# Nearly-tight VC-dimension bounds for piecewise linear neural networks

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#### **Abstract**

We prove new upper and lower bounds on the VC-dimension of deep neural networks with the ReLU activation function. These bounds are tight for almost the entire range of parameters. Letting W be the number of weights and L be the number of layers, we prove that the VC-dimension is  $O(WL\log(W))$  and  $\Omega(WL\log(W/L))$ . This improves both the previously known upper bounds and lower bounds. In terms of the number U of non-linear units, we prove a tight bound  $\Theta(WU)$  on the VC-dimension. All of these results generalize to arbitrary piecewise linear activation functions.

## 1 Introduction

Deep neural networks underlie many of the recent breakthroughs of applied machine learning, particularly in image and speech recognition. These successes motivate a renewed study of these networks' theoretical properties.

Classification is one of the learning tasks in which deep neural networks have been particularly successful, e.g., for image recognition. A natural foundational question that arises is: what are the theoretical limits on the classification power of these networks? The established way to formalize this question is by considering VC-dimension, as it is well known that this asymptotically determines the sample complexity of PAC learning with such classifiers [3].

In this paper, we prove nearly-tight bounds on the VC-dimension of deep neural networks in which the non-linear activation function is a piecewise linear function with a constant number of pieces. For simplicity we will henceforth refer to such networks as "piecewise linear networks". The most common activation function used in practice is, by far, the *rectified linear unit*, also known as *ReLU* [7, 8]. The ReLU function is defined as  $\sigma(x) = \max\{0, x\}$ , so it is clearly piecewise linear.

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It is particularly interesting to consider how the VC-dimension is affected by the various attributes of the network: the number W of parameters (i.e., weights and biases), the number U of non-linear units (i.e., nodes), and the number L of layers. Among all networks with the same size (number of weights), is it true that those with more layers have more classification power (i.e., larger VC-dimension)?

Such a statement is indeed true, and previously known, thereby providing some justification for the advantages of deep neural networks. However, a tight characterization of how depth affects VC-dimension was unknown prior to this work.

**Our results.** Our first main result is a new VC-dimension lower bound that holds even for the restricted family of ReLU networks.

**Theorem 1.1** (Main lower bound). There exists a universal constant C such that the following holds. Given any W, L with  $W > CL > C^2$ , there exists a ReLU network with  $\leq L$  layers and  $\leq W$  parameters with VC-dimension  $\geq WL \log(W/L)/C$ .

The proof appears in Section 3. Prior to our work, the best known lower bounds were  $\Omega(WL)$  [2, Theorem 2] and  $\Omega(W\log W)$  [10, Theorem 1]; we strictly improve both bounds to  $\Omega(WL\log(W/L))$ .

Our proof of Theorem 1.1 uses the "bit extraction" technique, which was also used in [2] to give an  $\Omega(WL)$  lower bound. We refine this technique to gain the additional logarithmic factor that appears in Theorem 1.1.

Unfortunately there is a barrier to refining this technique any further. Our next theorem shows the hardness of computing the mod function, implying that the bit extraction technique cannot yield a stronger lower bound than Theorem 1.1. Further discussion of this connection may be found in Remark 3.

**Theorem 1.2.** Assume there exists a piecewise linear network with W parameters and L layers that computes a function  $f : \mathbb{R} \to \mathbb{R}$ , with the property that  $|f(x) - (x \mod 2)| < 1/2$  for all  $x \in \{0, 1, \dots, 2^m - 1\}$ . Then we have  $m = O(L \log(W/L))$ .

The proof of this theorem appears in Section 4. One interesting aspect of the proof is that it does not use Warren's lemma [13], which is a mainstay of VC-dimension upper bounds [6, 2, 1].

Our next main result is an upper bound on the VC-dimension of neural networks with any piecewise linear activation function with a constant number of pieces. Recall that ReLU is an example of a piecewise linear activation function.

**Theorem 1.3** (Main upper bound). Consider a piecewise linear neural network with W parameters arranged in L layers. Let  $\mathcal{F}$  be the set of (real-valued) functions computed by this network. Then  $VCDim(sgn(\mathcal{F})) = O(WL \log W)$ .

The proof of this result appears in Section 5. Prior to our work, the best published upper bounds were  $O(W^2)$  [6, Section 3.1] and  $O(WL \log W + WL^2)$  [2, Theorem 1], both of which hold for piecewise polynomial activation functions; we strictly improve both bounds to  $O(WL \log W)$  for the special case of piecewise linear functions.

To compare our upper and lower bounds, let d(W, L) denote the largest VC-dimension of a piecewise linear network with W parameters and L layers. Theorems 1.1 and 1.3 imply there exist constants c, C such that

$$c \cdot WL \log(W/L) \le d(W, L) \le C \cdot WL \log W. \tag{1}$$

For neural networks arising in practice it would certainly be the case that L is significantly smaller than  $W^{0.99}$ , in which case our results determine the asymptotic bound  $d(W,L) = \Theta(WL\log W)$ . On the other hand, in the regime  $L = \Theta(W)$ , which is merely of theoretical interest, we also now have a tight bound  $d(W,L) = \Theta(WL)$ , obtained by combining Theorem 1.1 with results of [6]. There is now only a very narrow regime, say  $W^{0.99} \ll L \ll W$ , in which the bounds of (1) are not asymptotically tight, and they differ only in the logarithmic factor.

Our final result is a upper bound for VC-dimension in terms of W and U (the number of non-linear units, or nodes). This bound is tight in the case d=1, as discussed in Remark 3.

**Theorem 1.4.** Consider a neural network with W parameters and U units with activation functions that are piecewise polynomials of degree at most d. Let  $\mathcal{F}$  be the set of (real-valued) functions computed by this network. Then  $VCDim(sgn(\mathcal{F})) = O(WU \log(d+1))$ .

The proof of this result appears in Section 6. The best known upper bound before our work was  $O(W^2)$  (implicitly proven in [6, Section 3.1], for constant d). Our theorem improves this to the tight result O(WU).

**Related Work.** Recently there have been several theoretical papers that establish the power of depth in neural networks. Last year, two striking papers considered the problem of approximating a deep neural network with a shallower network. [12] shows that there is a ReLU network with L layers and  $U = \Theta(L)$  units such that any network approximating it with only  $O(L^{1/3})$  layers must have  $\Omega(2^{L^{1/3}})$  units; this phenomenon holds even for real-valued functions. [5] show an analogous result for a high-dimensional 3-layer network that cannot be approximated by a 2-layer network except with an exponential blow-up in the number of nodes.

Very recently, several authors have shown that deep neural networks are capable of approximating broad classes of functions. [11] show that a sufficiently non-linear  $C^2$  function on  $[0,1]^d$  can be approximated with  $\epsilon$  error in  $L_2$  by a ReLU network with  $O(\operatorname{polylog}(1/\epsilon))$  layers and weights, but any such approximation with O(1) layers requires  $\Omega(1/\epsilon)$  weights. [14] shows that any  $C^n$ -function on  $[0,1]^d$  can be approximated with  $\epsilon$  error in  $L_\infty$  by a ReLU network with  $O(\log(1/\epsilon))$  layers and  $O((\frac{1}{\epsilon})^{d/n}\log(1/\epsilon))$  weights. [9] show that a sufficiently smooth univariate function can be approximated with  $\epsilon$  error in  $L_\infty$  by a network with ReLU and threshold gates with  $O(\log(1/\epsilon))$  layers and  $O(\log\log(1/\epsilon))$  weights, but that  $O(\log(1/\epsilon))$  weights would be required if there were only  $O(\log(1/\epsilon))$  layers; they also prove analogous results for multivariate functions. Lastly, [4] draw a connection to tensor factorizations to show that, for non-ReLU networks, the set of functions computable by a shallow network have measure zero among those computable by a deep networks.

## 2 Preliminaries

A neural network is defined by an activation function  $\psi : \mathbb{R} \to \mathbb{R}$ , a directed acyclic graph, a weight for each edge of the graph, and a bias for each node of the graph. Let W denote the number of parameters (weights and biases) of the network, U denote the number of computation units (nodes), and U denote the number of layers.

The nodes at layer 0 are called input nodes, and simply output the real value given by the corresponding input to the network. For the purposes of this paper, we will assume that the graph has a single sink node, which is the unique node at layer L, (the output layer). In the jargon of neural networks, layers 1 through L-1 are called hidden layers.

The computation of a neural network proceeds as follows. For  $i=1,\ldots,L$ , the input into a computation unit u at layer i is  $w^{\top}x$  where x is the (real) vector corresponding to the output of the computational units with a directed edge to u and w is the corresponding edge weights. For layers  $1,\ldots,L-1$ , the output of u is  $\psi(w^{\top}x+b)$  where b is the bias parameter associated with u. For the output layer, we replace  $\psi$  with the identity. Since we consider VC-dimension, we will always take the sign of the output of the network, to make the output lie in  $\{0,1\}$  for binary classification. (Here, we define the sign function as  $\mathrm{sgn}(x)=\mathbf{1}[x>0]$ .)

A piecewise polynomial function with p pieces is a function f for which there exists disjoint intervals (pieces)  $I_1,\ldots,I_p$  and polynomials  $f_1,\ldots,f_p$  such that if  $x\in I_i$  then  $f(x)=f_i(x)$ . We assume that p is a constant independent of W,U and L. A piecewise linear function is a piecewise polynomial function in which each  $f_i$  is linear. The most common activation function used in practice is the rectified linear unit (ReLU) where  $I_1=(-\infty,0],\,I_2=(0,\infty)$  and  $f_1(x)=0,f_2(x)=x$ . We denote this function by  $\sigma(x):=\max\{0,x\}$ . The set  $\{1,2,\ldots,n\}$  is denoted [n].

### 3 Proof of Theorem 1.1

The proof of our main lower bound uses the "bit extraction" technique that was developed by [2] to prove a  $\Omega(WL)$  lower bound. We refine their technique in a key way — we partition the input bits into blocks and extract multiple bits at a time instead of a single bit at a time. This yields a more efficient bit extraction network, and hence a stronger VC-dimension lower bound.

We show the following result, which immediately implies Theorem 1.1.

**Theorem 3.1.** Let r, m, n be positive integers, and let  $k = \lceil m/r \rceil$ . There exists a ReLU network with 3 + 5k layers,  $2 + n + 4m + k((11 + r)2^r + 2r + 2)$  parameters, and  $m + 2 + k(5 \times 2^r + r + 1)$  computation units with VC-dimension  $\geq mn$ .

Remark. Choosing r=1 gives a network with W=O(m+n), U=O(m) and VC-dimension  $\Omega(mn)=\Omega(WU)$ . This implies that the upper bound O(WU) given in Theorem 1.4 is tight.

To prove Theorem 1.1, apply Theorem 3.1 with m = rL/8,  $r = \log_2(W/L)/2$ , and  $n = W - 5m2^r$ . In the rest of this section we prove Theorem 3.1.

Let  $S_n\subseteq\mathbb{R}^n$  denote the standard basis. We shatter the set  $S_n\times S_m$ . Given an arbitrary function  $f\colon S_n\times S_m\to\{0,1\}$ , we build a ReLU neural network that inputs  $(x_1,x_2)\in S_n\times S_m$  and outputs  $f(x_1,x_2)$ . Define n numbers  $a_1,a_2,\ldots,a_n\in\{\frac{0}{2^m},\frac{1}{2^m},\ldots,\frac{2^{m-1}}{2^m}\}$  so that the ith digit of the binary representation of  $a_j$  equals  $f(e_j,e_i)$ . These numbers will be used as the parameters of the network, as described below.

Given input  $(x_1, x_2) \in S_n \times S_m$ , assume that  $x_1 = e_i$  and  $x_2 = e_j$ . The network must output the ith bit of  $a_j$ . This "bit extraction approach" was used in [2, Theorem 2] to give an  $\Omega(WL)$  lower bound for the VC-dimension. We use a similar approach but we introduce a novel idea: we split the bit extraction into blocks and extract r bits at a time instead of a single bit at a time. This allows us to prove a lower bound of  $\Omega(WL\log(W/L))$ . One can ask, naturally, whether this approach can be pushed further. Our Theorem 1.2 implies that the bit extraction approach cannot give a lower bound better than  $\Omega(WL\log(W/L))$  (see Remark 3).

The first layer of the network "selects"  $a_j$ , and the remaining layers "extract" the *i*th bit of  $a_j$ . In the first layer we have a single computational unit that calculates

$$a_j = (a_1, \dots, a_n)^{\top} x_1 = \sigma ((a_1, \dots, a_n)^{\top} x_1).$$

This part uses 1 layer, 1 computation unit, and 1 + n parameters.

The rest of the network extracts all bits of  $a_j$  and outputs the *i*th bit. The extraction is done in k steps, where in each step we extract the r most significant bits and zero them out. We will use the following building block for extracting r bits.

**Lemma 3.2.** Suppose positive integers r and m are given. There exists a ReLU network with 5 layers,  $5 \times 2^r + r + 1$  units and  $11 \times 2^r + r2^r + 2r + 2$  parameters that given the real number  $b = 0.b_1b_2 \dots b_m$  (in binary representation) as input, outputs the (r+1)-dimensional vector  $(b_1, b_2, \dots, b_r, 0.b_{r+1}b_{r+2} \dots b_m)$ .

Figure 1 shows a schematic of the ReLU network in the above lemma.

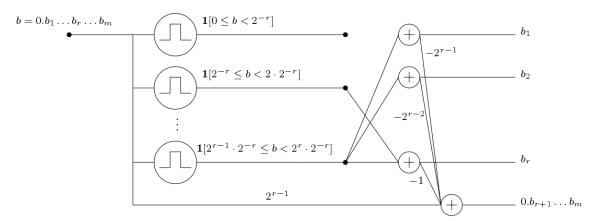


Figure 1: The ReLU network used to extract the most significant r bits of a number. Unlabelled edges indicate a weight of 1 and missing edges indicate a weight of 0.

*Proof.* Partition [0,1) into  $2^r$  even subintervals. Observe that the values of  $b_1, \ldots, b_r$  are determined by knowing which such subinterval b lies in. We first show how to design a two-layer ReLU network that computes the indicator function for an interval to any arbitrary precision. Using  $2^r$  of these networks in parallel allows us to determine which subinterval b lies in and hence, determine the bits  $b_1, \ldots, b_r$ .

For any  $a \leq b$  and  $\varepsilon > 0$ , observe that the function  $f(x) \coloneqq \sigma(1 - \sigma(a/\varepsilon - x/\varepsilon)) + \sigma(1 - \sigma(x/\varepsilon - b/\varepsilon)) - 1$  has the property that, f(x) = 1 for  $x \in [a,b]$ , and f(x) = 0 for  $x \notin (a - \varepsilon, b + \varepsilon)$ , and  $f(x) \in [0,1]$  for all x. Thus we can use f to approximate the indicator function for [a,b], to any desired precision. We will choose  $\varepsilon = 2^{-m-2}$  because we are working with m-digit numbers.

Thus, the values  $b_1,\ldots,b_r$  can be generated by adding the corresponding indicator variables. (For instance,  $b_1=\sum_{k=2^{r-1}}^{2^r-1}\mathbf{1}[b\in[k\cdot 2^{-r},(k+1)\cdot 2^{-r})]$ .) Finally, the remainder  $0.b_{r+1}b_{r+2}\ldots b_m$  can be computed as  $0.b_{r+1}b_{r+2}\ldots b_m=2^rb-\sum_{k=1}^r 2^{r-k}b_k$ .

Now we count the number of layers and parameters: we use  $2^r$  small networks that work in parallel for producing the indicators, each has 3 layers, 5 units and 11 parameters. To produce  $b_1, \ldots, b_r$  we need an additional layer,  $r \times (2^r + 1)$  additional parameters, and r additional units. For producing the remainder we need 1 more layer, 1 more unit, and r + 2 more parameters.

We use  $\lceil m/r \rceil$  of these blocks to extract the bits of  $a_j$ , denoted by  $a_{j,1},\ldots,a_{j,m}$ . Extracting  $a_{j,i}$  is now easy, noting that if  $x,y \in \{0,1\}$  then  $x \wedge y = \sigma(x+y-1)$ . So, since  $x_2 = e_i$ , we have

$$a_{j,i} = \sum_{t=1}^{m} x_{2,t} \wedge a_{j,t} = \sum_{t=1}^{m} \sigma(x_{2,t} + a_{j,t} - 1) = \sigma\left(\sum_{t=1}^{m} \sigma(x_{2,t} + a_{j,t} - 1)\right).$$

This calculation needs 2 layers, 1 + m units, and 1 + 4m parameters.

Remark. Theorem 1.2 implies an inherent barrier to proving lower bounds using the "bit extraction" approach of [2]. Recall that this technique uses n binary numbers with m bits to encode a function  $f \colon S_n \times S_m \to \{0,1\}$  to show an  $\Omega(mn)$  lower bound for VC-dimension, where  $S_k$  denotes the set of standard basis vectors in  $\mathbb{R}^k$ . The network begins by selecting one of the n binary numbers, and then extracting a particular bit of that number. [2] showed it is possible to take  $m = \Omega(L)$  and  $n = \Omega(W)$ , thus proving a lower bound of  $\Omega(WL)$  for the VC-dimension. In Theorem 1.1 we showed we can increase m to  $\Omega(L\log(W/L))$ , improving the lower bound to  $\Omega(WL\log(W/L))$ . Theorem 1.2 implies that to extract just the least significant bit, one is forced to have  $m = O(L\log(W/L))$ ; on the other hand, we always have  $n \leq W$ . Hence there is no way to improve the VC-dimension lower bound by more than a constant via the bit extraction technique. In particular, closing the gap for general piecewise polynomial networks will require a different technique.

## 4 Proof of Theorem 1.2

For a piecewise polynomial function  $\mathbb{R} \to \mathbb{R}$ , *breakpoints* are the boundaries between the pieces. So if the function has p pieces, it has p-1 breakpoints.

**Lemma 4.1.** Let  $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{R}$  be piecewise polynomial of degree D, and suppose the union of their breakpoints has size B. Let  $\psi : \mathbb{R} \to \mathbb{R}$  be piecewise polynomial of degree d with b breakpoints. Let  $w_1, \ldots, w_k \in \mathbb{R}$  be arbitrary. The function  $g(x) \coloneqq \psi(\sum_i w_i f_i(x))$  is piecewise polynomial of degree Dd with at most (B+1)(2+bD)-1 breakpoints.

*Proof.* Without loss of generality, assume that  $w_1 = \cdots = w_k = 1$ . The function  $\sum_i f_i$  has B+1 pieces. Consider one such interval  $\mathcal{I}$ . We will prove that it will create at most 2+bD pieces in g. In fact, if  $\sum_i f_i$  is constant on  $\mathcal{I}$ , g will have 1 piece on  $\mathcal{I}$ . Otherwise, for any point g, the equation  $\sum_i f_i(x) = g$  has at most g solutions on g. Let g be the breakpoints of g. Suppose we move along the curve g be the breakpoints of g. Suppose we move along the curve g be at most g be an end of g. Whenever we hit a point g be a for some g because g

**Theorem 4.2.** Assume there exists a neural network with W parameters and L layers that computes a function  $f: \mathbb{R} \to \mathbb{R}$ , with the property that  $|f(x) - (x \mod 2)| < 1/2$  for all  $x \in \{0, 1, \dots, 2^m - 1\}$ . Also suppose the activation functions are piecewise polynomial of degree at most  $d \ge 1$  in each piece, and have at most  $p \ge 1$  pieces. Then we have

$$m \le L \log_2(13pd^{(L+1)/2} \cdot W/L).$$

In the special case of ReLU functions, this gives  $m = O(L \log(W/L))$ .

*Proof.* For a node v of the network, let f(v) count the number of directed paths from the input node to v. Applying Lemma 4.1 iteratively gives that for a node v at layer  $i \geq 1$ , the number of breakpoints can be bounded by  $(6p)^i d^{i(i-1)/2} f(v) - 1$ . Let o denote the output node. Hence, o has at most  $(6p)^L d^{L(L-1)/2} f(o)$  pieces. The output of node o is piecewise polynomial of degree at most  $d^L$ . On the other hand, as we increase v from 0 to v 1, the function v 2 flips v 1 many times, which implies the output of v 2 becomes equal to v 2 at least v 2 times, thus we get

$$(6p)^{L}d^{L(L-1)/2}f(o) \times d^{L} \ge 2^{m} - 1.$$
(2)

Let us now relate f(o) with W and L. Suppose that, for  $i \in [L]$ , there are  $W_i$  edges between layer i and previous layers. By the AM-GM inequality,

$$f(o) \le \prod_{i} (1 + W_i) \le \left(\sum_{i} \frac{1 + W_i}{L}\right)^L \le (2W/L)^L.$$
 (3)

Combining Eqs. (2) and (3) gives the theorem.

[12] showed how to construct a function f which satisfies  $f(x) = (x \mod 2)$  for  $x \in \{0,1,\dots,2^m-1\}$  using a neural network with O(m) layers and O(m) parameters. By choosing  $m=k^3$ , Telgarsky showed that any function g computable by a neural network with  $\Theta(k)$  layers and  $O(2^k)$  nodes must necessarily have  $\|f-g\|_1 > c$  for some constant c>0.

Our theorem above implies a similar statement. In particular, if we choose  $m=k^{1+\varepsilon}$  then for any function g computable by a neural network with  $\Theta(k)$  layers and  $O(2^{k^{\varepsilon}})$  parameters, there must exist  $x\in\{0,1,\ldots,2^m-1\}$  such that |f(x)-g(x)|>1/2.

## 5 Proof of Theorem 1.3

The proof of this theorem is very similar to the proof of the upper bound for piecewise polynomial networks in [2, Theorem 1] but optimized for piecewise linear networks. Our proof requires the following lemma, which is a slight improvement of a result in [13].

**Lemma 5.1** (Theorem 8.3 in [1], Lemma 1 in [2]). Let  $p_1, \ldots, p_m$  be polynomials of degree at most d in  $n \leq m$  variables. Define

$$K := |\{(\operatorname{sgn}(p_1(x)), \dots, \operatorname{sgn}(p_m(x)) : x \in \mathbb{R}^n\}|,$$

i.e. K is the number of possible sign vectors given by the polynomials. Then  $K \leq 2(2emd/n)^n$ .

of Theorem 1.3. Suppose the activation function has p pieces, and suppose, for simplicity, that the pieces are  $(-\infty,t_1],(t_1,t_2],\ldots,(t_{p-1},+\infty)$ . (It is straightforward to generalize to pieces of arbitrary form.) For  $i\in [L]$ , let  $W_i$  denote the total number of parameters up to layer i, and let  $k_i$  denote the number of computational units at layer i. Note that  $W_L=W$ , and the total number of computational units is  $k=\sum_{i=1}^L k_i$ . Let m denote the VC-dimension of the network, let a be the number of input nodes, and let  $\{x_1,\ldots,x_m\}\subset\mathbb{R}^a$  be a shattered set. (Note that each  $x_i$  is a real vector in  $\mathbb{R}^a$ .) If  $m\leq W$  the theorem's conclusion already holds, so we may assume that m>W. Define

$$K := |\{(\operatorname{sgn}(f(x_1, w)), \dots, \operatorname{sgn}(f(x_m, w))) : w \in \mathbb{R}^W\}|.$$

In other words, K is the number of sign patterns that the neural network can output for the sequence of inputs  $(x_1, \ldots, x_m)$ . By definition of shattering, we have  $K = 2^m$ . We will prove geometric upper bounds for K, which will imply upper bounds for M. For the rest of the proof,  $x_1, \ldots, x_m$  are fixed, and we view the parameters of the network, denoted M, as a collection of M real variables.

To bound K, we will find a partition  $\mathcal{P}$  of  $\mathbb{R}^W$  such that, for all  $P \in \mathcal{P}$ , the functions  $f(x_1, \cdot), \dots, f(x_m, \cdot)$  are polynomials on P. Clearly,

$$K \le \sum_{P \in \mathcal{P}} |\{(\operatorname{sgn}(f(x_1, w)), \dots, \operatorname{sgn}(f(x_m, w))) : w \in P\}|.$$
(4)

We will define a sequence of partitions  $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_L=\mathcal{P}$  inductively, such that  $\mathcal{P}_i$  is a refinement of  $\mathcal{P}_{i-1}$ . Starting from layer 1, for  $\ell\in[m]$  and  $j\in[k_1]$ , let  $h_{1,j}(x_\ell,w)$  be the input into the jth computation unit in the first layer and let  $\mathcal{P}_1$  be a partition of  $\mathbb{R}^W$  such that the vector  $(\operatorname{sgn}(h_{1,j}(x_\ell,\cdot)-t_s))_{j\in[k_1],\ell\in[m],s\in[p-1]}$  is constant on each  $P\in\mathcal{P}_1$ . Since each  $h_{1,j}(x_\ell,\cdot)-t_s$  is a polynomial of degree 1, by Lemma 5.1, we

can take  $|\mathcal{P}_1| \leq 2(2ek_1mp/W_1)^{W_1}$ . Observe that on each  $P \in \mathcal{P}_1$ , the output of each computation unit in the first hidden layer is either a polynomial of degree 1 or the zero function.

Now, suppose the partitions  $\mathcal{P}_1,\ldots,\mathcal{P}_{i-1}$  have been defined. Let  $h_{i,j}(x_\ell,w)$  be the input into the jth computation unit in the ith layer and assume that on any fixed  $P\in\mathcal{P}_{i-1}$ , the function  $h_{i,j}(x_\ell,\cdot)$  is a polynomial of degree at most i. By Lemma 5.1, there exists a partition  $\mathcal{P}_{P,i}$  of P with  $|\mathcal{P}_{P,i}| \leq 2(2ek_imip/W_i)^{W_i}$  such that on each  $P'\in\mathcal{P}_{P,i}$ , the vector  $(\mathrm{sgn}(h_{i,j}(x_\ell,\cdot)-t_s))_{j\in[k_i],\ell\in[m],s\in[p-1]}$  is constant. We define  $\mathcal{P}_i:=\cup_{P\in\mathcal{P}_{i-1}}\mathcal{P}_{P,i}$ ; which is a partition of  $\mathbb{R}^W$  such that the vector  $(\mathrm{sgn}(h_{i,j}(x_\ell,\cdot)-t_s))_{j\in[k_i],\ell\in[m],s\in[p-1]}$  is constant on each  $P\in\mathcal{P}_i$ . Moreover, for any  $1< i\leq L$ , we have

$$|\mathcal{P}_i| \le 2 \left(\frac{2ek_imip}{W_i}\right)^{W_i} |\mathcal{P}_{i-1}|.$$

Recall that the last (output) layer has a single unit. Let  $\mathcal{P}_L$  be the partition of  $\mathbb{R}^W$  defined as above, such that on each  $P \in \mathcal{P}_L$ , the output of the network is a polynomial of degree at most L in W variables. By Lemma 5.1, each term of the sum in (4) is at most  $2(2emL/W)^W$ . Moreover, by our inductive construction, and since  $k_i \leq k$ ,  $W_i \geq 1, \sum_{i=1}^L W_i \leq WL$ , and  $i \leq L$ , we have

$$|\mathcal{P}_L| \le 2^L \prod_{i=1}^L \left(\frac{2ek_i mip}{W_i}\right)^{W_i} \le 2^L (2ekmLp)^{WL}.$$

Hence,

$$2^{m} = K \le 2(2emL/W)^{W} \times |\mathcal{P}_{L}| \le (2(2ekmLp)^{W})^{L+1}$$
.

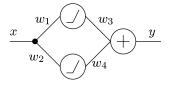
Taking logarithms and using the crude bounds  $k, L \leq W$  gives the condition

$$m < (L+1) + W(L+1)\log_2(2eW^2mp) < 4W(L+1)\log_2(2eWmp)$$

which implies  $m = O(WL \log(pW))$ , as required.

## 6 Proof of Theorem 1.4

The idea of the proof is that the sign of the output of a neural network can be expressed as a Boolean formula where each predicate is a polynomial inequality. For example, consider the following toy network, where the activation function of the hidden units is a ReLU.



The sign of the output of the network is  $sgn(y) = sgn(w_3\sigma(w_1x) + w_4\sigma(w_2x))$ . Define the following Boolean predicates:  $p_1 = (w_1x > 0)$ ,  $p_2 = (w_2x > 0)$ ,  $q_1 = (w_1x > 0)$ 

 $(w_3w_1x > 0)$ ,  $q_2 = (w_4w_2x > 0)$ , and  $q_3 = (w_3w_1x + w_4w_2x > 0)$ . Then, we can write

$$\operatorname{sgn}(y) = (\neg p_1 \wedge \neg p_2 \wedge 0) \vee (p_1 \wedge \neg p_2 \wedge q_1) \vee (\neg p_1 \wedge p_2 \wedge q_2) \vee (p_1 \wedge p_2 \wedge q_3).$$

A theorem of Goldberg and Jerrum states that any class of functions that can be expressed using a relatively small number of distinct polynomial inequalities has small VC-dimension.

**Theorem 6.1** (Theorem 2.2 of [6]). Let  $k, n \in \mathbb{N}$  and  $f : \mathbb{R}^n \times \mathbb{R}^k \to \{0, 1\}$  be a function that can be expressed as a Boolean formula containing s distinct atomic predicates where each atomic predicate is a polynomial inequality or equality in k + n variables of degree at most d. Let  $\mathcal{F} = \{f(\cdot, w) : w \in \mathbb{R}^k\}$ . Then  $VCDim(\mathcal{F}) \leq 2k \log_2(8eds)$ .

of Theorem 1.4. Consider a neural network with W weights and U computation units, and assume that the activation function  $\psi$  is piecewise polynomial of degree at most d with p pieces. To apply Theorem 6.1, we will express the sign of the output of the network as a boolean function consisting of less than  $2(1+p)^U$  atomic predicates, each being a polynomial inequality of degree at most  $\max\{U+1,2d^U\}$ .

Since the neural network graph is acyclic, it can be topologically sorted. For  $i \in [U]$ , let  $u_i$  denote the *i*th computation unit in the topological ordering. The input to each computation unit u lies in one of the p pieces of  $\psi$ . For  $i \in [U]$  and  $j \in [p]$ , we say " $u_i$  is in state j" if the input to  $u_i$  lies in the jth piece.

For  $u_1$  and any j, the predicate " $u_1$  is in state j" is a single atomic predicate which is the quadratic inequality indicating whether its input lies in the corresponding interval. So, the state of  $u_1$  can be expressed as a function of p atomic predicates. Conditioned on  $u_1$  being in a certain state, the state of  $u_2$  can be determined using p atomic predicates, which are polynomial inequalities of degree at most 2d+1. Consequently, the state of  $u_2$  can be determined using  $p+p^2$  atomic predicates, each of which is a polynomial of degree at most 2d+1. Continuing similarly, we obtain that for each i, the state of  $u_i$  can be determined using  $p(1+p)^{i-1}$  atomic predicates, each of which is a polynomial of degree at most  $d^{i-1} + \sum_{j=0}^{i-1} d^j$ . Consequently, the state of all nodes can be determined using less than  $(1+p)^U$  atomic predicates, each of which is a polynomial of degree at most  $d^{U-1} + \sum_{j=0}^{U-1} d^j \leq \max\{U+1,2d^U\}$  (the output unit is linear). Conditioned on all nodes being in certain states, the sign of the output can be determined using one more atomic predicate, which is a polynomial inequality of degree at most  $\max\{U+1,2d^U\}$ .

In total, we have less than  $2(1+p)^U$  atomic polynomial-inequality predicates and each polynomial has degree at most  $\max\{U+1,2d^U\}$ . Thus, by Theorem 6.1, we get an upper bound of  $2W\log(16e\cdot\max\{U+1,2d^U\}\cdot(1+p)^U)=O(WU\log((1+d)p))$  for the VC-dimension.

## References

[1] Martin Anthony and Peter Bartlett. *Neural network learning: theoretical foundations*. Cambridge University Press, 1999.

- [2] Peter Bartlett, Vitaly Maiorov, and Ron Meir. Almost linear VC-dimension bounds for piecewise polynomial networks. *Neural Computation*, 10(8):2159– 2173, Nov 1998.
- [3] A. Blumer, A. Ehrenfeucht, D. Haussler, and M. Warmuth. Learnability and the Vapnik-Chervonenkis dimension. *JACM*, 36(4), 1989.
- [4] N. Cohen, O. Sharir, and A. Shashua. On the expressive power of deep learning: A tensor analysis. In *COLT*, 2016.
- [5] Ronen Eldan and Ohad Shamir. The power of depth for feedforward neural networks. In *COLT*, 2016.
- [6] Paul W. Goldberg and Mark R. Jerrum. Bounding the Vapnik-Chervonenkis dimension of concept classes parameterized by real numbers. *Machine Learning*, 18(2):131–148, 1995.
- [7] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep Learning*. MIT Press, 2016. http://www.deeplearningbook.org.
- [8] Yann LeCun, Yoshua Bengio, and Geoffrey Hinton. Deep learning. *Nature*, 521(7553):436–444, 2015.
- [9] Shyu Liang and R. Srikant. Why deep neural networks?, October 2016.
- [10] Wolfgang Maass. Neural nets with superlinear VC-dimension. *Neural Computation*, 6(5):877–884, Sept 1994.
- [11] I. Safran and O. Shamir. Depth separation in relu networks for approximating smooth non-linear functions, October 2016. arXiv:1610.09887.
- [12] Matus Telgarsky. Benefits of depth in neural networks. In COLT, 2016.
- [13] Hugh E. Warren. Lower bounds for approximation by nonlinear manifolds. *Transactions of the American Mathematical Society*, 133(1):167–178, 1968.
- [14] Dmitry Yarotsky. Error bounds for approximations with deep ReLU networks, October 2016. arXiv:1610.01145.