CMPE 482 - Assignment I

Spring 2021

Due 07.04.2021, 20.00

1. Implement the Gram-Schmidt algorithm. Generate a 5×5 matrix called X, where each element is drawn from the unit Gaussian distribution. Use your Gram-Schmidt algorithm to obtain the decomposition $X = QR$ where Q has orthonormal columns and R is upper triangular. Print out the following quantities to show that your algorithm works as intended:
    a. $Q^T Q$
    b. QQ^T
    c. R
    d. $QR - X$
2. Consider the matrix $Y = \begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix}$. Draw the columns of this matrix as two vectors in $\mathbb{R}^2$. Then compute the QR-decomposition of this matrix using the algorithm you developed in Q1 and plot the columns of the resulting Q matrix in $\mathbb{R}^2$.
3. Let H be a 200×200 Hilbert matrix whose entries are given by the equation $H_{ij} = \dfrac{1}{i + j - 1}$ for $1 \leq i, j \leq 200$. Form a 200×200 matrix X by using the equation $X = 10^{-5} I + H$.
    a. Compute the QR-decomposition of X by using the Gram-Schmidt algorithm that you implemented in Q1.
    b. Implement the modified Gram-Schmidt algorithm (See Algorithm 8.1 from Trefethen & Bau, 1997, below), and compute the QR-decomposition of X again.
    c. Calculate an error matrix $E = I - Q^T Q$ for each decomposition. Then, compare the performance of the algorithms by reporting the value of the largest entry (in absolute value) in these error matrices.
4. Let $f(x) = x^2 + 2e^x$ defined in the range $x \in (-2, 2)$. Plot the original function and 0th, 1st, 2nd, and 3rd order Taylor approximations in this range, at $x_0 = 0.5$.
5. (Golub and van Loan, P.1.1.5) Suppose we have real n-by-n matrices C, D, E, and F. Show how to compute the real n-by-n matrices A and B with just three real n-by-n matrix multiplications so that

    $$A + iB = (C + iD)(E + iF)$$

    Note: The naive way of calculating this product would be using the identity $(C + iD)(E + iF) = (CE - DF) + i(CF + DE)$ as it is, where four matrix multiplications are needed.
6. (Boyd & Vandenberghe, 2018, Q2.4) The function $\phi : \mathbb{R}^3 \to \mathbb{R}$ satisfies

    $\phi(1, 1, 0) = -1$, $\phi(-1, 1, 1) = 1$, $\phi(1, -1, -1) = 1$.

    Choose one of the following and justify your choice: ϕ must be linear, ϕ could be linear, ϕ cannot be linear.

From Trefethen & Bau, 1997:

Algorithm 8.1. Modified Gram–Schmidt

for $i = 1$ **to** n
    $v_i = a_i$
for $i = 1$ **to** n
    $r_{ii} = \|v_i\|$
    $q_i = v_i / r_{ii}$
    for $j = i+1$ **to** n
        $r_{ij} = q_i^* v_j$
        $v_j = v_j - r_{ij} q_i$

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