Convection Diffusion with Linear Production

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The Convection - Diffusion - Production equation is one of the most frequently implemented equations for the description of physical phenomena across almost every scientific domain. For a homogeneous material with a linear source sink term, the general form of the equation is: phgh linear wiki

$$\alpha \frac{\partial \phi}{\partial t} = \beta \boldsymbol{\nabla}^2 \boldsymbol{\phi} - \gamma \nabla \cdot (\boldsymbol{v}\boldsymbol{\phi}) + R(\boldsymbol{\phi}) \tag{1}$$

- $R(\boldsymbol{\phi}) = \delta\boldsymbol{\phi} + \epsilon$ is a linear source ($R(\boldsymbol{\phi}) > 0$) or a linear sink ($R(\boldsymbol{\phi}) < 0$)
- α is the $\boldsymbol{capacity\ coefficient}$. It scales the time dependent term.
- β is the $\boldsymbol{diffusion\ coefficient}$. Generally diffusion is associated with transport phenomena that occur due to the medium molecular properties. It measures how easily does the property of interest disperse in the host medium.
- γ is the $\boldsymbol{convection\ coefficient}$. Convection is associated with transport phenomena due to a flow of velocity $\boldsymbol{v}$ inside and/or at the boundaries of the control volume.
- δ is the $\boldsymbol{dependent\ source\ term\ coefficient}$.
- ϵ is the $\boldsymbol{independent\ source\ term\ coefficient}$.

1 Weak Form - Spatial Discretization

The physical space is discretized in N elements with M nodes each. We assume that the solution takes the form:

$$\boldsymbol{\Phi} \cong \boldsymbol{N}\boldsymbol{\phi} = [N_1, ..., N_M] \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_M \end{bmatrix} = N_i \phi_i \tag{2}$$

where N_i are the weight functions and ϕ_i are the nodal parameters. By substituting (2) in (**??**) we deduce:

$$\alpha \frac{\partial}{\partial t} \boldsymbol{N}\boldsymbol{\phi} = \beta \boldsymbol{\nabla^2} \boldsymbol{N}\boldsymbol{\phi} - \gamma \boldsymbol{v} \boldsymbol{\nabla} \boldsymbol{N}\boldsymbol{\phi} + \delta \boldsymbol{N}\boldsymbol{\phi} + \epsilon \tag{3}$$

Multiply with weight functions vector and integrate over the element volume:

$$\alpha \iiint \boldsymbol{N^T} \frac{\partial}{\partial t} \boldsymbol{N}\boldsymbol{\phi} dV = \beta \iiint \boldsymbol{N^T} \boldsymbol{\nabla^2} \boldsymbol{N}\boldsymbol{\phi} dV - \gamma \boldsymbol{u} \iiint \boldsymbol{N^T} \nabla \boldsymbol{N}\boldsymbol{\phi} dV + \delta \iiint \boldsymbol{N^T} \boldsymbol{N}\boldsymbol{\phi} dV + \epsilon \iiint \boldsymbol{N^T} dV \tag{4}$$

Nodal parameter ϕ_i is independent of space and time so by integrating by parts and ignoring the boundary integrals (4) can be written as:

$$\alpha \frac{\partial \boldsymbol{\phi}}{\partial t} \iiint \boldsymbol{N^T} \boldsymbol{N} dV = \beta \boldsymbol{\phi} \iiint \boldsymbol{\nabla}(\boldsymbol{N^T}) \boldsymbol{\nabla} \boldsymbol{N} dV - \gamma \boldsymbol{v}\boldsymbol{\phi} \iiint \boldsymbol{N^T} \boldsymbol{\nabla} \boldsymbol{N} dV + \delta \phi \iiint \boldsymbol{N^T} \boldsymbol{N}\boldsymbol{\phi} dV + \epsilon \iiint \boldsymbol{N^T} dV \tag{5}$$

(5) can be simplified to:

$$\alpha \boldsymbol{M}\frac{\partial \boldsymbol{\phi}}{\partial t} + \beta \boldsymbol{C}\boldsymbol{\phi} - \gamma \boldsymbol{K}\boldsymbol{\phi} = \boldsymbol{F} \tag{6}$$

where

$$\boldsymbol{M} = \iint \boldsymbol{N}^{\boldsymbol{T}}\boldsymbol{N}dA$$

$$\boldsymbol{C} = \iint \boldsymbol{N}^{\boldsymbol{T}}\boldsymbol{\nabla}\boldsymbol{N}dA$$

$$\boldsymbol{K} = \iint \boldsymbol{\nabla}\boldsymbol{N}^{\boldsymbol{T}}\boldsymbol{\nabla}\boldsymbol{N}dA$$

$$\boldsymbol{F} = f\iint \boldsymbol{N}^{\boldsymbol{T}}dA$$