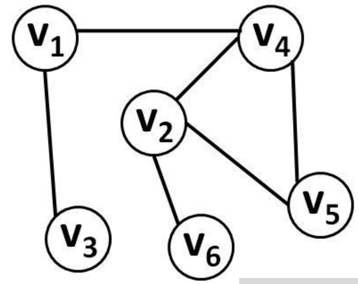
Graphs

Introduction

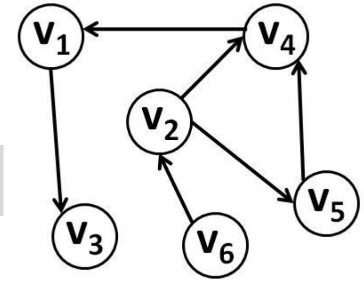
- Generalization of a tree.
- Collection of vertices (or nodes) and connections between them.

- No restriction on
  - The number of vertices.
  - The number of connections between the two vertices.
- Have several real life applications.

Definition



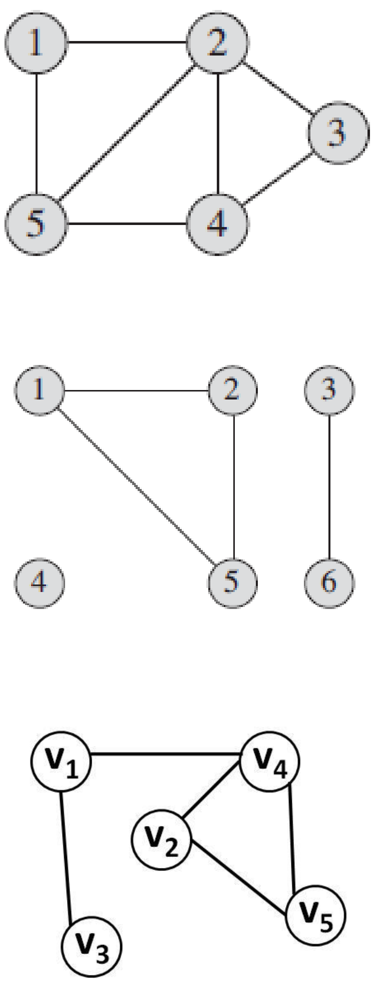
Undirected Graph



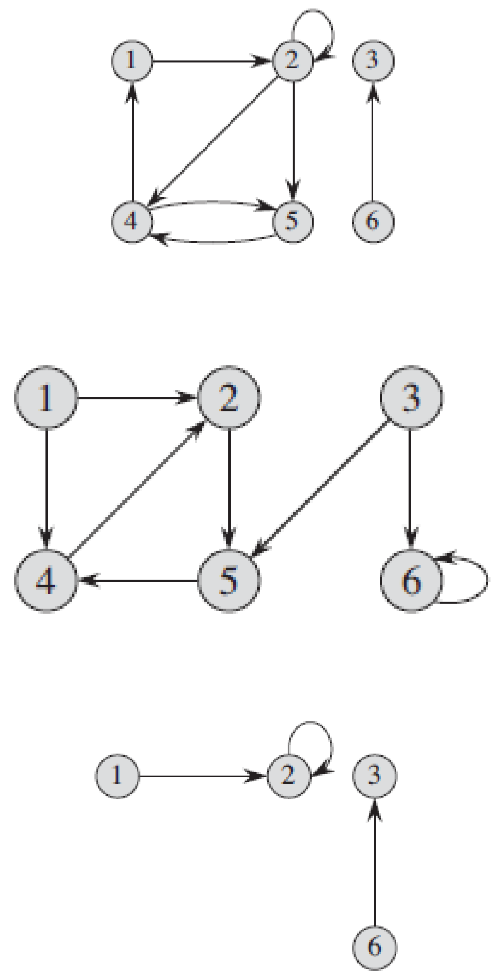
Directed Graph

- A graph ***G* = (*V*,*E*)** consists of a
  - Finite, non-empty set ***V*** of ***vertices*** and
  - Possibly empty set ***E*** of ***edges***. A binary relation on ***V***.
- **|*V*|** denotes number of vertices.
- **|*E*|** denotes number of edges.
- An edge (or arc) is a pair of vertices (v_i, v_j) from ***V***.
  - Simple or undirected graph $(v_i, v_j) = (v_j, v_i)$.
  - Digraph or directed graph $(v_i, v_j) \neq (v_j, v_i)$.
- An edge has an associated **weight** or **cost** as well.

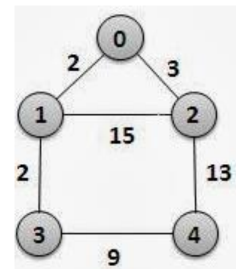
Contd...



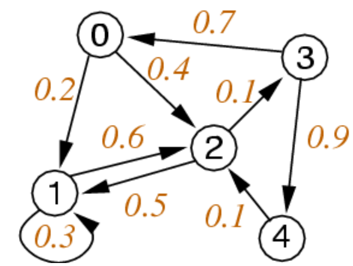
Undirected Graph



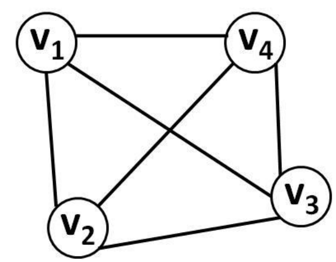
Directed Graph



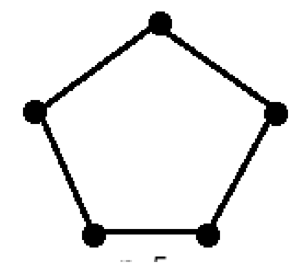
Weighted Undirected Graph



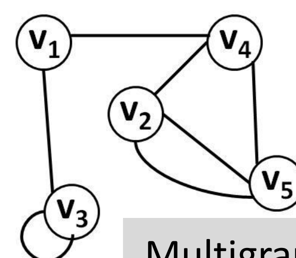
Weighted Directed Graph



Complete Graph



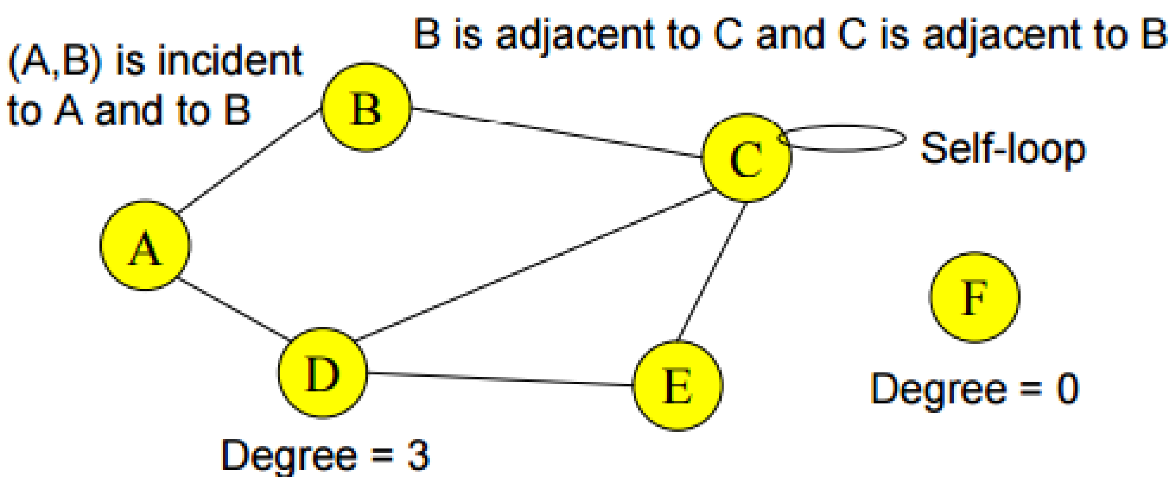
Cycle Graph



Multigraph

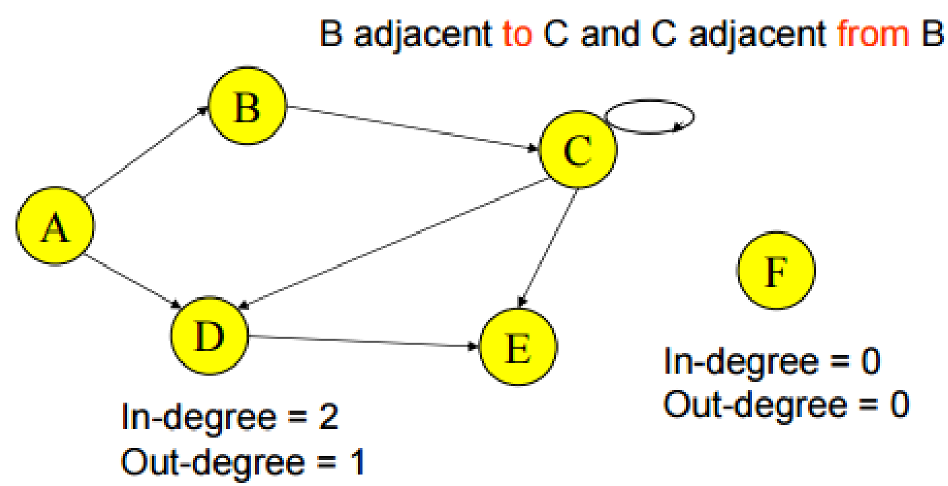
Terminology (Undirected)

- Two vertices u and v are adjacent if {u,v} is an edge in G.
  - Edge {u,v} is incident with vertex u and vertex v.
- Degree of a vertex is the number of edges incident with it.
  - A self-loop counts twice (both ends count).



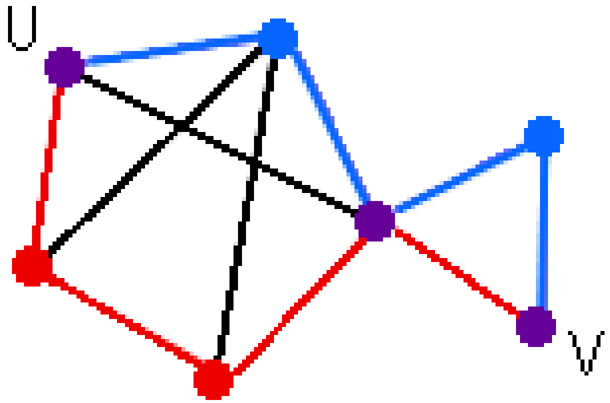
Terminology (Directed)

- Vertex u is adjacent to vertex v if (u,v) is an edge in G and vertex u is the initial vertex of (u,v).
- Vertex v is adjacent from vertex u, if vertex v is the terminal (or end) vertex of (u,v).
- A vertex has two types of degree.
  - in-degree: The number of edges with the vertex as the terminal vertex.
  - out-degree: The number of edges with the vertex as the initial vertex



Some Definitions

- Walk or Path
  - An alternating sequence of vertices and connecting edges.
  - Can end on the same vertex on which it began or on a different vertex.
  - Can travel over any edge and any vertex any number of times.
- Path or Simple Path
  - A walk that does not include any vertex twice, except that its first and last vertices might be the same.



Representations of Graphs

Representations of Graphs

- Two standard ways are:
  - Collection of adjacency lists.
  - Adjacency matrix.
- Applies to both directed and undirected graphs.
- Adjacency-list representation provides a compact way to represent sparse graphs ($|E| \ll |V|^2$).
  - Usually the method of choice.
- Adjacency-matrix representation is preferred when the graph is dense ($|E| \approx |V|^2$).

Representation – I

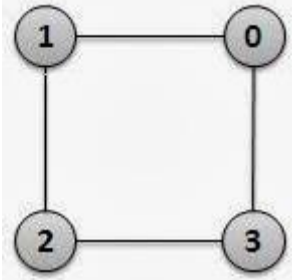
- Adjacency matrix
  - Adjacency matrix for a graph G = (V, E) is a two dimensional matrix of size |V| x |V| such that each entry of this matrix

$$a[i][j] = \begin{cases} 1 \text{ (or weight), if an edge } (v_i, v_j) \text{ exists.} \\ 0\text{, otherwise.} \end{cases}$$

  - For an undirected graph, it is always a symmetric matrix, as $(v_i, v_j) = (v_j, v_i)$.

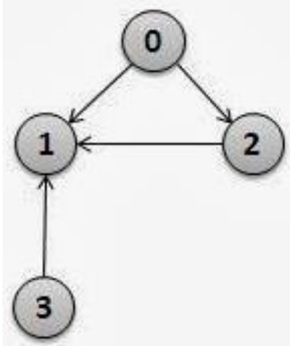
Adjacency matrix

- Undirected.
  - V = {0, 1, 2, 3}
  - E = {(0,1), (1,2), (2,3), (3,0)}



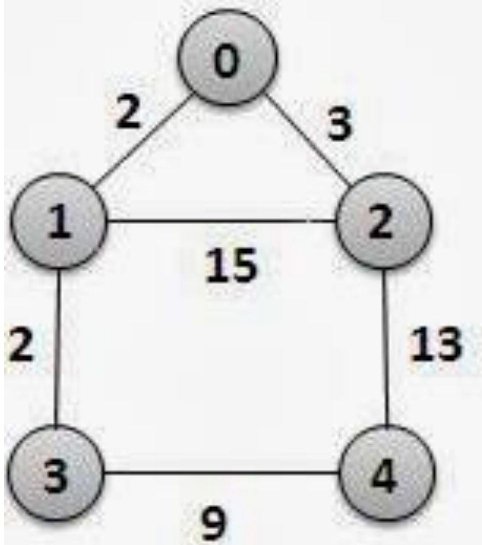
	0	1	2	3
0	0	1	0	1
1	1	0	1	0
2	0	1	0	1
3	1	0	1	0

- Directed.
  - V = {0, 1, 2, 3}
  - E = {(0,1), (0,2), (2,1), (3,1)}



	0	1	2	3
0	0	1	1	0
1	0	0	0	0
2	0	1	0	0
3	0	1	0	0

Contd… (weighted)

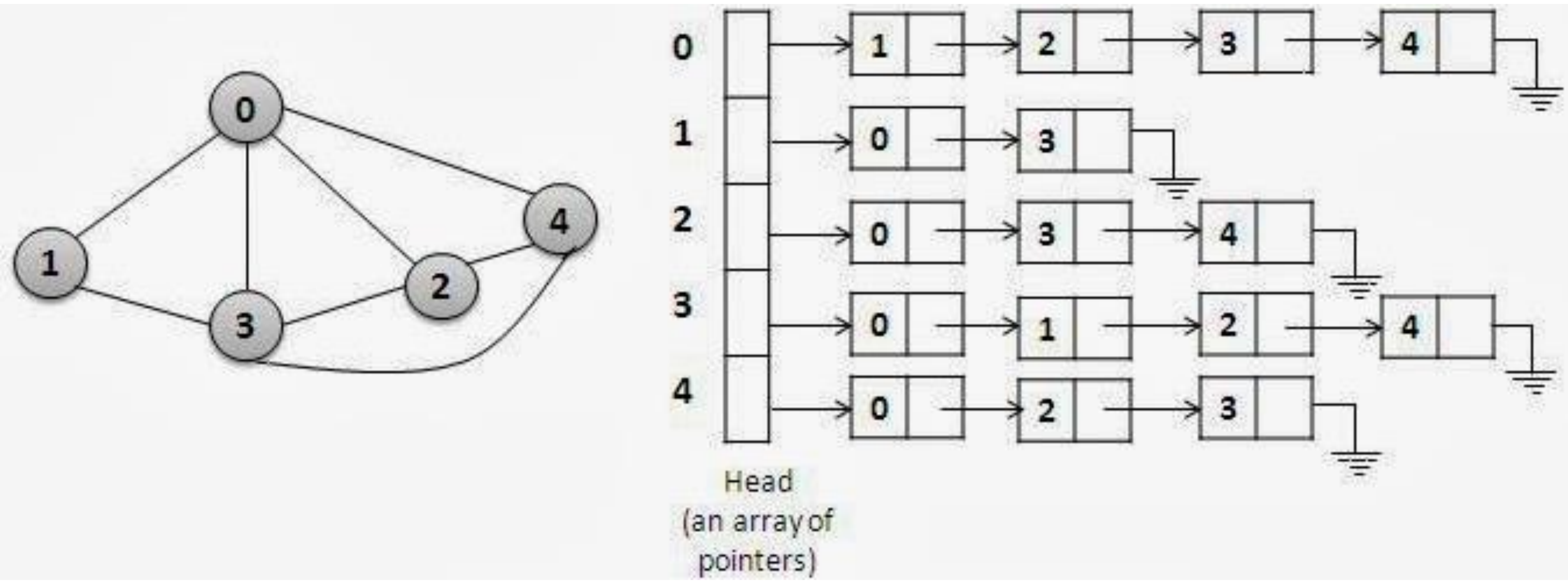


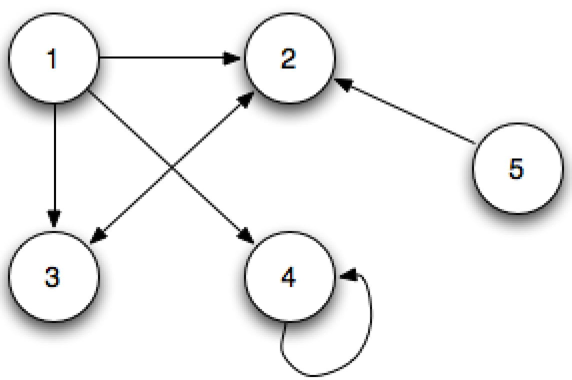
	0	1	2	3	4
0	0	2	3	0	0
1	2	0	15	2	0
2	3	15	0	0	13
3	0	2	0	0	9
4	0	0	13	9	0

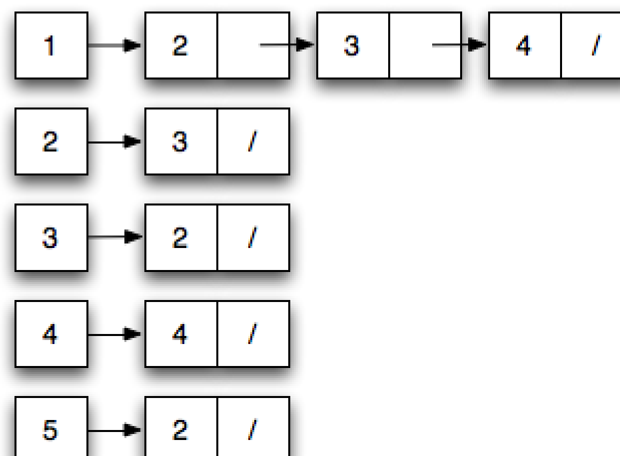
Representation – II

- Adjacency list
  - Uses an array of linked lists with size equals to **|V|**.
  - An i^{th} entry of an array points to a linked list of vertices adjacent to $\mathbf{v_i}$.
  - The weights of edges are stored in nodes of linked lists to represent a weighted graph.

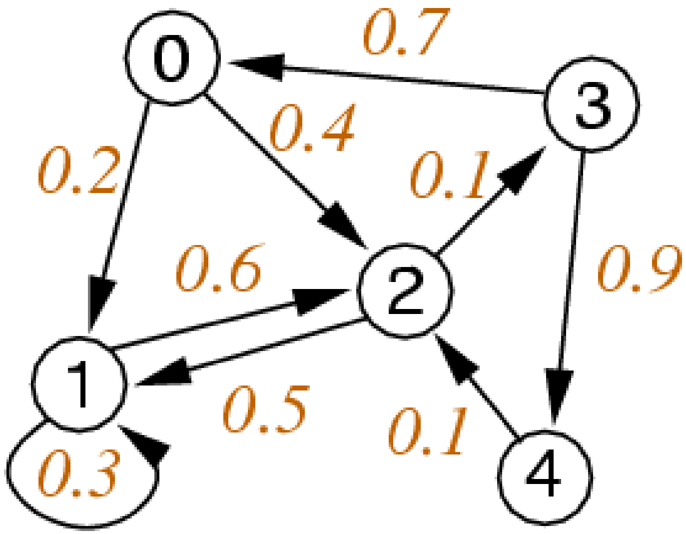
Adjacency List



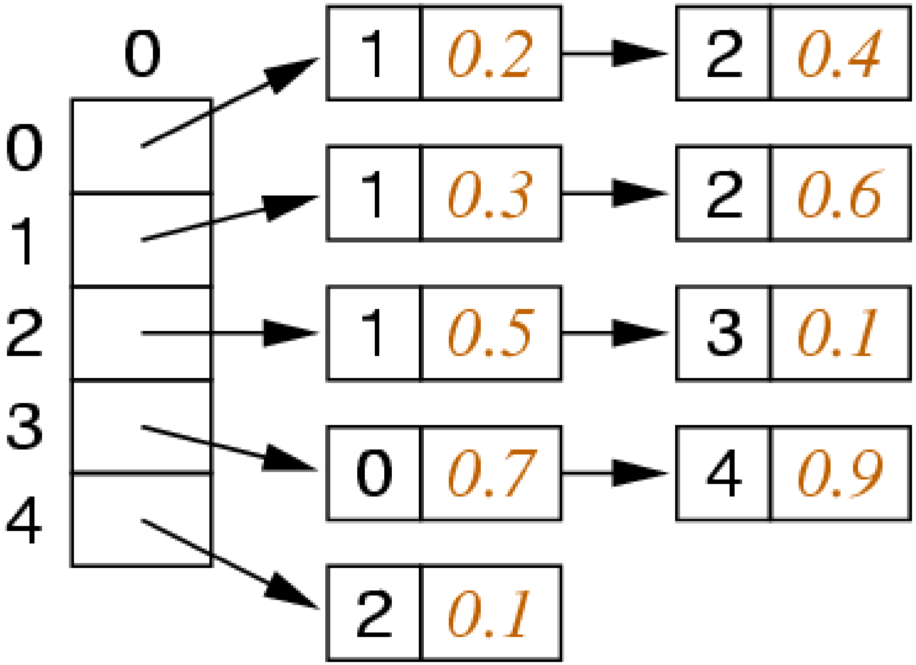




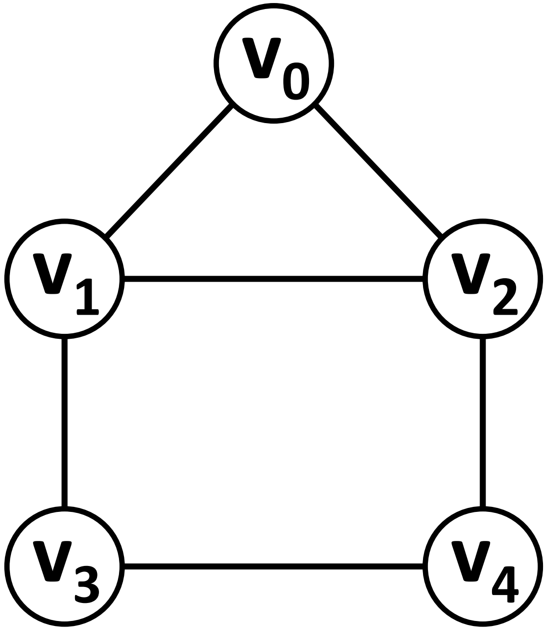
Contd…(weighted)



Weighted Digraph

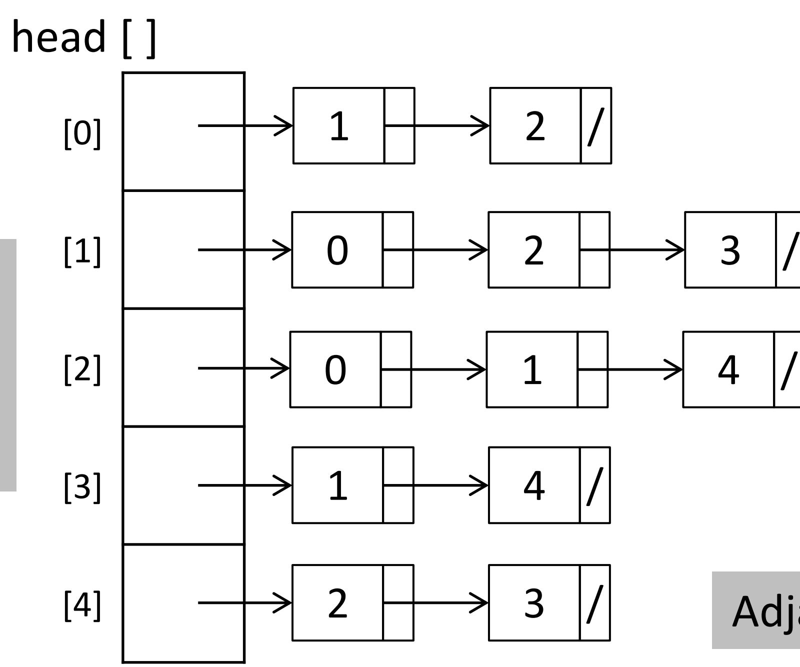


Adjacency Lists



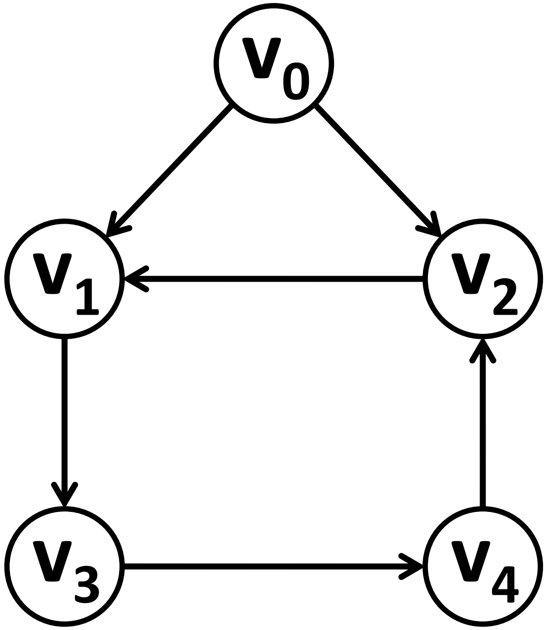
	V_0	V_1	V_2	V_3	V_4
V_0	0	1	1	0	0
V_1	1	0	1	1	0
V_2	1	1	0	0	1
V_3	0	1	0	0	1
V_4	0	0	1	1	0

Adjacency Matrix



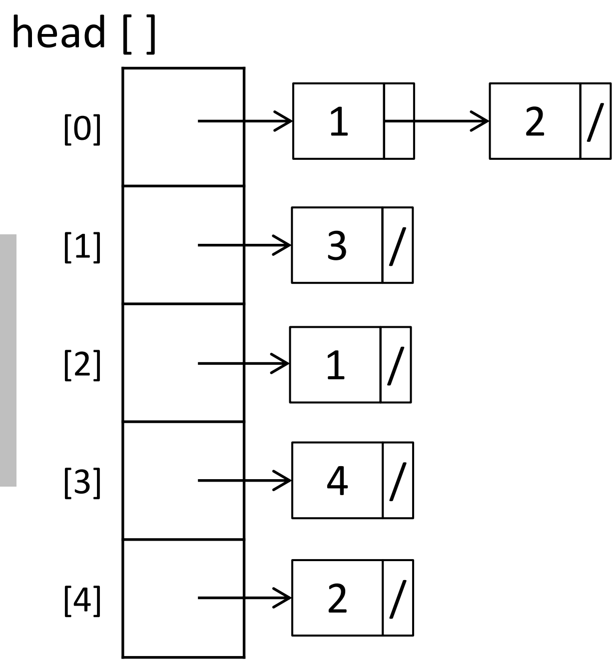

```
struct node
{ int v;
  struct node *next;
} *head[5];
```

Adjacency List



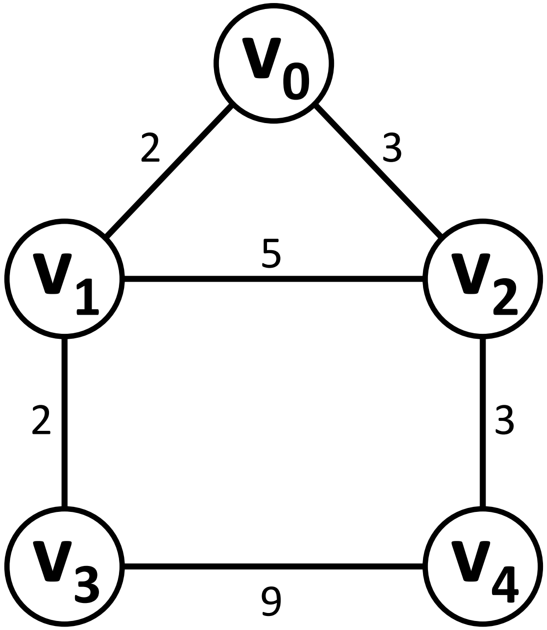
	V_0	V_1	V_2	V_3	V_4
V_0	0	1	1	0	0
V_1	0	0	0	1	0
V_2	0	1	0	0	0
V_3	0	0	0	0	1
V_4	0	0	1	0	0

Adjacency Matrix




```
struct node
{ int v;
  struct node *next;
} *head[5];
```

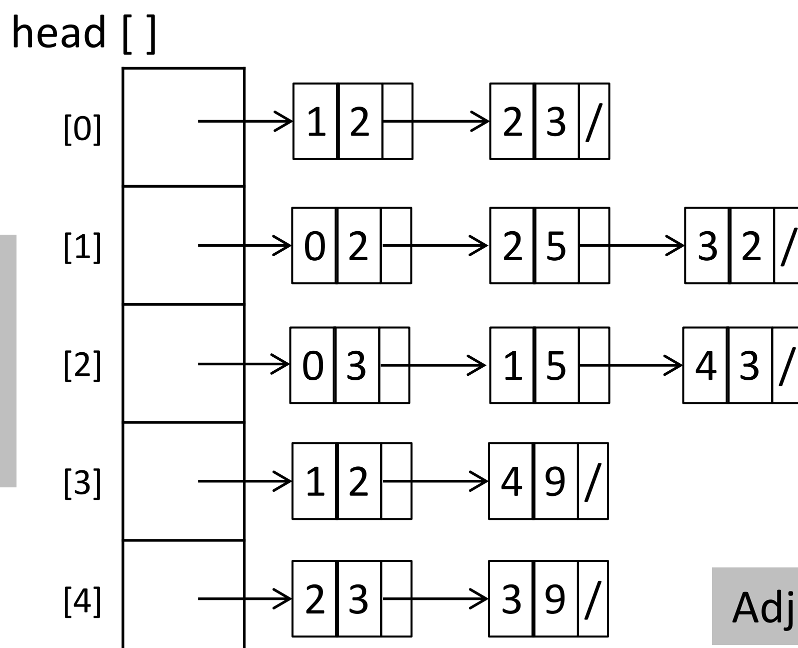
Adjacency List



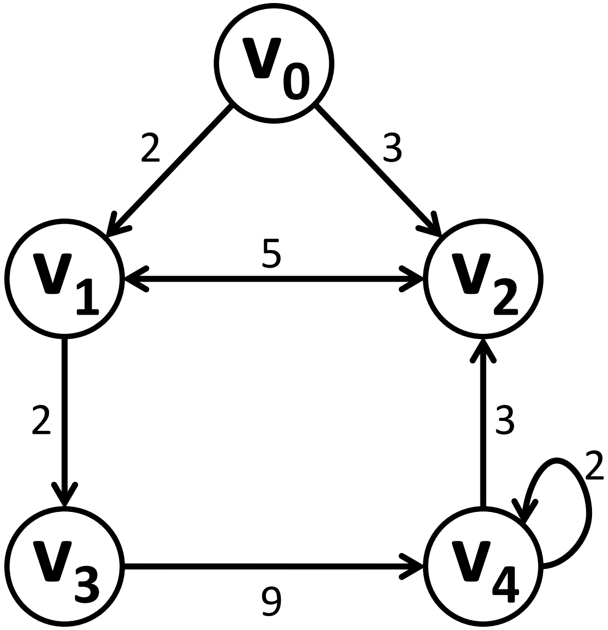
	V_0	V_1	V_2	V_3	V_4
V_0	0	2	3	0	0
V_1	2	0	5	2	0
V_2	3	5	0	0	3
V_3	0	2	0	0	9
V_4	0	0	3	9	0

Adjacency Matrix

```
struct node
{ int v, w;
  struct node *next;
} *head[5];
```

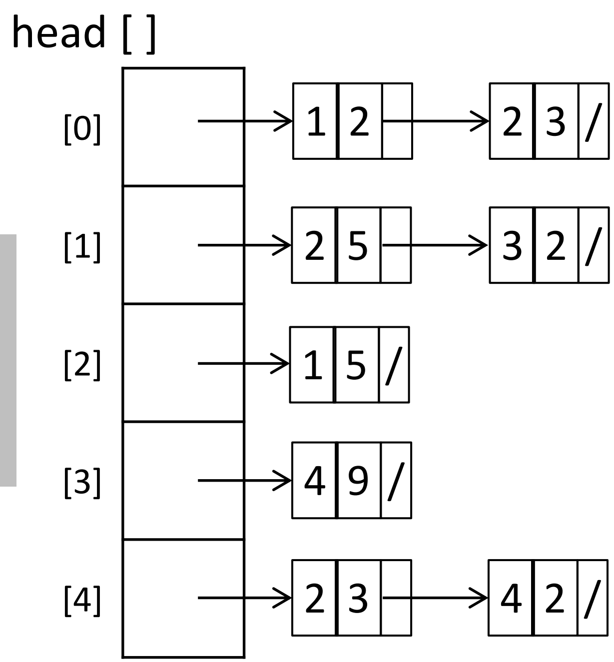


Adjacency List



	V_0	V_1	V_2	V_3	V_4
V_0	0	2	3	0	0
V_1	0	0	5	2	0
V_2	0	5	0	0	0
V_3	0	0	0	0	9
V_4	0	0	3	0	2

Adjacency Matrix




```
struct node
{ int v, w;
  struct node *next;
} *head[5];
```

Adjacency List

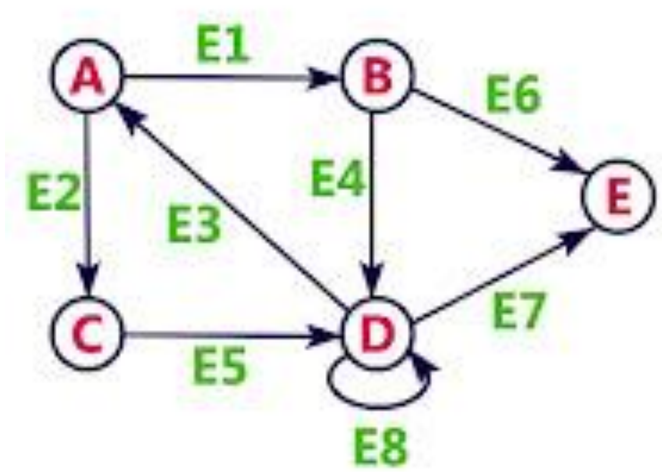
Representation – III

- Incidence Matrix

For directed graph:

M(i, j) = 1 if edge i is leading away from vertex j

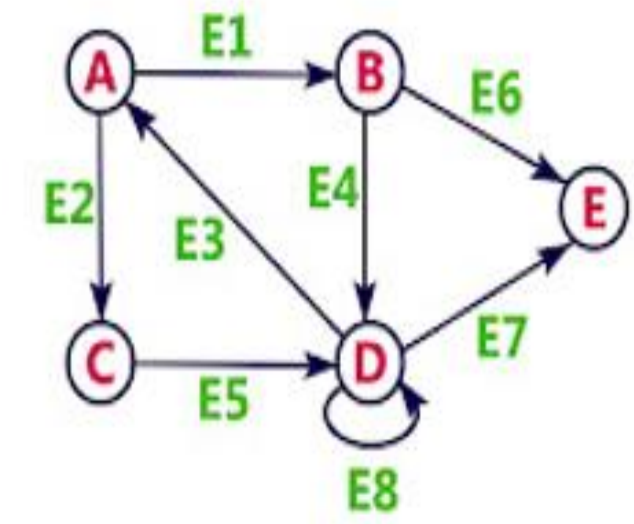
M(i, j) = -1 if edge i is leading to vertex j

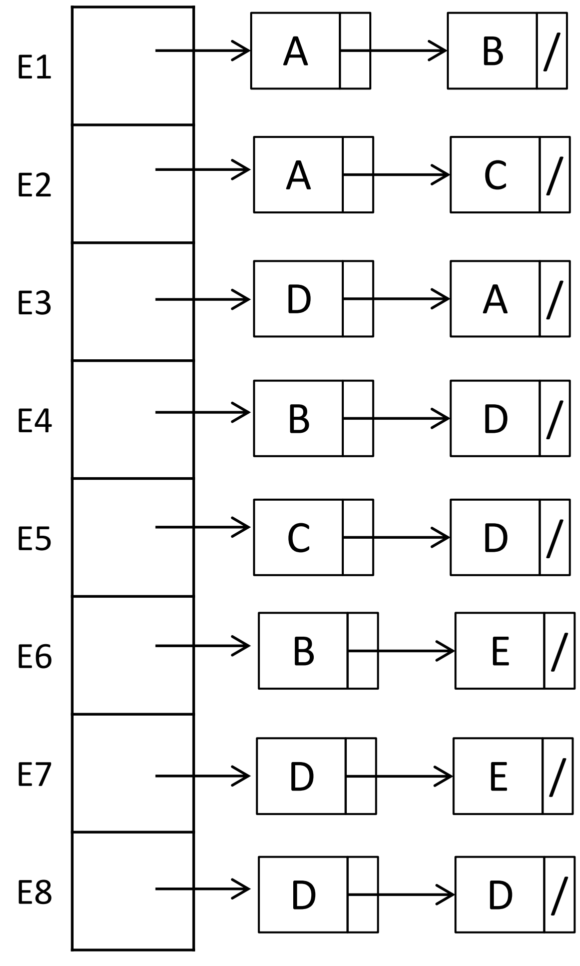


$$
\begin{array}{c|cccccccc}
 & E1 & E2 & E3 & E4 & E5 & E6 & E7 & E8 \\
\hline
A & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
B & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
C & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
D & 0 & 0 & 1 & -1 & -1 & 0 & 1 & 1 \\
E & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0
\end{array}
$$

Representation – IV

- Incidence List





Graph Searching

- Breadth-first search
- Depth-first search

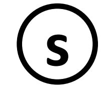
Breadth-first search (BFS)

- Given a graph G = (V,E) and a distinguished source vertex s, BFS systematically explores the edges of G to "discover" every vertex that is reachable from s.
- Discovers all vertices at distance k from a source vertex s before discovering any vertices at distance k + 1.
- It computes the distance (smallest number of edges) from s to each reachable vertex.
- It produces a "breadth-first tree" with root s that contains all reachable vertices.
- It works on both directed and undirected graphs.

Compute BFS - Undirected

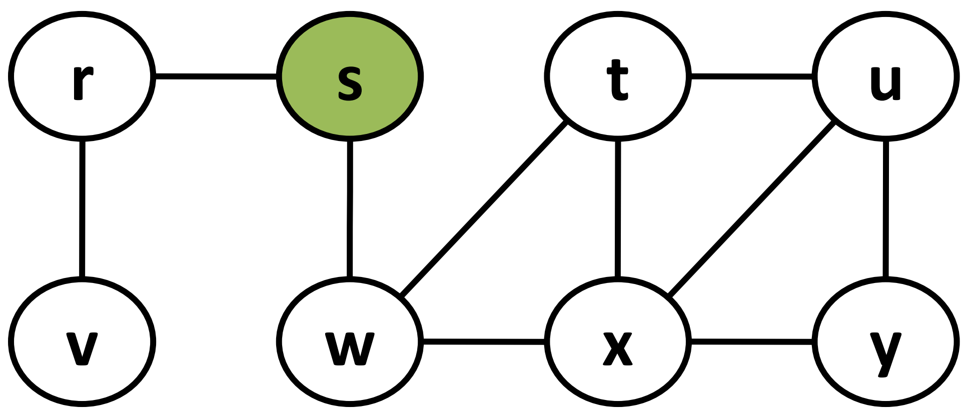
- BFS:

Predecessor sub-graph

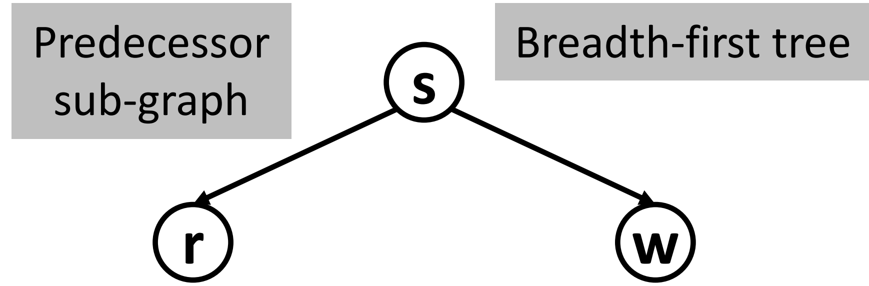


Breadth-first tree

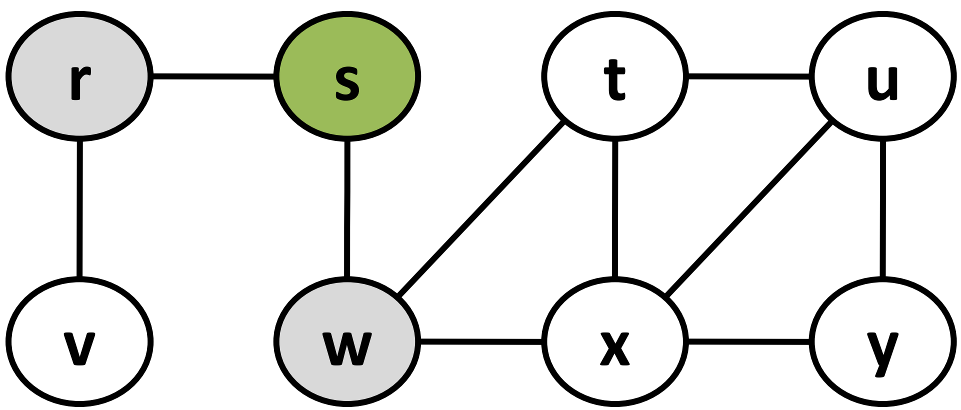
- Queue: $\boxed{s}$



Compute BFS - Undirected



- BFS: s
- Queue: **r** | **w**

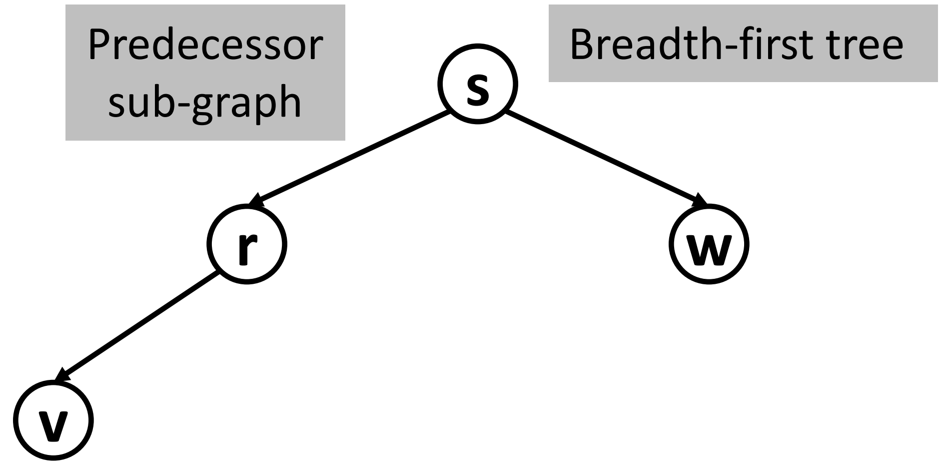


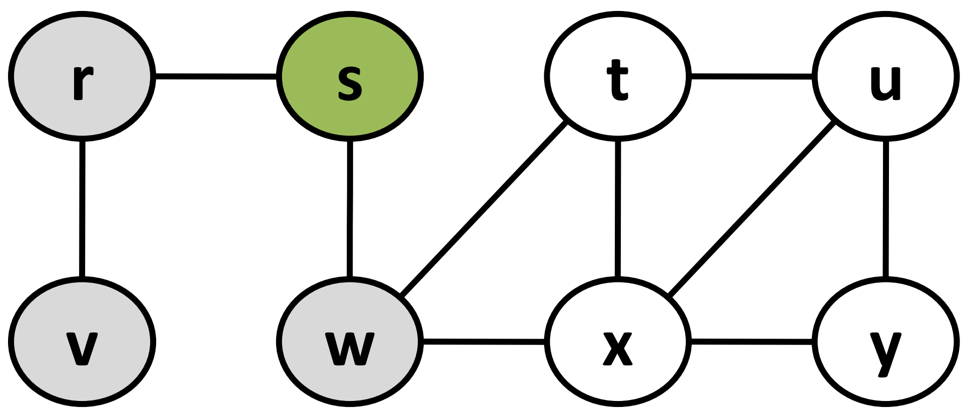
Compute BFS - Undirected

- BFS: s r

- Queue:

w	**v**

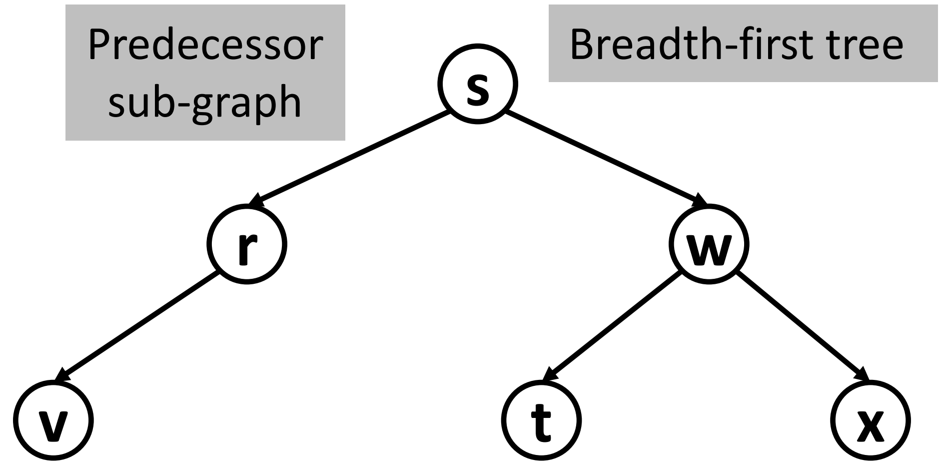


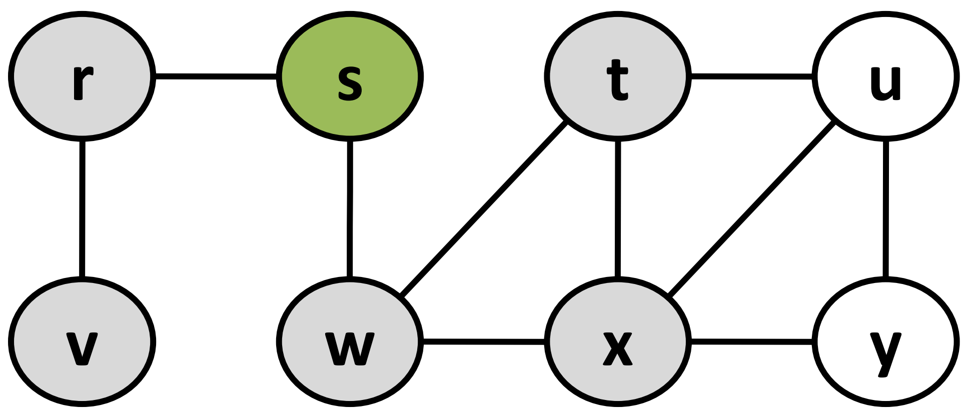


Compute BFS - Undirected

- BFS: s r w
- Queue:

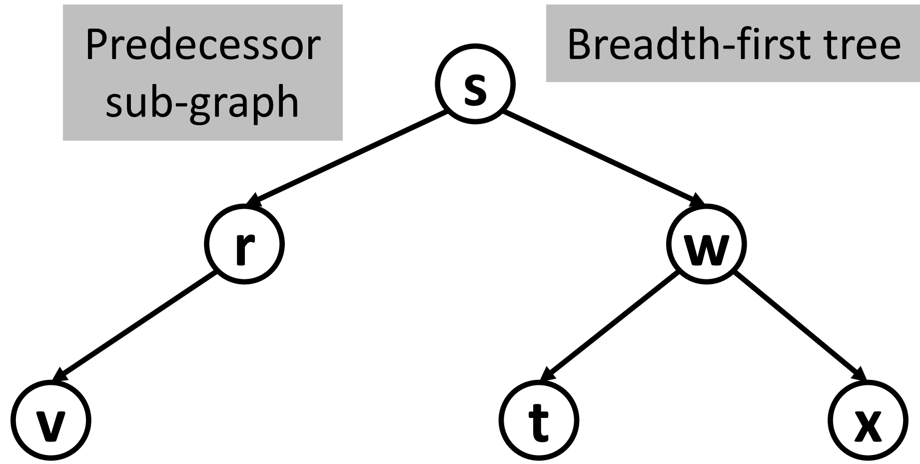
v	t	x

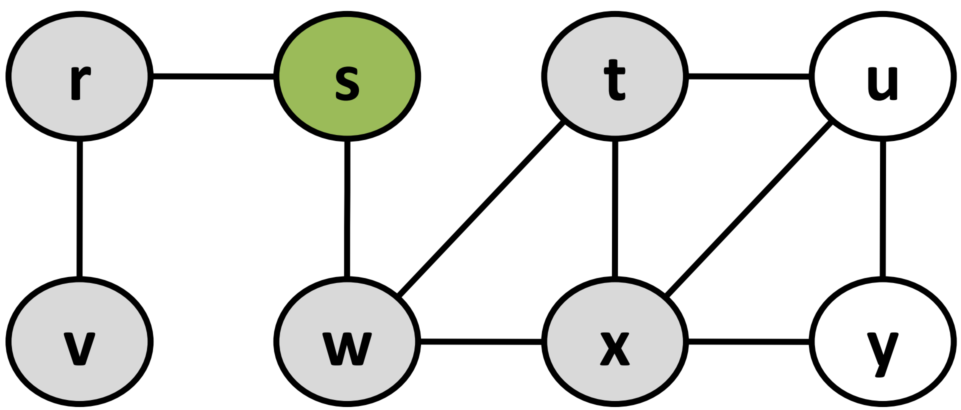




Compute BFS - Undirected

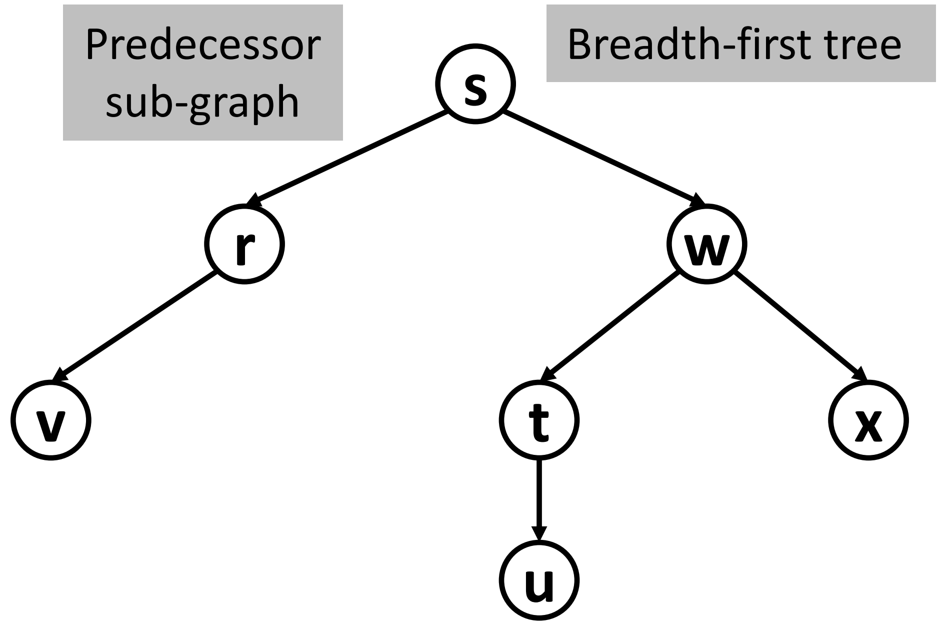
- BFS: s r w v
- Queue: | t | x |

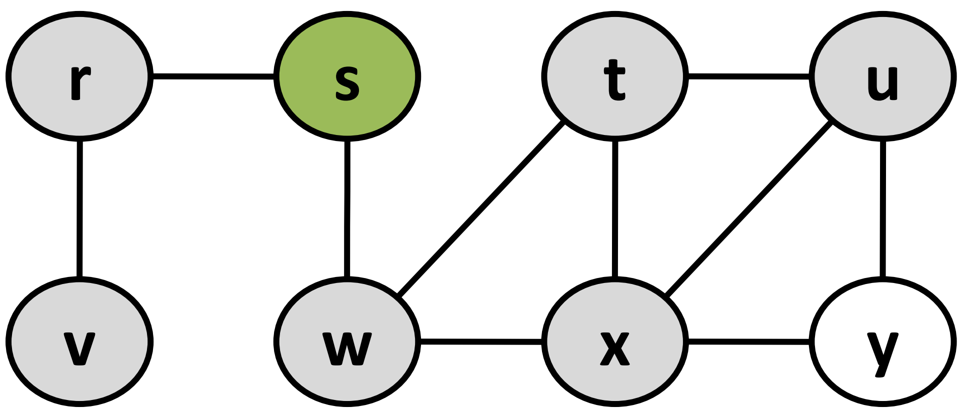




Compute BFS - Undirected

- BFS: s r w v t
- Queue: | x | u |

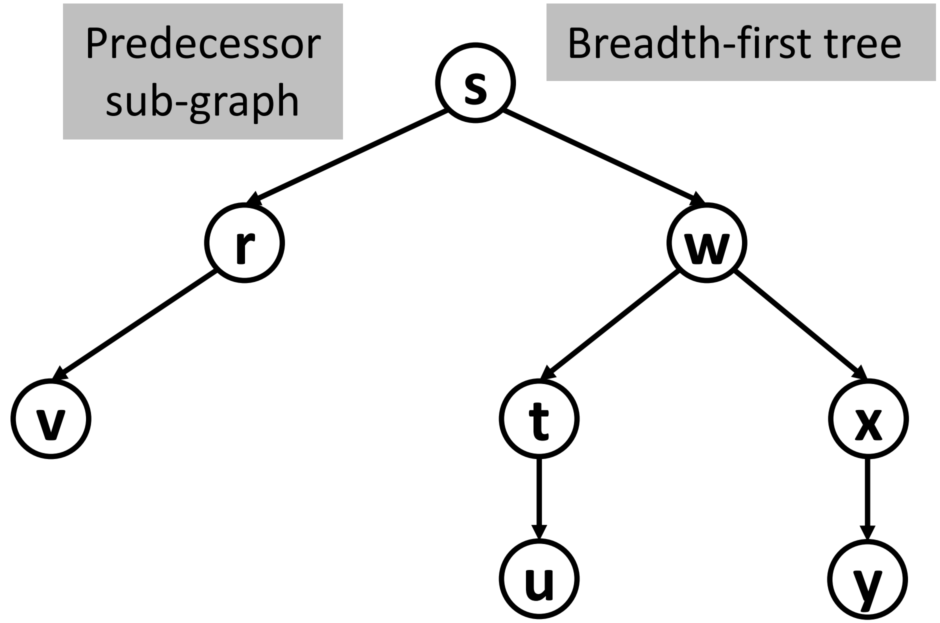


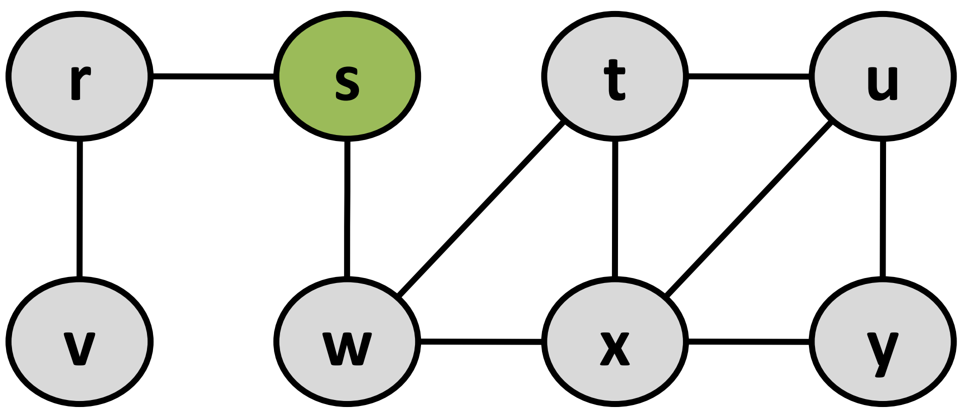


Compute BFS - Undirected

- BFS: s r w v t x

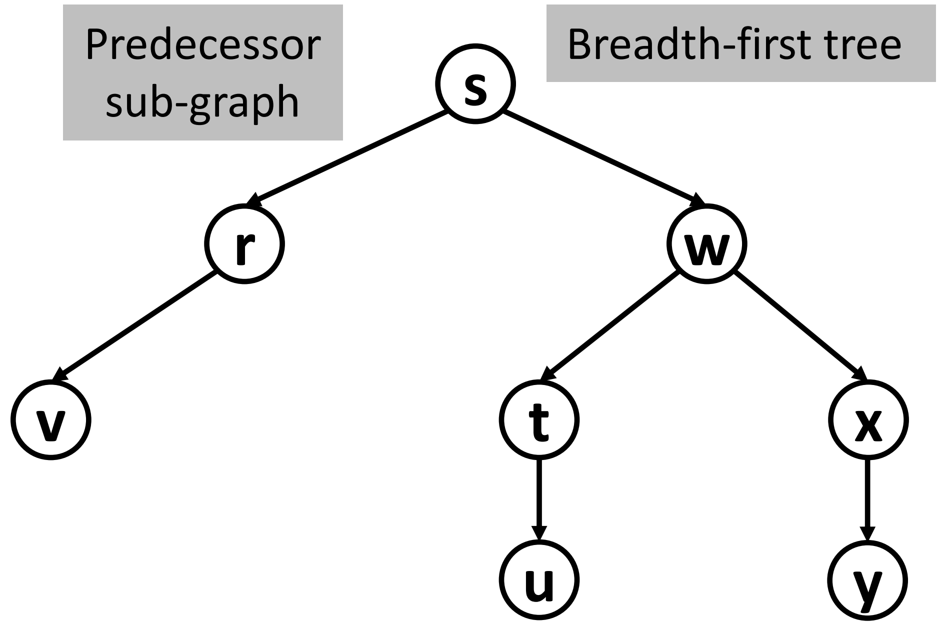
- Queue: | u | y |

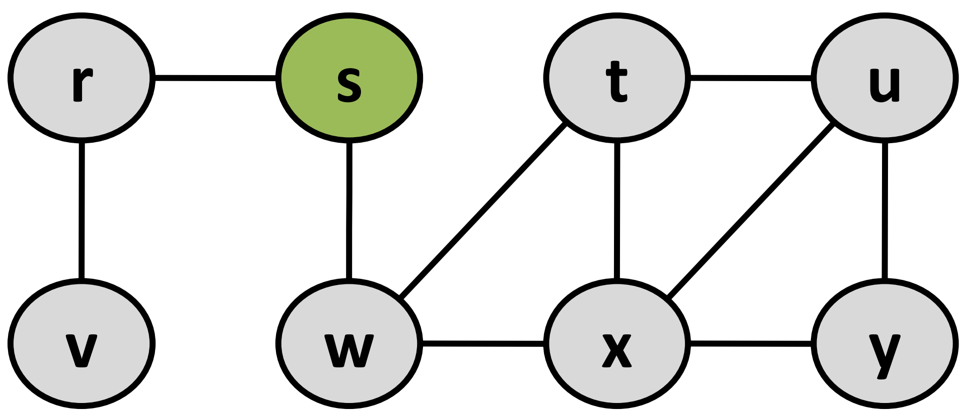




Compute BFS - Undirected

- BFS: s r w v t x u
- Queue: y

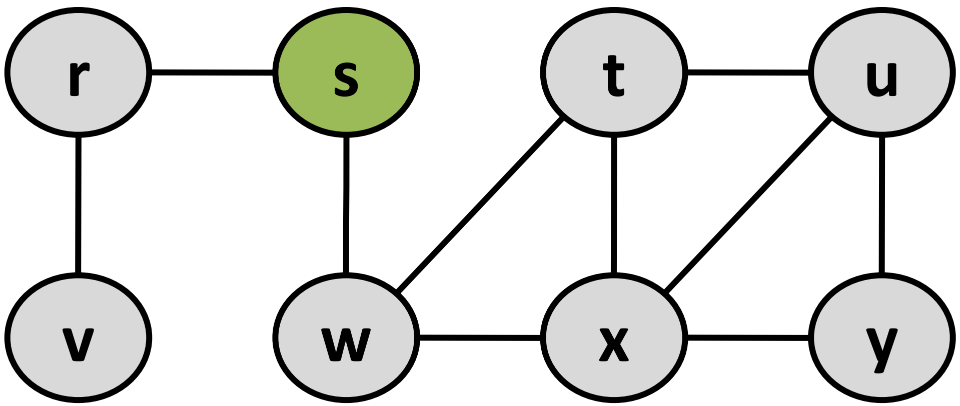




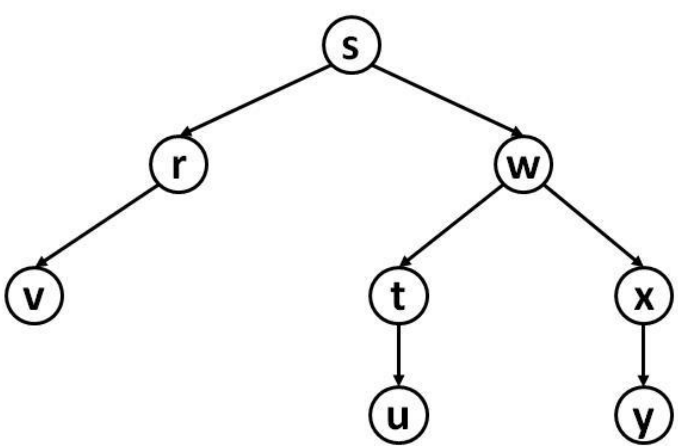
Compute BFS - Undirected

- BFS: s r w v t x u y

- Queue:

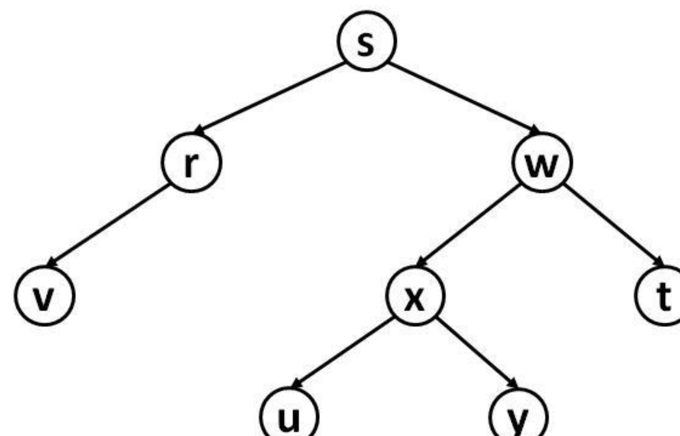


BFS	Queue
	s
s	w r
s w	r x t
s w r	x t v
s w r x	t v y u
s w r x t	v y u
s w r x t v	y u
s w r x t v y	u
s w r x t v y u	



Breadth-first tree

Predecessor sub-graph



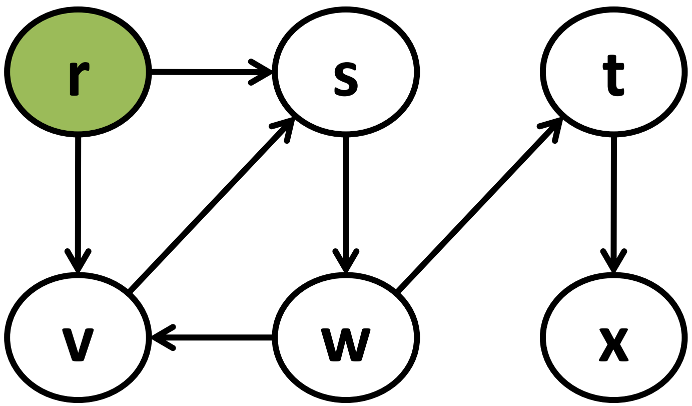
Compute BFS - Directed

- BFS:

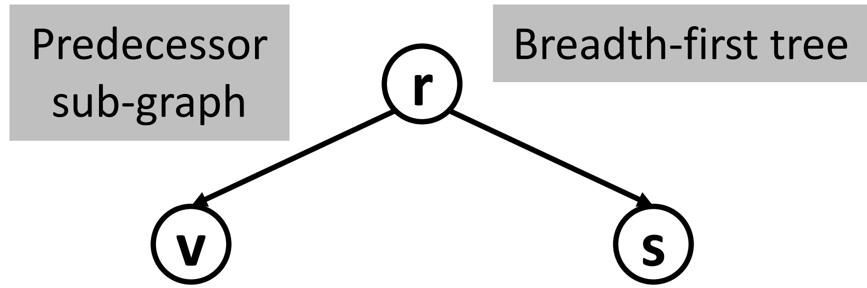
Predecessor sub-graph

Breadth-first tree

- Queue: r

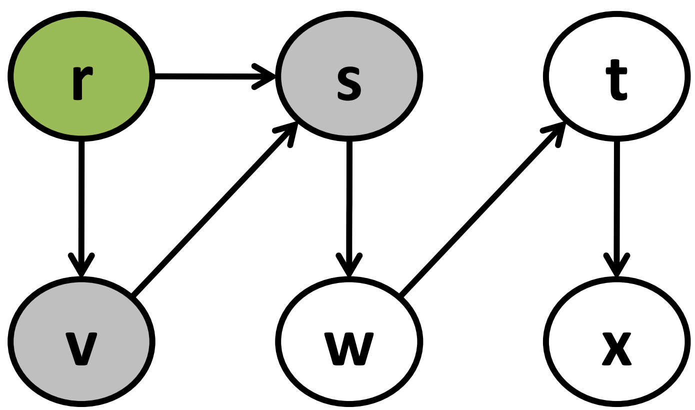


Compute BFS - Directed



- BFS: r
- Queue:

s	**v**



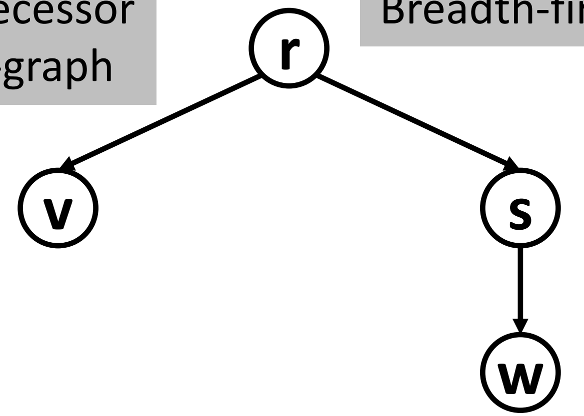
Compute BFS - Directed

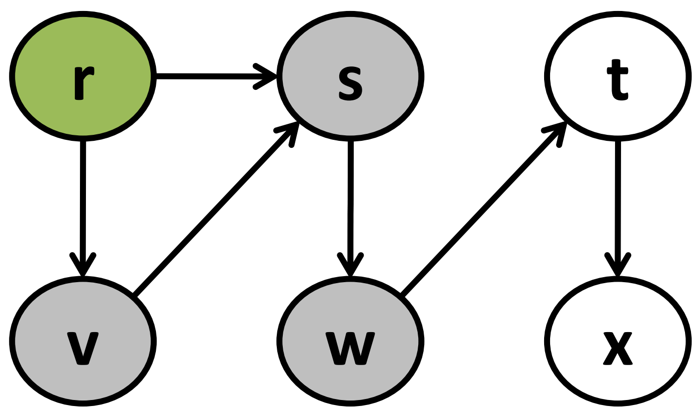
- BFS: r s
- Queue:

v	**w**

Predecessor sub-graph

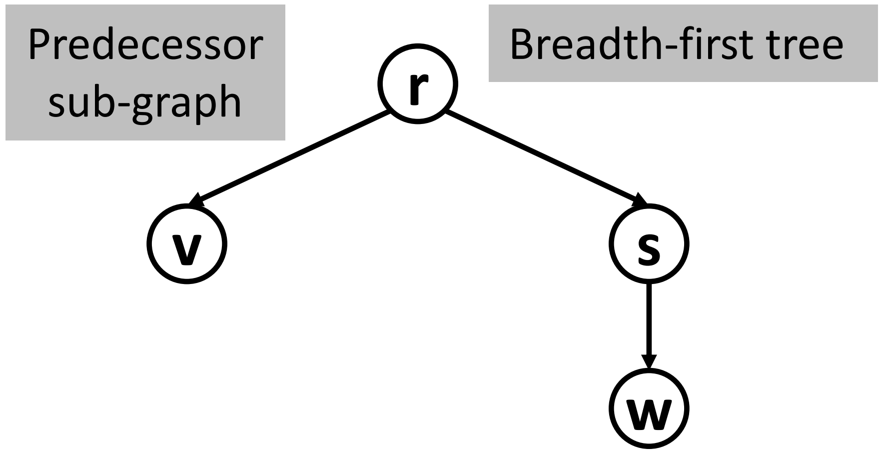
Breadth-first tree

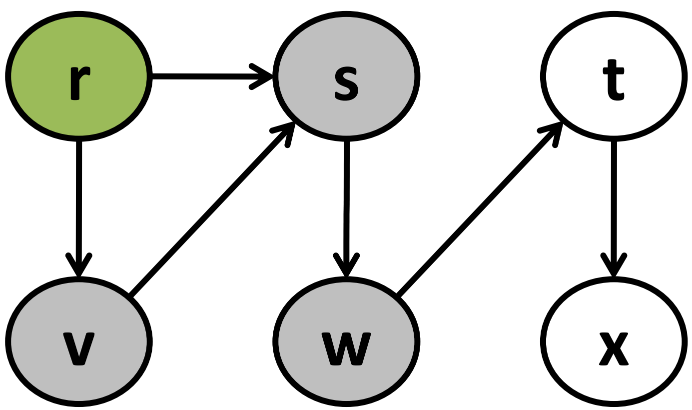




Compute BFS - Directed

- BFS: r s v
- Queue: w

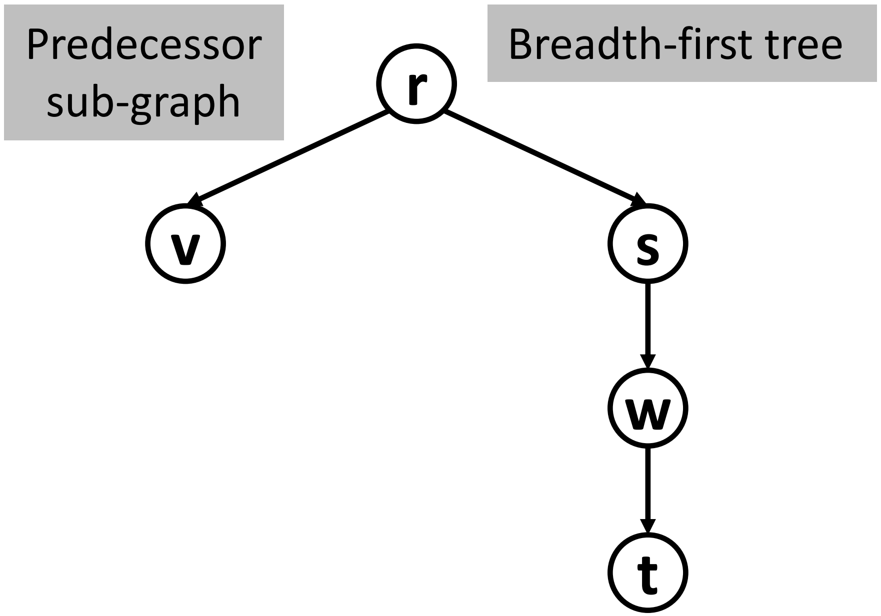


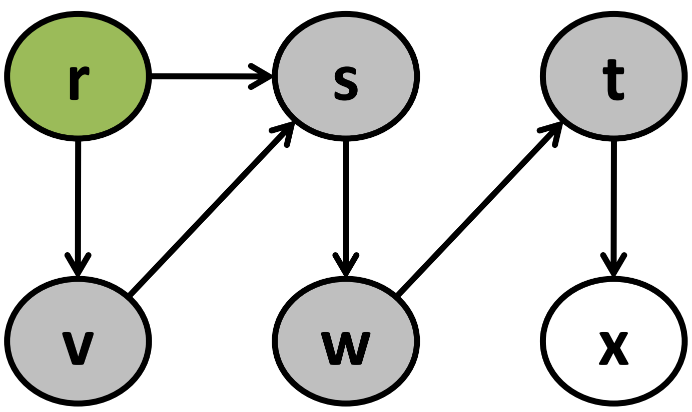


Compute BFS - Directed

- BFS: r s v w

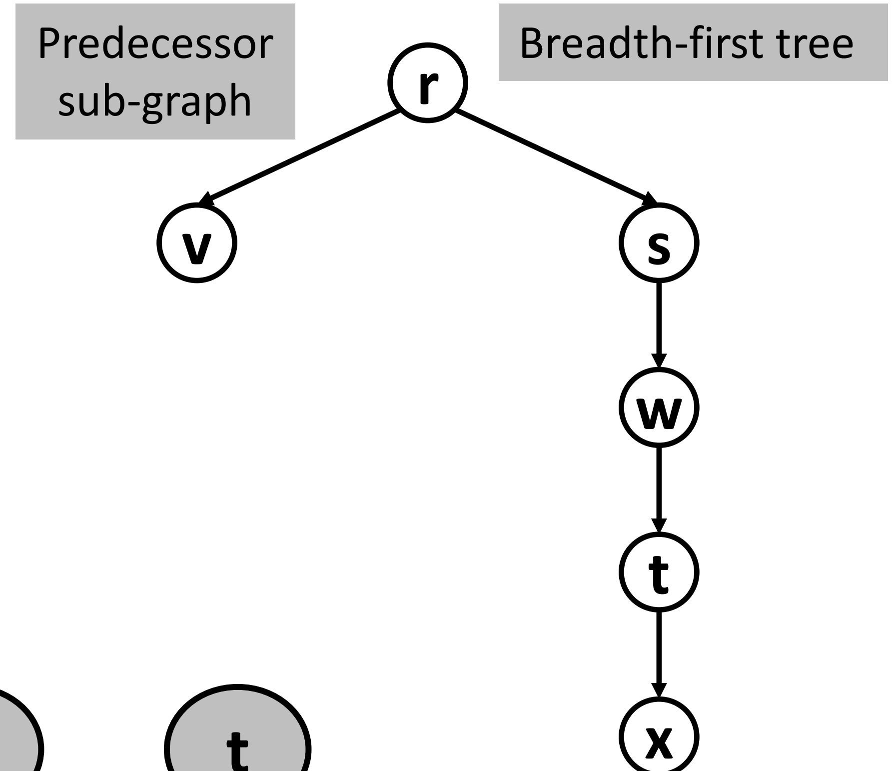
- Queue: t

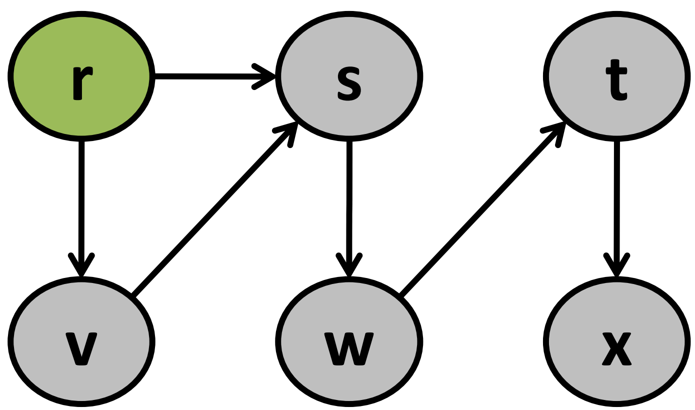




Compute BFS - Directed

- BFS: r s v w t
- Queue: $\boxed{x}$



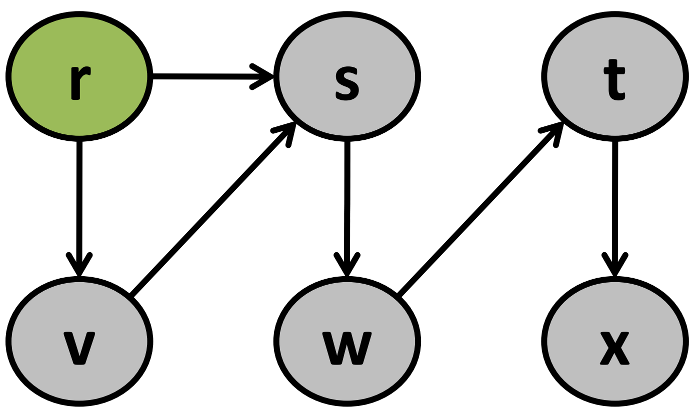


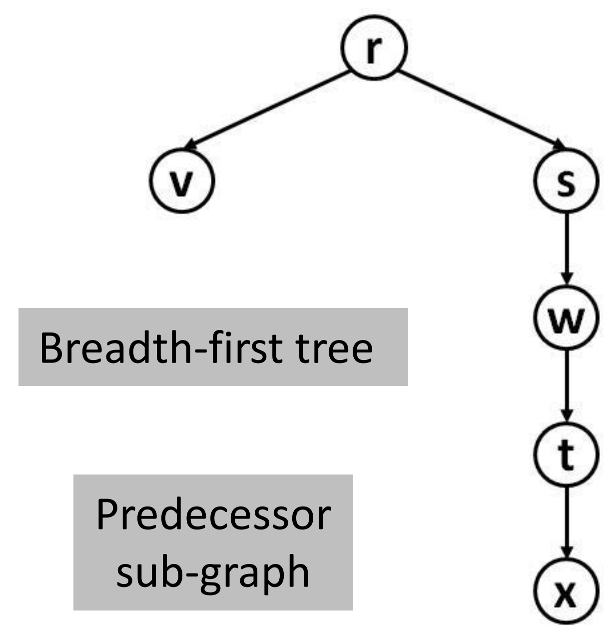
Compute BFS - Directed

- BFS: r s v w t x

- Queue:

BFS	Queue
	r
r	v s
r v	s
r v s	w
r v s w	t
r v s w t	x
r v s w t x	





Procedure BFS

- Assumptions:
  - The input graph G = (V,E) is represented using adjacency lists.
  - Each vertex in the graph has following additional attributes.
    - Color: Can be white (undiscovered), gray (may have some adjacent white vertices), or black (all adjacent vertices have been discovered).
    - π: predecessor of a vertex. Can be NIL.
    - d: The distance from the source vertex computed by the algorithm.
  - The queue Q is used to manage the set of gray vertices.

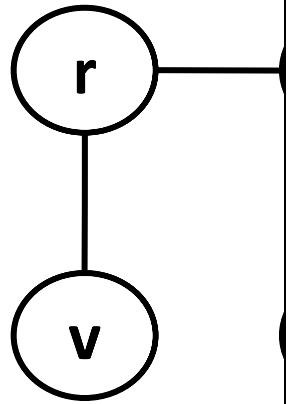
Contd...

$O(V+E)$

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Execution example

- s is the starting vertex.



for each vertex $u \in G.V - \{s\}$
    $u.color = \text{WHITE}$
    $u.d = \infty$
    $u.\pi = \text{NIL}$
$s.color = \text{GRAY}$
$s.d = 0$
$s.\pi = \text{NIL}$
$Q = \emptyset$
$\text{ENQUEUE}(Q, s)$

Vertex	Color	Distance (d)	Predecessor (π)
r	White	∞	NIL
s	**Gray**	**0**	**NIL**
t	White	∞	NIL
u	White	∞	NIL
v	White	∞	NIL
w	White	∞	NIL
x	White	∞	NIL
y	White	∞	NIL

r → s → v
s → r → w
t → u → w → x
u → t → x → y
v → r
w → s → t → x
x → t → u → w → y
y → u → x

Q: s

BFS:

Contd…

- s is the starting vertex.

Vertex	Color	Distance (d)	Predecessor (π)
r	White	∞	NIL
s	**Gray**	**0**	**NIL**
t	White	∞	NIL
u	White	∞	NIL
v	White	∞	NIL
w	White	∞	NIL
x	White	∞	NIL
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

$r \rightarrow s \rightarrow v$
$s \rightarrow r \rightarrow w$
$t \rightarrow u \rightarrow w \rightarrow x$
$u \rightarrow t \rightarrow x \rightarrow y$
$v \rightarrow r$
$w \rightarrow s \rightarrow t \rightarrow x$
$x \rightarrow t \rightarrow u \rightarrow w \rightarrow y$
$y \rightarrow u \rightarrow x$

Q: s

BFS:

Contd...

Vertex	Color	Distance (d)	Predecessor (π)
r	**Gray**	**1**	**s**
s	**Gray**	**0**	**NIL**
t	White	∞	NIL
u	White	∞	NIL
v	White	∞	NIL
w	White	∞	NIL
x	White	∞	NIL
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

r	$\to$ s	$\to$ v		
s	$\to$ r	$\to$ w		
t	$\to$ u	$\to$ w	$\to$ x	
u	$\to$ t	$\to$ x	$\to$ y	
v	$\to$ r			
w	$\to$ s	$\to$ t	$\to$ x	
x	$\to$ t	$\to$ u	$\to$ w	$\to$ y
y	$\to$ u	$\to$ x		

Q: r

BFS: s

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	**Gray**	**1**	**s**
s	**Gray**	**0**	**NIL**
t	White	∞	NIL
u	White	∞	NIL
v	White	∞	NIL
w	**Gray**	**1**	**s**
x	White	∞	NIL
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

r	$\rightarrow$ s $\rightarrow$ v
s	$\rightarrow$ r $\rightarrow$ w
t	$\rightarrow$ u $\rightarrow$ w $\rightarrow$ x
u	$\rightarrow$ t $\rightarrow$ x $\rightarrow$ y
v	$\rightarrow$ r
w	$\rightarrow$ s $\rightarrow$ t $\rightarrow$ x
x	$\rightarrow$ t $\rightarrow$ u $\rightarrow$ w $\rightarrow$ y
y	$\rightarrow$ u $\rightarrow$ x

Q: | r | w |

BFS: s

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	**Gray**	**1**	**s**
s	**Black**	**0**	**NIL**
t	White	∞	NIL
u	White	∞	NIL
v	White	∞	NIL
w	**Gray**	**1**	**s**
x	White	∞	NIL
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

r → s → v
s → r → w
t → u → w → x
u → t → x → y
v → r
w → s → t → x
x → t → u → w → y
y → u → x

Q: | r | w |

BFS: s

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	**Gray**	**1**	**s**
s	**Black**	**0**	**NIL**
t	White	∞	NIL
u	White	∞	NIL
v	**Gray**	**2**	**r**
w	**Gray**	**1**	**s**
x	White	∞	NIL
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

r	→	s	→	v	
s	→	r	→	w	
t	→	u	→	w	→ x
u	→	t	→	x	→ y
v	→	r			
w	→	s	→	t	→ x
x	→	t	→	u	→ w → y
y	→	u	→	x	

Q: | w | v |

BFS: s r

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	**Black**	**1**	**s**
s	**Black**	**0**	**NIL**
t	White	∞	NIL
u	White	∞	NIL
v	**Gray**	**2**	**r**
w	**Gray**	**1**	**s**
x	White	∞	NIL
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

r	$\rightarrow$ s $\rightarrow$ v
s	$\rightarrow$ r $\rightarrow$ w
t	$\rightarrow$ u $\rightarrow$ w $\rightarrow$ x
u	$\rightarrow$ t $\rightarrow$ x $\rightarrow$ y
v	$\rightarrow$ r
w	$\rightarrow$ s $\rightarrow$ t $\rightarrow$ x
x	$\rightarrow$ t $\rightarrow$ u $\rightarrow$ w $\rightarrow$ y
y	$\rightarrow$ u $\rightarrow$ x

Q: | w | v |

BFS: s r

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	**Black**	**1**	**s**
s	**Black**	**0**	**NIL**
t	**Gray**	**2**	**w**
u	White	∞	NIL
v	**Gray**	**2**	**r**
w	**Gray**	**1**	**s**
x	White	∞	NIL
y	White	∞	NIL

while $Q \neq \emptyset$
    $u = \text{DEQUEUE}(Q)$
    for each $v \in G.Adj[u]$
        if $v.color == \text{WHITE}$
            $v.color = \text{GRAY}$
            $v.d = u.d + 1$
            $v.\pi = u$
            $\text{ENQUEUE}(Q, v)$
    $u.color = \text{BLACK}$

r → s → v
s → r → w
t → u → w → x
u → t → x → y
v → r
w → s → t → x
x → t → u → w → y
y → u → x

Q: | v | t |

BFS: s r w

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	**Black**	**1**	**s**
s	**Black**	**0**	**NIL**
t	**Gray**	**2**	**w**
u	White	∞	NIL
v	**Gray**	**2**	**r**
w	**Gray**	**1**	**s**
x	**Gray**	**2**	**w**
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

r	$\to$ s $\to$ v
s	$\to$ r $\to$ w
t	$\to$ u $\to$ w $\to$ x
u	$\to$ t $\to$ x $\to$ y
v	$\to$ r
w	$\to$ s $\to$ t $\to$ x
x	$\to$ t $\to$ u $\to$ w $\to$ y
y	$\to$ u $\to$ x

Q: | v | t | x |

BFS: s r w

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	**Black**	**1**	**s**
s	**Black**	**0**	**NIL**
t	**Gray**	**2**	**w**
u	White	∞	NIL
v	**Gray**	**2**	**r**
w	**Black**	**1**	**s**
x	**Gray**	**2**	**w**
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

$r \rightarrow s \rightarrow v$
$s \rightarrow r \rightarrow w$
$t \rightarrow u \rightarrow w \rightarrow x$
$u \rightarrow t \rightarrow x \rightarrow y$
$v \rightarrow r$
$w \rightarrow s \rightarrow t \rightarrow x$
$x \rightarrow t \rightarrow u \rightarrow w \rightarrow y$
$y \rightarrow u \rightarrow x$

Q:	v	t	x

BFS: s r w

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	**Black**	**1**	**s**
s	**Black**	**0**	**NIL**
t	**Gray**	**2**	**w**
u	White	∞	NIL
v	**Black**	**2**	**r**
w	**Black**	**1**	**s**
x	**Gray**	**2**	**w**
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

$r \rightarrow s \rightarrow v$

$s \rightarrow r \rightarrow w$

$t \rightarrow u \rightarrow w \rightarrow x$

$u \rightarrow t \rightarrow x \rightarrow y$

$v \rightarrow r$

$w \rightarrow s \rightarrow t \rightarrow x$

$x \rightarrow t \rightarrow u \rightarrow w \rightarrow y$

$y \rightarrow u \rightarrow x$

Q: | t | x |

BFS: s r w v

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	**Black**	1	**s**
s	**Black**	**0**	**NIL**
t	**Gray**	2	**w**
u	**Gray**	3	**t**
v	**Black**	2	**r**
w	**Black**	1	**s**
x	**Gray**	2	**w**
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

$r \rightarrow s \rightarrow v$

$s \rightarrow r \rightarrow w$

$t \rightarrow u \rightarrow w \rightarrow x$

$u \rightarrow t \rightarrow x \rightarrow y$

$v \rightarrow r$

$w \rightarrow s \rightarrow t \rightarrow x$

$x \rightarrow t \rightarrow u \rightarrow w \rightarrow y$

$y \rightarrow u \rightarrow x$

Q: | x | u |

BFS: s r w v t

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	Black	1	s
s	Black	0	NIL
t	Black	2	w
u	Gray	3	t
v	Black	2	r
w	Black	1	s
x	Gray	2	w
y	White	∞	NIL

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

r	→ s → v
s	→ r → w
t	→ u → w → x
u	→ t → x → y
v	→ r
w	→ s → t → x
x	→ t → u → w → y
y	→ u → x

Q: | x | u |

BFS: s r w v t

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	Black	1	s
s	Black	0	NIL
t	Black	2	w
u	Gray	3	t
v	Black	2	r
w	Black	1	s
x	Gray	2	w
y	Gray	3	x

```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

r → s → v
s → r → w
t → u → w → x
u → t → x → y
v → r
w → s → t → x
x → t → u → w → y
y → u → x

Q: | u | y |

BFS: s r w v t x

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	Black	1	s
s	Black	0	NIL
t	Black	2	w
u	Gray	3	t
v	Black	2	r
w	Black	1	s
x	Black	2	w
y	Gray	3	x

```
while Q ≠ Ø
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

r → s → v
s → r → w
t → u → w → x
u → t → x → y
v → r
w → s → t → x
x → t → u → w → y
y → u → x

Q:	u	y

BFS: s r w v t x

Contd…

Vertex	Color	Distance (d)	Predecessor (π)
r	Black	1	s
s	Black	0	NIL
t	Black	2	w
u	Black	3	t
v	Black	2	r
w	Black	1	s
x	Black	2	w
y	Gray	3	x

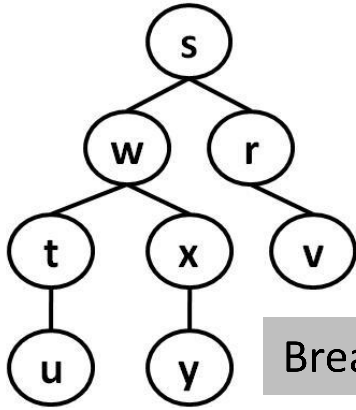
```
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

r	$\rightarrow$ s $\rightarrow$ v
s	$\rightarrow$ r $\rightarrow$ w
t	$\rightarrow$ u $\rightarrow$ w $\rightarrow$ x
u	$\rightarrow$ t $\rightarrow$ x $\rightarrow$ y
v	$\rightarrow$ r
w	$\rightarrow$ s $\rightarrow$ t $\rightarrow$ x
x	$\rightarrow$ t $\rightarrow$ u $\rightarrow$ w $\rightarrow$ y
y	$\rightarrow$ u $\rightarrow$ x

Q: y

BFS: s r w v t x u

Contd…



Breadth-first tree

Vertex	Color	Distance (d)	Predecessor (π)
r	Black	1	s
s	Black	0	NIL
t	Black	2	w
u	Black	3	t
v	Black	2	r
w	Black	1	s
x	Black	2	w
y	Black	3	x

while $Q \neq \emptyset$
  $u = \text{DEQUEUE}(Q)$
  for each $v \in G.Adj[u]$
    if $v.color == \text{WHITE}$
      $v.color = \text{GRAY}$
      $v.d = u.d + 1$
      $v.\pi = u$
      $\text{ENQUEUE}(Q, v)$
  $u.color = \text{BLACK}$

r	$\rightarrow$ s $\rightarrow$ v
s	$\rightarrow$ r $\rightarrow$ w
t	$\rightarrow$ u $\rightarrow$ w $\rightarrow$ x
u	$\rightarrow$ t $\rightarrow$ x $\rightarrow$ y
v	$\rightarrow$ r
w	$\rightarrow$ s $\rightarrow$ t $\rightarrow$ x
x	$\rightarrow$ t $\rightarrow$ u $\rightarrow$ w $\rightarrow$ y
y	$\rightarrow$ u $\rightarrow$ x

Q: ϕ

BFS: s r w v t x u y

Depth-first search (DFS)

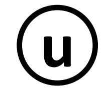
- Search “deeper” in the graph whenever possible.
- If any undiscovered vertices remain, then DFS selects one of them as a new-source, and it repeats the search from that source.
- The algorithm continues until it has discovered every vertex.
- It produces a “depth-first forest” comprising several “depth-first trees”.
- It works on both directed and undirected graphs.

Procedure DFS

- Assumptions:
  - The input graph G = (V,E) is represented using adjacency lists.
  - Each vertex in the graph has following additional attributes.
    - Color: Can be white (undiscovered), gray (when discovered), or black (all adjacent vertices have been examined completely).
    - π: predecessor of a vertex. Can be NIL.
    - d: Timestamp to record when the vertex is first discovered.
    - f: Timestamp to record when the vertex is examined completely.

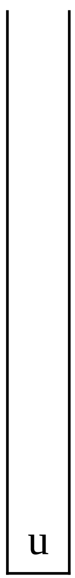
Compute DFS - Undirected

Predecessor sub-graph

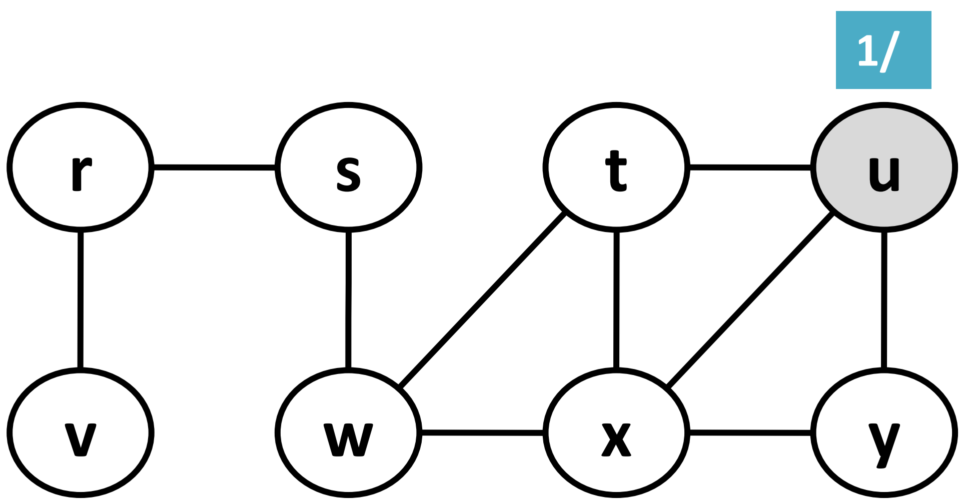


Depth-first forest

- DFS: u

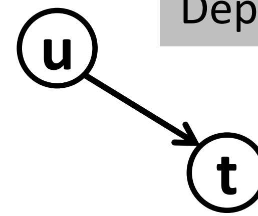


Stack



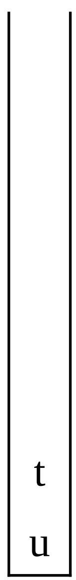
Compute DFS - Undirected

Predecessor sub-graph

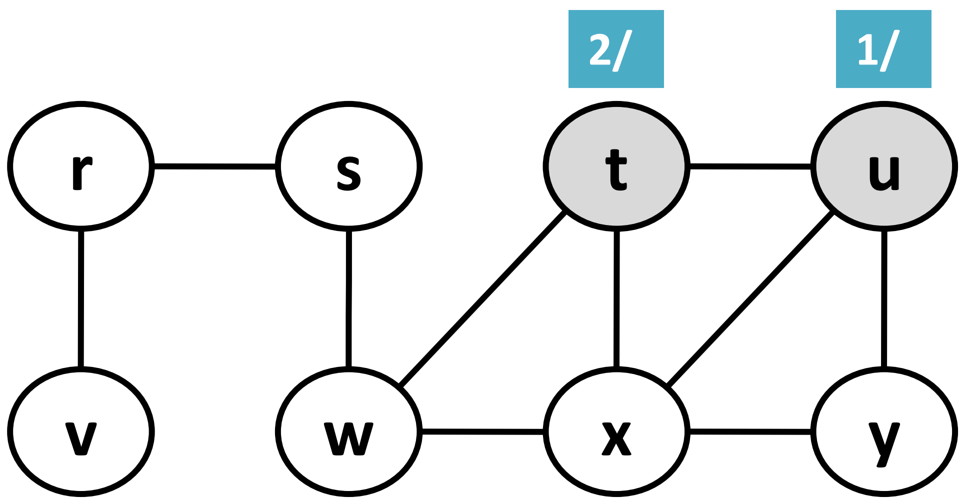


Depth-first forest

- DFS: u t

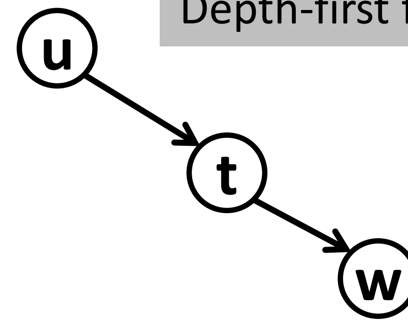


Stack



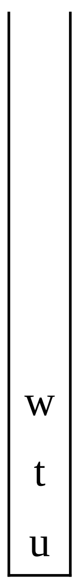
Compute DFS - Undirected

Predecessor sub-graph



Depth-first forest

- DFS: u t w



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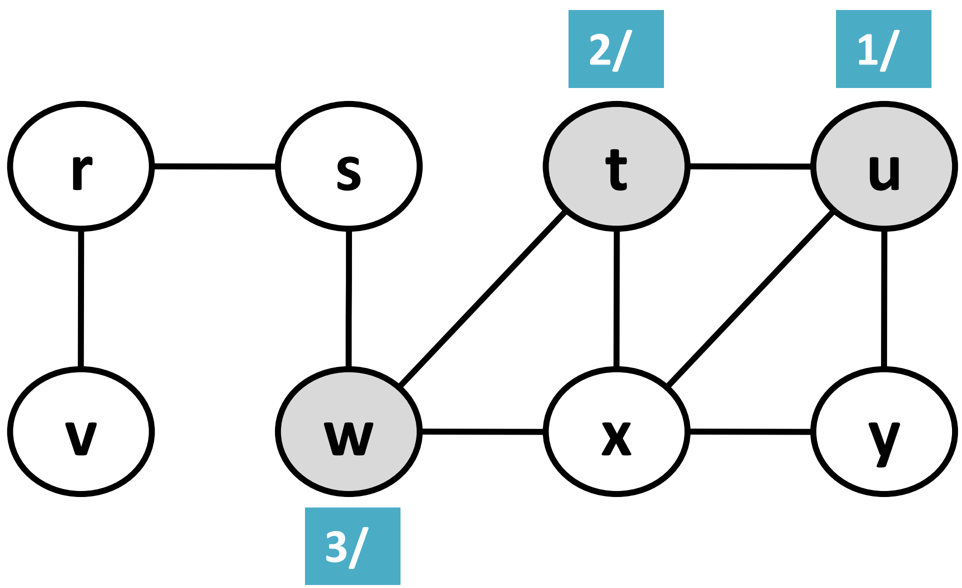
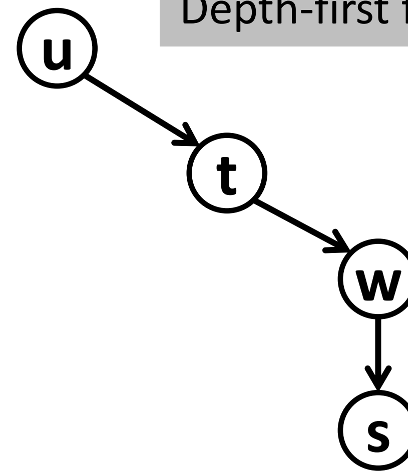


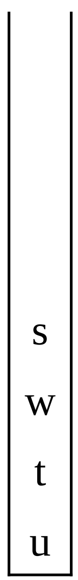
Compute DFS - Undirected

Predecessor sub-graph



Depth-first forest

- DFS: u t w s



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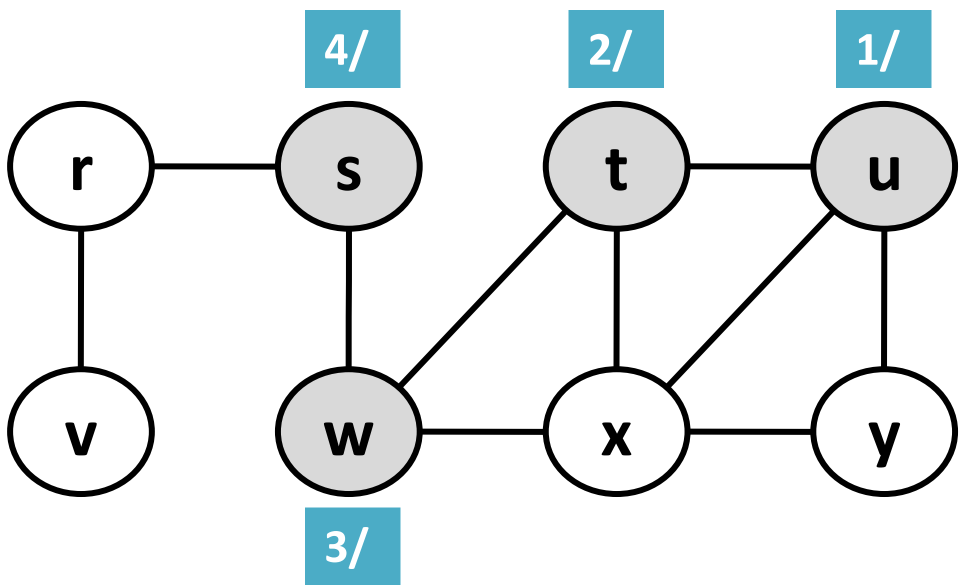
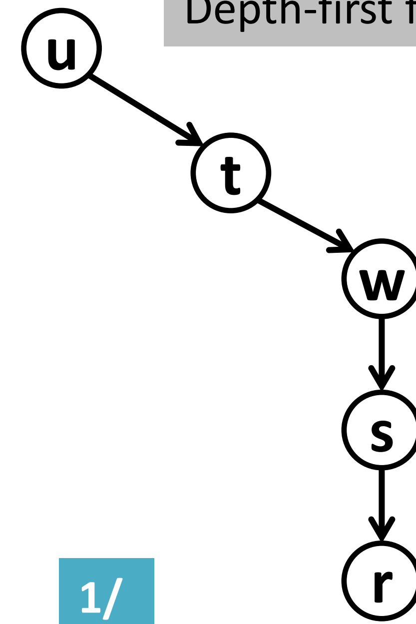


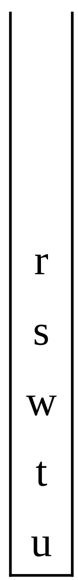
Compute DFS - Undirected

- DFS: u t w s r

Predecessor sub-graph

Depth-first forest





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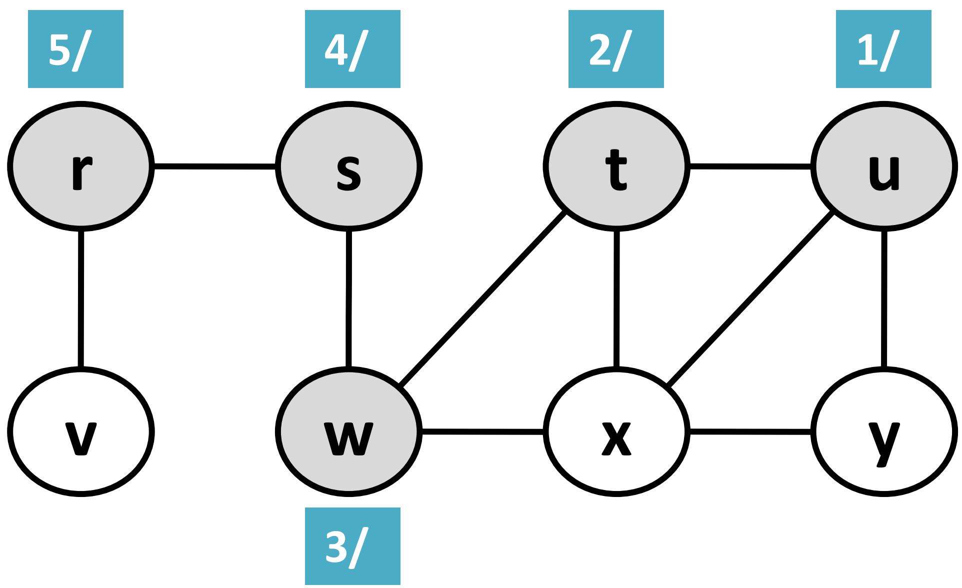
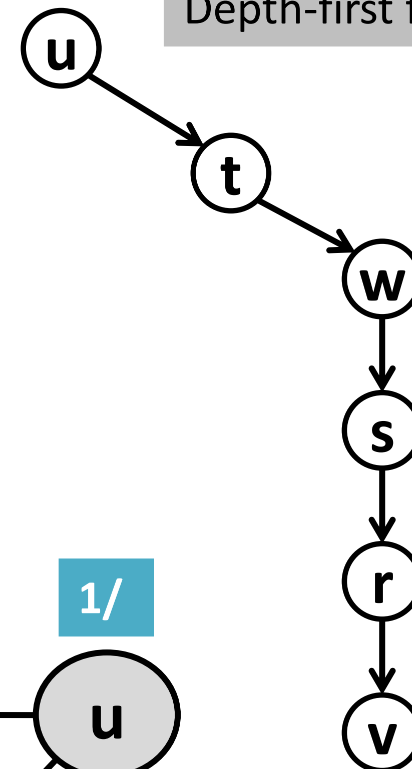


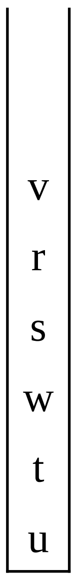
Compute DFS - Undirected

- DFS: u t w s r v

Predecessor sub-graph

Depth-first forest





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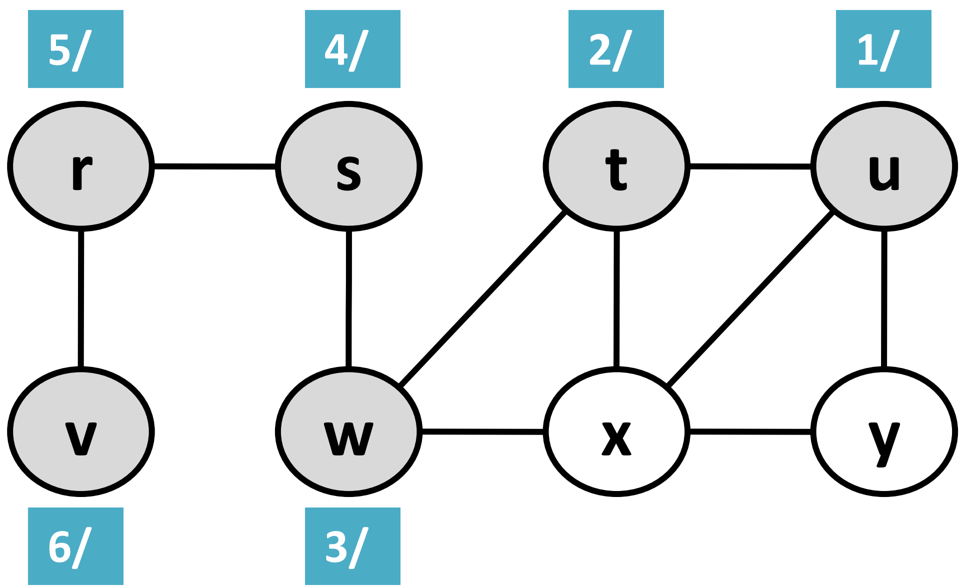
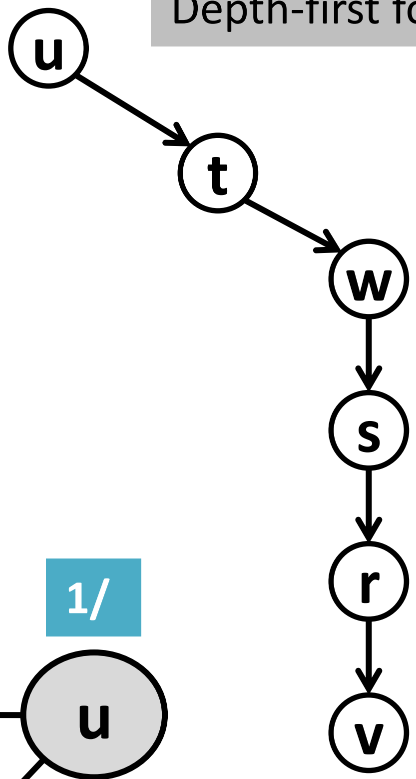


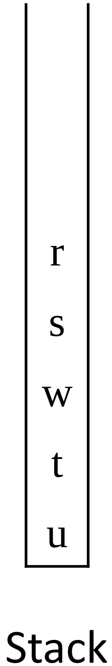
Compute DFS - Undirected

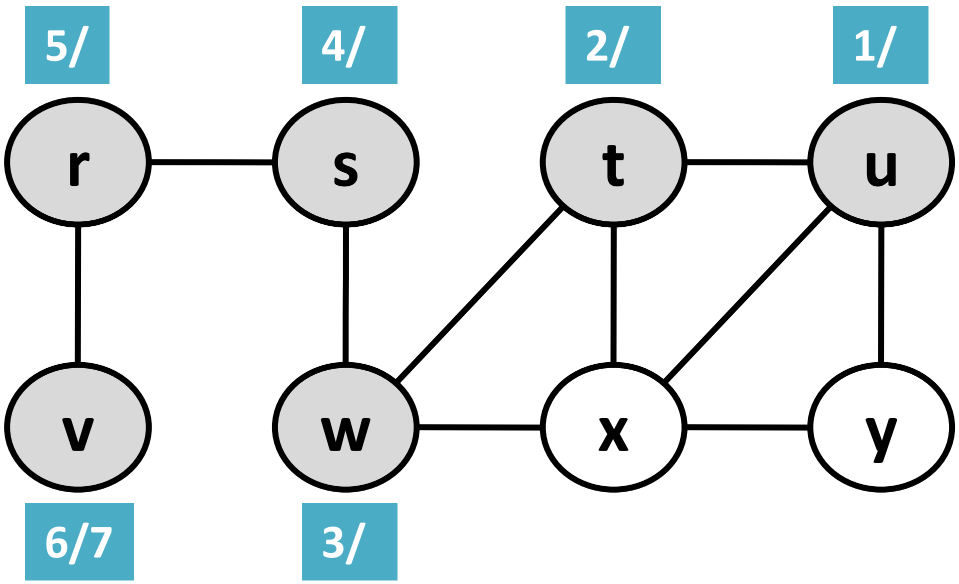
Predecessor sub-graph

Depth-first forest



- DFS: u t w s r v

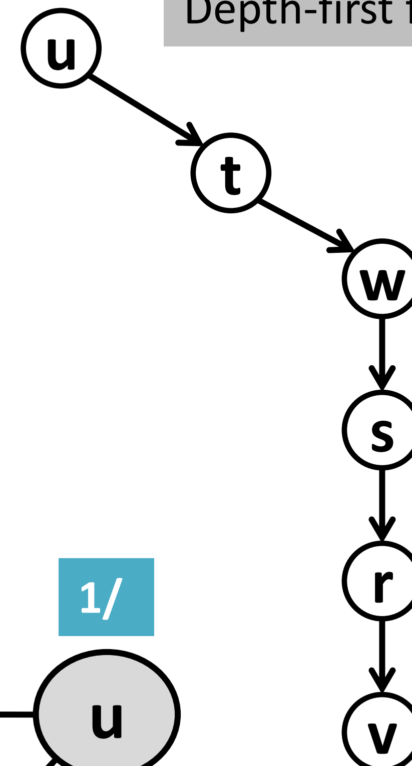




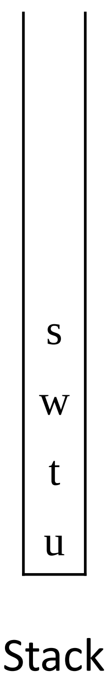
Compute DFS - Undirected

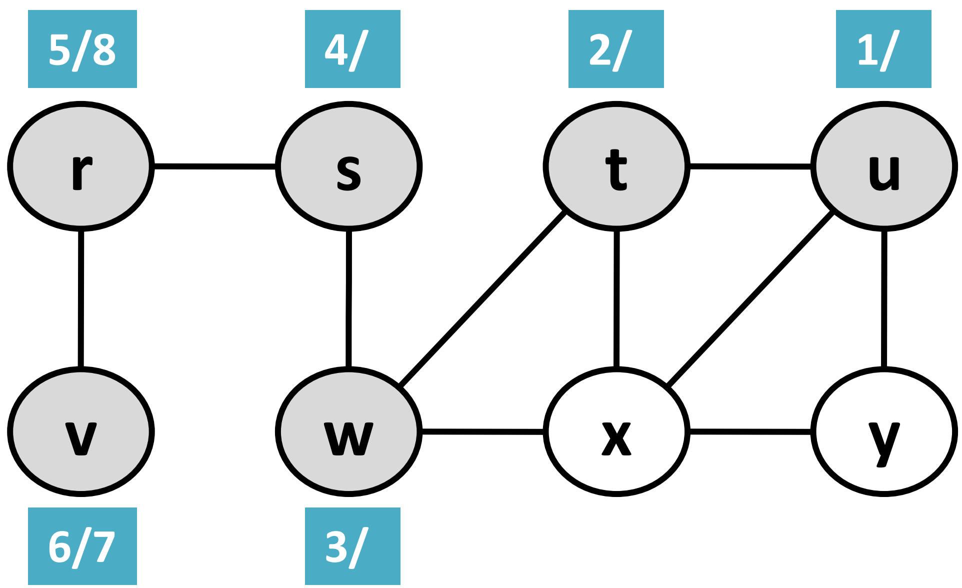
Predecessor sub-graph

Depth-first forest



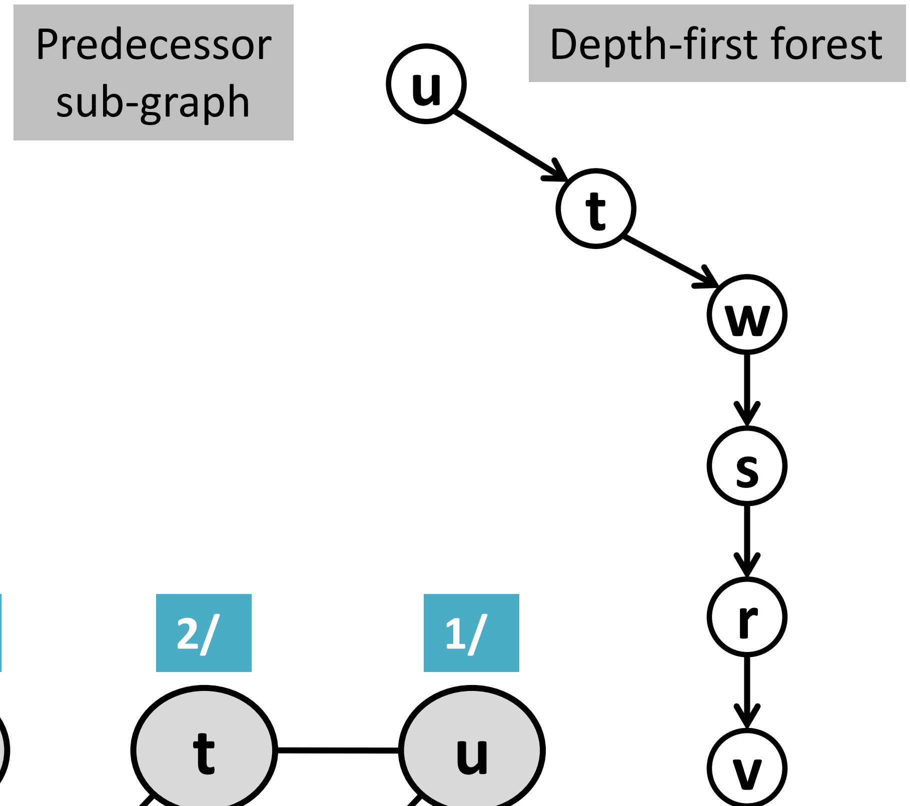
- DFS: u t w s r v

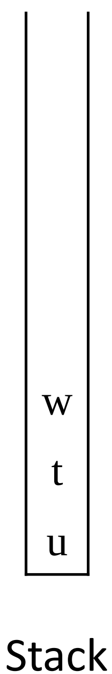


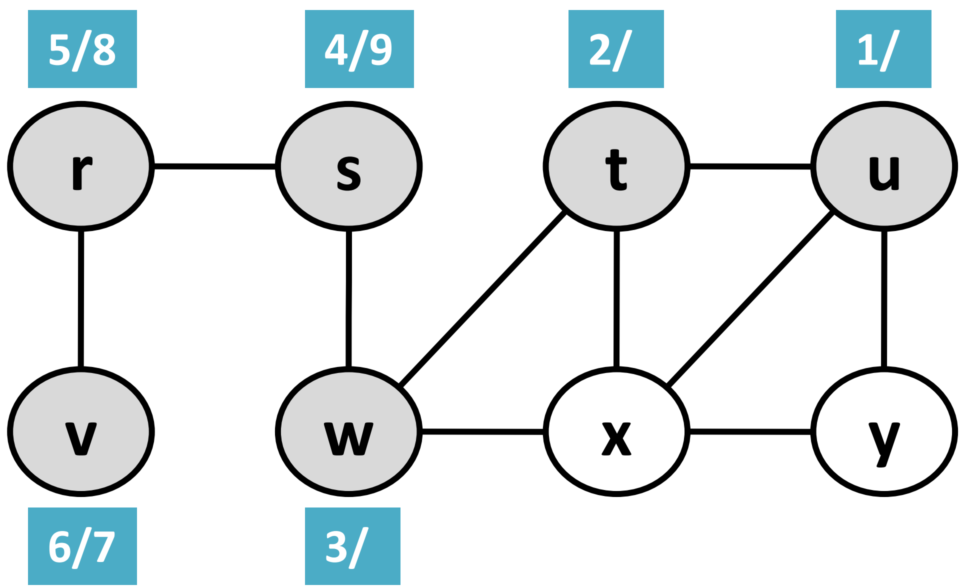


Compute DFS - Undirected

- DFS: u t w s r v



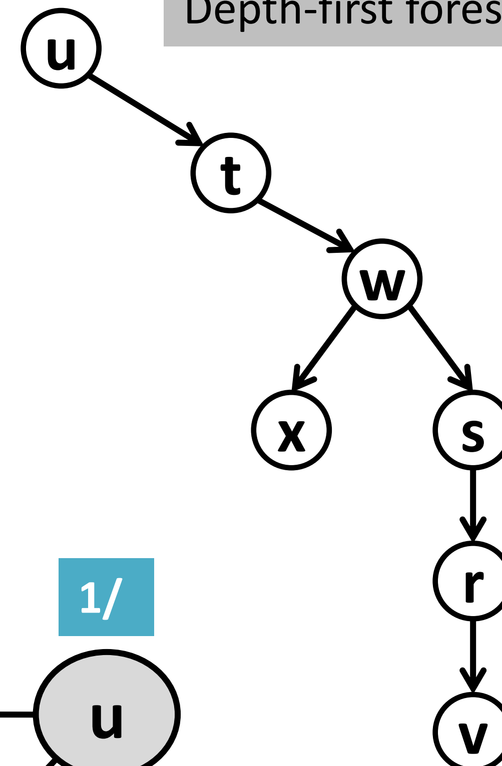




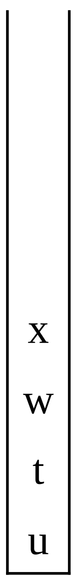
Compute DFS - Undirected

Predecessor sub-graph

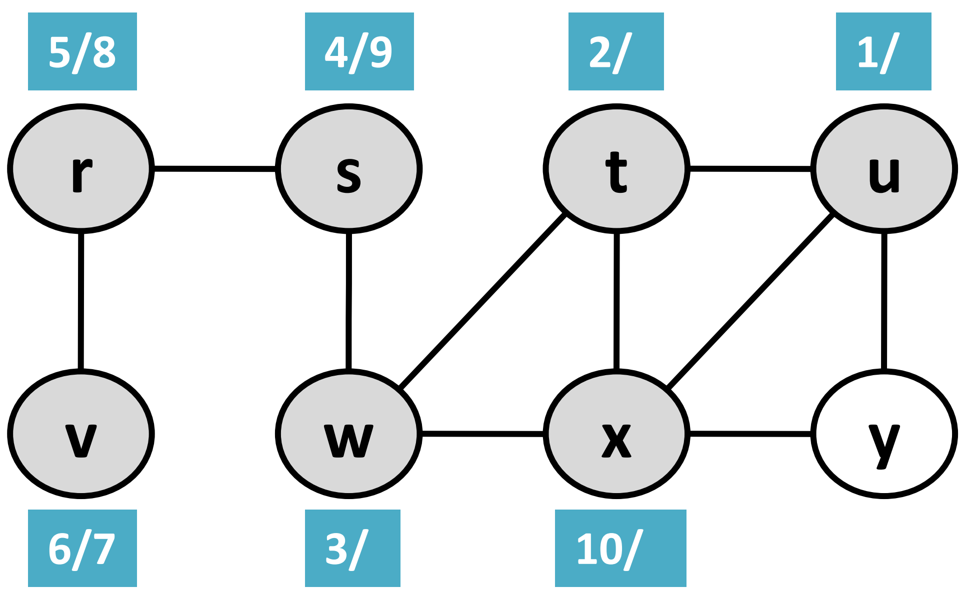
Depth-first forest



- DFS: u t w s r v x

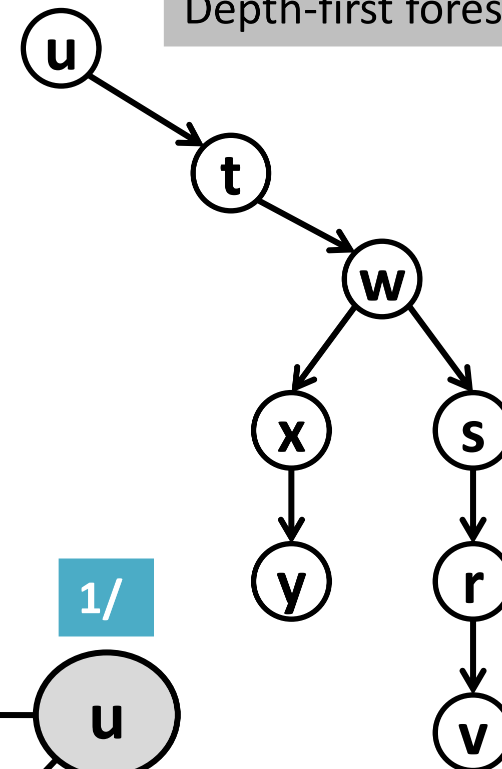


Stack



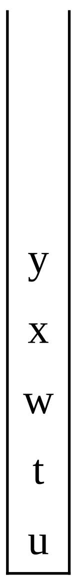
Compute DFS - Undirected

Predecessor sub-graph

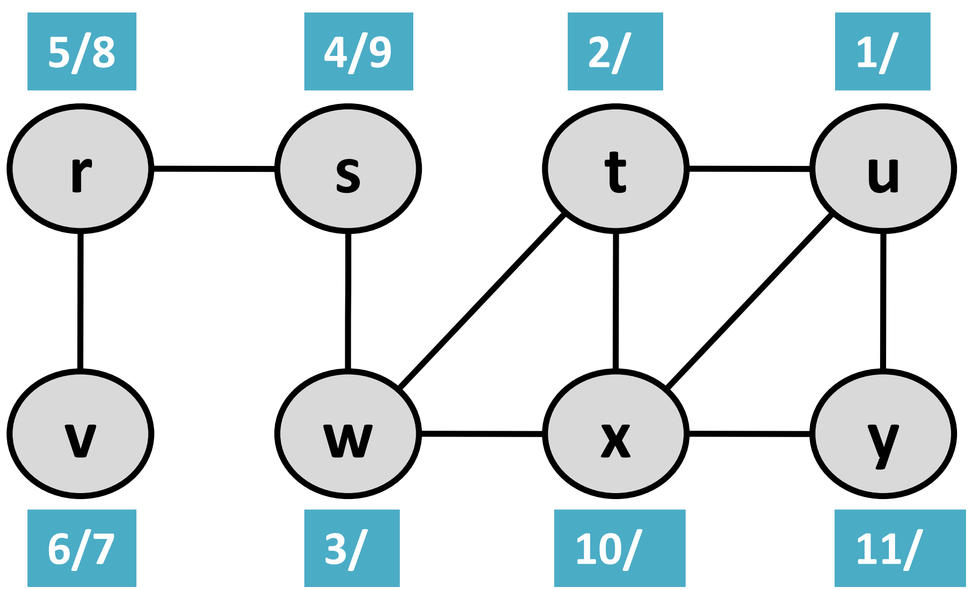


Depth-first forest

- DFS: u t w s r v x y



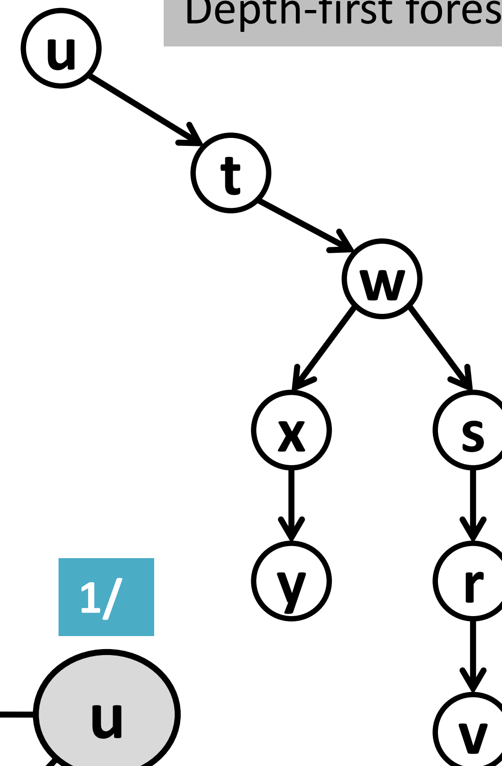
Stack



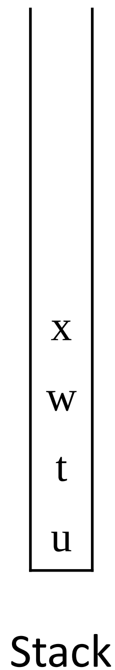
Compute DFS - Undirected

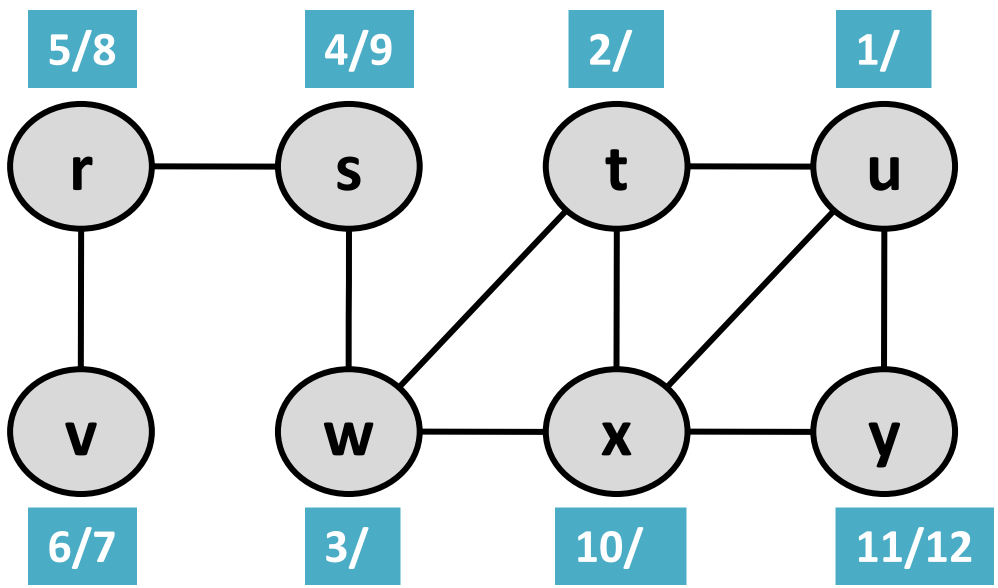
Predecessor sub-graph

Depth-first forest



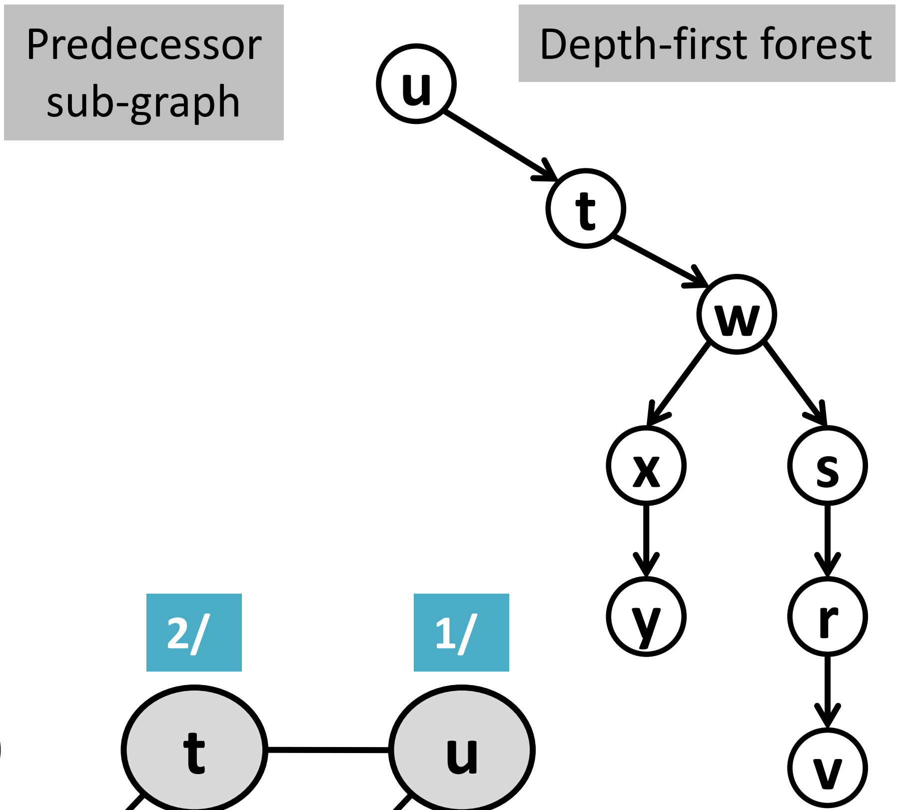
- DFS: u t w s r v x y

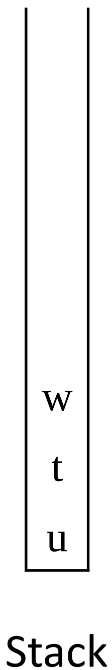


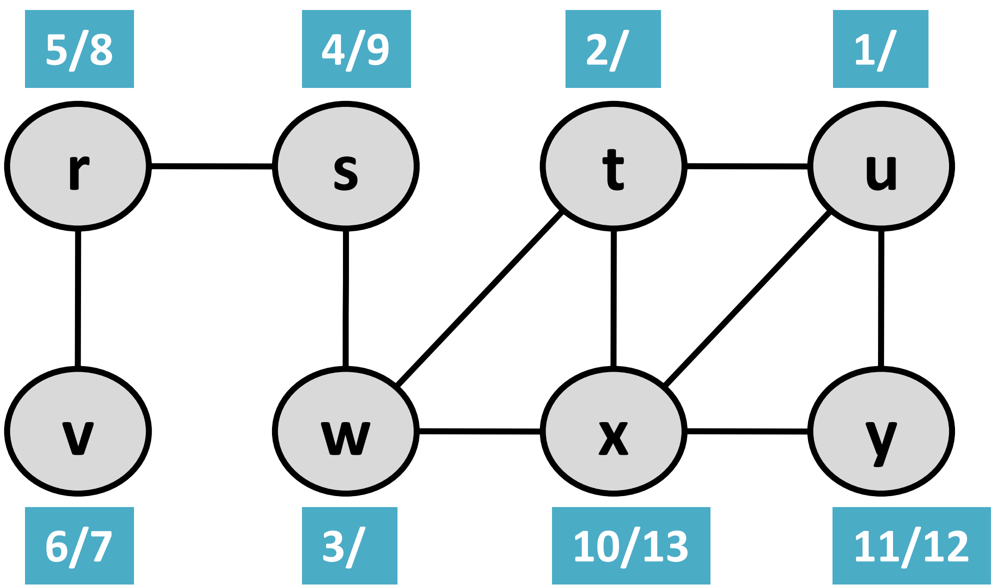


Compute DFS - Undirected

- DFS: u t w s r v x y

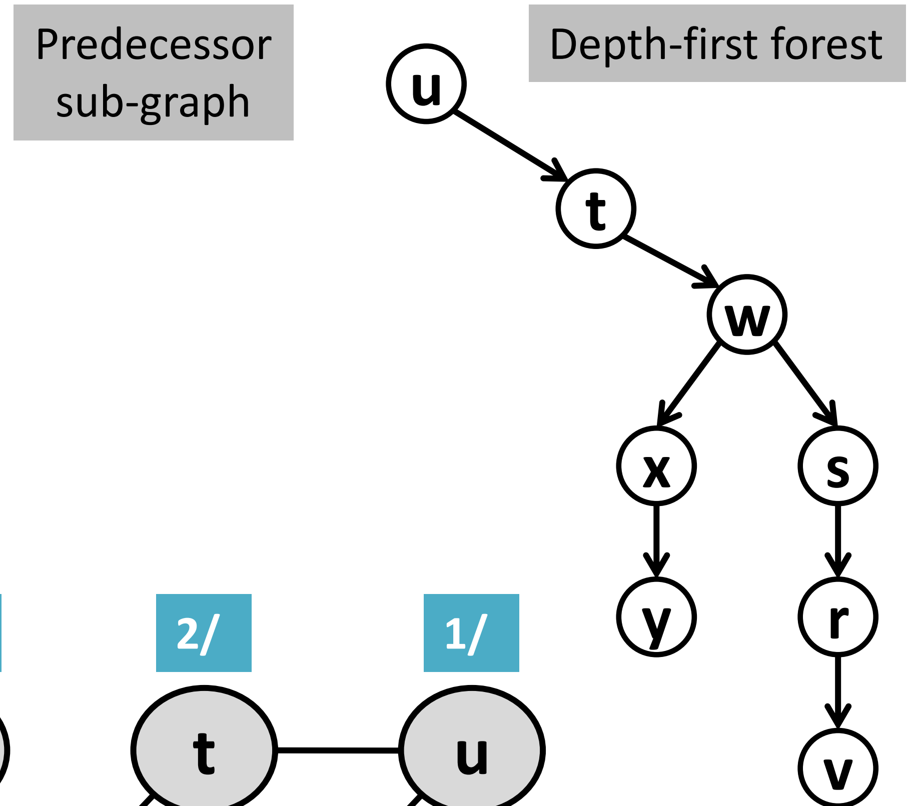


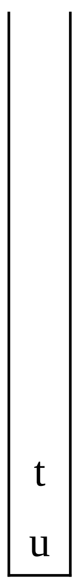




Compute DFS - Undirected

- DFS: u t w s r v x y





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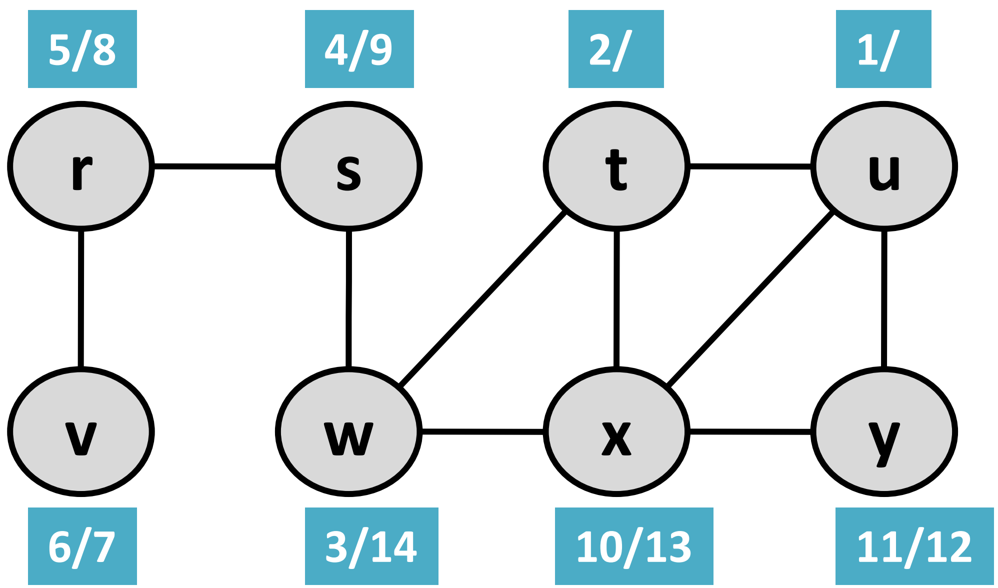
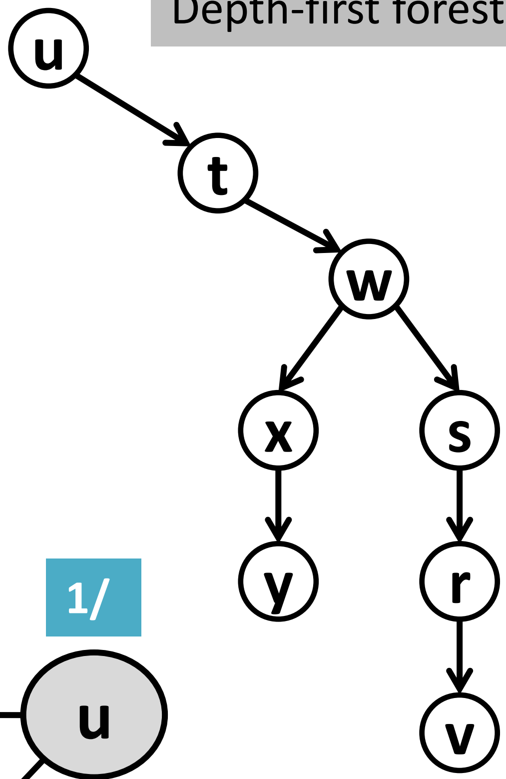


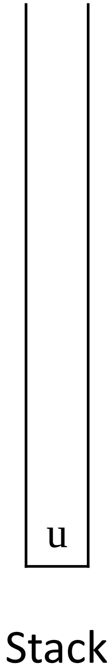
Compute DFS - Undirected

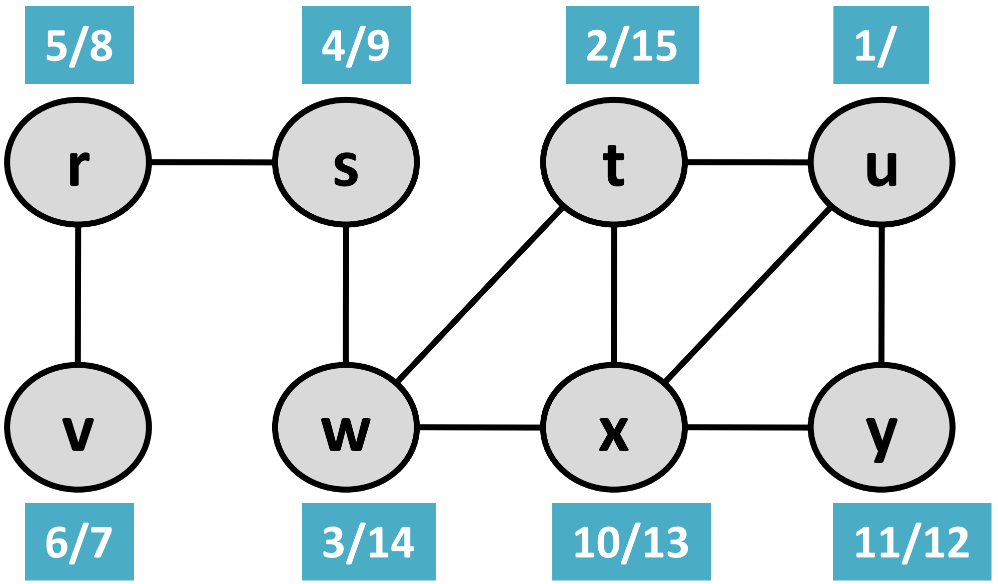
- DFS: u t w s r v x y

Predecessor sub-graph

Depth-first forest



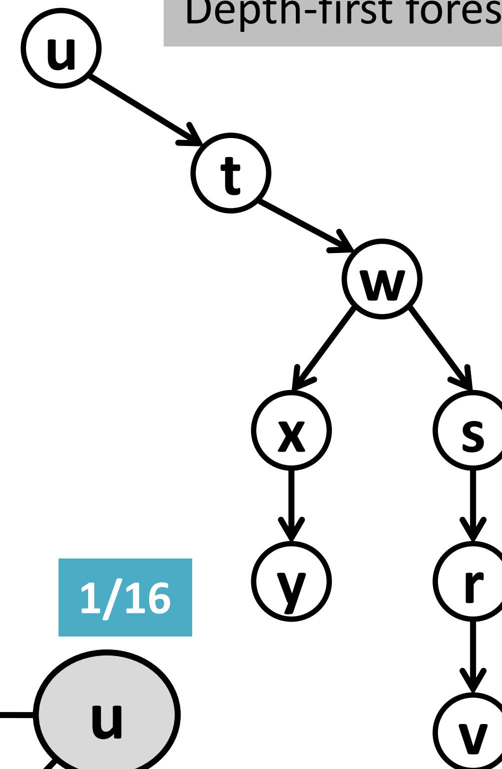




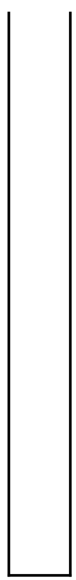
Compute DFS - Undirected

Predecessor sub-graph

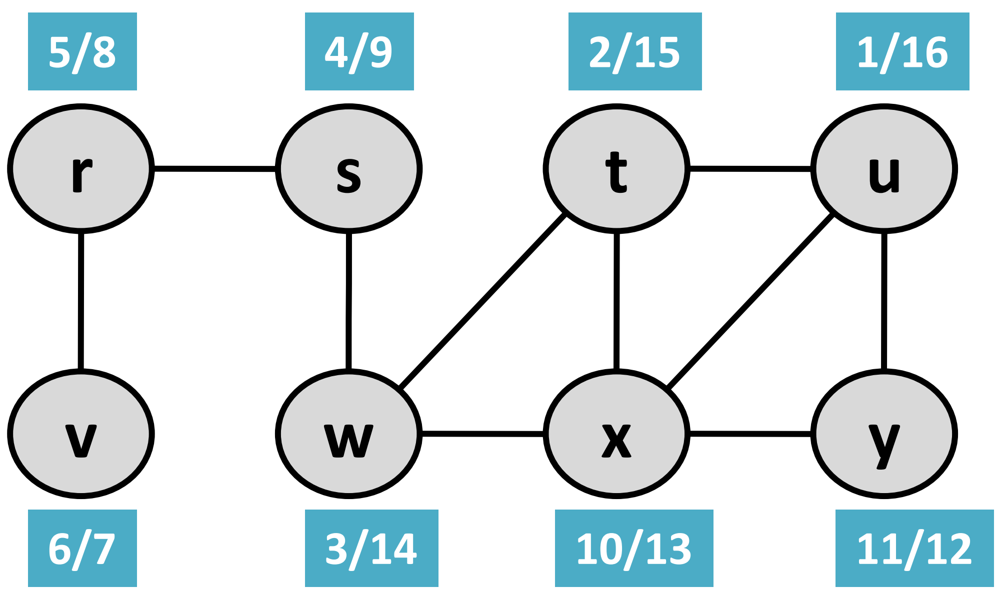
Depth-first forest



- DFS: u t w s r v x y



Stack

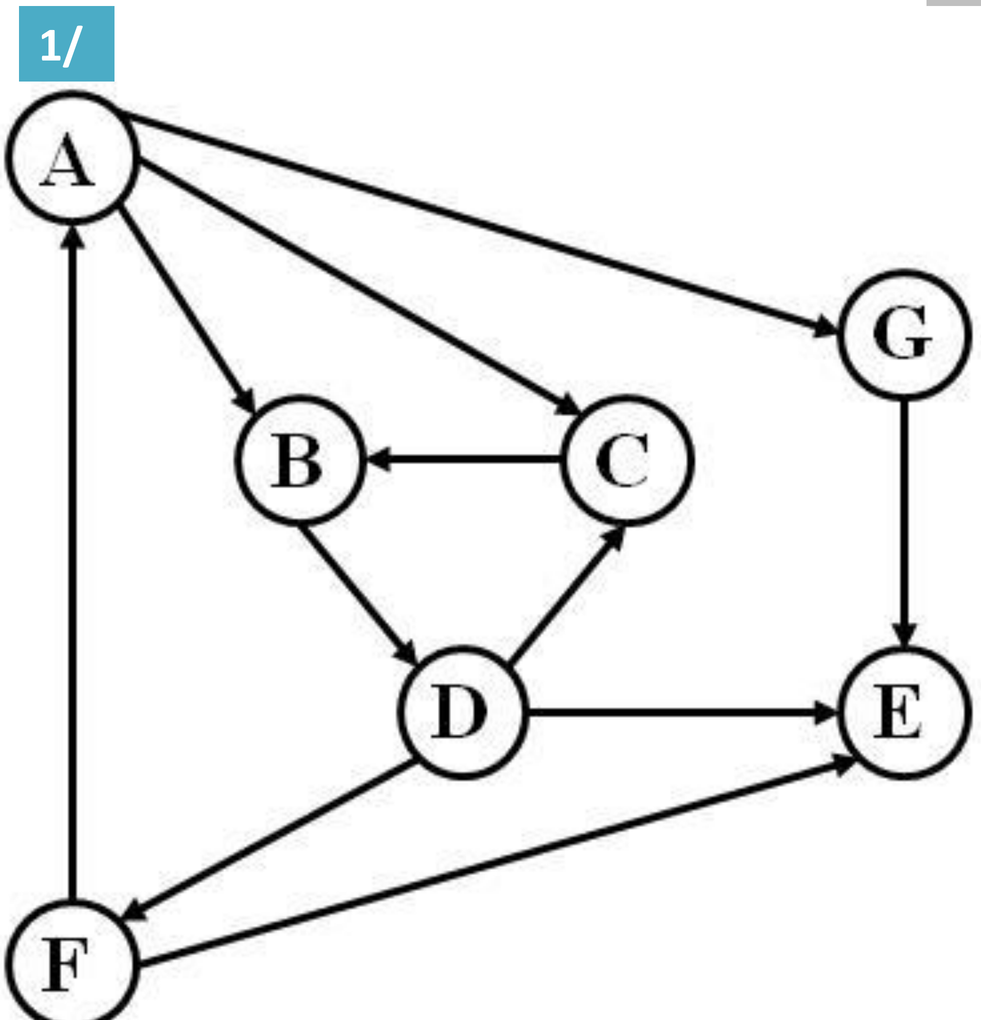


Compute DFS - Directed

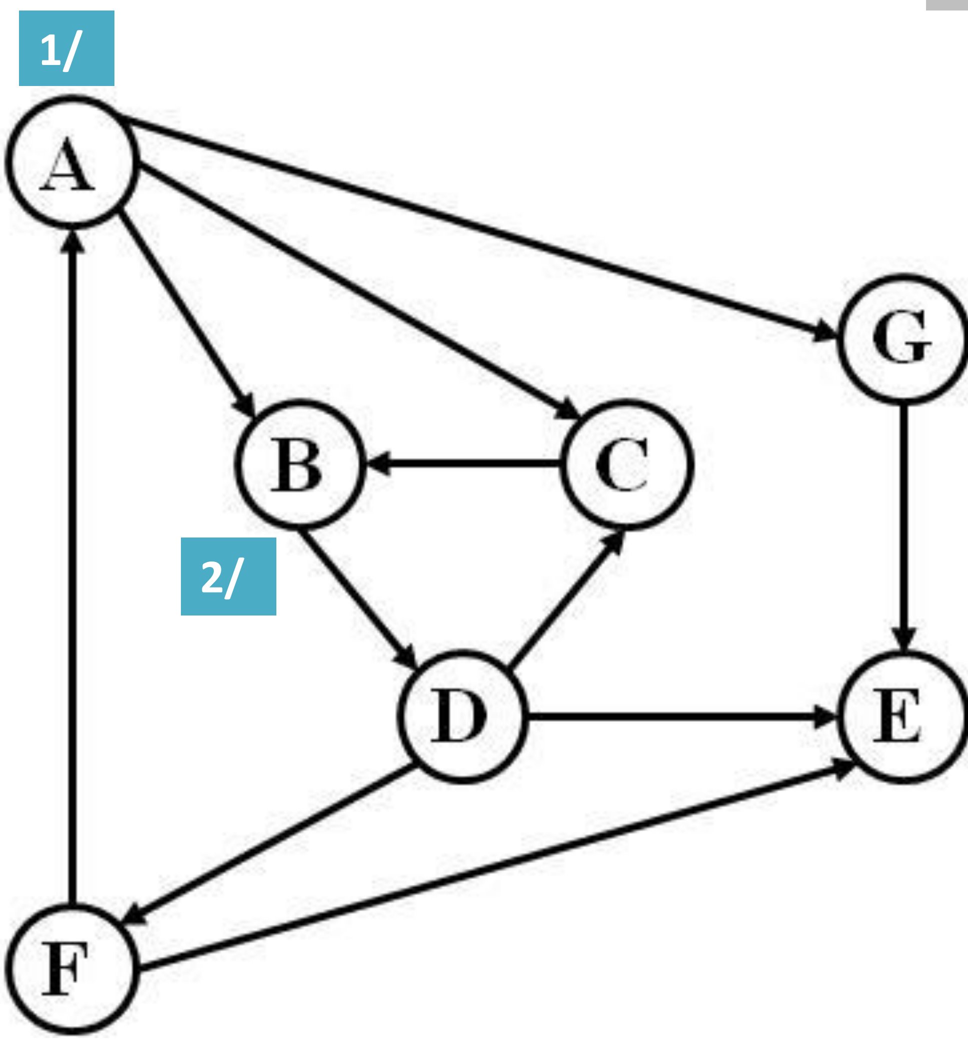
Predecessor sub-graph

A

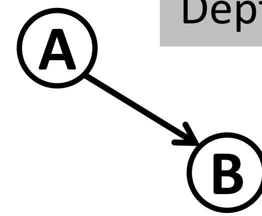
Depth-first forest



Compute DFS - Directed

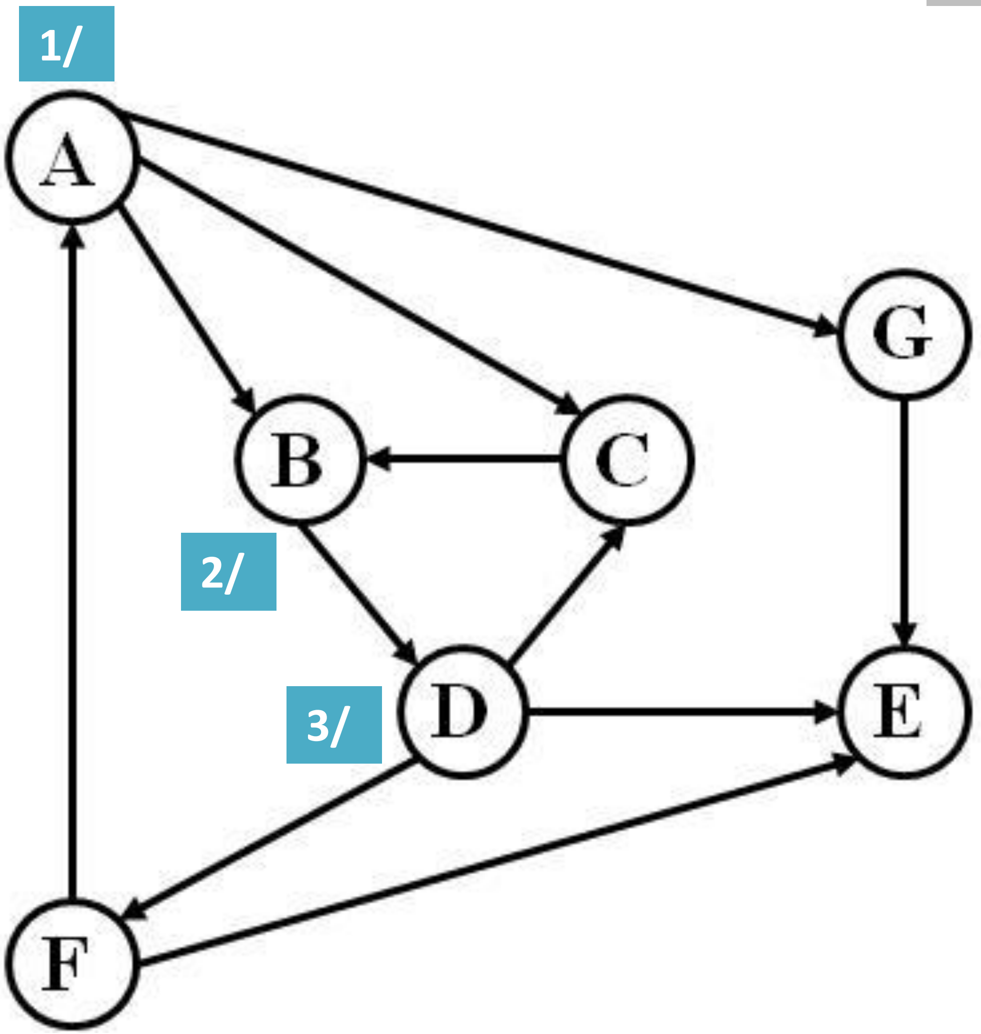


Predecessor sub-graph

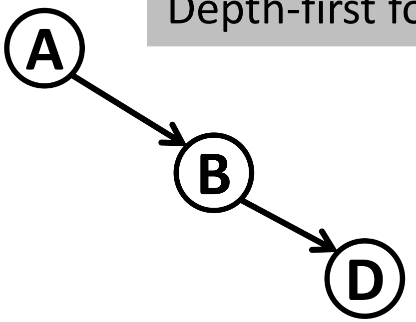


Depth-first forest

Compute DFS - Directed

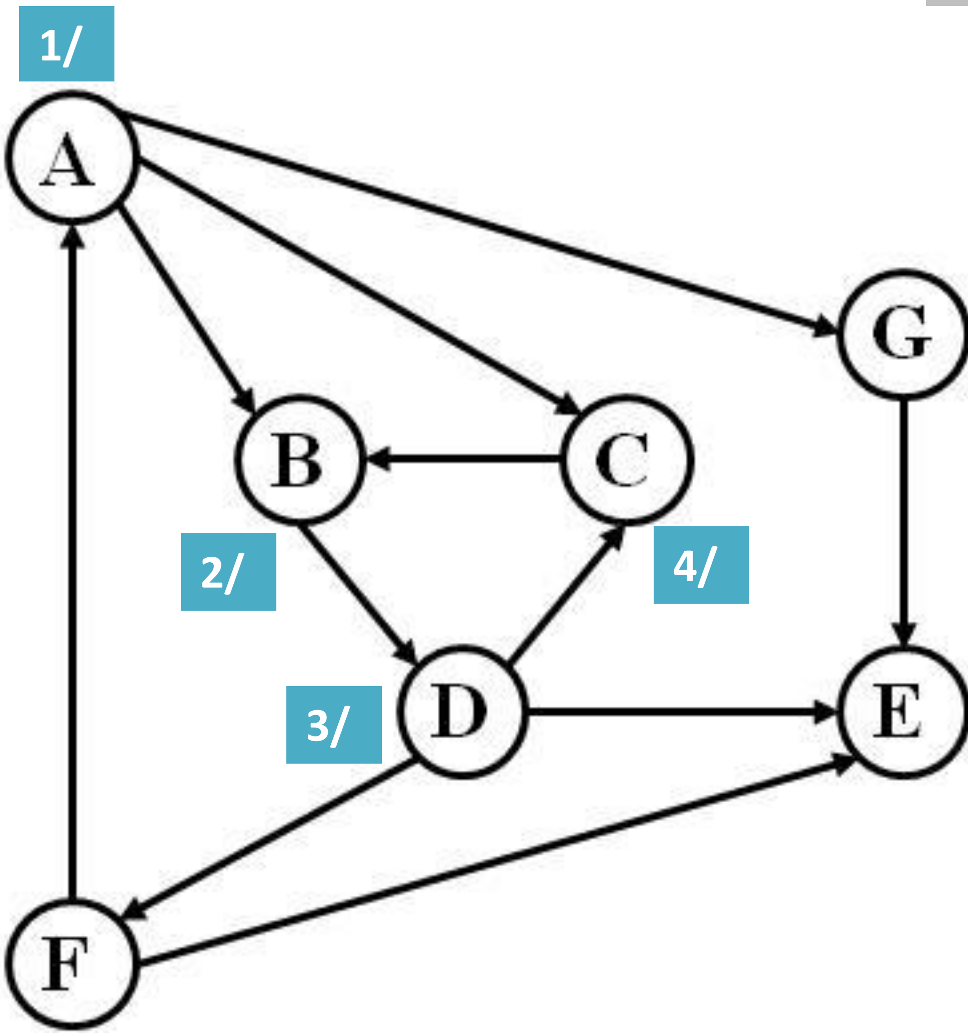


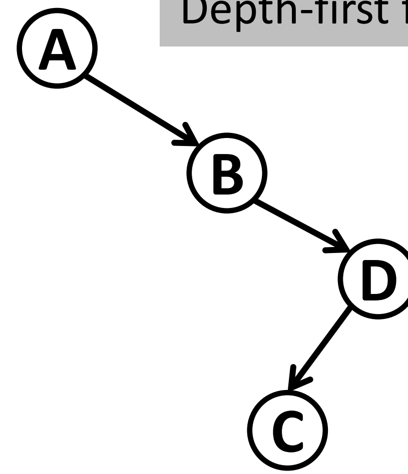
Predecessor sub-graph



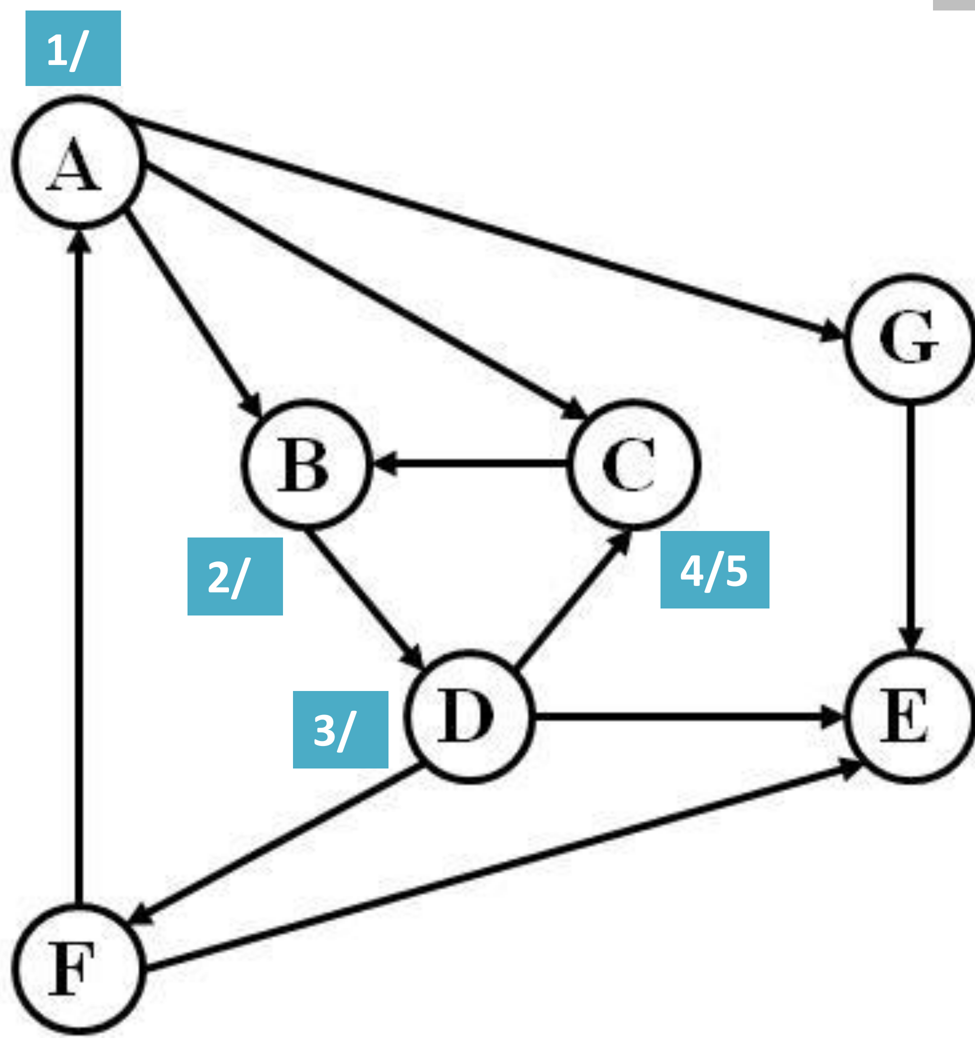
Depth-first forest

Compute DFS - Directed



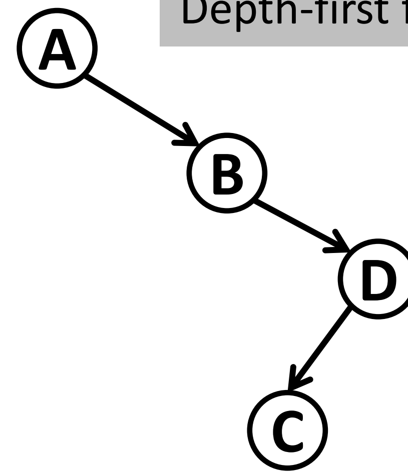


Compute DFS - Directed

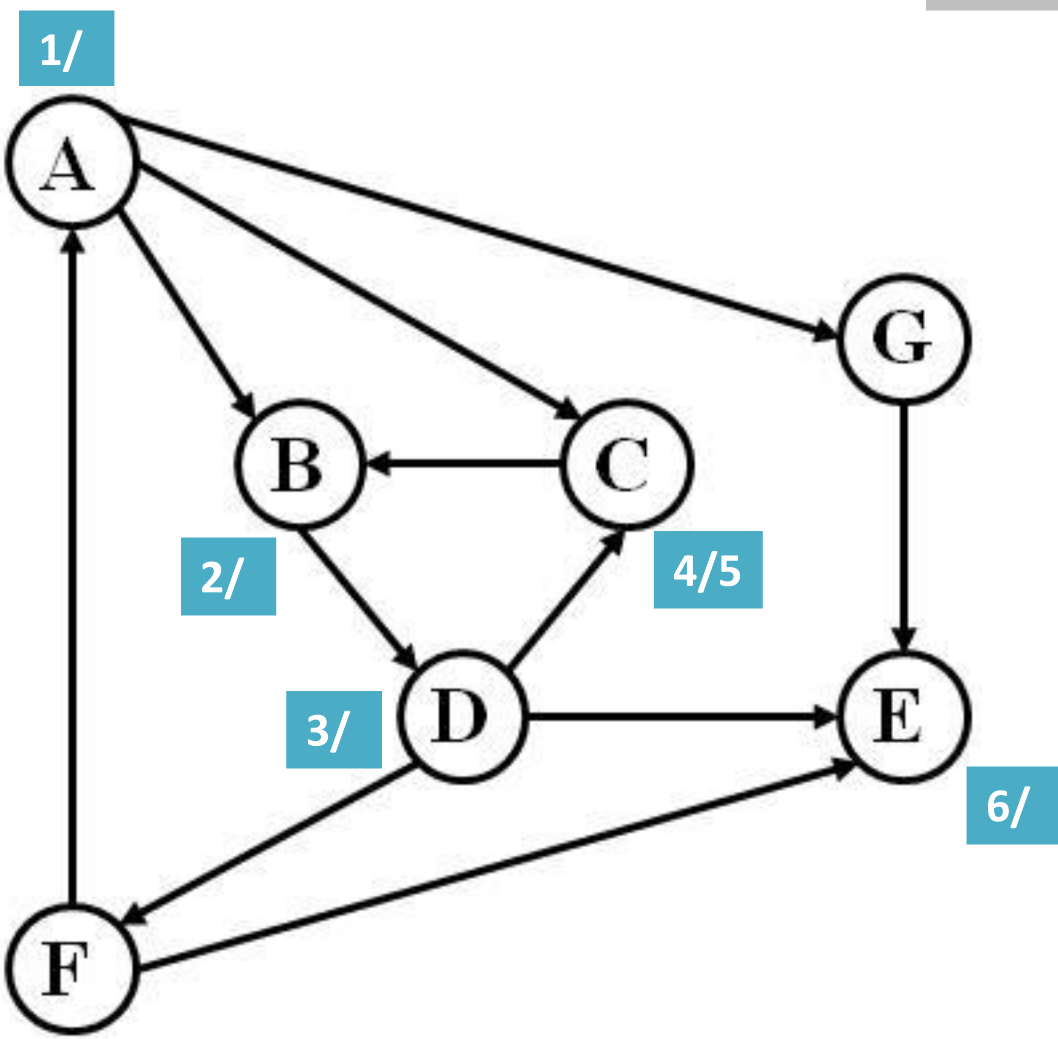


Predecessor sub-graph

Depth-first forest

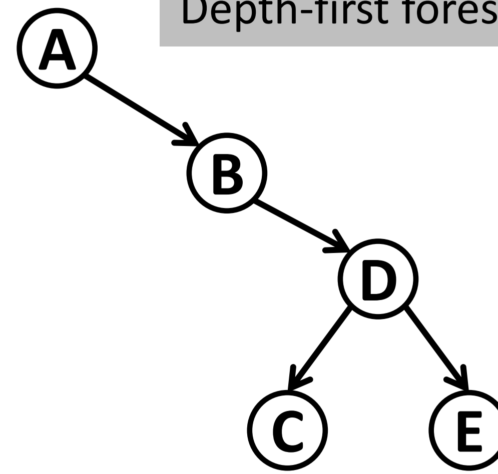


Compute DFS - Directed

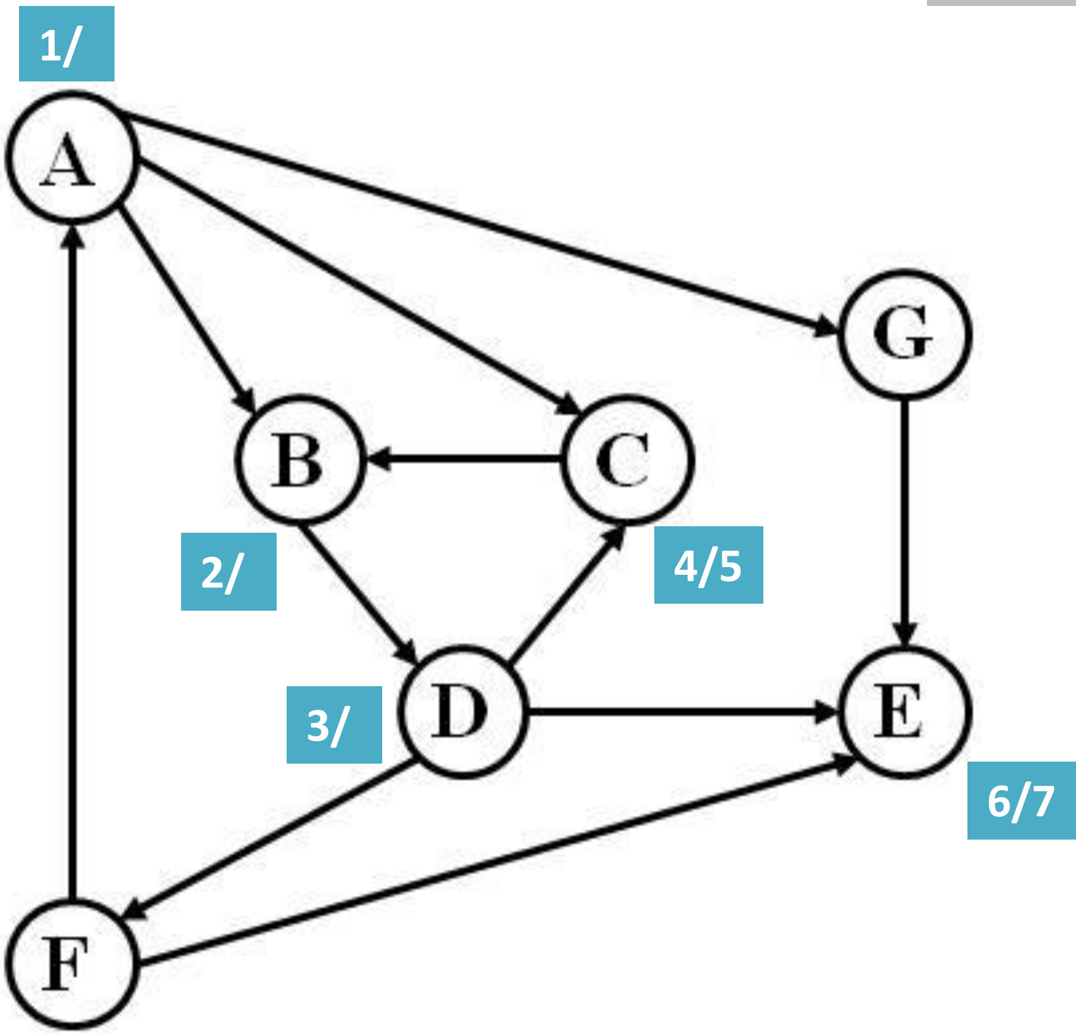


Predecessor sub-graph

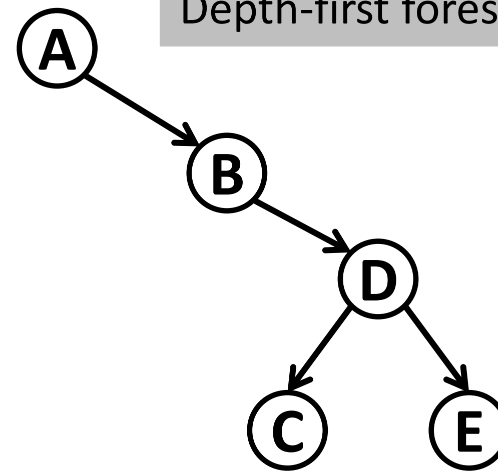
Depth-first forest



Compute DFS - Directed

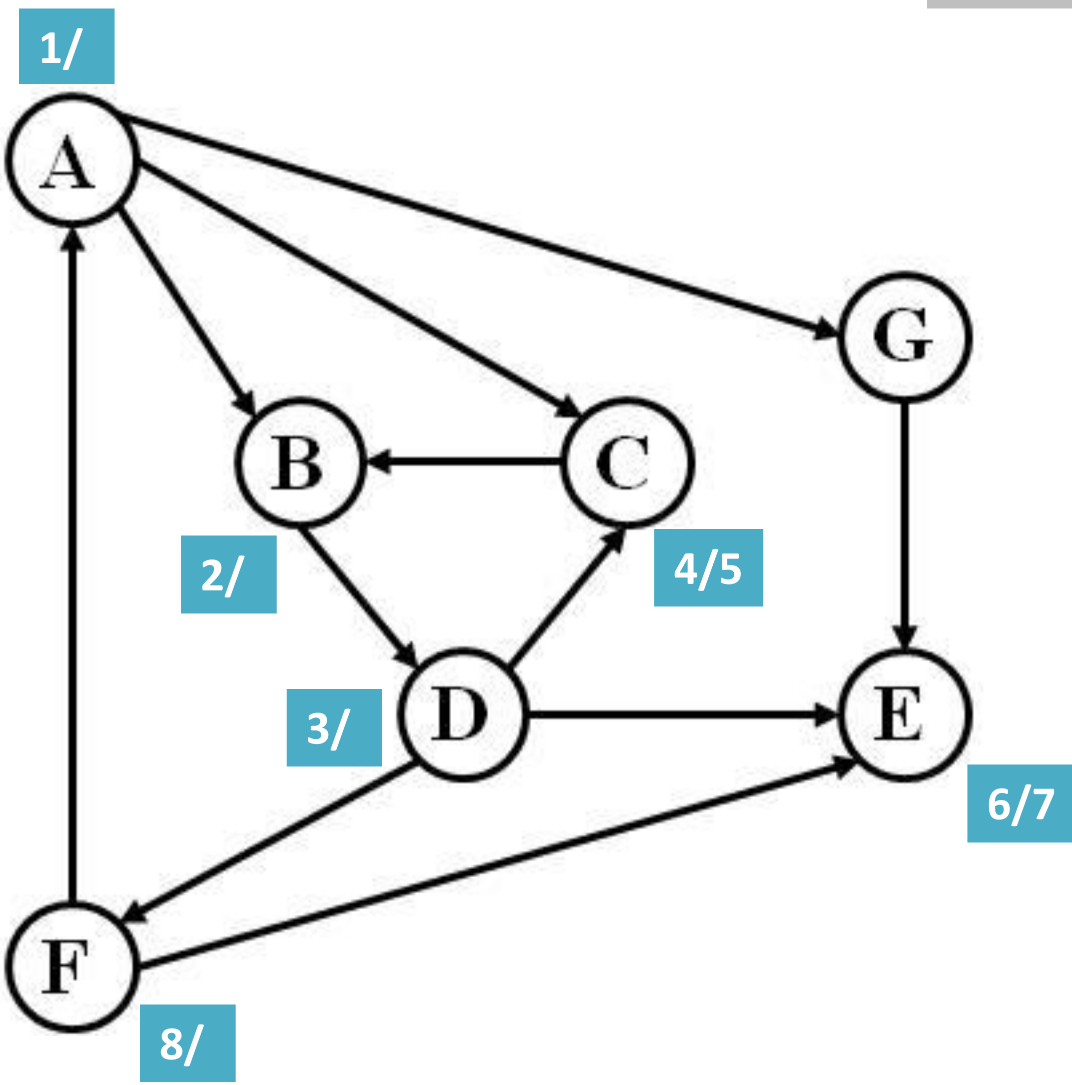


Predecessor sub-graph



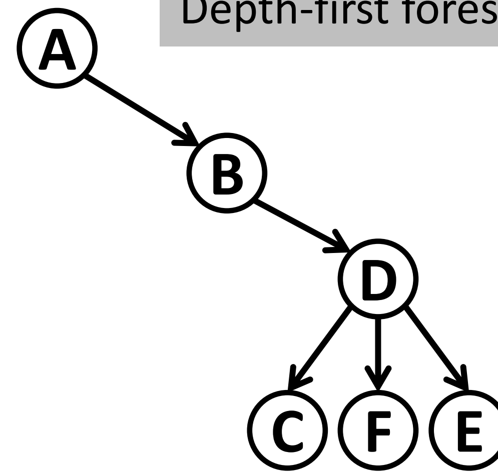
Depth-first forest

Compute DFS - Directed

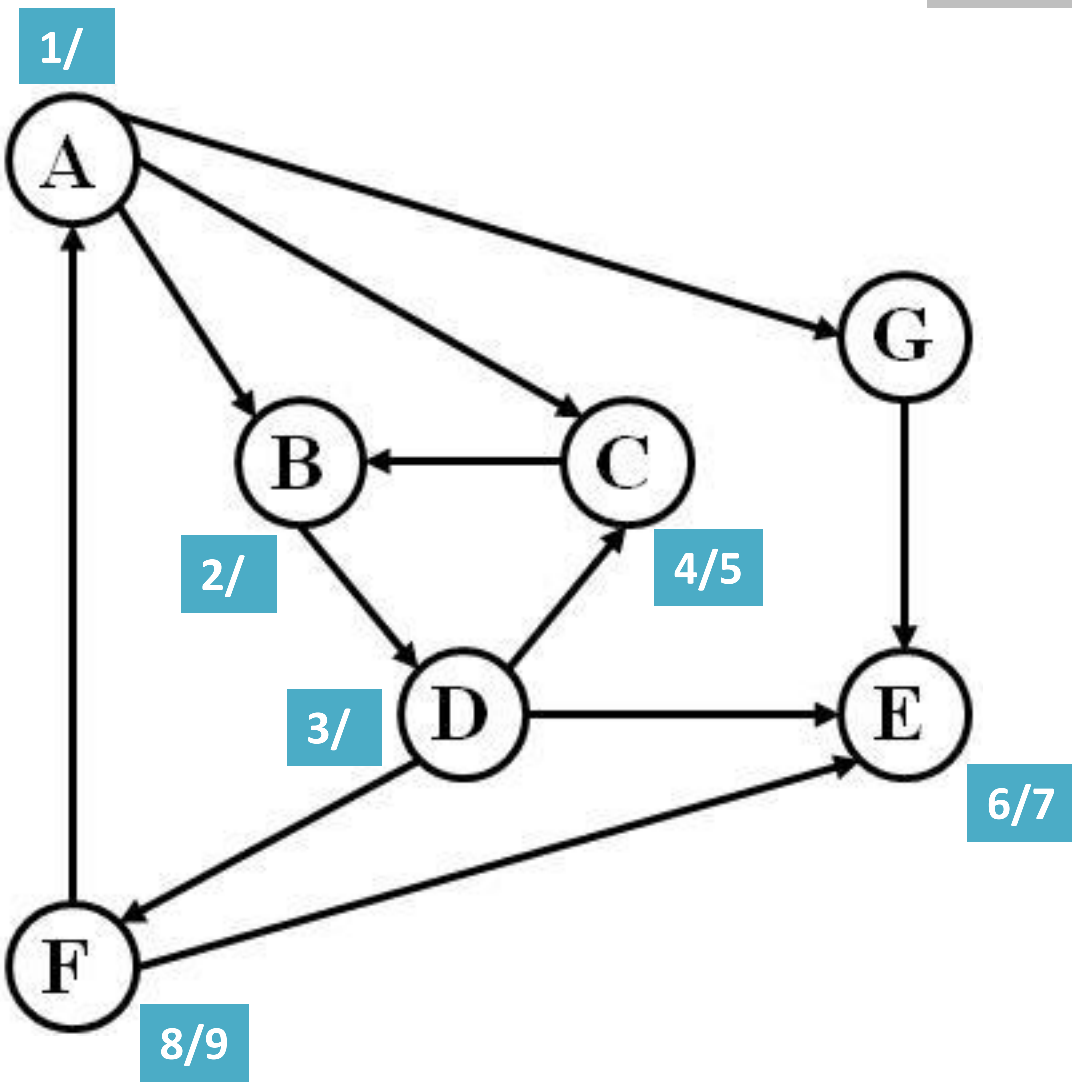


Predecessor sub-graph

Depth-first forest

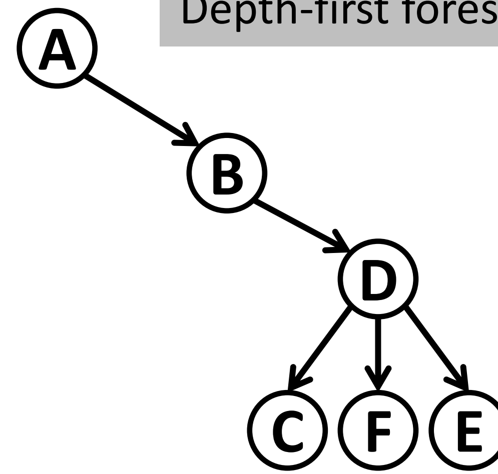


Compute DFS - Directed

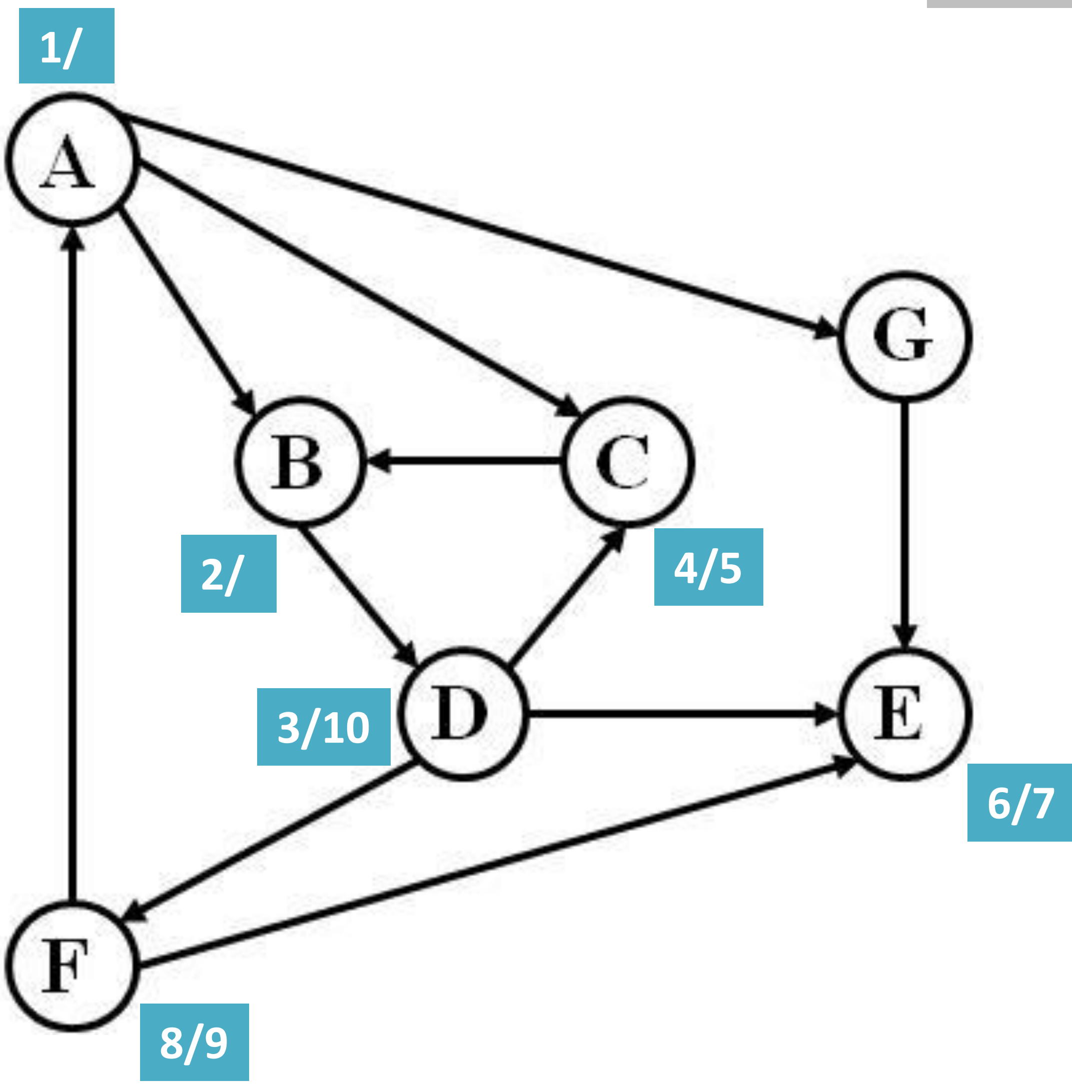


Predecessor sub-graph

Depth-first forest

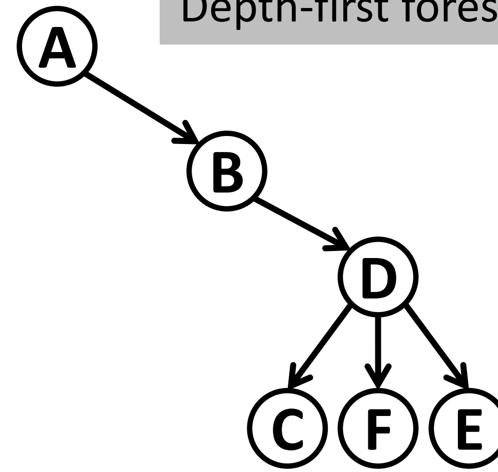


Compute DFS - Directed

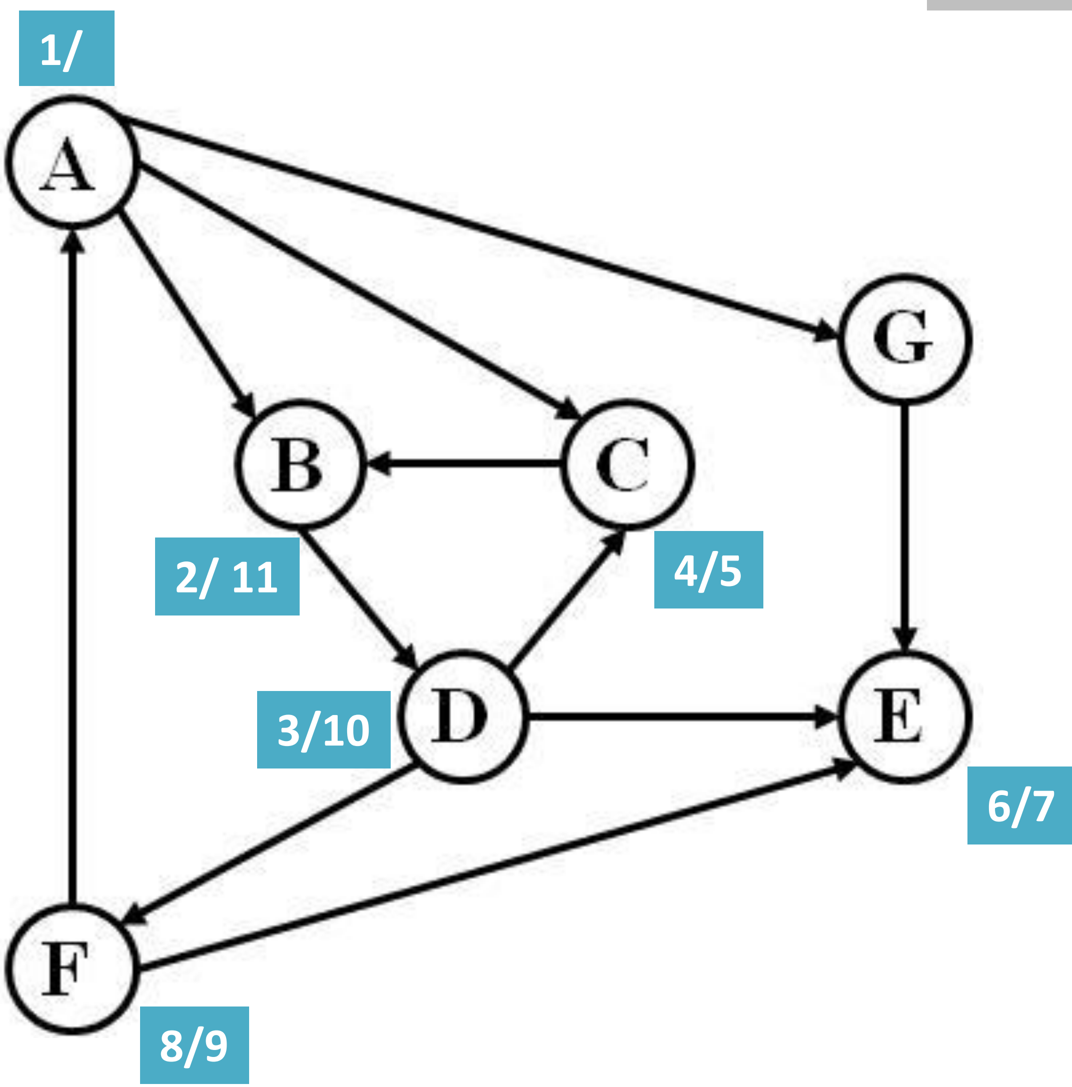


Predecessor sub-graph

Depth-first forest

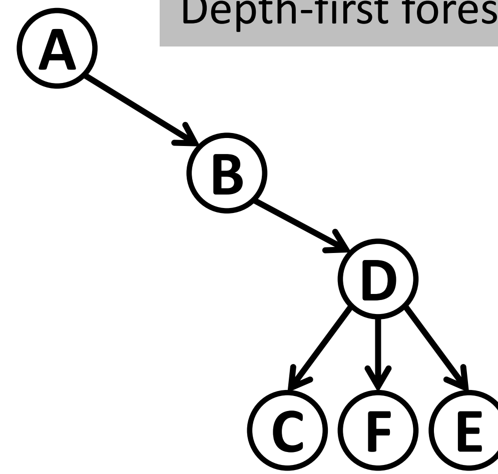


Compute DFS - Directed

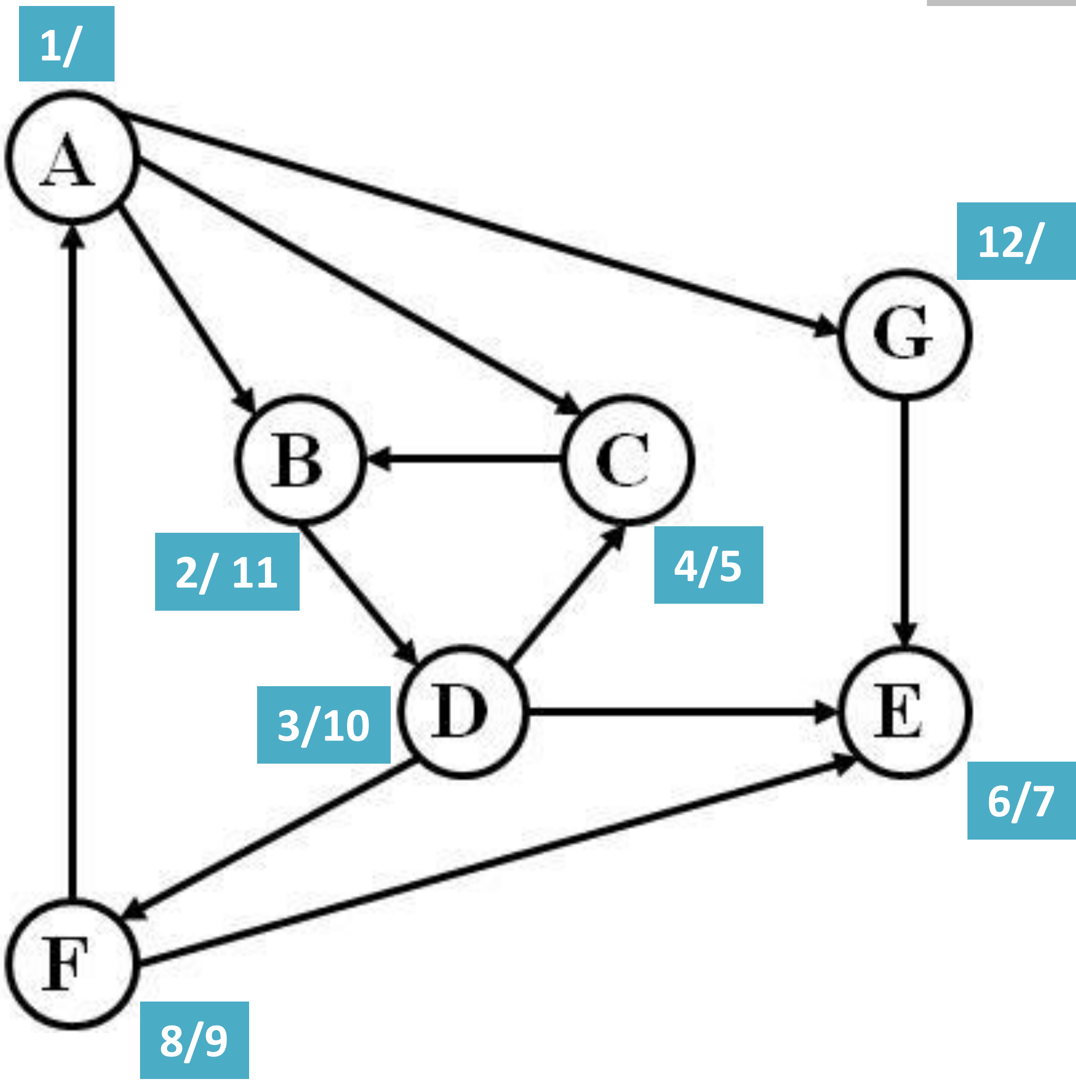


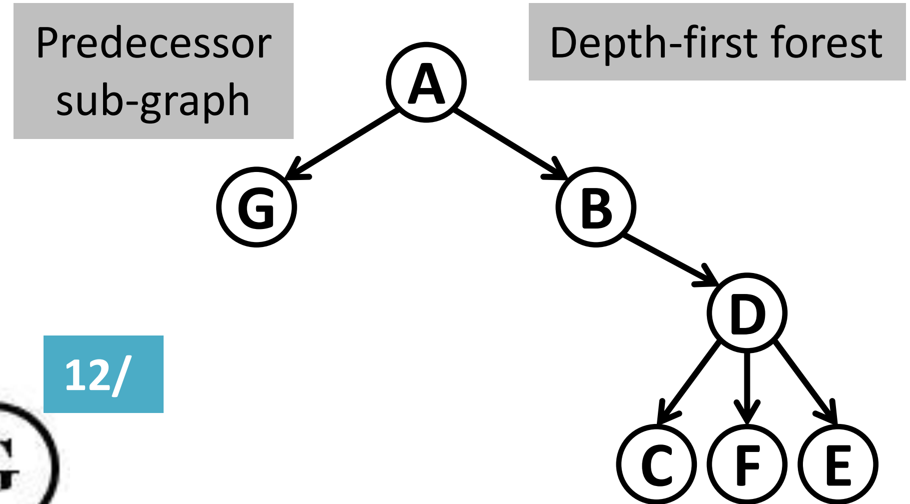
Predecessor sub-graph

Depth-first forest

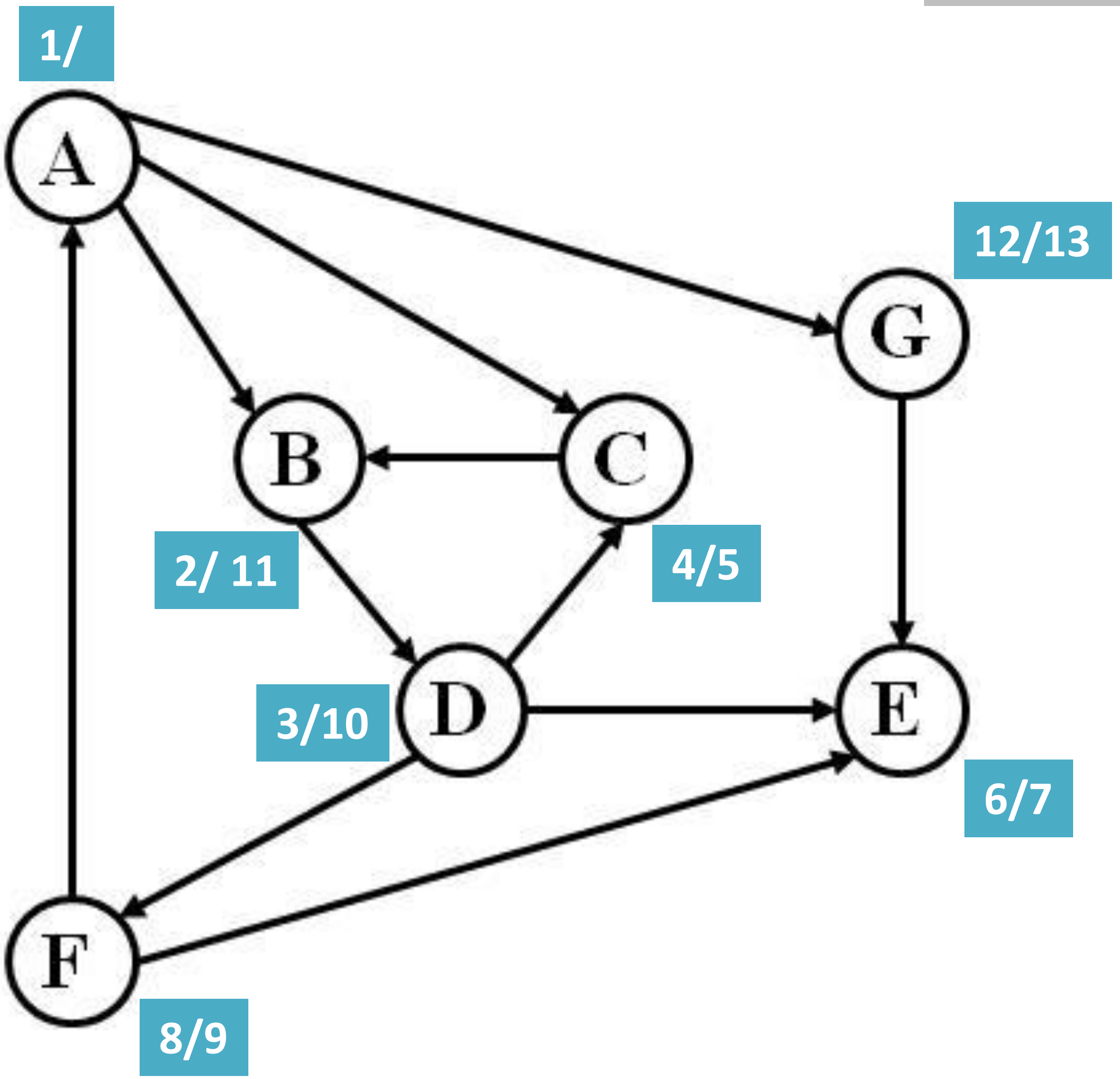


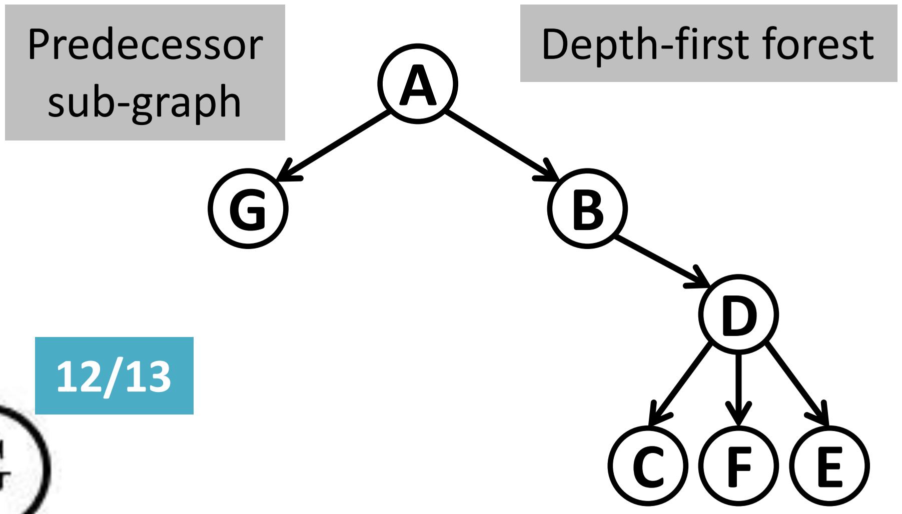
Compute DFS - Directed



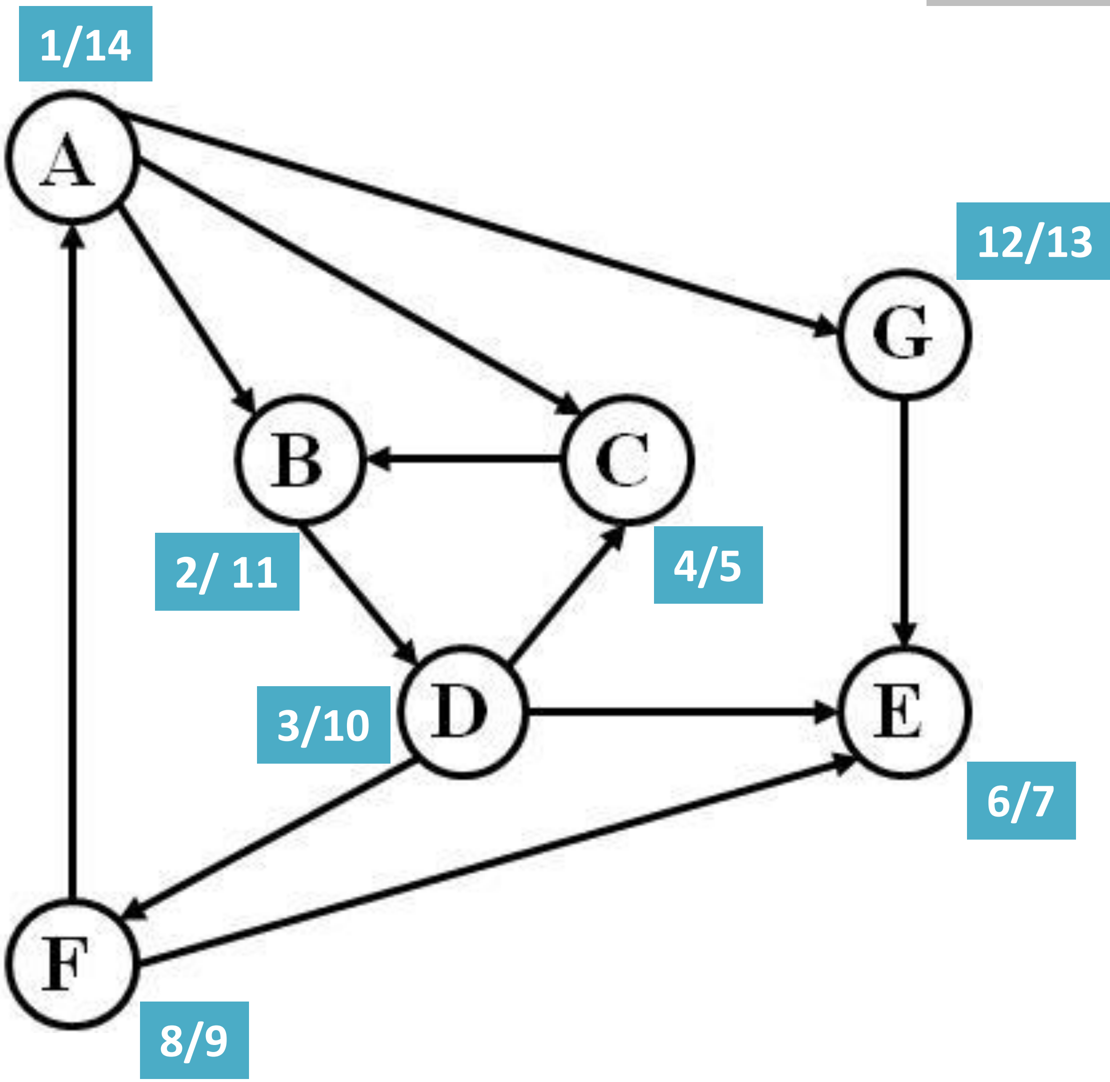


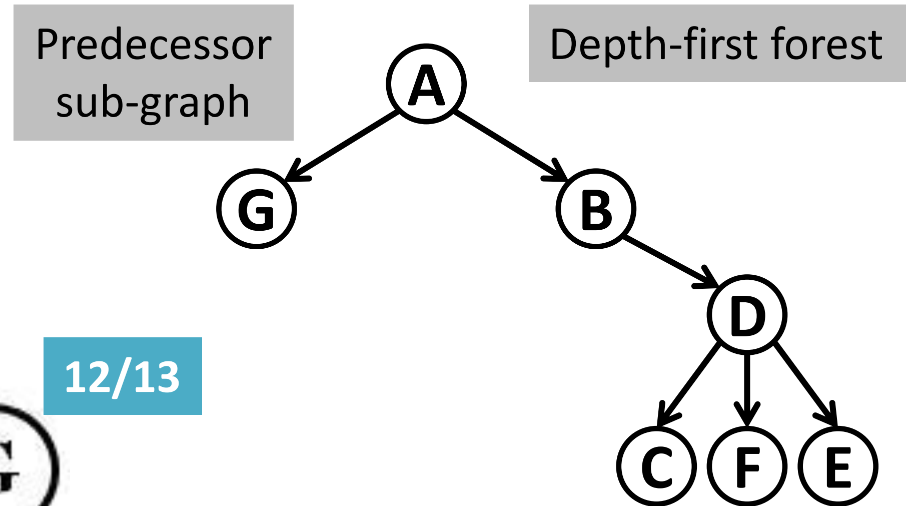
Compute DFS - Directed





Compute DFS - Directed





DFS: A B D C E F G

Procedure DFS

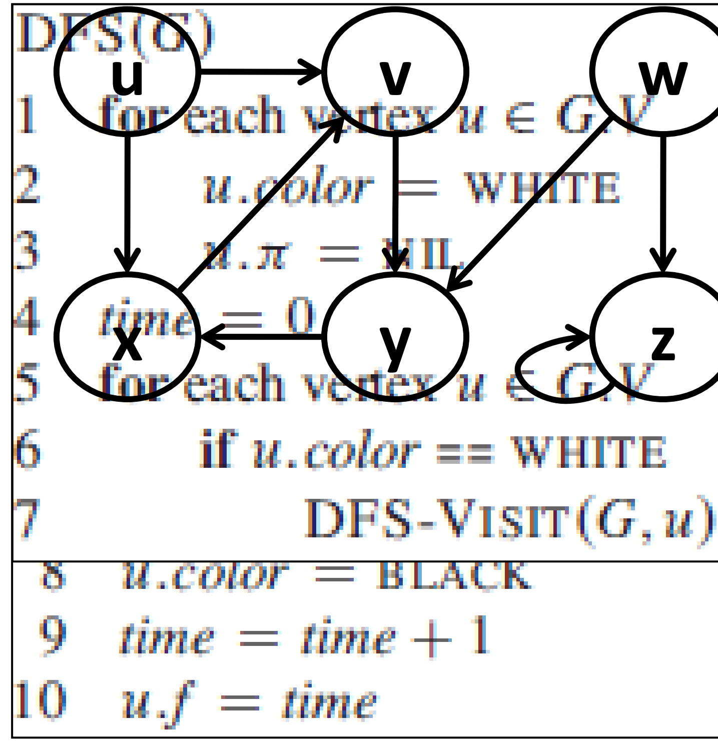
```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Execution example

- Let’s start with vertex u.

Vertex	Color	Timestamp		Predecessor (π)
		d	f	
u	White			NIL
v	White			NIL
w	White			NIL
x	White			NIL
y	White			NIL
z	White			NIL



u	→	v	→	x
v	→	y		
w	→	y	→	z
x	→	v		
y	→	x		
z	→	z		

time = 0

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
        if v.color == WHITE
                v.π = u
                DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp		Predecessor (π)
		d	f	
u	**Gray**	**1**		**NIL**
v	White			NIL
w	White			NIL
x	White			NIL
y	White			NIL
z	White			NIL

u → v → x
v → y
w → y → z
x → v
y → x
z → z

u = u   time = 1

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	**Gray**	**1**		**NIL**
v	**White**			**u**
w	White			NIL
x	White			NIL
y	White			NIL
z	White			NIL

u → v → x
v → y
w → y → z
x → v
y → x
z → z

u = u time = 1

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
        if v.color == WHITE
                v.π = u
                DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp		Predecessor (π)
		d	f	
u	**Gray**	**1**		**NIL**
v	**Gray**	**2**		**u**
w	White			NIL
x	White			NIL
y	White			NIL
z	White			NIL

u → v → x
v → y
w → y → z
x → v
y → x
z → z

u = v   time = 2

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp		Predecessor (π)
		d	f	
u	**Gray**	**1**		**NIL**
v	**Gray**	**2**		**u**
w	White			NIL
x	White			NIL
y	**White**			**v**
z	White			NIL

u	→ v → x
v	→ y
w	→ y → z
x	→ v
y	→ x
z	→ z

u = v   time = 2

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp d	f	Predecessor (π)
u	**Gray**	**1**		**NIL**
v	**Gray**	**2**		**u**
w	White			NIL
x	White			NIL
y	**Gray**	**3**		**v**
z	White			NIL

u	→ v → x
v	→ y
w	→ y → z
x	→ v
y	→ x
z	→ z

u = y time = 3

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	**Gray**	**1**		**NIL**
v	**Gray**	**2**		**u**
w	White			NIL
x	**White**			**y**
y	**Gray**	**3**		**v**
z	White			NIL

u → v → x
v → y
w → y → z
x → v
y → x
z → z

u = y   time = 3

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	**Gray**	**1**		**NIL**
v	**Gray**	**2**		**u**
w	White			NIL
x	**Gray**	**4**		**y**
y	**Gray**	**3**		**v**
z	White			NIL

u → v → x
v → y
w → y → z
x → v
y → x
z → z

u = x   time = 4

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
      if v.color == WHITE
            v.π = u
            DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	**Gray**	**1**		**NIL**
v	**Gray**	**2**		**u**
w	White			NIL
x	**Black**	**4**	**5**	**y**
y	**Gray**	**3**		**v**
z	White			NIL

u → v → x
v → y
w → y → z
x → v
y → x
z → z

u = x time = 5

Execution example

Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	**Gray**	**1**		**NIL**
v	**Gray**	**2**		**u**
w	White			NIL
x	**Black**	**4**	**5**	**y**
y	**Gray**	**3**		**v**
z	White			NIL

u = y time = 5

DFS-VISIT(G, u)

```
1   time = time + 1
2   u.d = time
3   u.color = GRAY
4   for each v ∈ G.Adj[u]
5       if v.color == WHITE
6           v.π = u
7           DFS-VISIT(G, v)
8   u.color = BLACK
9   time = time + 1
10  u.f = time
```

u	→	v	→	x
v	→	y		
w	→	y	→	z
x	→	v		
y	→	x		
z	→	z		

Execution example

Vertex	Color	Timestamp		Predecessor (π)
		d	f	
u	**Gray**	**1**		**NIL**
v	**Gray**	**2**		**u**
w	White			NIL
x	**Black**	**4**	**5**	**y**
y	**Black**	**3**	**6**	**v**
z	White			NIL

u = y time = 6

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

u → v → x
v → y
w → y → z
x → v
y → x
z → z

Execution example

Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	**Gray**	**1**		**NIL**
v	**Gray**	**2**		**u**
w	White			NIL
x	**Black**	**4**	**5**	**y**
y	**Black**	**3**	**6**	**v**
z	White			NIL

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
        if v.color == WHITE
                v.π = u
                DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

u → v → x
v → y
w → y → z
x → v
y → x
z → z

u = v   time = 6

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
        if v.color == WHITE
                v.π = u
                DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	**Gray**	**1**		**NIL**
v	**Black**	**2**	**7**	**u**
w	White			NIL
x	**Black**	**4**	**5**	**y**
y	**Black**	**3**	**6**	**v**
z	White			NIL

u → v → x
v → y
w → y → z
x → v
y → x
z → z

u = v   time = 7

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
        if v.color == WHITE
                v.π = u
                DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	**Gray**	**1**		**NIL**
v	**Black**	**2**	**7**	**u**
w	White			NIL
x	**Black**	**4**	**5**	**y**
y	**Black**	**3**	**6**	**v**
z	White			NIL

u → v → x
v → y
w → y → z
x → v
y → x
z → z

u = u time = 7

Execution example

Vertex	Color	Timestamp d	f	Predecessor (π)
u	**Black**	**1**	**8**	**NIL**
v	**Black**	**2**	**7**	**u**
w	White			NIL
x	**Black**	**4**	**5**	**y**
y	**Black**	**3**	**6**	**v**
z	White			NIL

u = u time = 8

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

u	→	v	→	x
v	→	y		
w	→	y	→	z
x	→	v		
y	→	x		
z	→	z		

Execution example

```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	**Black**	**1**	**8**	**NIL**
v	**Black**	**2**	**7**	**u**
w	White			NIL
x	**Black**	**4**	**5**	**y**
y	**Black**	**3**	**6**	**v**
z	White			NIL

u	→ v → x
v	→ y
w	→ y → z
x	→ v
y	→ x
z	→ z

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
        if v.color == WHITE
                v.π = u
                DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp		Predecessor (π)
		d	f	
u	**Black**	**1**	**8**	**NIL**
v	**Black**	**2**	**7**	**u**
w	**Gray**	**9**		**NIL**
x	**Black**	**4**	**5**	**y**
y	**Black**	**3**	**6**	**v**
z	White			NIL

u → v → x
v → y
w → y → z
x → v
y → x
z → z

u = w   time = 9

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp		Predecessor
		d	f	(π)
u	Black	1	8	NIL
v	Black	2	7	u
w	Gray	9		NIL
x	Black	4	5	y
y	Black	3	6	v
z	White			w

u	→ v → x
v	→ y
w	→ y → z
x	→ v
y	→ x
z	→ z

u = w time = 9

Execution example

Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	Black	1	8	NIL
v	Black	2	7	u
w	Gray	9		NIL
x	Black	4	5	y
y	Black	3	6	v
z	Gray	10		w

u = z time = 10

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
        if v.color == WHITE
                v.π = u
                DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

u → v → x
v → y
w → y → z
x → v
y → x
z → z

Execution example

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Vertex	Color	Timestamp		Predecessor (π)
		d	f	
u	Black	1	8	NIL
v	Black	2	7	u
w	Gray	9		NIL
x	Black	4	5	y
y	Black	3	6	v
z	Black	10	11	w

u	→	v	→ x
v	→	y	
w	→	y	→ z
x	→	v	
y	→	x	
z	→	z	

u = z   time = 11

Execution example

Vertex	Color	Timestamp		Predecessor (π)
		d	f	
u	Black	1	8	NIL
v	Black	2	7	u
w	Gray	9		NIL
x	Black	4	5	y
y	Black	3	6	v
z	Black	10	11	w

u = w time = 11

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

u → v → x
v → y
w → y → z
x → v
y → x
z → z

Execution example

Vertex	Color	Timestamp		Predecessor (π)
		d	f	
u	Black	1	8	NIL
v	Black	2	7	u
w	Black	9	12	NIL
x	Black	4	5	y
y	Black	3	6	v
z	Black	10	11	w

u = w time = 12

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

u → v → x
v → y
w → y → z
x → v
y → x
z → z

Execution example

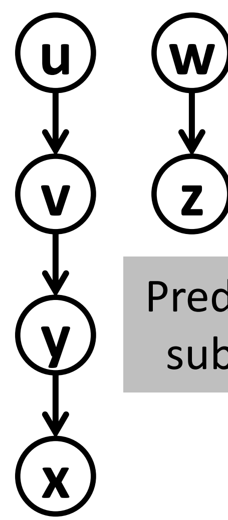
Vertex	Color	Timestamp d	Timestamp f	Predecessor (π)
u	Black	1	8	NIL
v	Black	2	7	u
w	Black	9	12	NIL
x	Black	4	5	y
y	Black	3	6	v
z	Black	10	11	w

```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

u → v → x
v → y
w → y → z
x → v
y → x
z → z

DFS: u v y x w z

Depth-first forest



Predecessor sub-graph