Case Study 2 and 3

Traffic flow modelling and Infectious disease modelling

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Outline

1. Trafic flow modelling
   - Linear scalar conservation laws
2. Infectious disease modelling
   - Procedure

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Linear scalar conservation law description

- $$\begin{aligned} u_t + a u_x &= 0, \qquad & x \in \mathbb{R}, t > 0, \\ u(x,0) &= u_0(x), \qquad & x \in \mathbb{R}, \end{aligned}$$ where $a > 0$.
- Now, we discretize the (x,t) plane.
  $x_i = ih \quad (i \in \mathbb{Z}), \quad t_n = nk \quad (n \in \mathbb{N}_0) \quad h, k > 0$ For simplicity, we take a uniform mesh with h and k constant.

Central difference scheme

- Scalar conservation laws, when central differnce scheme:
  $\frac{u_i^{n+1}-u_i^n}{k} = -a\frac{u_{i+1}^n-u_i^n}{2h}, \quad n \geq 0, i \in \mathbb{Z}$

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- It can be rewritten as: $u_i^{n+1} = u_i^n - \frac{ak}{2h}\left(u_{i+1}^n - u_{i-1}^n\right)$. This is explicit scheme.

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- It can also be rewritten into implicit form:
  $\frac{ak}{2h}u_{i+1}^{n+1} + u_i^{n+1} - \frac{ak}{2h}u_{i-1}^{n+1} = u_i^n$

Boundary conditions

- Periodic boundary conditions: $u\left(0,t\right)=u\left(N,h,t\right), \quad t>0$
- Discretized version: $u_0^n = u_N^n, \quad n \geq 0$
- Also by periodicity, we need to assume the following to use the schemes: $u_{-1}^n = u_{N-1}^n$ and $u_N^n = u_0^N$
- For continuous case: $u_0\left(x\right)=\sin\left(2\pi x\right), \quad 0 \leq x \leq 1$
- For discontinuous case: $u_0\left(x\right)=\begin{cases} 1: & 0 \leq x < 1/2 \\ 0: & 1/2 \leq x \leq 1 \end{cases}$
- h = 0.01 and x = 0 to 1
  k = 0.001 and t = 0 to 0.25

Lax Friedrich scheme

- Analytically, it could be written as:
  $\frac{1}{k}\left(u\left(x,t+k\right)-\frac{1}{2}\left(u\left(x+h,t\right)+u\left(x-h,t\right)\right)\right)$
- Here, the spatial derivative is approximated by the central difference scheme and thus the discretized version looks like:
  $u_i^{n+1}=\frac{1}{2}\left(u_{i+1}^n+u_{i-1}^n\right)-\frac{ak}{2h}\left(u_{i+1}^n-u_{i-1}^n\right)$

Downwind and Upwind scheme

- In downwind scheme: $u_i^{n+1} = u_i^n - \frac{ak}{2h}\left(u_{i+1}^n - u_i^n\right)$
- This scheme is unstable
- In upwind scheme: $u_i^{n+1} = u_i^n - \frac{ak}{2h}\left(u_i^n - u_{i-1}^n\right)$
- This scheme is almost exact.

LWR model for single lane traffic simulation

- LWR model for single lane:

$$
\begin{aligned}
u_t + f(u)_x &= 0, && x \in \mathbb{R}, t > 0, \\
u(x,0) &= u_0(x), && x \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R}.
\end{aligned}
$$

$$
f(\rho(x,t)) \equiv u_{max}\rho\left(1 - \frac{\rho}{\rho_{max}}\right)
$$

LWR model for single lane with traffic jam simulation

- LWR model for single lane:

$$
\begin{aligned}
u_t + f(u)_x &= 0, && x \in \mathbb{R}, t > 0, \\
u(x,0) &= u_0(x), && x \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R}.
\end{aligned}
$$

$$f(\rho(x,t)) \equiv u_{max}\rho\left(1 - \frac{\rho}{\rho_{max}}\right)$$

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Necessary things

- Start with a base model
- Parameter identification
- Sensitivity of the parameters
- Possibility of the solution
- Stability of the system
- Otherwise Lyapunov function's existence

Lyapunov function

- The equations describing a system could be written as: $\frac{d}{dt}\overline{X}(t) = A\overline{X}(t)$ and the initial conditions are given by: $\overline{X}(0) = \overline{X}_0$
- Now, here if $Re\,(eigenvalue\,(A)) < 0$, for all eigen values of A, then the system is stable, otherwise it is not stable.
- One more way of confirming the stability without knowing the solutions of the system, is by finding the Lyapunov function associated with the system. If a Lyapunov function exists, then the system is stable, otherwise, it is not. But, finding Lyapunov function for a system is not straightforward.

Lyapunov function

Let V: $\mathbb{R}^n \to \mathbb{R}$
be a continuous scalar function.
V is a Lyapunov-candidate-function if it is locally positive-definite, i.e.
$V(0) = 0$
$V(x) > 0 \; \forall x \in U0$
with U being a neighbourhood region around x=0 .
Let g:$\mathbb{R}^n \to \mathbb{R}^n$; $\dot{y} = g(y)$;
be an arbitrary autonomous dynamical system with equillibrium point: y* :
$0 = g(y*)$.
There always exists a coordinate transformation x=y-y* , such that:
$\dot{x} = \dot{y} = g(y) = g(x + y*) = f(x)$
$f(0) = 0$.
So the new system $f(x)$ has an equilibrium point at the origin.

Lyapunov function

Let x*=0 be an equilibrium of the autonomous system $\dot{x} = f(x)$.
And let $\dot{V}(x) = \frac{d}{dt}V(x(t)) = \frac{\partial V}{\partial x}.\frac{dx}{dt} = \nabla V.\dot{x} = \nabla V.f(x)$
be the time derivative of the Lyapunov-candidate-function V .
If the Lyapunov-candidate-function V is locally positive definite and the time derivative of the Lyapunov-candidate-function is locally negative semidefinite:
$\dot{V}(x) \leq 0 \forall x \in \mathcal{B}0$
for some neighborhood $\mathcal{B}$ of 0 , then the equilibrium is proven to be stable.
If the Lyapunov-candidate-function V is locally positive definite and the time derivative of the Lyapunov-candidate-function is locally negative definite:
$\dot{V}(x) < 0 \quad \forall x \in \mathcal{B} \setminus \{0\}$
for some neighborhood $\mathcal{B}$ of 0 , then the equilibrium is proven to be locally asymptotically stable.

Lyapunov function

If the Lyapunov-candidate-function V is globally positive definite, radially unbounded and the time derivative of the Lyapunov-candidate-function is globally negative definite:
$\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$,
then the equilibrium is proven to be globally asymptotically stable.
The Lyapunov-candidate function V(x) is radially unbounded if
$\|x\| \to \infty \Rightarrow V(x) \to \infty$.

System equations

$\frac{dV}{dt} = \alpha V - pVF$
$\frac{dF}{dt} = \beta C - \gamma pVF - aF$
$\frac{dC}{dt} = -\mu (c - c*) + g(m) kV_{t-T} F_{t-T}$
$\frac{dm}{dt} = \sigma V - \eta m(t)$

V - antigen count

F - antibody count

C - plasma cells

m - relative characteristic of the organ

α - the rate of antigen multiplication

p - the rate of contact between antigen and antibodies

β - the rate of antibody production by plasma cell

γ - amount of antibodies necessary for the neutralization of the antigens
a - mean lifetime of antibodies
μ - mean lifetime of plasma cells
k - the coefficient of immune system response
σ - the rate of injury by antigen
η - the rate of regeneration of the mass of the damaged
T - time taken for antigens to show response and die
$C*$ - normal level of plasma cells
$g(m)$ - describes dysfunction of immune system due to substantial organ damage