MA5710 Mathematical Modeling in Industry

July – November 2014

# CASE STUDY II: Traffic Flow (Numerical Aspects)

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# Scalar Conservation Law

$$\begin{aligned} u_t + f(u)_x &= 0, \qquad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}, \end{aligned} \qquad f : \mathbb{R} \to \mathbb{R}.$$

* *method of characteristics*

* solutions may develop discty after a finite time

## Weak Solution

*The function* $u : \mathbb{R} \times (0,T) \to \mathbb{R}$ *is called a* weak solution *if for all* $\phi \in C_0^1(\mathbb{R}^2)$

$$\int_0^\infty \int_{\mathbb{R}} (u\phi_t + f(u)\phi_x)\, dx\, dt = -\int_{\mathbb{R}} u_0(x)\phi(x,0)\, dx.$$

## Riemann Problem

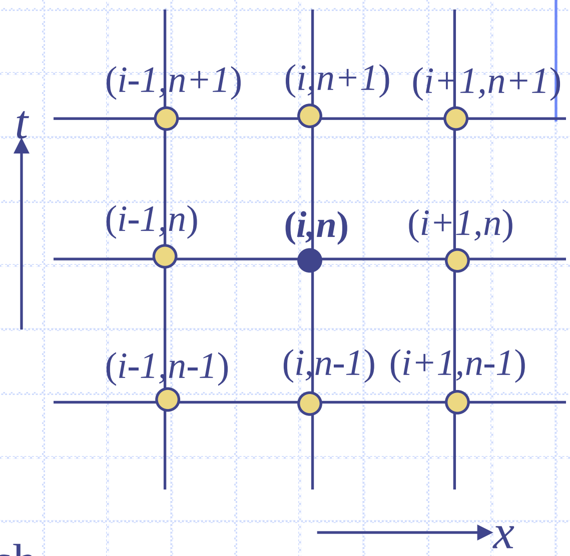
$$u_0(x) = \begin{cases} u_\ell & : \ x < 0 \\ u_r & : \ x \geq 0 \end{cases} \qquad u_\ell,\ u_r \in \mathbb{R}$$

# Numerical Approximation of Linear Scalar Conservation Laws

A simple linear scalar conservation law

$$\begin{aligned} u_t + a u_x &= 0, \qquad x \in \mathbb{R}, t > 0, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}, \end{aligned}$$

where $a > 0.$




Now, we discretize the ($x$,$t$) plane

$$x_i = ih \quad (i \in \mathbb{Z}), \qquad t_n = nk \quad (n \in \mathbb{N}_0) \quad h, k > 0$$

For simplicity we take a uniform mesh with $h$ and $k$ constant

The simplest of approximations to the solution at these grid points is the finite difference approximation i.e., to replace partial derivatives by difference quotients.

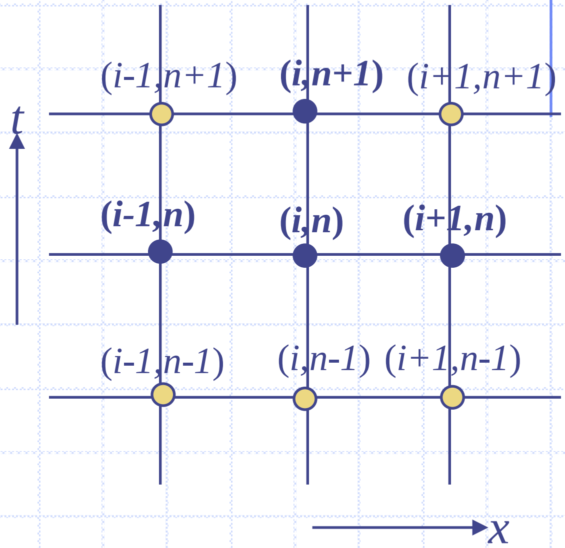
For example Taylor series expansion of the above equation

$$\frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{k} + \mathcal{O}(k) = -a \frac{u(x_{i+1}) - u(x_{i-1})}{2h} + \mathcal{O}(h^2)$$

# Central Difference Scheme

A simple linear scalar conservation law

$$\begin{aligned} u_t + a u_x &= 0, \qquad x \in \mathbb{R}, t > 0, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}, \end{aligned}$$




Using the Central Difference Approximation (in space only ??)

$$\frac{u_i^{n+1} - u_i^n}{k} = -a\,\frac{u_{i+1}^n - u_{i-1}^n}{2h}, \qquad n \geq 0,\ i \in \mathbb{Z}$$

Which can be re-written as $u_i^{n+1} = u_i^n - \frac{ak}{2h}\left(u_{i+1}^n - u_{i-1}^n\right).$

As we can compute $u_i^{n+1}$ from the data $u_i^n$ explicitly, this is known as *explicit scheme*

Equivalently, $\frac{ak}{2h}u_{i+1}^{n+1} + u_i^{n+1} - \frac{ak}{2h}u_{i-1}^{n+1} = u_i^n.$

This is an *implicit scheme*, where a linear system has to be solved

# Boundary Conditions

In Practice, we compute on a finite grid say *x* in *(0,a)* and we require appropriate Boundary Conditions.

Periodic Boundary Conditions $u(0,t) = u(Nh,t), \qquad t > 0,$

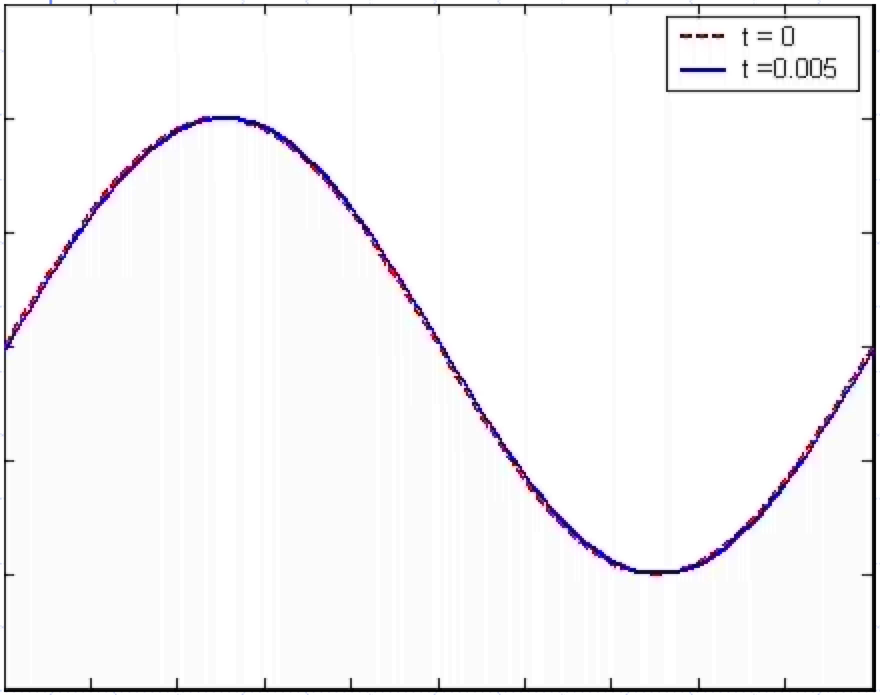
Discretized version $u_0^n = u_N^n, \qquad n \geq 0.$

Setting *i=0 or i=N*, we required to determine $u_{-1}^n$ or $u_{N+1}^n$ and we consider these points as artificial points with

$u_{-1}^n = u_{N-1}^n$ and $u_N^n = u_0^N,$ by periodicity

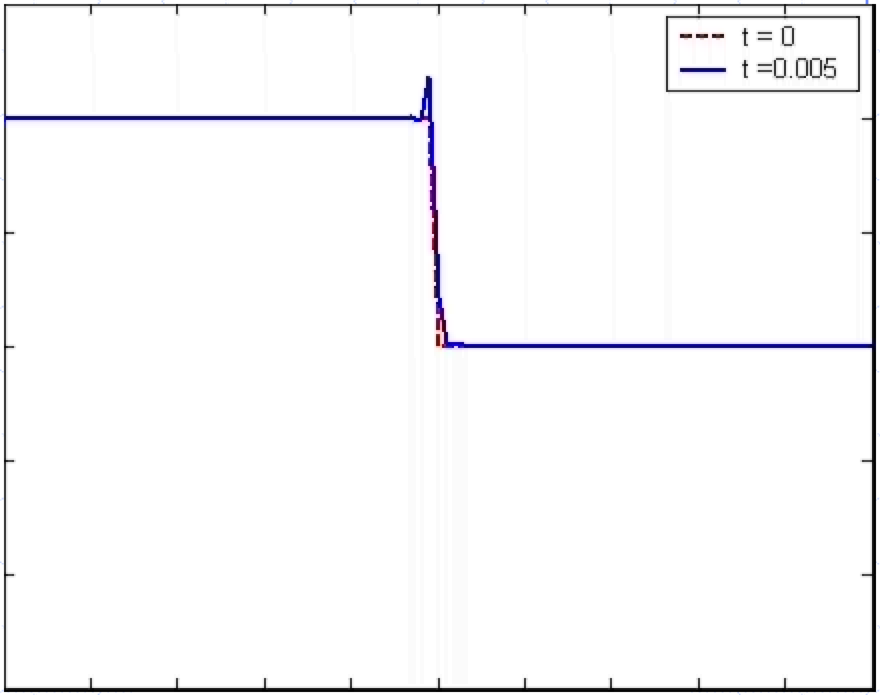
# Numerical Implementation

Continuous Initial Data




$$u_0(x) = \sin(2\pi x), \qquad 0 \le x \le 1$$

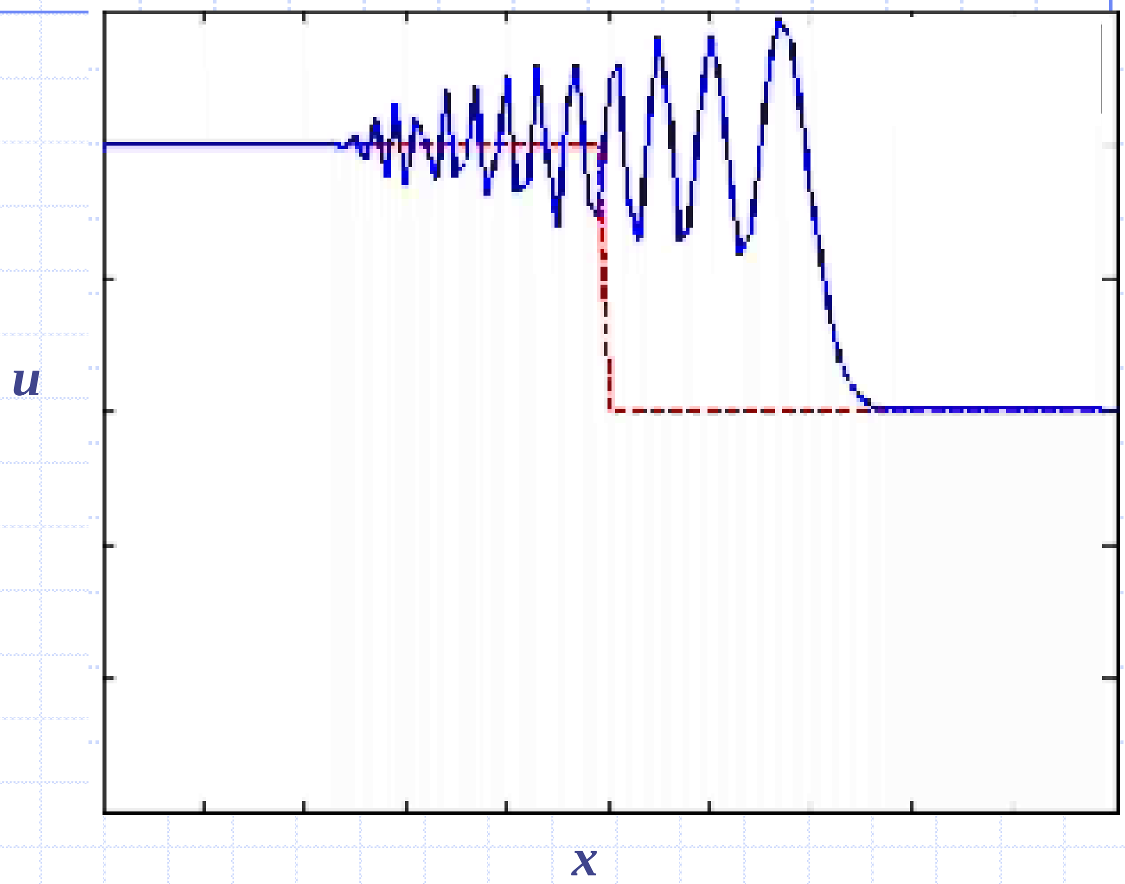
Discontinuous Initial Data




$$u_0(x) = \begin{cases} 1 & : \ 0 \le x < 1/2 \\ 0 & : \ 1/2 \le x \le 1 \end{cases}$$

*h* = 0.01     *x = 0 to 1*

*k* = 0.001     *t = 0 to 0.25*



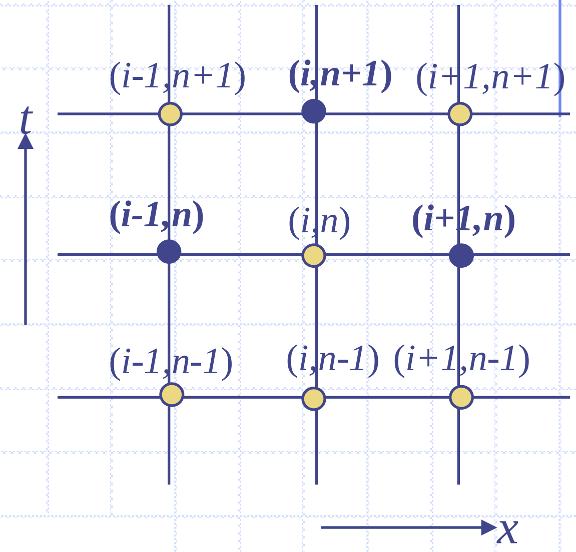
u
x

# Lax-Friedrich's Scheme

The time derivative is approximated using

$$\frac{1}{k}\left(u(x,t+k) - \frac{1}{2}(u(x+h,t)+u(x-h,t))\right)$$

And the spatial derivative is approximated using the central difference scheme



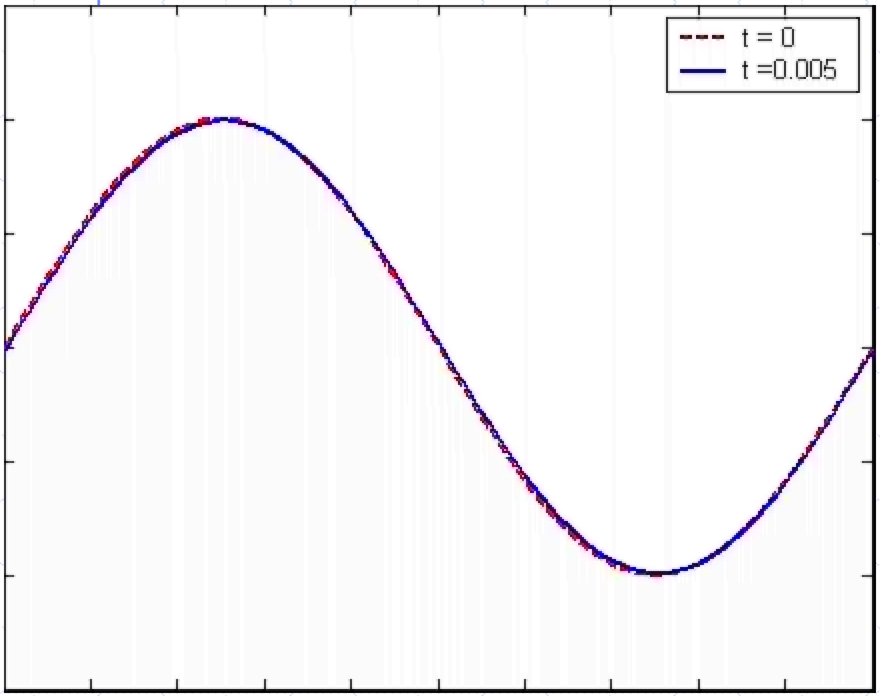

Hence, the scheme is

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{ak}{2h}(u_{i+1}^n - u_{i-1}^n), \qquad i = 1, \ldots, N-1.$$

We will see that the solution is smeared out, and this approximation becomes better and better for smaller *k>0*

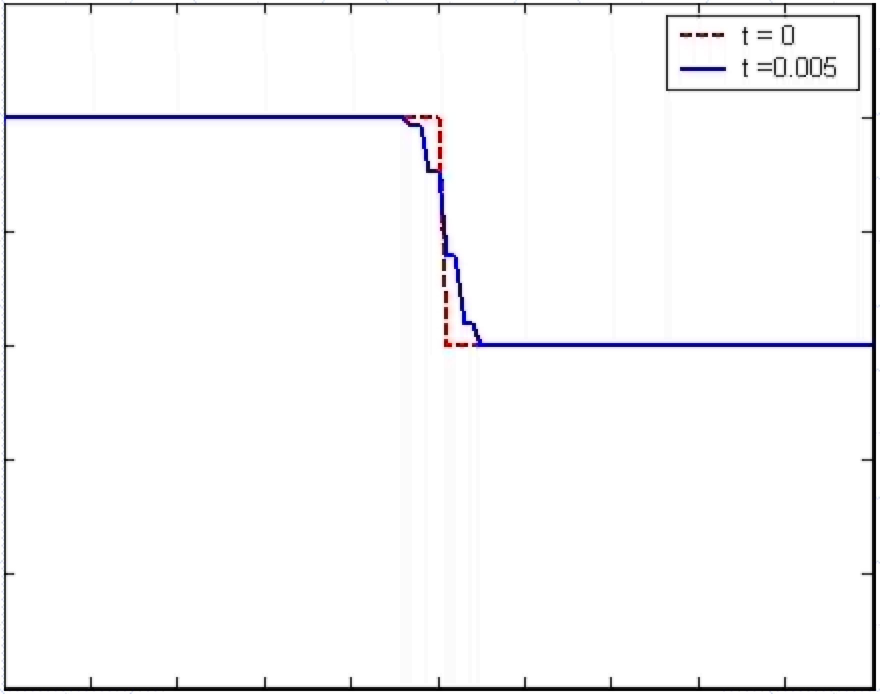
# Numerical Implementation

Continuous Initial Data




$$u_0(x) = \sin(2\pi x), \qquad 0 \le x \le 1$$

Discontinuous Initial Data




$$u_0(x) = \begin{cases} 1 & : \quad 0 \le x < 1/2 \\ 0 & : \quad 1/2 \le x \le 1 \end{cases}$$

*h* = 0.01 *x = 0 to 1*

*k* = 0.001 *t = 0 to 0.25*

# Down-Wind Scheme

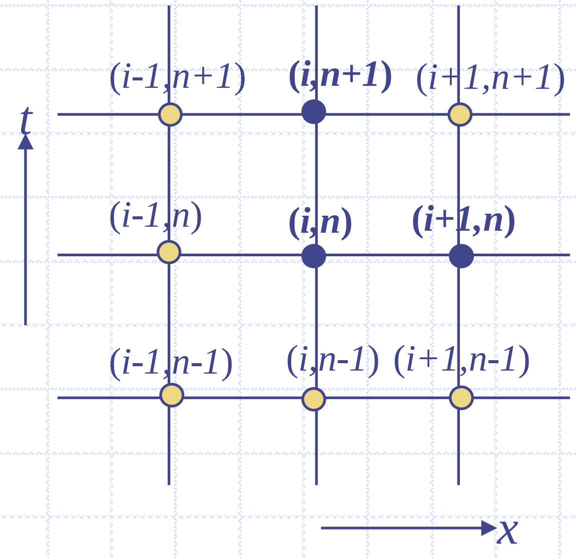
The Lax-Friedrich's scheme gives accurate approximations only if *k* is sufficiently small.

The Down-Wind scheme is described by

$$u_i^{n+1} = u_i^n - \frac{ak}{h}(u_{i+1}^n - u_i^n), \qquad i = 0, \ldots, N-1,$$

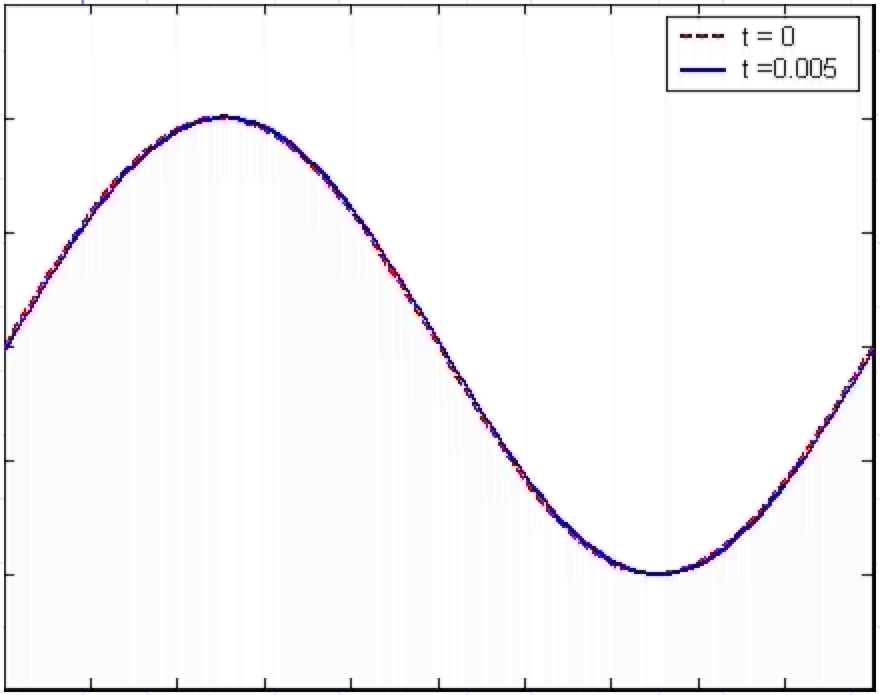
We will see that the numerical solution is unstable

The solution describes a wave from left to right.



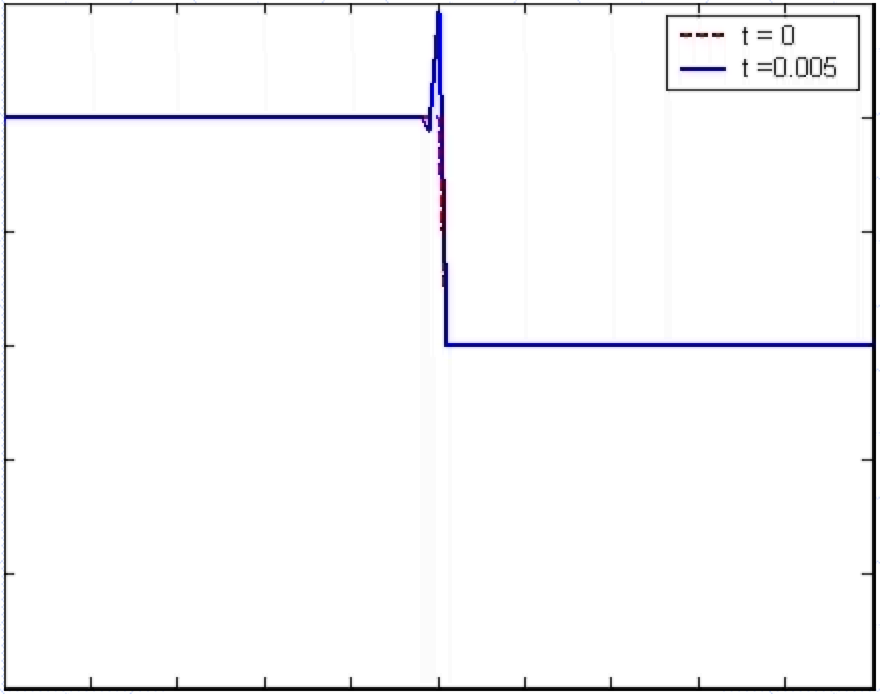
# Numerical Implementation

Continuous Initial Data




$$u_0(x) = \sin(2\pi x), \qquad 0 \leq x \leq 1$$

Discontinuous Initial Data




$$u_0(x) = \begin{cases} 1 & : \quad 0 \leq x < 1/2 \\ 0 & : \quad 1/2 \leq x \leq 1 \end{cases}$$

$h$ = 0.01    *x = 0 to 1*
$k$ = 0.001    *t = 0 to 0.25*

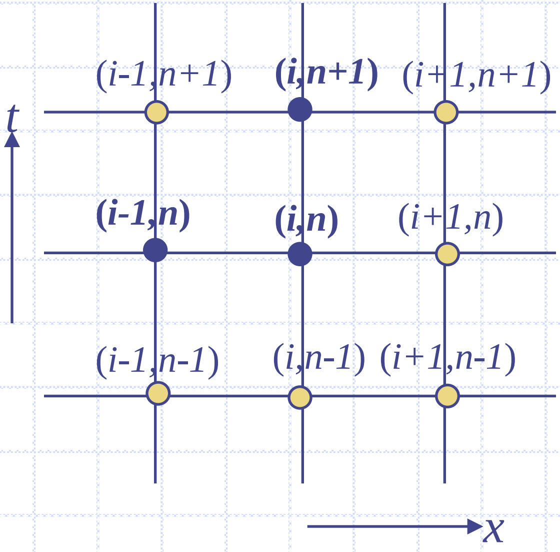
# Up-Wind Scheme

In the Down-Wind Scheme, the spatial derivative at $x_i$ uses the information at $x_{i+1}$ where the wave will go in the next time step, which does not make sense.

It would be more reasonable to use the information at $x_{i-1}$ where the wave comes from.

Hence, the Up-wind Scheme is described as

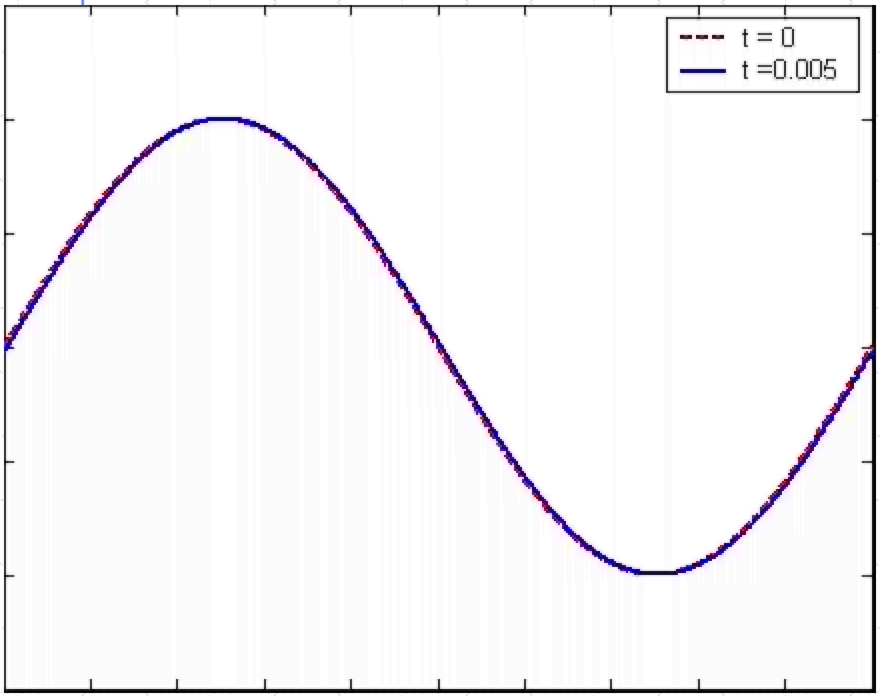
$$u_i^{n+1} = u_i^n - \frac{ak}{h}(u_i^n - u_{i-1}^n), \qquad i = 1, \ldots, N.$$




We will see that, the solution is *almost* exact

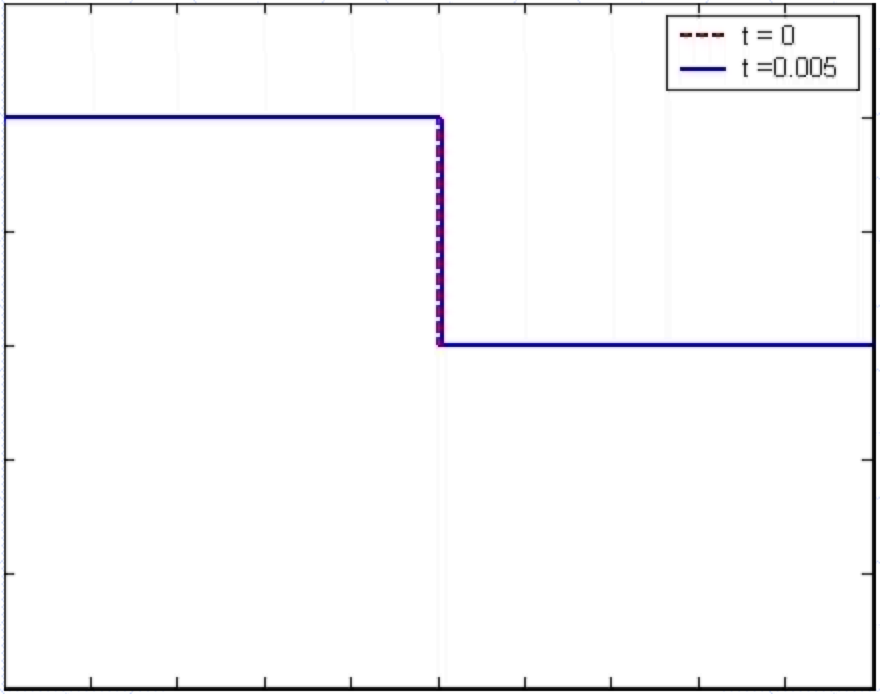
# Numerical Implementation

Continuous Initial Data




$$u_0(x) = \sin(2\pi x), \qquad 0 \leq x \leq 1$$

Discontinuous Initial Data



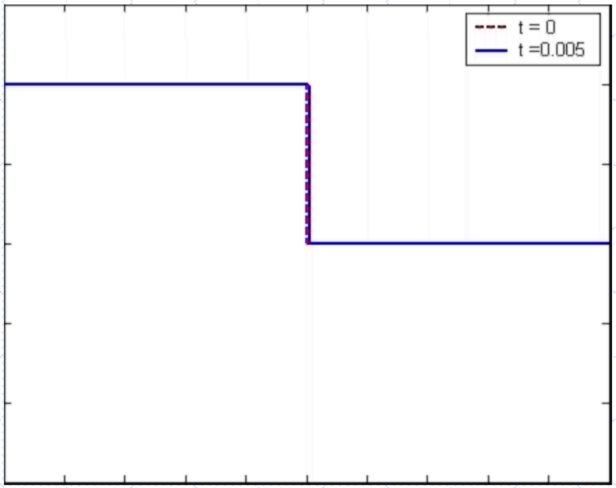

$$u_0(x) = \begin{cases} 1 & : \quad 0 \leq x < 1/2 \\ 0 & : \quad 1/2 \leq x \leq 1 \end{cases}$$

*h* = 0.01 *x = 0 to 1*

*k* = 0.001 *t = 0 to 0.25*

# Numerical Approximation of Non-Linear Scalar Conservation Laws

Up-Wind Scheme



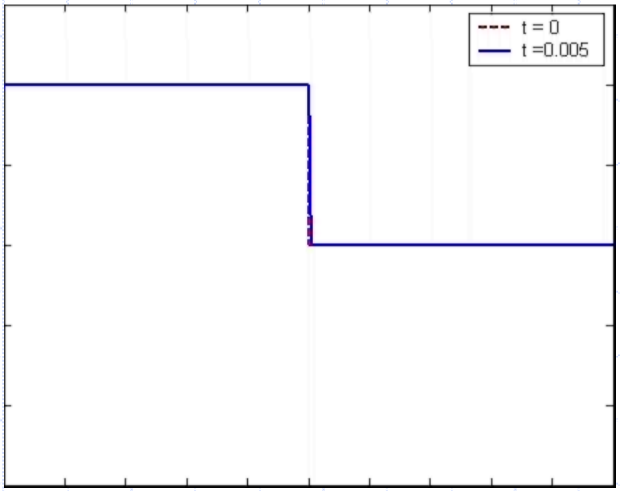

$$u_t + uu_x = 0, \qquad x \in \mathbb{R},\ t > 0,$$

$$u_i^{n+1} = u_i^n - \frac{k}{h} u_i^n (u_i^n - u_{i-1}^n),$$

$$i \in \mathbb{Z},\ n \geq 0.$$

Quasi-Linear equation

Lax-Friedrich's Scheme




$$u_t + \left(\frac{u^2}{2}\right)_x = 0. \qquad x \in \mathbb{R},\ t > 0,$$

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{k}{4h}\left((u_{i+1}^n)^2 - (u_{i-1}^n)^2\right)$$

$$i \in \mathbb{Z},\ n \geq 0.$$

Conservation form

# Method of Lines

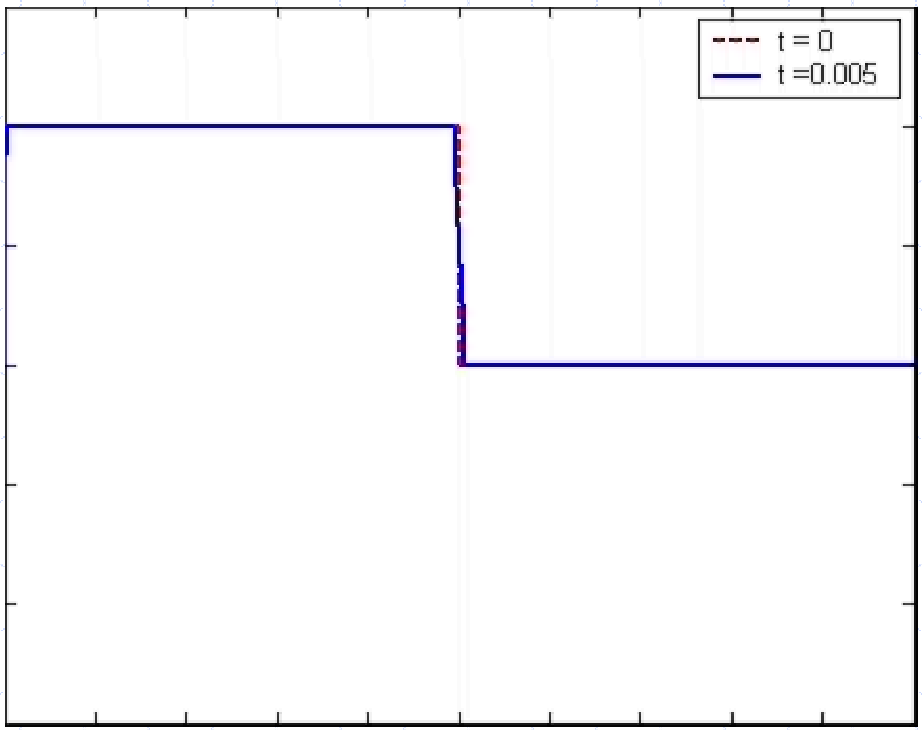
Suppose we want to solve $$\frac{\partial T}{\partial t} = \phi\left(x, t, T, \frac{\partial T}{\partial x}, \frac{\partial^2 T}{\partial x^2}\right)$$

We try to turn this into system of ODEs by approximating only in space

$$\frac{dT_j}{dt} = \phi(x_j, t, T_j(t), \frac{T_{j+1}(t) - T_{j-1}(t)}{2\triangle x}, \frac{T_{j+1}(t) - 2T_j(t) + T_{j-1}(t)}{\triangle x^2})$$

And very sophisticated methods are available to solve system of ODEs

# LWR Model for Single Lane- Simulation



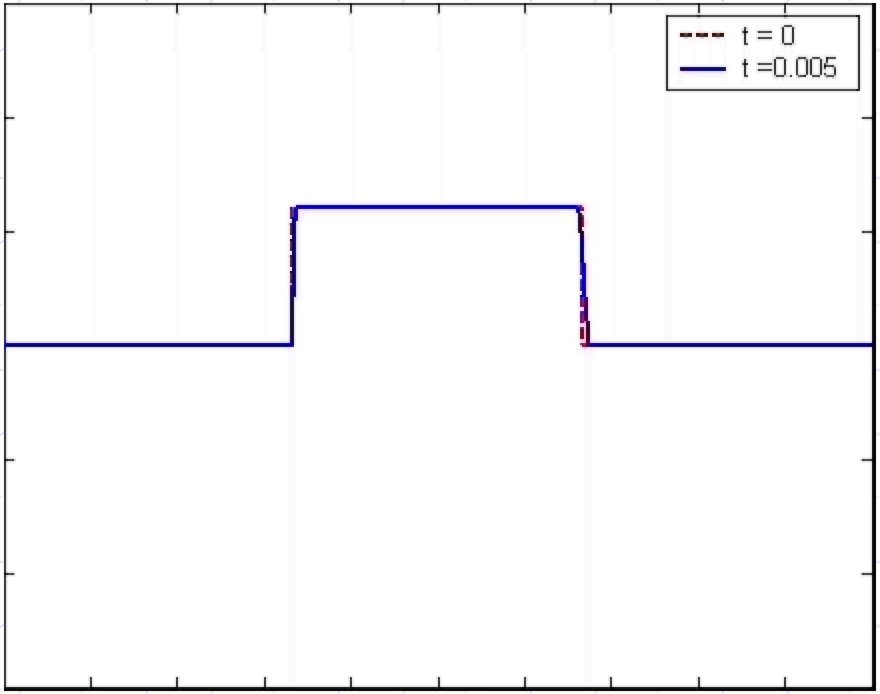

**LWR Model for single lane**

$$\begin{aligned} u_t + f(u)_x &= 0, \qquad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}, \end{aligned}$$

$$f : \mathbb{R} \to \mathbb{R}.$$

$$f(\rho(x,t)) \equiv u_{max}\rho(1 - \frac{\rho}{\rho_{max}})$$

# LWR Model for Single Lane With Traffic Jam- Simulation




## LWR Model for single lane

$$\begin{aligned} u_t + f(u)_x &= 0, \qquad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}, \end{aligned}$$

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