Nonlinear Control Design

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National Programme on Technology Enhanced Learning

A Joint initiative of IITs and IISc

Acknowledgements

I am indebted to all the students who took the course on Nonlinear analysis during 2007-2010. In particular, I thank my students T. Maruthi, M.Vijay, Anup Ekbote, Laxman Mawani, Sai Pushpak and Varun Tripuraneni who helped and offered constructive suggestions in preparing the lecture notes. I also thank IIT Madras for giving me an opportunity and assistance to prepare this lecture notes. I also thank the reviewers of this course material, whose suggestions have led to an improvement of the course contents.

Arun D. Mahindrakar

August 11, 2013

Nomenclature

Greek and Roman symbols

$I\!R$	Real numbers
$I\!R^+$	Set of positive real numbers
Z	Set of integers
Z^+	Set of positive integers
$I\!R^n$	Euclidean n-space
$\mathbb{S}^1$	unit circle
$\mathbb{S}^1 \times \mathbb{S}^1$	2-Torus
Q	configuration space
$B(x, \delta)$	Ball of radius δ centered at x

Course outline

This course deals with the analytical tools to analyze nonlinear systems. It covers

1. Mathematical preliminaries involving open and closed sets, compact set, dense set, Continuity of functions, Lipschitz condition, smooth functions, Vector space, norm of a vector, normed linear space, inner product space.

2. Well-posedness of ordinary differential equations, Lipschitz continuity and contraction mapping theorem.

3. An introduction to simple mechanical systems wherein the notion of degree-of-freedom, configuration space, configuration variables will be brought out. The state-space models of a few benchmark examples in nonlinear control will be derived using Euler-Lagrange formulation. The notion of equilibrium points and operating points leads to linearized models based on Jacobian linearization.

4. Second-order nonlinear systems occupy a special place in the study of nonlinear systems since they are easy to interpret geometrically in the plane. Here, the concept of a vector field, trajectories, vector field plot, phase-plane portrait and positively invariant sets are discussed. The classification of equilibrium points based on the eigenvalues of the linearized system will also be introduced and it will be seen why the analysis based on linearization fails in some cases.

5. Periodic solutions and the notion of limit cycles will lead us to the Bendixson's theorem and Poincaré-Bendixson criteria that provide sufficient conditions to rule-out and rule-in the existence of limit cycles respectively for a second-order system.

6. Stability is central to control system design and involves various notions of stability such as Lagrange stability, Lyapunov stability, asymptotic stability, global asymptotic stability, exponential stability and instability. The tools that we will use to

infer the stability properties include Lyapunov's direct and indirect method and La Salle's invariance property.

7. Two control design techniques, one based on Lyapunov function and the other on sliding mode are illustrated with examples in the final module.

Lecture-1

Introduction

All systems are inherently nonlinear in nature. This course deals with the analysis of nonlinear systems. The need for special tools to analyze nonlinear systems arises from the fact that the *principle of superposition* on which linear analysis is based, fails in the nonlinear case. This is just one reason for resorting to nonlinear analysis. Recall, the basic circuit analysis techniques such as the nodal, Thévenin, Norton etc. are applicable only for linear circuits.

The course subsumes some level of mathematical exposure to tools from set theory and calculus. The requisite mathematical preliminaries that are used in this course are brought out in the first module. Most physical systems can be modeled using ordinary differential equations. In the study of differential equations, it is natural to seek answers to questions on the existence, uniqueness etc. of solutions prior to finding a method to solve the equation. In the second module, we address these questions and seek answers to some of these them using the notion of Lipschitz continuity.

Many physical systems can be modeled accurately using the laws of Physics. With a bias towards mechanical systems, a plethora of simple mechanical systems are worked in the third Module. All the information needed to describe the motion of a system lies in the total energy, which is constituted of the kinetic and potential energies. The examples that are presented often serve as benchmark problems in control.

The study of second-order, time-invariant differential equations is given importance due to the easy visualization of the system behaviour on the plane. A major part of the study involves nature of trajectories in the vicinity of equilibrium points or the fixed points. The characterization of different equilibria and the study of the local behaviour by the method of linearization is recognized as an important and a first step in nonlinear analysis. In module 4, the focus is on second-order systems. Some nonlinear equations exhibit many interesting phenomena such as *limit cycles* and chaos. The former is the topic of study in the fifth module, which is devoted to a much general class of solutions, namely the periodic solutions.

Stability is central to control systems and has been the subject of study since the works of Alexander Lyapunov and James Clerk Maxwell. Maxwell, a Scottish physicist and mathematician, in 1868 analyzed the stability properties of the Watt's governor, 140 years after the invention of the steam engine. Much of nonlinear stability theory that is used extensively nowadays, called the second method of Lyapunov is attributed to the work by Lyapunov, a Russian mathematician who published his book "The General Problem of Stability of Motion" in 1892. The method makes use of energy-like functions that are sign-definite (positive) and looking at the sign of their gradients along the trajectories of the system. The last module is devoted the stability analysis of nonlinear systems.

Module 1

Mathematical Preliminaries

Objectives: To understand the basic definitions in set theory and their properties.

Lesson objectives

This module helps the reader in

– understanding an n-dimensional Euclidean space and the concept of length of a vector by way of *norm*.

– understanding the properties of open, closed, compact, dense and connected sets

– identifying a given set as an open, closed and connected sets.

– deducing how the sets are mapped using continuous functions

Suggested reading

- Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 1993.
- Principles of Mathematical Analysis by Walter Rudin, McGraw-Hill International Editions, 1976.

Lecture-2

We begin with a formal definition of real spaces.

Definition 2.0.1 *Let $n > 0$ be an integer. An ordered set of n real numbers $(x_1, x_2, \ldots, x_n)$ is called an n-dimensional point or a vector with n components. The number x_k is called the k^{th} coordinate of the point x or k^{th} component of the vector x. The set of all n-dimensional points is called n-dimensional Euclidean space or simply n-space and denoted by $\mathbb{R}^n$.*

The real line is an example of a 1-dimensional space and the real plane is a 2-dimensional space. A typical vector in $\mathbb{R}^2$ is denoted by $[x_1 \;\; x_2]^\top$.

The algebraic operations on $\mathbb{R}^n$ are carried out using the following rules.

1. For $x, y \in \mathbb{R}^n$, $x = y$ if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

2. For $x, y \in \mathbb{R}^n$,

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

3. For $\alpha \in I\!R$ and $x \in I\!R^n$,

$$\alpha\, x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}.$$

4. $0 = [0\; 0\; 0\; \cdots\; 0]^\top$.

5. For $x, y \in I\!R^n$,

$$x.y = \sum_{i=1}^{n} x_i y_i.$$

6. The length of a vector is captured by the norm of a vector. The norm function $||(\cdot)|| : I\!R^n \longrightarrow I\!R^+$ satisfies the following axioms:

   A1 $||\alpha x|| = |\alpha|\; ||x||$ (positive homogeneity).

   A2 $||x + y|| \leq ||x|| + ||y||$ (triangle inequality or subadditivity).

   A3 $||x|| = 0 \Longleftrightarrow x = 0$ and $||x|| > 0$ whenever $x \neq 0$ (positive definiteness).

   There is class of p-norms defined by

$$||x||_p = \left(|x_1|^p + \ldots + |x_n|^p\right)^{1/p}, \;\; 1 \leq p < \infty$$

   and

$$||x||_\infty = \max_i |x_i|.$$

$$||x||^2 =< x, x >= x_1^2 + x_2^2 + \ldots x_n^2$$

   where, the operator $< \cdot, \cdot >$: $I\!R^n \times I\!R^n \longrightarrow I\!R^+$ is used to denote the dot product between two vectors in $I\!R^n$. All p-norms are equivalent in the sense that if $||\cdot||_{\gamma_1}$ and $||\cdot||_{\gamma_2}$ are two different p-norms, then there exists two constants $c_1, c_2 > 0$ such that the following holds

$$c_1||x||_{\gamma_1} \leq ||x||_{\gamma_2} \leq c_2||x||_{\gamma_1}\; \forall\, x \in I\!R^n.$$

Lemma 2.0.2 *(Cauchy-Schwartz inequality) Let x, y belong to $I\!R^n$. Then $| < x, y > | \leq ||x||\; ||y||$ and $| < x, y > | = ||x||\; ||y||$ if and only if the elements x, y are linearly dependent.*

Instead of $I\!R^n$, we can consider a more general vector space X on which we can define the notion of distance. This leads to the definition of *normed linear space.*

Definition 2.0.3 *A normed linear space is an ordered pair $(X, ||\cdot||)$ where X is a linear vector space and $||\cdot|| : X \longrightarrow I\!R$ is real-valued function on X such that the following*

A1. $||x|| \geq 0 \ \ \forall\, x \in X$; $||x|| = 0$ if only if $x = 0$, *the zero vector in* X.

A2. $||\alpha x|| = |\alpha| ||x|| \ \forall\, x \in X$ *and* $\alpha \in I\!R$ or $\mathbb{C}$.

A3. $||x+y|| \leq ||x|| + ||y|| \ \ \forall\, x, y \in X$.

2.1 Matrix norms

Just as vector norms, one can define matrix norm. To begin, we proceed with induced norms, defined as follows.

Definition 2.1.1 *Let $||\cdot||$ be a given norm on $I\!R^n$. Then for each matrix $A \in I\!R^{n\times n}$, the quantity $||A||_i$ defined by*

$$||A||_i = \sup_{x\neq 0, x \in I\!R^n} \frac{||Ax||}{||x||} = \sup_{||x||=1} \frac{||Ax||}{||x||} = \sup_{||x||\leq 1} \frac{||Ax||}{||x||}$$

is called the induced matrix norm of A corresponding to the vector norm $||\cdot||$.

The induced norm can be interpreted as the least upper bound of the ratio $\frac{||Ax||}{||x||}$ as x varies over $I\!R^n$. In other words, let $\mathcal{C} = \{y \in I\!R^n : y = Ax, x \in \mathcal{B}\}$. Then $||A||_i$ is the smallest radius that completely covers the set $\mathcal{C}$. The induced norm function $||\cdot||_i : I\!R^n \times I\!R^n \longrightarrow [0, \infty)$ satisfies the following axioms:

1. $||A||_i \geq 0 \ \ \forall\, A \in I\!R^{n\times n}$.
2. $||\alpha A||_i = |\alpha| \; ||A||_i \ \ \forall\, A \in I\!R^{n\times n}, \alpha \in I\!R$.
3. $||A+B||_i \leq ||A||_i + ||B||_i \ \ \forall\, A, B \in I\!R^{n\times n}$.

Remark 2.1.2 *Corresponding to every vector norm on $I\!R^n$ there is a corresponding induced norm on $I\!R^{n\times n}$. The converse is not true as the following example shows.*

$$||A||_s = \max_{i,j} |a_{ij}|.$$

We drop the subscript i for the induced norm, since we will use only induced norms in the rest of the lectures. The norms for $p = 1, 2, \infty$ and the Frobenius norm can be computed as follows. Let $A \in \mathbb{R}^{m\times n}$.

- The one-norm , $||A||_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|$.
- The two-norm , $||A||_2 = \sqrt{\lambda_{\max}[A^*A]} = \sigma_{\max}[A]$, where $\lambda_{\max}$ is the maximum eigen value of $A^*A \geq 0$ and $\sigma_{\max}$ is the largest singular value of A.
- The ∞-norm , $||A||_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$.
- The Frobenius-norm , $||A||_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$.

Example 2.1.3 *Consider* $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$. *Then* $||A||_1 = 6, ||A||_2 = 4.4425, ||A||_\infty = 5$ *and* $||A||_F = 4.5826$.

Given any induced norm, and matrices A and B, the property $||AB||_p \leq ||A||_p||B||_p$ called the *submultiplicative* property holds. For any submultiplicative norm

$$||Ax||_p \leq ||A||_p||x||_p \;\forall\; x.$$

Lecture-3

In this lecture, we will discuss the notion of open set, closed set and compact set. We need the notion of open-ball and interior point to begin with. Let $a \in \mathbb{R}^n$ and r be a given positive number. The set of all points $x \in \mathbb{R}^n$ such that $||x - a|| < r$ is called an open n-ball of radius r and centered at a and denoted by $B(a, r)$. The ball $B(a, r)$ is also termed as the r-neighbourhood of a.

Let S be a subset of $\mathbb{R}^n$, and assume that $a \in S$. Then a is called an interior point of S if there is an open n-ball of some radius $r > 0$ with center a, all of whose points belong to S. The set of all interior points of S is called the interior of S and is denoted by *int* S. We can now define an open set as follows.

Definition 3.0.4 *A set $S \subset \mathbb{R}^n$ is called open if all its points are interior points ($S =$ int S).*

In $\mathbb{R}$, the simplest type of open sets are the open intervals of the form $(a, b) \stackrel{\triangle}{=} \{x \in \mathbb{R} : a < x < b\}$. Similarly, the open set in $\mathbb{R}^2$ is the open disk $\{(x_1, x_2) \in \mathbb{R}^2 : x^2 + y^2 < r\}$. In many texts, the word *neighbourhood* is used to denote an arbitrary open set.

Definition 3.0.5 *Let $x \in \mathbb{R}^n$. A neighbourhood of x is an open subset $N \subseteq \mathbb{R}^n$ containing x.*

A collection of open sets have the property that arbitrary union of open sets is open and finite intersection of open sets is open. The following example shows that arbitrary intersection of open sets need not be open.

Example 3.0.6 $S \stackrel{\triangle}{=} (\cap(-\frac{1}{n}, \frac{1}{n}), n = 1, 2, 3, \ldots) = \{0\}$, *which is not open.*

Definition 3.0.7 *A set S in $\mathbb{R}^n$ is said to be closed if its complement $\mathbb{R}^n \setminus S$ is open.*

The notation $A \setminus B$ is read as A minus B, consists of all points of A but excluding those that are common to A and B. A collection of closed sets have the property that a union of finite collection of closed sets is closed, and the intersection of an arbitrary collection of closed sets is closed. Closed set can be defined in terms of adherent points accumulation point/s, which are defined as follows.

Definition 3.0.8 *Let S be a subset of $I\!R^n$ and $x \in I\!R^n$, not necessarily in S. Then x is said to be adherent to S if for every $r > 0$ the n-ball $B(x, r)$ contains at least one point of S.*

The set of all adherent points of a set S is called the closure of S and is denoted by $\bar{S}$. Some points adhere to S because every ball contains points of S distinct from x and this leads to the notion of accumulation point.

Definition 3.0.9 *If A is a subset of $I\!R^n$ and if $x \in I\!R^n$, then x is said to be a limit point or accumulation point of A if every neighbourhood of x contains at least one point of A distinct from x.*

We say x is an accumulation point of S if x adheres to $S \setminus x$. Closed set can be alternatively defined as

- A set is closed if and only if it contains all its adherent points.
- A set is closed if and only if $S = \bar{S}$.
- A set $A \subset I\!R^n$ is closed if and only if, it contains all its accumulation points.

The following Lemma brings out the relation between the closure of a set and the set of its accumulation points.

Lemma 3.0.10 *Let A' denote the set of all accumulation points of a set A in $I\!R^n$, and $\bar{A}$ the set of all adherent points of A. Then, $\bar{A} = A \cup A'$.*

Proof: Let $x \in A \cup A'$. Then $x \in A$ or $x \in A'$. If $x \in A$, then $x \in \bar{A}$. since $A \subset \bar{A}$. Now if $x \in A'$, then every neighbourhood of x intersects A (in a point different from x). Therefore $x \in \bar{A}$ and thus $x \in A \cup A'$.

Conversely, if $x \in \bar{A}$. Then if $x \in A$, then it immediately follows that $x \in A \cup A'$. If $x \notin A$, then every neighbourhood $B(x)$ contains atleast one element of A, which implies $B(x) \cap A \neq \emptyset \implies x \in A'$. Therefore $x \in A \cup A'$. $\square$

On the real line, closed sets are denoted by closed intervals $[a, b] \triangleq \{x \in R : a \leq x \leq b\}$, while in $I\!R^2$ they are represented by closed disk $\{(x_1, x_2) : I\!R^2 : x_1^2 + x_2^2 \leq r\}$.

Example 3.0.11 *Let* $A = (0, 1] \subset I\!R$. *Then* 0 *is the limit point of* A *and so is every point of* $[0, 1]$ *a limit point of* A.

Example 3.0.12 *Consider the set* $A = \{1/n : n \in Z^+\}$, *where* Z^+ *is the set of positive integers. The only limit point of* A *is* 0.

Some sets are neither open nor closed, such as $[0, 1)$, which is called half-open or half-closed interval.

We next define a bounded set.

Definition 3.0.13 *A set* S *in* $I\!R^n$ *is said to be bounded if it lies entirely within an* n*-ball* $B(a, r)$ *for some* $r > 0$ *and some* $a \in I\!R^n$.

Lecture-4

4.1 Supremum and infimum of a set

Let S be a set of real numbers. If there is a real number b such that $x \leq b$ for every $x \in S$, then b is called an upper bound for S and we say that S is bounded above by b. Certainly any $c \geq b$ is an upper bound for S, but we are interested in the least among the upper bounds.

Definition 4.1.1 *Let S be a set of real numbers bounded above. A real number b is called a least upper bound for S, denoted by $b = \sup(S)$, if it has the following properties:*

- *b is an upper bound for S.*
- *No number less than b is an upper bound for S.*

Naturally, we can define the greatest lower bound, called the infimum of a set.

Definition 4.1.2 *Let S be a set of real numbers bounded below. A real number c is called the greatest lower bound for S, denoted by $c = \inf(S)$, if it has the following properties:*

- *c is an lower bound for S.*
- *No number greater than c is a lower bound for S.*

Example 4.1.3 *Consider $S = (-1, 3)$, then $\sup(S) = 3$ and $\inf(S) = -1$. Note that the $\sup$ and $\inf$ of a set need not belong to the set. When they do belong, they are the maximum and minimum elements of the set.*

Example 4.1.4 *Let $S = \{x : 3x^2 - 10x + 3 < 0\}$. Now $x \in S$ satisfies $(x-3)(3x-1) < 0$ which implies $\frac{1}{3} < x < 3$. Hence $\sup(S) = 3$ and $\inf(S) = 1/3$.*

The notion of closed and bounded sets leads to the definition of compact set. We first need the definition of *covering of a set*. A collection F of sets is said to be covering of a given set S if $S = \cup_{A \in F} A$. The collection F is also said to cover S. If F is a collection of open sets, then F is called an open covering of S.

Definition 4.1.5 *A set S in $\mathbb{R}^n$ is said to be compact if and only if every open covering of S contains a finite subcover that is, a finite subcollection which also covers S.*

The Heine-Borel theorem states that every closed and bounded set in $\mathbb{R}^n$ is compact.

We end this lecture with the definition of boundary of a set. A point $x \in \mathbb{R}^n$ is called a boundary point of $S \subset \mathbb{R}^n$ if every n-ball $B(x)$ contains at least one point of S and at least one point of $\mathbb{R}^n \setminus S$. The set of all boundary points of S is called the boundary of S and is denoted by $\partial S = \bar{S} \cap \overline{\mathbb{R}^n \setminus S}$ from which it follows that ∂S is a closed set in $\mathbb{R}^n$.

The notion of dense and connected sets are introduced here.

Definition 4.1.6 *Let $S \subset \mathbb{R}^n$. Then S is said to be dense in $\mathbb{R}^n$ if $\bar{S} = \mathbb{R}^n$.*

This is the case if and only if $S \cap B(x, r) \neq \emptyset$ for every $x \in \mathbb{R}^n$ and $r > 0$.

Example 4.1.7 *The set of rational numbers Q are dense in $\mathbb{R}$ and so are the set of irrationals.*

We next define an important concept of connectedness, for which we first need to say when are two sets are separated. Two subsets A and B of $\mathbb{R}^n$ are said to be separated if both $\bar{A} \cap B$ and $A \cap \bar{B}$ are empty. In other words, no point of A lies in the closure of B and no point of B lies in the closure of A.

Example 4.1.8 *Let $A = [0, 1]$ and $B = (1, 2)$. The sets A and B are disjoint but not separated.*

Example 4.1.9 *Let $A = (0, 1)$ and $B = (2, 3)$. The sets A and B are disjoint and separated.*

Example 4.1.10 *Let $A = [0, 1]$ and $B = [1, 2)$. The sets A and B are not disjoint and not separated.*

We can now define a connected set.

Definition 4.1.11 *A set $S \subset \mathbb{R}^n$ is said to be connected if S is not a union of two non-empty separated sets.*

Example 4.1.12 *Let $S = [0, 1] \cup (1, 2)$. Since S is a union of two non-empty sets that are not separated, it is connected.*

Lecture-5

A few solved examples and exercises are presented here.

P1. Find the limit point(s) of the set $C = \{0\} \cup (1, 2) \subset I\!R$.

Solution : Every point of the interval $[1, 2]$ is a limit point of $\mathbb{C}$.

P2. Determine if the following sets in $I\!R$ are open or closed (or neither).

a) All rational numbers Q.

b) All numbers of the form $\frac{(-1)^n}{(1+(1/n))}$, n=1,2,. . .

P3. If S and T are subsets of $I\!R^n$, prove that $(int\ S) \cap (int\ T) = int\ (S \cap T)$, where int refers to the interior of the set.

Solution : We have to show the following inclusions

$$int(S) \cap int(T) \subset int(S \cap T)$$
$$int(S \cap T) \subset int(S) \cap int(T)$$

to prove the equality. Let $x \in int(S) \cap int(T) \implies x \in int(S)$ and $x \in int(T)$. By the definition of interior point, there exists open n-balls $B_S(x), B_T(x)$ such that $x \in B_S(x) \subset S$ and $x \in B_T(x) \subset T \implies x \in B_s \cap B_T$. But $B_S \cap B_T$ is open in $S \cap T$ (recall that finite intersection of open sets is open). Hence $x \in int(S \cap T)$.

Conversely, let $x \in int(S \cap T)$. Then there exists an open n-ball $B(x) \subset S \cap T$. But $B(x)$ is open in both S and T, which further implies that $x \in int(S)$ and $x \in int(T)$. Hence $x \in int(S) \cap int(T)$.

P4. Give an example of a real-valued continuous function that maps an open set to a closed set.

Solution : Consider the function $f : (0, 20) \longrightarrow I\!R$ defined by $f(x) = \log(x) + \log(20 - x)$. The domain of f, denoted by D_f is $(0, 20)$, an open set. The range of f, denoted by R_f, is $(-\infty, 2]$ which is not an open set as 2 is not an interior point of $(-\infty, 2]$. However, $I\!R \setminus R_f = (2, +\infty)$, an open set implying that R_f is a closed set.

P5. The linear space $C[a, b]$ consists of continuous function on the real interval $[a, b]$ together with the norm defined as

$$||x|| \triangleq \max_{t\,\in[a,b]} |x(t)|.$$

Verify all the axioms of a norm hold for the proposed norm.

Solution It is clear that $||x|| \geq 0$ and is equal to zero only when the function is identically equal to zero. Next $||\alpha x|| = \max|\alpha x(t)| = |\alpha| \max|x(t)| = |\alpha|\ ||x||$. Finally, the triangle inequality follows from

$$\begin{aligned} ||x + y|| &= \max|x(t) + y(t)| \\ &\leq \max(|x(t)| + |y(t)|) = \max|x(t)| + \max|y(t)| = ||x|| + ||y||. \end{aligned}$$

P6. Prove the following identity

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2.$$

Solution : Expanding the LOS in terms of inner product, we have

$$\begin{aligned} ||x + y||^2 + ||x - y||^2 &= < x + y, x + y > + < x - y, x - y > \\ &= < x, x + y > + < y, x + y > + < x, x - y > - < y, x - y > \\ &= < x, x > + < x, y > + < y, x > + < y, y > - < x, y > \\ &\quad + < x, x > + < x, y > - < y, x > + < y, y > \\ &= 2||x||^2 + 2||y||^2 \end{aligned}$$

This equality is known as the Parallelogram law.

P7. Prove that $\partial A = \partial(I\!R^n - A)$.

P8. Find the sup and inf of the following set of real numbers.
$S = \{x : (x - a)(x - b)(x - c)(x - d) < 0\}$, where $a < b < c < d$.

P9. Consider the set $S = \{x \in I\!R : x = 2^{-n} + 5^{-m}, n, m = 1, 2, \ldots\}$. Show that S is neither open nor closed.

Solution : The maximum element of S is achieved when $m = n = 1$ and equals $x = 0.7 \in S$, but a neighbourhood of the form $(0.7-\epsilon, 0.7+\epsilon)$ does not belong to S since $(0.7+\epsilon)$ is not an element of S. Hence, S is not open. To prove that it is not closed, let $m = 2, n = 1$ which yields $x = 0.54 \in S$. Fix $x = 0.54 + \delta$, then it is easy to see that $0.54 + \delta \notin S$ since there exists no m and n such that $2^{-n} + 5^{-m} = 0.54 + \delta$. Therefore $0.54 + \delta \in I\!R \setminus S$. Now it can be observed that for any $\epsilon > \delta$, the interval $(0.54+\delta-\epsilon, 0.54+\delta+\epsilon)$ is not contained in $I\!R \setminus S$ because this interval includes the element 0.54 which is a member of S and hence does not belong to $I\!R \setminus S$ implying $(0.54 + \delta - \epsilon, 0.54 + \delta + \epsilon) \cap S \setminus 0.54 = \emptyset$.

Exercise problems

1. Consider the set $S = (-\infty, -100) \cup (-1, 4) \cup \{4\} \cup (6, 8) \subset I\!R$. Find $\bar{S}, S', \partial S, S^c$.

2. Find the supremum and infimum of $S = \{x \in I\!R : (1 + x^2)^{-1} > \frac{1}{2}\}$. State whether they are in S.

3. In the proof of the **Contraction mapping theorem**, at which step one cannot proceed further if the Lipschitz constant $\rho > 1$.

4. Let $x, y \in I\!R^n$. Show that $||x|| - ||y|| \leq ||x - y||$.

5. Show that the norm function $|| \cdot || : I\!R^n \longrightarrow I\!R$ is a continuous function. (You have to use the ϵ, δ definition)

Module 2

Well-posedness of ordinary differential equations

Objectives: To study the notion of Lipschitz continuity and its usefulness in establishing the existence and uniqueness of solution of a differential equation.

Lesson objectives

This module helps the reader in answering questions related to

– existence and uniqueness of solutions by identifying the continuity property associated with the right-hand side of the differential equation

– Contraction mapping theorem and its applications

– Flows, orbits and positive invariance.

Advanced reading

- Theory of Ordinary Differential Equations by E. A. Coddington and N. Levinson.
- Ordinary differential equations by V. I. Arnold

Lecture-6

We begin with the definition of continuity of a function at a point.

Definition 6.0.13 *A function $f : I\!\!R^n \longrightarrow R^m$ is continuous at a point $x \in I\!\!R^n$ if, given $\epsilon > 0$, there is $\delta > 0$ (dependent only on ϵ) such that*

$$||x - y|| < \delta \Rightarrow ||f(x) - f(y)|| < \epsilon.$$

A function is continuous on a set S if it is continuous at every point of S, and it is uniformly continuous on S if, given $\epsilon > 0$ there is $\delta(\epsilon, x) > 0$ such that the inequality holds for $x, y \in S$. Note that uniform continuity is defined on a set, while continuity is defined at a point. We next define a property of functions that is stronger than continuity, but weaker than differentiability, called the *Lipschitz* continuity. Let D be a domain in $I\!\!R^n$.

Definition 6.0.14 *A function $f : D \subset I\!\!R^n \to I\!\!R^n$ is said to be Lipschitz continuous at x_0 if there exists a constant $L = L(x_0) > 0$ and a neighbourhood $N \subset D$ such that*

$$||f(x) - f(y)|| \leq L||x - y|| \tag{6.1}$$

for all $x, y \in N$. The constant $L > 0$ is called the Lipschitz constant.

A function f is said to be Lipschitz on a domain D if f is Lipschitz continuous at every point in D. f is said to be uniformly Lipschitz continuous on D if there exists a constant $L > 0$ such that

$$||f(x) - f(y)|| \leq L||x - y||, \ \ \forall \ \ x, y \in D. \tag{6.2}$$

A Lipschitz function on a domain D is not necessarily uniformly Lipschitz on D, since the Lipschitz condition may not hold uniformly (with the same constant L) for all points in D. However, a Lipschitz function on a domain D is uniformly Lipschitz on every compact subset of D.

Definition 6.0.15 *A function f is said to be globally Lipschitz if it is Lipschitz on $\mathbb{R}^n$.*

Example 6.0.16 *Consider the function $f : [a, b] :\longrightarrow \mathbb{R}$ defined by $f(x) = x^2$. For $x, y \in [a, b]$, we have*

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y|\ |x + y| \leq M|x - y|$$

where, $M \triangleq 2\max\{|a|, |b|\}$. Hence f, is uniformly Lipschitz with Lipschitz constant $L = M$.

Example 6.0.17 *Consider $f(x) = |x|$. Then for $x, y \in \mathbb{R}$,*

$$|f(x) - f(y)| = |\ |x| - |y|\ | \leq |x - y|.$$

Hence, f is globally Lipschitz with Lipschitz constant $L = 1$.

Example 6.0.18 *Consider $f(x) = -|x|^\beta \mathrm{sign}(x), \beta \geq 1$. For x satisfying $|x| \leq 1$, we have*

$$|f(x) - f(0)| = |\ -\mathrm{sign}(x)|x|^\beta|\ | = |x|^\beta \leq |x|.$$

Hence, f is uniformly Lipschitz on the compact set $D_1 \triangleq \{x \in \mathbb{R} : |x| \leq 1\}$.

Definition 6.0.19 *A function f:$\mathbb{R}^n \rightarrow \mathbb{R}^n$ is piecewise continuous if there exists a finite collection of disjoint, open, and connected sets $D_1, D_2, ..., D_m \subset \mathbb{R}^n$, whose closures cover $\mathbb{R}^n$, that is*

$$\mathbb{R}^n = \bigcup_{k=1}^{m} \bar{D}_k$$

such that for all $k = 1, 2, ..., m$, f is continuous on D_k.

Definition 6.0.20 *A sequence $x_1, x_2, x_3, \ldots$ in a normal linear space $(X, \|.\|)$ is said to be Cauchy, if for every $\epsilon > 0$ there exists an integer $N = N(\epsilon)$ such that*
$\|x_i - x_j\| < \epsilon\ \forall\ i, j \geq N$.

Definition 6.0.21 *A sequence $x_1, x_2, x_3, \ldots$ in a normal linear space $(X, \|.\|)$ is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists an integer $N = N(\epsilon)$ such that $\|x_i - x\| < \epsilon\ \forall\ i \geq N$.*

Example 6.0.22 *Consider the sequence* $\{\frac{1}{n}\}_{n=1}^{\infty}$. *The claim is that* $x = 0$ *is a limit of the sequence. Let* $\epsilon > 0$, *then* $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ *if and only if* $n > \frac{1}{\epsilon}$. *Hence, letting* $N(\epsilon) > \frac{1}{\epsilon}$, *it follows that* $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ *for all* $n > N$.

A sequence is said to be divergent, if it is not convergent. The sequence $\{(-1)^n\}_{n=0}^{\infty}$ is a divergent sequence.

Facts:

- A Cauchy sequence is bounded.
- Every Cauchy sequence in a complete normed linear space is a convergent sequence.
- If a sequence converges, its limit is unique.

Lecture-7

In the study of ordinary differential equations of the form

$$\begin{aligned} \dot{x} &= f(t, x), \quad \forall\, t \geq 0 \\ x(t_0) &= x_0 \end{aligned} \tag{7.1}$$

where, $f : I\!R^n \longrightarrow I\!R^n$, we are interested in the answer to the following questions:

1. Does (7.1) has at least one solution? This is related to the existence of a solution.
2. If there exists a solution to (7.1), then, is it unique?
3. Is the unique solution defined for all time $t \in [0, \infty)$?
4. Does the unique solution that is defined for all $t \geq 0$ depend continuously on the initial condition x_0?

If the answer to the last question is affirmative, then we have say that the problem (7.1) is *well-posed*. In the following examples, we show that based on the nature of f, the ODE's could have or have not an affirmative answer to the question of existence, uniqueness and continuous dependence.

Example 7.0.23 *The differential equation defined on* $I\!R$

$$\begin{aligned} \dot{x} &= -\text{sign}(x) \\ x(0) &= 0 \end{aligned} \tag{7.2}$$

where the sign function is defined by

$$\mathbf{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

does not admit any continuously differentiable function $x(t)$ *such that* (7.2) *holds.*

Example 7.0.24

$$\begin{aligned} \dot{x} &= x^{1/3} \\ x(0) &= 0. \end{aligned} \tag{7.3}$$

This ODE also admits two solutions, namely $x(t) \equiv 0$ *and* $x(t) = \left(\frac{2t}{3}\right)^{3/2}$ $\forall\, t \geq 0$.

In the following example we see the local existence of a solution.

Example 7.0.25

$$\begin{aligned} \dot{x} &= 1 + x^2 \\ x(0) &= 0. \end{aligned} \tag{7.4}$$

The solution of (7.4) *is given by* $x(t) = \tan t$, *which is defined for all* $t \in [0, \pi/2)$. *Because the solution* $x(t) \longrightarrow \infty$ *at* $t = \pi/2$, *the ODE* (7.4) *is said to possess finite escape time property.*

Another example that has the same property is given by

Example 7.0.26

$$\begin{aligned} \dot{x} &= -x^2 \\ x(0) &= -1. \end{aligned} \tag{7.5}$$

The solution of (7.5) *is given by* $x(t) = \frac{1}{t-1}$, *which is defined for all* $t \in [0, 1)$.

Finally, an example that possess a unique solution that is defined for all $t \geq t_0$.

Example 7.0.27

$$\begin{aligned} \dot{x} &= -x^3 \\ x(t_0) &= x_0. \end{aligned} \tag{7.6}$$

The solution of (7.6) *is given by*

$$x(t) = \operatorname{sign}(x_0)\sqrt{\frac{x_0^2}{(1 + 2x_0^2(t - t_0))}}$$

Note that the right-hand side of (7.3) is continuous, and yet the solution is not unique. Therefore the continuity of $f(\cdot)$ does not guarantee the existence of a unique solution. It is natural to ask then under what conditions on $f(\cdot)$, a solution to (7.1) exists and is unique. The following theorem uses the Lipschitz condition on $f(\cdot)$ to show local existence and uniqueness.

Theorem 7.0.28 *Let $f(t, x)$ be piecewise continuous in t and satisfy the Lipschitz condition*

$$||f(t, x) - f(t, y)|| \leq ||x - y|| \;\; \forall\, x, y \in \bar{B}(x_0, r), \;\; \forall\, t \in [t_0, t_1].$$

Then, there exists some $\gamma > 0$ such that (7.1) has a unique solution over $[t_0, t_0 + \gamma]$.

As an application of theorem 7.0.28, the right-hand side of the ODE in (7.4), (7.5) and (7.6) are locally Lipschitz in x and hence guarantee a unique solution over some finite time-interval.

Lecture-8

Many equations that we encounter in practice are not solvable explicitly. In such a case, the given equation can be recast as $f(x) = x$ form and under certain hypotheses on f and the underlying space, the existence of a point, called the fixed point, denoted by x^* and defined as $f(x^*) = x^*$ can be found. A powerful result that provides the required hypotheses is the *contraction mapping theorem*, stated with proof here.

Theorem 8.0.29 *Let $(X, ||\cdot||)$ be a complete normed linear space and $T : X \longrightarrow X$. Suppose there exists a fixed constant $\rho < 1$ such that*

$$||Tx - Ty|| \leq \rho||x - y|| \ \ \forall \ x, y \in X.$$

Under these conditions, there exists exactly one $x^* \in X$ such that $Tx^* = x^*$. For each $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ in X defined by $x_{n+1} = Tx_n$ converges to x^*. Moreover, the estimate of the rate of convergence is given by

$$||x^* - x_n|| \leq \frac{\rho^n}{\rho - 1}||Tx_0 - x_0||.$$

Proof:
Let $x_0 \in X$ and define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$. We first show that $\{x_n\}$ is a Cauchy sequence. For each $n \geq 0$ it follows that

$$\begin{aligned} ||x_{n+1} - x_n|| &= ||Tx_n - Tx_{n-1}|| \leq \rho||x_n - x_{n-1}|| \\ &\leq \rho^2||x_{n-1} - x_{n-2}|| \ldots \leq \rho^n||x_1 - x_0|| = \rho^n||Tx_0 - x_0||. \end{aligned}$$

Suppose $m = n + r$, $r \geq 0$ is given, then by triangle inequality

$$\begin{aligned}
||x_m - x_n|| &= ||x_{n+r} - x_n|| \leq ||x_{n+r} - x_{n+r-1}|| + ||x_{n+r-1} - x_n|| \\
&\leq ||x_{n+r} - x_{n+r-1}|| + ||x_{n+r-1} - x_{n+r-2}|| + ||x_{n+r-2} - x_n|| \\
&\leq ||x_{n+r} - x_{n+r-1}|| + \ldots + ||x_{n+1} - x_n|| \\
&= \rho^{n+r-1}||Tx_0 - x_0|| + \rho^{n+r-2}||Tx_0 - x_0|| + \ldots + \rho^n||Tx_0 - x_0|| \\
&= \sum_{i=0}^{r-1} \rho^{n+i}||Tx_0 - x_0|| \leq \sum_{i=0}^{\infty} \rho^{n+i}||Tx_0 - x_0||.
\end{aligned}$$

By noting that the series $(1 + \rho + \rho^2 + \ldots)$ is a Geometric series whose sum is $\frac{1}{1-\rho}$, we have $\rho^n + \rho^{n+1} + \rho^{n+2} + \ldots = \frac{\rho^n}{1-\rho}$. Thus

$$||x_m - x_n|| \leq \frac{\rho^n}{1-\rho}||Tx_0 - x_0||.$$

As $n \longrightarrow \infty$, $\rho^n \longrightarrow 0$. Therefore $||x_m - x_n|| \longrightarrow 0$ as $n \longrightarrow \infty$, thereby proving the sequence $\{x_n\}$ to be Cauchy. Since X is a complete space, x_n converges to a limit $x^* \in X$. Now $Tx^* = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*$. Hence x^* is a fixed point of T.

Next, we show that x^* is unique. Let $\bar{x}$ be also a fixed point of T. Then

$$||\bar{x} - x^*|| = ||T\bar{x} - Tx^*|| \leq \rho||\bar{x} - x^*||$$

if and only if $\bar{x} = x^*$. Finally, the estimate of the rate of convergence is given by

$$||x_n - x^*|| = ||x_n - \lim_{m\to\infty} x_m|| = \lim_{m\to\infty} ||x_n - x_m|| \leq \frac{\rho^n}{1-\rho}||Tx_0 - x_0||.$$

□

We illustrate the application of contraction mapping theorem through the following examples.

Example 8.0.30 *Consider $f : [0,1] \longrightarrow [0,1]$, where $f(x) = \cos(x)$. Find $x \in [0,1]$ s.t $f(x) = x$.*

We use the fact that the closed interval $[0,1]$ is a complete normed space and then show that $\cos(\cdot)$ is a contraction on $[0,1]$. From mean-valued theorem

$$\cos(1) - \cos(0) = -\sin(c)(1-0)$$

where, $c \in (0,1)$. Therefore

$$|\cos(1) - \cos(0)| = |sin(c)|\;|(1-0)|.$$

Now, $|\sin(c)| \leq \sin(1) \approx 0.8417$. Hence $\cos(\cdot)$ is a contraction on $[0,1]$. Thus there exists a unique x^* such that $\cos(x^*) = x^*$. By iteration, $x^* = 0.739$.

Fixed point theorems are often used to prove the existence of solutions to differential equations, such as the Picard's theorem about the existence of solutions to an initial value problem for a first order ODEs.

Lecture-9

The notion of *positive invariance* of a set is a powerful analytical tool that finds use in stability studies.

Consider a dynamical system described by

$$\dot{x}(t) = f(x) \tag{9.1}$$

where $f : D \longrightarrow I\!R^n$ is locally Lipschitz on the open set $D \subseteq I\!R^n$. Let $\psi(t, x_0)$ denote the solution of (9.1) that exists for all $t \in [0, \infty)$ and satisfies the initial condition $x(0) = x_0$. This leads to the map $\psi : [0, \infty) \times D \longrightarrow D$ satisfying $\psi(0, x_0) = x_0$ and possesses the semi-group property $\psi(t_1, \psi(t_2, x)) = \psi(t_1 + t_2, x)$ for all $t_1, t_2 \geq 0$ and $x \in D$.

Definition 9.0.31 *The orbit $\mathcal{O}_x$ of a point $x \in D$ is the set $\{\psi(t, x) : t \geq 0\}$.*

Definition 9.0.32 *A set $\mathcal{U} \subseteq \mathbb{R}^n$ is positively invariant if $\psi_t(\mathcal{U}) \subseteq \mathcal{U}$ for all $t \geq 0$. The set $\mathcal{U}$ is negatively invariant if, for every $z \in \mathcal{U}$ and every $t \geq 0$, there exists $x \in \mathcal{U}$ such that $\psi(t, x) = z$ and $\psi(\tau, x) \in \mathcal{U}$ for all $\tau \in [0, t]$. Hence, if $\mathcal{U}$ is negatively invariant, then $\mathcal{U} \subseteq \psi_t(\mathcal{U})$ for all $t \geq 0$. Finally, the set $\mathcal{U}$ is invariant if $\psi_t(\mathcal{U}) = \mathcal{U}$ for all $t \geq 0$.*

For a linear system, the positive invariance of a set is easily established, as illustrated through the following example.

Example 9.0.33 *Consider the following linear system*

$$\begin{aligned} \dot{x}_1 &= 2x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{aligned} \tag{9.2}$$

We wish to show that the set $\mathcal{N} = \{x \in I\!R^2 \mid x_1 + x_2 = 0\}$ is positively invariant. We will show that if $x_1(0) = -x_2(0)$, then $x_1(t) = -x_2(t)\ \forall\ t > 0$. A straight forward calculation *yields the solution of* (9.2) *as*

$$
\begin{aligned}
x_1(t) &= e^{-t/2}\left[\left(\cosh(3t/2) + \frac{1}{3}\sinh(3t/2)\right)x_1(0) + \frac{4}{3}\sinh(3t/2)x_2(0)\right] \\
x_2(t) &= e^{-t/2}\left[(\cosh(3t/2) - \frac{1}{3}\sinh(3t/2))x_2(0) + \frac{2}{3}\sinh(3t/2)x_1(0)\right]
\end{aligned}
$$

Since $x_1(0) = -x_2(0)$, *on rearranging, we have*

$$
\begin{aligned}
x_1(t) &= e^{-t/2}x_2(0)\left[-\cosh(3t/2) + \sinh(3t/2)\right] \\
x_2(t) &= -e^{-t/2}x_2(0)\left[-\cosh(3t/2) + \sinh(3t/2)\right] \\
&= -x_1(t)\ \forall\, t \geq 0.
\end{aligned}
$$

□

Module 3

Modeling and state-space formulation of nonlinear systems

Objectives: To be able to model simple mechanical and electrical systems, write their state-space formulation and find the equilibrium set.

Lesson objectives

This module helps the reader in

– developing mathematical models of physical systems with emphasis on simple mechanical systems

– writing down the state-space representation that is amenable for further analysis.

In this module, models of pendulum-like systems will be derived using the total energy of the system. Examples include the pendulum on a cart, rotary pendulum, spherical pendulum and the inertia wheel pendulum. These systems have been studies extensively and continue to generate interest among Physicists, Mathematicians and control theorists.

Suggested reading

- A mathematical introduction to robotic manipulation by Richard M. Murray, Zexiang Li and S. Shankar Sastry

Lecture-10

> Of all the motions that may bring a system of material particles from a certain initial position to a given final position (the total energy remaining constant), the actual motion is that for which the action is a minimum. The virtual motions must, therefore, satisfy the principle of energy. They may, on the other hand, take any arbitrary time. According to this conception, the path of a particle is that along which it will reach its final position in the shortest time, if it move with constant velocity, and if frictional forces be absent. Thus, the path is the line of shortest length, that is, for a free particle a straight line. –J. L. Lagrange (1760).

Euler-Lagrange formulation of mechanical systems

In this module we present the equations of motion of well-known benchmark problems in control systems.

The mechanical systems that we study here consist of rigid/flexible linkages which are interconnected in some way. They are given the structure of a set whose points are in one-to-one correspondence with the set of configurations of the system. We denote the configuration set, also called as configuration space by Q. The number of configuration variables depends upon the degrees-of-freedom of the system, defined as follows.

Definition 10.0.34 *The degree-of-freedom of a mechanical system is the number of independent coordinates required to describe the pose or the configuration of the system.*

In many mechanical systems, the space Q is non-Euclidean , for example, the configuration space of a simple pendulum is the Unit circle, that is $Q = \mathbb{S}^1$).

Definition 10.0.35 *For a system of m particles with k constraints of the form $g_j(r_1,\ldots,r_m)$, $j=1,\ldots,m$ and $g_j's$ are smooth, we find a set of $n=3m-k$ variables $(q_1,\ldots,q_n)$ and smooth functions $f_1,\ldots,f_m$ such that $r_i = f_i(q_1,\ldots,q_n) \iff g_j(r_1,\ldots,r_m) = 0, i = 1,\ldots,m, j=1,\ldots,k$. The $q_i's$ are called as generalized coordinates for the system.*

The number of generalized coordinates is equal to the number of degrees-of-freedom of the system. When the external forces acting on the system are expressed in terms of components along the generalized coordinates, they are termed generalized forces.

There are many methods of modeling mechanical systems viz. using Newton's laws, Euler-Lagrange and Hamiltonian based equations of motion. Nevertheless, all are equivalent and hence yield the same dynamics. We focus on the Euler-Lagrange equations for which we first define the Lagrangian as $\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - V(q)$, where $T(q,\dot{q})$ is the total stored kinetic energy associated with the masses and moments of inertia and the total potential energy $V(q)$ associated with gravitational and stiffness elements.

Theorem 10.0.36 *The equations of motion for a mechanical system with generalized coordinates and velocities $(q,\dot{q}) \in I\!R^n \times I\!R^n$ and Lagrangian $\mathcal{L}(q,\dot{q})$ are given by*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = F_i, i = 1,\ldots,n$$

where F_i is the external force acting on the i^{th} generalized coordinate.

State-space formulation and equilibrium points

An n^{th}-order nonlinear system with m control inputs $u_1,u_2,\ldots,u_m$ can be rewritten as n first-order ODEs as

$$\begin{aligned} \dot{x} &= f(x) + \textstyle\sum_{i=1}^m g_i(x)u_i \\ x(t_0) &= x_0 \end{aligned} \tag{10.1}$$

where, $x = (x_1,x_2,\ldots,x_n)$ are termed the states of the system, $f(x) \in I\!R^n$ is termed the drift vector field and $g_i(x) \in I\!R^n$ are the control vector fields. The vector fields f, g_i are said to be smooth if its components are smooth functions. The underlying space on which the states evolve is termed the state-space of the system. If the system has no control inputs ($u_i = 0, i = 1,\ldots,m$), then it is said to be unforced. An important notion associated with unforced system is the equilibrium point denoted by x_e. If $x_0 = x_e$, then the solution $x(t)$ of (10.1) satisfies $x(t) = x_e$ for all $t \geq t_0$. For autonomous systems (where the vector fields f does not depend explicitly on time t) the equilibrium set $\mathcal{E}$

of (10.1) is given by $\mathcal{E} = \{x \in I\!R^n : f(x) = 0\}$. If $\mathcal{E}$ has a singleton (single point), then the system (10.1) is said to possess an unique equilibrium point. The notion of isolated equilibrium is brought out through the following definition.

Definition 10.0.37 *An equilibrium point $x_e \in I\!R^n$ of* (10.1) *is said to be isolated if there exists an $\epsilon > 0$ such that $B(x_e, \epsilon)$ contains no equilibrium points other than x_e.*

If the set $\mathcal{E}$ consist of multiple points and further if each equilibrium point has no other equilibrium in its vicinity, then the system is said to possess multiple isolated equilibrium points. If the multiple equilibrium points are not isolated, then the system is said to have a continuum of equilibrium points. A test to check for isolated equilibrium point is stated as follows.

Proposition 10.0.38 *If $\frac{\partial f}{\partial x}\Big|_{x=x_e}$ is non-singular, then the equilibrium point x_e of* (10.1) *is an isolated equilibrium point.*

We next present nonlinear models of physical systems along with their state-space formulation and equilibrium sets.

Lecture-11

The pendulum on a cart system is well-know example in the study of nonlinear control. Various control strategies have been devised to swing-up the pendulum and stabilize it at the upward vertical position (analogous to balancing a stick on your palm).

Pendulum on a cart

We consider a pendulum pivoted on a cart as depicted in figure 11.1. The system consists of a pendulum of length l and of point mass m pivoted on a cart of mass M and making an angle of ϕ with the vertical. The cart is actuated with a force F and its displacement is denoted by x. The reference for the potential energy is the horizontal line passing through the origin of the inertial frame. Here, the configuration space is $Q = \mathbb{R} \times \mathbb{S}^1$ and is parameterized by (x, ϕ). Note that, in topology, the space $\mathbb{R} \times \mathbb{S}^1$ is a cylinder. The

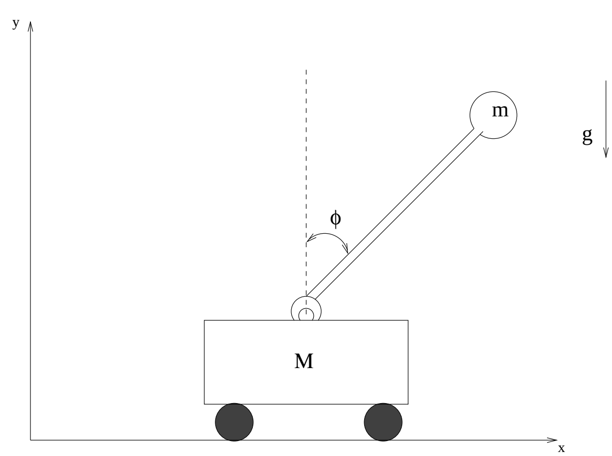

Figure 11.1: Pendulum on a cart system

Lagrangian of the system, is given by

$$\mathcal{L} = \frac{M}{2}\dot{x}^2 + \frac{m}{2}v^\top v - mgl\cos(\phi)$$

where,

$$
\begin{aligned}
v &= \frac{d}{dt}\begin{bmatrix} x + l\sin(\phi) \\ l\cos(\phi) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \dot{x} + l\,\dot{\phi}\cos(\phi) \\ -l\,\dot{\phi}\sin(\phi) \\ 0 \end{bmatrix}.
\end{aligned}
$$

By collecting the inertial parameters in the following constants $a_1 \triangleq (M+m), a_2 \triangleq ml^2, a_3 \triangleq ml, a_4 \triangleq mgl$ and defining the control input as $u = F$, the Euler-Lagrange equations are given by

$$
\underbrace{\begin{bmatrix} a_1 & a_3\cos\phi \\ a_3\cos\phi & a_2 \end{bmatrix}}_{M}\begin{bmatrix} \ddot{x} \\ \ddot{\phi} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -a_3\dot{\phi}\sin\phi \\ 0 & 0 \end{bmatrix}}_{C}\begin{bmatrix} \dot{x} \\ \dot{\phi} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -a_4\sin\phi \end{bmatrix}}_{h} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{G}u. \quad (11.1)
$$

In robotics, the matrix M in (11.1) is termed the inertia-matrix, the matrix C acting on the vector of generalized velocities $[\dot{x} \quad \dot{\phi}]^\top$, in general, consists of the Coriolis and centripetal forces, the vector h represents the gradient of the potential energy and G is the vector of input forces/torques. It is a well-known fact that for rigid-body systems, the following structural properties hold.

- $M = M^\top$ and $M > 0$.
- $\dot{M} - 2C$ is skew-symmetric.

These properties are useful in stability analysis of robotic systems.

With the states defined as $x = [x_1 \ x_2 \ x_3 \ x_4]^\top = [x \quad \phi \quad \dot{x} \quad \dot{\phi}]^\top$ and $u = F$, the state-space model is given by

$$
\dot{x} = f(x) + g(x)u \qquad (11.2)
$$

where,

$$
f(x) = \begin{pmatrix} x_3 \\ x_4 \\ \frac{1}{\Delta}(a_2a_3x_4^2\sin x_2 - a_3a_3\sin x_2\cos x_2) \\ \frac{1}{\Delta}(-a_3x_4^2\sin x_2\cos x_2 + a_1a_4\sin x_2) \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 0 \\ \frac{a_2}{\Delta} \\ \frac{-a_3}{\Delta}\cos x_2 \end{pmatrix},
$$

where $\Delta = a_1a_2 - a_3^2\cos x_2 > 0$. The equilibrium set is given by $\mathcal{E} = \{x : (x_1, x_2, x_3, x_4) = (*, 0, 0, 0)\} \cup \{x : (x_1, x_2, x_3, x_4) = (*, \pi, 0, 0)\}$. Here $*$ indicates that the cart position is arbitrary.

Points to ponder

- Verify the structural properties of the system?
- Is it possible to compute the equilibrium points of a rigid-body system by examining its potential function? Hint: Look at the gradient of the potential function.

Lecture-12

12.1 Furuta pendulum

We consider the Furuta pendulum, which is an inverted pendulum on a rotary arm as depicted in Figure 12.1, where l is the length of the pendulum, R is the radius of the arm, q_2 is the angle made by the pendulum from the vertical, q_1 is the angle of the mass M from a fixed vertical plane, m is the point mass of the pendulum, g is the acceleration due to gravity and u is the torque applied at the shaft joined to the arm. The total kinetic

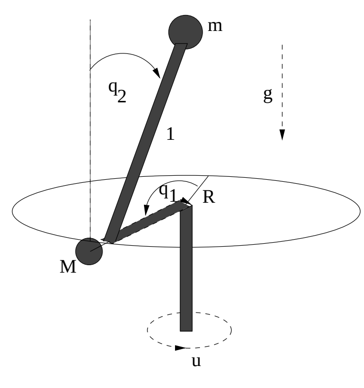

Figure 12.1: Furuta pendulum

energy is given by

$$\begin{aligned} K.E &= K.E_{arm} + K.E_{pendulum} \\ &= \frac{M}{2}R^2\dot{q}_1^2 + \frac{m}{2}v^\top v \end{aligned}$$

where, v (see figure 12.2) is given by

$$v = \begin{bmatrix} -R\dot{q}_1 \sin q_1 \\ R\dot{q}_1 \cos q_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{q}_1 & -\dot{q}_2 \sin q_1 \\ \dot{q}_1 & 0 & \dot{q}_2 \cos q_1 \\ \dot{q}_2 \sin q_1 & -\dot{q}_2 \cos q_1 & 0 \end{bmatrix} \begin{bmatrix} (l \sin q_2) \sin q_1 \\ -(l \sin q_2) \cos q_1 \\ l \cos q_2 \end{bmatrix}.$$

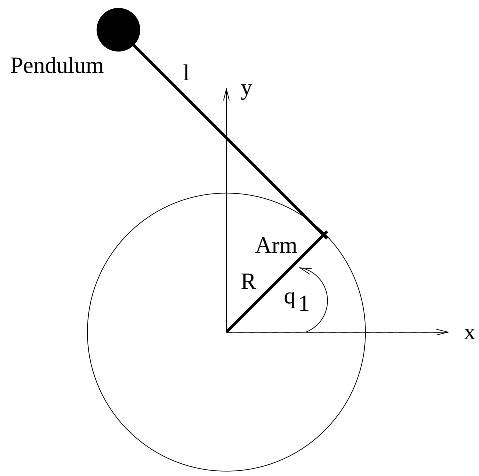

Figure 12.2: Top view of the setup

Collecting the inertial parameters in the constants $a_i, i = 1, \ldots, 4$ defined by

$$a_1 = (M+m)R^2 \qquad a_2 = ml^2$$
$$a_3 = mRl \qquad a_4 = mgl.$$

the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(a_1 + a_2 \sin {q_2}^2)\dot{q}_1^2 + a_3\dot{q}_1\dot{q}_2 \cos q_2 + \frac{a_2}{2}\dot{q}_2^2 - a_4 \cos q_2.$$

The Euler-Lagrange equations are given by

$$\begin{bmatrix} a_1 + a_2 \sin {q_2}^2 & a_3 \cos q_2 \\ a_3 \cos q_2 & a_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 2a_2\dot{q}_1\dot{q}_2 \sin q_2 \cos q_2 - a_3\dot{q}_2^2 \sin q_2 \\ -a_2\dot{q}_1^2 \sin q_2 \cos q_2 - a_4 \sin q_2 \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

Inertia Wheel Pendulum

The inertia wheel pendulum consists of an unactuated pendulum to which an inertia wheel is attached at the free end (see Figure 12.3). The pendulum angle is θ_1 and the inertia wheel angle is θ_2 (both angles are measured clockwise). With the states defined as $x = [x_1\ x_2\ x_3\ x_4]^\top = [\theta_1\ \ \theta_2\ \ \dot{\theta}_1\ \ \dot{\theta}_2]^\top$ and $u = \tau$, the state-space model is given by

$$\dot{x} = f(x) + g(x)u \tag{12.1}$$

where,

$$f(x) = \begin{pmatrix} x_3 \\ x_4 \\ c_1c_3 \sin x_1 \\ -c_1c_3 \sin x_1 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 0 \\ -c_3 \\ \frac{c_2c_3}{I_w} \end{pmatrix},$$

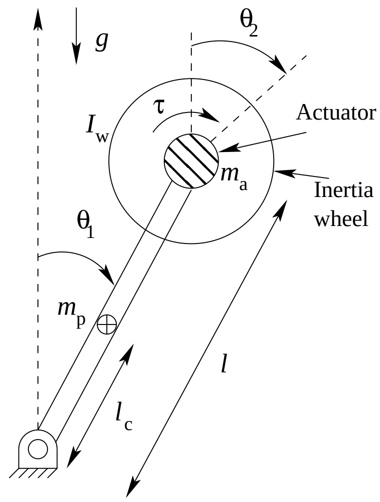

Figure 12.3: Schematic of the inertia wheel pendulum system

where, $c_1 \triangleq (m_p l_c + (m_a + m_w)l)g$, $c_2 \triangleq I_1 + I_w + (m_w + m_a)l^2$, $c_3 \triangleq \frac{I_w}{c_2 I_w - Iw^2}$, m_p is mass of the pendulum, l_c is center of mass of the pendulum, l is length of the pendulum, m_a is mass of the actuator, m_w is mass of the inertia wheel, I_1 is inertia of the pendulum about its end and I_w is inertia of the inertia wheel about the wheel center. The equilibrium set is given by $\mathcal{E} = \{x : (x_1, x_2, x_3, x_4) = (2n\pi, 2m\pi, 0, 0)\}$, $(n, m) \in \mathbb{Z}$.

Points to ponder

- Write the configuration space on which the Furuta pendulum evolves. How is it different from the configuration space of the pendulum on a cart system?

Lecture-13

Ball on a beam system

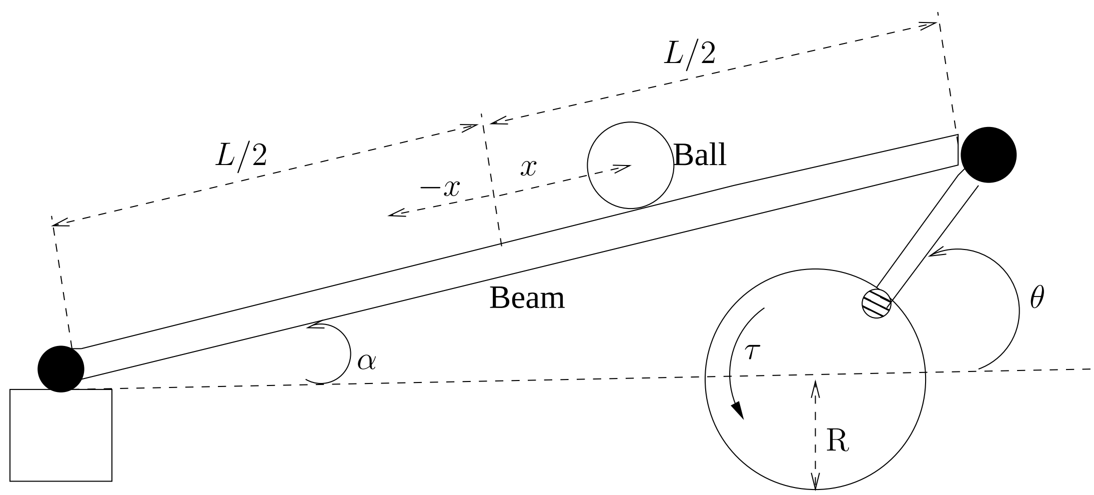

Figure 13.1: Ball and beam system

The system (see figure 13.1) consists of a ball that is free to move on a horizontal beam. The beam angle can be varied to control the position of the ball. Let α be the angle subtended by the beam with the horizontal and x denote the position of the ball measured with respect to the center of the beam. The connecting rod connects the gear to the right end of the beam (which is a rotary joint). Let θ be the angular distance covered by the gear, measured with respect to the horizontal. The ball is assumed to roll on the beam without slipping. It should also be noted that the approximation $R\theta = L\alpha$ holds for small angles.

Equations of Motion

The parameters of the system are depicted in table 13.1. The system has two degrees-

Parameter	Symbol
Length of the beam	L
Radius of the ball	r_b
Mass of the ball	M_b
Moment of inertia of the ball	J_b
Mass of the rod	M_r
Connecting rod displacement or radius of gear	R
Moment of inertia of the rod (about its pivot point)	J_r
Moment of inertia of the gear or flywheel	J_{fw}
Motor shaft torque	τ_m
Motor torque constant	K_t
Motor back-emf constant	K_m
Load torque	u_1
Motor input voltage	V_m
Motor current	I_m
Gear ratio	K_g
Gear efficiency	η_g
Motor armature resistance	R_m
Motor moment of inertia	J_m
Viscous damping coefficient on the load side	B_{eq}
Motor efficiency	η_m

Table 13.1: System parameters

of-freedom and is parameterized by the configuration variables (x, α). The Lagrangian $\mathcal{L}(x, \alpha, \dot{x}, \dot{\alpha})$, defined as the difference between the kinetic and potential energy is given by

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}J_r\dot{\alpha}^2 + \tfrac{1}{2}M_b\dot{x}^2 + \tfrac{1}{2}J_b{\omega_b}^2 + \tfrac{1}{2}M_b\left(\tfrac{L}{2}+x\right)^2\dot{\alpha}^2 + \tfrac{1}{2}J_{fw}\dot{\theta}^2 \\ &\quad -M_r g\tfrac{L}{2}\sin(\alpha) - M_b g\left(\tfrac{L}{2}+x\right)\sin(\alpha).\end{aligned}$$

The corresponding Euler-Lagrange equations of motion are given by

$$\begin{aligned}&a_1\ddot{\alpha} + a_2\left(\tfrac{L}{2}+x\right)^2\ddot{\alpha} + 2a_2\left(\tfrac{L}{2}+x\right)\dot{\alpha}\dot{x} + a_3\cos(\alpha)\\ &+a_2 g\left(\tfrac{L}{2}+x\right)\cos(\alpha) = u_1 - \left(\tfrac{B_{eq}L}{R}\right)\dot{\alpha}\end{aligned} \tag{13.1}$$

and

$$\frac{7}{5}\ddot{x} - \left(\frac{L}{2}+x\right)\dot{\alpha}^2 + g\sin(\alpha) = 0 \tag{13.2}$$

where, the constants $\frac{J_r R}{L} + \frac{J_{fw}L}{R} = a_1$, $\frac{M_b R}{L} = a_2$, $\frac{M_r R g}{2} = a_3$. The torque provided by the motor is represented by u_1.

Motor dynamics

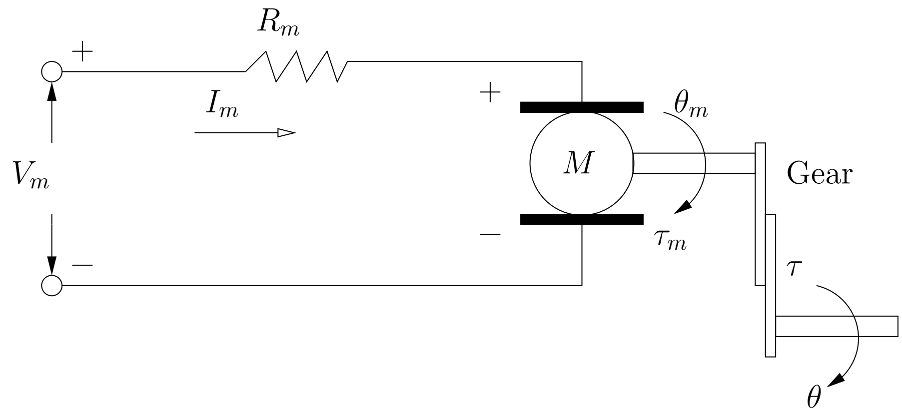

Figure 13.2: Motor circuit

The dynamics of the motor used to actuate the gear mechanism will be brought into the framework described by (13.1) and (13.2). This will enable one to apply a voltage to the motor as input rather than attempting to force the motor to provide the required torque u_1. The dynamics of the motor is given by

$$\begin{aligned}J_m\ddot{\theta}_m &= \tau_m - \tfrac{u_1}{\eta_g K_g}\\ \tau_m &= \eta_m K_t I_m\\ I_m &= \tfrac{V_m - K_m\dot{\theta}_m}{R_m}\\ \theta_m &= K_g\theta.\end{aligned} \tag{13.3}$$

From (13.3) and by re-arranging terms, the following is obtained

$$u_1 = c_1 V_m - c_2\dot{\alpha} - c_3\ddot{\alpha} \tag{13.4}$$

where, $c_1 = \frac{\eta_g K_g \eta_m K_t}{R_m}$, $c_2 = \frac{\eta_g K_t K_m {K_g}^2 L}{R R_m}$ and $c_3 = \frac{L \eta_g {K_g}^2 J_m}{R}$. Equation (13.4) is substituted in (13.1) to yield

$$\begin{aligned} &a_1\ddot{\alpha} + a_2\left\{\frac{L}{2}+x\right\}^2\ddot{\alpha} + 2a_2\left\{\frac{L}{2}+x\right\}\dot{\alpha}\dot{x} + a_3\cos(\alpha) \\ &+a_2 g\left\{\frac{L}{2}+x\right\}\cos(\alpha) = c_1 V_m - c_2\dot{\alpha} - c_3\ddot{\alpha} - \left\{\frac{B_{eq}L}{R}\right\}\dot{\alpha}. \end{aligned} \tag{13.5}$$

State-space formulation

The dynamics of the system given by (13.2) and (13.5) is expressed in the state-space form by choosing the state variables as $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \triangleq (x, \dot{x}, \alpha, \dot{\alpha})$. The resulting state-space model is given by

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})V_m$$

where, the drift and control vector fields are given by

$$f = \begin{bmatrix} x_2 \\ \frac{5}{7}\left\{\frac{L}{2}+x_1\right\}{x_4}^2 - \frac{5}{7}g\sin(x_3) \\ x_4 \\ \frac{-\left\{c_2+\frac{B_{eq}L}{R}\right\}x_4 - 2a_2\left\{\frac{L}{2}+x_1\right\}x_2x_4 - \left\{a_3+a_2 g\left\{\frac{L}{2}+x_1\right\}\right\}\cos(x_3)}{a_1+c_3+a_2\left\{\frac{L}{2}+x_1\right\}^2} \end{bmatrix}; g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{c_1}{a_1+c_3+a_2\left\{\frac{L}{2}+x_1\right\}^2} \end{bmatrix}.$$

Points to ponder

- Does the system have an equilibrium point?
- Derive the equations of motion of a centrally pivoted ball-beam system? Does this configuration have an equilibrium point?

Lecture-14

Ball on a circular beam system

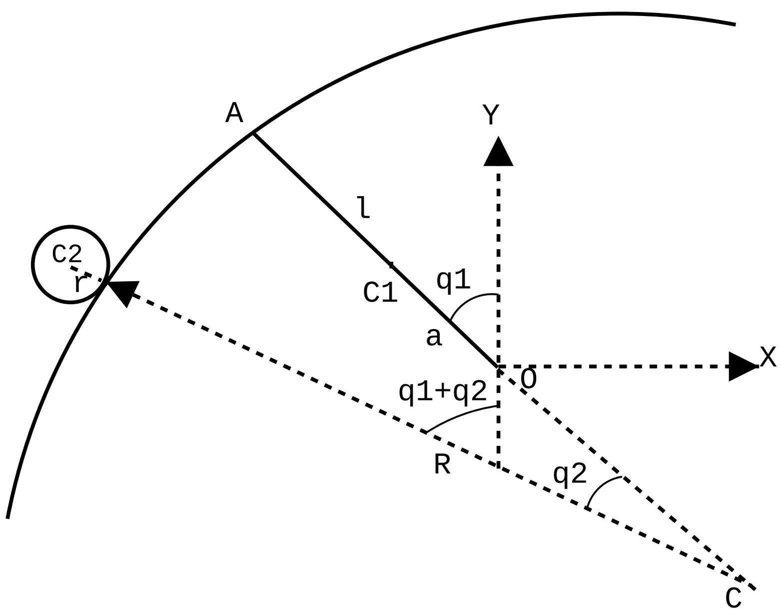

Figure 14.1: Schematic of circular beam and ball system

The circular beam and ball system consists of a circular beam with center C, radius R and a ball on it with center C_2 and radius r, as in Figure 14.1. The beam is actuated at the holder end O. C_1 is the center of mass of the beam with it's holder OA. Let m_1, ρ_1 and m_2, ρ_2 be the mass and radii of inertia of the beam with it's holder and ball respectively. Let l be the length of the holder, $OC_1 = a$ and g be the acceleration due to gravity. The kinetic energy is given by

$$K.E = \frac{m_1}{2} v_1^\top v_1 + \frac{m_2}{2} v_2^\top v_2$$

where,

$$
\begin{aligned}
v_1 &= \frac{d}{dt}\begin{pmatrix} -a\sin q_1 \\ a\cos q_1 \\ 0 \end{pmatrix} \\
v_2 &= \frac{d}{dt}\begin{pmatrix} -(R+r)\sin q_1+q_2+(R-l)\sin q_1 \\ (R+r)\cos q_1+q_2-(R-l)\cos q_1 \\ 0 \end{pmatrix}
\end{aligned}
$$

Defining the constants as,

$$
\begin{aligned}
a_1 &= m_1\rho_1^2+m_2(R-l)^2 \\
a_2 &= m_2(R+r)^2 \\
a_3 &= m_2(R+r)(R-l) \\
a_4 &= m_2(\rho_2R/r)^2 \\
b_1 &= (m_1a-m2(R-l))g \\
b_2 &= m_2(R+r)g
\end{aligned}
$$

the Lagrangian is given by

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}((a_1+a_2-2a_3\cos q_2)\dot{q_1}^2+(a_2+a_4)\dot{q_2}^2+2(a_2-a_3\cos q_2)\dot{q}_1\dot{q}_2) \\
&\quad -b_1\cos q_1+b_2\cos(q_1+q_2).
\end{aligned}
$$

Defining the states as $x=(x_1,x_2,x_3,x_4)\stackrel{\triangle}{=}(q_1,q_2,\dot{q}_1,\dot{q}_2)$ the equations of motion can be expressed as

$$\dot{x}=f(x)+g(x)u$$

where the drift and control vector fields are given by,

$$f=\begin{pmatrix} x_3 \\ x_4 \\ f_3 \\ f_4 \end{pmatrix};g=\begin{pmatrix} 0 \\ 0 \\ g_3 \\ g_4 \end{pmatrix}$$

with

$$
\begin{aligned}
f_3 &= \tfrac{1}{\delta(x_2)}\{(a_2+a_4)(-2a_3x_3x_4\sin x_2 + a_3x_4^2\sin x_2 - b_1\sin x_1 - b_2\sin(x_1+x_2)) \\
&\quad -(a_2 - a_3\cos(x_2))(a_3x_3^2\sin x_2 - b_2\sin(x_1+x_2))\} \\
f_4 &= \tfrac{1}{\delta(x_2)}\{(a_2 - a_3\cos x_2)(2a_3x_3x_4\sin x_2 - a_3x_4^2\sin x_2 + b_1\sin x_1 + b_2\sin(x_1+x_2)) \\
&\quad +(a_1a_2 - 2a_3\cos x_2)(a_3x_3^2\sin x_2 - b_2\sin(x_1+x_2))\} \\
g_3 &= (a_2+a_4)/\delta(x_2) \\
g_4 &= (-a_2 + a_3\cos x_2)/\delta(x_2) \\
\delta(x_2) &= a_1a_2 + a_1a_4 + a_2a_4 - 2a_3a_4\cos x_2 - a_3^2\cos {x_2}^2
\end{aligned}
$$

and $u = \tau$.

The cart on a beam system

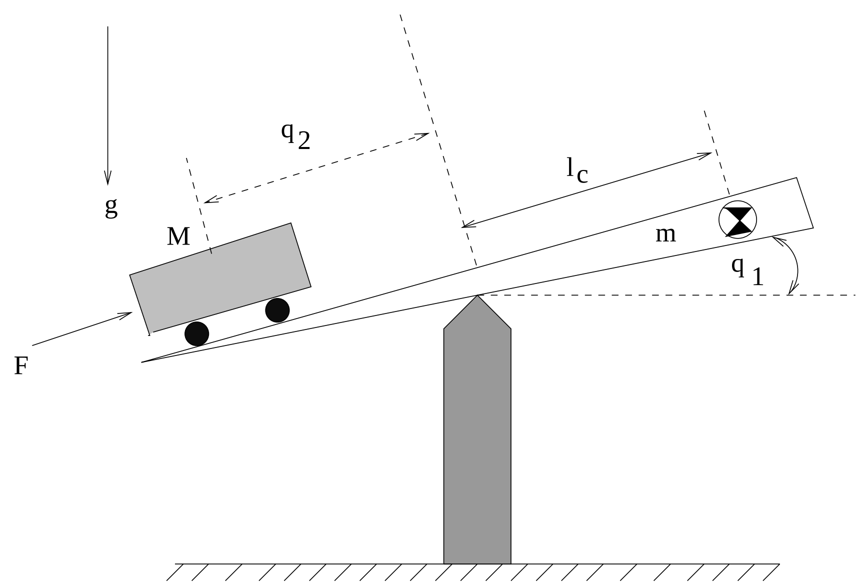

Figure 14.2: Schematic of the cart-beam system

The system consists of a beam of length l and mass m and the center of the beam is free to rotate on a pivot under the influence of gravity as shown in figure 14.2. The beam geometry is such that the center-of-mass is at a distance l_c from the center of the beam. A cart of mass M, actuated by a force F, traverses on the beam without slipping. The total energy of the CB system is given by

$$E(q,p) = \frac{1}{2}\dot{q}^\top D(q_2)\dot{q} + \left(q_2 + \frac{ml_c}{M}\right) M\ g\sin q_1 \tag{14.1}$$

where, $D(q) \triangleq \text{diag}\{J + Mq_2^2, M\}$, J denotes the moment-of-inertia of the beam about its pivot, q_1 is the angle made by the beam with the horizontal, $\dot{q}_1$ is the angular velocity, q_2 is the cart position, $\dot{q}_2$ is the cart velocity and g is the acceleration due to gravity. By choosing the states as $x \triangleq [x_1\ x_2\ x_3\ x_4]^\top = [q_1 \quad q_2 + \frac{ml_c}{M} \quad \dot{q}_1 \quad \dot{q}_2]^\top$, the Euler-Lagrange formulation of the CB system yields the following state-space representation

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{14.2}$$

where,

$$f(x) = \begin{pmatrix} x_3 \\ x_4 \\ \frac{(-2Mx_3x_4\alpha(x_2) - Mgx_2\cos x_1)}{\beta(x_2)} \\ x_3^2\alpha(x_2) - g\sin x_1 \end{pmatrix}; g(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where, $u \triangleq \frac{F}{M}, \alpha(x_2) \triangleq (x_2 - ml_c/M), \beta(x_2) \triangleq J + M\alpha^2(x_2) > 0\ \forall\ x_2$. If the beam angle is restricted to $(-\pi/2, \pi/2)$ to keep the cart in the upward configuration, then the system has a unique equilibrium point $x_e \triangleq (0, 0, 0, 0)$.

Points to ponder

- Linearize the system about the origin and find its eigen values. Compare the number of positive eigenvalues for the ball on a horizontal beam system with the ball on a circular beam system.
- Rewrite the dynamics for cart on a beam system when the center-of-mass of the beam coincides with the pivot point. Does the equilibrium point shift?

Lecture-15

Acrobot

The acrobot is a two-link manipulator that moves in a vertical plane under the influence of gravity. The manipulator is actuated by a single actuator at the elbow joint as shown in Figure 15.1. The equations of motion of the acrobot configuration space $Q = \mathbb{S}^1 \times \mathbb{S}^1$

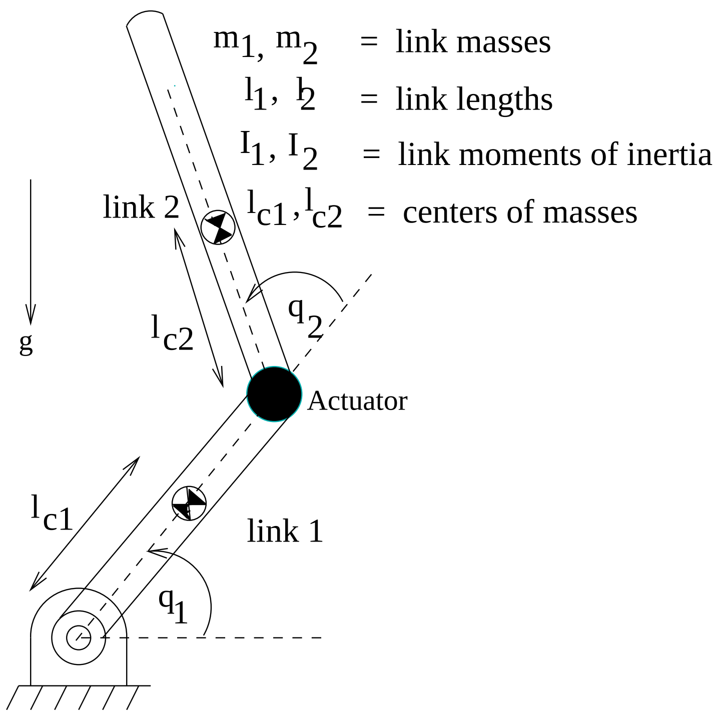

Figure 15.1: The Acrobot

are given by

$$
\begin{aligned}
d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + h_1 + \psi_1 &= 0 \\
d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + h_2 + \psi_2 &= \tau_2
\end{aligned}
\tag{15.1}
$$

where

$$
\begin{aligned}
d_{11} &= c_1 + c_2 + 2c_3 \cos q_2 \\
d_{12} &= c_2 + c_3 \cos q_2 \\
d_{21} &= d_{12} \\
d_{22} &= c_2 \\
h_1 &= -c_3(2\dot{q}_1\dot{q}_2 + \dot{q}_2{}^2) \sin q_2 \\
h_2 &= c_3\dot{q}_1{}^2 \sin q_2 \\
\psi_1 &= c_4 g \cos q_1 + c_5 g \cos(q_1 + q_2) \\
\psi_2 &= c_5 g \cos(q_1 + q_2).
\end{aligned}
$$

and

$$
\begin{aligned}
c_1 &= m_1 l_{c1}^2 + m_2 l_1^2 + I_1 \\
c_2 &= m_2 l_{c2}^2 + I_2 \\
c_3 &= m_2 l_1 l_{c2} \\
c_4 &= m_1 l_{c1} + m_2 l_1 \\
c_5 &= m_2 l_{c2}.
\end{aligned}
$$

Equation (15.1) can be rewritten as Defining the state vector as $x = (x_1, x_2, x_3, x_4) \triangleq ((q_1 - \frac{\pi}{2}), q_2, \dot{q}_1, \dot{q}_2)$, the dynamics can be represented as

$$
\dot{x} = f(x) + g(x)u \tag{15.2}
$$

where the state-space manifold is $M = \mathbb{S}^1 \times \mathbb{S}^1 \times I\!R^2$ and the smooth vector fields f and g are given by

$$
f = \begin{pmatrix} x_3 \\ x_4 \\ f_3 \\ f_4 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 0 \\ \frac{-(c_2 + c_3 \cos x_2)}{\gamma(x_2)} \\ \frac{(c_1 + c_2 + 2c_3 \cos x_2)}{\gamma(x_2)} \end{pmatrix}
$$

where

$$
\begin{aligned}
f_3 &= \frac{1}{\gamma(x_2)}\{c_2c_3(x_3+x_4)^2\sin x_2 + c_2c_4g\sin x_1 + c_3^2x_3^2\cos x_2\sin x_2 \\
&\quad -c_3c_5g\sin(x_1+x_2)\cos x_2\} \\
f_4 &= \frac{1}{\gamma(x_2)}\{-(c_2+c_3\cos x_2)(2x_3x_4+x_4^2)c_3\sin x_2 - c_2c_4g\sin x_1 \\
&\quad -c_3c_4g\cos x_2\sin x_1 + c_1c_5g\sin(x_1+x_2) + c_3c_5g\cos x_2\sin(x_1+x_2) \\
&\quad -(c_1+c_2+2c_3\cos x_2)x_3^2c_3\sin x_2\} \\
\gamma(x_2) &= c_1c_2 - c_3^2\cos^2 x_2
\end{aligned}
$$

and $u = \tau_2$. The system has four equilibrium points given by

$$\{x_{ij}^e \in M : x_1 = -i\pi, x_2 = j\pi, x_3 = 0, x_4 = 0\} \quad i,j = 0,1.$$

A variant of the acrobot is the pendubot, the difference being that the pendubot is actuated only at the shoulder joint.

PPR manipulator

This manipulator has two prismatic joints which are actuated, while the third joint is a revolute passive joint, hence the name PPR. The two prismatic joints are orthogonal. Let (x, y) denote the position of the free joint and θ the orientation of the third link. It is clear that the configuration space of a PPR manipulator is $Q = I\!R \times I\!R \times \mathbb{S}^1$ and is parameterized by the coordinates $q = [x \quad y \quad \theta]$. Let F_1 and F_2 be the control inputs to the two prismatic joints. Also, denote by l the distance between the center of mass of the third link and the third joint. The equations of motion using the Newtonian approach are

$$
\begin{aligned}
m_x\ddot{x} - m_3l\dot{\theta}^2\cos\theta - m_3l\ddot{\theta}\sin\theta &= F_1 \\
m_y\ddot{y} - m_3l\dot{\theta}^2\sin\theta + m_3l\ddot{\theta}\cos\theta &= F_2 \\
I\ddot{\theta} + m_3l\ddot{y}\cos\theta - m_3l\ddot{x}\sin\theta &= 0,
\end{aligned} \tag{15.3}
$$

where $m_x = m_1 + m_2 + m_3, m_y = m_2 + m_3$ and $I = I_3 + m_3l^2$. Clearly all points $q \in Q$ constitute the equilibrium configuration at zero control inputs.

Points to ponder

- Identify the Coriolis and centripetal forces?

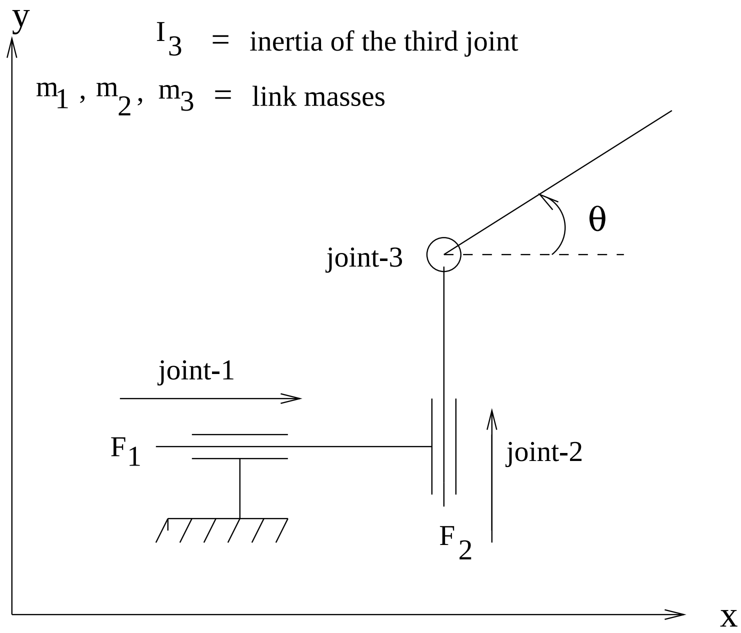

Figure 15.2: PPR planar manipulator

- Institutively, which of the equilibrium points is stable?
- Write the input vector G for the pendubot?
- Linearize the PPR system at the origin. Check for controllability of the linearized system using Kalman's rank condition. You will observe that the rank condition fails and hence the system is not completely controllable. What is the cause for the loss of controllability for the linearized system?

Lecture-16

Mobile robot

The mobile robot is a popular robotic system that finds application in a multitude of tasks and is also a challenging problem from a control point-of-view. The kinematic model of the mobile robot as shown in figure 16.1 can be expressed as

$$\begin{aligned}\dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega\end{aligned} \tag{16.1}$$

where, the triple (x, y, θ) denotes the position and the orientation of the vehicle with respect to the inertial frame and v, ω are the linear and angular velocities of the mobile robot. The dynamic model is obtained using the following relations.

$$\begin{aligned}M\dot{v} &= F \\ I\dot{\omega} &= \tau\end{aligned} \tag{16.2}$$

where, M is the mass of the vehicle, I is moment-of-inertia, $\tau = \frac{L}{r}(\tau_1 - \tau_2)$, $F = \frac{1}{r}(\tau_1 + \tau_2)$, with L being the distance between the center-of-mass and the wheel, τ_1, τ_2, the left and right wheel motor torques and r is the radius of the rear wheel. Two assumptions are made to simplify the model: the center-of-mass and the rear-axis center coincide and the wheels do not slide. The second assumption leads to a velocity level nonholonomic constraint $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$. With the following choice of state-vector

$$\begin{aligned}x_1 &= \theta \\ x_2 &= x\cos\theta + y\sin\theta \\ x_3 &= (\theta y - 2x)\sin\theta + (2y + \theta x)\cos\theta \\ x_4 &= \omega \\ x_5 &= v - \omega(x\sin\theta - y\cos\theta),\end{aligned}$$

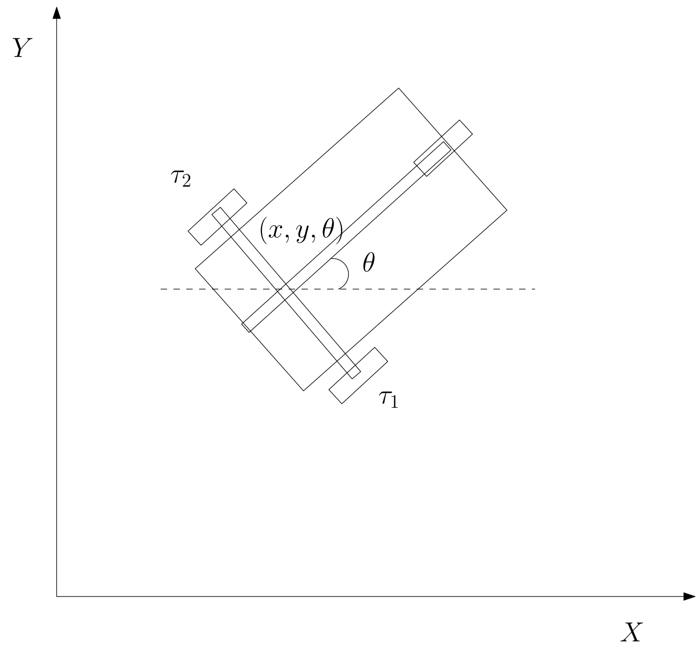

Figure 16.1: Mobile robot

equations (16.1) - (16.2) can be combined to yield a state-space representation on the manifold $S^1 \times I\!R^4$ as

$$\dot{x} \;=\; f(x) + g_1(x_1)\tau + g_2(x)F \tag{16.3}$$

where

$$f(x) = \begin{pmatrix} x_4 \\ x_5 \\ x_1x_5 - x_2x_4 \\ 0 \\ -x_4^2x_2 \end{pmatrix}; g_1(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ k_1 \\ -\frac{k_1(x_1x_2-x_3)}{2} \end{pmatrix}; g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ k_2 \end{pmatrix}, k_1 \triangleq \frac{1}{I}, k_2 \triangleq \frac{1}{M}.$$

Points to ponder

- Rewrite the kinematic model of the mobile robot assuming that the robot has a differential drive. The model should have the wheel angular velocities (ω_r, ω_l) as control inputs.
- For the differential drive model derived, analyze the motion of the robot for the following control inputs. a.) When $\omega_r = -\omega_l$, and $|\omega_r|$ is a constant.
  b.)When $\omega_r - \omega_l = 0$, and $|\omega_r|$ and $|\omega_l|$ are constants, but unequal.

Lecture-17

Bioreactor Model

A bioreactor is a tank in which several biological reactions occur simultaneously in a liquid medium. In the simplest reactor two components are considered, biomass and substrate. Biomass consists of cells that consume the substrate. Consider the schematic of a biochemical reactor in Figure 17.1, where X and S are the cell and substrate concentrations, F is the feed flow rate of substrate (assuming no biomass in feed) and V the volume of the reactor. The continuous time representation of the bioreactor system is:

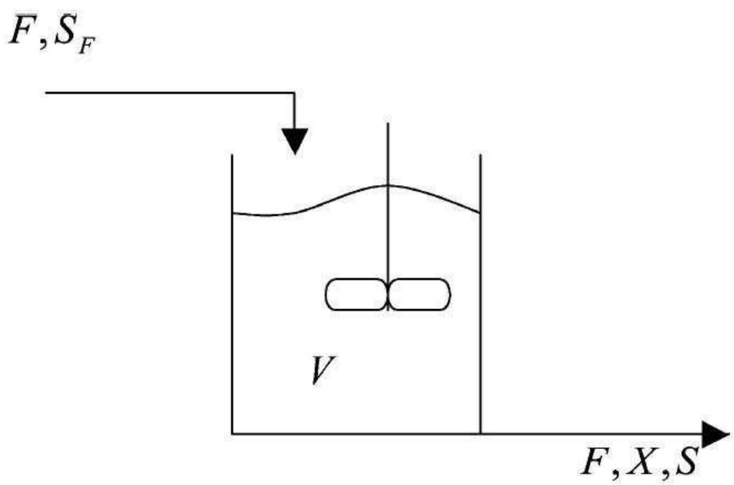

Figure 17.1: Bioreactor

$$\begin{aligned} \frac{dX}{dt} &= -\frac{F}{V}X + \mu(S) \\ \frac{dS}{dt} &= -\frac{F}{V}(S_F - S) - \sigma(S)X \end{aligned} \tag{17.1}$$

where S is the substrate concentration and X is the cell concentration. The growth model where all of the biomass cells are "lumped together" in a single (one-hump) model

$$\mu(S) = KSe^{-\frac{S}{K}} \tag{17.2}$$

is used to represent the substrate growth rate as a function of biomass with K as the growth model coefficient. The specific substrate consumption rate, σ, is related to the yield coefficient Y in the form

$$Y(S) = \frac{\mu(S)}{\sigma(S)} = a + bS \tag{17.3}$$

where a and b are positive constants used to approximate the increasing portion of the yield curve. Combining (17.1)-(17.3), the dimensionless cell and substrate mass balances can be represented as

$$\begin{aligned} \frac{dC_1}{d\tau} &= -C_1 + Da(1 - C_2)e^{\frac{c_2}{\gamma}}C_1 \\ \frac{dC_2}{d\tau} &= -C_2 + Da\frac{1+\beta}{1+\beta-C_2}(1 - C_2)e^{\frac{c_2}{\gamma}}C_1 \end{aligned} \tag{17.4}$$

where C_1 and C_2 are the dimensionless cell mass and substrate conversion, respectively. Brengel and Seider transformed (17.4) to yield the following system of equations incorporating the manipulated input variable, u which is the dimensionless feed flow rate.

$$\begin{aligned} \frac{dX_1}{d\tau_S} &= -(C_1S + X_1)(1 + u) + Da_S(1 - X_2 - C_{2S})e^{\frac{(c_{2S}+X_2)}{\gamma}}(C_{1S} + X_1) \\ \frac{dX_2}{d\tau_S} &= -(C_2S + X_2)(1 + u) + Da_S(1 - X_2 - C_{2S})e^{\frac{(c_{2S}+X_2)}{\gamma}}\frac{1+\beta}{1+\beta-X_2-C_{2S}}(C_{1S} + X_1) \end{aligned} \tag{17.5}$$

where τ_S represents dimensionless time under steady state conditions Da_S, the Damkohler number at the steady state, X_1 the dimensionless cell mass deviation from steady state and X_2 is the corresponding dimensionless substrate conversion deviation.

DC motor

The traditional model of DC motor is a second-order linear one, which ignores the dead nonlinear zone of the motor. Unfortunately, the dead zone caused by the nonlinear friction would bring great effect to servo systems. Therefore, it is vital to model the dead zone of DC motor accurately in order to improve the performance of servo system. In order to simplify applications and reflect the real nonlinear friction of the motor accurately, a simplified friction model was proposed by Cong S. et al. , which is expressed as

$$T_f = T_c\mathrm{sgn}(\omega) + (T_s - T_c)e^{(-\alpha|\omega|)}\mathrm{sgn}(\omega) \tag{17.6}$$

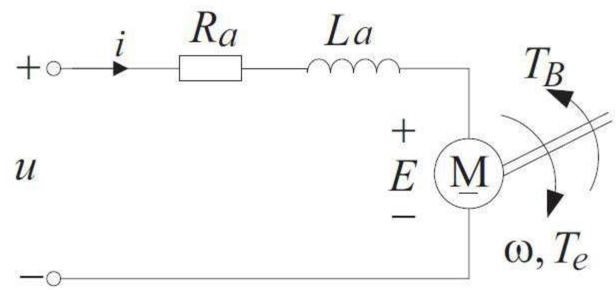

Figure 17.2: DC motor

Where, T_c is the Coulomb friction torque (Nm); T_s is static friction torque (Nm); α is time constant; ω is the angular speed of the rotor (rad/s). The DC motor equivalent circuit under rating excitation is as shown in Figure 17.2. The voltage balance equation of DC motor armature circuit is expressed as

$$u = R_a i + L_a \frac{di}{dt} + K_e \omega$$

where, i is armature current (A); L_a is equivalent inductance of armature circuit (H); R_a is equivalent resistance of armature circuit (Ω); u is terminal voltage of armature circuit (V); K_e is voltage coefficient of DC motor (Vs/rad). The torque balance equation of DC motor is expressed as

$$K_t i - B\omega = J \frac{d\omega}{dt}$$

where, J is the inertia moment of the rotor (Kgm^2); K_t is the torque coefficient of DC motor (Nm/A). B is viscous friction coefficient of DC motor (Nms/rad). The linear DC motor model is obtained as

$$\begin{aligned} J\dot{\omega} &= -B\omega + K_t i \\ L_a \dot{i} &= -K_e \omega - R_a i + u \end{aligned}$$

Considering the effect of nonlinear friction, the nonlinear friction torque is applied to model the dead zone of DC motor. Then the nonlinear DC motor model is shown as

$$\begin{aligned} J\dot{\omega} &= -B\omega + K_t i - T_c \mathrm{sgn}(\omega) - (T_s - T_c) e^{(-\alpha|\omega|)} \mathrm{sgn}(\omega) \\ L_a \dot{i} &= -K_e \omega - R_a i + u \end{aligned} \tag{17.7}$$

The DC motor model is transformed into second-order nonlinear model from traditional second-order linear one due to the presence of nonlinear friction torque, which makes the model of DC motor more accurate.

Exercise problems

1. Using suitable states, represent the Furuta pendulum dynamics in state-space form and find the equilibrium set.

2. Find the equilibrium set for the ball on a circular beam system.

3. The dynamics of a DC motor can be described by

$$\begin{aligned} v_f &= R_f i_f + L_f \dot{i_f} \\ v_a &= c_1 i_f \omega + L_a \dot{i_a} + R_a i_a \\ J\dot{\omega} &= c_2 i_f i_a - c_3 \omega \end{aligned}$$

   The first equation is for the field circuit with v_f, R_f and L_f being its voltage, current, resistance and inductance. The variables v_a, R_a and L_a are the corresponding variables for the armature circuit described by the second equation. The third equation is a torque equation for the shaft, with J as the rotor inertia and c_3 as a damping coefficient. The term $c_1 i_f \omega$ is the back e.m.f induced in the armature circuit, and $c_2 i_f i_a$ is the torque produced by the interaction of the armature current with the field circuit flux.

   a. For a separately excited D.C motor, the voltages v_a and v_f are independent control inputs. Choose appropriate state variables and find the state equation.

   b. Specialize the state equation of part (a) to the field controlled DC motor, where v_f is the control input while v_a is held constant.

   c. Specialize the state equation of part (a) to the armature controlled DC motor, where v_a is the control input while v_f is held constant. Can you reduce the order of the model in this case?

d. In a shunt wound DC motor, the field and armature windings are connected in parallel and an external resistance R_x is connected in series with the field winding to limit the field flux; that is, $v = v_a = v_f + R_x i_f$. With v as the control input, write down the state equation.

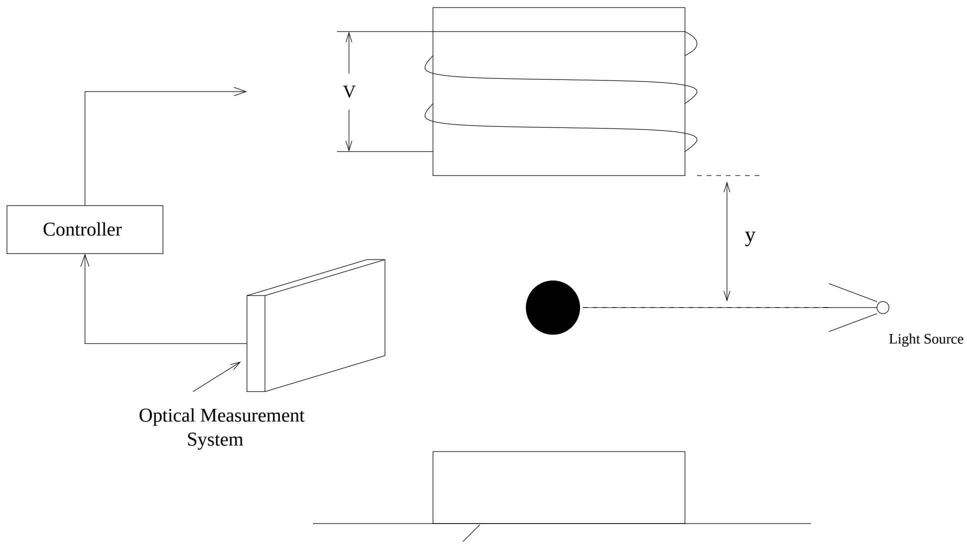

Figure 17.3: Magnetic suspension system

4. The schematic diagram of a magnetic suspension system (see Figure (17.3)), where a ball of magnetic material is suspended by means of an electromagnet whose current is controlled by feedback from the optically measured ball position. This system has the basic ingredients of systems constructed to levitate mass, used in gyroscopes, accelerometers and fast trains. The equation of motion of the ball is $m\ddot{y} = -k\dot{y} + mg + F(y, i)$, where m is the mass of the ball, $y \geq 0$ is the vertical (downward) position of the ball measured from a reference point ($y = 0$ when the ball is next to the coil), k is a viscous friction coefficient, g is the acceleration due to gravity, $F(y, i)$ is the force generated by the electromagnet, and i is its electric current. The inductance of the electromagnet depends on the position of the ball and can be modeled as $L(y) = L_1 + L_0/(1 + y/a)$ where L_1, L_0 and a are positive constants. This model represents the case that the inductance has its highest value when the ball is next to the coil and decreases to a constant value as the ball is removed to $y = \infty$. With $E(y, i) = 0.5L(y)i^2$ as the energy stored in the electromagnet, the force $F(y, i)$ is given by $F(y, i) = -\frac{L_0 i^2}{(2a(1+y/a)^2)}$. When the electric circuit of the coil is driven by a voltage source with voltage v, Kirchoff's voltage law gives the relationship $v = \dot{\phi} + Ri$, where R is the series resistance of the circuit and $\phi = L(y)i$

is the magnetic flux linkage.

a. Using $x_1 = y, x_2 = \dot{y}, x_3 = i$ as state variables and $u = v$ as control input, find the state equation.

b. Suppose it is desired to balance the ball at a certain position $r > 0$. Find the steady state values I_{ss} and V_{ss} of i and v, respectively, which are necessary to maintain such balance.

5. Consider a magnetic beam balance (MBB) system, that consists of an off-centered beam of length l free to rotate on a pivot, under the influence of gravity, as shown in Figure 17.4.

   The beam geometry is such that the center-of-mass is at a distance l_c from the center of the beam. An electromagnet A is placed below one end of the beam and a dummy electromagnet B is placed at the other end to incorporate physical constraints into the system and obtain a symmetrical arrangement.

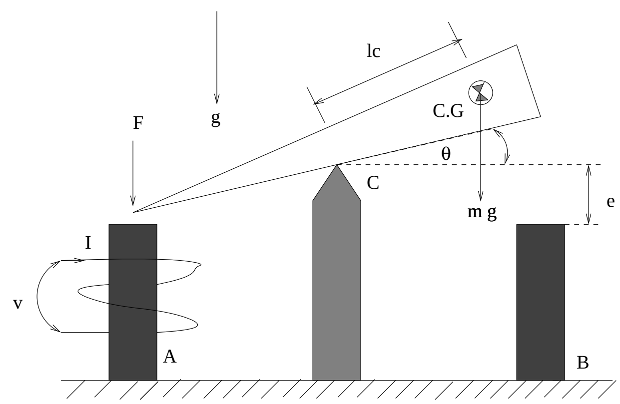

Figure 17.4: Schematic of the beam-balance system

Let J, m, l_c denote the moment-of-inertia, mass and distance between the pivot and center-of-mass of the beam, θ is the angle made by the beam with the horizontal, $\dot{\theta}$ is the angular velocity, g is the acceleration due to gravity and F is the force created by the electromagnet. Further, let l denote the length of the beam and e the air-gap. The electromagnetic circuit is given by

$$v = L\dot{I} + I\dot{L} + IR \tag{17.8}$$

where R and L are the resistance and inductance of the coil respectively, I is the current and v is the voltage applied across the coil terminals. The inductance L is related to the beam angle as $L = \frac{L_0 \theta_m}{\theta_m + \theta}$, where L_0 is the inductance of the coil when $\theta = 0$ and θ_m is the maximal angle that is reached when one end of the beam touches the electromagnet and is given by $\theta_m = \arctan(2\, e/l)$. Finally the expression for the force F is given by

$$F = \lambda \left(\frac{\theta_m I}{\theta_m + \theta} \right)^2 \tag{17.9}$$

where λ is a positive constant. In arriving at the above equations, it is assumed that the permeability of the magnetic material is constant over the operating range and the flux density is uniformly distributed across the air-gap. Also, the effect of eddy current, stray flux and flux leakage are assumed to be small that they can be neglected. Find the state-space representation of the system with a) with F as the control input , (b) with the voltage v as the current input.

6. What is the number of isolated equilibrium points for an n-link manipulator moving in a vertical plane under the influence of gravity?

Module 4

Second-order systems

Objectives: To anlayze second-order systems through the process of linearization about an equilibrium point and to classify the nature of the equilibrium point based on the eigen spectrum of the system matrix.

Lesson objectives

This module helps the reader in

- appreciating the first approximation in the Taylor series expansion of a nonlinear function as the first tool in nonlinear analysis
- obtaining Canonical transformation of linear systems
- classifying the various types of equilibrium points by computing the eigen values of the linearized system matrix.

Lecture-18

The study of second-order systems is of importance as it is easy to visualize the behaviour in the plane. A second-order time-invariant system is represented as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \tag{18.1}$$

Let $x(t) = (x_1(t), x_2(t))$ be the unique solution of (18.1) with the initial condition $(x_1(0), x_2(0)$. The plane formed by taking the variable x_1 as the x-axis and x_2 as the y-axis is called as the *state-plane*. The plot of $x_1(t)$ versus $x_2(t)$ as t varies over $[0, \infty)$ is called *state-plane trajectory* or *orbit* of the system (18.1). The family of all trajectories or orbits in the state-plane is called the *phase-portrait* of (18.1). In practice, an approximate phase-portrait is drawn by choosing a large number of initial conditions spread over the (x_1, x_2) plane.

The time t is suppressed in a trajectory and as a result it is not possible to recover the trajectory $(x_1(t), x_2(t))$ from the phase-portrait. Hence, the phase-portrait gives only a qualitative behaviour of the trajectory. To know the quantitative behaviour of the trajectory, we need to explicitly know the solution $x(t)$. In much of the nonlinear analysis, we are interested in the qualitative behaviour of the system.

Linearization of nonlinear system

A first step in nonlinear analysis involves the linearization a given nonlinear system about an equilibrium point of interest and then study the resulting linear system. The linear models are obtained by collecting the linear terms in the Taylor series expansion of the right-hand side of the differential equation. Denote by x_e the isolated equilibrium point of interest of (18.1). The linearization of (18.1) about an equilibrium point (x_e) is given by

$$\dot{z} = Az \tag{18.2}$$

where $z \triangleq x - x_e$ and $A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x_e}$. This process is also termed as Jacobian linearization since the partial of a vector f w.r.t x is called the Jacobian of f w.r.t x. In the following sequel, we will assume without loss of generality, the equilibrium x_e as the origin.

Example 18.0.1 *Find the equilibrium set of the following system and linearize the system about each of the equilibrium points. Also compute the eigenvalues of the linearized system matrix.*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_1 x_2 \end{aligned} \tag{18.3}$$

It is easy to compute that the system has an isolated equilibrium point at the origin. The linearized model about the origin is

$$\dot{x} = \left.\begin{bmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} \\ \frac{\partial(-x_1+x_1x_2)}{\partial x_1} & \frac{\partial(-x_1+x_1x_2)}{\partial x_2} \end{bmatrix}\right|_{x_e=(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The eigenvalues of the system matrix are given by $\lambda_1 = i, \lambda_2 = -i$.

Lecture-19

Depending on the eigenvalue of A, denoted by λ_1, λ_2, there exists a coordinate transformation $x = M^{-1}z$, such that in the coordinate x, the linear system $\dot{x} = \hat{A}x$, where $\hat{A} \triangleq M^{-1}AM$ takes on the following canonical forms:

Case 1: λ_1 and λ_2 are real and distinct

In this case the transformation M is given by $[v_1 \;\; v_2]$, where v_1 and v_2 are the eigenvectors associated with the eigenvalues λ_1 and λ_2 respectively. The corresponding canonical form is

$$\dot{x} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x. \tag{19.1}$$

The solution of (19.1) to the initial condition $(x_1(0), x_2(0))$ is given by

$$\begin{aligned} x_1(t) &= e^{\lambda_1 t} x_1(0) \\ x_2(t) &= e^{\lambda_2 t} x_2(0) \end{aligned}$$

We can solve for time t from the solution $x(t)$ as

$$x_2(t) = k x_1(t)^{\lambda_2/\lambda_1} \tag{19.2}$$

where, $k \triangleq \frac{x_2(0)}{x_1(0)^{\lambda_2/\lambda_1}}$. A family of curves can be generated using (19.2) by varying k over $\mathbb{R}$. The direction of the curves is based on the signs of the eigenvalues, which naturally leads to the following sub-cases:

Case 1a : $\lambda_2 < \lambda_1 < 0$

It is clear that the trajectory $x(t)$ approaches zero as time t approaches infinity. Since λ_2 is more negative than λ_1, λ_2 is termed as fast eigenvalue while λ_1 is termed as slow eigenvalue. In other words, $x_2(t)$ approaches zero faster than $x_1(t)$. The slope of the curve (19.2) is given by

$$\frac{dx_2}{dx_1} = k\frac{\lambda_2}{\lambda_1}x_1^{((\lambda_2/\lambda_1)-1)} \tag{19.3}$$

where $\frac{\lambda_2}{\lambda_1} > 1$ which implies $((\lambda_2/\lambda_1) - 1) > 0$. The slope (19.3) approaches zero as $|x_1| \longrightarrow 0$ and approaches ∞ as $|x_1| \longrightarrow \infty$ and thus all trajectories approach the origin a shown in Figure 19.1. The equilibrium point $x = 0$ is called a *stable node*. When

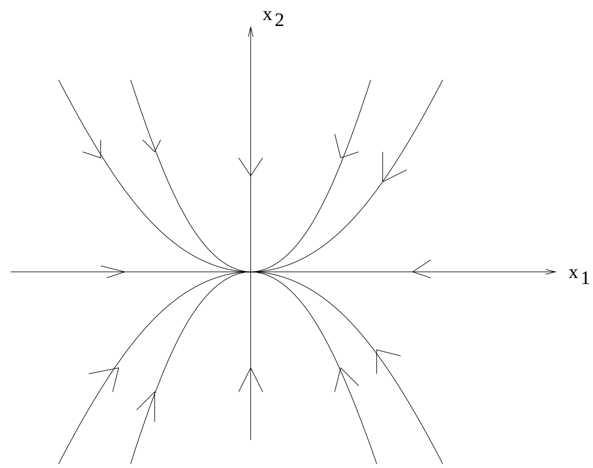

Figure 19.1: Phase-portrait of a stable node

the eigenvalues are both positive, say, $\lambda_2 > \lambda_1 > 0$, the trajectories approach infinity as $t \longrightarrow \infty$ and the origin is a *unstable node* as shown in Figure 19.2.

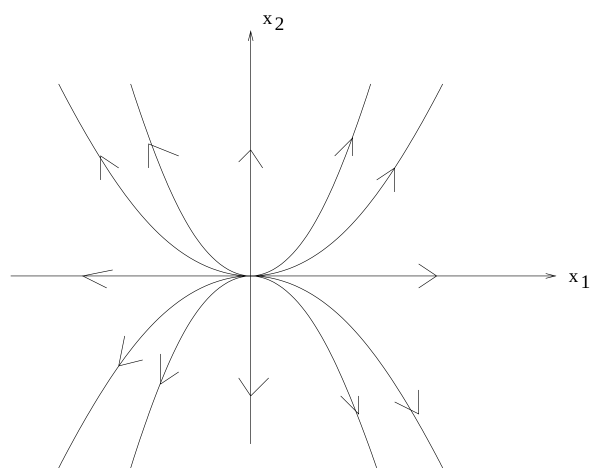

Figure 19.2: Phase-portrait of a unstable node

Case 1b : $\lambda_2 < 0 < \lambda_1$

In this case, λ_2 is a stable eigenvalue and λ_1 is unstable. Thus, $\lim_{t \longrightarrow \infty} x_1(t) \longrightarrow \infty$ and $\lim_{t \longrightarrow \infty} x_2(t) \longrightarrow 0$, that is, the trajectories have hyperbolic shapes. From (19.2), in view of $(\lambda_2/\lambda_1) < 0$, trajectories originating in the $I\!R^2 \setminus \{x_1 = 0, x_2 = 0\}$ become tangent to x_1-axis as $|x_1| \longrightarrow \infty$ and tangent to x_2-axis as $|x_1| \longrightarrow 0$. Along the x_2-axis, the trajectories approach the origin, while along the x_1-axis, the trajectories approach infinity. In this case, the origin, which is unstable, is termed as *saddle* node. A typical vector field plot about a saddle node is shown in Figure 19.3.

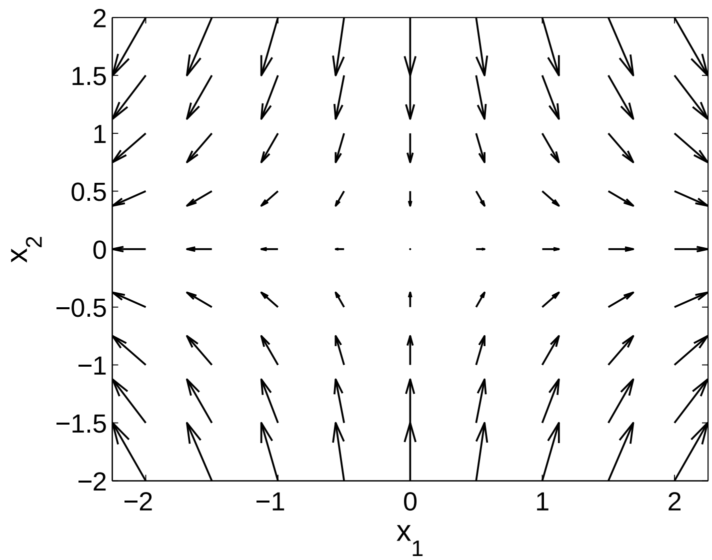

Figure 19.3: Phase-portrait of a saddle node

Case 2 : $\lambda_1 = \lambda_2 = \lambda$ are real

We will consider the case when the algebriac multiplicity (AM) is two and the geometric multiplicity (GM) is one (when AP=GP, we are lead to the case where the normal form is diagonal). Here, the transformation M is given by $[v_1 \;\; \bar{v}_1]$, where v_1 is the eigenvector associated with the eigenvalues λ and $\bar{v}_1$ is the generalized eigenvector associated with λ. The corresponding canonical form, with $\hat{A}$ termed the Jordon matrix is

$$\dot{x} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} x. \tag{19.4}$$

The solution is given by

$$\begin{aligned} x_1(t) &= x_{10}e^{\lambda t} + x_{20}te^{\lambda t} \\ x_2(t) &= x_{20}e^{\lambda t} \end{aligned}$$

The time t can be suppressed in the previous expression to obtain

$$x_1 = \frac{x_{10}}{x_{20}}x_2 + \frac{x_2}{\lambda}\ln\left(\frac{x_2}{x_{20}}\right)$$

If $\lambda > 0$, then $\lim_{t\longrightarrow\infty} x(t) \longrightarrow \infty$ and hence the equilibrium is an unstable node. A typical vector field plot of an unstable system with repeated eigenvalue at $\lambda = 2$ is shown in Figure 19.4.

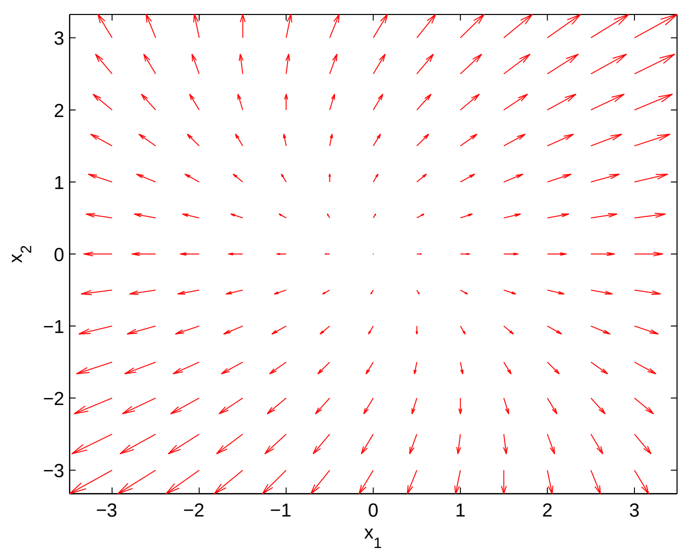

Figure 19.4: Vector field plot of a unstable node

Lecture-20

Case 3: λ_1 and λ_2 are complex

Consider (18.2) with complex eigen values $\alpha \pm j\beta$. It is first desired to find a suitable transformation $z = M^{-1}x$, such that the transformed system $\dot{x} = \tilde{A}x$, where $\tilde{A} = MAM^{-1}$ has the form $\tilde{A} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, then

$$\begin{aligned} x_1 &= m_{11}z_1 + m_{12}z_2 \\ x_2 &= m_{21}z_1 + m_{22}z_2 \end{aligned} \tag{20.1}$$

Differentiating (20.1), we have

$$\begin{aligned} m_{11}a_{11} + m_{12}a_{21} - \alpha m_{11} - \beta m_{21} &= 0 \\ m_{11}a_{12} + m_{12}a_{22} - \alpha m_{12} - \beta m_{22} &= 0 \\ m_{21}a_{11} + m_{22}a_{21} - \alpha m_{21} + \beta m_{11} &= 0 \\ m_{21}a_{12} + m_{22}a_{22} - \alpha m_{22} + \beta m_{12} &= 0. \end{aligned}$$

The above equations can be expressed in matrix form as

$$\begin{bmatrix} (a_{11}-\alpha) & a_{21} & -\beta & 0 \\ a_{12} & (a_{22}-\alpha) & 0 & -\beta \\ \beta & 0 & (a_{11}-\alpha) & a_{21} \\ 0 & \beta & a_{12} & (a_{22}-\alpha) \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{21} \\ m_{22} \end{bmatrix} = 0.$$

The homogeneous equation (3) can be rewritten as

$$\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \begin{bmatrix} M1 \\ M2 \end{bmatrix} = 0 \tag{20.2}$$

where,

$$X = \begin{bmatrix} (a_{11}-\alpha) & a_{21} \\ a_{12} & (a_{22}-\alpha) \end{bmatrix}$$

$Y = \beta I$; $M_1 = [m_{11} \ \ m_{12}]^\top$ and $M_2 = [m_{21} \ \ m_{22}]^\top$. Note that $Y^{-1} = \frac{1}{\beta} I$ and from we have from (20.2) $M_2 = Y^{-1} X M_1$ and $(\beta^2 I + X^2) M_1 = 0$.

In other words, $M_1 \in Kernel(\beta^2 I + X^2)$. Finally, using the invariance of the eigenvalues under similarity transformation, $|\lambda I - A| = |\lambda I - \tilde{A}|$, we can show after some algebraic manipulation that $(\beta^2 I + X^2) = 0$. Hence, $M_1 \in I\!R^2$ is a non-trivial arbitrary vector and M_2 is obtained as $M_2 = Y^{-1} X M_1 = \frac{1}{\beta} X M_1$. Thus, in canonical representation, the system $\dot{x} = \tilde{A}x$ is given by

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 + \beta x_2 \\ \dot{x}_2 &= \alpha x_2 - \beta x_1. \end{aligned} \tag{20.3}$$

Using polar coordinates $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$, (20.3) reduces to

$$\begin{aligned} \dot{r} &= \alpha r \\ \dot{\theta} &= -\beta \end{aligned} \tag{20.4}$$

whose solution is easily obtained as

$$\begin{aligned} r(t) &= r_0 e^{\alpha t} \\ \theta(t) &= \theta_0 - \beta t. \end{aligned} \tag{20.5}$$

It follows that if $\alpha < 0$, then $\lim_{t \longrightarrow \infty} r(t) \longrightarrow 0$ and since $\beta > 0$, $\theta(t)$ monotonically increases with time. Thus in the $x_1 - x_2$ plane, the trajectories are logarithmic spirals converging to the origin as shown in Figure 20.1. The origin is then termed as stable focus. If $\alpha > 0$, the trajectories are logarithmic spirals diverging from the origin as shown in Figure 20.2. When $\alpha = 0$, the trajectories are closed orbits about the origin as shown in Figure 20.2, and the origin in this case is termed as *center*.

The various types of equilibrium for a second-order system are summarized in the Table 20.1. A natural question that arises is what can be said about the nature of the equilibrium point (origin) of the nonlinear system (18.1). The answer is provided in the following fact: "If the linearized system (18.2) does not have any eigen-values with zero real part (non-hyperbolic), then the trajectories of the nonlinear system (18.1) in the neighbourhood of $(x_1, x_2) = (0, 0)$ have the same characteristic shape as the trajectories of the linearized system (18.2) in the neighbourhood of $(z_1, z_2) = (0, 0)$". This fact is summarized in Table 20.2.

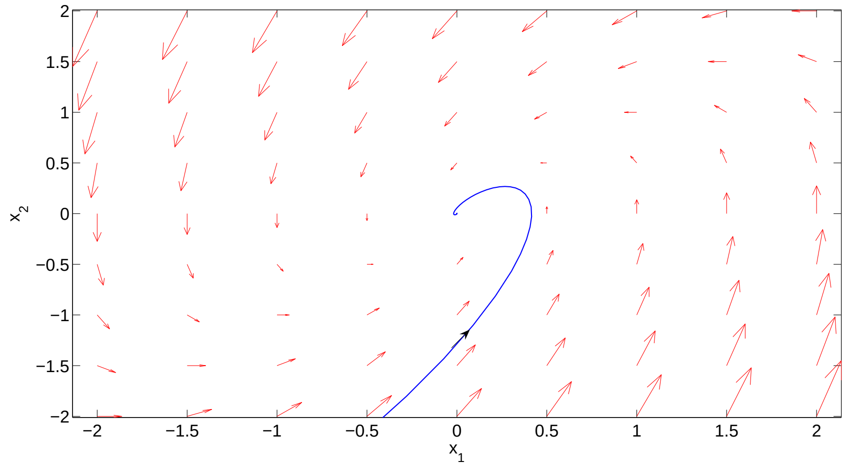

Figure 20.1: Vector field plot of a stable focus

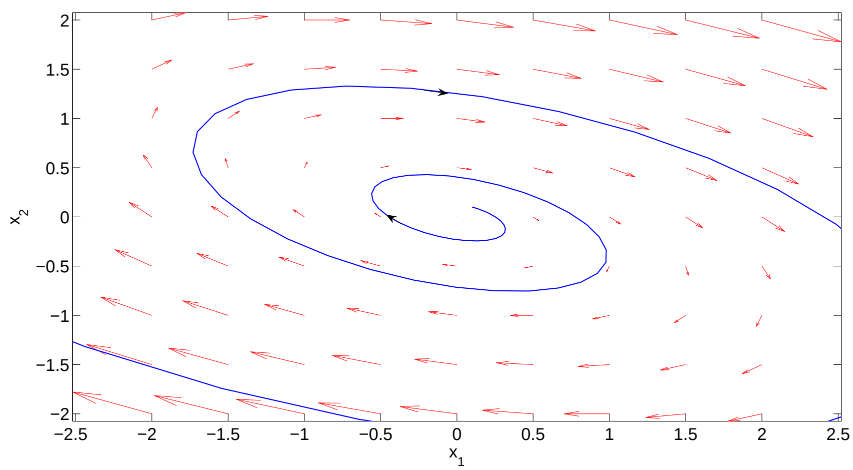

Figure 20.2: Vector field plot of a unstable focus

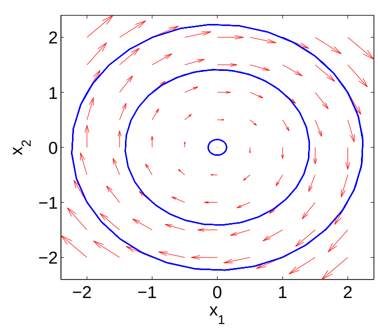

Figure 20.3: Vector field plot of a center

Eigen values of A	Type of equilibrium
λ_1, λ_2, real, both negative	Stable node
λ_1, λ_2, real, both positive	Unstable node
λ_1, λ_2, real and $\lambda_1\lambda_2 < 0$	Saddle
$\lambda_{1,2} = \alpha \pm j\beta$, $\alpha < 0$	Stable focus
$\lambda_{1,2} = \alpha \pm j\beta$, $\alpha > 0$	Unstable focus
$\lambda_{1,2} = \alpha \pm j\beta$, $\alpha = 0$	Center

Table 20.1: Types of equilibria

Equilibrium of linearized system	Equilibrium of nonlinear system
Stable node	Stable node
Unstable node	Unstable node
Saddle	Saddle
Stable focus	Stable focus
Unstable focus	Unstable focus
Center	Undefined

Table 20.2: Nature of equilibrium for nonlinear system

Lecture-21

Numerical construction of approximate phase-portrait

Consider a second-order system represented by (18.1) and it is required to get an approximate phase-portrait. We first need to draw vector-field plot since the vector field f at a point $x = (x_1, x_2)$ in the plane represents the tangent vector to the trajectory of (18.1) emanating from x. Once the vector field plot is obtained, the trajectories are obtained by fitting a smooth curve to the tangent vectors. The construction of the vector field plot is illustrated through the following example.

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= -x_1 + x_1^2 x_2 \end{aligned} \tag{21.1}$$

The first step involves finding all the equilibrium points of (21.1). A straightforward calculation shows that the origin is the unique equilibrium point. Next, the nature of the origin is inferred by linearizing the system (18.1) about the equilibrium point and evaluating the eigenvalues of the linearized system matrix. The linearized system about the origin is given by

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} x$$

and the corresponding eigenvalues are $\lambda_{1,2} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$, hence the origin is an unstable focus. Since, there is a unique equilibrium point, we can choose a bounding box such that $x_1, x_2 \in [-2, 2]$. We pick a point $x_0 = (1, 1)$ in the first quadrant. At x_0, a small arrow is drawn with the arrow head pointing away from x_0 and making an angle $\theta = \tan^{-1}(f_2(x_0)/f_1(x_0)) = 0^\circ$ with the x_1-axis. The tip of the arrow yields the new point, which is approximately equal to $(1.2, 1)$. Again a small arrow is drawn at this point with an orientation $\theta = \tan^{-1}(f_2(x)/f_1(x))|_{x=(1.2,1)} = 6.2^{\circ}$. The process is completed till the trajectory reaches close to the bounding box, which we know from the nature of the equilibrium point. Vector fields are drawn with different initial conditions, say, from different quadrants and a few initial conditions close to the origin. To obtain the vector fields in the neighbourhood of the unstable equilibrium point, it helps to consider the solution of (18.1) in reverse time through the change of time variable $\tau = -t$. It then follows that the solution in reverse time of (18.1) is equivalent to the solution in forward time of (21.2). It should be noted that the arrowhead for a point x_0 on the reverse trajectory is placed heading into x_0. For example, consider a point $x_0 = (-0.01, 0.01)$. In forward time, the vector field has an orientation $\theta = +90^{\circ}$, while in reverse time, it is $\theta = -90^{\circ}$ with the arrow heads as in Figure 21.1.

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = -f(x). \tag{21.2}$$

A computer-aided vector field plot for (21.1) is depicted in Figure 40.1.

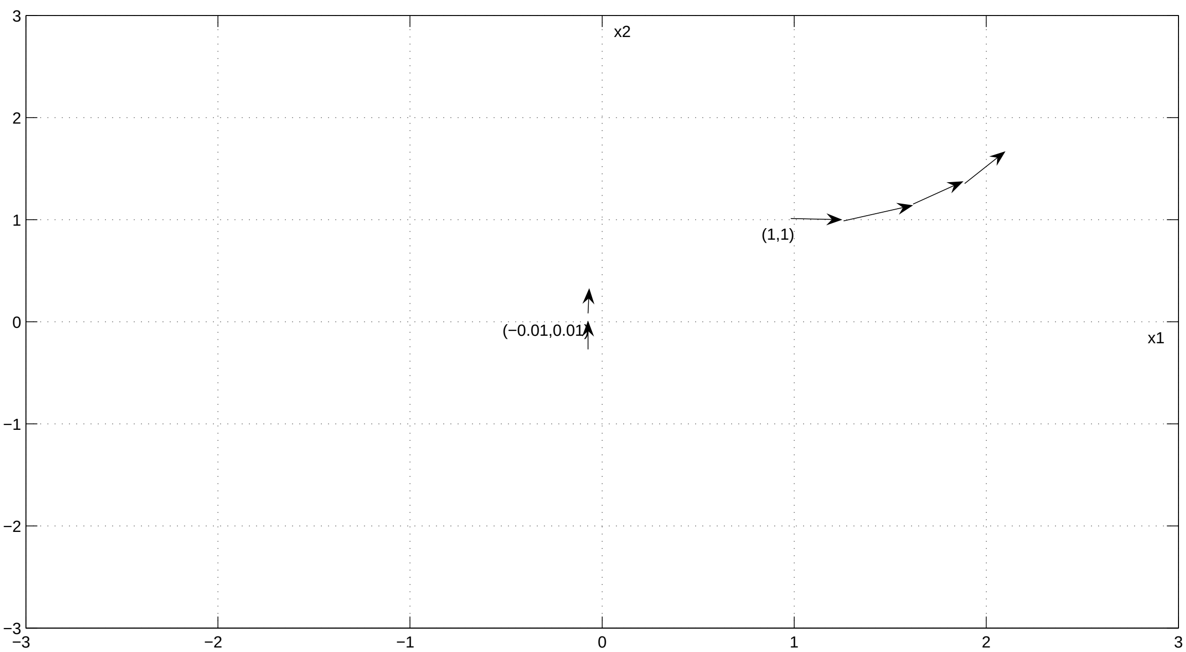

Figure 21.1: Graphical representation of a trajectory

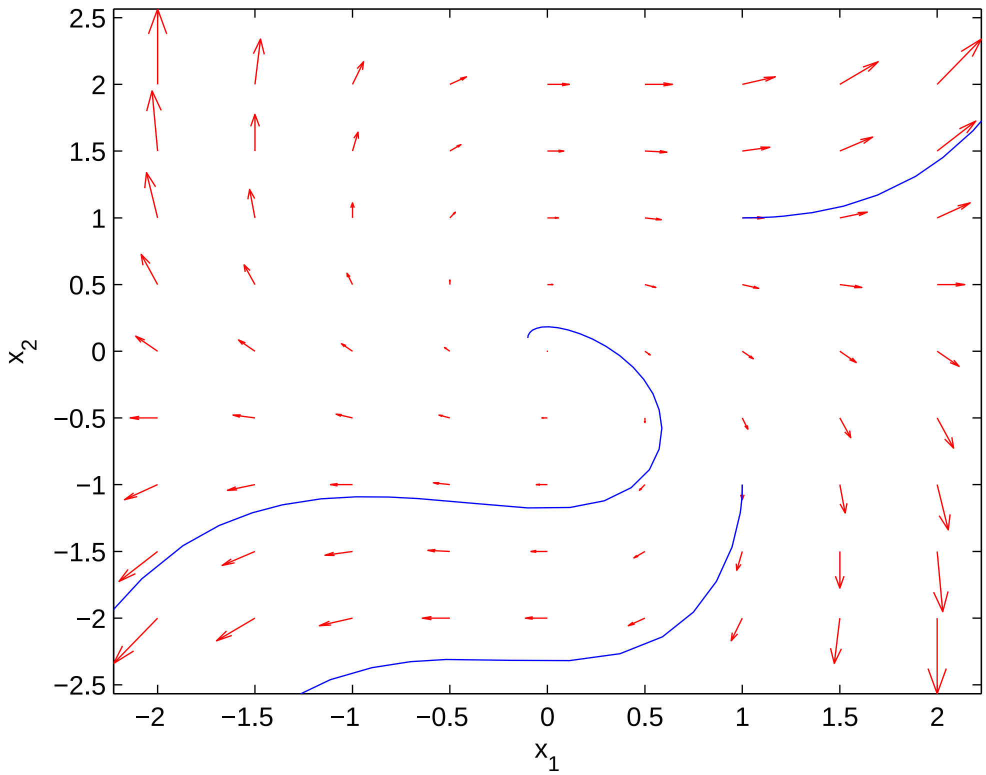

Figure 21.2: Vector field plot of Ex.3

Lecture-22

Example 22.0.2 *Consider the following system*

$$\begin{aligned}\dot{x}_1 &= 2x_1 - x_1 x_2\\ \dot{x}_2 &= 2x_1^2 - x_2.\end{aligned}$$

Find all equilibrium points of the system and obtain a linear model about each of them. Further, transform the linear system into appropriate canonical form and comment on the nature of the equilibrium point.

The system has three equilibrium points given by $\{(0,0), (-1,2), (1,2)\}$. *The Jacobian matrix is given by* $\frac{\partial f}{\partial x} = \begin{bmatrix} 2 - x_2 & -x_1 \\ 4x_1 & -1 \end{bmatrix}$. *The linear model about the equilibrium points* $(0,0)$ *is obtained as*

$$A_1 = \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

The eigenvalues of A_1 *are* $\lambda_1 = 2, \lambda_2 = -1$ *and moreover, the linear system is already in diagonal form. Since* $\lambda_1 \lambda_2 < 0$, *the equilibrium point* $(0,0)$ *is a saddle.*

Next, the linear model about the equilibrium points $(-1,2)$ *is obtained as*

$$A_2 = \left.\frac{\partial f}{\partial x}\right|_{(-1,2)} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix}$$

The eigenvalues of A_2 *are* $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}$. *Since the real part of the complex eigenvalues are negative, the equilibrium point* $(-1,2)$ *is a stable focus. The transformation that casts* A_2 *into canonical form is given by* $M = \begin{bmatrix} 1 & 1 \\ -1.8074 & 0.2582 \end{bmatrix}$. *A similar result holds for the equilibrium* $(1,2)$.

We next analyze the local behaviour of a simple pendulum system.

Example 22.0.3 *The motion of the simple pendulum has been the subject matter for physicists and mechanists since centuries. Its approximate isochronism, discovered by Galileo, makes it an accurate and simple time-keeper. In the hands of Newton, the earliest evidence that inertial and gravitational masses are proportional was established. The study of simple pendulum is ubiquitous, since it manifests as in various systems such as:*

1. *Synchronous generator connected to an infinite bus.*

2. *The model of a phase-locked loop.*

3. *The model of a fuel-slosh phenomena in space vehicles.*

The pendulum is modeled taking into account the friction at the pivot. The rod is assumed to be rigid and has zero mass. The length of the rod is denoted by l and the mass of the bob by m. The pendulum is free to swing in the vertical plane.

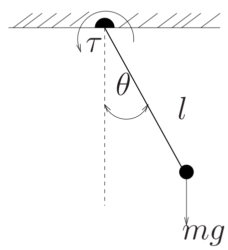

Figure 22.1: Simple pendulum

The equations of motion are

$$ml^2\ddot{\theta} = -mgl\sin\theta - k_f l\dot{\theta} \tag{22.1}$$

where $k_f > 0$ is the friction coefficient. Defining the states as $x = (x_1 = \theta, x_2 = \dot{\theta})$, the dynamics (22.1) are rewritten as

$$\dot{x} = \begin{pmatrix} x_2 \\ \frac{-g}{l}\sin x_1 - \frac{k_f x_2}{ml} \end{pmatrix}.$$

The equilibrium set $\mathcal{E}$ is obtained as

$$\begin{aligned} \mathcal{E} &= \{x \in \mathrm{S}^1 \times I\!R \mid f(x) = 0\} \\ &= \{(0,0), (\pi, 0)\}. \end{aligned}$$

The linear model is given by

$$\dot{x} = Ax$$

where

$$A = \left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array}\right]\Bigg|_{x\in\mathcal{E}} = \left[\begin{array}{cc} 0 & 1 \\ \frac{-g\cos x_1}{l} & \frac{-k_f}{ml} \end{array}\right]\Bigg|_{x\in\mathcal{E}}$$

The eigenvalues of the linearized model reveals the nature of the two equilibrium points $(0,0),(\pi,0)$, *with and without friction, as shown in Tables 22.1 and 22.2. The corresponding vector field plots are shown in Figures 22.2 and 22.3. When friction is considered, the lower equilibrium position of the pendulum is no longer a center, but a stable node/focus.*

Equilibrium point	Eigen values of A	Type
$(0,0)$	$\lambda_{1,2} = \pm i\sqrt{g/l}$	Center
$(\pi,0)$	$\lambda_{1,2} = \pm\sqrt{g/l}$	Saddle point

Table 22.1: Nature of equilibria without friction

Equilibrium point	discriminant	Eigen values of A	Type
$(0,0)$	$\left((\frac{k_f}{ml})^2 - \frac{4g}{l}\right) \geq 0$	$\lambda_1 < 0, \lambda_2 < 0$	Stable node
	$\left((\frac{k_f}{ml})^2 - \frac{4g}{l}\right) < 0$	$\alpha \pm j\beta, \alpha < 0$	Stable focus
$(\pi,0)$	$\left((\frac{k_f}{ml})^2 + \frac{4g}{l}\right) > 0$	$\lambda_1\lambda_2 < 0$	Saddle point

Table 22.2: Nature of equilibria with friction

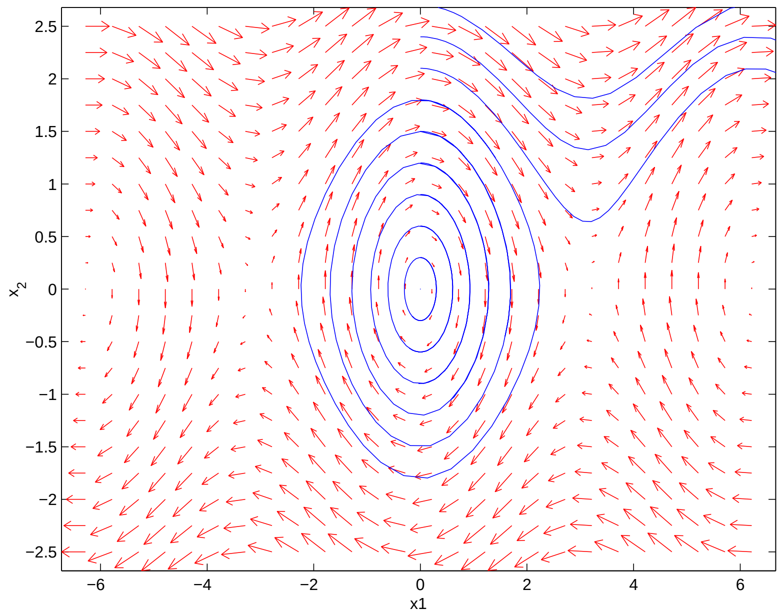

Figure 22.2: Vector field plot with no friction

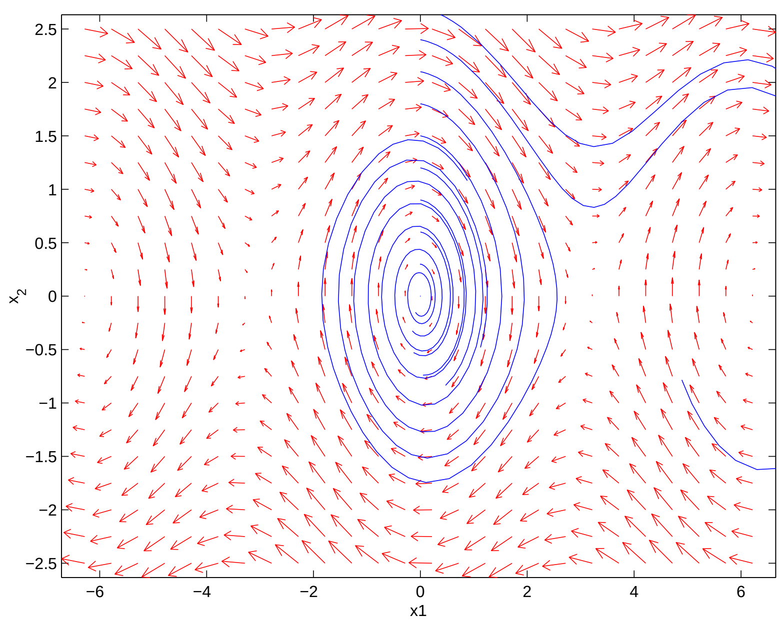

Figure 22.3: Vector field plot with friction considered ($k_f = 0.1$)

Exercise problems

1. Hand sketch the state trajectory for the following system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= -x_1 + x_1^2 x_2\end{aligned}$$

   with the initial condition $x(0) = (0, 2)$. Comment on the nature of the equilibrium point.

2. Draw the phase-portrait for the following system

$$\begin{aligned}\dot{x}_1 &= -x_1 - \frac{x_2}{\ln\sqrt{x_1^2+x_2^2}} \\ \dot{x}_2 &= -x_2 + \frac{x_1}{\ln\sqrt{x_1^2+x_2^2}}\end{aligned}$$

   near the origin. You may transform the system into polar coordinates.

3. For each of the following matrices transform the system into appropriate canonical form and classify the equilibrium $(0, 0)$ to its type.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}; \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & 5 \\ -1 & -1 \end{bmatrix}; \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}; \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}$$

4. Find all equilibrium points and the nature of the equilibrium points in each of the following examples. Also sketch the approximate phase-portrait.

   i.

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1^3 + x_1 x_2^2 \\ \dot{x}_2 &= -x_2 + x_2^3 + x_1^2 x_2\end{aligned}$$

ii.

$$
\begin{aligned}
\dot{x}_1 &= x_2 \cos(x_1) \\
\dot{x}_2 &= \sin(x_1)
\end{aligned}
$$

iii.

$$
\begin{aligned}
\dot{x}_1 &= x_2 + 2x_1(1 - x_1^2 - x_2^2) \\
\dot{x}_2 &= -x_1 + 2x_2(1 - x_1^2 - x_2^2)
\end{aligned}
$$

Module 5

Periodic Solutions

Objectives: To understand the notion of periodic solutions; their existence and non-existence.

Lesson objectives

This module helps the reader in

– identifying periodic solutions and *limit* cycles

– applying the sufficient conditions to rule-in and rule-out the existence of periodic solutions in $\mathbb{R}^2$

– finding analytical expression for closed-orbits for a few selected problems.

Lecture-23

A periodic solution or orbit corresponds to a special type of solution for a dynamical system, namely one which repeats itself in time. Many dynamical systems exhibit periodic orbits: the motion of planets around the Sun, the small oscillations of a simple pendulum and an oscillator circuit. Consider a system defined on the domain $D \subset I\!R^2$

$$\begin{aligned} \dot{x} &= f(x), \quad x \in \mathbb{R}^n \\ x(0) &= x_0. \end{aligned} \tag{23.1}$$

Definition 23.0.4 *A non-trivial solution $x(t, x_0)$ of* (23.1) *is said to be periodic if there exists a time $T > 0$ such that*

$$x(t + T, x_0) = x(t, x_0) \; \forall \; t \geq 0. \tag{23.2}$$

Solutions of (23.1) with initial condition belonging to the equilibrium set correspond to trivial solutions that satisfy (23.2) for all $t \geq 0$ and are ruled out from the definition of a periodic solution. Periodic solutions correspond to closed trajectories in the phase portrait. Such a trajectory is called a *periodic orbit* or a *closed orbit*, defined as follows:

Definition 23.0.5 *A set $\mathcal{O}$ is a periodic orbit of* (23.1) *if*

$$\mathcal{O} = \{x \in D : x = x(t, x_0), t \in I\!R\}$$

for some periodic solution $x(t, x_0)$ of (23.1).

Consider the simple harmonic oscillator given by the equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned} \tag{23.3}$$

The phase portrait of this system shows continuum of closed orbits and hence it can be concluded that the system has periodic solutions. In Figure 23.1, the closed orbits correspond to two different initial conditions. It can be observed that a linear oscillator, apart from the fact that it cannot be practically realized, suffers from two major drawbacks:

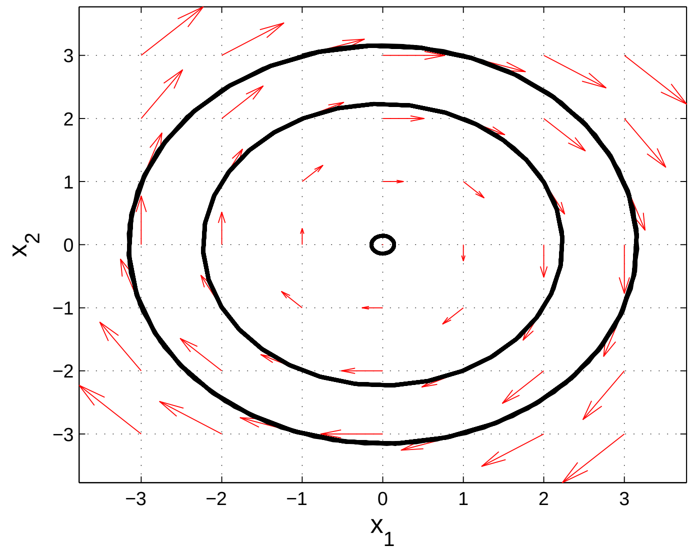

Figure 23.1: Vector field plot of linear harmonic oscillator

- The amplitude of oscillation depends on the initial condition.
- The frequency of oscillation is structurally unstable.

It is thus natural to seek an oscillator that can be practically realized, is structurally stable and whose amplitude of oscillation is invariant with respect to the initial condition. One such oscillator is the Van der Pol oscillator given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2 \end{aligned} \tag{23.4}$$

where $\epsilon > 0$. A typical vector-field plot along with the closed orbit is depicted in Figure 23.2. Observe that the Van der Pol oscillator (23.4) exhibits a unique periodic solution. Further, all non-trivial trajectories originating within the closed orbit converge to it, and all trajectories originating outside the periodic solution also converge to it. Such a closed orbit is termed stable and in later chapters we will in a precise manner, define stable closed orbits. An isolated periodic orbit is referred to by the special name *limit cycle*. Further, a limit cycle is compact and invariant.

We end this lecture with the notion of attracting set. It helps to know what is meant by the neighbourhood of a set. An ϵ-neighbourhood N of a set $M \subset D$ is $N \triangleq \{x \in D : ||x - y|| < \epsilon, y \in M\}$ for some $\epsilon > 0$.

Definition 23.0.6 *A closed invariant set $M \subset D$ is an attracting set of* (23.1) *if there*

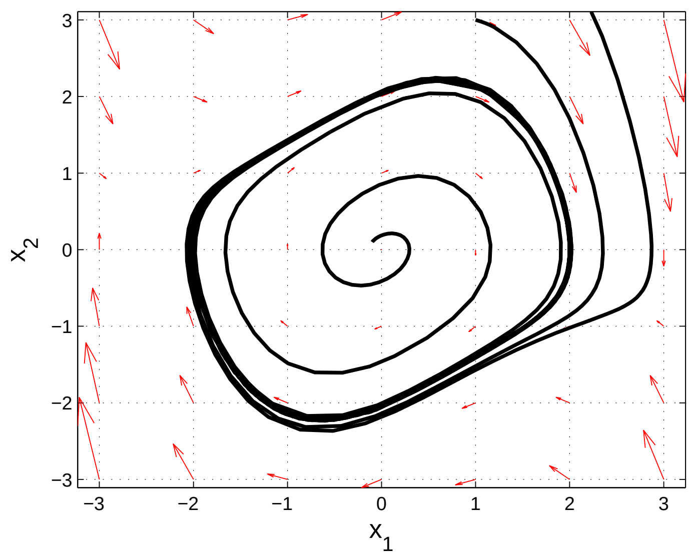

Figure 23.2: Vector field plot of a Van der Pol oscillator with $\epsilon = 0.5$

exists a neighbourhood N of M such that, for all $x \in N$, $\phi(t,x) \in N \quad \forall\ t \geq 0$ and $\phi(t,x) \longrightarrow M$ as $t \longrightarrow \infty$.

Definition 23.0.7 *A set $M \subset D$ is an attractor of* (23.1) *if M is an attracting set of D and contains a dense orbit.*

The limit cycle of the Van der Pol Oscillator (23.2) is an example of an attractor.

Lecture-24

Periodic solutions on $I\!R^2$

When confined to $I\!R^2$, sufficient conditions exist to *guarantee* or *rule out* the existence of periodic solutions. These theorems are based on the concept of bounded trajectories and positive invariance of a set. We now have the requisite tools to understand the following Lemma.

Existence of periodic solutions

Lemma 24.0.8 The Poincaré-Bendixson criterion
Let P be a closed, bounded subset of $\mathbb{R}^2$. Poincaré-Bendixson criteria states that P contains a periodic solution of (23.1) *if it satisfies the following conditions:*

1. *P contains no equilibrium points or contains only unstable equilibrium points (unstable node/focus).*

2. *P is positive invariant with respect to the trajectories of* (23.1).

Note that the above criteria is only a *sufficient* condition and further it does not say anything about the uniqueness of the periodic solutions nor about the number of periodic solutions. Thus it cannot be used to rule-out the existence of limit cycles. While the first criteria is easy to check, the second one requires the test of positive invariance. For nonlinear systems, one way is to characterize the region enclosed by a simple closed curve defined by $V(x) = c, c > 0$ where, V is a smooth function. Differentiating V along the trajectory of (23.1), we obtain

$$\dot{V} = \frac{\partial V}{\partial x} f(x) = < \frac{\partial V}{\partial x}, f(x) >$$

where, $\frac{\partial V}{\partial x}$ is the gradient of V and $f(x)$ the tangent vector at x. Recall that the gradient of a scalar function is a vector field which points in the direction of the greatest rate of increase of the scalar function. Thus the direction of $\frac{\partial V}{\partial x}$ is as shown in Figure 24.1. The inner product formula $<\frac{\partial V}{\partial x}, f(x)>= \cos\theta||\frac{\partial V}{\partial x}||\ ||f(x)||$ allows us to determine the

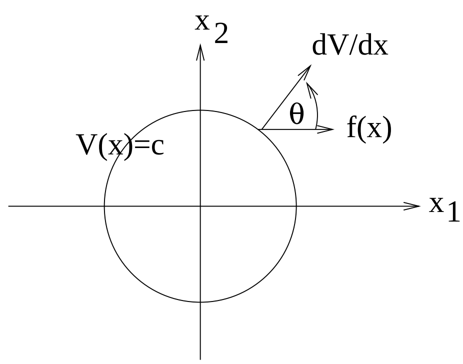

Figure 24.1: Gradient of V and the tangent vector at a point x on the closed curve $V(x) = c$

direction of $f(x)$ with respect to the gradient of V. We have

- $\dot{V} \leq 0$ whenever $f(x)$ points inwards at a point x on the curve $V(x) = c$, implying trajectories are trapped inside the set $P \triangleq \{x \in I\!R^2 : V(x) \leq c\}$.
- $\dot{V} \geq 0$ whenever $f(x)$ points outwards at a point x on the curve $V(x) = c$, implying trajectories leave the set P.
- $\dot{V} = 0$ whenever $f(x)$ is tangent to the curve $V(x) = c$ at x.

For the region bounded by the curve $V(x) = c$ to be positively invariant the tangent vector should point inwards or be tangential to $V(x) = c$ at all points. The test for the existence of periodic solutions using Poincaré Bendixson's criteria is applied to the following examples.

Example 24.0.9 *Consider the system given by*

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{aligned} \qquad (24.1)$$

It can be seen that $(0, 0)$ *is the unique equilibrium point of this system. Further, the linearized model about the origin is given by*

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} x$$

and has the eigen spectrum $\{1 \pm i\sqrt{2}\}$ *and thus the origin is an unstable focus of the nonlinear system. Let* $V(x) = x_1^2 + x_2^2 = c$. *Then*

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x}\dot{x} \\ &= 2c - 2c^2 - 2x_1x_2 \\ &\leq 3c - 2c^2 \end{aligned}$$

where we have used the inequality $|2x_1x_2| \leq x_1^2 + x_2^2$. *Hence the region* $P \triangleq \{x \in I\!R^2 : V(x) \leq c\}$ *is positive invariant provided* $c \geq 3/2$. *By Poincaré Bendixson's criteria,* P *contains periodic solutions. The phase-portrait in Figure 24.2 reveals a stable limit cycle.*

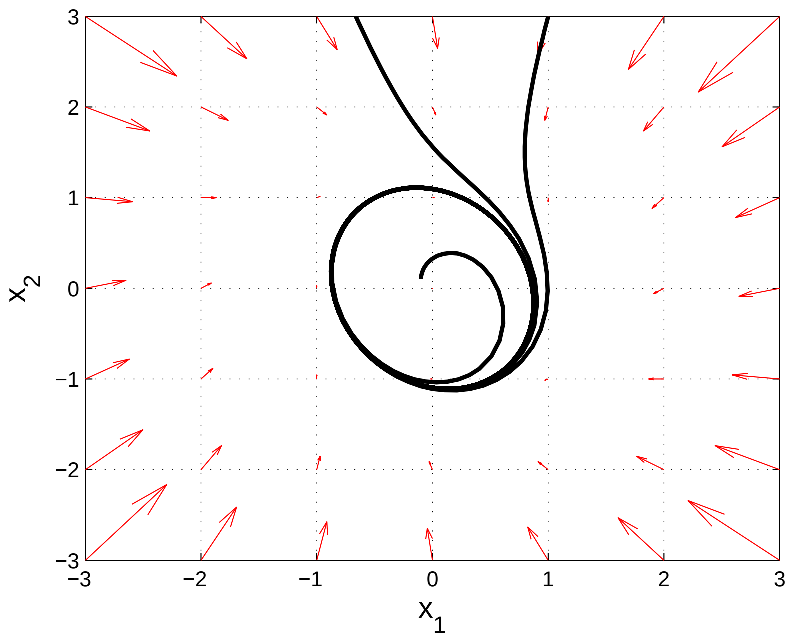

Figure 24.2: A stable limit cycle

Example 24.0.10 *Consider the system given by*

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1 - x_2 \end{aligned} \tag{24.2}$$

Consider the region P *bounded by the curve* $V(x) = x_1^2 + x_2^2 = c$. *The derivative of* $V(x)$ *along the trajectories of the system yields*

$$\dot{V} = 2x_1(-x_1 + x_2) + 2x_2(-x_1 - x_2) = -2c < 0.$$

Hence P is positive invariant and the second condition of the Poincaré Bendixson criterion is satisfied. The system has an unique equilibrium point at $(0,0)$ and is a stable focus, thus violating the first condition. Hence the criterion is not applicable. In fact it can be seen that all trajectory entering P will converge to this stable focus. This example brings out the importance of the set P not containing any stable equilibrium points even if it is positively invariant.

Example 24.0.11 *Consider the linear oscillator given by*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

The system has a center at $(0,0)$. Hence we consider a region in the form of an annular ring about the origin given by $P = \{x \in \mathbb{R}^2 : c_1 \leq V(x) \leq c_2, c_1, c_2 > 0\}$ where, $V(x) = x_1^2 + x_2^2$. This choice of P excludes the non-hyperbolic equilibrium from it. Differentiating $V(x)$ along the trajectories of the system yields $\dot{V} = 0$. Hence the vector $f(x)$ is tangential to all points of the curve $V(x) = c$ implying the annular ring is positively invariant, establishing the existence of periodic solution inside it. Recall, the linear oscillator exhibits a continuum of periodic solutions.

Lecture-25

Analytical expressions for closed-orbits

In certain cases the periodic solution of a system can be obtained in an analytic form as show by the following examples.

Example 25.0.12 *Considering again the linear oscillator* (23.1), *let us assume* $V(x)$ *of the form* $V(x) = h_1(x_1) + h_2(x_2)$*. Differentiating* $V(x)$ *along the trajectories of* (23.1), *we require*

$$\begin{aligned} \dot{V} &= \frac{\partial h_1}{\partial x_1}x_2 + \frac{\partial h_2}{\partial x_2}(-x_1) = 0 \\ &\Rightarrow \frac{1}{x_1}\frac{\partial h_1}{\partial x_1} = \frac{1}{x_2}\frac{\partial h_2}{\partial x_2} = c \end{aligned}$$

where, c *is a constant. Therefore* $\frac{\partial h_1}{\partial x_1} = cx_1$ *and* $\frac{\partial h_2}{\partial x_2} = cx_2$*. Integrating, we obtain* $h_1 = cx_1^2$ *and* $h_2 = cx_2^2$*. Hence we see that periodic solutions are of the form* $x_1^2 + x_2^2 = c$*.*

Example 25.0.13 *Consider the simple pendulum given by*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{l}\sin x_1 \end{aligned}$$

Continuing with the same $V(x)$ *as in the previous example, we have*

$$\dot{V} = \frac{\partial h_1}{\partial x_1}x_2 + \frac{\partial h_2}{\partial x_2}(-\frac{g}{l}\sin x_1) = 0$$

which implies $\frac{\partial h_1}{\partial x_1} = c\frac{g}{l}\sin x_1$ *and* $\frac{\partial h_2}{\partial x_2} = cx_2$*. On integrating, we obtain* $h_1 = -c\frac{g}{l}\cos x_1, h_2 = cx_2^2$*. Thus,* $V = -\frac{g}{l}\cos x_1 + x_2^2 = a$ *constant. The expression for* V *correspond to closed-orbits of constant energy.*

Lecture-26

We discuss conditions under which the existence of periodic solutions in a plane are ruled out.

Non-existence of periodic solutions

We need the following definitions before stating the main result.

Definition 26.0.14 *A curve $\gamma : [a, b] \to I\!R$ is said to be closed if $\gamma(a) = \gamma(b)$. Further the curve γ is said to be a simple closed curve if γ is one-to-one on the interval $[a, b)$.*

The region in a plane about which the non-existence of periodic solutions need to be verified should possess the following property.

Definition 26.0.15 *A region D is said to be simply connected if for every simple closed curve γ in D, the inner region of γ is also a subset of D.*

An annular region is not simply connected, while the inner region of a circle is simply connected. We are now in a position to interpret the following criteria.

Lemma 26.0.16 Bendixson's criteria *If on a simply connected region D of $I\!R^2$, $\operatorname{div} f \triangleq \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically equal to zero and does not change sign then the system (23.1) has no periodic solutions lying entirely within D.*

Proof:

On any orbit of (23.1), the following holds

$$\frac{dx_2}{dx_1} = \frac{f_2}{f_1}$$

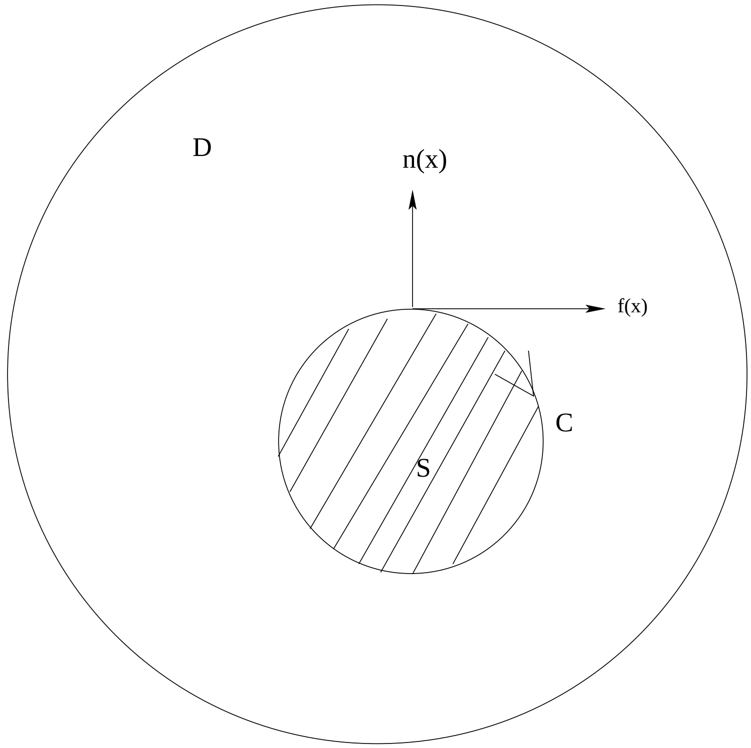

Figure 26.1: various sets in the proof of Bendixson's theorem

Therefore on any closed orbit C (see Figure 26.1) of (23.1),

$$\oint_C < f(x), n(x) > dC = \oint_C f_2 dx_1 - f_1 dx_2 = 0$$

where, $n(x)$ is the outward normal to $f(x)$ at $x \in C$. Applying *Green's* theorem [1],

$$\oint_C f_2 dx_1 - f_1 dx_2 = 0 = \iint_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1 dx_2 = 0$$

where S is the region enclosed by C. From the last identity, we can conclude that $\mathrm{div}(f) = 0$ or changes sign, the negation of which is the hypothesis of the criteria. □

Example 26.0.17 *Consider the system given by*

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 x_2^2 \\ \dot{x}_2 &= -x_1 + x_1^2 x_2 \end{aligned} \tag{26.1}$$

We have $\mathrm{div} f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_2^2 + x_1^2 > 0 \quad , \forall\, x \neq 0$. *By Bendixson's criteria the system has no periodic solutions in the entire plane.*

[1] $\oint_C M(x_1, x_2) dx_1 + N(x_1, x_2) dx_2 = 0 = \iint_D \left(\frac{\partial N}{\partial x_1} - \frac{\partial M}{\partial x_2}\right) dA$, where D is the region enclosed by the closed curve C.

Example 26.0.18 *For the system*

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= g(x_1) + a x_2;\ \ a \neq 1
\end{aligned}
\tag{26.2}
$$

A straightforward calculation yields $\mathrm{div} f = -1 + a$. *Thus the system also has no periodic solutions in the plane.*

Example 26.0.19

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_1^3 + x_1 x_2^2 \\
\dot{x}_2 &= -x_2 + x_2^3 + x_2 x_1^2
\end{aligned}
\tag{26.3}
$$

We have $\mathrm{div} f = -2 + 4(x_1^2 + x_2^2) < 0$ *on* $D \triangleq \{x : x_1^2 + x_2^2 < 1/2\}$. *Since* D *is simply connected there are no periodic solutions in* D.

Lecture-27

Example 27.0.20 *Consider the system*

$$\begin{aligned} \dot{x}_1 &= ax_1 - x_1x_2 \\ \dot{x}_2 &= bx_1^2 - cx_2 \end{aligned} \tag{27.1}$$

where a, b, c are positive constants with $c > a$. Let $D = \{x \in I\!R^2 : x_2 \geq 0\}$.
(a.) Show that D is positively invariant.
(b.) Show that there can be no periodic orbits through any point $x \in D$.
The system has three equilibrium points $(0,0), (\sqrt{ac}/b, a), (-\sqrt{ac}/b, a)$. The origin is a saddle point while the other two equilibrium points are stable foci. The trajectory along the x_2-axis approaches the origin. Further, at $x_2 = 0, \dot{x}_1 = ax_1, \dot{x}_2 = bx_1^2$. Hence D, in Figure 27.1 is positively invariant, simply connected and $\mathrm{div} f = -(c-a) - x_2 < 0$ *on D. Hence there are no periodic solutions that are entirely contained in D and nothing further can be inferred.*

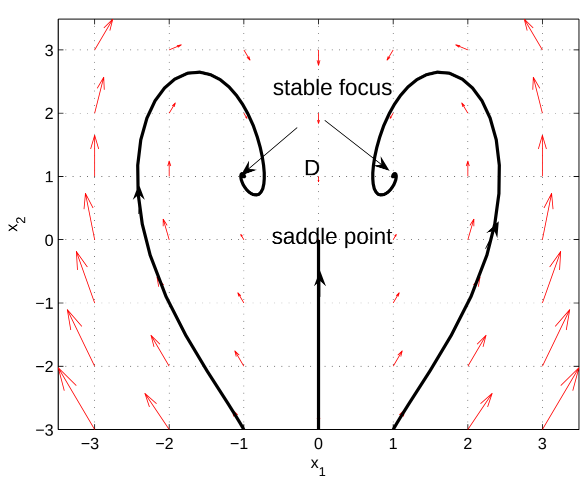

Figure 27.1: The positive invariant set D

Example 27.0.21 *Consider the system governed by*

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1 h(x) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2 h(x)\end{aligned} \tag{27.2}$$

where, $h(x) = \max\{|x_1|, |x_2|\}$. Clearly, $(0,0)$ is the equilibrium point and the characteristic equation of the linearized system about the origin is $\lambda^2 - 2\lambda + 1 = 0$, implying the origin is unstable. Now, consider the curve $V = x_1^2 + \frac{x_2^2}{2}$, then $\dot{V} = -(2x_1^2 + x_2^2)(h(x) - 1) <$ 0 on the set $\mathbb{R}^2 \setminus D$, where $D \triangleq \{x \in \mathbb{R}^2 : h(x) < 1\}$ is depicted by the shaded portion in Figure 27.2. Hence, $\mathbb{R}^2 \setminus D$ contains periodic solutions. Further $\nabla f = 2(1 - h(x) > 0$ on

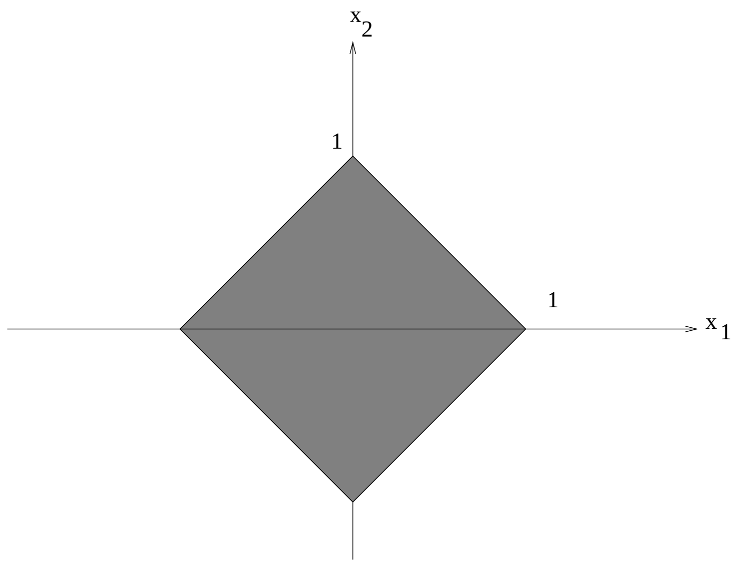

Figure 27.2: The set D is the unshaded region

D, which is simply connected. Hence there are no periodic solutions in D. It is obvious, there is a limit cycle characterized by $h(x) = 1$.

Example 27.0.22 *Show that there are no limit cycles in the following system.*

$$\begin{aligned}\dot{x}_1 &= x_2 \cos x_1 \\ \dot{x}_2 &= \sin x_1\end{aligned} \tag{27.3}$$

Solution*:*

We first compute the divergence of f, that yields $\operatorname{div} f = -\sin x_1 x_2$. Using Bendixson's criteria, we can say that there are no periodic solutions in the sets $D_1 \triangleq \{x \in S^1 \times \mathbb{R} :$ $x_2 > 0, x_1 \in (0, \pi)\}$ or $D_2 \triangleq \{x \in S^1 \times \mathbb{R} : x_2 < 0, x_1 \in (\pi, 2\pi)\}$. Therefore by Bendixson's criteria we can only say that there are no periodic orbits contained entirely within D_1 and

D_2. Since the question is about completely ruling out limit cycles, we explore if there is a continuum of periodic solutions.

Consider $V(x) = h_1(x_1) + h_2(x_2) = a$ constant. Differentiating V along the trajectories of (27.3), *we have*

$$\dot{V} = \frac{\partial h_1}{\partial x_1} x_2 \cos x_1 + \frac{\partial h_2}{\partial x_2} \sin x_1 = 0$$

which implies $\frac{\partial h_1}{\partial x_1} = -c \tan x_1$ and $\frac{\partial h_2}{\partial x_2} = cx_2$, where c is a constant. On integrating, we obtain $h_1 = c \log(\cos x_1), h_2 = cx_2^2$. Thus, $V = \log(\cos x_1) + x_2^2/2 = a$ constant. The expression for V does NOT correspond to closed-orbits as shown in Figure 27.3. The surface $V(x) = c$ is a saddle. The same fact can be verified by computing the Hessian of V as

$$\nabla_x^2 V = \begin{bmatrix} -\sec^2 x_1 & 0 \\ 0 & 1 \end{bmatrix}$$

and when evaluated at the origin is sign-indefinite. A careful observation of the vector field plot of (27.3) *(see shown Figure 27.4) reveals that the two sets D_1, D_2 are positively invariant. Further, all trajectories, except the trivial trajectories (the equilibrium points* $(0,0)$ *and* $(\pi, 0)$*), originating in the complement $S^1 \times I\!R \setminus \{D_1 \cup D_2\}$ exit from it. Trajectories originating from D_1 approach $(\pi/2, \infty)$, while those from D_2 approach $(\frac{3\pi}{2}, -\infty)$ as $t \longrightarrow \infty$. Hence, there are no periodic solutions, ruling out the possibility of limit cycles.*

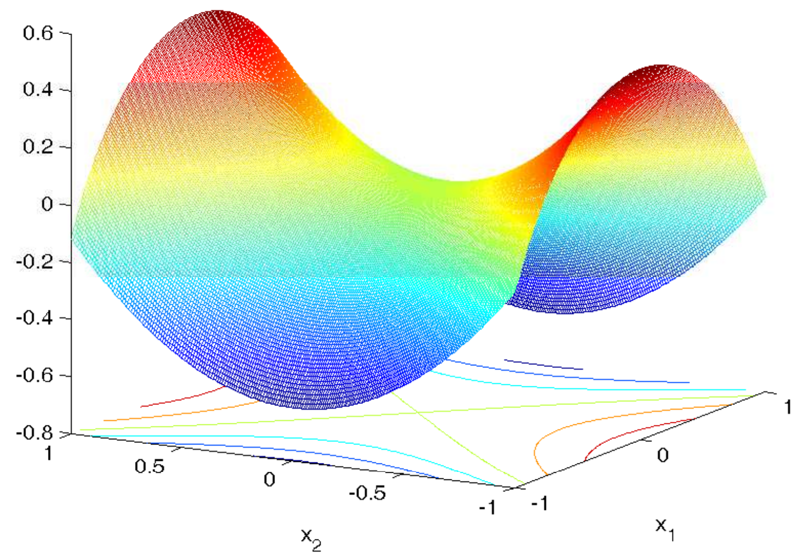

Figure 27.3: Profile of $V = \log(\cos x_1) + x_2^2/2 =$ a constant

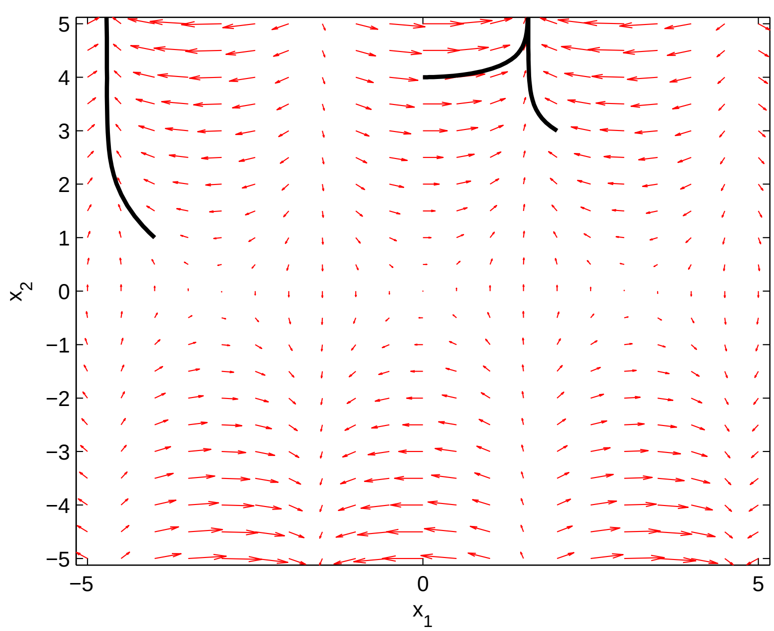

Figure 27.4: Vector field plot

Exercise problems

1. For the following system, use Poincare-Bendixson's criteria to show the existence of periodic orbit.

$$\ddot{y} + y = \epsilon\dot{y}(1 - y^2 - \dot{y}^2).$$

2. Consider the system described by

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g(x_1)\end{aligned}$$

Under what conditions on $g(x_1)$ there is a continuum of periodic solutions. Find the expression for closed trajectories.

3. Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2(x_1 + a) - b \\ \dot{x}_2 &= -cx_1(x_1 + a)\end{aligned}$$

where a, b, c are positive constants with $b > a$. Let

$$D = \left\{ x \in I\!R^2 : x_1 < -a \text{ and } x_2 < \frac{x_1 + b}{x_1 + a} \right\}$$

a. Show that every trajectory starting in D stays in D for all future time.

b. Show that there can be no periodic orbits through any point $x \in D$.

- For the dynamical system below

$$\begin{aligned}\dot{x}_1 &= 4x_1^2 x_2 - g_1(x_1)(x_1^2 + 2x_2^2 - 4) \\ \dot{x}_2 &= -2x_1^3 - g_2(x_2)(x_1^2 + 2x_2^2 - 4)\end{aligned}$$

where, $x_1 g_1(x_1) > 0, x_1 \neq 0$, $x_2 g_2(x_2) > 0, x_2 \neq 0$, $g_1(0) = 0$, and $g_2(0) = 0$.

a.) Show that the set defined by $S = \{x \in I\!R^2 : (x_1^2 + 2x_2^2 - 4) = 0\}$ is invariant, closed and bounded.

c). Find weather the closed trajectory on the set S moves clockwise or anti-clockwise with justification. Please mark the direction on a neat sketch of the set.

4. Show that the following system has a periodic orbit.

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + 5x_2^2)\\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + 5x_2^2).\end{aligned}$$

5. Show that the following system has a periodic orbit.

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1(|x_1| + |x_2|)\\ \dot{x}_2 &= -2x_1 + x_2 - x_2(|x_1| + |x_2|).\end{aligned}$$

6. Consider

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_1 - x_1^3 - \alpha x_2 + x_1^2 x_2.\end{aligned}$$

Does the system possess a periodic orbit for $\alpha > 0$?

Module 6

Stability Analysis

Objectives: To understand the various notions of stability and Lyapunov based analysis in establishing them.

Lesson objectives

This module helps the reader in

– understanding the various notions of nonlinear stability such as stability, asymptotic stability, global asymptotic stability and exponential statbility

– the use of Lyapunov stability theory in proving stability of equilibrium points

– estimating the region of attraction of an equilibrium point

– the use of La Salle's invariance principle to infer asymptotic stability

– establishing the stability of linear systems by computing the eigen-spectrum of the linearized system matrix

– establishing the stability of linear systems by solving the Lyapunov equation

– instability result.

Suggested reading

- Nonlinear System Analysis: M. Vidyasagar
- Nonlinear Systems: H. K. Khalil
- Linear Systems Theory: Joao P. Hespanha

Lecture-28

Stability definitions

Consider a system described by

$$\dot{\boldsymbol{x}} = f(x) \tag{28.1}$$

where $x \in I\!R^n$, $f : D \longrightarrow I\!R^n$ is locally Lipschitz in x on D, and $D \subset I\!R^n$ is a domain that contains the origin $x = 0$. Let $x_e \in D$ be an equilibrium point, that is, $f(x_e) = 0$. Without loss of generality, we assume $x_e = 0$, the origin. We next state the various notions of stability for the system (28.1).

Definition 28.0.23 *The equilibrium point $x = 0$ of (28.1) is stable if, for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that*

$$||x(0)|| < \delta \implies ||x(t)|| < \epsilon,\ \forall\, t \geq 0. \tag{28.2}$$

The notion of stability is depicted in Figure 28.1, where a trajectory originating from δ-ball with initial condition $x(t_0)$ is confined to ϵ-ball for all $t \geq 0$. Note that, there is no requirement that trajectories approach the origin as time evolves.

Example 28.0.24 *Let us revisit the linear oscillator system*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned} \tag{28.3}$$

and we know that the origin is the unique equilibrium point and further the trajectories of the system (28.3) *describe concentric circles centered at the origin. Thus, give any $\epsilon > 0$, there exists a $\delta > 0$ such the condition* (29.1) *holds. In fact, any δ satisfying $\delta \leq \epsilon$ meets the requirement. Hence, the origin is a stable equilibrium point.*

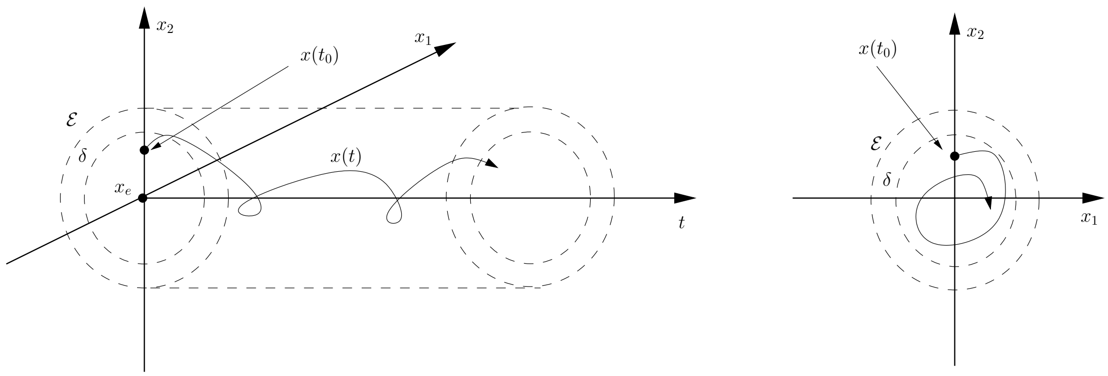

Figure 28.1: Stability: Evolution of trajectory

Definition 28.0.25 *The equilibrium point $x = 0$ of (28.1) is attractive if there is a $\eta > 0$ such that $||x(0)|| < \eta \implies ||x(t)|| \longrightarrow 0$ as $t \longrightarrow \infty$.*

Example 28.0.26 *The following example (Vinograd's equation), shows (see figure 28.2) that attractivity does not imply stability.*

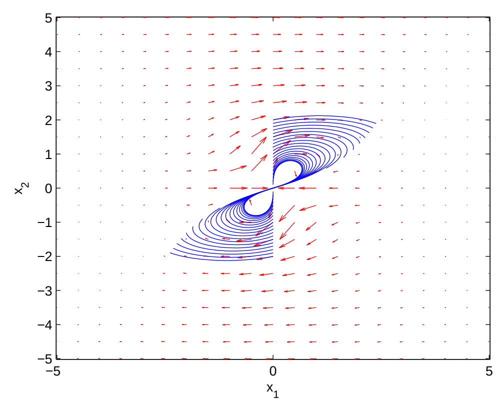

Figure 28.2: Phase-portrait of Vinograd's equation

$$
\begin{aligned}
\dot{x}_1 &= \frac{x_1^2(x_2 - x_1) + x_2^5}{x_1^2 + x_2^2(1 + (x_1^2 + x_2^2)^2)} \\
\dot{x}_2 &= \frac{x_2^2(x_2 - 2x_1)}{x_1^2 + x_2^2(1 + (x_1^2 + x_2^2)^2)}.
\end{aligned}
$$

Definition 28.0.27 *The equilibrium point $x = 0$ of (28.1) is asymptotically stable if it is Lyapunov stable and there is a positive constant c such that*

$$x(t) \longrightarrow 0 \ \text{ as } \ t \longrightarrow \infty, \ for \ \ all \ ||x(0)|| < c. \tag{28.4}$$

and globally asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ *is satisfied for any* $x(0)$.

The notion of asymptotic stability is depicted in Figure 28.3, where the trajectories eventually approach the origin.

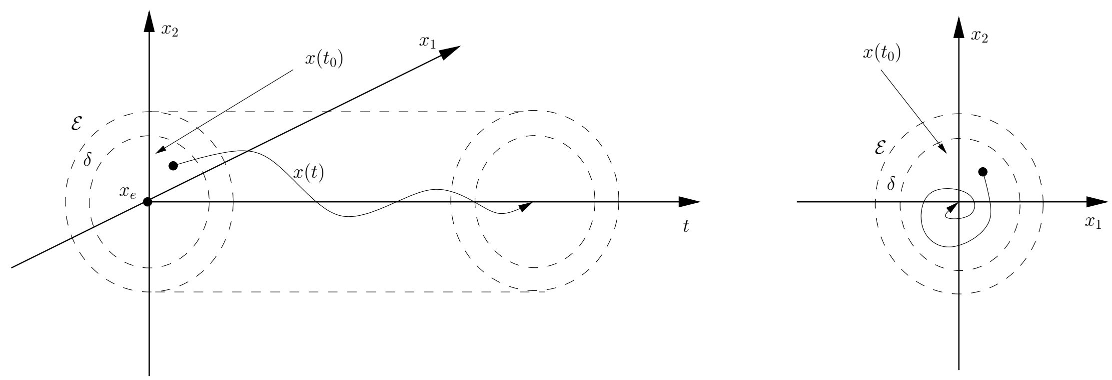

Figure 28.3: Asymptotic Stability: Evolution of trajectory

Definition 28.0.28 *The equilibrium point* $x = 0$ *of (28.1) is exponentially stable if there exist positive constants* c, k *and* λ *such that*

$$||x(t)|| \leq k||x(0)||e^{-\lambda\, t}, \ \forall \ ||x(0)|| < c \tag{28.5}$$

and globally exponentially stable if (28.5) is satisfied for any $x(0)$.

Lecture-29

We next present Lyapunov's stability theorem concerning the stability of the equilibrium point $x = 0$ of (28.1).

Theorem 29.0.29 *Let $V : D \longrightarrow \mathbb{R}$ be a continuously differentiable function such that V is positive-definite on D and $\dot{V} \leq 0$ in D. Then $x = 0$ is stable. Further if $\dot{V} < 0$ in $D \setminus 0$, then $x = 0$ is asymptotically stable.*

Proof:
We have to establish a δ-Ball such that for all initial conditions in it, the corresponding trajectories of (28.1) are bounded by ϵ for all $t \geq 0$. Choose $r \in (0, \epsilon]$ such that $B_r \triangleq \{x \in \mathbb{R}^n : ||x|| \leq r\} \subset D$. Let $\alpha = \min\limits_{||x||=r} V(x)$. Since $V(x) > 0$ for $x \neq 0$, $\alpha > 0$. Take $\beta \in (0, \alpha)$. and define $\Omega_\beta \triangleq \{x \in B_r : V(x) \leq \beta\}$. Then Ω_β is in the interior of B_r. This claim is proved by contradiction. Suppose there is a point $p \in \Omega_\beta$ such that $p \in \partial B_r$. Then $V(p) \geq \alpha > \beta$, which is not true. Hence, the claim. The set Ω_β along with various other sets are depicted in Figure 29.1.

Since $\dot{V} \leq 0$ on D, the following holds $V(x(t)) \leq V(x(0)) \leq \beta$, which further implies that if $x(0) \in \Omega_\beta$ then $x(t) \in \Omega_\beta$ for all $t \geq 0$. This establishes the positive invariance of Ω_β.

Next, Ω_β is closed and bounded (since it is contained in B_r). Thus Ω_β is compact and the solution of (28.1) is unique and is defined for all $t \geq 0$ whenever $x(0) \in \Omega_\beta$. This follows from the fact that a locally Lipschitz function on a domain D is Lipschitz on a compact subset of D.

The continuity of V implies that there exists a $\delta > 0$ such that $||x|| < \delta \implies ||V(x)|| = V(x) < \beta$. Therefore $B_\delta \subset \Omega_\beta \subset B_r$ and $x(0) \in B_\delta \implies x(0) \in \Omega_\beta \implies x(t) \in \Omega_\beta \implies x(t) \in B_r$ for all $t \geq 0$. Therefore, $||x(0)|| < \delta \implies ||x(t)|| < r \leq \epsilon$ for all $t \geq 0$, thus establishing the stability of $x = 0$.

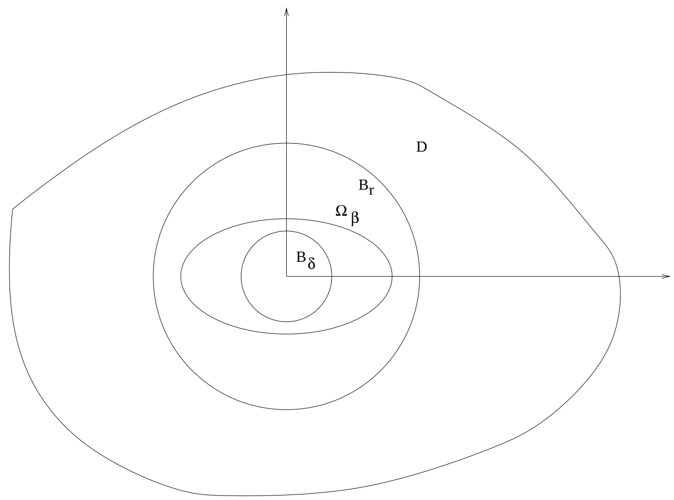

Figure 29.1: Geometric representation of sets in a plane

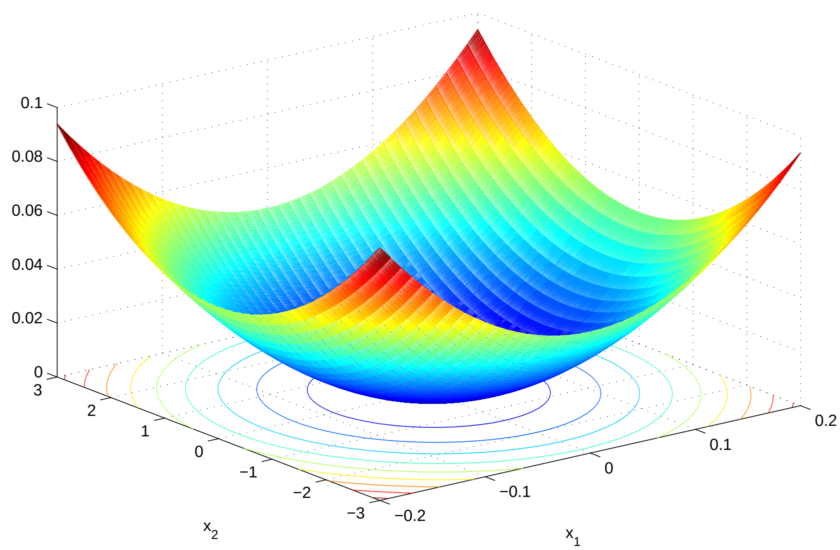

Figure 29.2: Level sets of a Lyapunov function V

Now given $\dot{V} < 0$, we have to show that the origin is asymptotically stable. Since we have already established stability, we are left to show that $x(t) \longrightarrow 0$ and $t \longrightarrow \infty$ or equivalently, for every $a > 0$, there is a $T > 0$ such that $||x(t)|| < a$ for all $t > T$. By the same argument, for every $a > 0$, we can choose a $b > 0$ such that $\Omega_b \subset B_a$. Therefor it suffices to show that $V(x(t)) \longrightarrow 0$ as $t \longrightarrow \infty$. Since $V(x(t))$ is monotonically decreasing and bounded from below ($V(0) = 0$) by zero, $V(x(t)) \longrightarrow c \geq 0$ as $t \longrightarrow \infty$. We show that $c = 0$ by contradiction. Suppose $c > 0$. By continuity of V there is a $d > 0$ such that $B_d \subset \Omega_c$. The limit $\lim\limits_{t \longrightarrow \infty} V(x) = c$ implies that the trajectory $x(t)$ lies outside the ball B_d

for all $t \geq 0$. Let $-\gamma = \max\limits_{d \leq ||x|| \leq r} \dot{V}(x)$. The existence of γ is ensured because of the fact that a continuous function (V) on a compact set ($\{x : d \leq ||x|| \leq r\}$) achieves its maximum and minimum. Since $\dot{V} < 0$, $-\gamma < 0$. Since $V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \leq V(x(0)) - \gamma t$, $V(x(t))$ will eventually become negative and hence contradicts $V(x(t)) \longrightarrow c \geq 0$. Thus $c = 0$. $\square$

Remark 29.0.30 *Stability of the equilibrium is also referred to as stability in the sense of Lyapunov or simply Lyapunov stable.*

Lecture-30

The use of Lyapunov's theorem 29.0.29 will be shown through the following examples. The theorem provides only sufficient conditions under which stability holds. It means that if the conditions are not met, then one cannot infer anything about the stability result and further investigation is required to arrive at a Lyapunov function. Even in the plane, the search for Lyapunov functions is nontrivial and several methods like the *variable gradient*, have been proposed to provide a way of arriving at a Lyapunov function. If the energy of the system is known, then it serves as the candidate Lyapunov function as can be seen through the simple pendulum example.

Example 30.0.31 *Consider the simple pendulum system*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 \end{aligned} \tag{30.1}$$

Consider the candidate Lyapunov function $V = a(1-\cos x_1) + x_2^2/2$. *Then* $\dot{V} = 0$. *Therefore the lower equilibrium point* $(0,0)$ *is Lyapunov stable. If friction is considered in the model, then the equations of motion are*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - bx_2 \end{aligned} \tag{30.2}$$

where, $b = kl$ *and* k *being the friction co-efficient. If we use the same* V, *then* $\dot{V} = -bx_2^2 \leq 0$, *that is* $\dot{V}$ *is negative semi-definite. Therefore, we cannot infer about the asymptotic stability of the origin, which the system possesses from the physics of the system or from the phase-portrait. Hence, we search for candidate Lyapunov function such that* $\dot{V}$ *is negative-definite. Consider* $V(x) = \frac{1}{2}x^\top Px + a(1-\cos x_1)$, *where* $P = P^\top > 0$ *should be selected in a manner that* $\dot{V} < 0$. *The derivative of* V *along the trajectories of* (30.2) *is given by*

$$\dot{V} \;=\; a(1-p_{22})x_2\sin x_1 - ap_{12}x_1\sin x_1 + (p_{11}-p_{12}b)x_1x_2 + (p_{12}-p_{22}b)x_2^2.$$

To eliminate the sign indefinite terms, we set $p_{22} = 1$ *and* $p_{11} = p_{12}b$. *To retain the sign-definite terms, let* $p_{12} = \frac{bp_{22}}{2}$. *Then* $\dot{V} = -\frac{ab}{2}x_1 \sin x_1 - \frac{b}{2}x_2^2 < 0$ *in* $D \stackrel{\triangle}{=} \{x \in I\!R^2 : x_1 \in (-\pi, \pi)\}$. *The corresponding* $V = x^\top \begin{bmatrix} b^2 & \frac{b}{2} \\ \frac{b}{2} & 1 \end{bmatrix} x + a(1 - \cos x_1) > 0$ *in* D. *Thus the origin is locally asymptotically stable.*

Example 30.0.32

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 - x_2^3 \end{aligned} \tag{30.3}$$

Clearly $(0,0)$ *is the unique equilibrium point. Consider the candidate Lyapunov function* $V = \frac{x_1^4}{4} + \frac{x_2^2}{2}$. *Then* $\dot{V} = -x_2^4 \leq 0$. *Therefore* V *is a Lyapunov function and the origin is Lyapunov stable.*

Example 30.0.33

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1(x_1) - x_2 - h_2(x_3) \\ \dot{x}_3 &= x_2 - x_3. \end{aligned} \tag{30.4}$$

where, h_1 *and* h_2 *are locally Lipschitz functions that satisfy* $h_i(0) = 0$ *and* $zh_i(z) > 0$, *i=1,2, for all* $z \neq 0$. *Clearly* $(0,0,0)$ *is the unique equilibrium point. Consider the candidate Lyapunov function* $V = \int_0^{x_1} h_1(z)dz + \int_0^{x_3} h_2(z)dz + \frac{x_2^2}{2}$. *Then* $\dot{V} = h_1\dot{x}_1 + h_2\dot{x}_3 + x_2\dot{x}_2 = -x_2^2 - x_3h_2(x_3) \leq 0$. *Therefore* V *is a Lyapunov function and the origin is Lyapunov stable.*

The construction of Lyapunov function is not straightforward for nonlinear systems, especially when the system does not possess energy-like function. A few approaches to constructing Lyapunov functions are variable gradient method, Krasovski's, Zubov's and Energy-Casimir methods. We focus on the variable gradient method.

Lecture-31

Variable gradient method

In this method, we assume a structure for the gradient $\frac{\partial V}{\partial x}$ and fix its components to render $\dot{V} < 0$. Let V be the candidate Lyapunov function whose gradient we assumed to have the form

$$\frac{\partial V}{\partial x}^{\top} = g(x)$$

where, $g(x) = [g_1(x)\ g_2(x)\ \ldots g_n(x)]^{\top}$. Poincaré Lemma gives us the condition under which a given vector is a gradient of some scalar function.

Lemma 31.0.34 *Given a smooth vector field $v(x)$, $v : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, there exists a smooth function $h(x)$, $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\nabla_x h = v(x)$ if and only if $\nabla_x v = (\nabla_x v)^{\top}$.*

The construct $g(x)$ is such that $\dot{V}$ is negative definite on a domain D. Finally, the Lyapunov function V is extracted from the line integral

$$V(x) = \int_0^x g^{\top}(s)ds = \int_0^x \sum_{i=1}^{n} g_i(s)ds_i. \tag{31.1}$$

Using the fact that the line integral of a gradient vector $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is path independent, the line integration in (31.1) can be taken along any path joining the origin to x. Choosing the path made up of line segments parallel to the coordinates axes, (31.1) becomes

$$\begin{aligned} V(x) &= \int_0^{x_1} g_1(s_1, 0, 0, \ldots, 0)ds_1 + \int_0^{x_2} g_2(x_1, s_2, 0, \ldots, 0)ds_2 \\ &\quad + \ldots + \int_0^{x_n} g_n(x_1, x_2, x_3, \ldots, s_n)ds_n. \end{aligned}$$

This method is illustrated through the following pendulum-like example.

Example 31.0.35

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2, \quad a > 0 \end{aligned} \tag{31.2}$$

where, $h(\cdot)$ is locally Lipschitz, $h(0) = 0$ and $yh(y) > 0$ for all $y \in (-b, c)$ for some $b, c > 0$. Assume

$$g(x) = \begin{bmatrix} x_1\alpha(x) + x_2\beta(x) \\ x_1\gamma(x) + x_2\delta(x) \end{bmatrix}$$

Thus, the Jacobian of $g(x)$ is symmetric iff

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

which yields

$$\beta(x) + x_2\frac{\partial\beta(x)}{\partial x_2} + x_1\frac{\partial\alpha(x)}{\partial x_2} = \gamma(x) + x_1\frac{\partial\gamma(x)}{\partial x_1} + x_2\frac{\partial\delta(x)}{\partial x_1} \tag{31.3}$$

Now the derivative of V along the trajectories of (31.2) is

$$\dot{V} = g(x)^\top f(x) = \alpha(x)x_1x_2 + \beta(x)x_2^2 - h(x_1)x_1\gamma(x) - h(x_1)x_2\delta(x) - ax_1x_2\gamma(x) - ax_2^2\delta(x) \tag{31.4}$$

To eliminate the sign-indefinite terms in (31.4), we set

$$\alpha(x)x_1 - h(x_1)\delta(x) - ax_1\gamma(x) = 0 \tag{31.5}$$

and (31.4) reduces to

$$\dot{V} = \beta(x)x_2^2 - h(x_1)x_1\gamma(x) - ax_2^2\delta(x).$$

Now fix $\delta(x) = \delta > 0$, $\beta(x) = \beta$ and $\gamma(x) = \gamma > 0$, as constants. Then the symmetry condition (31.3) reduces to $\beta + x_1\frac{\partial\alpha(x)}{\partial x_2} = \gamma$ which implies that $\alpha(x) = \alpha(x_1)$ and $\beta = \gamma$. Next,

$$\dot{V} = \gamma h(x_1)x_1 - x_2^2(a\delta - \gamma)$$

is negative-definite for $0 < \gamma < a\delta$ and $x_1 \in (-b, c)$. Finally, we extract V as

$$\begin{aligned} V &= \int_0^x (\delta h(y_1) + a\gamma y_1)dy_1 + (\gamma y_1 + \delta y_2)dy_2 \\ &= \int_0^{x_1} (\delta h(y_1) + a\gamma y_1)dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)dy_2 \\ &= \delta \int_0^{x_1} h(y_1)dy_1 + \frac{a\gamma}{2}y_1^2 + \gamma x_1x_2 + \frac{\delta}{2}y_2^2 \\ &= \delta \int_0^{x_1} h(y_1)dy_1 + \frac{1}{2}x^\top \underbrace{\begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}}_{>0} x. \end{aligned}$$

It is clear that V *is positive-definite in* $D \triangleq \{x : x_1 \in (-b, c)\}$ *and as already claimed* $\dot{V}$ *is negative-definite. Hence the origin is asymptotically stable.*

Lecture-32

Global asymptotic stability

The existence of a Lyapunov function $V(x)$ which proves the origin to be asymptotically stable does not guarantee that all trajectories originating from D will converge to the origin. For the origin to be *globally asymptotically stable* the Lyapunov function should be 'radially unbounded'.The Lyapunov function $V(x)$ is said to be radially unbounded if $V(x) \to \infty$ as $x \to \infty$, uniformly in x.

Define level sets of $V(x)$ as $\Omega_c = \{x \in D : V(x) \leq c\}$. If Ω_c is closed and bounded for any $c > 0$ (assured by radial unboundedness) and if $D = \mathbb{R}^n$ then the origin $x = 0$ is globally asymptotically stable. Note that the notion of GAS is applicable only to systems with isolated unique equilibrium point.

Region/Domain of attraction

Definition 32.0.36 *Let $\Phi(t, x_0)$ be the flow associated with a system. Then the region/domain of attraction R_A is defined as*

$$R_A = \{x : \Phi(t, x) \text{ is defined for all } t \geqslant 0 \text{ and } \Phi(t, x) = 0 \text{ as } t \to \infty\}$$

Lemma 32.0.37 *The set R_A is open, connected and positive invariant. Moreover, the boundary of R_A is formed by trajectories.*

Example 32.0.38 *Consider the system*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 4(x_1 + x_2) - h(x_1 + x_2) \end{aligned} \tag{32.1}$$

where $h(0) = 0$ *and* $h(p)p \geq 0 \ \forall \ |p| \leq 1$. *The origin* $(0,0)$ *is the unique equilibrium point of the system. We will show that it is asymptotically stable and estimate the region of attraction. Consider* $V(x) = x^\top P x$ *where* $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} > 0$. *The derivative of* V *along the trajectories of* (32.1) *is*

$$\begin{aligned} \dot{V} &= -x^\top \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x - 2h(x_1 + x_2)(x_1 + x_2) \\ &\leqslant -x^\top \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x. \end{aligned}$$

It is clear that $\dot{V}$ *is negative definite on* $D \triangleq \{x : |(x_1 + x_2)| \leq 1\}$. *Hence* $(0,0)$ *is asymptotically stable. From the phase portrait it can be seen that any trajectory within the strip* $|(x_1 + x_2)| \leqslant 1$ *converges to the origin.*

Finding the exact region of attraction is an onerous task even for second-order systems. However, an estimate of the region of attraction still provides guaranteed stability. One way of obtaining the estimate is to use the closed and bounded level sets of the Lyapunov function itself. A closed form solution for the estimates can be found when the Lyapunov function is quadratic in x.

Lecture-33

Estimate of R_A for quadratic Lyapunov functions

Let the quadratic Lyapunov function for a system be given as $V = x^\top Px; P = P^\top > 0$. Further, it is assumed that $\frac{\partial V}{\partial x} f(x) < 0$ in D. We can estimate the region/domain of attraction for such systems depending on the constraints on D.

Case 1: For constraints of the form $\Omega_c \subseteq D \stackrel{\triangle}{=} \{x \in \mathbb{R}^n : \| x \|_2 < r\}$ choose

$$c \;<\; \min_{\|x\|=r} x^\top Px = \lambda_{min}(P)r^2$$

where $\lambda_{\min}(P)$ is the minimum eigen value of P.

Case 2: For constraints of the form $\Omega_c \subseteq D \triangleq \{x \in \mathbb{R}^n : |b^\top x|_2 \leqslant r\}$, choose

$$c \;<\; \min_{|b^\top x| \leqslant r} x^\top Px$$

which can be solved using the necessary conditions for optimality. Setting the cost function as $L = x^\top Px + \lambda\{(b^\top x)^2 - r^2\}$ where $\lambda \in I\!R$ is the Lagrange multiplier. The necessary conditions for optimality are $\frac{\partial L}{\partial x} = 0$ and $\frac{\partial L}{\partial \lambda} = 0$. Now,

$$\frac{\partial L}{\partial x} \;=\; 2Px + 2\lambda(b^\top x)b = 0$$

$$\Rightarrow Px \;=\; -\lambda(b^\top x)b \tag{33.1}$$

$$\frac{\partial L}{\partial \lambda} \;=\; (b^\top x)^2 - r^2$$

$$\Rightarrow b^\top x = \pm r. \tag{33.2}$$

Solving (33.1) and (33.2), we obtain the optimum values as x^* and λ^* as

$$x^* = \pm\frac{rP^{-1}b}{b^\top P^{-1}b} \; ; \; \lambda^* = \frac{-1}{b^\top P^{-1}b}.$$

Hence $J^* = x^{*\top}Px^* = \frac{r^2}{b^\top P^{-1}b}$ and we choose $c < J^*$. If the set D is characterized by multiple constraints, the optimization problem becomes $\min\limits_{|b_i^\top x| \leqslant r_i} x^\top Px$ for $i = 1$ to m. It is a known fact that $J = \min(J_i, ..., J_m)$ where $J_i = \frac{r_i^2}{b_i^\top P^{-1}b_i}$.

Example 33.0.39 *Consider the system*

$$\begin{aligned} \dot{x}_1 &= -x_1(1-(x_1^2+x_2^2)) \\ \dot{x}_2 &= -x_2(1-(x_1^2+x_2^2)). \end{aligned} \tag{33.3}$$

We need to first show that the origin is asymptotically stable. Taking $V(x) = \frac{x_1^2+x_2^2}{2}$ *results in* $\dot{V}(x) = -(x_1^2+x_2^2)(1-(x_1^2+x_2^2))$. *Thus* $\dot{V} < 0$ *in* $D \triangleq \{(x_1,x_2) : x_1^2+x_2^2 < 1\}$. *Hence* $(0,0)$ *is asymptotically stable and an estimate of the domain of attraction is given by* $\{x \in I\!R^2 : x_1^2 + x_2^2 \leq 1\}$.

Example 33.0.40 *Consider the system*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 - (2x_2+x_1)(1-x_2^2) \end{aligned} \tag{33.4}$$

It is required to show that the origin is asymptotically stable and estimate its region of attraction. The asymptotic stability of the origin is established by considering $V(x) = x^\top \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} x$ *and noting that* $\dot{V} < 0$ *in* $D \triangleq \{x \in I\!R^2 : |x_2| \leq 1\}$. *The estimate of the domain of attraction is given by* $\Omega_c = \{x \in I\!R^2 : V(x) \leq c\}$, *where*

$$c < \frac{1}{[0 \;\; 1]\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}^{-1}[0 \;\; 1]^\top} = \frac{9}{5}$$

.

Example 33.0.41 *Consider a second-order system* $\dot{x} = f(x)$ *with asymptotically stable origin. Let* $V(x) = x_1^2 + x_2^2$, *and* $D = \{x \in I\!R^2 : |x_2| < 1, |x_1 - x_2| < 1\}$. *Suppose that* $\frac{\partial V}{\partial x}f(x)$ *is negative definite in* D. *Then, the estimate of the domain of attraction is given* *by* $\Omega_c = \{x \in I\!R^2 : V(x) \leq c\}$, *where* $c < \min(c_1, c_2)$ *with*

$$c_1 = \frac{1^2}{[1 \quad -1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}[1 \quad -1]^\top} = \frac{1}{2}$$

$$c_2 = \frac{1^2}{[10 \;\; 1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}[0 \;\; 1]^\top} = 1.$$

The estimate is $\{x_1^2 + x_2^2 \leq r^2\}$ *where* $r = \frac{1}{\sqrt{2}}$ *as shown Figure 33.1.*

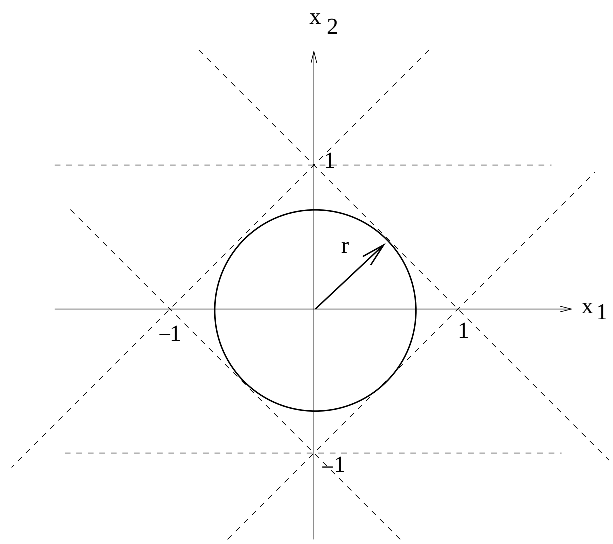

Figure 33.1: Estimate of the domain of the attraction of the origin

Lecture-34

Example 34.0.42 *Consider the system*

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1^3 + x_2 \\ \dot{x}_2 &= 3x_1 - x_2 \end{aligned} \tag{34.1}$$

a. *Find all equilibrium points of the system.*

b. *Using linearization, study the stability of the each equilibrium point.*

c. *Using a quadratic Lyapunov function, estimate the region of attraction of each asymptotically stable equilibrium point. Try to make your estimate as large as possible.*

d. *Construct the phase-portrait of the system and show on it the exact regions of attraction as well as your estimates.*

Solution*:*
The system has three equilibrium points given by $\mathcal{E} = \{(x_1, x_2) : (0,0), (2,6), (-2,-6)\}$. *The linearization about each of these equilibrium points is given by* $\dot{x} = A_i x, i = 1,2,3$ *where, the matrices* A_1, A_2 *and* A_3 *are given by*

$$A_1 = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}; A_2 = \begin{bmatrix} -11 & 1 \\ 3 & -1 \end{bmatrix}; A_3 = \begin{bmatrix} -11 & 1 \\ 3 & -1 \end{bmatrix}$$

The corresponding eigenvalue are given by $\mathrm{eig}(A_1) = \{2, -2\}$, $\mathrm{eig}(A_1) = \mathrm{eig}(A_2) = \{-11.2915, -0.7085\}$. *Therefore, the equilibrium point* $(0,0)$ *is a saddle and equilibrium points* $(2,6)$ *and* $(-2,-6)$ *are stable nodes.*

Next, we perform a coordinate change so that the equilibrium point $(2,6)$ *is the origin of the new coordinate system and the objective is establish the stability using a quadratic*

Lyapunov function. Define $z_1 = x_1 - 2, z_2 = x_2 - 6$, *then the resulting equations* (34.1) *in the new coordinates is given by*

$$\begin{aligned} \dot{z}_1 &= -11z_1 + z_2 - 6z_1^2 - z_1^3 \\ \dot{z}_2 &= 3z_1 - z_2 \end{aligned} \tag{34.2}$$

The equilibrium points are transformed as follows in z*-coordinates.*

$$\begin{aligned} (2,6) &\mapsto (0,0) \\ (0,0) &\mapsto (-2,-6) \\ (-2,-6) &\mapsto (-4,-12) \end{aligned}$$

Now consider a quadratic candidate Lyapunov function $V(z) = \frac{1}{2}z^\top Pz$, *where*

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} > 0.$$

The derivative of V *along the trajectories of* (34.2) *is given by*

$$\begin{aligned} \dot{V} = {} & z_1^2(-11p_{11} + 3p_{12}) + z_1z_2(p_{11} - 12p_{12} + 3p_{22}) + z_2^2(p_{12} - p_{22}) \\ & -p_{11}z_1^4 - 6p_{11}z_1^3 - p_{12}z_2z_1^3 - 6p_{12}z_1^2z_2. \end{aligned} \tag{34.3}$$

To eliminate the sign-indefinite terms, we set $p_{12} = 0$. *Then* (34.3) *can be expressed as*

$$\dot{V} = -z^\top Q(z_1)z \tag{34.4}$$

where,

$$Q(z_1) = \begin{bmatrix} p_{11}(z_1^2 + 6z_1 + 11) & \frac{(p_{11}+3p_{22})}{2} \\ \frac{(p_{11}+3p_{22})}{2} & p_{22} \end{bmatrix}.$$

It is easy to see that for $p_{11}, p_{22} > 0$, *the matrix* $Q(z_1)$ *is locally positive-definite around* $z_1 = 0$. *Fix,* $p_{11} = 3, p_{22} = 1$ *(see subsection 34.0.42). Then* $Q(z_1) > 0$ *in the set* $D_1 \triangleq \{(z_1, z_2) : z_1 \in (-2, \infty)\}$. *Thus* $\dot{V} < 0$ *in* D_1 *and it follows that* $(z_1, z_2) = (0, 0)$ *is asymptotically stable.*

To compute the region of attraction R_A *and compare it with the estimate of the domain of attraction* Ω_c, *we plot the vector field plot of* (34.2) *as shown in Figure 34.1. A local analysis about the saddle point shows that the stable eigenvectors satisfy* $z_2 - z_1 = 0$ *and the unstable eigenvectors satisfy* $3z_1 + z_2 = 0$. *The region of attraction is given by* $R_A = \{(z_1, z_2) \in I\!R^2 : 3z_1 + z_2 > 0\}$. *The estimate of the domain of attraction of the origin* $(z_1 = 0, z_2 = 0)$ *is given by* $\Omega_c \triangleq \{(z_1, z_2) \in (3z_1+z_2 > 0) \cap D_1 : 3z_1^2 + z_2^2 \leq c, c > 0\}$ *is compact.*

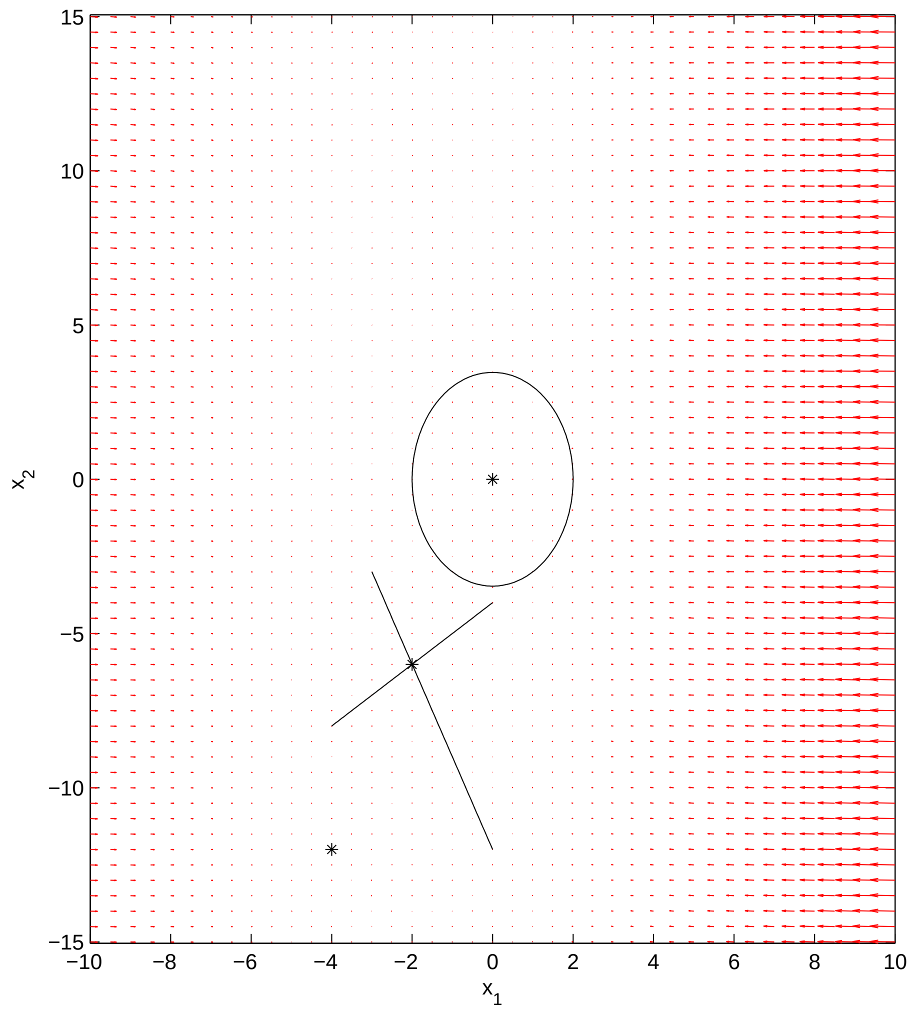

Figure 34.1: Vector field plot with various equilibria marked by $*$

Finding the largest D_1

To render $Q(z_1) > 0$*, we first note that* $p_{11}(z_1^2 + 6z_1 + 11) > 0$*, and thus it remains to satisfy* $h(z_1) \triangleq \det(Q(z_1)) > 0$ *where,*

$$h(z_1) = p_{11}p_{22}(z_1^2 + 6z_1 + 11) - \left(\frac{p_{11} + 3p_{22}}{2}\right)^2. \tag{34.5}$$

The function $h(z_1)$ has minima at $z_1 = -3$ for all values of p_{11} and p_{22}. So by substituting $z_1 = -3$ in (34.5)

$$\begin{aligned} h(-3) &= p_{11}p_{22}(9-18+11) - \left(\frac{p_{11}+3p_{22}}{2}\right)^2 \\ &= -p_{11}^2 - 9p_{22}^2 + 2p_{11}p_{22}. \end{aligned}$$

In order to render $Q(z_1) > 0$, the $det(Q(z_1)) > 0$, this implies $h(-3) \geqslant 0$, hence

$$-p_{11}^2 - 9p_{22}^2 + 2p_{11}p_{22} \geqslant 0.$$

Let us set the minimum value of $h(z_1) = 0$, thus

$$\begin{aligned} -p_{11}^2 - 9p_{22}^2 + 2p_{11}p_{22} &= 0 \\ \Rightarrow \left(\frac{p_{11}}{p_{22}}\right)^2 - 2\left(\frac{p_{11}}{p_{22}}\right) + 9 &= 0. \end{aligned} \tag{34.6}$$

Substituting $\frac{p_{11}}{p_{22}} = y$, we obtain imaginary roots of (34.6), thus for no positive real values of p_{11} and p_{22} the function $h(z_1)$ can achieve a positive minima. Further, by putting $z_1 = -2$ in (34.5) and solving we get

$$\begin{aligned} h(-2) &= p_{11}p_{22}(4-12+11) - \left(\frac{p_{11}+3p_{22}}{2}\right)^2 \\ &= -p_{11}^2 - 9p_{22}^2 + 6p_{11}p_{22} \end{aligned}$$

hence for $f(-2) \geqslant 0$,

$$-p_{11}^2 - 9p_{22}^2 + 6p_{11}p_{22} \geqslant 0.$$

Let us set the minimum value of $h(z_1) = 0$, thus

$$\begin{aligned} -p_{11}^2 - 9p_{22}^2 + 6p_{11}p_{22} &= 0 \\ \Rightarrow \left(\frac{p_{11}}{p_{22}}\right)^2 - 6\left(\frac{p_{11}}{p_{22}}\right) + 9 &= 0. \end{aligned} \tag{34.7}$$

Again, substituting $\frac{p_{11}}{p_{22}} = y$ and solving we get roots of (34.7) as $y = 3, 3$ which are real, positive and equal. Thus for $z_1 \leqslant -2$, $h(z_1)$ cannot have positive minima for all positive real values of p_{11} and p_{22}. Hence we have $\frac{p_{11}}{p_{22}} = y = 3 \Rightarrow p_{22} = 1, p_{11} = 3$.

Lecture-35

La Salle's invariance principle

In many problems, it is easy to establish the negative semi-definiteness of $\dot{V}$ rather than the negative definite property. In such a case, Lyapunov's theorem only establishes stability and nothing can be said about asymptotic stability. La Salle's invariance principle aids in establishing asymptotic stability property of the origin even when $\dot{V}$ is negative semi-definite.

Theorem 35.0.43 *Consider the system $\dot{x} = f(x)$ and let Ω be a compact set that is invariant with respect to this system. Let $V : D \to \mathbb{R}$ be continuously differentiable such that $\dot{V} \leq 0$ in Ω. Let E be the set of points in Ω such that $\dot{V} = 0$. Further let M be the largest invariant set in E. Then every solution starting in Ω converges to M as $t \to \infty$.*

Note that in theorem 35.0.43, the only requirement on V is that it be continuously differentiable and $\dot{V} \leq 0$ on a compact set Ω. Since we deal with positive-definite V, the choice $\Omega = \Omega_c$ ensures the compactness of Ω, at least for sufficiently small $c > 0$. The extension of La Salle's invariance principle to systems that possess a positive definite V is stated in the following Corollary, also called as the theorem of Barbashin and Krasovskii. The La Salle's invariance principle is illustrated through the following examples.

Corollary 35.0.44 *Consider the system $\dot{x} = f(x)$ and let $x = 0$ be the equilibrium point. Let $V : D \to \mathbb{R}$ be continuously differentiable that is positive definite and $\dot{V} \leq 0$ on D. Let $S = \{x \in \Omega_c : \dot{V} = 0\}$. If no solution of $\dot{x} = f(x)$ can stay identically in S except the trivial solution $x(t) = 0$ then the origin is asymptotically stable.*

Example 35.0.45 *For the pendulum system*

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -a\sin x_1 - bx_2 \end{aligned} \tag{35.1}$$

Take $V(x) = a(1-\cos x_1) + \frac{x_2^2}{2} > 0$ *on* $D = \{(x_1, x_2) : x_1 \in (-\pi, \pi)\}$. *It is seen that* $\dot{V} = -bx_2^2 \leqslant 0 \Rightarrow$ *is negative semi-definite. Let* $S = \{x \in \Omega_c, \dot{V} = 0\}$. *Then*

$$x(t) \in S \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x_2} \equiv 0 \Rightarrow \sin x_1 \equiv 0 \Rightarrow x_1 \equiv 0$$

Hence $(0,0)$ *being the only point in* M, *is asymptotically stable.*

Example 35.0.46

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 - x_2^3 \end{aligned} \tag{35.2}$$

Taking $V(x) = \frac{x_1^4}{4} + \frac{x_2^2}{2}$, $\dot{V} = -x_2^4$ *which is negative semi-definite. Let* $S = \{x \in \Omega_c, \dot{V} = 0\}$. *Then* $x(t) \in S \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x_2} \equiv 0 \Rightarrow -x_1^3 - x_2^3 \equiv 0 \Rightarrow x_1 \equiv 0 \ \Rightarrow M = (0,0)$. *Hence the origin is asymptotically stable.*

Example 35.0.47

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 - x_2|x_2| \end{aligned} \tag{35.3}$$

Taking $V(x) = \frac{x_1^2 + x_2^2}{2}$ *yields* $\dot{V} = -x_2^2(1 + |x_2|) \leqslant 0$. *Let* $S = \{x \in \Omega_c, \dot{V} = 0\}$. *Then* $x(t) \in S \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x_2} \equiv 0 \Rightarrow \dot{x_1} \equiv 0 \Rightarrow x_1 \equiv 0$, *thereby proving the asymptotic stability of the origin.*

Example 35.0.48

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1(x_1) - x_2 - h_2(x_3) \\ \dot{x}_3 &= x_2 - x_3 \end{aligned} \tag{35.4}$$

where $h_i(u)u \geqslant 0; i = 1, 2$. *Taking* $V(x) = \int_0^{x_1} h_1(z)dz + \int_0^{x_2} h_2(z)dz$ *we get* $\dot{V} = -x_2^2 - h_2(x_3)x_3$ *which is negative semi-definite. Then*

$$x(t) \in S \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x_2} \equiv 0 \Rightarrow \dot{x_3} \equiv 0 \Rightarrow x_3 \equiv 0$$

Hence, the origin $(0,0,0)$ *is asymptotically stable.*

Lecture-36

Stability of Linear systems

Consider the linear system given by $\dot{x} = Ax$ and a corresponding $V(x) = x^\top Px$ where $P^\top = P > 0$. Then,

$$\begin{aligned}
\dot{V} &= \dot{x}^\top Px + x^\top P\dot{x} \\
&= x^\top A^\top Px + x^\top PAx \\
&= x^\top (A^\top P + PA)x
\end{aligned}$$

Hence for asymptotic stability we get the Lyapunov equation given by

$$A^\top P + PA = -Q$$

for some $Q > 0$.

Definition 36.0.49 *A Matrix A is said to be Hurwitz if all its eigen values satisfy $Re(\lambda_i) < 0$.*

Theorem 36.0.50 *A matrix 'A' is Hurwitz and only if given any positive definite matrix Q, there exists a symmetric, positive definite matrix P such that $A^\top P + PA = -Q$ holds. Moreover the matrix P is unique.*

Example 36.0.51 *Consider the system*

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} x \tag{36.1}$$

Choosing $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ *and* $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$*, solving the Lyapunov equation we obtain*

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{12} \\ P_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} P_{11} \\ P_{12} \\ P_{22} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$P = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$ *is symmetric, positive definite and unique. From Theorem 36.0.50* $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ *is Hurwitz and hence* $(0,0)$ *is stable.*

Lyapunov's indirect method

Theorem 36.0.52 *Consider a system*

$$\dot{x} = f(x), \qquad x \in I\!R^n, \tag{36.2}$$

Let $x = 0$ *be the equilibrium point i.e* $f(0) = 0$*. Further assume* $f : D \to \mathbb{R}^n$ *is continuously differentiable on a domain* D *in the neighbourhood of the origin. Let* $A = \frac{\partial f}{\partial x}|_{x=0}$*. If the linearized system* (36.2) *is exponentially stable, then there exists a ball* $B \subset D$ *and constants* $c, \lambda > 0$ *such that for every solution* $x(t)$ *to the nonlinear system* (36.2) *that starts at* $x(0) \in B$*, we have*

$$\| x(t) \| \leq ce^{-\lambda(t)} \| x(0) \|, \quad \forall\, t \geq 0. \tag{36.3}$$

For proving the above theorem we require the following results:

1. If Q is positive definite then the following inequality holds

$$\lambda_{min}(Q) \| x \|_2^2 \leqslant x^\top Q x \leqslant \lambda_{max}(Q) \| x \|_2^2$$

Lemma 36.0.53 *Comparison Lemma Let* $V : I\!R^n \to I\!R$ *be a differentiable function such that* $\frac{\partial V}{\partial x}^\top f(x) \leq \mu V(x)$ *for some constant* $\mu \in I\!R$*. Then* $V(x(t)) \leq e^{\mu(t)} V(x(0)), \forall\, t \geq 0$*.*

The proof of the main theorem now follows.

Proof: Since f in (36.2) is twice differentiable, we know from Taylor's theorem that

$$r(x) \triangleq f(x) - \big(f(x_e) + Ax\big) = f(x) - Ax = O(\| x \|^2).$$

which means that there exists a constant c and a ball $\bar{B}$ around $x = 0$ for which

$$\| r(x) \| \leq c \| x \|^2, \qquad \forall x \in \bar{B}. \tag{36.4}$$

Since the linearized system is exponentially stable, there exists a positive-definite-matrix P for which

$$A'P + PA = -I.$$

Define a real-valued continuously differentiable function

$$V \triangleq x^\top P x$$

and its derivative along trajectories of (37.1) is given by

$$\begin{aligned} \dot{V} &= f(x)^\top Px + x^\top P f(x) \\ &= \big(Ax + r(x)\big)^\top Px + x^\top P\big(Ax + r(x)\big) \\ &= x^\top (A^\top P + PA)x + 2x\top Pr(x) \\ &= - \| x \|^2 + 2x^\top Pr(x) \\ &\leq - \| x \|^2 + 2 \| P \| \| x \| \| r(x) \| . \end{aligned} \tag{36.5}$$

Let $\epsilon > 0$ be sufficiently small so that the ellipsoid centered at $x = 0$ satisfies the following:

- $$\mathcal{E} \triangleq \{x \in I\!R^n : x^\top Px \leq \epsilon\} \subset \bar{B}.$$
- $$1 - 2c \| P \| \| x \| \geq \frac{1}{2} \implies \| x \| \leq \frac{1}{4\, c \| P \|}.$$

Thus, for $x \in \mathcal{E}$,

$$\begin{aligned} \dot{V} &\leq - \| x \|^2 + 2c \| P \| \| x \|^3 \\ &= -\big(1 - 2c \| P \| \| x \| \big) \| x \|^2 \\ &= -\tfrac{1}{2} \| x \|^2 \\ &< 0, x \neq 0. \end{aligned} \tag{36.6}$$

For this choice of ϵ, the set $\mathcal{E}$ is positively invariant and the origin is asymptotically stable. Further, from (36.6) and the fact that $x^\top Px \leq \| P \| \| x \|^2$, it follows that if $x(0)$ starts

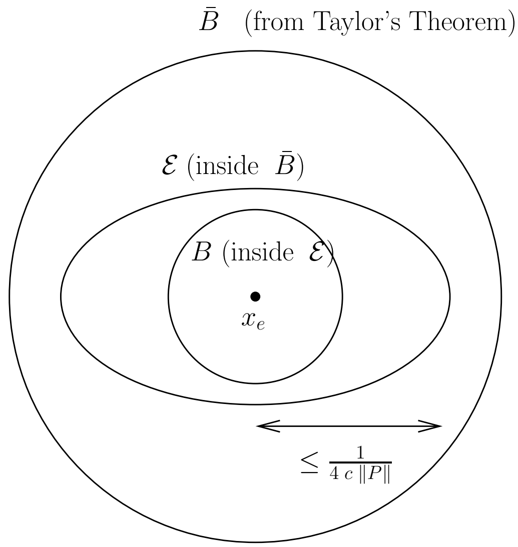

Figure 36.1: Construction of ball B

inside $\mathcal{E}$,

$$\dot{V} \leq -\frac{V}{2 \parallel P \parallel}$$

and therefore, by Comparison lemma 36.0.53, V and consequently x decrease to zero exponentially fast. The ball B around the origin in the statement of the theorem can be any ball inside $\mathcal{E}$.

Lecture-37

Instability

An equilibrium is unstable if it is not stable. There are several results establishing instability, but we shall use the result by Chetayev, which is stated as follows.

Theorem 37.0.54 *Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : D \longrightarrow \mathbb{R}$ be a continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $||x_0||$. Define a set $U = \{x \in B_r : V(x) > 0\}$, where $B_r = \{x \in D : ||x|| \leq r\}$ and suppose that $\dot{V}(x) > 0$ in U. Then $x = 0$ is unstable.*

Example 37.0.55 *Consider the following nonlinear system*

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= x_1^6 - x_2^3 \end{aligned} \tag{37.1}$$

Show the region $\Gamma = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\}$ is a positively invariant set and further show that the origin is unstable.

Note the system has two equilibrium points at $(0,0)$ and $(1,1)$. The region Γ is bounded by the curves (as shown in Figure 37.1) $\gamma_1 \stackrel{\triangle}{=} x_2 - x_1^2$ and $\gamma_2 \stackrel{\triangle}{=} x_2 - x_1^3$. The derivative of γ_1 along the trajectorioes of (37.1) is,

$$\begin{aligned} \dot{\gamma}_1 &= x_1^6 - x_2^3 - 2x_1(-x_1^3 + x_2) \\ &= (x_1^2 - x_2)^3 + 3x_1^2 x_2(x_1^2 - x_2) + 2x_1(x_1^3 - x_2) \\ &= 2x_1(x_1^3 - x_1^2) \quad \leq \quad 0 \end{aligned}$$

Hence, the direction of the vector field f on the curve γ_1 points into Γ. Similarly, the derivative of γ_2 along trajectories of (37.1) is,

$$\begin{aligned}\dot{\gamma}_2 &= x_1^6 - x_2^3 - 3x_1^2(-x_1^3 + x_2) \\ &= (x_1^2 - x_2)^3 + 3x_1^2 x_2(x_1^2 - x_2) + 3x_1^2(x_1^3 - x_2) \\ &= (x_1^2 - x_2)^3 + 3x_1^2 x_2(x_1^2 - x_2) \quad \geq \quad 0\end{aligned}$$

Hence, the direction of the vector field f on the curve γ_2 also points into Γ. Thus all the trajectories eminating from a neighbourhood of Γ are trapped in it, that is, Γ is positively invariant set.

Next, consider a ball centered at origin defined by $B_{2/3} = \{(x_1, x_2) \in \mathbb{R}^2 : ||x||_2 \leq \frac{2}{3}\}$. Define $U = \{x \in B_{2/3} : x_2 \geq x_1^3, x_2 \leq x_1^2\}$. Let $V = x_1^2 - x_1^3$ be the continuously differentiable function. $V(x) = 0$ on the boundary of U and since the origin also belongs to the boundary of U, $V(0) = 0$. There exists $x_0 \in U$, arbitrarily close to the origin, such that $V(x_0) > 0$ (for instance, consider x_0 as (0.2,0.02), then $V(x_0) = 0.032$). The

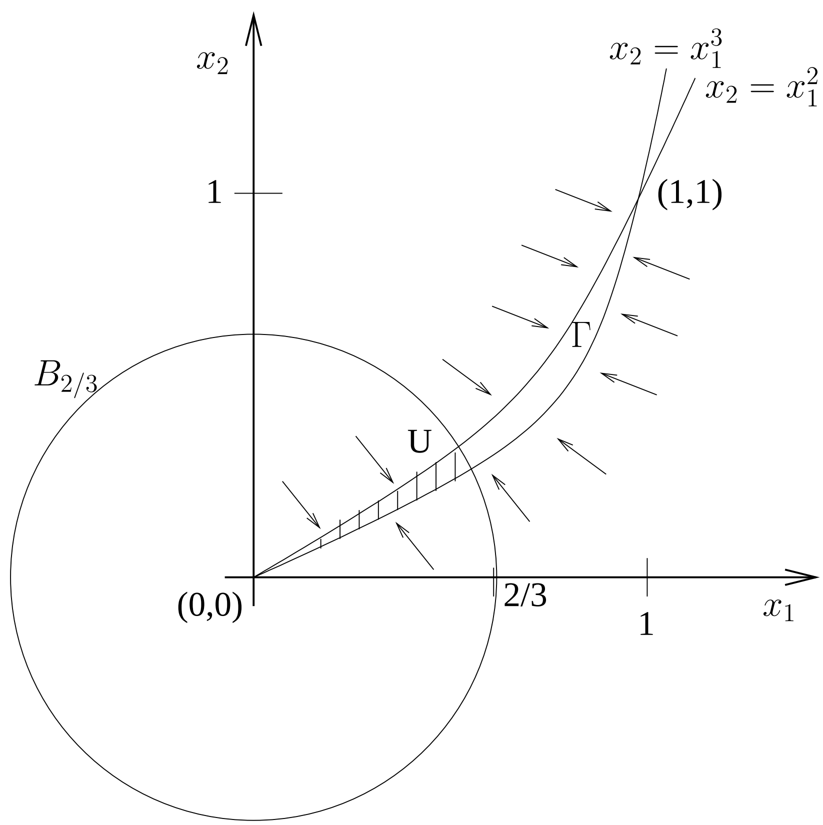

Figure 37.1: Regions Γ, U, $B_{2/3}$

derivative of V along the trajectories of (37.1) is,

$$\begin{aligned}\dot{V} &= 2x_1(-x_1^3+x_2) - 3x_1^2(-x_1^3+x_2) \\ &= (x_2 - x_1^3)x_1(2-3x_1) \quad \geq \quad 0 \qquad \text{for all} \quad x \in U.\end{aligned}$$

All the hypothesis of Chetayev's theorem hold, thereby proving the origin to be unstable.

Exercise problems

1. Use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable:

   a.

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2\\ \dot{x}_2 &= -x_2\end{aligned}$$

   b.

$$\begin{aligned}\dot{x}_1 &= x_2(1 - x_1^2)\\ \dot{x}_2 &= -(x_1 + x_2)(1 - x_1^2)\end{aligned}$$

2. Euler equations for a rotating rigid spacecraft are given by

$$J\dot{\omega} + \omega \times J\omega = u$$

   or

$$\begin{aligned}J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1\\ J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_1\omega_3 + u_2\\ J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3\end{aligned} \tag{37.2}$$

   where $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity vector ω along the principal axes, u_1, u_2, u_3 are the torque inputs applied about the principal axes, and $J_1 > J_2 > J_3 > 0$ are the principal moments of inertia. The rotational kinetic energy of the spacecraft is given by $T(\omega) = \frac{1}{2}(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2)$ while the total angular momentum of the body is given by $P(\omega) = \sqrt{J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2}$.

   a. Find all the equilibria of the rotational dynamics.

b. Compute $\dot{T}$ and $\dot{P}$ along the trajectories of (37.2). What can you conclude about stability?

c. Show that with $u_i = 0, i = 1, 2, 3$, the origin $\omega = 0$ is stable. Is it asymptotically stable?

d. Use the Lyapunov function $V(\omega) = (T(\omega) - T(\omega_e))^2 + [(P(\omega))^2 - (P(\omega_e))^2]^2$ to show that every equilibrium of the form $\omega_e = [\Omega \;\; 0 \;\;\; 0]^\top$ is Lyapunov stable.

e. Repeat the above step for the equilibria of the form $\omega_e = [0 \;\; 0 \;\;\; \Omega]^\top$.

f. Use the function $V(\omega) = \omega_1\omega_3$ to show that every equilibrium point of the form $\omega_e = [0 \;\;\; \Omega \;\;\; 0]^\top$ is unstable. The result can be proved as follows. Define the states as $(x_1, x_2, x_3) = (\omega_1, \omega_2 - \Omega, \omega_3)$. Then the rotating rigid body dynamics can be written as

$$\begin{aligned}
\dot{x}_1 &= \tfrac{1}{J_1}(J_2 - J_3)(x_2x_3 + x_3\Omega) \\
\dot{x}_2 &= \tfrac{1}{J_2}(J_3 - J_1)(x_1x_3) \\
\dot{x}_3 &= \tfrac{1}{J_3}(J_1 - J_2)(x_1x_2 + x_1\Omega)
\end{aligned} \tag{37.3}$$

The derivative of smooth function $V = x_1x_3$ along the trajectories of (37.3) is given by

$$\dot{V} = \frac{(J_2 - J_3)}{J_1}(x_2x_3^2 + \Omega x_2^2) + \frac{(J_1 - J_2)}{J_3}(x_1^2x_2 + \Omega x_1^2) > 0 \;\; \forall \; x_2 > 0.$$

Let U be a closed ball of radius r around the origin and define the region D_1 in U as $D_1 = \{x \in U : x_1x_3 > 0, x_2 > 0\}$. The boundary of D_1 is the surface $V(x) = 0$ and the sphere $||x|| = r$. Since $V(0) = 0$, the origin lies on the boundary of D_1. Further, in D_1, both V and $\dot{V} > 0$. By Chetayev's theorem, the origin is unstable.

g. Suppose that the torque inputs apply the feedback control $u_i = -k_1\omega_i$, where $k_i, i = 1, 2, 3$ are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

3. Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(x_1 + x_2) - h(x_1 + x_2)
\end{aligned}$$

where h is continuously differentiable and $zh(z) > 0 \; \forall \; z \in I\!R$. Using the variable gradient method, find a Lyapunov function that shows that the origin is globally asymptotically stable.

4. Show that the origin of

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 - x_2^3\end{aligned}$$

is globally asymptotically stable.

5. Consider the linear system $\dot{x} = (A - BR^{-1}B^\top P)x$, where $P = P^\top > 0$ and satisfies the Riccati equation

$$PA + A^\top P + Q - PBR^{-1}B^\top P = 0$$

where, $R = R^\top > 0$, and $Q = Q^\top \geq 0$. Using $V(x) = x^\top Px$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable when
(1) $Q > 0$.
(2) $Q = C^\top C$ and the pair (A, C) is observable.

6. Consider the tunnel diode circuit as shown in figure 1. The tunnel diode has a nonlinear voltage-controlled constitutive relationship that contains the region of negative resistance. This characteristic made it possible to construct bistable circuits which were used in switching or memory elements in early computers. The circuit equations are given by

$$\begin{aligned}L\dot{I} &= E - RI - V \\ C\dot{V} &= I - \hat{I}(V).\end{aligned}$$

Using the Lyapunov function,

$$\begin{aligned}P(I,V) &= \frac{C}{2}(E - RI - V)^2 + \frac{L}{2}(I - \hat{I}(V))^2 + \lambda[\frac{1}{2}RI^2 - EI \\ &\quad +IV - \int_0^V \hat{I}_g(V_g)dV_g]\end{aligned}$$

where $\lambda \in I\!R$ is free, find conditions on λ for asymptotic stability. Also find the condition on the steepness of slope of the negative resistance region of the tunnel diode.

7. Consider the following system

$$\begin{aligned}\dot{x}_1 &= (x_2 - 1)x_1^3 \\ \dot{x}_2 &= -\frac{x_1^4}{(1 + x_1^2)^2} - \frac{x_2}{(1 + x_2^2)}\end{aligned}$$

(a) Show that the origin is asymptotically stable. (b) Is V radially unbounded ?

8. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1^3 + x_2\\ \dot{x}_2 &= 3x_1 - x_2\end{aligned}$$

   c. Using a quadratic Lyapunov function, show that the equilibrium point $(-2, -6)$ is asymptotically stable. Clearly define the set D where $\dot{V} < 0$.

   d. On the plane, show the region of attraction R_A as well as the estimate of the region of attraction Ω_c.

9. Show that the origin of the system $\dot{x} = Ax$, where $A = \begin{bmatrix} -2 & 3 \\ -1 & -4 \end{bmatrix}$ is asymptotically stable using the Lyapunov equation $A^\top P + PA = -Q$, where $Q > 0$.

10. The dynamics of a Spring-mass system with viscous and Coulomb friction and zero external force is given by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

where, M is the mass, k is the spring constant, $c_1, c_2 > 0$ are friction coefficients and y is the displacement of the mass from a reference position. Show that the origin is globally asymptotically stable.

Module 7

Nonlinear control design

Objectives: To design nonlinear stabilizing control laws.

Lesson objectives

This module helps the reader in

- using Lyapunov function to extract control laws that guarantee stability.
- the use of sliding mode control technique to achieve robust stability.
-

Lecture-38

Control Lyapunov function

Consider the system $\dot{x} = f(x) + g(x)u$ where u is the control input. Our objective is to stabilize the origin/equilibirium point of interest. Considering a positive definite $V(x)$ we see

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u$$

We could choose a $u = \alpha(x)$ such that $\dot{V} \leqslant 0$. V is then called a *control Lyapunov function.*

Example 38.0.56 *Consider the problem of stabilizing the simple pendulum at a desired angle θ_d. By taking $x_1 = \theta - \theta_d; x_2 = \dot{\theta}$ we obtain the equations*

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -K\sin(x_1 + \theta_d) + c\tau \end{aligned} \tag{38.1}$$

Let $V(x) = \frac{x_1^2 + x_2^2}{2}$. Then

$$\begin{aligned} \dot{V} &= x_1 x_2 + x_2(-K\sin(x_1 + \theta_d) + c\tau) \\ &= x_2(x_1 - K\sin(x_1 + \theta_d) + c\tau) \end{aligned}$$

To force $\dot{V} = -K_1 x_2^2$ we have to make $x_1 - K\sin(x_1 + \theta_d) + c\tau = -K_1 x_2$. Hence we obtain

$$\tau = \frac{-[x_1 + K_1 x_2 - K\sin(x_1 + \theta_d)]}{c}$$

This would make $\dot{V}$ only negative semidefinite. Hence to confirm stability we have to rely on LaSalle's invariance principle.

$$x_2 \equiv 0 \Rightarrow \dot{x_2} \equiv 0 \Rightarrow -K\sin(x_1 + \theta_d) + c\tau \equiv 0 \Rightarrow x_1 \equiv 0 \Rightarrow \theta = \theta_d$$

Hence $M = (\theta_d, 0)$ and stability follows.

Example 38.0.57

$$\begin{aligned}\dot{x_1} &= -x_1 + x_2^2 \\ \dot{x_2} &= -x_1x_2 - x_2^2x_3 \\ \dot{x_3} &= -x_1^2x_3^2 + u\end{aligned}$$

The objective is to stabilize the origin. Taking $V(x) = \frac{x_1^2+x_2^2+x_3^2}{2}$ *we obtain* $\dot{V} = -x_1^2 - x_3[x_2^3 + x_1^2x_3^2 + u]$. *Proceeding as in the above example we obtain*

$$u = -kx_3 - x_2^2 - x_1^2x_3^2$$

To prove stability we again use LaSalle's invariance principle which shows that $(0,0,0)$ *is asymptotically stable.*

Swing-up control of the pendulum on a cart system

In Lecture 11, we presented the nonlinear model of the pendulum on a cart systems. We use the model so obtained to design a swing-up control law based on Lyapunov function. The control objective is to swing-up the pendulum from the vertically downward position to the vertically upward equilibrium position and then balance it at that position; termed as the swing-up phase and the capture phase. We address the swing-up problem through the following analysis. The balancing of the pendulum can be achieved using a linear feedback control and is not discussed here. The equations of motion of the pendulum on a cart system are given by

$$\dot{x} = f(x) + g(x)u \tag{38.2}$$

where,

$$f(x) = \begin{pmatrix} x_3 \\ x_4 \\ \frac{1}{\Delta}(a_2a_3x_4^2\sin x_2 - a_3a_3\sin x_2\cos x_2) \\ \frac{1}{\Delta}(-a_3x_4^2\sin x_2\cos x_2 + a_1a_4\sin x_2) \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 0 \\ \frac{a_2}{\Delta} \\ \frac{-a_3}{\Delta}\cos x_2 \end{pmatrix},$$

Consider the candidate Lyapunov function V defined by

$$V(x) = \frac{1}{2}(k_px_1^2 + k_dx_3^2 + k_e\hat{E}^2(x)) \tag{38.3}$$

where k_p, k_d, k_e are strictly positive constants and $\hat{E}(x) = E(x) - E_d$. The motivation for this choice of V achieves the following tasks:

- Pump-in energy $E(x) = E_d$, where $E_d = mgl$ is the potential energy of the system at the upward equilibrium position.
- Bring the cart to rest at the origin.

Note that V is positive-definite. Differentiating (38.3) along the trajectories of (38.2), we have

$$\begin{aligned}\dot{V} &= k_p x_1 x_3 + k_d x_3 f_3 + k_d x_3 g_3 u + k_e \hat{E}(x) x_3 u \\ &= x_3(k_p x_1 + k_d f_3 + (k_d g_3 + k_e \hat{E}(x))u). \end{aligned} \tag{38.4}$$

In taking the derivative of $\hat{E}$, we have used the passivity property of robotic system, in other words, $\dot{E}$ is the inner-product of the vector of generalized velocities and input vector. Since, the pendulum is not actuated, we have $\dot{E} = x_3 u$. To render the rate of change of V to be negative semidefinite, we set

$$k_p x_1 + k_d f_3 + (k_d g_3 + k_e \hat{E}(x))u = -x_3 \tag{38.5}$$

which results in

$$\dot{V} = -x_3^2. \tag{38.6}$$

Equation (38.5) can be rewritten as

$$\{k_p x_1 + k_d x_3 + (g_3(x) + k_e \hat{E}(x))u\} = -x_3 \tag{38.7}$$

From (38.7) we extract the control law u as

$$u = \frac{-\{x_3 + k_p x_1 + k_d f_3(x)\}}{k_d g_3(x) + k_e \hat{E}(x)} \tag{38.8}$$

The control law (38.8) will not encounter any singularities provided

$$(k_d g_3(x) + k_e \hat{E}(x)) \neq 0.$$

Points to ponder

- To avoid singularities in the control law (38.8), the initial energy of the system has to be restricted, which imposes a restriction on the set of initial conditions that the pendulum can swing-up. Obtain the initial condition set in terms of level sets of the Lyapunov function V that guarantees the swing-up of the pendulum.
- The $\dot{V}$ in (38.6) is negative semi-definite. Using LaSalle's invariance analysis, show that the control law (38.8) indeed meets the set objectives.

Lecture-39

Sliding mode control

Sliding mode is a nonlinear control design technique that has been used in almost every control application. It ensures robustness in the presence of uncertainties and external disturbances. For certain systems that cannot be smoothly stabilized (viz. Mobile robot, autonomous underwater vehicle etc.), sliding mode offers a possible solution in view of the discontinuous nature of the control law. Verification of stability is restricted to a reduced-order as compared to a Lyapunov based control technique.

Sliding mode terminology

Consider a system described by

$$\dot{x} = f(x) \tag{39.1}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

- $S(\boldsymbol{x})$ called *sliding function* such that $S : \mathbb{R}^n \longrightarrow \mathbb{R}$
- The set $\{\boldsymbol{x} \in \mathbb{R}^n : S(\boldsymbol{x}) = 0\}$ is called *sliding surface*.
- $\mathcal{O} \subset \mathbb{R}^n$ termed as *switching surface* such that it satisfies the following properties.
    a. $\mathcal{O}$ is connected and contains the equilibrium point $\boldsymbol{x}_e$.
    b. The closed-loop system confined to this set $\mathcal{O}$, called *equivalent dynamics*, is stable.

In a typical sliding mode control, the control objective is

- Design of a proper *switching surface* such that the closed-loop system behaves as desired.
- Reach the switching surface in *finite time* and maintain it there (positive invariance).

The design of sliding surface is greatly simplified if an affine in control system of the form $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$ can be represented in a regular form:

$$\begin{aligned}\dot{\boldsymbol{x}}_1 &= f_1(\boldsymbol{x}_1, \boldsymbol{x}_2)\\ \dot{\boldsymbol{x}}_2 &= f_2(\boldsymbol{x}_1, \boldsymbol{x}_2) + \textstyle\sum_1^m \bar{g}_j(\boldsymbol{x}_1, \boldsymbol{x}_2)u_j\end{aligned}$$

where
$\boldsymbol{x}_1 \in I\!R^{n-m}$, $\boldsymbol{x}_2 \in I\!R^{m}$ and $\bar{g}_j$ is related to g_j as $g_j = [0_{1\times(n-m)} \;\; \bar{g}_j^\top]^\top$.

Design outline

- Aim is to restrict the motion of the system to $\boldsymbol{x}_1$ dynamics
- Let
$$\mathcal{O} \triangleq \{\boldsymbol{x} \in I\!R^n : f_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = l(\boldsymbol{x}_1)\}$$
- The equivalent dynamics
$$\dot{\boldsymbol{x}}_1 = l(\boldsymbol{x}_1)$$

Case 1:

- Assign a linear dynamics to $l(\boldsymbol{x}_1)$ of the form
$$l(\boldsymbol{x}_1) = -K\boldsymbol{x}_1$$
where $K \in I\!R^{(n-m)\times(n-m)} > 0$. Equivalent dynamics $\dot{\boldsymbol{x}}_1 = -K\boldsymbol{x}_1$
- Difficulties
If $\mathcal{O}$ is the intersection of $(n-m)$ sliding surfaces and $n-m > m$, then it is difficult to reach using m control inputs.

Case 2:

- Consider the following candidate Lyapunov function $V : I\!R^{n-m} \longrightarrow I\!R$ defined as $V(\boldsymbol{x}_1) = \frac{1}{2}\boldsymbol{x}_1^\top \boldsymbol{x}_1$
- Construct $l(\boldsymbol{x}_1)$ such that $\dot{V}(\boldsymbol{x}_1) \leq 0$ on $\mathcal{O}$
- $\mathcal{O}$ is the intersection of m sliding surfaces.

Finite-time reachability

Definition 39.0.58 *The switching surface $\mathcal{O}$ is said to be finite-time reachable if for any $\boldsymbol{x}(0) \in \mathcal{U} \subseteq I\!R^n$, there exists $T \in [0, \infty)$ and an admissible control $u : [0, T] \longrightarrow I\!R^m$ with the property that $\boldsymbol{x}(T) \in \mathcal{O}$.*

- Assume that m sliding functions $S_j(\boldsymbol{x})$ such that $\mathcal{O} = \bigcap_{j=1}^{m} (S_j(\boldsymbol{x}) = 0)$.
- The dynamics obtained after differentiating each S_j into r_j times

$$\begin{pmatrix} S_1^{(r_1)} \\ S_2^{(r_2)} \\ \vdots \\ S_m^{(r_m)} \end{pmatrix} = R(\boldsymbol{x}) + Q(\boldsymbol{x})\boldsymbol{u}$$

where,

$$\begin{aligned} R(\boldsymbol{x}) &= \begin{bmatrix} L_f^{r_1} S_1 \\ L_f^{r_2} S_2 \\ \vdots \\ L_f^{r_m} S_m \end{bmatrix}; \\ Q(\boldsymbol{x}) &= \begin{bmatrix} L_{g_1} L_f^{r_1-1} S_1 & L_{g_2} L_f^{r_1-1} S_1 & \dots & L_{g_m} L_f^{r_1-1} S_1 \\ L_{g_1} L_f^{r_2-1} S_2 & L_{g_2} L_f^{r_2-1} S_2 & \dots & L_{g_m} L_f^{r_2-1} S_2 \\ \vdots & \vdots & \vdots & \vdots \\ L_{g_1} L_f^{r_m-1} S_m & L_{g_2} L_f^{r_m-1} S_m & \dots & L_{g_m} L_f^{r_m-1} S_m \end{bmatrix} \end{aligned}$$

Then term $L_f S_j(x) = \frac{\partial S_j}{\partial x}^{\top} f(x)$ refers to the Lie derivative of a real-valued function $S_i(x)$ along the vector field f. In coordinates, it is the inner-product of the gradient of S_i with f. In a similar way, the iterated Lie-derivative $L_f^{r_i} S_j(x) \triangleq \frac{\partial S_j}{\partial x} L_f^{r_i - 1}$.

- By the definition of well-defined vector relative degree , $Q(\boldsymbol{x})$ is invertible at $\boldsymbol{x}_e$ and thus for all $\boldsymbol{x}$ in the neighbourhood of $\boldsymbol{x}_e$, we have

$$\boldsymbol{u} = Q^{-1}(\boldsymbol{x}) \left\{ P(\boldsymbol{x}) - R(\boldsymbol{x}) \right\}.$$

- The choice of $P(\boldsymbol{x}) \in I\!R^m$ depends on the relative degree of the sliding functions. For example, if the vector relative degree is $\{1, 2\}$, then a few possible choices of

$P(\boldsymbol{x}) = [P_1(\boldsymbol{x}) \;\; P_2(\boldsymbol{x})]^\top$ are

$$P_1(\boldsymbol{x}) = \begin{cases} -K_1 \operatorname{sign}(S_1) \\ -S_1^\lambda \\ -K_1 \operatorname{sign}(S_1) - K_2 S_1 \\ -K_1 |S_1|^\lambda \operatorname{sign}(S_1) \end{cases}$$

$$P_2(\boldsymbol{x}) = \begin{cases} -\operatorname{sign}(S_2)|S_2|^a - \operatorname{sign}(\dot{S}_2)|\dot{S}_2|^b \\ -\operatorname{sign}(\dot{S}_2)|\dot{S}_2|^{1/3} - \operatorname{sign}\Big[(S_2 + \frac{3}{5}\dot{S}_2^{5/3})\Big]\Big|\Big(S_2 + \frac{3}{5}\dot{S}_2^{5/3}\Big)\Big|^{1/5} \\ -\gamma_1 \operatorname{sign}(\dot{S}_2 + \gamma_2 |S_2|^{1/2} \operatorname{sign}(S_2)) \end{cases}$$

where, $K_1 > 0, \lambda, b \in (0,1), a > \frac{b}{2-b}, \gamma_1, \gamma_2 > 0.$

A general form of the reaching law is given by

$$\dot{S} = -k_1 \operatorname{sign}(S) - k_2 h(S) \tag{39.2}$$

where, $k_1, k_2 > 0$ and $Sh(S) > 0, h(0) = 0$. Three special cases of (39.2) are:

1. Constant rate reaching law

$$\dot{S} = -k_1 \operatorname{sign}(S).$$

Large reaching time and severe chattering for large k_1.

2. Constant plus proportional rate reaching law

$$\dot{S} = -k_1 \operatorname{sign}(S) - k_2 S.$$

3. Power rate reaching law

$$\dot{S} = -k_1 |S|^\alpha \operatorname{sign}(S), \quad \alpha \in (0,1).$$

Eliminates chattering !

4. Fractional power reaching law

$$\dot{S} = -K_1 S^\alpha$$

Continuous, bounded, robust to disturbance and uncertainties.

Lecture-40

Sliding mode control: Example

The robustness property of sliding-mode technique is illustrated through the following example.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= h(x) + g(x)u \end{aligned} \tag{40.1}$$

where h and g are unknown nonlinear functions and $g(x) \geq g_0 > 0 \ \forall \ x$. The control objective is to stabilize the origin $(x_1, x_2) = (0, 0)$.

- Choose $S(x) = x_2 + kx_1, \ \ k > 0$.
- Assumption: h and g satisfy $|\frac{kx_2+h}{g(x)}| \leq \rho(x) \ \ \forall \ x$.
- The derivative of $V = \frac{S^2}{2}$ w.r.t time along the trajectories of (40.1) is

$$\dot{V} = S\dot{S} = S(x)(h(x) + kx_2 + g(x)u) \leq g(x)\rho(x)|S(x)| + g(x)S(x)u$$

- Choose $u = -\beta(x)\text{sign}(S(x))$, where $\beta(x)$ is free.
- Then, $\dot{S} = -(\beta(x) - \rho(x))g(x)|S|$.
- To render $\dot{V}$ negative definite, choose $\beta(x) \geq \rho(x) + \beta_0, \ \beta_0 > 0$, which implies

$$\dot{V} \leq -g(x)\beta_0|S| \leq -g_0\beta_0|S|$$

.

- The equivalent dynamics on the set $\{(x_1, x_2) : x_2 = -kx_1\}$ is given by $\dot{x}_1 = -kx_2$, which is exponentially stable.

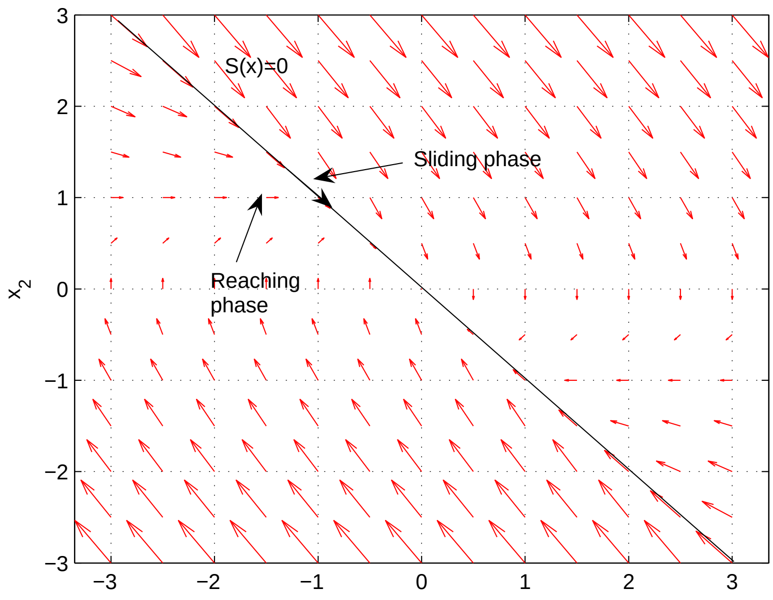

Figure 40.1: Vector field plot of a closed-loop second-order system