Course : Algorithmic Game Theory and Its Applications

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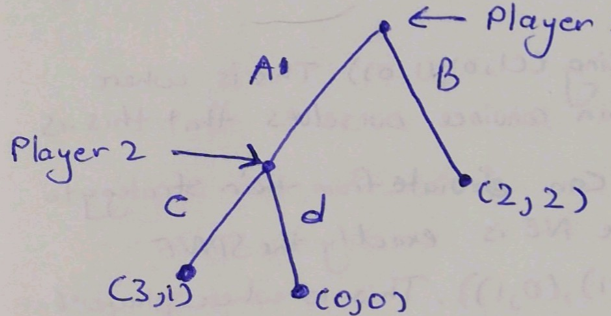
Question 1

a) Give an example of a pure NE which is not a SPNE, for a finite extensive form game of perfect information.

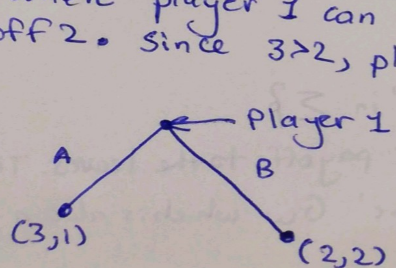
Let's consider the two player extensive form game shown below:



First, we will find ~~that~~ the SPNE of this game by using 'Backwards induction' and then we will calculate the NE of the strategic game. Then, we will show that there exists a pure NE in the strategic game that is not a SPNE.

We can notice that in this game there are exactly 2 subgames. Those are the subgame starting at the node where Player 2 can choose an action and the game itself.

We can see that there is only one action that Player 2 can make to maximise his outcome. In this case he plays action C and this is because of the fact that $1 > 0$. As a result we can reduce the tree to the figure below where player 1 can either play A with payoff 3 or B with payoff 2. Since $3 > 2$, player 1 will always play A



Question 1

a) By using backwards induction we have found the SPNE of the games that is player 1 plays A and player 2 plays c, (A, c)

Now we need to find the NE of this game. This game is given in strategic form by the following table

	c	d
A	(3, 1)	(0, 0)
B	(2, 2)	(2, 2)

Player 1 is row player and player 2 is column player

We can easily find the NE of this game, these are precisely the strategies that neither player can increase their payoff by deviating to anothes strategy.
In this game there are exactly 2 pure NE:

1. The first pure strategy profile being ((1, 0), (1, 0)). This is where player 1 plays A and player 2 plays c. We can convince ourselves that this is a pure Nash equilibrium since neither player can deviate from their strategy to increase their payoff. We can notice that pure NE is exactly the SPNE
2. The second pure strategy profile is ((0, 1), (0, 1)). This is where player 1 plays B and player 2 plays d. Same reasoning as point 1 is applied here.

As a result, we have found a NE, that is player 1 plays B with probability 1 and player 2 plays d with probability 1 which is not a SPNE

b) Suppose that we have some pure profile, $s = (S_1, \dots, S_n)$ that is a pure SPNE.
To prove that this is unique if we are in a finite extensive game of perfect information where there are no chance nodes and where no player gets the same payoff at any two distinct leaves, we will introduce some definitions. For a game G with game tree T and for $w \in T$, we define the subtree $T_w \in T$, by:

$$T_w = \{w' \in T' \mid w' = ww'' \text{ for } w'' \text{ in } \Sigma\}$$

Since tree is finite, we can just associate payoffs to the leaves. Thus, the subtree T_w, in an obvious way, defines a 'sub game' G_w which is also a perfect information game.

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b) The depth of a node w in T is its length $|w|$ as a string. The depth of tree T is the maximum depth of any node in T. The depth of a game G is the depth of its game tree.

To prove this, we will use proof by induction. We will show that every Subgame G_w has a unique pure strategy $s^w = (s_1^w, \dots s_n^w)$ therefore if we show that a game with the maximum depth of the tree has a unique pure strategy we have proved the statement.

Base case: Depth 0 : In this case we are at a leaf w. There is nothing to prove: This is because each player i will get payoff $U_i(w)$, and the strategies in the SPNE s^0 are "empty", this is because it doesn't matter which player's node w is, since there are no actions to take. Therefore for depth 0 there must be a unique SPNE which is exactly the 'empty' strategy for all players.

Base case : Depth 1 : We also need to define the base case when we are in depth one - This is to show that there must be a unique SPNE. Let Action $(1) = \{a_1', \dots a_r'\}$ be the set of actions available at the root of G_1. We need to show that the subtrees $T_{1a_j'}$ for $j = 1 \dots r$ each define a perfect information subgame $G_{1a_j'}$ of depth 1 which also have a unique SPNE $s^{1a_j} = (s_1^{1a_j'} \dots s_n^{1a_j'})$.

We know that since all the payoffs are unique since all payoff as distinct for any two leaves there must be exactly one pure SPNE for the depth 1, this is because there must be a leaf which has a strictly larger payoff than all the others. As a result, the players can not deviate from ~~this as~~ their strategies of picking the largest payoff since this payoff is unique. Therefore, there must be a unique action in Action(1) which maximises the payoff $U_i(1a_j'))$ for all players at this depth. As a result, we have found the unique SPNE for depth 1.

Inductive step: Suppose depth of G_w is $k+1$. Let Action $(w) = \{a_1', \dots a_r'\}$ be the set of actions available at the root of G_w the subtrees $T_{wa_j'}$ for $j = 1, \dots, r$; each define a perfect information subgame $G_{wa_j'}$ of depth $\leq k$ which have a unique SPNE $s^{wa_j} = (s_1^{wa_j}, \dots, s_n^{wa_j})$ by induction.

Now we need to show that this holds for the game G_w at depth $k+1$. Since there are no chance nodes we only need to consider the action that each player can take.

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b) Let $w \in PI_i$ $i>0$ that is the root, w of T_w belongs to player i, where PI_i is the perfect information ~~game~~ set for player i.
For $a \in Action(w)$, let $h_i^{wa}(s^{wa})$ be the expected payoff to player i in the subgame G_{wa}.

Let $a' = \arg\max_{a \in Action(w)} h_i^{wa}(s^{wa})$. That is the action that maximises the payoff to player i when we are at node w. However, we know that since we can reduce each sub-game G_{wa_j} to a single leaf and we know that all the payoffs are unique, then it must be the case that a' is unique for all players i, that is there is only one action that can be taken since deviating will result in a smaller payoff.

Therefore, we can define for all players $i' \neq i$, the $S_{i'}^{w}$ for player i' be the union $\bigcup_{a \in Action(w)} S_{i'}^{wa}$ of its pure strategies in each of the subgames, which we know that all the pure strategies in the subgames are unique by induction.

Likewise, we can define for player i the S_i^w to be $\bigcup_{a \in Action(w)} S_i^{wa} \cup \{w \to a'\}$ that is the union of its pure strategies in each of the subgames including the final action they take at the node w, which ~~is~~ in this case is a' which is a unique action.

Therefore we have defined a unique SPNE given by $s^w = (S_1^w, \dots, S_n^w)$, where the strategies S_i^w have been defined above. Since, we have shown it is true for $k+1$, and we know it is true for the base cases then by Induction we have shown it is true for all k, thus completing the proof.

c) A subgame perfect pure Nash Equilibrium is a SPNE where all the strategies are pure. Therefore, if we find a SPNE where the only NE are mixed strategies, but the whole game is a NE, then we have given an example. In this question, let's deal with the finite extensive form game given below:

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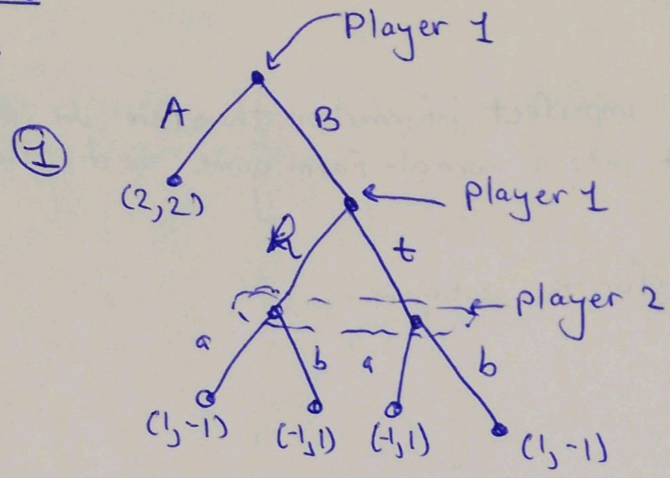
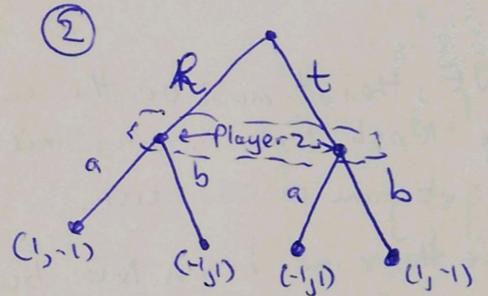
c)



We can note that this is a game of imperfect information as we can see this from player's 2 information set. Furthermore, we can divide the problem into subgames, the game itself and the subgame which starts where player 1 has to decide between h or t. A graph of this is provided below:



if we consider the first normal form game that is the normal form representation of the whole extensive form game (①), we have the following payoff table:

	a	b
Ah	(2,2)	(2,2)
At	(2,2)	(2,2)
Bh	(1,-1)	(-1,1)
Bt	(-1,1)	(1,-1)

We can see that we have several pure ~~strategies~~ NE in particular: {(Ah, a), (Ah, b), (At, a), (At, b)}; this is because no player can deviate from their strategies to increase their expected payoff. Therefore, we have shown that this game does indeed have at least one pure NE.

Now, we can use 'backwards induction' to find SPNE. We will start by considering the subgame in ②.

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c) We can notice that we have imperfect information therefore we need to solve this by first converting it into a normal-form game and then finding the NE of the game.

Below we have payoff table for this subgame:

	a	b
h	(1,-1)	(-1,1)
t	(-1,1)	(1,-1)

We can see that this game has no pure NE since it is always the case that one of two players can deviate from their strategies to increase their expected payoff. In fact, this game is the matching pennies game. This problem has only one NE and it is the mixed NE where both players play either tactic with probability 1/2.

Therefore, since this sub-game has no pure NE, then it must be the case that there cannot exist a subgame perfect pure Nash Equilibrium, since by definition it must have a pure NE at every subgame of the tree.

Therefore, we have found such an example where the game has a NE but no subgame perfect pure Nash Equilibrium.