Course: Algorithmic Game Theory and Its Appliations

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Coursework 1

Question 2

a) By using the provided payoff matrix, we can define the following LP problem in order to compute minimax profile. The below are defined for player 1:

Maximize v

subject to

$$3x_1 + 7x_2 + x_3 + x_4 + 4x_5 \geq v$$
$$3x_1 + 8x_2 + 2x_3 + 4x_4 + 7x_5 \geq v$$
$$9x_1 + 4x_2 + 5x_3 + 4x_4 + 7x_5 \geq v$$
$$6x_1 + 5x_2 + 6x_3 + 5x_4 + 8x_5 \geq v$$
$$2x_1 + 3x_2 + 4x_3 + 9x_4 + 3x_5 \geq v$$
$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

This is equivalent to

Maximize v

subject to

$$v - 3x_1 - 7x_2 - x_3 - x_4 - x_5 \leq 0$$
$$v - 3x_1 - 8x_2 - 2x_3 - 4x_4 - 7x_5 \leq 0$$
$$v - 9x_1 - 4x_2 - 5x_3 - 4x_4 - 7x_5 \leq 0$$
$$v - 6x_1 - 5x_2 - 6x_3 - 5x_4 - 8x_5 \leq 0$$
$$v - 2x_1 - 3x_2 - 4x_3 - 9x_4 - 3x_5 \leq 0$$
$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

Question 2

a)

In primal form this is equal to

Maximize $\underline{c}^T \underline{x}$

Subject to:

$(B\,\underline{x})_i \leq b_i$ for $i = 1,2,3,4,5$

$(B\,\underline{x})_6 = b_6$

$x_i \geq 0$ for $i = 1,2,3,4,5$

where $\underline{c} = [0,0,0,0,0,1]^T$, $\underline{x} = [x_1, x_2, x_3, x_4, x_5]^T$

$\underline{b} = [0,0,0,0,0,1]^T$ and

$$B = \begin{bmatrix} -3 & -7 & -1 & -1 & -4 & 1 \\ -3 & -8 & -2 & -4 & -7 & 1 \\ -9 & -4 & -5 & -4 & -7 & 1 \\ -6 & -5 & -6 & -5 & -8 & 1 \\ -2 & -3 & -4 & -9 & -3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Therefore by using the LP duality theorem we can get the LP problem to find the maxminimizer strategy profile that is the strategy for player 2 we need to use the LP duality theorem, to find dual LP:

Minimize $\underline{b}^T \underline{y}$

Subject to

$(B^T \underline{y})_j \geq c_j$, $j = 1,2,3,4,5$

$(B^T \underline{y})_6 = c_6$

$y_j \geq 0$, $j = 1,2,3,4,5$

where in this case $\underline{y} = [y_1, y_2, y_3, y_4, y_5, v]^T$

By Minimax Theorem we have that $\underline{b}^T \underline{y} = \underline{c}^T \underline{x}$

The results were computed in Matlab using linprog package

Minimax Value: $14/3$, Minimax Profile:

Player 1 $\quad x_1 = [0, \frac{1}{2}, 0, \frac{5}{18}, \frac{2}{9}]^T$

Player 2 $\quad x_2 = [\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}]^T$

Question 2

b)

$$B = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix}$$

Please note that the value of any symmetric 2 player zero sum game must be equal to 0. This implies, by the minimax theorem that $Bw \le 0$. Suppose for contradiction that $z = 0$. Then we have the following:

$$Bw = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix} \begin{bmatrix} y^* \\ x^* \\ 0 \end{bmatrix} = \begin{bmatrix} Ax^* \\ -A^T y^* \\ b^T y^* - c^T x^* \end{bmatrix} \le 0$$

We can deduce the following inequalities:

- $Ax^* \le 0$
- $-A^T y^* \le 0 \iff A^T y^* \ge 0$
- $b^T y^* - c^T x^* \le 0 \iff b^T y^* \le c^T x^*$

Now suppose $y^* \neq 0$ then we have that $(y^*)^T (Ax' - b) < 0$ because $Ax' < 0$ and $y^* > 0$

Notice that by weak duality we must have $c^T x^* \le b^T y^*$, therefore putting this together with with $b^T y^* \le c^T x^*$ we have that $b^T y^* \neq 0$ as a result this must mean that $c^T x^* \neq 0$ so we have that $x^* \neq 0$.

Then it follows that $(x^*)^T (A^T y' - c) > 0$ since we are given that $A^T y' > c$.

~~By~~ We can put all of this together and using the fact that $c^T x^* = b^T y^*$ since by strong duality they must be the same since they are optimal solutions:

① $(y^*)^T (Ax' - b) < 0 < (x^*)^T (A^T y' - c)$

② $(y^*)^T (Ax' - b) < (x^*)^T (A^T y' - c)$

Question 2

b) ③ $(y^*)^T A x' - (y^*)^T b < (x^*)^T A^T y' - (x^*)^T c$

④ $(y^*)^T A x' < (x^*)^T A^T y' - (x^*)^T c + (y^*)^T b$

⑤ $(y^*)^T A x' < (x^*)^T A^T y'$

This is because ~~x~~ $(x^*)^T c = (y^*)^T b$

⑥ $(y^*)^T A x' < ((x^*)^T A^T y')^T$

We can do this since $(x^*)^T A^T y'$ is 1×1 matrix so trasposing it does not change anything

⑦ $(y^*)^T A x' < (y')^T A x^*$

However, in the last step we see that $(y^*)^T A x' < (y')^T A x^*$ is a contradiction since we have that y^* is an optimal ~~solution~~ response. That means that there must be some x' and y' such that this equality holds - we know this by Minimax Theorem. We also know that x' and y' are feasible solutions to the problem but since ⑦, we will never have equality. Hence we have proved by contradiction that ~~that~~ $z > 0$.