Finding Eigenvalues

Matrix $A = \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix}$

1. Eigenvalue Definition:

$$Av = \lambda v$$

$$(A - \lambda I)v = 0$$

nontrivial solution if $\det(A - \lambda I) = 0$

2. Determinant

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & 13 & -14 & -13-\lambda \end{vmatrix}$$

3. Using Bareiss Algorithm.

first pivot: $a_{1,1} = -\lambda + 4$ as pivot

$$\begin{matrix} \lambda^2 + 5\lambda - 20 & \lambda^2 + 5\lambda - 20 & \lambda + 25 & -2\lambda + \dots \\ 0 & 0 & 2\lambda - 10 & 4\lambda - \dots \\ 0 & 0 & -\lambda^3 + 25\lambda & -10\lambda^2 - \dots \\ 0 & 0 & -14\lambda^2 - 43\lambda + 175 & -\lambda^3 - 15\lambda^2 \dots \end{matrix}$$

4. final pivot

$$\det(A - \lambda I) = \lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500$$

finding eigenvalues

$$\lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500 = 0$$

Values

$\lambda_1 = -21.125$

$\lambda_2 = -5.604$

$\lambda_3 = 2.675$

$\lambda_4 = 11.054$

Vector for first eigenvalue $\lambda = -21.125$

1. Augmented Matrix:

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 12.761 & -2.080 & -4.159 & 0 \\ 0 & 10 & 26.125 & -10 & 0 \\ -1 & 13 & -14 & 8.125 & 0 \end{array}\right)$$

2. Row operations

1. Add R_1 to R_4

$$R_4 \leftarrow R_4 + R_1$$

$$\begin{matrix} 1 & 0.318 \\ 0 & 12.761 \\ 0 & 26.125 \\ 0 & -14.040 \end{matrix} \quad \left[\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 12.761 & -2.080 & -4.159 & 0 \\ 0 & 10 & 26.125 & -10 & 0 \\ 0 & -12.682 & -14.040 & 8.045 & 0 \end{array}\right]$$

2. Normalize R_2

$$R_2 \leftarrow \frac{R_2}{12.761}$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & -0.163 & -0.326 & 0 \\ 0 & 10 & 26.125 & -10 & 0 \\ 0 & -12.682 & -14.040 & 8.045 & 0 \end{array}\right)$$

3. Eliminate R_3

$$R_3 \leftarrow R_3 - 10 \cdot R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & -0.163 & -0.326 & 0 \\ 0 & 0 & 27.754 & -6.741 & 0 \\ 0 & -12.682 & -14.040 & 8.045 & 0 \end{array}\right)$$

4. Eliminate R_4

$$R_4 \leftarrow R_4 + 12.682 \cdot R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & -0.163 & -0.326 & 0 \\ 0 & 0 & 27.754 & -6.741 & 0 \\ 0 & 0 & -16.106 & 3.912 & 0 \end{array}\right)$$

5. Normalize R_3

$$R_3 \leftarrow \frac{R_3}{27.754}$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & -0.163 & -0.326 & 0 \\ 0 & 0 & 1 & -0.243 & 0 \\ 0 & 0 & -16.106 & 3.912 & 0 \end{array}\right)$$

6. Eliminate R_4: $R_4 \leftarrow R_4 + 16.106 \cdot R_3$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & -0.163 & -0.326 & 0 \\ 0 & 0 & 1 & -0.243 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

3. Back Substitution

from third row $X_3 - 0.243 X_4 = 0 \Rightarrow X_3 = 0.243 X_4$

from second row $x_2 - 0.163 x_3 - 0.326 x_4 = 0 \Rightarrow x_2 = 0.16$
$+ 0.$
$= 0$

From first row:

$x_1 + 0.040 x_3 - 0.080 x_4 = 0 \Rightarrow x_1 = -0.040(0.243 x_4) + 0.080 x_4$

Solution

$$\begin{aligned} x_1 &= -0.027 x_4 \\ x_2 &= 0.365 x_4 \\ x_3 &= 0.243 x_4 \\ x_4 &= x_4 \text{ (free Variable)} \end{aligned}$$

Manual Eigenvalue Calculation for λ_2

Step 1: Set up the Characteristic Equation

We need to find $\det(A-\lambda I)=0$

$$A=\begin{bmatrix} 4 & 8 & -2 \\ -2 & -9\lambda & -4 \\ 0 & 10 & -10 \\ -1 & -13 & -13 \end{bmatrix} \qquad \lambda=\begin{bmatrix} -9 & -9 & -1 \\ -2 & -9-\lambda & -2 \\ 0 & 10 & -10 \\ -1 & -18 & -13 \end{bmatrix} - \begin{matrix} -2 \\ -2\lambda \end{matrix}$$

Step 2: Calculate the Determinant

Using cofactor expansion along their first row:

$$\det(A-\lambda)=(4-\lambda)+8C_{12}+(-1)+2C_4)-2C_4$$

$$\text{Where } C_{1j} = -2\cdot(390+10\lambda$$
$$= 390+10\lambda-4^2$$

$$\text{Calculore } C_{13} = 175+22\lambda-\lambda^2$$

$$175+22\lambda-\lambda^2-\lambda^2$$

$$175+22\lambda-\lambda^2-\lambda^2-\lambda^2$$
$$=175+22\lambda-\lambda^2-\lambda^2$$

Step 3: Combine and Simplify

After full expansion (this requires careful algebraic manuification):

$$\lambda^{24}+13b^2+42^2+30=0=\lambda^2(313k^6+42\lambda+30)=0$$

Testing $\lambda_4=-0$

$$(-2)^2+13(-2)^2+42(-2)+30=0$$

Dividing them $s-2$

$$\lambda^2+132-23+30=(a+2)\,(\lambda+2)(\lambda^2+11\lambda+30)$$

$$\lambda^2+11\lambda+10=(\lambda+1)\,(\lambda+10)$$

Step 6: Final Eigenvalues

The complete factorization: $\lambda_4\,(\lambda+2)\,(\lambda+1\text{-}10)=0$

$$\lambda_2=-1$$

$$\lambda_3 \approx 2.675$$

$$A-\lambda_3 I \approx \begin{pmatrix} 1.325 & 8 & -1 & -2 \\ -2 & -11.675 & -2 & -4 \\ 0 & 10 & 2.325 & -10 \\ -1 & -13 & -14 & -15.675 \end{pmatrix}$$

$$A_v = \lambda v^*$$

$$(A-\lambda I)\cdot v = 0$$

So we have a homogeneous system of linear equations, we solve it by Gaussian Elimination.

$$\left(\begin{array}{cccc|c} 1.325 & 8 & -1 & -2 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array}\right) \quad \times 0.754 \quad R_1/(1.325) \to R_1 \quad \left(\begin{array}{cccc|c} 1 & 6.036 & -0.754 & -1.509 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & 13 & -14 & -15.675 & 0 \end{array}\right)$$

$$\times(2)\ R_2-(-2)\cdot R_1 \to R_2$$

$$\left(\begin{array}{cccc|c} 1 & 6.036 & -0.754 & -1.509 & 0 \\ 0 & 0.397 & -3.509 & -7.018 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array}\right) \quad \times(1)\ R_4-(-1)\cdot R_1 \to R_4 \quad \left(\begin{array}{cccc|c} 1 & 6.036 & -0.754 & -1.509 & 0 \\ 0 & 0.397 & -3.509 & -7.018 & 0 \\ 0 & 10 & +2.325 & -10 & 0 \\ 0 & -6.964 & -14.754 & -17.184 & 0 \end{array}\right)$$

$$\times(2.518)\ R_2/(0.397) \to R_2$$

$$\left(\begin{array}{cccc|c} 1 & 6.036 & -0.754 & -1.509 & 0 \\ 0 & 1 & -8.834 & -17.669 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ 0 & -6.964 & -14.754 & -17.184 & 0 \end{array}\right) \quad \times(-10)\ R_3-10\cdot R_2 \to R_3 \quad \left(\begin{array}{cccc|c} 1 & 6.036 & -0.754 & -1.509 & 0 \\ 0 & 1 & -8.834 & -17.669 & 0 \\ 0 & 0 & 90.669 & 166.688 & 0 \\ 0 & -6.964 & -14.754 & -17.184 & 0 \end{array}\right)$$

$$\times(6.964)\ R_4-(-6.964)\cdot R_2 \to R_4$$

$$\left(\begin{array}{cccc|c} 1 & 6.036 & -0.754 & -1.509 & 0 \\ 0 & 1 & -8.834 & -17.669 & 0 \\ 0 & 0 & 90.669 & 166.688 & 0 \\ 0 & 0 & -76.278 & -140.231 & 0 \end{array}\right) \quad \times(0.011)\ R_3/(90.669) \to R_3 \quad \left(\begin{array}{cccc|c} 1 & 6.036 & -0.754 & -1.509 & 0 \\ 0 & 1 & -8.834 & -17.669 & 0 \\ 0 & 0 & 1 & 1.838 & 0 \\ 0 & 0 & -76.278 & -140.231 & 0 \end{array}\right)$$

$$\times(76.278)\ R_4-(-76.278)\cdot R_3 \to R_4$$

$$\left(\begin{array}{cccc|c} 1 & 6.036 & -0.754 & -1.509 & 0 \\ 0 & 1 & -8.834 & -17.669 & 0 \\ 0 & 0 & 1 & 1.836 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \times(8.834)\ R_2-(-8.834)R_3 \to R_2 \quad \left(\begin{array}{cccc|c} 1 & 6.036 & -0.754 & -1.509 & 0 \\ 0 & 1 & 0 & -1.428 & 0 \\ 0 & 0 & 1 & 1.838 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\times(0.754)\ R_1-(-0.754)\cdot R_3 \to R_1$$

$$\left(\begin{array}{cccc|c} 1 & 6.036 & 0 & -0.122 & 0 \\ 0 & 1 & 0 & -1.428 & 0 \\ 0 & 0 & 0 & 1.838 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \times(-6.036)\ R_1-6.036\cdot R_2 \to R_1 \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & 8.494 & 0 \\ 0 & 1 & 0 & -1.428 & 0 \\ 0 & 0 & 1 & 1.838 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 \quad + 8.494 \cdot x_4 = 0 \\ x_2 \quad - 1.428 \cdot x_4 = 0 \\ x_3 \quad + 1.838 \cdot x_4 = 0 \end{cases} \quad (1)$$

$\equiv$

- Variable x_3 from eq 3 of system (1)

$$x_3 = -1.838 x_4$$

- Variable x_2 from eq 2 of system (1)

$$x_2 = 1.428 x_4$$

- Variable x_1 from eq 1 of system (1)

$$x_1 = -8.494 x_4$$

Therefore Ans.

$$x_1 = -8.494 x_4$$
$$x_2 = 1.428 x_4$$
$$x_3 = -1.838 x_4$$
$$x_4 = x_4$$

Solution : $X = \begin{pmatrix} -8.494 x_4 \\ 1.428 x_4 \\ -1.838 x_4 \end{pmatrix}$

$\equiv$

Solution set $= \left\{ x_4 \begin{pmatrix} -8.494 \\ 1.428 \\ -1.838 \\ 1 \end{pmatrix} \right.$

Let $x_4 = 1$, $v_3 = \begin{pmatrix} -8.494 \\ 1.428 \\ -1.838 \\ 1 \end{pmatrix}$

$\equiv$

$$A - \lambda_4 I \simeq \begin{pmatrix} -7.054 & 8 & -1 & -2 \\ -2 & -20.054 & -2 & -4 \\ 0 & 10 & -6.054 & -10 \\ -1 & -13 & -14 & -24.054 \end{pmatrix} \quad , \quad \begin{array}{l} Av = \lambda v \\ (A - \lambda I) \cdot v = 0 \end{array}$$

Using Gaussian Elimination:

$$\left(\begin{array}{cccc|c} -7.054 & 8 & -1 & -2 & 0 \\ -2 & -20.054 & -2 & -4 & 0 \\ 0 & 10 & -6.054 & -10 & 0 \\ -1 & 13 & -14 & -24.054 & 0 \end{array}\right) \times (-0.142) \Rightarrow \frac{R_1}{(-7.054)} \to R_1 \left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ -2 & -20.054 & -2 & -4 & 0 \\ 0 & 10 & -6.054 & -10 & 0 \\ -1 & 13 & -14 & -24.054 & 0 \end{array}\right)$$

$$R_2 - (-2) \cdot R_1 \to R_2 \left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & -22.322 & -1.716 & -3.433 & 0 \\ 0 & 10 & -6.054 & -10 & 0 \\ -1 & -13 & -14 & -24.054 & 0 \end{array}\right) \times (1) \quad R_4 - (-1) \cdot R_1 \to R_4$$

$$\left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & -22.322 & -1.716 & -3.403 & 0 \\ 0 & 10 & -6.054 & -10 & 0 \\ 0 & -14.134 & -13.858 & -23.771 & 0 \end{array}\right) \times (-0.045) \quad R_2 / (-22.322) \to R_2 \left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & 1 & 0.077 & 0.154 & 0 \\ 0 & 10 & -6.054 & -10 & 0 \\ 0 & -14.134 & -13.858 & -23.771 & 0 \end{array}\right)$$

$$R_3 - 10 \cdot R_2 \to R_3 \left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & 1 & 0.077 & 0.154 & 0 \\ 0 & 0 & -6.823 & -11.538 & 0 \\ 0 & -14.134 & -13.858 & -23.771 & 0 \end{array}\right) \times (14.134) \quad R_4 - (-14.134) \cdot R_2 \to R_4$$

$$\left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & 1 & 0.077 & 0.154 & 0 \\ 0 & 0 & -6.823 & -11.538 & 0 \\ 0 & 0 & -12.771 & -21.597 & 0 \end{array}\right) \times (-0.147) \quad R_3 / (-6.823) \to R_3 \left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & 1 & 0.077 & 0.154 & 0 \\ 0 & 0 & 1 & 1.691 & 0 \\ 0 & 0 & -12.771 & -21.597 & 0 \end{array}\right) \times (12.771)$$

$$R_4 - (-12.771) \cdot R_3 \to R_4 \left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & 1 & 0.077 & 0.154 & 0 \\ 0 & 0 & 1 & 1.691 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \times (-0.077) \quad R_2 - 0.077 \cdot R_3 \to R_2$$

$$\left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & 1 & 0 & 0.024 & 0 \\ 0 & 0 & 1 & 1.691 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \times (-0.142) \quad R_1 - 0.142 \cdot R_3 \to R_1 \left(\begin{array}{cccc|c} 1 & -1.134 & 0 & 0.044 & 0 \\ 0 & 1 & 0 & 0.024 & 0 \\ 0 & 0 & 1 & 1.691 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \times (1.134) \quad R_1 - (-1.134) \cdot R_2 \to$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0.071 & 0 \\ 0 & 1 & 0 & 0.024 & 0 \\ 0 & 0 & 1 & 1.691 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + 0.071 x_4 = 0 \\ x_2 + 0.024 x_4 = 0 \\ x_3 + 1.691 x_4 = 0 \end{array} \Rightarrow \begin{array}{l} x_3 = -1.691 x_4 \\ x_2 = -0.024 x_4 \\ x_1 = -0.071 x_4 \\ x_4 = x_4 \end{array}$$

$$\text{Finally} \quad X = \begin{pmatrix} -0.071 x_4 \\ -0.024 x_4 \\ -1.691 x_4 \\ x_4 \end{pmatrix} \quad \Big| \quad \text{Solution set: } \left\{ x_4 \cdot \begin{pmatrix} -0.071 \\ -0.024 \\ -1.691 \\ 1 \end{pmatrix} \right\}$$

$$\text{Let } x_4 = 1 \,, \quad v_4 = \begin{pmatrix} -0.071 \\ -0.024 \\ -1.691 \\ 1 \end{pmatrix}$$