CMPE 482 - Assignment 2

Spring 2021

Due 21.04.2021, 20.00

1. (2 pts.) Consider the vector $v = (4, -1)$.
    a. Create a matrix R that rotates this vector 30 degrees counterclockwise. Print the matrix.
    b. Create a matrix P that projects this vector onto the line through the vector $l = (1, -1)$. Print the matrix. (*Hint*: You can use the formulation on pg. 130)
    c. Plot the original vector and the rotated and the projected vector.
    d. Compute the norm of the original vector, the rotated vector, and the projected vector. How do they compare?
    e. Plot columns of R and P. What properties (or lack thereof) of R and P lead to this? (*Hint*: What property of their columns can explain this?)

2. (2 pts.) Create and fix a vector $v \in \mathbb{R}^5$ by randomly sampling each entry from the unit Gaussian. (*Hint*: Do no resample v once you fixed it as it will invalidate your work that follows)
    a. Design a matrix S that sorts v in an ascending order, creating the sorted vector v_s. Print S, v, and v_s.
    b. Create a matrix K that reverses the order of the *sorted* vector v_s, creating a vector called v_k. Print K and v_k.
    c. Using S and K, this time first *reverse* the original vector v and *then* apply the sorting, creating the new vector v_z. Are v_z and v_k the same or are they different?
    d. Use K to reverse every column of S to create a new sorting matrix $\hat{S}$. Then sort v using $\hat{S}$ to create v_h. Are v_h and v_k the same or are they different?
    e. What properties questions 2d and 2e demonstrate about matrix multiplication?

3. (3 pts). Given a vector $v = (1, -4, 3, 7, 0, -3) \in \mathbb{R}^6$,
    a. Create a matrix T that produces a smoothed vector $\hat{v} = Tv$, $\hat{v} \in \mathbb{R}^4$ such that $\hat{v}_i = 0.1v_i + 0.3v_{i+1} + 0.6v_{i+2}$ for $i \in 1, 2, 3, 4$. Print T and $\hat{v}$.
    b. Does T correspond to a linear function? Demonstrate.
    c. Imagine $\hat{T}$ such that $\hat{T}\hat{v} = v$. Can such a $\hat{T}$ exist? If yes, please print the matrix. If not, please explain why. (Hint: Is T invertible?)
    d. Express the relationship between $\hat{T}$, $\hat{v}$, and v as a system of linear equations (even if a perfect solution is not feasible). Is this an overdetermined or underdetermined system? Why?
    e. Apply least-squares method to create a best approximation of v from $\hat{v}$, called $\tilde{v}$. Print $\tilde{v}$.
    f. Is $\hat{T}$ a (pseudo-)inverse of T? If yes, which one?

4. (2 pts). [*IALA*, Q11.12] *Combinations of invertible matrices*. Suppose the $n \times n$ matrices A and B are both invertible. Determine whether each of the matrices given below is invertible, without any further assumptions about A and B.
    a. $A + B$

b. $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$

c. $\begin{bmatrix} A & 0 \\ A+B & B \end{bmatrix}$

d. ABA

5. (3 pts). Implement the following steps:
   a. Generate 20 equispaced points in the interval $[-1, 1]$, i.e. with equal intervals among them. (*Hint:* you can use the function np.linspace(-1,1,20)).
   b. For all 20 points compute $y = \sin(2\pi x/T) + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$ with $\sigma = 0.1$. Create a scatter plot of x vs y for the 20 points you generated for period parameter $T = 1.5$ (see slide 13.15 for scatter plot examples - you may ignore the fitted lines).
   c. For approximating y with a polynomial of degree at most 10, construct the Vandermonde matrix V based on x, where V has the form $V_{ij} = x_i^j$ (see slide 13.14).
   d. Compute the QR decomposition of V. Let $q_0, \ldots, q_{10}$ be the columns of Q, Plot each of the first 5 individual columns of the matrix Q against x as a scatter plot, similar to that of in Q5b.
   e. We wish to compute a series of successive approximations to y, by computing $Qc^{(j)}$ where $j \in 0, \ldots, 10$. Here, $c^{(j)}$ corresponds to the vector $[c_0, c_1, \ldots, c_j, 0, 0, , 0]^T$. How can we find the best $c^{(j)}$ that minimizes the $\|y - Qc^{(j)}\|$ for each j? Plot x vs y as well as the approximations for $j = 0, 5$, and 10 (which we can name $y^{(0)}, y^{(5)}, y^{(10)}$) in a scatter plot, comparing the original relationship to the three approximations.
   f. Repeat the previous approximation experiment with $y = Vc^{(j)}$, again compute the approximations and plot x vs y as well as $y^{(0)}, y^{(5)}, y^{(10)}$. Compare these approximations with those in question 5e. Which approximation worked better? Why do you think this is the case? Do the matrices V and Q span different subspaces?