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Approaching Chaplygin pressure limit of solutions to the Aw–Rascle model ☆

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ABSTRACT

This paper studies the limit of solutions to the Aw–Rascle model as the pressure tends to the Chaplygin gas pressure. For concreteness, the pressure is taken as a modified Chaplygin gas pressure. Firstly, the Riemann problem for the Aw–Rascle model with the modified Chaplygin gas pressure is solved constructively. Secondly, it is shown that as the pressure tends to the Chaplygin gas pressure, some Riemann solutions containing a shock and a contact discontinuity tend to a delta-shock solution, whose propagation speed and strength are different from those of delta-shock solution to the Aw–Rascle model with a Chaplygin gas pressure. Besides, it is also proven that the rest Riemann solutions converge to a two-contact-discontinuity solution, which is exactly the solution to the Aw–Rascle model with a Chaplygin gas pressure. Thirdly, some numerical results are presented to exhibit the process of formation of delta-shocks.

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1. Introduction

Consider the Aw–Rascle model of traffic flow in the conservative form [2]

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \big(\rho(u+P)\big)_t + \big(\rho u(u+P)\big)_x = 0, \end{cases} \tag{1.1}$$

where $\rho \geqslant 0$ represents the traffic density, $u \geqslant 0$ the traffic velocity, and P is the velocity offset and called as the "pressure" inspired from gas dynamics. The model (1.1) is now widely used to study the formation and dynamics of traffic jams. It was proposed by Aw and Rascle [2] to remedy the deficiencies of second order models of car traffic pointed out by Daganzo [9] and had also been independently derived by Zhang [30]. Since its introduction, it had received extensive attention (see [12,1,14,22], etc.). Recently, Pan and Han [19] introduced the Chaplygin gas pressure [5,26]

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$$P = -\frac{B}{\rho}, \tag{1.2}$$

where $B > 0$ is constant. Interestingly, delta-shocks appear in solutions, which may be used to explain the serious traffic jam.

Delta-shock is a very interesting topic in the theory of systems of conservations laws. It is a generalization of an ordinary shock. Speaking informally, it is a kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a weighted Dirac delta function with the discontinuity as its support. From the physical point of view, it represents the process of concentration of the mass. For related researches of delta-shocks, we refer the readers to [24,25,13,15,23,16,27,10,21,8,28,6,7,29].

In this paper, we are interested in the limit of solutions to (1.1) as the pressure tends to the Chaplygin gas pressure. For this purpose, we assume that the pressure $P = P(\rho, A)$ is a function of density ρ and parameter $A > 0$, and it tends to the Chaplygin gas pressure as $A \to 0$, that is,

$$\lim_{A\to 0} P(\rho, A) = -\frac{B}{\rho}. \tag{1.3}$$

For concreteness, we focus on the following pressure for a modified Chaplygin gas

$$P = A\rho - \frac{B}{\rho}, \tag{1.4}$$

which can be regarded as a combination of the perfect fluid and Chaplygin gas. The modified Chaplygin gas was introduced by Benaoum in [3] as a suitable kind of candidates of dark energy and then attracted great interest (see [4,20,11,17]). Especially, when $B = 0$, (1.4) reduces to $P = A\rho$, which is just the standard equation of state of perfect fluid.

Firstly, we solve the Riemann problem for (1.1) and (1.4) with initial data

$$(u,\rho)(t=0,x) = \begin{cases} (u_-,\rho_-), & x < 0, \\ (u_+,\rho_+), & x > 0. \end{cases} \tag{1.5}$$

Because one eigenvalue is genuinely nonlinear and the other is linearly degenerate, the elementary waves consist of rarefaction wave, shock and contact discontinuity. The curves of elementary waves divide the phase-plane into four domains. By the analysis method in phase plane, we establish the existence and uniqueness of Riemann solutions with two different structures $R + J$ and $S + J$. Secondly, we investigate the limit as $A \to 0$ of Riemann solutions to (1.1), (1.4) and (1.5). It is shown that when $u_+ < u_- - \frac{B}{\rho_-}$, the Riemann solution $S + J$ converges to a delta-shock solution, the density between S and J tends to an extreme concentration in the form of a weighted Dirac delta function, which results in the formation of a delta-shock mathematically. However, the propagation speed and the strength of delta-shock in the limit are different from those of delta-shock solution to (1.1), (1.2) and (1.5). Besides, it is also proven that when $u_+ \geqslant u_- - \frac{B}{\rho_-}$, the Riemann solutions $R + J$ and $S + J$ converge to a two-contact-discontinuity solution, which is exactly the solution to (1.1), (1.2) and (1.5). Thirdly, we examine the process of formation of delta-shock with numerical results as A decreases.

From the point of view of hyperbolic conservation laws, following the discussion in the present paper, one can find that hyperbolicity of the limiting system is preserved, however, the phenomena of concentration in the limit process still occurs in solutions. In this regard, it is different from that in Chen and Liu [6,7], etc. In any case, the phenomena of concentration can be regarded as phenomena of resonance between two characteristic fields. This is just the mechanism of occurrence of delta shock waves.

The organization of this paper is as follows. In Section 2, we review the solution to (1.1), (1.2) and (1.5). Section 3 solves the Riemann problem (1.1), (1.4) and (1.5) by analysis in phase-plane. Sections 4 and 5 investigate the limit of solutions to (1.1), (1.4) and (1.5). Finally, the processes of formation of delta-shocks are examined by some numerical results in Section 6.

2. Solutions to Riemann problem (1.1), (1.2) and (1.5)

In this section, we briefly review the Riemann problem (1.5) for (1.1) with (1.2), for which the detailed investigations can be found in [19]. The eigenvalues are

$$\lambda_1 = u - \frac{B}{\rho}, \qquad \lambda_2 = u \tag{2.1}$$

and the corresponding right eigenvectors are

$$r_1 = \left(1, -\frac{B}{\rho^2}\right)^T, \qquad r_2 = (1,0)^T$$

satisfying

$$\nabla\lambda_1 \cdot r_1 \equiv 0, \qquad \nabla\lambda_2 \cdot r_2 \equiv 0. \tag{2.2}$$

Therefore this system is strictly hyperbolic and fully linearly degenerate.

Looking for the self-similar solution

$$(u,\rho)(t,x) = (u,\rho)(\xi), \qquad \xi = x/t,$$

we obtain a two-point boundary value problem

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi\left\{\rho\left(u - \frac{B}{\rho}\right)\right\}_\xi + \left\{\rho u\left(u - \frac{B}{\rho}\right)\right\}_\xi = 0, \end{cases} \tag{2.3}$$

and

$$(u,\rho)(\pm\infty) = (u_\pm, \rho_\pm). \tag{2.4}$$

Besides the constant state

$$(u,\rho)(\xi) = \mathit{constant}, \tag{2.5}$$

it can be checked that the classical elementary waves only consist of contact discontinuities

$$J_1\colon \quad \begin{cases} \sigma = u - \frac{B}{\rho} = u_- - \frac{B}{\rho_-}, \\ u - \frac{B}{\rho} = u_- - \frac{B}{\rho_-}, \end{cases} \tag{2.6}$$

and

$$J_2\colon \quad \begin{cases} \sigma = u, \\ u = u_-. \end{cases} \tag{2.7}$$

The curves in the first quadrant of the (u,ρ)-plane, the phase-plane, expressed by the second equation in (2.6) and (2.7) are called as the 1-contact discontinuity curve and 2-contact discontinuity curve, respectively (see Fig. 2.1).

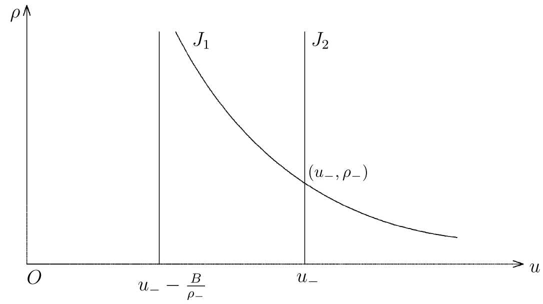

Fig. 2.1. Curves of elementary waves.

By the analysis method in phase-plane, it can be concluded that when $u_+ > u_- - \frac{B}{\rho_-}$, the Riemann solution consists of two contact discontinuities J_1 and J_2 with the intermediate state (u_*, ρ_*) besides the constant states (u_-, ρ_-) and (u_+, ρ_+), where

$$u_* = u_+, \qquad \frac{B}{\rho_*} = u_+ - u_- + \frac{B}{\rho_-}. \tag{2.8}$$

However, when $u_+ \leqslant u_- - \frac{B}{\rho_-}$ (when $u_- - \frac{B}{\rho_-} < 0$, this case does not exist; in this paper, we assume $u_- - \frac{B}{\rho_-} > 0$), the singularity cannot be a jump with finite amplitude; that is, there is no solution which is piecewise smooth and bounded. Hence a solution containing a weighted δ-measure (i.e., delta-shock) supported on a line should be constructed in order to establish the existence in a space of measures from the mathematical point of view.

To define the measure solutions, the weighted δ-measure $w(s)\delta_L$ supported on a smooth curve L parameterized as $t = t(s)$, $x = x(s)$ $(c \leqslant s \leqslant d)$ is defined by

$$\langle w(s)\delta_L, \phi\rangle = \int_c^d w(s)\phi\big(t(s), x(s)\big)\, ds \tag{2.9}$$

for all test functions $\phi \in C_0^\infty(R^2)$.

For the case $u_+ \leqslant u_- - \frac{B}{\rho_-}$, the Riemann solution should be the following delta-shock solution

$$(u,\rho)(t,x) = \begin{cases} (u_-, \rho_-), & x < \sigma t, \\ (\sigma, w(t)\delta(x - x(t))), & x = \sigma t, \\ (u_+, \rho_+), & x > \sigma t \end{cases} \tag{2.10}$$

with

$$\frac{B}{\rho} = \begin{cases} \frac{B}{\rho_-}, & x < \sigma t, \\ 0, & x = \sigma t, \\ \frac{B}{\rho_+}, & x > \sigma t, \end{cases} \tag{2.11}$$

where the weight $w(t)$ and velocity σ satisfy the generalized Rankine–Hugoniot relation

$$\begin{cases} \dfrac{dw(t)}{dt} = -\sigma[\rho] + [\rho u], \\ \dfrac{dw(t)\sigma}{dt} = -\sigma\left[\rho\left(u - \dfrac{B}{\rho}\right)\right] + \left[\rho u\left(u - \dfrac{B}{\rho}\right)\right] \end{cases} \tag{2.12}$$

and entropy condition

$$u_+ \leqslant \sigma \leqslant u_- - \frac{B}{\rho_-}, \tag{2.13}$$

where and after, $[a] = a_- - a_+$ is the jump of a across the discontinuities. Under (2.13), solving (2.12) with initial data $w(0) = 0$ gives

$$w(t) = \rho_-(u_- - u_+)t, \qquad \sigma = \frac{u_+ + u_- - \frac{B}{\rho_-}}{2} \tag{2.14}$$

as $\rho_- = \rho_+$, and

$$w(t) = w_0 t, \qquad \sigma = \frac{\rho_- u_- - \rho_+ u_+ - w_0}{\rho_- - \rho_+} \tag{2.15}$$

as $\rho_- \neq \rho_+$, where

$$w_0 = \sqrt{\rho_-\rho_+(u_- - u_+)\left\{\left(u_- - \frac{B}{\rho_-}\right) - \left(u_+ - \frac{B}{\rho_+}\right)\right\}}.$$

Theorem 2.1. *For Riemann problem (1.1), (1.2) and (1.5), there exists a unique entropy solution, which consists of two (just one) different contact discontinuities when $u_+ > u_- - \frac{B}{\rho_-}$ or a delta-shock when $u_+ \leqslant u_- - \frac{B}{\rho_-}$.*

3. Solutions to Riemann problem (1.1), (1.4) and (1.5)

In this section, we solve the elementary waves and then construct the solutions to the Riemann problem (1.1), (1.4) and (1.5) by the analysis method in phase-plane. The system has two eigenvalues

$$\lambda_1 = u - A\rho - \frac{B}{\rho}, \qquad \lambda_2 = u \tag{3.1}$$

with right eigenvectors

$$r_1 = \left(1, -A - \frac{B}{\rho^2}\right)^T, \qquad r_2 = (1, 0)^T$$

satisfying

$$\nabla\lambda_1 \cdot r_1 = -2A, \qquad \nabla\lambda_2 \cdot r_2 \equiv 0. \tag{3.2}$$

Thus this system is strictly hyperbolic. The first characteristic field is genuinely nonlinear and the associated wave is either shock or rarefaction wave; the second characteristic field is always linearly degenerate and the associated wave is only contact discontinuity.

Seeking the self-similar solution

$$(u, \rho)(t, x) = (u, \rho)(\xi), \qquad \xi = x/t,$$

we reach

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi\Big(\rho\Big(u + A\rho - \dfrac{B}{\rho}\Big)\Big)_\xi + \Big(\rho u\Big(u + A\rho - \dfrac{B}{\rho}\Big)\Big)_\xi = 0, \end{cases} \tag{3.3}$$

and

$$(u,\rho)(\pm\infty) = (u_\pm, \rho_\pm). \tag{3.4}$$

For any smooth solution, (3.3) is equivalent to

$$\begin{pmatrix} u-\xi & \rho \\ 0 & u - A\rho - \frac{B}{\rho} - \xi \end{pmatrix} \begin{pmatrix} d\rho \\ du \end{pmatrix} = 0, \tag{3.5}$$

which provides either the general solution (constant state)

$$(u,\rho)(\xi) = constant, \tag{3.6}$$

or rarefaction wave, which is a wave of the first characteristic family,

$$R: \quad \begin{cases} \xi = \lambda_1 = u - A\rho - \dfrac{B}{\rho}, \\ du + \Big(A + \dfrac{B}{\rho^2}\Big)\, d\rho = 0, \end{cases} \tag{3.7}$$

or singular solution, which is a wave of the second characteristic family,

$$\begin{cases} \xi = \lambda_2 = u, \\ du = 0. \end{cases} \tag{3.8}$$

Noticing

$$\frac{d\lambda_1}{d\rho} = \frac{\partial\lambda_1}{\partial u}\frac{du}{d\rho} + \frac{\partial\lambda_1}{\partial\rho} = -\Big(A + \frac{B}{\rho^2}\Big) - \Big(A - \frac{B}{\rho^2}\Big) = -2A < 0, \tag{3.9}$$

we integrate (3.7) and take the requirement $\lambda_1(u,\rho) > \lambda_1(u_-,\rho_-)$ into account to obtain

$$R: \quad \begin{cases} \xi = \lambda_1 = u - A\rho - \dfrac{B}{\rho}, \\ u = -A\rho + \dfrac{B}{\rho} + u_- + A\rho_- - \dfrac{B}{\rho_-}, \quad \rho < \rho_-. \end{cases} \tag{3.10}$$

Integrating (3.8) leads to

$$\begin{cases} \xi = \lambda_2 = u, \\ u = u_-, \end{cases} \tag{3.11}$$

which is actually a contact discontinuity (see (3.14)).

For a bounded discontinuity at $\xi = \sigma$, the Rankine–Hugoniot relation

$$\begin{cases} -\sigma[\rho] + [\rho u] = 0, \\ -\sigma\Big[\rho\Big(u + A\rho - \dfrac{B}{\rho}\Big)\Big] + \Big[\rho u\Big(u + A\rho - \dfrac{B}{\rho}\Big)\Big] = 0 \end{cases} \tag{3.12}$$

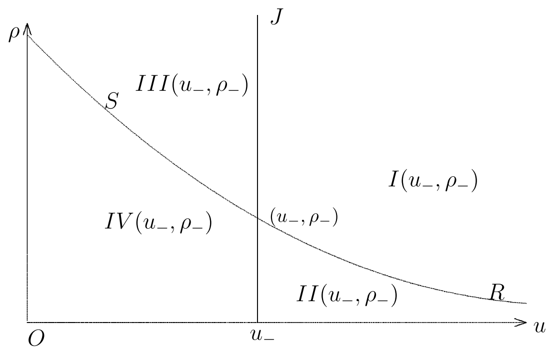

Fig. 3.1. Curves of elementary waves.

holds. By solving (3.12), we obtain the shock, which is a wave of the first characteristic family,

$$S\colon\quad \begin{cases} \sigma = u - \dfrac{B}{\rho} - A\rho_- = u_- - \dfrac{B}{\rho_-} - A\rho, \\ u = -A\rho + \dfrac{B}{\rho} + u_- + A\rho_- - \dfrac{B}{\rho_-}, \end{cases} \tag{3.13}$$

and the contact discontinuity, which is a wave of the second characteristic family

$$J\colon\quad \begin{cases} \sigma = u, \\ u = u_-. \end{cases} \tag{3.14}$$

Using the Lax entropy inequalities, the shock should satisfy

$$\sigma < \lambda_1(u_-, \rho_-) < \lambda_2(u_-, \rho_-), \qquad \lambda_1(u, \rho) < \sigma < \lambda_2(u, \rho), \tag{3.15}$$

which means that three of the characteristic lines on both sides of shock, two λ_1 and one λ_2 are incoming with respect to the shock, while the remaining one λ_2 is outgoing (i.e., three incoming, one outcoming). Furthermore, it can be checked that the entropy inequalities (3.15) are equivalent to $\rho > \rho_-$. Therefore the shock can be expressed as

$$S\colon\quad \begin{cases} \sigma = u - \dfrac{B}{\rho} - A\rho_- = u_- - \dfrac{B}{\rho_-} - A\rho, \\ u = -A\rho + \dfrac{B}{\rho} + u_- + A\rho_- - \dfrac{B}{\rho_-}, \quad \rho > \rho_-. \end{cases} \tag{3.16}$$

The curves in the first quadrant of the (u, ρ)-plane expressed by the second equation in (3.10), (3.16) and (3.14) are called as the rarefaction wave, shock and contact discontinuity curve, respectively. The contact discontinuity curve $u = u_-$ is a straight line parallel to the ρ-axis. The rarefaction wave curve and shock curve have the same expression $u = -A\rho + \frac{B}{\rho} + u_- + A\rho_- - \frac{B}{\rho_-}$, which means the system belongs to the Temple type. Due to $u_\rho = -A - B\rho^{-2} < 0$ and $u_{\rho\rho} = 2B\rho^{-3} > 0$, the rarefaction wave curve and shock curve are monotonic decreasing and convex. Moreover, it can be verified that $\lim_{\rho\to 0^+} u = +\infty$, which implies that the rarefaction wave curve has the u-axis as the asymptote. It can also be proved that $\lim_{\rho\to+\infty} u = -\infty$, which implies the shock curve interacts with the ρ-axis at some point. Fixing a left state (u_-, ρ_-), the first quadrant of the (u, ρ)-plane can be divided into four regions by the rarefaction wave curve, shock curve and contact discontinuity curve, denoted by $I(u_-, \rho_-)$, $II(u_-, \rho_-)$, $III(u_-, \rho_-)$ and $IV(u_-, \rho_-)$, respectively (see Fig. 3.1).

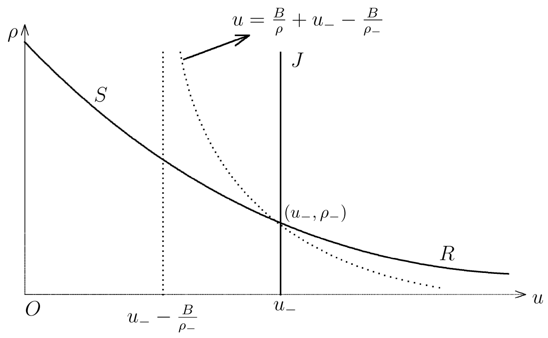

Fig. 3.2. Limit of curves of elementary waves.

Then by the analysis method in phase-plane, according to the right state (u_+, ρ_+) in the different region, one can construct the unique global Riemann solutions connecting two constant states (u_-, ρ_-) and (u_+, ρ_+), which contain a rarefaction wave and a contact discontinuity when $(u_+, \rho_+) \in I(u_-, \rho_-) \cup II(u_-, \rho_-)$, that is $u_+ > u_-$, a shock and a contact discontinuity when $(u_+, \rho_+) \in III(u_-, \rho_-) \cup IV(u_-, \rho_-)$, i.e. $u_+ < u_-$.

Finally, it can be seen that for a given (u_-, ρ_-), as $A \to 0$, the elementary wave curves $u = -A\rho + \frac{B}{\rho} + u_- + A\rho_- - \frac{B}{\rho_-}$ tends to $u = \frac{B}{\rho} + u_- - \frac{B}{\rho_-}$ (see Fig. 3.2).

4. Limit of Riemann solutions when $u_+ \leqslant u_- - \frac{B}{\rho_-}$

This section is devoted to studying the limit as $A \to 0$ of the Riemann solutions to (1.1) with (1.4) in the case $u_+ \leqslant u_- - \frac{B}{\rho_-}$.

4.1. Limit behaviors of the Riemann solutions

For any $A > 0$, the solution of Riemann problem (1.5) of (1.1) with (1.4) is a shock S followed by a discontinuity J with the intermediate state (u_*^A, ρ_*^A) besides two constant states (u_-, ρ_-) and (u_+, ρ_+). They have the following relations

$$S: \quad \begin{cases} \sigma^A = u_*^A - \dfrac{B}{\rho_*^A} - A\rho_-, \\ u_*^A = -A\rho_*^A + \dfrac{B}{\rho_*^A} + u_- + A\rho_- - \dfrac{B}{\rho_-}, \quad \rho_*^A > \rho_- \end{cases} \tag{4.1}$$

and

$$J: \quad \tau^A = u_*^A = u_+, \tag{4.2}$$

where σ^A and τ^A are the propagation speeds of S and J, respectively. Then we have the following lemmas.

Lemma 4.1. $\lim_{A\to 0} \rho_*^A = +\infty$.

Proof. From (4.1) and (4.2), we easily get

$$u_- - \frac{B}{\rho_-} - u_+ = A\rho_*^A - \frac{B}{\rho_*^A} - A\rho_-, \quad \rho_*^A > \rho_-,$$

from which we have

$$\rho_*^A = \frac{u_- - \frac{B}{\rho_-} - u_+ + A\rho_- + \sqrt{(u_- - \frac{B}{\rho_-} - u_+ + A\rho_-)^2 + 4AB}}{2A}. \tag{4.3}$$

Therefore $\lim_{A\to 0}\rho_*^A = +\infty$. □

Lemma 4.2. $\lim_{A\to 0}\sigma^A = \lim_{A\to 0}\tau^A = u_+$.

Proof. From (4.1) and (4.2), we have

$$\lim_{A\to 0}\sigma^A = \lim_{A\to 0}\left(u_*^A - \frac{B}{\rho_*^A} - A\rho_-\right) = \lim_{A\to 0}u_*^A = u_+. \tag{4.4}$$

The proof is finished. □

Lemmas 4.1–4.2 show that when A drops to zero, S and J coincide, the intermediate density ρ_*^A becomes singular.

Lemma 4.3.

$$\lim_{A\to 0}\int_{\sigma^A}^{\tau^A}\rho_*^A\,d\xi = \rho_-(u_- - u_+) \neq 0. \tag{4.5}$$

Proof. From the Rankine–Hugoniot relation (3.12) for S and J, we have

$$-\sigma^A[\rho] + [\rho u] = 0$$

with $[a] = a_- - a_*^A$ and

$$-\tau^A[\rho] + [\rho u] = 0$$

with $[a] = a_*^A - a_+$. Then, we easily reach

$$\lim_{A\to 0}\rho_*^A\left(\tau^A - \sigma^A\right) = \rho_-(u_- - u_+) \tag{4.6}$$

which gives

$$\lim_{A\to 0}\int_{\sigma^A}^{\tau^A}\rho_*^A\,d\xi = \lim_{A\to 0}\rho_*^A\left(\tau^A - \sigma^A\right) = \rho_-(u_- - u_+).$$

The proof is finished. □

Lemma 4.3 shows that the limit of ρ_*^A has the same singularity as a weighted Dirac delta function at $\xi = \sigma$.

4.2. Weighted delta-shocks

Now, we show the theorem characterizing the limit as $A \to 0$ for the case $u_+ \leqslant u_- - \frac{B}{\rho_-}$.

Theorem 4.4. *Let $u_+ \leqslant u_- - \frac{B}{\rho_-}$ and assume $(u^A, \rho^A)(t,x)$ is the Riemann solution $S + J$ to (1.1), (1.4) and (1.5) constructed in Section 3. Then*

$$\lim_{A\to 0} u^A(t,x) = \begin{cases} u_-, & x < u_+ t, \\ u_+, & x = u_+ t, \\ u_+, & x > u_+ t, \end{cases} \tag{4.7}$$

and $\rho^A(t,x)$ converges in the sense of distributions, and the limit function is the sum of a step function and a Dirac delta function supported on $x = u_+ t$ with weight $\rho_-(u_- - u_+)t$.

Proof. 1. Set $\xi = x/t$. Then for each $A > 0$, the Riemann solution $S + J$ can be expressed as

$$(u^A, \rho^A)(\xi) = \begin{cases} (u_-, \rho_-), & \xi < \sigma^A, \\ (u_*^A, \rho_*^A), & \sigma^A < \xi < \tau^A, \\ (u_+, \rho_+), & \xi > \tau^A. \end{cases} \tag{4.8}$$

From (3.3), we have weak formulations

$$-\int_{-\infty}^{+\infty} \rho^A (u^A - \xi)\phi' \, d\xi + \int_{-\infty}^{+\infty} \rho^A \phi \, d\xi = 0, \tag{4.9}$$

for any $\phi \in C_0^1(-\infty, +\infty)$. The limit (4.7) can be directly obtained from (4.8).

2. The first integral on the left hand side of (4.9) can be decomposed into

$$\int_{-\infty}^{+\infty} \rho^A (u^A - \xi)\phi' \, d\xi = \left(\int_{-\infty}^{\sigma^A} + \int_{\sigma^A}^{\tau^A} + \int_{\tau^A}^{+\infty} \right) \rho^A (u^A - \xi)\phi' \, d\xi. \tag{4.10}$$

The limit of the sum of the first and last term of (4.10) is

$$\begin{aligned} &\lim_{A\to 0} \int_{-\infty}^{\sigma^A} \rho^A (u^A - \xi)\phi' \, d\xi + \lim_{A\to 0} \int_{\tau^A}^{+\infty} \rho^A (u^A - \xi)\phi' \, d\xi \\ &= \lim_{A\to 0} \int_{-\infty}^{\sigma^A} \rho_- (u_- - \xi)\phi' \, d\xi + \lim_{A\to 0} \int_{\tau^A}^{+\infty} \rho_+ (u_+ - \xi)\phi' \, d\xi \\ &= \rho_-(u_- - u_+)\phi(u_+) + \int_{-\infty}^{+\infty} H(\xi - u_+)\phi \, d\xi \end{aligned} \tag{4.11}$$

with

$$H(x) = \begin{cases} \rho_-, & x < 0, \\ \rho_+, & x > 0. \end{cases}$$

The limit of the second term of (4.10) is

$$
\begin{aligned}
&\lim_{A\to 0}\int_{\sigma^A}^{\tau^A}\rho^A\big(u^A-\xi\big)\phi'\,d\xi\\
&\quad=\lim_{A\to 0}\int_{\sigma^A}^{\tau^A}\rho_*^A\big(u_*^A-\xi\big)\phi'\,d\xi\\
&\quad=\lim_{A\to 0}\rho_*^A\big(\tau^A-\sigma^A\big)\left(\frac{\phi(\tau^A)-\phi(\sigma^A)}{\tau^A-\sigma^A}u_*^A-\frac{\tau^A\phi(\tau^A)-\sigma_-^A\phi(\sigma^A)}{\tau^A-\sigma^A}+\frac{1}{\tau^A-\sigma^A}\int_{\sigma^A}^{\tau^A}\phi\,d\xi\right)\\
&\quad=\rho_-(u_--u_+)\big(u_+\phi'(u_+)-u_+\phi'(u_+)-\phi(u_+)+\phi(u_+)\big)\\
&\quad=0.
\end{aligned}
\tag{4.12}
$$

Returning to (4.9), we immediately obtain that

$$
\lim_{A\to 0}\int_{-\infty}^{+\infty}\rho^A\phi\,d\xi=\rho_-(u_--u_+)\phi(u_+)+\int_{-\infty}^{+\infty}H(\xi-u_+)\phi\,d\xi.
\tag{4.13}
$$

4. Then we study the limit of ρ^A by tracking the time-dependence of the weights of the δ-measures as $A\to 0$. Taking (4.13) into account, we have for any $\psi\in C_0^\infty(R\times R^+)$

$$
\begin{aligned}
&\lim_{A\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho^A(x/t)\psi(x,t)\,dx\,dt\\
&\quad=\lim_{A\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho^A(\xi)\psi(\xi t,t)\,d(\xi t)\,dt\\
&\quad=\lim_{A\to 0}\int_0^{+\infty}t\left(\int_{-\infty}^{+\infty}\rho^A(\xi)\psi(\xi t,t)\,d\xi\right)dt\\
&\quad=\int_0^{+\infty}t\left(\rho_-(u_--u_+)\psi(u_+t,t)+\int_{-\infty}^{+\infty}H(\xi-u_+)\psi(\xi t,t)\,d\xi\right)dt\\
&\quad=\int_0^{+\infty}\rho_-(u_--u_+)t\psi(u_+t,t)\,dt+\int_0^{+\infty}t\left(\int_{-\infty}^{+\infty}H(\xi-u_+)\psi(\xi t,t)\,d\xi\right)dt\\
&\quad=\int_0^{+\infty}\rho_-(u_--u_+)t\psi(u_+t,t)\,dt+\int_0^{+\infty}\int_{-\infty}^{+\infty}H(x-u_+t)\psi(x,t)\,dx\,dt,
\end{aligned}
\tag{4.14}
$$

in which by definition (2.9)

$$\int_0^{+\infty} \rho_-(u_- - u_+)t\psi(u_+t,t)\,dt = \langle w(\cdot)\delta_L, \psi(\cdot,\cdot)\rangle$$

with

$$w(t) = \rho_-(u_- - u_+)t.$$

These finish the proof. $\square$

From Theorem 4.4, it can be seen that the delta-shock in the limit is very different from the delta-shock (2.10) with (2.14) or (2.15) constructed in Section 2.

5. Limit of Riemann solutions when $u_+ > u_- - \frac{B}{\rho_-}$

In this section, we turn to the limit as $A \to 0$ of the Riemann solutions to (1.1) with (1.4) in the case $u_+ > u_- - \frac{B}{\rho_-}$.

Case 5.1. $u_- - \frac{B}{\rho_-} < u_+ < u_-$.

For any $A > 0$, the solution of Riemann problem (1.5) for (1.1) with (1.4) is a shock S followed by a contact discontinuity J with the intermediate state (u_*^A, ρ_*^A) besides two constant states (u_-, ρ_-) and (u_+, ρ_+). They satisfy

$$S\colon \quad \begin{cases} \sigma^A = u_*^A - \dfrac{B}{\rho_*^A} - A\rho_-, \\ u_*^A = -A\rho_*^A + \dfrac{B}{\rho_*^A} + u_- + A\rho_- - \dfrac{B}{\rho_-}, \quad \rho_*^A > \rho_- \end{cases} \tag{5.1}$$

and

$$J\colon \quad \tau^A = u_*^A = u_+. \tag{5.2}$$

Then ρ_*^A is determined by

$$u_+ = -A\rho_*^A + \frac{B}{\rho_*^A} + u_- + A\rho_- - \frac{B}{\rho_-}, \quad \rho_*^A > \rho_-, \tag{5.3}$$

which gives

$$\rho_*^A = \frac{u_- - \frac{B}{\rho_-} - u_+ + A\rho_- + \sqrt{(u_- - \frac{B}{\rho_-} - u_+ + A\rho_-)^2 + 4AB}}{2A}. \tag{5.4}$$

Denote $(u_*, \rho_*) = \lim_{A\to 0}(u_*^A, \rho_*^A)$, then with the L'Hôspital rule in mathematical analysis, one easily obtains

$$\frac{B}{\rho_*} = u_+ - u_- + \frac{B}{\rho_-}, \qquad u_* = u_+. \tag{5.5}$$

Letting $A \to 0$ in (5.1), then S degenerates into a contact discontinuity as follows

$$J_1\colon \quad \tau = u_* - \frac{B}{\rho_*} = u_- - \frac{B}{\rho_-}. \tag{5.6}$$

Letting $A \to 0$ in (5.2), then J becomes the contact discontinuity as follows

$$J_2\colon \quad \tau = u_* = u_+. \tag{5.7}$$

Case 5.2. $u_+ > u_-$.

For any $A > 0$, the solution of Riemann problem (1.5) for (1.1) with (1.4) is a rarefaction wave R followed by a contact discontinuity J with the intermediate state (u_*^A, ρ_*^A) besides two constant states (u_-, ρ_-) and (u_+, ρ_+). The following relations are satisfied:

$$R\colon \quad \begin{cases} \xi = \lambda_1 = u - A\rho - \dfrac{B}{\rho}, \\ u = -A\rho + \dfrac{B}{\rho} + u_- + A\rho_- - \dfrac{B}{\rho_-}, \quad \rho < \rho_- \end{cases} \tag{5.8}$$

and

$$J\colon \quad \tau^A = u_*^A = u_+. \tag{5.9}$$

Then ρ_*^A satisfies

$$u_+ = -A\rho_*^A + \frac{B}{\rho_*^A} + u_- + A\rho_- - \frac{B}{\rho_-}, \quad \rho_*^A < \rho_-. \tag{5.10}$$

Let $(u_*, \rho_*) = \lim_{A\to 0}(u_*^A, \rho_*^A)$, then

$$\frac{B}{\rho_*} = u_+ - u_- + \frac{B}{\rho_-}, \qquad u_* = u_+. \tag{5.11}$$

Taking the limit $A \to 0$ in (5.8), then R degenerates into the following contact discontinuity

$$J_1\colon \quad \tau = u_* - \frac{B}{\rho_*} = u_- - \frac{B}{\rho_-}. \tag{5.12}$$

Taking the limit $A \to 0$ in (5.9), then J turns into the following contact discontinuity

$$J_2\colon \quad \tau = u_* = u_+. \tag{5.13}$$

In summary, for the case $u_+ > u_- - \frac{B}{\rho_-}$, we have the limit

$$\lim_{A\to 0}\bigl(u^A, \rho^A\bigr)(t,x) = \begin{cases} (u_-, \rho_-), & -\infty < x/t < u_- - \frac{B}{\rho_-}, \\ (u_+, \frac{B}{u_+ - u_- + B/\rho_-}), & u_- - \frac{B}{\rho_-} < x/t < u_+, \\ (u_+, \rho_+), & u_+ < x/t < +\infty, \end{cases} \tag{5.14}$$

which is exactly the solution to (1.1) and (1.2) with the same initial data (1.5).

6. Numerical simulations

In this section, we present some representative numerical results to examine the processes of formation of delta-shocks studied in Section 4. To discretize the system, we employ the Nessyahu–Tadmor scheme [18] with 500×500 cells and CFL $= 0.475$. We take $B = 1$ and the following initial data

$$(u, \rho)(t = 0, x) = \begin{cases} (5, 1), & x < 0, \\ (2, 1), & x > 0. \end{cases}$$

The numerical simulations for different choices of A are presented in Figs. 6.1–6.4, respectively.

One can see clearly from these above numerical results that, when A decreases, the location of the shock and contact discontinuity becomes closer and closer, and the density of the intermediate state increases dramatically. Therefore, the numerical simulations are in complete agreement with the theoretical analysis.

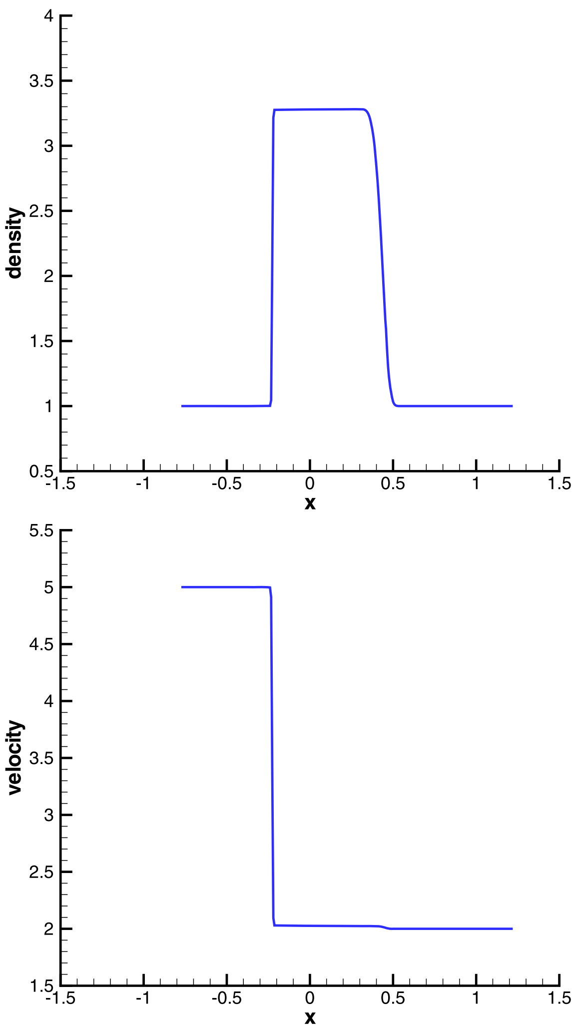

Fig. 6.1. Density and velocity for $A = 1$ at $t = 0.5$.

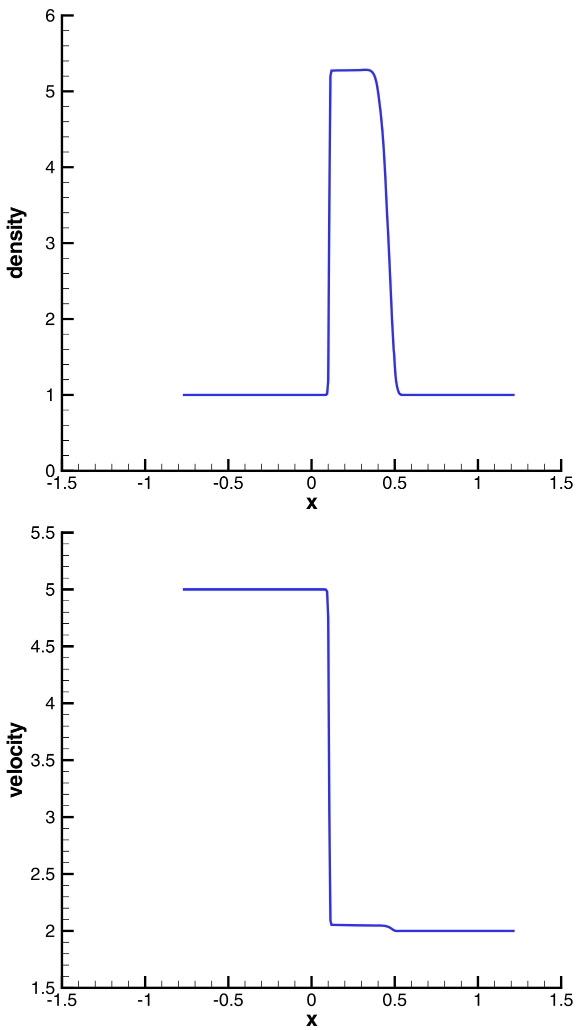

Fig. 6.2. Density and velocity for $A = 0.5$ at $t = 0.5$.

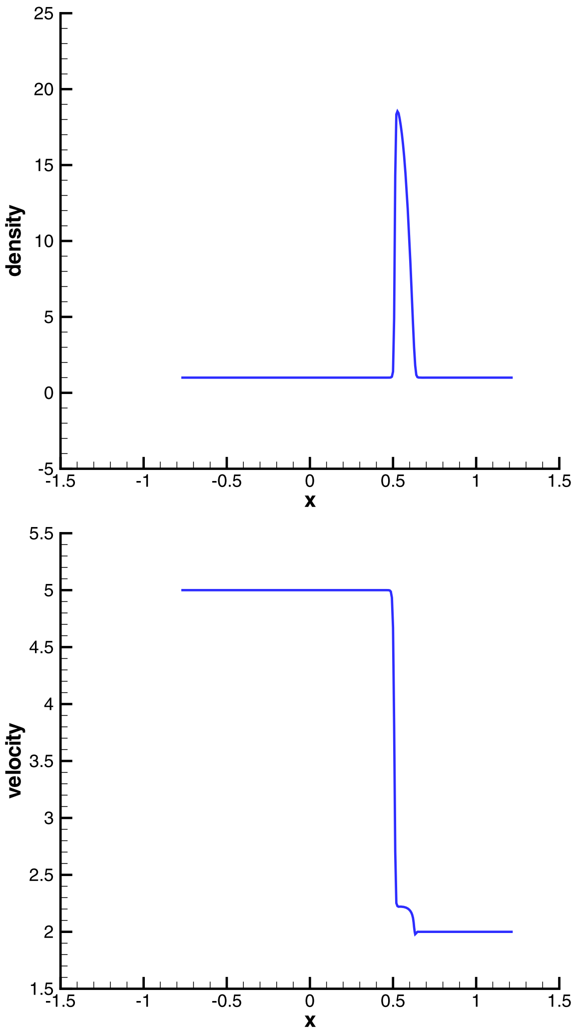

Fig. 6.3. Density and velocity for $A = 0.1$ at $t = 0.5$.

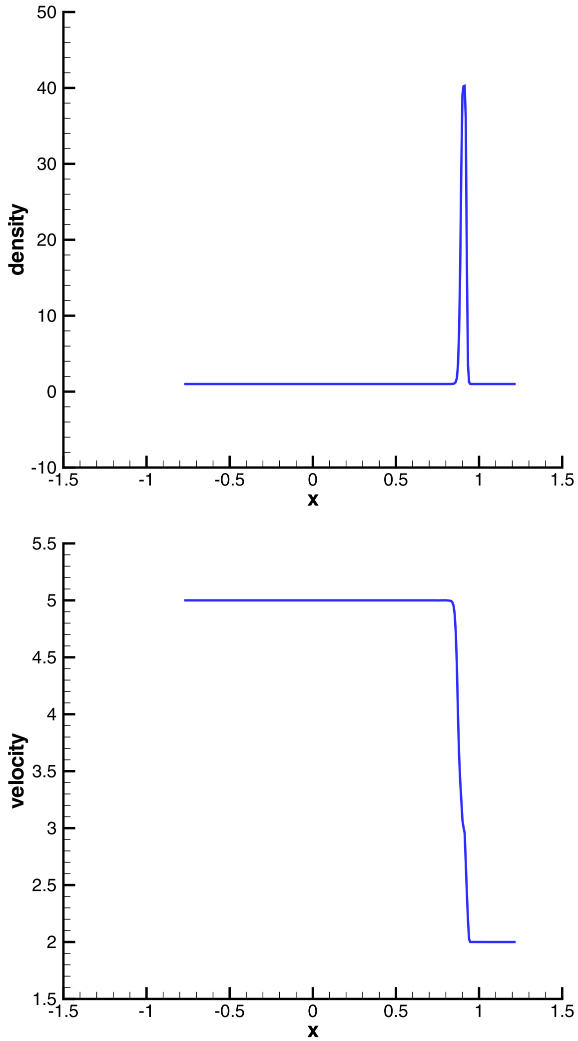

Fig. 6.4. Density and velocity for $A = 0.0001$ at $t = 0.5$.

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