ELEC 7410 Solution: Homework Assignment #3 Sept. 16, 2025

1. (**Problem 30 in Gubner, Chapter 3**.)

$$X \sim \text{Poisson}(\lambda) \;\Rightarrow\; P(X=n) = \frac{\lambda^n e^{-\lambda}}{n!}, \quad n = 0, 1, \cdots$$

$$Y\Big|_{X=n} \sim \text{Bernoulli}(\frac{1}{n+1}) \;\Rightarrow\; P(Y=1|X=n) = \frac{1}{n+1}.$$

$$\begin{aligned} P(Y=1) &= \sum_{n=0}^{\infty} P(Y=1|X=n)P(X=n) = \sum_{n=0}^{\infty} \frac{1}{n+1} \times \frac{\lambda^n e^{-\lambda}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{(n+1)!} \overset{m=n+1}{=} \sum_{m=1}^{\infty} \frac{\lambda^{m-1} e^{-\lambda}}{m!} = \frac{1}{\lambda}\Big[\underbrace{\sum_{m=0}^{\infty} \frac{\lambda^m e^{-\lambda}}{m!}}_{=1} - e^{-\lambda}\Big] = \frac{1-e^{-\lambda}}{\lambda}. \end{aligned}$$

By Bayes rule,

$$\begin{aligned} P(X=n|Y=1) &= \frac{P(Y=1|X=n)P(X=n)}{P(Y=1)} = \frac{\frac{1}{n+1}\frac{\lambda^n e^{-\lambda}}{n!}}{\frac{1-e^{-\lambda}}{\lambda}} \\ &= \frac{\lambda^{n+1}}{(e^{\lambda}-1)(n+1)!}, \quad n = 0, 1, \cdots \end{aligned}$$

2. (**Problem 7 (+ part (d)) in Gubner, Chapter 4**.) Given

$$\begin{aligned} X &\sim \exp(\lambda) \;\Rightarrow\; f_X(x) = \lambda e^{-\lambda x} u(x) \\ Y &\sim \exp(\mu) \;\Rightarrow\; f_Y(y) = \mu e^{-\mu y} u(y) \\ f_{XY}(x,y) &= f_X(x) f_Y(y) \end{aligned}$$

(a) $P(Y \le 2) = \int_{-\infty}^{2} f_Y(y)\,dy = \int_0^2 \mu e^{-\mu y}\,dy = \mu \frac{e^{-\mu y}}{-\mu}\Big|_0^2 = 1 - e^{-2\mu}$

(b) $P(X \le 12, Y \le 12) = P(X \le 12)P(Y \le 12) = (1-e^{-12\lambda})(1-e^{-12\mu})$

(c) Use DeMorgan's law: $P(\{X \le 12\} \cup \{Y \le 12\}) = 1 - P(X > 12, Y > 12) = 1 - P(X > 12)P(Y > 12) = 1 - e^{-12(\lambda+\mu)}$

(d) Given $\mu^{-1} = 48$ months and $\lambda^{-1} = 24$. We have

$$\begin{aligned} P(X > Y) &= \int\int_{\{(x,y)\,:\,x>y\}} f_{XY}(xy)\,dx\,dy = \int_0^{\infty} \Big[\underbrace{\int_y^{\infty} \lambda e^{-\lambda x}\,dx}_{=e^{-\lambda y}}\Big] \mu e^{-\mu y}\,dy \\ &= \int_0^{\infty} \mu e^{-(\lambda+\mu)y}\,dy = \frac{\mu}{\lambda+\mu} = \frac{48^{-1}}{24^{-1}+48^{-1}} = \frac{1}{3} = 0.333... \end{aligned}$$

3. (**Problem 35 in Gubner, Chapter 4**.) Let X_i denote the number of passengers in the ith flight. Given $X_i \sim \exp(\lambda)$, $\lambda^{-1} = 20$, and X_i's are independent. There are five flights a day. Need to calculate the probability of the set

$$\{\text{At least one flight a day makes money}\} = \bigcup_{i=1}^{5} \{X_i > 25\}.$$

We have

$$P(\bigcup_{i=1}^{5}\{X_i > 25\}) = 1 - P(\bigcap_{i=1}^{5}\{X_i \le 25\}) = 1 - \prod_{i=1}^{5} P(X_i \le 25)$$
$$= 1 - \prod_{i=1}^{5}\left(1 - e^{-25\lambda}\right) = 1 - (1 - e^{-1.25})^5 = 0.8151$$

4. (**Problem 40 in Gubner, Chapter 4**.) Given

$$X \sim \text{uniform}(0,1) = \text{U}(0,1) \;\Rightarrow\; f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$
$$Y = \ln(1/X) = -\ln(X)$$

Need to show that the moment generating function (MGF) $M_Y(s)$ of Y is the same as that of an exp(1) random variable. Then $Y \sim$ exp(1). We have

$$M_Y(s) = E\{e^{sY}\} = E\{e^{-s\ln(X)}\} = E\{e^{\ln(X^{-s})}\} = E\{X^{-s}\}$$
$$= \int_{-\infty}^{\infty} x^{-s} f_X(x)\,dx = \int_0^1 x^{-s}\,dx = \frac{x^{-s+1}}{1-s}\Big|_0^1 = \frac{1}{1-s} \text{ if } s-1<0$$

The MGF of $Z \sim$ exp(1) is

$$M_Z(s) = E\{e^{sZ}\} = \int_0^{\infty} e^{sz}e^{-z}\,dx = \frac{e^{(s-1)z}}{s-1}\Big|_0^{\infty} = \frac{1}{1-s} \text{ if } s-1<0$$

Compare the two expressions to conclude that $Y = \ln(1/X) \sim$ exp(1)

5. (**Problem 67 in Gubner, Chapter 4**.) Given $X \sim$ exp$(1) \Rightarrow f_X(x) = e^{-x}u(x)$. Then $E\{X\} = 1$, $E\{X^2\} = 2$, $F_X(a) = 1 - e^{-a}$.

   - **Exact value**: $P(X \ge a) = 1 - F_X(a) = e^{-a}$
   - **Markov Inequality**: $P(X \ge a) \le \frac{E\{X\}}{a} = \frac{1}{a}$
   - **Chebyshev Inequality**: $P(X \ge a) \le \frac{E\{X^2\}}{a^2} = \frac{2}{a^2}$
   - **Chernoff Bound**: For any $a > 0$,

$$P(X \ge a) \le \min_{s\ge 0}\left[e^{-as}E\{e^{sX}\}\right]$$

We have

$$E\{e^{sX}\} = \int_0^\infty e^{sx}e^{-x}\,dx = \frac{e^{(s-1)x}}{s-1}\Big|_{x=0}^{\infty} = \frac{1}{1-s} \text{ if } 1-s>0$$

Now $P(X \geq a) \leq \min_{0\leq s<1} g(s)$ where $g(s) = e^{-as}/(1-s)$. We have

$$\frac{dg(s)}{ds} = \frac{(1-s)(-ae^{-as}) - e^{-as}(-1)}{(1-s)^2} = \frac{e^{-as}(1-a(1-s))}{(1-s)^2} = 0$$

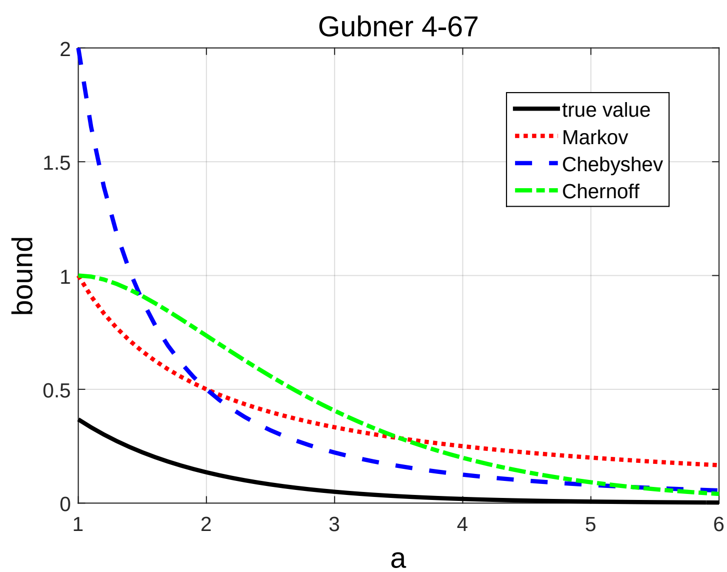
$$\Rightarrow 1 - a(1-s) = 0 \Rightarrow s = \frac{a-1}{a}$$

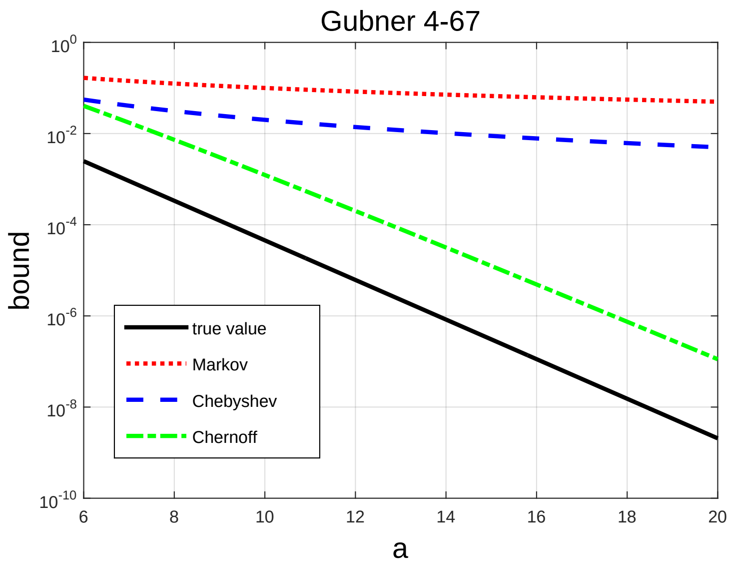
Hence, $P(X \geq a) \leq g(\frac{a-1}{a}) = e^{-as}/(1-s)\Big|_{s=(a-1)/a}$. But the bound expression is valid only for $0 \leq s < 1$, therefore, must have $a \geq 1$ for $s \geq 0$.
For $a \geq 1$, the Chernoff bound is

$$P(X \geq a) \leq \frac{e^{-a(a-1)/a}}{1-(a-1)/a} = ae^{1-a}\,.$$

(a) For Markov inequality to be smaller than the Chebyshev inequality, must have $\frac{1}{a} < \frac{2}{a^2}$, that is, $a < 2$.

(b) Plots (linear and semilog scales) are for $a \geq 1$ since the Chernoff bound does not exist for $a < 1$.
The Markov bound is the smallest for $a \leq 2$. The Chebyshev bound is the smallest for approximately $2 < a < 5$.





6. (**Problem 14 in Gubner, Chapter 5**.) Let $Z = \max(X, Y)$ where $X, Y \sim \exp(\lambda)$ and X and Y are independent. Therefore, $F_X(x) = F_Y(x) = (1 - e^{-\lambda x})u(x)$ where $u(x)$ is the unit step function. We have

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(\{X \le z\} \cap \{Y \le z\}) = P(X \le z)P(Y \le z) \\ &= F_X(z)F_Y(z) = \left(1 - e^{-\lambda z}\right)^2 u(z)\,. \end{aligned}$$

The pdf is given by

$$f_Z(z) = \frac{dF_Z(z)}{dz} = 2\lambda e^{-\lambda z}\left(1 - e^{-\lambda z}\right)u(z)\,.$$

Therefore

$$\begin{aligned} E\{Z\} &= \int_0^\infty 2z\lambda e^{-\lambda z}\left(1 - e^{-\lambda z}\right)dz = 2\underbrace{\int_0^\infty z\lambda e^{-\lambda z}\,dz}_{=1/\lambda} - \underbrace{\int_0^\infty z(2\lambda e^{-2\lambda z})\,dz}_{=1/(2\lambda)} \\ &= \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}\,. \end{aligned}$$