# Math 344 Lecture #18
# 3.4 QR with Householder Transformations

We detail a different algorithm for finding a $QR$-decomposition of a matrix through left multiplication by special orthonormal matrices knows as Householder transformations.

## 3.4.1 The Geometry of Householder Transformations

**Definition 3.4.1.** For a nonzero vector $v \in \mathbb{F}^n$, the orthogonal complement of $v$, denoted by $v^\perp$, is the set of all vectors $x \in \mathbb{F}^n$ that are orthogonal to $v$, i.e.,

$$v^\perp = \{x \in \mathbb{F}^n : \langle v, x\rangle = 0\}.$$

For each $x \in \mathbb{F}^n$, the residual $x - \mathrm{proj}_v(x)$ is orthogonal to $v$, and so

$$x - \mathrm{proj}_v(x) \in v^\perp.$$

This means that the map $x \mapsto x - \mathrm{proj}_v(x)$ is the projection of $x$ onto the orthogonal complement $v^\perp$, i.e.,

$$\mathrm{proj}_{v^\perp}(x) = x - \mathrm{proj}_v(x) = x - \frac{v}{\|v\|^2}\langle v, x\rangle = x - \frac{vv^{\mathrm{H}}x}{v^{\mathrm{H}}v} = \left(I - \frac{vv^{\mathrm{H}}}{v^{\mathrm{H}}v}\right)x.$$

The idea of a Householder transformation is to double the vector $\mathrm{proj}_v(x)$ subtracted from $x$, resulting in a reflection of vectors across an orthogonal complement of a vector.

**Definition 3.4.3.** For a fixed nonzero $v \in \mathbb{F}^n$ the Householder transformation of $x \in \mathbb{F}^n$ is the map $H_v$ given by

$$H_v(x) = x - 2\mathrm{proj}_v(x) = \left(I - 2\frac{vv^{\mathrm{H}}}{v^{\mathrm{H}}v}\right)x.$$

**Proposition 3.4.4.** Assume $v \in \mathbb{F}^n$ is nonzero. Then,

(i) $H_v(x)$ or an orthonormal transformation, and

(ii) $H_v(x) = x$ for all $x \in v^\perp$.

The proof of these is HW (Exercise 3.20).

## 3.4.2 Computing the QR Decomposition via Householder

The first step in finding a $QR$ decomposition of a matrix is to find for each nonzero $x \in \mathbb{F}^n$ the nonzero $v \in \mathbb{F}^n$ for which $H_v(x) \in \mathrm{span}(e_1)$, where $e_1$ is the first standard basis vector of $\mathbb{F}^n$.

**Lemma 3.4.5.** For $x \in \mathbb{R}^n$, if $v = x + \|x\|e_1$ is nonzero, then $H_v(x) = -\|x\|e_1$, and if $v = x - \|x\|e_1$ is nonzero, then $H_v(x) = \|x\|e_1$.

Proof. The argument for the two cases is similar; we show the argument for $v = x + \|x\|e_1 \neq 0$. For $x_1$ the first entry of $x$ we have

$$\begin{aligned} v^Tv &= (x + \|x\|e_1)^T(x + \|x\|e_1) \\ &= x^Tx + \|x\|x^Te_1 + \|x\|e_1^Tx + \|x\|^2e_1^Te_1 \\ &= 2\|x\|^2 + 2\|x\|x_1 = 2\|x\|(\|x\| + x_1), \end{aligned}$$

and

$$\begin{aligned} vv^Tx &= (x + \|x\|e_1)(x + \|x\|e_1)^Tx \\ &= xx^Tx + \|x\|xe_1^Tx + \|x\|e_1x^Tx + \|x\|^2e_1e_1^Tx \\ &= \|x\|^2x + \|x\|x_1x + \|x\|^3e_1 + \|x\|^2x_1e_1 \\ &= \|x\|(\|x\|x + x_1x + \|x\|^2e_1 + \|x\|x_1e_1) \\ &= \|x\|\big((\|x\| + x_1)x + \|x\|(\|x\| + x_1)e_1\big) \\ &= \|x\|(\|x\| + x_1)(x + \|x\|e_1). \end{aligned}$$

Then

$$\begin{aligned} H_v(x) &= x - 2\frac{vv^Tx}{v^Tv} \\ &= x - 2\frac{\|x\|(\|x\| + x_1)(x + \|x\|e_1)}{2\|x\|(\|x\| + x_1)} \\ &= x - (x + \|x\|e_1) \\ &= -\|x\|e_1. \end{aligned}$$

This gives the result. □

The complex sign of a complex number $z$ is $\text{sign}(z) = z/|z|$ when $z \neq 0$, and $\text{sign}(0) = 1$.

Lemma 3.4.7. Let $x_1$ be the first entry of a nonzero $x \in \mathbb{C}^n$. If $v = x + \text{sign}(x_1)\|x\|e_1$, then

$$H_v(x) = -\text{sign}(x_1)\|x\|e_1.$$

The proof of this is HW (Exercise 3.21).

Remark 3.4.8. The vector $v = x + \text{sign}(x_1)\|x\|e_1$ is never zero unless $x = 0$. For if $x + \text{sign}(x_1)\|x\|e_1 = 0$ when $x \neq 0$, then $x = -\text{sign}(x_1)\|x\|e_1$ which implies that $x_1 = -\text{sign}(x_1)|x_1|$, but since $x_1 = \text{sign}(x_1)|x_1|$ we would get $\text{sign}(x_1) = -\text{sign}(x_1)$, which implies that $\text{sign}(x_1) = 0$, which is a contradiction.

The algorithm for computing a $QR$ decomposition by means of Householder transformations is repeated applications of Lemma 3.4.5 and/or Lemma 3.4.7.

Theorem 3.4.9. For $A \in M_{m\times n}(\mathbb{F})$ with $m \geq n$ there are vectors $v_1, v_2, \ldots, v_l \in \mathbb{F}^n$ where $l = \min\{m-1, n\}$ such that $R = H_{v_l}\cdots H_{v_2}H_{v_1}A$ is upper triangular matrix and $Q = H_{v_1}^{\mathrm{H}}H_{v_2}^{\mathrm{H}}\cdots H_{v_l}^{\mathrm{H}}$ is an orthonormal matrix that satisfy $A = QR$.

The proof of this is quite long and involved. We will demonstrate the proof by way of an example.

### 3.4.3 A Complete Worked Example

Recall that previously we found a $QR$ decomposition for

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

by means of the Gram-Schmidt process, obtaining

$$Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

To find a $QR$ decomposition of $A$ by means of Householder transformations, we start with the first column $x_1$ of $A$, set

$$v_1 = x_1 - \operatorname{sign}(x_{11})\|x_1\|e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

and get the Householder transformation

$$H_{v_1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}.$$

Multiplying $A$ on the left by $H_{v_1}$ gives

$$H_{v_1}A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -5 & 2 \end{bmatrix}.$$

Notice that the first column of $H_{v_1}A$ is a positive scalar multiple of $e_1$ as Lemma 3.4.5 predicted.

We now consider the second column $x_2$ of $H_{v_1}A$ which can we written as

$$x_2 = x_2' + x_2'' = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -5 \end{bmatrix}$$

where $x_2' \in \operatorname{span}(\{e_1\})$ and $x_2'' \in e_1^{\perp}$.

Taking $x_{22}''$ to be the second entry of $x_2''$ we set

$$v_2 = x_2'' - \operatorname{sign}(x_{22}'')\|x_2''\|e_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -5 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 0 \\ -5 \end{bmatrix}.$$

The Householder transformation associated to $v_2$ is

$$H_{v_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Since $x_2''$ and $e_2$ are both orthogonal to $e_1$, the vector $v_2$ is orthogonal to $e_1$.

Since $x_2' \in \text{span}(\{e_1\})$, we obtain by Proposition 3.4.4 part (ii) that $H_{v_2}(x_2') = x_2'$.

Thus $H_{v_2}(x_2) = H_{v_2}(x_2' + x_2'') = x_2' + H_{v_2}(x_2'')$.

By the appropriate adaptation of Lemma 3.4.7 the vector $H_{v_2}(x_2'')$ belongs to $\text{span}(\{e_2\})$.

Thus $H_{v_2}(x_2)$ belongs to $\text{span}(\{e_1, e_2\})$, and so the second column of $H_{v_2}(H_{v_1}A)$ belongs to $\text{span}(\{e_1, e_2\})$.

We verify this:

$$H_{v_2}(H_{v_1}A) = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is $R$ and

$$Q = (H_{v_2}H_{v_1})^T = H_{v_1}^T H_{v_2}^T = \begin{bmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}.$$

[All the matrix computations, including the entries of the Householder matrices, were performed in Maple.]