Couse: Algorithmic Game Theory and Its Applications
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Coursework 1
Question 4

a) First lets start by showing that if an x exists such that $Ax \leq b$, there is no vector y such that $y \geq 0$, $y^T A = 0$ and $y^T b < 0$. Assume there is an x that solves the linear equation above. Now if such a y exists:

$$Ax \leq b$$
$$y^T A x \leq y^T b \quad (\text{since } y \geq 0)$$
$$y^T A x < 0 \quad (\text{Since } y^T b < 0)$$
$$0x < 0 \quad (\text{since } y^T A = 0)$$
$$0 < 0$$

This is an obvious contradiction as a number cannot be smaller than itself. Therefore, in the case where an x exists that solves $A\underline{x} \leq b$, there cannot be a y satisfying $y \geq 0$ and $y^T A = 0$ and $y^T b \leq 0$.

Next it must be shown that (1) if there does not exist a solution x for the system $Ax \leq b$, there must exist a y satisfying $y \geq 0$ and $y^T A = 0$ and $y^T b \leq 0$. This will be done using induction on the number of columns in A using Fourier-Motzkin elimination.

Base Case: There is only one column in A. This implies $x = \{x_1\}$ and $A = [a_1\ a_2\ \dots\ a_m]^T$

Let's consider variables a_i. If any a_i is 0, then an unsolveable system can be created by setting $b_i < 0$ (so that $0 \leq b_i < 0$). A suitable y for such a system can be found by letting $y_i = 1$ and $y_j = 0$ $(i \neq j)$

Then $y^T A = 0$ since $a_i = 0$ and $y^T b < 0$ since $b_i < 0$

Now let's consider all a_i are non-zero. Let's denote by a_j any positive element of A and a_k any negative element.

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a) All inequalities then take one of the two forms:

$a_k x_1 \le b_k$ or $a_j x_1 \le b_k$

$(\Rightarrow)$

$x_1 \le \frac{b_j}{a_j}$ $\qquad$ $x_1 \ge \frac{b_k}{a_k}$ ← since $a_k < 0$

For there to be a solution to $Ax \le b$ the smallest j fraction must be greater or equal than biggest k fraction:

$$\max \frac{b_k}{a_k} \le \min \frac{b_j}{a_j}$$

In order to form an unsolveable system, let some $\frac{b_{k'}}{a_{k'}}$ be greater than some $\frac{b_{j'}}{a_{j'}}$. Let $y_{k}' = a_j'$, $y_j' = -a_k'$ and all other $y_i = 0$

Solving for $y^T A$:

$$y^T A = y_{k'} a_{k'} + y_{j'} a_{j'} = a_{j'} a_{k'} + (-a_{k'}) a_{j}' = 0$$

Next solving for $y^T b$ ($a_{k'} < 0$ and $|a_{j'} b_{k'}| < |a_{k'} b_{j'}|$):

$$y^T b = y_{k'} b_{k'} + y_{j'} b_{j'} = a_{j'} b_{k'} + a_{k'} b_{j'} < 0$$

Therefore, for any formation of a one column matrix A where there is no solution x, there exists a y such that $y \ge 0$ and $y^T A = 0$ and $y^T b \le 0$

Base Case holds.

Inductive Step: Assume that for less than n columns statement 1 holds. Let's show that based on this assumption, it also holds for n columns.

For a n-column matrix A, the inequality $Ax \le b$ expands to:

$$\sum_{j=0}^{n} a_{i,j} x_j \le b_i \quad \text{for } i = 1, \dots, m$$

or row-wise:

$$a_{1,1} x_1 + \dots + a_{1,n} x_n \le b_1$$
$$\vdots$$
$$a_{m,1} x_1 + \dots + a_{m,n} x_n \le b_m$$

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a) To remove x_n using Fourier-Motzkin elemination, the following steps are performed

- Re-write all constraints for which $a_{i,n} x_n \neq 0$ as either $x_n \leq \frac{\dots}{a_{i,n}}$ or $x_n \geq \frac{\dots}{a_{i,n}}$. Second case occurs when $a_{i,n} < 0$. Let's call first set of indices P and second set of indices N.

- Generate all pairs of P and N such that $\frac{\dots}{a_{i,n}} \leq \frac{\dots}{a_{j,n}}$ for $i \in N, j \in P$. Add these constraints to the system

Let's consider how fourier-Motzkin inequalities will look like:

$$\frac{b_i - a_{i,1} x_1 - \dots - a_{i,n-1} x_{n-1}}{a_{i,n}} \leq \frac{b_j - a_{j,1} x_1 - \dots - a_{j,n-1} x_{n-1}}{a_{j,n}}$$

for $i \in N, j \in P$

$\Leftrightarrow$ also $a_{i,n} < 0$

$$a_{j,n} (b_i - a_{i,1} x_1 - \dots a_{i,n-1} x_{n-1}) \geq a_{i,n} (b_j - a_{j,1} x_1 - \dots - a_{j,n-1} x_{n-1})$$

$$a_{j,n} b_i - a_{j,n} a_{i,1} x_1 - \dots - a_{j,n} a_{i,n-1} x_{n-1} \geq a_{i,n} b_j - a_{i,n} a_{j,1} x_1 - \dots - a_{i,n} a_{j,n-1} x_{n-1}$$

Rearranging we get:

$$a_{j,n} a_{i,1} x_1 + \dots a_{j,n} a_{i,n-1} x_{n-1} - a_{i,n} a_{j,1} x_1 - \dots - a_{i,n} a_{j,n-1} x_{n-1} \leq a_{j,n} b_i - a_{i,n} b_j$$

Multiplying any inequality in a system of linear inequalities by a positive number will not change whether or not there is a solution and will not change existence of y. Therefore it can be assummed without loss of generality that all $a_{i,n} < 0$ are -1 and all $a_{j,n} > 0$ are 1.

Making this assumption simplifies the inequality to:

$$a_{i,1} x_1 + \dots a_{i,n-1} x_{n-1} + a_{j,1} x_1 + \dots + a_{j,n-1} x_{n-1} \leq b_i + b_j$$

$$(a_{i,1} + a_{j,1}) x_1 + \dots + (a_{i,n-1} + a_{j,n-1}) x_{n-1} \leq b_i + b_j$$

So the new constraints in the system will have right hand sides of the form $b_i + b_j$ for $i \in N, j \in P$. Call the modified constraint vector b'

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a) The new system A' has $n-1$ columns and no solution. (Since the original n-column system did not have one) Therefore, the induction hypothesis holds for it - there is y' such that $y' \geq 0$, $y'^T A' = 0$ and $y'^T b' < 0$

Now we expand $y'^T b' < 0$, remembering that b' consists of the unchanged indices U and the new rows $b_i + b_j$:

$$y'^T b' < 0$$

$$\sum_{u \in U} y_u' b_u + \sum_{i \in N, j \in P} y_{i,j}' (b_i + b_j) < 0$$

$$\sum_{u \in U} y_u' b_u + \sum_{i \in N, j \in P} y_{ij}' b_j + y_{i,j}' b_i < 0$$

$$\sum_{u \in U} y_u' b_u + \sum_{i \in N, j \in P} y_{ij}' b_j + \sum_{i \in N, j \in P} y_{ij}' b_i < 0$$

$$\sum_{u \in U} y_u' b_u + \sum_{j \in P} \Big(\sum_{i \in N} y_{ij}' \Big) b_j + \sum_{i \in N} \Big(\sum_{j \in P} y_{ij}' \Big) b_i < 0$$

Next we expand $y'^T A' = 0$ noting that A' consists of the unchanged rows $A_i, i \in U$ and new rows $A_{ij}' = A_i + A_j$

$$y'^T A' = 0$$

$$\sum_{u \in U} y_u' A_{u,c} + \sum_{i \in N, j \in P} y_{ij}' (A_{j,c} + A_{i,c}) = 0 \text{ for columns } c = 1, \dots, n-1$$

$$\sum_{u \in U} y_u' A_{u,c} + \sum_{i \in N, j \in P} (y_{ij}' A_{j,c} + y_{i,j}' A_{i,c}) = 0$$

$$\sum_{u \in U} A_{u,c} + \sum_{i \in N, j \in P} y_{ij}' A_{j,c} + \sum_{i \in N, j \in P} y_{ij}' A_{i,c} = 0$$

$$\sum_{u \in U} A_{u,c} + \sum_{j \in P} \Big(\sum_{i \in N} (y_{ij}') \Big) A_{j,c} + \sum_{i \in N} \Big(\sum_{j \in P} y_{ij}' \Big) A_{i,c} = 0$$

Those 2 expansions give a y for the original n column system:

Let $y_i = y_i'$ for $i \in U$, $y_j = \sum_{i \in N} y_{ij}'$ and $y_i = \sum_{j \in P} y_{ij}'$

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a) Such y causes the above expansions to turn into:

$$\sum_{u \in U} y'_u b_u + \sum_{j \in P} \left(\sum_{i \in N} y'_{ij} \right) b_j + \sum_{i \in N} \left(\sum_{j \in P} y'_{ij} \right) b_i < 0$$

$$\sum_{i=0}^{m} y_i b_i < 0$$

$$y^T b < 0$$

Similar logic can be applied to $y^T A = 0$

Therefore there exists a y for n column system such that $y \geq 0$, $y^T A = 0$ and $y^T b \leq 0$, which completes the inductive proof.

b) Let A be defined as:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

we let $x^T = [x_1, x_2]$, $y^T = [y_1, y_2]$, $b^T = [0, -1]$, $c^T = [1, 0]$

The primal and dual form for the following system are equal to:

Maximize $c^T x$

Subject to

$(Ax)_i \leq b_i$ for $i = 1, 2$

$x_i \geq 0$ for $i = 1, 2$

Minimize $b^T y$

Subject to:

$(A^T y)_j \geq c_j$ for $j = 1, 2$

$y_j \geq 0$ for $j = 1, 2$

Now if we substitute values for primal we get

Maximize x_1

Subject to

$x_1 - x_2 \leq 0$

$-x_1 + x_2 \leq -1$

$x_i \geq 0$ for $i = 1, 2$

Note that we are trying to optimize

$x_1 - x_2 \leq 0$ and $-x_1 + x_2 \leq -1$.

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b) This is the same as:

$x_1 - x_2 \leq 0$ and $x_1 - x_2 \geq 1$

As a result there does not exist a solution such that $x_1 - x_2 \leq 0$ and $x_1 - x_2 \geq 1$. This is clearly infeasible. We have shown that primal is infeasible, let's show that the dual is too

When we substitute the values for the dual we get:

Minimize $-y_2$

Subject to:

$y_1 - y_2 \geq 1$

$-y_1 + y_2 \geq 0$

$y_j \geq 0$ for $i = 1, 2$

We are trying to optimize

$y_1 - y_2 \geq 1$ and $-y_1 + y_2 \geq 0$

This is the same as

$y_1 - y_2 \geq 1$ and $y_1 - y_2 \leq 0$

There does not exist an expression $S = y_1 - y_2$ such that $S \geq 1$ and $S \leq 0$. So the dual is infeasible as well.

Since we have shown that both primal and dual are infeasible, we have completed the task.