CP(2) Notes (Implementation Summary)

Fields and Geometry

We simulate the 2D CP(2) model on a square lattice with volume $V = L_x L_y$. The field at each site is a complex vector

$$z(x) \in \mathbb{C}^3, \qquad z(x)^\dagger z(x) = 1, \tag{1}$$

representing a point in $\mathrm{CP}(2) \simeq \mathrm{SU}(3)/\mathrm{U}(2)$. We use the rank–1 projector

$$P(x) = z(x) z(x)^\dagger, \tag{2}$$

which satisfies $P^2 = P$ and $\operatorname{Tr} P = 1$.

Lattice Action

We use the quartic (projector) action

$$S[z] = \beta\Big(N_d V - \sum_{x,\mu} \big|z(x)^\dagger z(x+\hat{\mu})\big|^2\Big), \tag{3}$$

so aligned fields give $S = 0$. This action corresponds to the standard $\mathrm{CP}(N-1)$ lattice formulation with a compact U(1) link variable $U_{x,\mu}$ coupling neighboring z fields; integrating out the link yields a quartic action in z.[1]

We also use the unshifted energy

$$S_0[z] = -\beta \sum_{x,\mu} \big|z(x)^\dagger z(x+\hat{\mu})\big|^2 \tag{4}$$

for monitoring fluctuations.

Lie-Derivative Force

Let T_a be the anti-Hermitian generators of $\mathfrak{su}(3)$ (normalization $\mathrm{Tr}[T_a T_b] = -\frac{1}{2}\delta_{ab}$), as in `LieGroups/su3.py`. The Lie derivative is

$$\partial_x^a z(y) = T_a \, \delta_{xy} \, z(y). \tag{5}$$

Define

$$M(x) = \sum_\mu \Big(P(x+\hat\mu) + P(x-\hat\mu)\Big), \tag{6}$$

then the force components are

$$F_a(x) = -\sum_a T_a \, \partial_x^a S = \beta \, \mathrm{Tr}\Big(T_a [P(x), M(x)]\Big). \tag{7}$$

We implement both the 8-component real force $F_a(x)$ and the matrix force $F(x) = \sum_a F_a(x) T_a$.

Momenta and Kinetic Energy

We use two equivalent representations for momenta:

1. Components $p_a(x) \in \mathbb{R}^8$, with kinetic energy

$$K = \tfrac{1}{2} \sum_{x,a} p_a(x)^2. \tag{8}$$

2. Matrix momenta $P(x) = \sum_a p_a(x) T_a$ (anti-Hermitian, traceless), with

$$K = -\sum_x \mathrm{Tr}\,[P(x)^2]. \tag{9}$$

The two are equivalent by the generator normalization.

Molecular Dynamics Update

The field update uses the group exponential

$$z \leftarrow \exp\big(\Delta t \, P\big) \, z, \tag{10}$$

with the SU(3) exponential as implemented in `LieGroups/su3.py`.

Topological Charge

U(1) plaquette definition. The induced compact U(1) links are

$$u_\mu(x) = \frac{z(x)^\dagger z(x+\hat{\mu})}{|z(x)^\dagger z(x+\hat{\mu})|}, \tag{11}$$

and the plaquette phase is

$$U_p(x) = u_x(x)\, u_y(x+\hat{x})\, u_x(x+\hat{y})^*\, u_y(x)^*. \tag{12}$$

The lattice topological charge is

$$Q = \frac{1}{2\pi} \sum_x \operatorname{Arg} U_p(x). \tag{13}$$

The plaquette product and its logarithm define the compact U(1) field strength in terms of link variables,[2] and the topological charge as a sum of plaquette angles follows directly from $F_{12} = \operatorname{Arg} U_p$ in 2D.[2]

Continuum-inspired discretization. We also compute

$$Q_{\text{cont}} = \frac{1}{2\pi i} \sum_x \operatorname{Tr}\Big(P\,[\partial_x P, \partial_y P]\Big), \tag{14}$$

using forward differences on the lattice. This is the standard continuum expression for the CP$(N-1)$ topological charge in terms of the projector.[3]

Two-Point Function and Correlation Length

Define the traceless projector field

$$X(x) = P(x) - \tfrac{1}{3} I. \tag{15}$$

We measure a two-point function along the x direction

$$C(r) = \langle \operatorname{Tr}[X(x) X(x + r\hat{x})] \rangle, \tag{16}$$

averaged over sites and configurations. We also compute a second-moment correlation length using

$$\xi^2 = \frac{1}{4\sin^2(\pi/L)} \left(\frac{\chi}{C_{2p}} - 1 \right), \tag{17}$$

where $\chi = V\,\langle |\bar{X}|^2 \rangle$ is the zero-momentum susceptibility and C_{2p} is the correlator at momentum $(2\pi/L, 0)$. This mirrors the standard O(N) second-moment estimator.

Autocorrelation (Madras–Sokal)

We provide a helper in `analysis/autocorr.py` to compute the integrated autocorrelation time with the Madras–Sokal automatic window, and the corresponding effective sample size.[4, 5]

Visualization

We visualize z configurations in four complementary ways:

1. RGB from the projector diagonal: $\mathrm{diag}(P) = \big(|z_0|^2, |z_1|^2, |z_2|^2\big)$.
2. Same RGB with brightness modulated by $\arg(P_{01})$.
3. HSV mapping with hue $\arg(P_{01})$ and value $|P_{01}|$.
4. Plaquette-angle map $\theta(x) = \arg U_p(x)$.

References

[1] F. Bruckmann, C. Gattringer, T. Kloiber, T. Sulejmanpasic, "Dual lattice representations for O(N) and CP(N-1) models with a chemical potential," Phys. Lett. B 749 (2015) 495–501.

[2] S. Onoda, "'t Hooft Line in 4D U(1) Lattice Gauge Theory and a Microscopic Description of Dyon's Statistics," Prog. Theor. Exp. Phys. 2026, 013B04 (2026).

[3] B. Douçot, R. Moessner, D. L. Kovrizhin, "Topological electrostatics," J. Phys.: Condens. Matter 35, 074001 (2023) (arXiv:2107.10700).

[4] N. Madras and A. D. Sokal, "The pivot algorithm: A highly efficient Monte Carlo method for the self-avoiding walk," J. Stat. Phys. 50, 109 (1988).

[5] "pyhmc.integrated_autocorr6 documentation," pythonhosted.org/pyhmc.