PDE Based Image Filters: Modeling Aspects

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Outline

1. Introduction

2. PDE based approach

Image Restoration

- The process of removing or diminishing the effects of degradation is called restoration.

Major Approaches in Image processing:

- Stochastic modeling
- Wavelets
- Partial Differential Equations (PDE)

Why PDE's?

- PDE's are closely related to physical world.
- Theory for PDE's is well established.
- Reasoning in the continuous framework makes the understanding of physical realities easier and provides the intution to propose new models.

The physical background of Diffusion

- Diffusion is a physical process that equilibrates concentration differences without creating or destroying mass.
- The equilibration property is expressed by Fick's law:

$$j = -D.\nabla u \tag{1}$$

which states that a concentration gradient ∇u will cause a flux j of concentrations which aims at compensating the gradient. where D is the diffusion tensor.

- Preservation of mass is expressed by the continuity equation

$$u_t = -div(j) \tag{2}$$

where t is time.

from (1) and (2)

$$u_t = div(D.\nabla u). \tag{3}$$

- The case where j and ∇u are parallel is called isotropic. Then we replace the diffusion tensor by a positive scalar-valued diffusivity g.
- If j and ∇u are not parallel, then we say that diffusion process is anisotropic.
- In the context of heat transfer (3) is called heat equation.
- In Image processing we identify the concentration with the gray value at certain location.
- If the diffusion tensor is a function of the differential structure of the evolving image (i.e. u) then we call (3) as nonlinear diffusion.

Linear Diffusion filtering
Motivation

- Let a gray-scale image f be represented by a real-valued mapping $f \in L^1(\mathbb{R}^2)$.
- A widely-used way to smooth f is by calculating the convolution:

$$(k_\sigma * f)(x) = \int_{\mathbb{R}^2} k_\sigma(x-y) f(y) dy \tag{4}$$

  where k_σ denotes the two-dimensional Gaussian of width (standard deviation) $\sigma > 0$ and is given by $k_\sigma(x) = \frac{1}{2\pi\sigma^2} exp(\frac{-|x|^2}{2\sigma^2})$.
- This operating is called low-pass filtering.
- Note that $k_\sigma \in C^\infty(\mathbb{R}^2) \Rightarrow \; k_\sigma * f \in C^\infty(\mathbb{R}^2)$ even if f is absolutely integrable.

PDE Interpretation

- For any bounded $f \in C(\mathbb{R}^2)$ the linear diffusion process

$$u_t = u_{xx} + u_{yy}; \qquad u(x,0) = f(x) \tag{5}$$

- The explicit solution is

$$u(x,t) = \int_{\mathbb{R}^2} k_{\sqrt{2t}}(x-y) f(y) dy \tag{6}$$

- Therefore solving heat equation is equivalent to carrying out a Gaussian linear filtering with $\sigma = \sqrt{2t}$.
- Hence, smoothing structures of order σ requires to stop the diffusion process at time $T = \frac{\sigma^2}{2}$

- The solution (6) is unique, whenever

$$|u(x,t)| \leq Mexp(a|x|^2) \;\; (M, a > 0). \tag{7}$$

- This solution depends continuously on the initial image f with respect to $||.||_{L^\infty(\mathbb{R}^2)}$ and it fulfils the maximum-minimum principle:

$$inf_{\mathbb{R}^2} f \leq u(x,t) \leq sup_{\mathbb{R}^2} f \;\; on \;\; \mathbb{R}^2 \times [0, \infty). \tag{8}$$

Scale-Space

- The whole embedding of the original image into a one-parameter family of simplified images is called scale-space.
- Let T_t, $t > 0$ be the family of scale operators from $L^1(\Omega)$ into $L^1(\Omega)$ defined by $(T_t u_0)(x) = u(x, t)$, where $u_0(x)$ is the initial image (f), Ω is the domain $\subset \mathbb{R}^2$and $u(x, t)$ is the unique solution of (5).

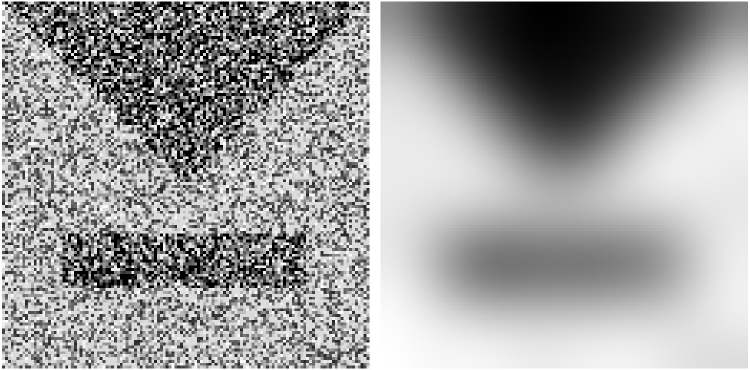
Properties of T_t

- Gray level shift invariance:
  $T_t(0) = 0$ and $T_t(u_0 + c) = T_t u_0 + c$, for any constant c.
- Translation invariance:
  $T_t(\tau u_0) = \tau(T_t u_0)$, where τ_h is the translation $\tau(f)(x) = f(x + h)$.
- Scale invariance:
  $T_t(H_\lambda u_0) = H_\lambda(T_{t'} u_0)$ with $t^{'} = t\lambda^2$, where $H_\lambda f(x) = f(\lambda x)$
- Isometry invariance:
  $T_t(Ru_0) = R(T_t u_0)$, for any orthogonal transformation R of $\mathbb{R}^2$ where $Rf(x) = f(Rx)$.
- Conservation of average value:
  $T_t(Mu_0) = M(T_t u_0)$, where $Mf = \frac{1}{|\Omega|} \int_\Omega f(x)dx$

- Semigroup Property:
  $T_{t+s}(u_0) = T_t(T_s u_0)$
- Comparison Principle:
  if $u_0 \leq v_0$ then $(T_t u_0) \leq (T_t v_0)$

Question: Are these properties sufficient to ensure correct qualitative properties for $T_t u$?
Ans: No.

Linear diffusion example



Drawbacks

- Smoothing does not depend on the image, and it is same in all directions. In particular edges are not preserved.
- For two arbitrary orthonormal directions D1 and D2, we have $\Delta u = u_{D1D1} + u_{D2D2}$.
- Rewriting this equality with the directions $D1 = N = \frac{\nabla u}{|\nabla u|}$ and $D2 = T$ with $T.N = 0$, $|T| = 1$, then
$$\Delta u = u_{NN} + u_{TT}$$
Hence the diffusion is same in two directions.
- It also dislocates edges when moving from finer to coarser scales.

The instantaneous regularity is not a desirable property, since edges can be lost or severely blurred.

Now the Goal is to develop a model which will remove noise but preserve edges at best.

IDEA: Apply a process which depends on local properties of the image. Look for an inhomogeneous process that reduces the diffusivity at those locations which have larger likelihood to be edges. This likelihood is measured by $|\nabla u|^2$.

Perona Malik model

Perona and Malik[1] introduced the following model

$$u_t = div(c(|\nabla u|^2)\nabla u) \quad in \quad \Omega\times(0,T), \tag{9}$$

$$\frac{\partial u}{\partial N} = 0 \text{ on } \partial\Omega \times (0,T),$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega$$

with $c(s) : [0,\infty) \to (0,\infty)$
choose $c(s)$ as a decreasing function satisfying $c(0) = 1$ and $\lim_{s\to\infty} c(s) = 0$.
with this choice:

- Near the regions boundaries, where the magnitude of the gradient is large, the regularization is stopped and the edges are preserved.
- Inside the regions where the gradient of u is weak, (9) acts like the heat equation, resulting in isotropic smoothing.

[1] P.Perona, J.Malik; Scale-space and edge detection using anisotropic diffusion , *IEEE Tran. Pattern Anal. Machine intelligence*, **12(7)**, (1990), 629-639.

Interpreting the divergence operator using T, N associated to the image, we get

$$u_t = c(|\nabla u|^2)u_{TT} + b(|\nabla u|^2)u_{NN}$$

where $b(s) = c(s) + 2sc^{'}(s)$.
It is preferable to smooth more in the tangential direction T than in the normal direction N.
Thus

$$lim_{s\to\infty}\frac{b(s)}{c(s)} = 0,$$

By the definition of b,

$$lim_{s\to\infty}\frac{sc^{'}(s)}{c(s)} = -\frac{1}{2}$$

$$\Rightarrow \quad c(s) \approx \frac{1}{\sqrt{s}} \text{ as } s \to \infty$$

- A well-adapted framework to study (9) is nonlinear semigroup theory.
- Perona-Malik used the following diffusivity:

$$c(|\nabla u|^2) = \frac{1}{1 + \frac{|\nabla u|^2}{\lambda^2}} \qquad (10)$$

  where λ is the contrast parameter.
- This model is both smoothing and enhancing.

Let the flux $\phi(\nabla u) = c(|\nabla u|^2)\nabla u$.
Restricting to the 1D-case, denoting $s = u_x$

$$\Rightarrow \quad \phi(s) = c(s^2).s$$

$$\Rightarrow \quad \phi^{'}(s) \geq 0 \text{ for } |s| \leq \lambda$$

$$\phi^{'}(s) < 0 \text{ for } |s| > \lambda$$

In 1D-case (5) becomes,

$$u_t = div(\phi(s))$$

$$\Rightarrow \; u_t = \phi^{'}(u_x)u_{xx}.$$

- Perona-Malik model is forward parabolic type for $|s| \leq \lambda$ as here $\phi^{'}(s) \geq 0$ and backward parabolic type for $|s| > \lambda$ as here $\phi^{'}(s) < 0$.
- Forward diffusion for $|s| \leq \lambda$ smooths the contrast, whereas backward diffusion for $|s| > \lambda$ enhances contrast.

Drawbacks of PM model

- It will misinterpret noise as small-scale edges and hence enhance the noise.
- There is no theory that guarantees well-posedness of the problem.

Regularization of the PM model

- Regularized nonlinear diffusion equation proposed by Catte[2]

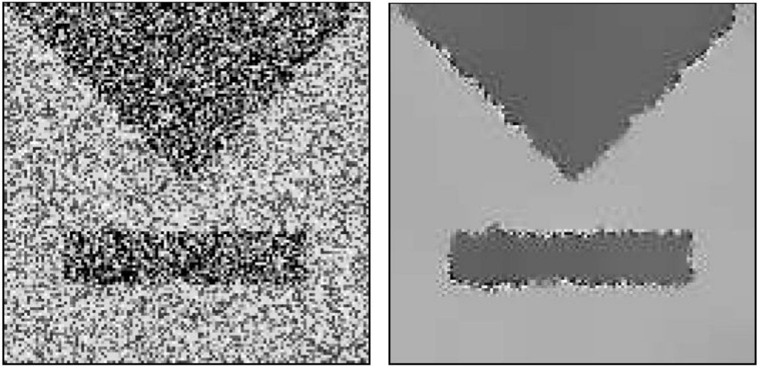
$$u_t = div(c(|\nabla u_\sigma|^2)\nabla u) \tag{11}$$

where σ is a regularized parameter.

- existence, uniqueness and regularity of a solution is proved.

[2] F.Catte, P.L.Lions, J.M. Morel, and T.Coll; Image selective smoothing and edge detection by nonlinear diffusion , *SIAM Journal on Numerical Analysis*, **29(1)**, (1992), 182-193.

Perona-Malik diffusion example



Open Questions

- PM model is successfully used in many numerical experiments even though there is no rigorous mathematical theory. This phenomenon is still unexplained.
- Does (11) converge to (9) as $\sigma \to 0$?
- choice of parameter σ. This depends related to other parameters, for example in the defining the function $c(s)$.

Edge enhancing diffusion (EED)

This model is proposed by Weickert[3]
Image $\longleftrightarrow$ Concentration distribution
Average gray value $\longleftrightarrow$ mass.

- construct orthonormal eigen vectors v_1, v_2 of D such that $v_1 \parallel \nabla u_\sigma$ and $v_2 \perp \nabla u_\sigma$.
- choose the corresponding eigen values

$$\lambda_1 = c(|\nabla u_\sigma|^2)$$

$$\lambda_2 = 1.$$

- With eigen vectors v_1 and v_2 as

$$v_1 = \frac{\nabla u_\sigma}{|\nabla u_\sigma|},$$

$$v_2 = \frac{\nabla u_\sigma^\perp}{|\nabla u_\sigma|}$$

where $\nabla u_\sigma = k_\sigma * \begin{pmatrix} u_x \\ u_y \end{pmatrix}$, $\nabla u_\sigma^\perp = k_\sigma * \begin{pmatrix} -u_y \\ u_x \end{pmatrix}$

[3] J.Weickert; Anisotropic diffusion in image processing. Stuttgart, Germany: Teubner, 1998.

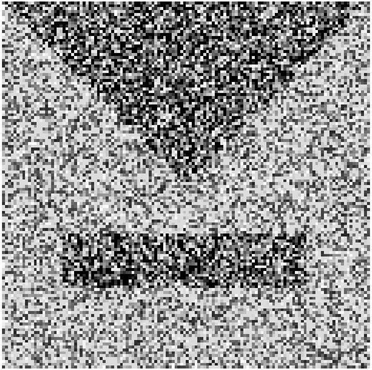
- By Spectral theorem, the EED diffusion tensor is

$$D = D(\nabla u_\sigma) = (v_1, v_2) diag(\lambda_1, \lambda_2) \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$$

- The EED is

$$\partial_t u = div(D(\nabla u_\sigma)\nabla u)$$

EED example



comparing all models