Course: Algorithmic Game Theory and Its Applications
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Question 4

(a) We assume that the bidders bid their true valuations (which we can, since in the VCG mechanism bidding the true valuations is a dominant strategy for all bidders). Then given these valuations, the VCG mechanism firstly picks an outcome that maximises the sum total valuation of all the bidders, i.e. maximises the total social welfare of the outcome. This is given by:

$$c' \in \operatorname{argmax}_{c \in C} \sum_{k \in V} v_k(c)$$

where we have a set $V = \{1, \dots, n\}$ of n agents declare their valuation functions, $v_k : C \to \mathbb{R}_{\geq 0}$, $i \in V$, over the set C of possible outcomes. To find outcomes that maximise this function we ~~set~~ fix the outcome for player S, and therefore we just need to find the combinations of the remaining outcomes for players L and B such that they maximise the sum total valuation. Below are the results when we fix the outcomes for player S.

Player S	Sum Total Valuation	Outcome
$\{\phi\}$	55	$S=\{\phi\}, L=\{T_1,T_3\}, B=\{T_2\}$
$\{T_1\}$	55	$S=\{T_1\}, L=\{\phi\}, B=\{T_2,T_3\}$
$\{T_2\}$	53	$S=\{T_2\}, L=\{T_1,T_3\}, B=\{\phi\}$
$\{T_3\}$	51	$S=\{T_3\}, L=\{T_1,T_2\}, B=\{\phi\}$
$\{T_1,T_2\}$	41	$S=\{T_1,T_2\}, L=\{T_3\}, B=\{\phi\}$
$\{T_1,T_3\}$	54	$S=\{T_1,T_3\}, L=\{\phi\}, B=\{T_2\}$
$\{T_2,T_3\}$	39	$S=\{T_2,T_3\}, L=\{\phi\}, B=\{T_1\}$
$\{T_1,T_2,T_3\}$	54	$S=\{T_1,T_2,T_3\}, L=\{\phi\}, B=\{\phi\}$

Now let's go over them one by one:

1. Player S outcome is $\{\phi\}$

if this is the case then we know that Players L and B must have between them a combination of all outcomes $\{T_1,T_2,T_3\}$:

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- Player L outcome: ϕ and players B outcome: T_1, T_2, T_3:

Sum of total valuation is $V_S(\{\phi\}) + V_L(\{\phi\}) + V_B(\{T_1, T_2, T_3\}) = 0 + 0 + 54 = 54$

- Player B outcome: ϕ and player L outcome: T_1, T_2, T_3:

Sum of total valuation $= V_S(\{\phi\}) + V_B(\{\phi\}) + V_L(\{T_1, T_2, T_3\}) = 53$

- Player L outcome: T_1 and player B outcome: T_2, T_3:

Similarly, sum of total valuation $= 43$

- Player L outcome: T_2 and player B outcome: T_1, T_3:

Sum of total valuation is 35

- Player L outcome: T_3 and player B outcome T_1, T_2:

Sum of total valuation is 51

- Player B outcome is T_1 and player L outcome: T_2, T_3:

Sum of total valuation is 47

- Player B outcome is T_2 and player L outcome is: T_1, T_3:

Sum of total valuation is 55

- Player B outcome is T_3 and player L outcome is T_1, T_2:

Sum of total valuation is 42

The outcome that maximises sum total valuation is when Player S gets $\{\phi\}$ is when player 2 outcome is $\{T_2\}$ and player L outcome: $\{T_1\}$, $\{T_3\}$ where the valuation is 55. This is the entry in the first row.

2. Player S outcome is $\{T_1\}$. Then we know that in this case that players L and B must have as an outcome some combinations of $\{T_2, T_3\}$

- Player L outcome is ϕ and Player B outcome: T_2, T_3

Sum total valuation is $V_S(\{T_1\}) + V_L(\{\phi\}) + V_B(\{T_2, T_3\}) = 55$

- Player B outcome is ϕ and player L outcome: T_2, T_3

Sum of total valuation is $V_S(\{T_1\}) + V_B(\{\phi\}) + V_L(\{T_2, T_3\}) = 53$

- Player L outcome is T_2 and player B outcome is T_3

Sum of total valuation is 27

- Player L outcome is T_3 and player B outcome is T_2

Sum of total valuation is 46

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a) As a result the outcome that maximises sum total valuation when Player S gets $\{T_1\}$ is when Player B outcome is $\{T_2, T_3\}$ and player L outcome is $\{\phi\}$. Sum total valuation is 55. This is entry in second row of the table

3. Player S outcome is $\{T_2\}$. Then we know players L and B must have as outcome some combination of $\{T_1, T_3\}$.

- Player L outcome = $\{\phi\}$ and player B outcome is $\{T_1, T_3\}$

Sum total valuation = $V_S(\{T_2\}) + V_L(\{\phi\}) + V_B(\{T_1, T_3\}) = 44$

- Player B outcome is $\{\phi\}$ and player L outcome is $\{T_1, T_3\}$

Sum total valuation is 53

- Player L outcome is $\{T_1\}$ and player B outcome is $\{T_3\}$

Sum total valuation is 24

- Player L outcome is $\{T_3\}$ and player B outcome is $\{T_1\}$

Sum total valuation is 38

Outcome that maximises sum total valuation when Player S gets $\{T_2\}$ is when player L outcome is $\{T_1, T_3\}$ and player B outcome is $\{\phi\}$ with a sum total valuation of 53.

4. Player S outcome is $\{T_3\}$. Player L and B will get a combination of $\{T_1, T_2\}$.

- Player L outcome is $\{\phi\}$ and player B outcome $\{T_1, T_2\}$:

Sum total valuation is $V_S(\{T_3\}) + V_L(\{\phi\}) + V_B(\{T_1, T_2\}) = 44$

- Player B outcome is $\{\phi\}$ and player L outcome is $\{T_1, T_2\}$:

Sum total valuation is 51

- Player L outcome is T_1 and player B outcome is T_2

Sum total valuation is 35

- Player L outcome is T_2 and player B outcome is T_1

Sum total valuation is 24

Outcome that maximises sum total valuation ~~is~~ when player S gets $\{T_3\}$ is when player L outcome: $\{T_1, T_2\}$ and player B outcome is $\{\phi\}$ where the valuation is 51.

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5. Player S outcome is $\{T_1, T_2\}$. T_1 and T_3 will get a combination of $\{T_3\}$

- Player L outcome : ϕ and Player B outcome : $\{T_3\}$:

Sum total valuation is $V_S(\{T_1, T_2\}) + V_L(\{\phi\}) + V_B(\{T_3\}) = 33$

- Player L outcome is T_3 and Player B outcome is $\{\phi\}$:

Sum total valuation is 41

As a result the outcome that maximises sum total valuation when Player S gets $\{T_1, T_2\}$ is when L outcome $\{T_3\}$ and B outcome : $\{\phi\}$ where the valuation is 41

6. Player S outcome is $\{T_1, T_3\}$

Then we know that in this case we have that players L and B must have as an outcome some combination of $\{T_2\}$:

- Player L outcome : $\{\phi\}$ and player B outcome : $\{T_2\}$

Sum of total valuation is $V_S(\{T_1, T_3\}) + V_L(\{\phi\}) + V_B(\{T_2\}) = 54$

- Player L outcome : $\{T_2\}$ and player B outcome : $\{\phi\}$

Sum of total valuation is 43

As a result, the outcome that maximises the sum total valuation when Player $S = \{T_1, T_3\}$ is when player L outcome is $\{\phi\}$ and player B outcome : $\{T_2\}$ where valuation is 54

7. Player S outcome is $\{T_2, T_3\}$. Players L and B will get a combination of $\{T_1\}$

- Player L outcome : $\{\phi\}$ and player B outcome is $\{T_1\}$

Sum of total valuation is $V_S(\{T_2, T_3\}) + V_L(\{\phi\}) + V_B(\{T_1\}) = 39$

- Player B outcom : $\{\phi\}$ and player L outcome is $\{T_1\}$

Sum of total valuation is 33

As a result outcome that maximises the sum total valuation when player S gets $\{T_2, T_3\}$, is when player L outcome is $\{\phi\}$ and player B outcome is $\{T_1\}$ with sum total valuation of 39

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a) 8. Player S outcome is $\{T_1, T_2, T_3\}$. In this case both player L and B will have outcome of ϕ and sum of total valuation is

$$V_S(\{T_1, T_2, T_3\}) + V_L(\{\phi\}) + V_B(\{\phi\}) = 54$$

As a result, we have found all the outcomes that maximise the function while fixing outcomes of player S.
We have found 2 cases where the sum total valuation is the biggest and is equal to 55. VCG mechanism picks one of the possible outcomes and then asks the bidders to pay the VCG payments associated with that outcome
VCG mechanism does not specify which of these outcomes is to be preferred.
So, I will use the outcome $S = \{T_1\}$, $L = \{\phi\}$, $B = \{T_2, T_3\}$ to calculate the price that each player will pay for each painting. To do this, we need to calculate the price given by:

$$P_i(c') = \left(\max_{c \in C} \sum_{k \in V/\{i\}} V_k(c)\right) - \sum_{k \in V/\{i\}} V_k(c')$$

The price paid for each player when the outcome c' is given by $S = \{T_1\}$, $L = \{\phi\}$, $B = \{T_2, T_3\}$ is calculated by:

1. Price paid by Player S when outcome is c':

$$P_S(c') = \left(\max_{c \in C} \sum_{k \in V/\{S\}} V_k(c)\right) - \sum_{k \in V/\{S\}} V_k(c')$$
$$= (V_L(\{T_1, T_3\}) + V_B(\{T_2\})) - (V_L(\{\phi\}) + V_B(\{T_2, T_3\}))$$
$$= 37 + 18 - (0 + 39) = 16$$

2. Price paid by player L when outcome is c':

$$P_L(c') = \left(\max_{c \in C} \sum_{k \in V/\{L\}} V_k(c)\right) - \sum_{k \in V/\{L\}} V_k(c')$$
$$= (V_S(\{T_1\}) + V_B(\{T_2, T_3\})) - (V_S(\{T_1\}) + V_B\{T_2, T_3\})$$
$$= 16 + 39 - (16 + 39)$$
$$= 0$$

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a) 3. Price paid by player B when outcome is c'

$$P_B(c') = \left(\max_{c \in C} \sum_{k \in V/\{B\}} V_k(c)\right) - \sum_{k \in V/\{B\}} V_k(c')$$

$$= (V_S(\{T_1, T_2, T_3\}) + V_L(\{\phi\})) - (V_S(\{T_1\}) + V_L\{\phi\})$$

$$= (54+0) - (16+0)$$

$$= 38$$

As a result, we have found VCG outcome for this auction. In particular, we have that:

1. Player S will get painting T_1 and will pay a price of 16×10^5 pounds
2. Player L will get no painting and pay 0 pounds
3. Player B will get paintings T_2, T_3 and pay a price of 38×10^5 pounds

b) As I have shown in the previous section the VCG outcome that I calculated in the previous section is not unique. This is because we have found another VCG outcome that has same sum total valuation, namely 55.

In particular, the other auction that has this valuation is when $S = \{\phi\}$, $L = \{T_1, T_3\}$, $B = \{T_2\}$, where we can calculate the sum total valuation as:

$$V_S(\{\phi\}) + V_L(\{T_1, T_3\}) + V_B(\{T_2\}) = 0 + 37 + 18 = 55$$

So we need to check if the prices paid by each player are uniquely determined. To do this, we need to calculate the price that each player pays under this new VCG outcome.

So the price paid for each player is where outcome x' is given by $S = \{\phi\}$, $L = \{T_1, T_3\}$, $B = \{T_2\}$:

. The price paid by player S when outcome is x':

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$$P_S(x') = \left(\max_{x \in C} \sum_{k \in V / \{S\}} V_k(x)\right) - \sum_{k \in V / \{S\}} V_k(x')$$

$$= (V_L(\{T_1, T_3\}) + V_B(\{T_2\})) - (V_L(\{T_1, T_3\}) + V_B(\{T_2\}))$$

$$= 37 + 18 - (37 + 18) = 0$$

2. Price paid by Player L when outcome is x':

$$P_L(x') = \left(\max_{x \in C} \sum_{k \in V / \{L\}} V_k(x)\right) - \sum_{k \in V / \{L\}} V_k(x')$$

$$= V_S(\{T_1\}) + V_B(\{T_2, T_3\}) - (V_S(\{\phi\}) + V_B(\{T_2\}))$$

$$= 16 + 39 - (0 + 18)$$

$$= 37$$

3. Price paid by Player B when the outcome is c':

$$P_B(x') = \left(\max_{x \in C} \sum_{k \in V / \{B\}} V_k(x)\right) - \sum_{k \in V / \{B\}} V_k(x')$$

$$= (V_S(\{T_1, T_2, T_3\}) + V_L(\{\phi\}) - (V_S(\{\phi\}) + V_L(\{T_1, T_3\}))$$

$$= 54 + 0 - (0 + 37) = 17$$

As a result, we have found VCG outcome for this auction. In particular we have that:

1. Player S will get no painting and pay 0 pounds
2. Player L will get painting T_1 and T_3 and pay 37×10^5 pounds
3. Player B will get painting T_2 and pay 17×10^5 pounds

So to answer the second question, we say that VCG prices are uniquely determined if with every VCG outcome for that setting, every one of those n players has to pay exactly the same VCG price, as with any VCG outcome.

We can see that we have found two VCG outcomes but the players pay a different amount so as a result the VCG prices paid by the players are not uniquely determined.

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c) We have leaved from the notes that the VCG mechanism is incentive compatible (i.e. strategy proof). This implies that declaring their true valuation function V_i is a (weakly) dominant strategy for all players i. However, in many cases a bidders valuation may not be independent. This leads for example to Winner's Curse. As a result, information about other people's valuation matters a lot. This is the case because if we know what other people are expected to value, then the bidders might change their valuation. That means that It is vulnerable to bidder collusion. if all bidders in Vickrey auction reveal their valuations to each other, they can lower some or all of their valuations, while preserving who wins the auction.

The problem of using VCG is that we have to compute an outcome c' that maximises the total value $\sum_{i \in p} V_i(c)$. That is an NP hard problem to solve and as a result we will not easily find such a solution. So I would not suggest using the VCG mechanism for this auction

An alternative that can be used instead of VCG mechanism is the Generalized Second Price Auction. In this case truth telling is not a dominant strategy and we are ensured that the minimum equilibrium in which payoffs correspond precisely to the VCG, however if we end up at any other equilibrium then the revenue of the seller is higher than VCG. Thus the auction house can ensure a payoff at least as good as VCG.