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1) $X \sim \text{Poisson}(\lambda)$, Given $X = n$, $Y \sim \text{Bernoulli}(1/(n+1))$,
Find $P(X=n \mid Y=1)$

$$P(X=n \mid Y=1) = \frac{P(Y=1 \mid X=n)\, P(X=n)}{P(Y=1)}$$

$$= \frac{\frac{1}{n+1}\left(\frac{\lambda^n}{n!}e^{-\lambda}\right)}{\sum_{n=0}^{\infty} P(Y=1 \mid X=n) P(X=n)}$$

$$= \frac{\frac{1}{n+1}\left(\frac{\lambda^n}{n!}e^{-\lambda}\right)}{\sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{\lambda^n}{n!}e^{-\lambda}\right)}$$

$$= \frac{\frac{1}{n+1}\left(\frac{\lambda^n}{n!}e^{-\lambda}\right)}{e^{-\lambda}\sum_{n=0}^{\infty} \frac{\lambda^n}{n!(n+1)}}$$

$$= \frac{\frac{1}{n+1}\left(\frac{\lambda^n}{n!}e^{-\lambda}\right)}{e^{-\lambda}\sum_{n=0}^{\infty} \frac{\lambda^n}{(n+1)!}}$$

$$= \frac{\frac{1}{n+1}\left(\frac{\lambda^n}{n!}e^{-\lambda}\right)}{e^{-\lambda}\sum_{Y=1}^{\infty} \frac{\lambda^{Y-1}}{Y!}}$$

$$= \frac{\frac{1}{n+1}\left(\frac{\lambda^n}{n!}e^{-\lambda}\right)}{\frac{e^{-\lambda}}{\lambda}\sum_{Y=1}^{\infty} \frac{\lambda^Y}{Y!}}$$

= Taylor Series Expansion w/out x_0 (e^x)

$$\frac{\frac{1}{n+1}\left(\frac{\lambda^n}{n!}e^{-\lambda}\right)}{e^{-\lambda}\left(\frac{e^{\lambda}-1}{\lambda}\right)}$$

Simplify

$$\boxed{P(X=n \mid Y=1) = \frac{\lambda^{n+1}e^{-\lambda}}{(n+1)!\,(1-e^{-\lambda})}}$$

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2) A) FIND PROB. THAT Y FAILS IN 2 MONTHS

$Y \sim \exp(\mu)$

$$P(y \leq 2) = \int_0^2 \mu e^{-\mu x} dx$$

$$= \left[-e^{-\mu x}\right]_0^2 = \left[-e^{-2\mu} - (-1)\right]$$

$$\boxed{P(y \leq 2) = 1 - e^{-2\mu}}$$

B) FIND PROB THAT BOTH X & Y FAIL WITHIN A YEAR

$$P(X \leq 12 \cap Y \leq 12) = \underset{\text{INDEPENDENT}}{P(X \leq 12) P(Y \leq 12)} \qquad X \sim \exp(\lambda)$$

$$\boxed{P(X \leq 12 \cap Y \leq 12) = \text{FOLLOWING FORM SOLVED IN A)} \quad (1 - e^{-12\lambda})(1 - e^{-12\mu})}$$

C) FIND PROB. EITHER X OR Y FAIL WITHIN A YEAR

$$P(X \leq 12 \cup Y \leq 12) = 1 - P(X > 12 \cap Y > 12)$$

$$= 1 - P(X > 12) P(Y > 12)$$

$$= 1 - e^{-12\lambda} e^{-12\mu}$$

$$\boxed{= 1 - e^{-12(\lambda - \mu)}}$$

$$P(x > 12) = \int_{12^+}^{\infty} \lambda e^{-\lambda x} dx$$

$$= \left[-e^{-\lambda x}\right]_{12^+}^{\infty}$$

$$= 0 - (-e^{-12\lambda})$$

$$= e^{-12\lambda}$$

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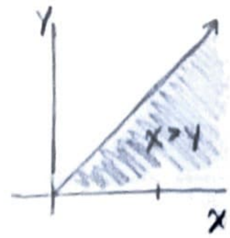
D) FIND PROB IF X FAILS AFTER Y. GIVEN $1/\mu = 48$; $1/\lambda = 24$

SINCE INDEPENDENT $f_{XY}(x,y) = f_X(x) f_Y(y)$

$$= \lambda e^{-\lambda x} \mu e^{-\mu y}$$

$$= \lambda \mu e^{-\lambda x} e^{-\mu y}$$



$$P(X>Y) = \int_{y=0}^{\infty} \int_{x=y}^{\infty} \lambda \mu e^{-\lambda x} e^{-\mu y} \, dx \, dy$$

$$= \int_{y=0}^{\infty} \mu e^{-(\lambda+\mu)y} \, dy$$

$$= \frac{\mu}{-(\lambda+\mu)} e^{-(\lambda+\mu)y} \Big]_{y=0}^{\infty}$$

$$= \frac{\mu}{(\lambda+\mu)}$$

$$= \frac{(1/48)}{(1/24 + 1/48)}$$

$$\boxed{P(X>Y) = 1/3}$$

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3) FIND PROB AT LEAST ONE FLIGHT A DAY MAKES MONEY

$$\left.\begin{array}{l} X \sim \exp(\lambda) \\ E[X] = 1/\lambda = 20 \end{array}\right\} X \sim \exp(1/20)$$

$$P\left(\bigcup_{i=1}^{5} (X_i > 25)\right) = 1 - P\left(\bigcap_{i=1}^{5} (X_i \leq 25)\right)$$

$$\overset{\text{INDEPENDENT}}{=} 1 - \prod_{i=1}^{5} P(X_i \leq 25)$$

$$= 1 - (1 - e^{-5/4})^5$$

$$\boxed{P\left(\bigcup_{i=1}^{5} (X_i > 25)\right) = 0.815}$$

$$P(X \leq 25) = \int_0^{25} \tfrac{1}{20} e^{-x/20}\, dx$$

$$= \left[-e^{-x/20}\right]_0^{25}$$

$$= \left(-e^{-25/20} - (-1)\right)$$

$$= 1 - e^{-5/4}$$

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4) SHOW MOMENT GENERATING FUNCTION OF Y IS IDENTICAL TO $X \sim EXP(1)$

$$M_x = \int_{-\infty}^{\infty} e^{sx} e^{-x}\,dx = \int_{0}^{\infty} e^{(s-1)x}\,dx = \left[\frac{e^{(s-1)x}}{s-1}\right]_0^{\infty}$$

$$M_x = \frac{1}{1-s}$$

$X \sim$ UNIFORM $(0,1)$ $\quad Y = \ln(1/x) = -\ln(x)$

$$M_Y = \int_{-\infty}^{\infty} e^{sy} f_x(x)\,dx = \int_{-\infty}^{\infty} e^{-s\ln x} f_x(x)\,dx$$

$$= f_x(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{ELSE} \end{cases}$$

$$\int_0^1 x^{-s} \cdot 1\,dx$$

$$= \left.\frac{x^{(-s+1)}}{(-s+1)}\right]_0^1$$

$$= \frac{1-0}{1-S}$$

$$M_Y = \frac{1}{1-S}$$

$$\boxed{M_Y = M_x \Rightarrow Y \sim X \sim \exp(1)}$$

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5) A) $X \sim \exp(1)$, COMPUTE MARKOV INEQUALITY, CHEBYSHEV BOUND, CHERNOFF BOUND ON $P(X \geq a)$. FIND $P(X \geq a)$

MARKOV INEQUALITY

$$E\{X\} = 1/\lambda = 1$$

$$P(X \geq a) \leq \frac{E\{X\}}{a} = \frac{1}{a} \Rightarrow \boxed{P(X \geq a) \leq \frac{1}{a}}$$

CHEBYSHEV INEQUALITY

$$\sigma_x^2 = 1/\lambda^2 = 1$$

$$P(X \geq a) \leq P(|X - E\{X\}| \geq a - E\{X\}) = \frac{\sigma_x^2}{(a - E\{X\})^2}$$

$$\boxed{P(X \geq a) \leq \frac{1}{(a-1)^2}}$$

CHERNOFF BOUND

M.G.F SOLVED IN QUESTION #4

$$P(X \geq a) \leq \min_{s \geq 0} \left[e^{-as} \underbrace{E\{e^{sX}\}} \right]$$

$$\leq \min_{s \geq 0} \left[\underbrace{e^{-as} \left(\frac{1}{1-s} \right)}_{y(s)} \right]$$

$$\frac{dy(s)}{ds} = -ae^{-as} \left(\frac{1}{1-s} \right) + e^{-as} \left(\frac{1}{(1-s)^2} \right) = 0$$

$$\frac{-a}{1-s} + \frac{1}{(1-s)^2} = 0$$

$$\frac{-a + as + 1}{(1-s)^2} = 0$$

$$s = \frac{a-1}{a} = 1 - \frac{1}{a}$$

$$P(X \geq a) \leq y\left(\frac{a-1}{a}\right) = e^{-a+1} \left(\frac{1}{1/a} \right)$$

$$\boxed{P(X \geq a) = ae^{-a+1}}$$

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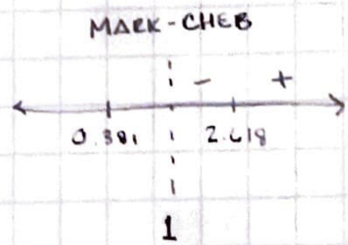
$P(X \geq a)$ TRUE

$$P(X \geq a) = \int_a^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_a^{\infty}$$
$$= 0 - (-e^{-a})$$
$$\boxed{P(X \geq a) = e^{-a}}$$

WHERE DOES MARKOV < CHEBYSHEV

$$\frac{1}{a} = \frac{1}{(a-1)^2}$$
$$a = a^2 - 2a + 1$$
$$a^2 - 3a + 1 = 0$$
$$a = \frac{3 \pm \sqrt{9 - 4(1)(1)}}{2}$$



$$a = \frac{3+\sqrt{5}}{2}, \quad \frac{3-\sqrt{5}}{2}$$

MUST SATISFY $a > 1$ FOR CHEBYSHEV

$$\boxed{a > \frac{3+\sqrt{5}}{2} = 2.618}$$

B) MATLAB PLOTS SHOWN ON NEXT PAGE

	RANGE OF a	
TYPE	$0 \leq a \leq 6$	$6 \leq a \leq 20$
MARKOV	$(0, 2.618)$	N/A
CHEBYSHEV	$(2.618, 6]$	$[6, 6.024)$
CHERNOFF	N/A	$(6.024, 20]$ *SMALLEST*

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6) IF X & Y ARE INDEPENDENT ~ exp(λ), FIND $E[\max(x,y)]$

CDF. $Z = \max(x,y)$

$$F_Z(z) = P(\max(x,y) \le z) = P(X \le z \cap Y \le z)$$

= INDEPENDENT

$$P(x \le z) P(x \le z)$$

$$= (1 - e^{-\lambda z})(1 - e^{-\lambda z})$$

$$F_z(z) = (1 - e^{-\lambda z})^2$$

$$\frac{dF_z(z)}{dz} = f_z(z) = 2(-\lambda)(-e^{\lambda z})(1 - e^{-\lambda z})$$

$$f_z(z) = 2\lambda e^{-\lambda z}(1 - e^{-\lambda z}) \quad z \ge 0$$

$$E[Z] = \int_0^\infty z f_z(z)\, dz = \int_0^\infty z(2\lambda e^{-\lambda z}(1 - e^{-\lambda z}))\, dz$$

$$= 2\lambda \left[\int_0^\infty z e^{-\lambda z} dz - \int_0^\infty z e^{-2\lambda z} dz \right]$$

* SOLVED W/ WOLFRAM ALPHA *

$$= 2\lambda \left[1/\lambda^2 - 1/4\lambda^2 \right]$$

$$= 2\lambda \left[3/4\lambda^2 \right]$$

$$\boxed{E[\max(X,Y)] = \frac{3}{2}\lambda}$$