first find the pdf of X, since we do not know $f_{X|Y}(x|y)$. But using iterated expectation, we have

$$
E\{X\} = E_Y\left\{E_X\{X \mid Y\}\right\} = E\{Y^2\} = \int_0^1 y^2\,dy = \frac{1}{3}
$$

Probability as Expectation

- Indicator Function Let $A \subseteq \mathcal{X}$. The indicator function of a subset A of a set $\mathcal{X}$ is a function $\mathbf{1} : A \to \{0, 1\}$ defined as

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A . \end{cases}$$

- In general, $\mathbf{1}_A(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in A$, $= 0$ otherwise, where $\boldsymbol{x}$ is a vector (or denotes multiple variables).
- Therefore, $P(X \in A) = \int_{x \in A} f_X(x)\, dx = E\{\mathbf{1}_A(X)\}$. For instance,

$$P(x_1 < X \leq x_2) = E\{\mathbf{1}_{(x_1, x_2]}(X)\} = \int_{x_1}^{x_2} f_X(x)\, dx .$$

- Suppose we need $P\{(X, Y) \in A\}$. Then $P\{(X, Y) \in A\} = E\{\mathbf{1}_A(X, Y)\}$. Using iterated expectation

$$\begin{aligned} P\{(X, Y) \in A\} &= E_Y\left\{E_X\{\mathbf{1}_A(X, Y) \,|\, Y\}\right\} \\ &= \int_{y \in A} \left[\int_{(x,y) \in A} f_{X|Y}(x|y)\, dx \right] f_Y(y)\, dy . \end{aligned}$$

Uncorrelation

- X and Y are said to be uncorrelated if $\mathrm{Cov}(X, Y) = 0$, equivalently, if $E\{XY\} = E\{X\}E\{Y\}$.
- If X and Y are independent then they are uncorrelated, since

$$
\begin{aligned}
E\{XY\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x, y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)\,dx\,dy \\
&= \int_{-\infty}^{\infty} yf_Y(y)\Big(\int_{-\infty}^{\infty} xf_X(x)\,dx\Big)dy \\
&= E\{X\}\int_{-\infty}^{\infty} yf_Y(y)\,dy = E\{X\}E\{Y\}
\end{aligned}
$$

- X and Y uncorrelated does not necessarily imply that they are independent.

- Example: Let $X, Y \in \{-2, -1, +1, +2\}$ such that

$$P_{XY}(x, y) = \begin{cases} \frac{2}{5} & (x, y) = (1, 1), (-1, -1) \\ \frac{1}{10} & (x, y) = (2, -2), (-2, 2) \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent? Are they uncorrelated?
Solution: It is easy to see that $P_X(-2) = 1/10 = P_Y(-2)$, $P_X(-1) = 2/5 = P_Y(-1)$, $P_X(1) = 2/5 = P_Y(1)$ and $P_X(2) = 1/10 = P_Y(2)$. Since $P_{XY}(2, -2) = \frac{1}{10} \neq P_X(2)P_Y(-2) = \frac{1}{10^2}$, X and Y are not independent. Let us check their covariance:

$$\begin{aligned} E\{X\} &= \frac{-2}{10} + \frac{-2}{5} + \frac{2}{5} + \frac{2}{10} = 0 \\ E\{Y\} &= \frac{-2}{10} + \frac{-2}{5} + \frac{2}{5} + \frac{2}{10} = 0 \\ E\{XY\} &= \frac{2}{5} + \frac{2}{5} + \frac{-4}{10} + \frac{-4}{10} = 0 \end{aligned}$$

Thus, $\text{Cov}(X, Y) = 0$, and X and Y are uncorrelated!

Correlation Coefficient

- The correlation coefficient of X and Y is defined as

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

- Fact: $|\rho_{XY}| \leq 1$.

$$E\left\{\left(\frac{X - E(X)}{\sigma_X} \pm \frac{Y - E(Y)}{\sigma_Y}\right)^2\right\} \geq 0$$

$$\Rightarrow \frac{E\{(X - E(X))^2\}}{\sigma_X^2} + \frac{E\{(Y - E(Y))^2\}}{\sigma_Y^2} \pm 2\frac{E\{(X - E(X))(Y - E(Y))\}}{\sigma_X \sigma_Y} \geq$$

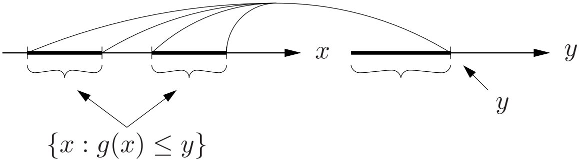
$$\Rightarrow 1 + 1 \pm 2\rho_{XY} \geq 0 \;\Rightarrow\; -2 \leq 2\rho_{XY} \leq 2 \;\Rightarrow\; |\rho_{XY}| \leq 1$$

- It follows from the proof that $\rho_{XY} = \pm 1$ iff $\frac{X-E(X)}{\sigma_X} = \pm \frac{Y-E(Y)}{\sigma_Y}$, i.e., iff $X - E(X)$ is a linear function of $Y - E(Y)$.

Functions of a Random Variable

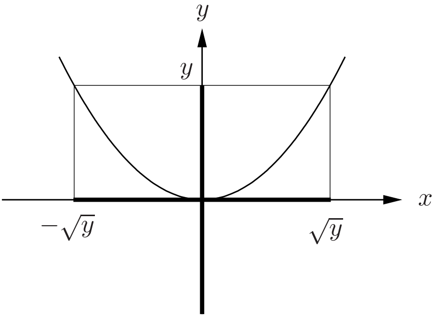
- Suppose we are given an r.v. X with known cdf $F_X(x)$ (or pdf $f_X(x)$) and a function $y = g(x)$. What is the cdf (or pdf) of the random variable $Y = g(X)$?
- Approach 1: Use

$$F_Y(y) = P(Y \leq y) = P(x : g(x) \leq y)$$



- Then $f_Y(y) = \frac{dF_Y(y)}{dy}$.

Example: Square law detector. Let $X \sim \mathcal{N}(0, \sigma^2)$ and $Y = X^2$. Find $F_Y(y)$ and $f_Y(y)$.



If $y < 0$, then clearly $F_Y(y) = 0$, hence, $f_Y(y) = 0$. Consider $y \geq 0$:

$$F_Y(y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) + \underbrace{P(X = -\sqrt{y})}_{=0}$$

$$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$$

- Approach 2: Suppose $X \sim f_X(x)$, $Y = g(X)$, and $g(x)$ is differentiable.
- For a fixed y, solve $g(x) = y$ for x. If there exists no real-valued solution, then $f_Y(y) = 0$.
- Else, let there be n solutions $x_1, x_2, \cdots, x_n$ satisfying $g(x_i) = y$. Then

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{\left|g'(x_i)\right|}$$

  if $g'(x_i) \neq 0$ for any i, where $g'(x_i) = \frac{d\,g(x)}{dx}\Big|_{x=x_i}$.
- This method fails if $g'(x_i) = 0$

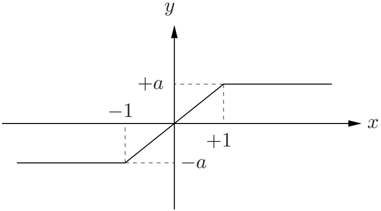
Example: Square law detector. Let $X \sim \mathcal{N}(0, \sigma^2)$ and $Y = X^2$. Find $F_Y(y)$ and $f_Y(y)$.

- If $y < 0$, then there is no real-valued solution to $x^2 = y$, hence, $f_Y(y) = 0$.
- Consider $y > 0$. Then $x = \pm\sqrt{y}$ satisfies the quadratic $x^2 = y$. Let $x_1 = \sqrt{y}$ and $x_2 = -\sqrt{y}$. We have $g(x) = x^2 \Rightarrow g'(x) = 2x$ Therefore,

$$f_Y(y) = \frac{f_X(x_1)}{|2x_1|} + \frac{f_X(x_2)}{|2x_2|} = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$$

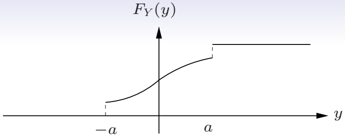
- When $y = 0$, $x = x_1 = 0$ but $g'(x_1) = 0$, so can't use the formula. But since X is a continuous RV, we can ignore $f_Y(y)$ at $y = 0$ and set it to either $0 = f_Y(0^-)$ or $f_Y(0^+)$.

Example: Limiter. Let $X \sim \text{Laplace}(1)$, i.e., $f_X(x) = \frac{1}{2}e^{-|x|}$, and Y be defined by the function of X shown in the figure. Find $F_Y(y)$ and $f_Y(y)$.



Consider the following cases:

- $\boxed{y < -a}$ Clearly $F_Y(y) = 0$
- $\boxed{y = -a}$ $F_Y(-a) = F_X(-1) = \int_{-\infty}^{-1} 0.5e^{x}\,dx = 0.5e^{-1}$
- $\boxed{-a < y < a}$ $F_Y(y) = P(Y \le y) = P(aX \le y) = F_X(y/a) = 0.5e^{-1} + \int_{-1}^{y/a} 0.5e^{-|x|}\,dx$
- $\boxed{y \ge a}$ $F_Y(y) = F_X(\infty) = 1$



We have

$$F_Y(y) = \begin{cases} 0 & y < -a \\ 0.5e^{-1} = 0.1839 & y = -a \\ 0.5e^{y/a} & -a < y \leq 0 \\ 1 - 0.5e^{-y/a} & 0 < y < a \\ 1 & y \geq a \end{cases}$$

Thus, there is a jump of $0.5e^{-1}$ at $y = -a$ as well as at $y = a$. The pdf of Y is given by

$$f_Y(y) = \begin{cases} 0 & y < -a \\ 0.5e^{-1}\delta(y+a) & y = -a \\ \frac{1}{2a}e^{y/a} & -a < y \leq 0 \\ \frac{1}{2a}e^{-y/a} & 0 < y < a \\ 0.5e^{-1}\delta(y-a) & y = a \\ 0 & y > a \end{cases}$$

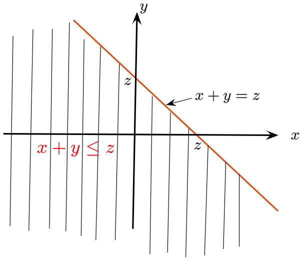
Functions of Two Random Variables

- Suppose we are given two Rvs X and Y with known joint cdf $F_{XY}(x, y)$ (or pdf $f_{XY}(x, y)$) and a function $z = g(x, y)$. What is the cdf (or pdf) of the random variable $Z = g(X, Y)$?
- Use

$$\begin{aligned} F_Z(z) =& P(Z \leq z) = P((x, y) : g(x, y) \leq z) \\ &= \int\int_{\{(x,y):g(x,y)\leq z\}} f_{XY}(x, y)\, dx\, dy \end{aligned}$$

- Then $f_Z(z) = \frac{dF_Z(z)}{dz}$.

Example: Sum of Two RVs. Let $Z = X + Y$ and $(X, Y) \sim f_{XY}(x, y)$. Find $f_Z(z)$.



$$F_Z(z) = \int\int_{\{(x,y)\,:\,x+y\leq z\}} f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z-y} f_{XY}(x,y)\,dx\right]dy$$

$$\Rightarrow\ f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} f_{XY}(z-y,y)\,dy$$

If X and Y are independent, then $f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)\,dy$. This is convolution of $f_X(.)$ with $f_Y(.)$