# Motif Counting Algorithm

Ori Cohen

February 25, 2019

**Abstract**

This report aims to describe the algorithm used to count all the motifs of a graph in an Optimal manner. The algorithm is optimal in the sense that each motif is only counted once, resulting in an algorithm that is linear in the number of motifs in the graph. The algorithm takes advantage of the combinatorics of the 3- and 4-motifs to explicitly iterate over them. This report further describes the new C++ kernel that has been implemented to facilitate work on large graphs, which can run both on CPU and GPU devices.

This report presents a top-down approach to the algorithm and the theory behind it.

All the code described in this report can be found as an open source library at the GitHub repository[1].

[1] https://github.com/oricc/graph_measures

# Contents



# List of Algorithms

# 1 Algorithm Description

## 1.1 The Short Version

This section aims to describe the essence of the algorithm without going into the details, for those who are only interested in a high level view.

There are three main principles in the algorithm:

1. **Divide and conquer**. As we discussed above, we sort the nodes by their degree, from the node with the highest degree to that with the lowest. This allows us to quickly separate the graph into smaller, unconnected sub-graphs.

2. **Known motif structure.** As we can see in figure 2, we can list all of the options for the different motif structures. The algorithm is build so that it enumerates all the motifs of a certain structure before moving on to the next one, allowing us to make assumptions about the next motifs we should count.

3. **Conditions to stop double-counting.** Following the previous principle, we can make assumptions about the order in which the motifs are counted, and so we can write specific conditions that stop the code from counting certain motifs twice.

All motif building algorithms are a combination of these principles, allowing us to efficiently count all motifs in a graph. The rest of this report describes the algorithms with all details.

## 1.2 Overview

There are two main ideas implemented in this algorithm.

The first is that of explicit counting - since we know all the possible ways a motif can look, we can iterate over them directly, thus enabling us to enumerate all of them in an efficient fashion, both in memory and in time.

The second is the well known concept of divide and conquer. We use this concept by removing from the graph each node we have gone over. This concept is strengthened by a simple preprocessing of the graph, sorting all the nodes by their degree from highest to lowest. Having the nodes sorted in this way allows us to deal with the "heavier" nodes (the ones with a higher degree) first, and in so doing break up the graph into smaller, independent sub-graphs.

These two ideas give us the general shape of the algorithm, which can be seen, from above, as an iteration over the sorted nodes.

The *CreateNodeSubtree* function is where most of the hard work occurs, as that is the function where the actual motif counting is performed.

## 1.3 Motif combinatorics

Counting all motifs containing a given node is done using a **breadth first approach.** For each motif, we can define a *depth*. The depth of a motif is

**Algorithm 1** Motif Counter Overview

**function** MOTIFCOUNTER(g,level)
▷ $g$ is a graph (either directed or undirected)
▷ *level* is either 3 or 4, the level of motif to count
**global** *motifCounter* ← list of motif counters
▷ A motif counter is a dictionary containing a counter for each motif
*sortdNodes* ← g.vertices
sort *sortedNodes* by node degree
**for** *node* in *sortedNodes* **do**
**if** *level* == 3 **then**
CREATENODESUBTREE3($g$,*node*)
**else** ▷ level is 4
CREATENODESUBTREE4($g$,*node*)
**end if**
*g.remove*(*node*)
▷ We remove the node after iterating over it so as to not repeat the motifs that were already seen.
**end for**
**end function**

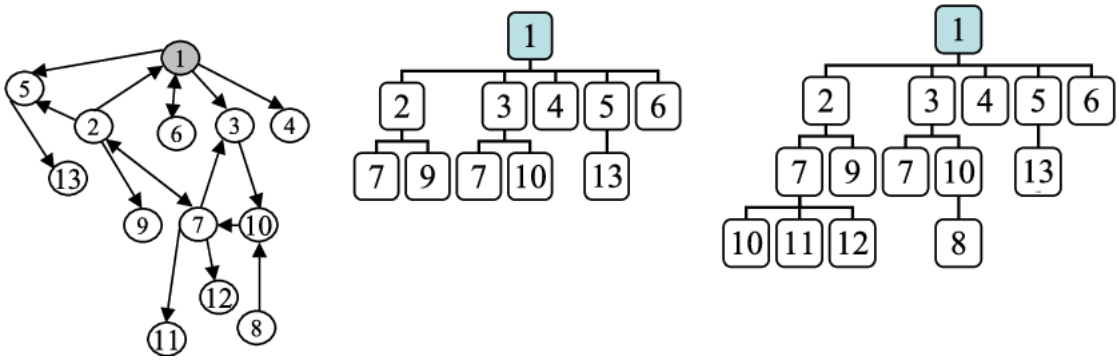



Figure 1: Node order
This figure describes the order of counting all motifs with node 1, which is also the node with the highest rank and so is the first node to be the root for motif counting.
From left to right: the graph itself, the order of counting 3-motifs and the order of counting 4-motifs.

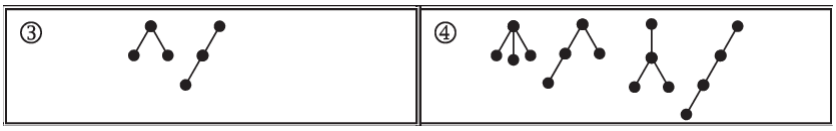



Figure 2: 3- and 4-motif variations
From left to right: 3-motif of depths 1and 2, 4-motifs of depth 1, 2 (option 1), 2 (option 2), and 3.

defined as the distance of the furthest vertex in the motif from the root node. This can also be though of as the furthest ring of neighbors that needs to be accessed to build the motif (for example, for a motif a depth 2 we must first select a vertex from the root's first neighbors, and then proceed to selecting a node from that vertex's neighbors, totaling two neighbor rings that we needed to see to build the motif). The second view is the one employed by us in implementing this algorithm. We separate the motifs by their *depth* and count all motifs of the same depth together.

It is obvious that the depth of a motif is bound by the number of nodes in the motif:

- A 3-motif can only be of depth 1 or 2
- A 4-motif can be of depth 1,2 (with two variations) or 3

In order to count all of the motifs of a certain depth, we must understand how each motif looks.

It can be seen that we can easily iterate over the necessary neighbors (whether they are first degree or second degree neighbors) to construct all the possible motifs containing the root vertex.

## 1.4 Subtree Construction

A *subtree* is considered to be the vicinity of the node where motifs containing is could appear. The *CreateNodeSubtree* is responsible for going over all the possible depths and counting all the motifs in each. For readability, we have separated the depths into separate functions.

**It is important to note that when we say we iterate over all of a node's neighbors, we iterate over the neighbors in the *full graph*, i.e. we consider both the vertices connected to the node and the ones the node is connected to (which, in a directed graph, are not equivalent).**

**Algorithm 2** Create Node Subtree for 3-motifs

```
function CreateNodeSubtree3(g,root)
                                    ▷ g is a graph (either directed or undirected)
                                    ▷ root is the vertex that must be in all motifs
    for n_1 in root's neighbors do
        Mark that we saw n_1
    end for
    Count3MotifsDepth2(g,root)
    Count3MotifsDepth1(g,root)
end function
```

**Algorithm 3** Create Node Subtree for 4-motifs

```
function CreateNodeSubtree4(g,root)
                                    ▷ g is a graph (either directed or undirected)
                                    ▷ root is the vertex that must be in all motifs
    for n_1 in root's first neighbors do
        Mark n_1 as seen at level 1
    end for
    Count4MotifsDepth1(g,root)
    Count4MotifsDepth2(g,root)
    Count4MotifsDepth3(g,root)
end function
```

## 1.5 Motif Counting

### 1.5.1 Motif Counter updating

While each is slightly different, all motif counting function are essentially the same.

All of these functions, broadly speaking, have similar behavior:

- Iterate over all the motifs of the specified depth that you can build
  - For each of those motif update the relevant motif counters

The way of iterating over the motifs is the main difference between each of the functions.

Updating the motif counters is done by calculating the motif index (as discussed in the next section), and is relatively straightforward:

**Algorithm 4** Updating Motif Counters

```
function UpdateMotifCount(g,motif)
                                  ▷ g is a graph (either directed or undirected)
                                    ▷ motif is the motif that is being updated
    motifIndex ← GetMotifIndex(g,motif)
    for n doode in motif
        motifCounters[node][motifIndex] + +
    end for
end function
```

### 1.5.2 3-Motif building algorithms

As we discussed in the previous section, 3-motifs can only be of depth 1 or 2. For both depths, we know what their form looks like:

- A 3-motif of depth 1 must be the root node connected to two of it's first neighbors.
- A 3-motif of depth 2 can only be the root, a first neighbor and a second neighbor.

The in which the motifs are counted is critical, as it directly affects which conditions we need to check to avoid double counting. For 3-motifs, we count the motifs in the following order:

1. Motifs of depth 2; then
2. Motifs of depth 1

**Algorithm 5** Motif 3 builders

```
function Count3MotifsDepth2(g,root)
                                    ▷ g is a graph (either directed or undirected)
                                    ▷ root is the vertex that must be in all motifs
    for n_1 in root's neighbors do
        for n_2 in n_1's neighbors do
            if n_2 is a neighbor of root's then
                if we saw n_1 before n_2 then          ▷ Count the pair only once
                    UpdateMotifCount(g,[root, n_1, n_2])
                end if
            else
                Mark that we saw n2
                UpdateMotifCount(g,[root, n_1, n_2])
            end if
        end for
    end for
end function


function Count3MotifsDepth1(g,root)
                                    ▷ g is a graph (either directed or undirected)
                                    ▷ root is the vertex that must be in all motifs
    neighborCombinations ← all 2-combinations of root's neighbors
    for tuple in neighborCombinations do
        n_1 ← tuple.first
        n_2 ← tuple.second
        if we saw n_1 before n_2 and they aren't neighbors then (*)
  ▷ We only want to count (n_1, n_2) once, and if they are neighbors, we would
                    have discovered the motif as a motif of depth 2 (see figure 3)
            UpdateMotifCount(g,[root, n_1, n_2])
        end if
    end for
end function
```

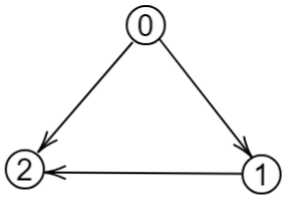



Figure 3: Motif 3 special cases

This is an example of a case where the condition marked **(*)** in $Count3MotifsDepth1$ stop us from double counting a motif.
The motif above can be counted as both a depth-1 and a depth-2 motif, both starting at node 0. depth 1: $0 \rightarrow 1$and $0 \rightarrow 2$., or depth 2: $0 \rightarrow 1 \rightarrow 2$.
As the motif is counted first as a depth 2 motif, the condition in the depth 1 counter stops the motif from being counted twice, since they are neighbors.

### 1.5.3 4-Motif Building algorithms

Like 3-motifs, we can separate the different 4-motifs into groups by their respective depths

- **Depth 1** are 4- motifs constructed from the root and three of it's first neighbors
- **Depth 2** can be one of two options:
  - The root, two of it's first neighbors and one of it's second neighbors (which is a neighbor of one of the first neighbors)
  - The root, one first neighbors, and two second neighbors who are connected to the first neighbor.
- **Depth 3** can only be a chain of the root, and a first, second and third neighbors.

The order between the depths is doubly important for 4-motifs. The order is as they were listed above. It is important to note that **all** motifs of a certain depth are counted before moving on to the next depth.

**Algorithm 6** Motif 4 depth 1 and 2 builders

```
function Count4MotifsDepth1(g,root)
                                        ▷ g is a graph (either directed or undirected)
                                        ▷ root is the vertex that must be in all motifs
    neighborCombinations ← all 3-combinations of root's neighbors
    for tuple in neighborCombinations do
        n11 ← tuple.first
        n12 ← tuple.second
        n13 ← tuple.third
        UpdateMotifCount(g,[root, n11, n12, n13])
    end for
end function

function Count4MotifsDepth2(g,root)
                                        ▷ g is a graph (either directed or undirected)
                                        ▷ root is the vertex that must be in all motifs
    for n1 in root's neighbors do
        for n2 in n1's neighbors do
            if n2 was not seen at level 1 then
                Mark n2 as seen at level 2
            end if

            ▷ Motifs of the form root, two first neighbors and a second neighbor
            for n11 in root's neighbors do
                if n2 was seen at level 2 and n1 ≠ n11 then (*)
                    edgeExists ← (n2, n11) ∈ E or (n11, n2) ∈ E
                    if not edgeExists or (edgeExists and n1⟨n11) then (**)
                                        ▷ If there is an edge between n1 and n11,
                                        we would have already counted the motif
                                        as a motif of depth 2. If no such edge exists,
                                        we only want to count the motif once.
                        UpdateMotifCount(g,[root, n1, n11, n2])
                    end if
                end if
            end for
        end for
    end for

        ▷ Motifs of the form root, a first neighbors and two second neighbors
    secondNeighborCombinations ← all 2-combinations of n1's neighbors
    for combination in secondNeighborCombinations do
        n21 ← secondNeighborCombinations.first
        n22 ← secondNeighborCombinations.second
        if n21 and n22 were seen at level 2 then (***)
      ▷ If both nodes were seen at level 2 then the motif wasn't counted as a
                                                                    depth 1 motif
            UpdateMotifCount(g,[root, n1, n21, n22])
        end if
    end for
end function
```

**Algorithm 7** Motif 4 depth 3 builder

```
function Count4MotifsDepth3(g,root)
                                                ▷ g and root are as before
    for n_1 in root's neighbors do
        for n_2 in n_1's neighbors do
            if n_2 was seen at level 1 then (†)
            ▷ If n_2 was seen at level 1 then we can't complete a depth-3 motif
                continue
            end if
            for n_3 in n_2's neighbors do
                if n_3 was not seen yet then
                    Mark n_3 as seen at level 3
                    if n_2 was seen at level 2 then
                        UpdateMotifCount(g,[root, n_1, n_2, n_3])
                    end if
                else
                    if n_3 was seen at level 1 then (††)
                     ▷ This motif was already counted as a depth-1 or -2 motif
                        continue
                    end if
                    edgeExists ← (n_1, n_3) ∈ E or (n_1, n_3) ∈ E
                    if n_3 was seen at level 2 and not edgeExists then († † †)
                                                ▷ If an edge exists we already counted
                                                  this motif as a depth-1 or -2 motif
                        UpdateMotifCount(g,[root, n_1, n_2, n_3])
                    end if
                    if n_3 was seen at level 3 and n_2 was seen at level 2 then
                        UpdateMotifCount(g,[root, n_1, n_2, n_3])
                    end if
                end if
            end for
        end for
    end for
end function
```

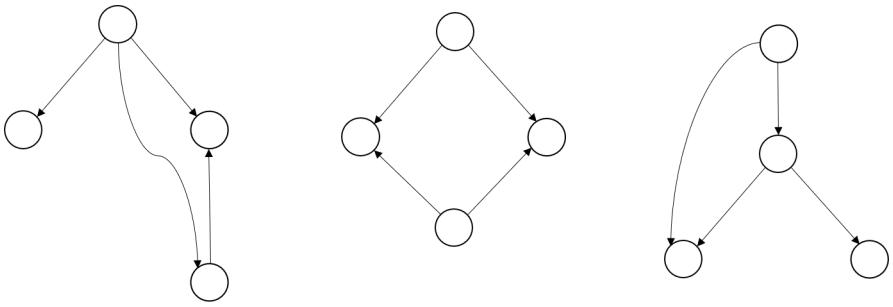


Figure 4: Motif 4 depth 2 special cases
There are three conditions we must check for 4-motifs of depth 2 so that they won't be double counted. The conditions in the code are marked with *.
From left to right, these motifs are examples of conditions **(*)** , **(**)** and **(***)**.

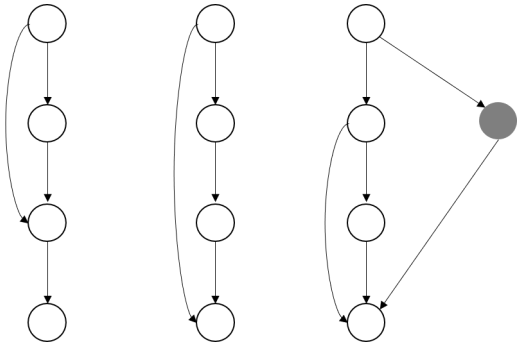


Figure 5: Motif 4 depth 3 special cases
There are three conditions we must check for 4-motifs of depth 3 so that they won't be double counted. The conditions in the code are marked with †.
From left to right, these motifs are examples of conditions (†), (††) and († † †).
The case of († † †) is delicate. The gray node is one not included in the motif, but it causes $n_3$ to be marked as a level 2 node. By itself, this is not enough to stop us from counting the motif, for example if the internal link $(n_1 \to n_3)$ didn't exists. Only when the internal link exists can we be sure we have already counted the motif.

## 1.6 Motif representation

Another important idea contributing to the efficiency of this method is that we can generate a list of all possible motifs and their corresponding isomorphisims beforehand. We can then build a list of all motifs and the motif index they correspond to, and access that list at run-time. The full list can be found in appendix B

In order to build such a list we must be able to represent each motif in a fashion that is both unique and concise. The representation we use is that of a bit array of adjacency. Since each motif contains only 3 or 4 nodes,we can build a miniature adjacency matrix which represents the motif. If we then read the matrix as a single string of bits and convert it to an unsigned int - we have effectively represented the motif using a single, easy to interpret, number.

There are two important points to note about this representation method:

- The connections between a node and itself is not considered when transforming the adjacency matrix of the motif into a number. The reason for that is simply that we assume we are dealing with a graph **without self-loops**. As this is a common assumption, it is considered legitimate in this case.

- The way of building the adjacency matrix for directed and undirected graphs is slightly different. While for a motif in a directed graph a link of $a \rightarrow b$ does not necessarily imply a connection in the other direction ($b \rightarrow a$), In an **undirected** graph we do know this, and therefor can limit ourselves to writing only *half* of the adjacency matrix.

  Algorithmically, this distinction is implemented by using different combinatorical functions.

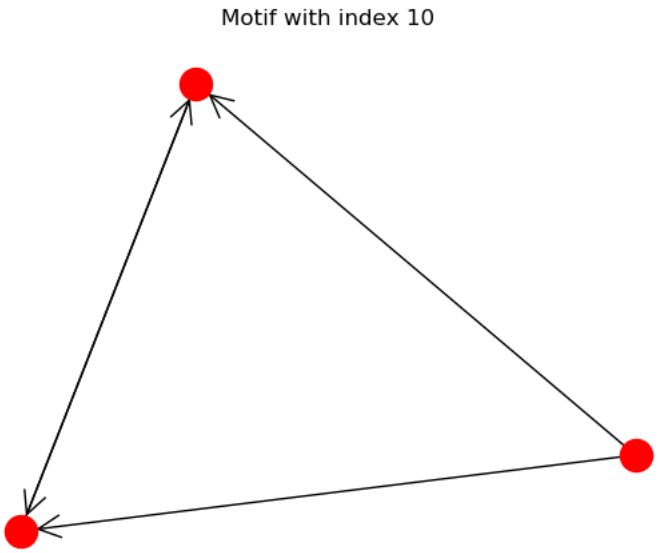
  - For a directed graph we use the *permutations* function
  - For an undirected graph we use the *combinations* function.

  For a further explanation about these functions, refer to Appendix A.

The Transformation itself consists of 4 steps.

1. The motif must be transformed into it's adjacency matrix.
2. The matrix is flattened into a bitstring.
3. The bitstring is read as an unsigned int representation and transformed into such a number.
4. The number is then looked up in the variations table to find the corresponding motif index.

The motif indices and their variations can be found in Appendix B.




$$matrix = \begin{pmatrix} - & 1 & 1 \\ 0 & - & 1 \\ 0 & 1 & - \end{pmatrix} \Rightarrow bitstring = 110101 \Rightarrow number = 53 \Rightarrow index = 10$$

Figure 6: Motif to number transformation example
An example of the entire motif transformation, from the motif to it's corresponding index.

**Algorithm 8** Motif to number representation

```
function GetMotifIndex(g, motif)
                                        ▷ g is a graph (either directed or undirected)
                                        ▷ motif is the motif that is being represented
    edges is an empty list of booleans
    if g is directed then
        options ← permutations(motif)
    else
        options ← combinations(motif)
    end if
    for tuple in options do
        n_1 ← tuple.first
        n_2 ← tuple.second
        neighbors ← whether n_1 and n_2 are neighbors in g
        append neighbors to edges
    end for
    motifNumber ← the unsigned int represented by edges
    motifIndex ← the index corresponding to the motif number
                ▷ The index is read from the index file we created ahead of time
    return motifIndex
end function
```

## 1.7 Motif Visualization

To assist with both understanding and debugging, a motif visualization script was written. The script can be found under the *measure_tests* directory with the name *draw_motifs.py*.

Running the script is simply a matter of changing the values given in the *main* section on the bottom of the script. The scripts outputs an image using networkx and matplotlib of the motif with the given index. You can also change the motif level and toggle whether the graph is directional. An example output is the motif given in figure 1.

# 2 C++ Kernel

## 2.1 Overview

As we have reached the optimal algorithmic performance of motif counting (we count each motif only once), in order to facilitate analyzing motifs in larger graphs, we must write faster **code**.

Previously, the code to count the motifs has been written in Python. Python, while easy to use and develop code in, is inherently slow, as the language itself is interpreted. Faster code performance is obtained through the use of code in C++, both running on the CPU and on the GPU. Running the algorithms using a C++ kernel requires two additional changes:

1. The graph must be saved in a LIL (List of Lists) format.
2. The algorithm must be modified so as to that the nodes are not actually removed from the graph, as that is inefficient using this format.

## 2.2 Algorithm changes

### 2.2.1 Graph Format

In the C++ kernel, the graph is saved using a graph format which is commonly used for sparse graph, but can be used to our advantage in this case.

As our format is aimed at increasing the efficiency of our code by leveraging the computer's internal cache mechanism, we call this structure a CacheGraph.

The CacheGraph is composed of two arrays:

1. An array which is composed of all the neighborhood lists placed back to back (Neighbors).
2. An array which holds the starting index of each node's neighbor list (Indices).

There follows an example of the conversion routine for a simple graph. The given graph is the one composed of these edges: (0->1, 0->2,0->3,2->0,3->1,3->2).

The behavior of the conversion now depends on whether the graph is directed or undirected.

- If the graph is directed, the result is as follows: Indices: [0, 3, 3, 4, 6], Neighbors: [1, 2, 3, 0, 1, 2]
- Else, the results for the undirected graph are: Indices: [0, 3, 5, 7, 10], Neighbors: [1, 2, 3, 0, 3, 0, 3, 0, 1, 2]

The CacheGraph object is designed around the principle of cache-awareness. The most important thing to remember when accessing the graph in the C++ code is that we are aiming to accelerate the computations by loading sections of the graph into the cache ahead of time for quick access. When using the CacheGraph, this comes into effect when we iterate over the offset vector first

and then access the blocks of neighbor nodes in the graph vector. By doing this, we are pulling the entire list of a certain node's neighbors into the cache, allowing us to iterate over them extremely quickly.

A concrete example of a good use and bad use case of the CacheGraph are the two popular search strategies BFS and DFS. When using BFS we are utilizing the full power of the cache-aware storage mechanism by pulling in the entire list of a node's neighbors to be processed at once, which is fast due to the fact that the neighbors are in the cache. In contrast, when using DFS, we are not going over all of a node's neighbors at once but jumping from one node to the other, which completely nullifies the speed advantage that the cache can give us, as the contents of the cache need to be constantly swapped out by the processor.

### 2.2.2 Removal Index

One disadvantage of the CacheGraph is that it can't be modified at runtime, as that is an extremely costly operation. Instead, we will hold an array where each node has a corresponding "removal index", which will be the first iteration where the node no longer exists in the graph. As the algorithm removes each node immediately after it iterates it's subtree, computing the removal indices of all nodes is simply a matter of "flipping" the list which contains the order in which the nodes will be traversed, i.e. the nodes as ordered by their degree. "Flipping" the list entails saving each node's position in the nodes list as the value corresponding to that node in the removal indices array.

For example, if the nodes were traversed in this order : [2,0,1], then the corresponding removal indices array will be [1,2,0] (i.e. node 0 is removed at iteration 1, node 1 at iteration 2, etc.)

We now need to further modify our code to check each node we use, to verify that it's removal index of the node is lower than the removal index of the root of the current subtree.

### 2.2.3 GPU Changes

To further accelerate the motif counting code, a version of it was written for the GPU using the CUDA library. Writing the code for the GPU means that we will run all of the motif subtree calculations for each node in parallel, which in turn means we must be able to run each of those calculations independently of the others. Luckily, using the removal indices we can already make the subtrees independent of one another, as the removal indices were pre-computed.

As the programming model we can use on the GPU itself is quiet limited (in essence, we are restricted to basic C code i.e. we must use the malloc and free function together with pointers), and as we want to limit the amount of memory used, we must make further modifications to the code.

1. **GPU functions** are written outside the main bodies of code (MotifCalculator,MotifUtils, CacheGraph), as they cannot be member functions. They are instead re-written as stand-alone functions within the file.

2. **Global variables** are used to communicate between the class methods, which provide the CPU-side pre- and post-processing of the results, and the GPU functions.

3. **Atomic add** is used to update the motif counters, so that the GPU threads won't interfere with one another.

4. **All data** which is used in the GPU must be copied into GPU memory. Additionally, the data is prefetched asynchronously to improve the code's performance.

5. **Visited vertices** (which saves which nodes were already visited) is no longer a map, as those can't be used in the GPU. Instead, it is an matrix of size $n^2$ in which each row corresponds to one node. The type of the matrix changes depending on the motif level. For level 3 a matrix of booleans is used, and for level 4 a matrix of short ints.

6. **No vectors** can be used in the GPU. In order to provide a unified interface, CPU code at the end of the calculation converts the feature from a matrix to the vector of vectors returned by the CPU motif calculator.

Furthermore, the GPU's memory itself is limited, and so we can't run graphs of any size on it. The GPU code, because of it's parallel nature, requires $n^2$ memory space, and so is unsuited for large graphs.

## 2.3 GPU requirements

In order to run the GPU code, some requirements must be met, in both software and hardware.

- The GPU must be an NVIDIA GPU of compute capability 3.5 or higher.
- Version 8.0 of the CUDA Toolkit must be installed and in the PATH of the system. Higher versions may work, but were not tested.
- The GPU drivers must be compatible with the CUDA Toolkit.

Specifically, the code was tested on a system with the following specs:

- Centos 7 Linux distribution
- A GeForce GTX 1080 Ti with compute capability 6.1
- CUDA Toolkit version 8.0
- GPU drivers version 396.26

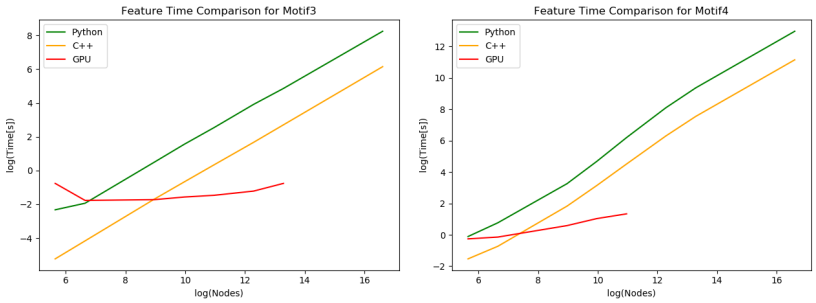



Figure 7: Run time comparison for different kernels

The benchmarks were ran on regular graphs with a regular degree of 20, which is why the graphs are linear in nature.

## 2.4 Performance comparison

The C++ kernel gives us code that runs several times faster than the original python code, and the GPU version gives us an additional 10X factor on top of the regular C++ code. Below are graph comparing the run-time of the various kernels on Erdos-renyi graphs of varying size. Figure 7 shows the comparisons between the various kernels. The raw benchmarks and code to generate the plots can be found in the repository features_algorithms/accelerated_graph_features/src/ accelerated_graph_features/benchmarks directory.

## 2.5 Future work

We are yet to implement one variation of the motif code (which exists in python). This variation is that of counting motifs for edges, rather than nodes.

This change requires minimal work, as the bulk of the code is already written for the node-based motif algorithm. Basically, the algorithm must accept the motifs found by the subtree functions and update the counters for each edge in the motif.

Further work could also include an implementation of the algorithm with the nodes used in chunks. Some of the time limitations we are faced with today could be solved if the code could be run on several CPUs/GPUs together. In order to enable this, the algorithm must be able to divide the nodes into smaller groups, perform the algorithm on each group and combine the results.

One more change that could possible increase the number of cases in which the GPU can be used is a sparse-graph implementation of the GPU algorithm. In contrast to the python/C++ algorithms, the GPU can't utilize expandable structures like dictionaries or hashmaps, instead relying on an array with a size of the number of nodes to store the neighbors of each node we iterate over, as we iterate over all nodes concurrently, this results in $O(n^2)$ memory. For sparse graphs, this approach is wasteful, and could be improved.

# A Combinatorical functions

## A.1 Permutations

The *permutations* function is used to create all the possible tuples of a given size which are comprised of elements of the original set **with respect to order**. In our case we only use 2-permutations, which give us all the possible edges that can exist between a given set of nodes.

For example, assume our motif is comprised of nodes 1,2 and 3. The possible 2-permutations in this case would be:

$$perm(\{1,2,3\}) = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$$

And since the order matters, we can see that, for example, both $(1,2)$ and $(2,1)$ appear in the resulting list.

A more intuitive way to view this function, and the one employed by us, is as looking at all of the elements of a matrix, without the main diagonal:

$$\begin{matrix} 0 & X & X \\ X & 0 & X \\ X & X & 0 \end{matrix}$$

Where X is an element accessed by the permutations, 0 is one which was not.

## A.2 Combinations

The *combinations* function is very similar to the *permutations* function in that it gives us a list of tuples derived from the original sets, but differs from it in that it **does not respect order**, and so only gives us a single instance of each tuple:

$$comb(\{1,2,3\}) = \{(1,2),(1,3),(2,3)\}$$

In the context of adjacency matrices, the function only iterated over the top half of the matrix:

$$\begin{matrix} 0 & X & X \\ 0 & 0 & X \\ 0 & 0 & 0 \end{matrix}$$

# B Motif variations

A core part of this algorithm is it's ability to use a precomputed table of the various motif variations and their corresponding motif index. As was discussed, each motif is represented using a single integer which is read from it's representing bit matrix. The following tables map each such integer to a "motif index", which is shared among all isomorphic motifs.

There are four tables in total, for 3- and 4-motifs in both directed and undirected cases. The code for generating these tables, along with pickle files containing the dictionaries which can be used in the python code can be found in the GitHub repository[2].

The actual implementation of the tables is reverse than the one displayed here. In the code, each motif number is mapped to it's index for quick access. The following tables are meant for quick reference and some examples, and so each index is mapped to all of the motif numbers corresponding to it.

## B.1 3-Motif Variations

### B.1.1 Undirected

| | |
|---|---|
| 0 | 3,5,6 |
| 1 | 7 |
| None | 0,1,2,4 |

### B.1.2 Directed

| | |
|---|---|
| 0 | 3,12,48 |
| 1 | 6,9,17,24,34,36 |
| 2 | 7,13,19,44,50,56 |
| 3 | 10,20,33 |
| 4 | 11,14,28,35,49,52 |
| 5 | 15,51,60 |
| 6 | 21,22,26,37,41,42 |
| 7 | 23,45,58 |
| 8 | 25,38 |
| 9 | 27,29,39,46,54,57 |
| 10 | 30,43,53 |
| 11 | 31,47,55,59,61,62 |
| 12 | 63 |
| None | 0,1,2,4,5,8,16,18,32,40 |

[2] https://github.com/oricc/graph_measures/tree/master/features_algorithms/motif_variations

## B.2 4-Motif Variations

### B.2.1 Undirected

| | |
|---|---|
| 0 | 7,25,42,52 |
| 1 | 13,14,19,22,26,28,35,37,41,44,49,50 |
| 2 | 15,23,27,29,39,43,46,53,54,57,58,60 |
| 3 | 30,45,51 |
| 4 | 31,47,55,59,61,62 |
| 5 | 63 |
| None | 0,1,2,3,4,5,6,8,9,10,11,12,16,17,18,20,21,24,32,33,34,36,38,40,48,56 |

### B.2.2 Directed

| | |
|---|---|
| 0 | 7,56,448,3584 |
| 1 | 14,49,69,168,336,386,515,1048,1792,2240,2592,3076 |
| 2 | 15,57,71,184,450,464,519,1080,2496,3588,3616,3840 |
| 3 | 21,28,35,42,112,134,196,224<br>259,280,321,392,560,896,1030,1344<br>1538,1552,2053,2088,2561,2688,3080,3136 |
| 4 | 22,38,50,52,104,133,261,296<br>328,352,385,388,536,704,1027,1216<br>1600,1664,2051,2072,2568,2576,3073,3074 |
| 5 | 23,39,58,60,120,135,263,312<br>449,452,456,480,568,960,1031,1472<br>2055,2104,3585,3586,3592,3600,3648,3712 |
| 6 | 29,43,198,240,323,408,1542,1584,2565,2944,3112,3392 |
| 7 | 30,46,51,53,197,232,325,344<br>368,387,390,424,1539,1560,1856,1920<br>2563,2600,2608,2752,3077,3078,3096,3264 |
| 8 | 31,47,59,61,199,248,327,440<br>451,454,472,496,1543,1592,2567,3008<br>3128,3520,3589,3590,3624,3632,3904,3968 |
| 9 | 54,360,389,1728,2584,3075 |
| 10 | 55,62,376,391,453,488,1984,2616,3079,3587,3608,3776 |
| 11 | 63,455,504,3591,3640,4032 |
| 12 | 76,161,274,522,577,800,1041,1160,1284,2084,2128,2178 |

| | |
|---|---|
| 13 | 77,78,169,177,338,402,523,526<br>579,581,1049,1073,1176,1192,1796,1824<br>2242,2256,2384,2434,2596,2848,3108,3332 |
| 14 | 79,185,466,527,583,1081,1208,2498,2512,3620,3844,3872 |
| 15 | 84,97,140,162,266,273,529,546<br>642,672,769,784,1034,1044,1092,1104<br>1282,1288,2060,2081,2113,2144,2180,2184 |
| 16 | 85,113,142,170,337,394,547,561<br>771,898,1038,1052,1304,1360,1794,1808<br>2117,2216,2244,2272,2593,2720,3084,3140 |
| 17 | 86,105,141,178,330,401,537,550<br>706,773,1035,1076,1232,1320,1604,1696<br>2115,2200,2400,2436,2572,2832,3105,3330 |
| 18 | 87,121,143,186,458,465,551,569<br>775,962,1039,1084,1336,1488,2119,2232<br>2500,2528,3596,3617,3652,3744,3842,3856 |
| 19 | 92,163,204,225,275,282,554,816<br>833,928,1045,1286,1348,1416,1546,1553<br>2085,2092,2160,2182,2625,2690,3152,3208 |
| 20 | 93,171,206,241,339,410,555,835<br>1053,1432,1550,1585,1798,1840,2246,2288<br>2597,2629,2946,2976,3116,3240,3396,3408 |
| 21 | 94,179,205,233,346,403,558,837<br>1077,1448,1547,1561,1860,1952,2416,2438<br>2604,2627,2754,2864,3109,3224,3280,3334 |
| 22 | 95,187,207,249,467,474,559,839<br>1085,1464,1551,1593,2502,2544,2631,3010<br>3256,3536,3621,3628,3846,3888,3908,4000 |
| 23 | 99,156,212,226,267,281,533,646<br>688,912,1066,1094,1136,1346,1556,1570<br>2061,2089,2369,2440,2692,2817,3168,3336 |

| | |
|---|---|
| 24 | 100,164,268,276,289,290,530,592<br>608,641,648,776,1042,1089,1096,1154<br>1156,1281,2058,2065,2068,2082,2120,2177 |
| 25 | 101,172,270,305,340,418,531,643<br>680,848,1050,1093,1112,1410,1793,1800<br>2062,2097,2241,2248,2594,2656,3092,3204 |
| 26 | 102,180,269,297,354,404,534,645<br>664,720,1074,1091,1128,1218,1632,1668<br>2059,2073,2376,2433,2580,2824,3106,3329 |
| 27 | 103,188,271,313,468,482,535,647<br>696,976,1082,1095,1144,1474,2063,2105<br>2497,2504,3604,3618,3680,3716,3841,3848 |
| 28 | 106,114,149,150,329,393,540,564<br>708,900,1059,1062,1248,1376,1602,1680<br>2307,2309,2328,2344,2569,2704,3081,3138 |
| 29 | 107,157,214,242,331,409,541,710<br>1067,1264,1574,1588,1606,1712,2371,2456<br>2573,2821,2948,2960,3113,3368,3394,3424 |
| 30 | 108,165,278,306,332,417,538,705<br>808,864,1043,1224,1285,1412,1601,1672<br>2086,2100,2136,2179,2570,2640,3089,3202 |
| 31 | 109,173,334,342,433,434,539,707<br>1051,1240,1605,1704,1797,1832,2243,2264<br>2574,2598,2896,2912,3121,3124,3458,3460 |
| 32 | 110,181,333,370,406,425,542,709<br>1075,1256,1603,1688,1888,1924,2392,2435<br>2571,2612,2768,2856,3097,3110,3266,3333 |
| 33 | 111,189,335,441,470,498,543,711<br>1083,1272,1607,1720,2499,2520,2575,3024<br>3129,3522,3622,3636,3845,3880,3936,3972 |
| 34 | 115,158,213,234,345,395,565,902<br>1070,1392,1564,1571,1858,1936,2373,2472<br>2601,2736,2756,2819,3085,3142,3296,3352 |

| | |
|---|---|
| 35 | 116,166,277,298,353,396,562,736<br>792,897,1046,1220,1283,1352,1616,1666<br>2076,2083,2152,2181,2577,2696,3082,3137 |
| 36 | 117,174,341,369,398,426,563,899<br>1054,1368,1795,1816,1872,1922,2245,2280<br>2595,2609,2728,2784,3086,3100,3141,3268 |
| 37 | 118,182,361,362,397,405,566,901<br>1078,1384,1730,1732,1744,1760,2408,2437<br>2585,2588,2712,2840,3083,3107,3139,3331 |
| 38 | 119,190,377,399,469,490,567,903<br>1086,1400,1986,2000,2501,2536,2617,2744<br>3087,3143,3612,3619,3780,3808,3843,3864 |
| 39 | 122,151,457,572,964,1063,1504,2311,2360,3593,3650,3728 |
| 40 | 123,159,215,250,459,473,573,966<br>1071,1520,1575,1596,2375,2488,2823,3012<br>3384,3552,3597,3625,3654,3760,3906,3984 |
| 41 | 124,167,279,314,460,481,570,824<br>961,992,1047,1287,1476,1480,2087,2108<br>2168,2183,3594,3601,3649,3664,3714,3720 |
| 42 | 125,175,343,442,462,497,571,963<br>1055,1496,1799,1848,2247,2296,2599,3040<br>3132,3524,3598,3633,3653,3752,3920,3970 |
| 43 | 126,183,378,407,461,489,574,965<br>1079,1512,1988,2016,2424,2439,2620,2872<br>3111,3335,3595,3609,3651,3736,3778,3792 |
| 44 | 127,191,463,471,505,506,575,967<br>1087,1528,2503,2552,3599,3623,3641,3644<br>3655,3768,3847,3896,4034,4036,4048,4064 |
| 45 | 220,227,283,944,1350,1557,1578,2093,2694,2881,3184,3464 |
| 46 | 221,222,235,243,347,411,1565,1579<br>1582,1589,1862,1968,2605,2758,2883,2885<br>2950,2992,3117,3312,3398,3440,3480,3496 |
| 47 | 223,251,475,1583,1597,2887,3014,3512,3568,3629,3910,4016 |
| 48 | 228,284,291,624,904,1158,1345,1554,2069,2090,2689,3144 |

| | |
|---|---|
| 49 | 229,236,286,307,348,419,880,936<br>1349,1414,1555,1562,1857,1928,2094,2101<br>2602,2672,2691,2753,3093,3160,3206,3272 |
| 50 | 230,244,285,299,355,412,752,920<br>1222,1347,1558,1586,1648,1670,2077,2091<br>2581,2693,2945,2952,3114,3176,3393,3400 |
| 51 | 231,252,287,315,476,483,952,1008<br>1351,1478,1559,1594,2095,2109,2695,3009<br>3192,3528,3605,3626,3696,3718,3905,3976 |
| 52 | 237,350,435,1563,1861,1960,2606,2755,2928,3125,3288,3462 |
| 53 | 238,245,349,371,414,427,1566,1587<br>1859,1904,1926,1944,2603,2613,2757,2800<br>2947,2984,3101,3118,3270,3304,3397,3416 |
| 54 | 239,253,351,443,478,499,1567,1595<br>1863,1976,2607,2759,3011,3056,3133,3320<br>3526,3544,3630,3637,3909,3952,3974,4008 |
| 55 | 246,363,413,1590,1734,1776,2589,2949,2968,3115,3395,3432 |
| 56 | 247,254,379,415,477,491,1591,1598<br>1990,2032,2621,2951,3000,3013,3119,3399<br>3448,3560,3613,3627,3782,3824,3907,3992 |
| 57 | 255,479,507,1599,3015,3576,3631,3645,3911,4024,4038,4080 |
| 58 | 292,584,1153,2066 |
| 59 | 293,294,300,308,356,420,600,616<br>712,840,1155,1157,1217,1409,1608,1665<br>2067,2070,2074,2098,2578,2632,3090,3201 |
| 60 | 295,316,484,632,968,1159,1473,2071,2106,3602,3656,3713 |
| 61 | 301,358,436,728,1219,1640,1669,2075,2582,2888,3122,3457 |
| 62 | 302,309,357,372,422,428,744,856<br>1221,1411,1624,1667,1864,1921,2078,2099<br>2579,2610,2664,2760,3094,3098,3205,3265 |
| 63 | 303,317,359,444,486,500,760,984<br>1223,1475,1656,1671,2079,2107,2583,3016<br>3130,3521,3606,3634,3688,3717,3912,3969 |
| 64 | 310,364,421,872,1413,1729,1736,2102,2586,2648,3091,3203 |

| | |
|---|---|
| 65 | 311,318,380,423,485,492,888,1000<br>1415,1477,1985,1992,2103,2110,2618,2680<br>3095,3207,3603,3610,3672,3715,3777,3784 |
| 66 | 319,487,508,1016,1479,2111,3607,3642,3704,3719,4033,4040 |
| 67 | 365,366,374,429,437,438,1731,1733<br>1752,1768,1896,1925,2587,2590,2614,2776<br>2904,2920,3099,3123,3126,3267,3459,3461 |
| 68 | 367,445,502,1735,1784,2591,3032,3131,3523,3638,3944,3973 |
| 69 | 373,430,1880,1923,2611,2792,3102,3269 |
| 70 | 375,381,431,446,494,501,1912,1927<br>1987,2008,2615,2619,2808,3048,3103,3134<br>3271,3525,3614,3635,3781,3816,3928,3971 |
| 71 | 382,439,493,1989,2024,2622,2936,3127,3463,3611,3779,3800 |
| 72 | 383,447,495,503,509,510,1991,2040<br>2623,3064,3135,3527,3615,3639,3643,3646<br>3783,3832,3960,3975,4035,4037,4056,4072 |
| 73 | 511,3647,4039,4088 |
| 74 | 585,586,588,804,1161,1169,1185,1316,2130,2194,2322,2340 |
| 75 | 587,589,590,1177,1193,1201,1828,2258,2386,2450,2852,3364 |
| 76 | 591,1209,2514,3876 |
| 77 | 593,609,650,673,778,786,801,802<br>1100,1105,1162,1164,1292,1297,1298,1300<br>2124,2129,2132,2148,2186,2209,2210,2212 |
| 78 | 594,612,649,780,1097,1170,1188,1313,2122,2193,2324,2338 |
| 79 | 595,613,651,681,782,850,1101,1113<br>1178,1196,1329,1426,1804,1825,2126,2225<br>2250,2257,2388,2466,2660,2850,3236,3348 |
| 80 | 596,610,652,657,658,676,777,788<br>1098,1106,1121,1124,1172,1186,1289,1314<br>2121,2146,2185,2196,2314,2316,2321,2337 |
| 81 | 597,611,654,689,779,914,1102,1137<br>1180,1194,1305,1362,1812,1826,2125,2217<br>2260,2274,2385,2442,2724,2849,3172,3340 |

| | |
|---|---|
| 82 | 598,614,653,665,722,781,1099,1129<br>1202,1204,1234,1321,1636,1700,2123,2201<br>2378,2402,2449,2452,2828,2836,3361,3362 |
| 83 | 599,615,655,697,783,978,1103,1145<br>1210,1212,1337,1490,2127,2233,2506,2513<br>2516,2530,3684,3748,3852,3860,3873,3874 |
| 84 | 601,617,714,805,806,842,1163,1165<br>1233,1324,1332,1425,1612,1697,2131,2134<br>2202,2226,2404,2468,2636,2834,3233,3346 |
| 85 | 602,620,713,812,844,868,1171,1189<br>1225,1317,1441,1444,1609,1673,2138,2195<br>2326,2342,2354,2356,2634,2642,3217,3218 |
| 86 | 603,621,715,846,1179,1197,1241,1457<br>1613,1705,1829,1836,2259,2266,2390,2482<br>2638,2854,2898,2916,3249,3380,3474,3492 |
| 87 | 604,618,716,820,841,932,1173,1187<br>1249,1318,1380,1417,1610,1681,2162,2198<br>2323,2330,2341,2348,2633,2706,3154,3209 |
| 88 | 605,619,718,843,1181,1195,1265,1433<br>1614,1713,1830,1844,2262,2290,2387,2458<br>2637,2853,2962,2980,3241,3372,3410,3428 |
| 89 | 606,622,717,845,1203,1205,1257,1449<br>1611,1689,1892,1956,2394,2418,2451,2454<br>2635,2770,2860,2868,3225,3282,3365,3366 |
| 90 | 607,623,719,847,1211,1213,1273,1465<br>1615,1721,2515,2518,2522,2546,2639,3026<br>3257,3538,3877,3878,3884,3892,3940,4004 |
| 91 | 625,803,906,1166,1308,1361,1810,2133,2218,2276,2721,3148 |
| 92 | 626,628,740,796,905,908,1174,1190<br>1252,1315,1353,1377,1618,1682,2154,2197<br>2325,2332,2339,2346,2697,2705,3145,3146 |

| | |
|---|---|
| 93 | 627,629,907,910,1182,1198,1369,1393<br>1820,1827,1874,1938,2261,2282,2389,2474<br>2729,2737,2788,2851,3149,3150,3300,3356 |
| 94 | 630,909,1206,1385,1746,1764,2410,2453,2713,2844,3147,3363 |
| 95 | 631,911,1214,1401,2002,2517,2538,2745,3151,3812,3868,3875 |
| 96 | 633,807,970,1167,1340,1489,2135,2234,2532,3660,3745,3858 |
| 97 | 634,636,828,969,972,996,1175,1191<br>1319,1481,1505,1508,2170,2199,2327,2343<br>2362,2364,3657,3658,3666,3721,3729,3730 |
| 98 | 635,637,971,974,1183,1199,1497,1521<br>1831,1852,2263,2298,2391,2490,2855,3044<br>3388,3556,3661,3662,3753,3761,3922,3986 |
| 99 | 638,973,1207,1513,2020,2426,2455,2876,3367,3659,3737,3794 |
| 100 | 639,975,1215,1529,2519,2554,3663,3769,3879,3900,4050,4068 |
| 101 | 659,684,852,1114,1125,1442,1801,2249,2318,2353,2658,3220 |
| 102 | 661,662,668,692,724,916,1123,1126<br>1130,1138,1250,1378,1634,1684,2315,2317<br>2329,2345,2377,2441,2708,2825,3170,3337 |
| 103 | 663,700,980,1127,1146,1506,2319,2361,2505,3682,3732,3849 |
| 104 | 666,677,721,790,809,866,1107,1132<br>1226,1293,1330,1428,1633,1676,2137,2150<br>2187,2228,2380,2465,2644,2826,3234,3345 |
| 105 | 667,685,723,854,1115,1133,1242,1458<br>1637,1708,1805,1833,2251,2265,2382,2481<br>2662,2830,2900,2914,3252,3377,3476,3490 |
| 106 | 669,726,1131,1266,1638,1716,2379,2457,2829,2964,3369,3426 |
| 107 | 670,693,725,918,1134,1139,1258,1394<br>1635,1692,1890,1940,2381,2393,2443,2473<br>2740,2772,2827,2857,3174,3298,3341,3353 |
| 108 | 671,701,727,982,1135,1147,1274,1522<br>1639,1724,2383,2489,2507,2521,2831,3028<br>3385,3554,3686,3764,3853,3881,3938,3988 |
| 109 | 674,785,1108,1290,2145,2188 |

| | |
|---|---|
| 110 | 675,682,787,817,849,930,1109,1116<br>1294,1306,1364,1418,1802,1809,2149,2161<br>2190,2220,2252,2273,2657,2722,3156,3212 |
| 111 | 678,690,738,789,793,913,1110,1140<br>1236,1291,1322,1354,1620,1698,2147,2153<br>2189,2204,2401,2444,2700,2833,3169,3338 |
| 112 | 679,698,791,825,977,994,1111,1148<br>1295,1338,1482,1492,2151,2169,2191,2236<br>2508,2529,3668,3681,3724,3746,3850,3857 |
| 113 | 683,851,1117,1434,1806,1841,2254,2289,2661,2978,3244,3412 |
| 114 | 686,691,853,915,1118,1141,1370,1450<br>1803,1817,1876,1954,2253,2281,2417,2446<br>2659,2732,2786,2865,3173,3228,3284,3342 |
| 115 | 687,699,855,979,1119,1149,1466,1498<br>1807,1849,2255,2297,2510,2545,2663,3042<br>3260,3540,3685,3756,3854,3889,3924,4002 |
| 116 | 694,917,1142,1386,1748,1762,2409,2445,2716,2841,3171,3339 |
| 117 | 695,702,919,981,1143,1150,1402,1514<br>2004,2018,2425,2447,2509,2537,2748,2873<br>3175,3343,3683,3740,3796,3810,3851,3865 |
| 118 | 703,983,1151,1530,2511,2553,3687,3772,3855,3897,4052,4066 |
| 119 | 729,730,813,870,1227,1235,1325,1460<br>1641,1644,1677,1701,2139,2203,2406,2484<br>2646,2838,2890,2892,3250,3378,3473,3489 |
| 120 | 731,1243,1645,1709,1837,2267,2894,2902,2918,3505,3506,3508 |
| 121 | 732,948,1251,1382,1642,1685,2331,2349,2710,2889,3186,3465 |
| 122 | 733,734,1259,1267,1643,1646,1693,1717<br>1894,1972,2395,2459,2774,2861,2891,2893<br>2966,2996,3314,3373,3430,3442,3481,3497 |
| 123 | 735,1275,1647,1725,2523,2895,3030,3513,3570,3885,3942,4020 |
| 124 | 737,794,810,818,865,929,1228,1299<br>1301,1302,1356,1420,1617,1674,2140,2156<br>2164,2211,2213,2214,2641,2698,3153,3210 |

| | |
|---|---|
| 125 | 739,795,945,946,1244,1307,1358,1366<br>1621,1706,1813,1834,2157,2221,2268,2275<br>2702,2726,2897,2913,3185,3188,3466,3468 |
| 126 | 741,798,882,937,1260,1331,1357,1430<br>1619,1690,1889,1932,2158,2229,2396,2467<br>2676,2699,2769,2858,3161,3238,3274,3349 |
| 127 | 742,754,797,921,1238,1268,1323,1355<br>1622,1652,1702,1714,2155,2205,2403,2460<br>2701,2837,2956,2961,3177,3370,3402,3425 |
| 128 | 743,799,953,1010,1276,1339,1359,1494<br>1623,1722,2159,2237,2524,2531,2703,3025<br>3193,3530,3700,3750,3861,3882,3937,3980 |
| 129 | 745,814,858,869,1229,1333,1427,1452<br>1625,1675,1868,1953,2142,2227,2420,2470<br>2643,2668,2762,2866,3226,3237,3281,3350 |
| 130 | 746,821,857,934,1237,1326,1396,1419<br>1628,1699,1866,1937,2163,2206,2405,2476<br>2665,2738,2764,2835,3158,3213,3297,3354 |
| 131 | 747,859,1245,1435,1629,1707,1838,1845<br>1870,1969,2270,2291,2669,2766,2899,2917<br>2982,2994,3245,3313,3414,3444,3482,3500 |
| 132 | 748,860,884,940,1253,1381,1443,1446<br>1626,1683,1865,1929,2334,2350,2355,2357<br>2666,2674,2707,2761,3162,3221,3222,3273 |
| 133 | 749,862,1261,1459,1627,1691,1869,1893<br>1961,1964,2398,2483,2670,2763,2771,2862<br>2930,2932,3253,3289,3290,3381,3478,3494 |
| 134 | 750,861,1269,1451,1630,1715,1867,1908<br>1945,1958,2419,2462,2667,2765,2802,2869<br>2963,2988,3229,3286,3305,3374,3418,3429 |
| 135 | 751,863,1277,1467,1631,1723,1871,1977<br>2526,2547,2671,2767,3027,3058,3261,3321<br>3542,3546,3886,3893,3941,3956,4006,4012 |

| | |
|---|---|
| 136 | 753,811,867,922,1230,1309,1363,1436<br>1649,1678,1814,1842,2141,2219,2278,2292<br>2645,2725,2954,2977,3180,3242,3404,3409 |
| 137 | 755,923,1246,1371,1653,1710,1821,1835<br>1878,1970,2269,2283,2733,2790,2901,2915<br>2958,2993,3181,3316,3406,3441,3484,3498 |
| 138 | 756,924,1254,1379,1650,1686,2333,2347,2709,2953,3178,3401 |
| 139 | 757,926,1262,1395,1651,1694,1891,1906<br>1942,1948,2397,2475,2741,2773,2804,2859<br>2955,2985,3182,3302,3306,3357,3405,3417 |
| 140 | 758,925,1270,1387,1654,1718,1750,1766<br>1778,1780,2411,2461,2717,2845,2957,2965<br>2969,2972,3179,3371,3403,3427,3433,3434 |
| 141 | 759,927,1278,1403,1655,1726,2006,2034<br>2525,2539,2749,2959,3001,3029,3183,3407<br>3449,3562,3814,3828,3869,3883,3939,3996 |
| 142 | 761,815,871,986,1231,1341,1468,1491<br>1657,1679,2143,2235,2534,2548,2647,3018<br>3258,3537,3692,3749,3862,3890,3916,4001 |
| 143 | 762,829,985,998,1239,1327,1483,1524<br>1660,1703,2171,2207,2407,2492,2839,3020<br>3386,3553,3670,3689,3725,3762,3914,3985 |
| 144 | 763,987,1247,1499,1661,1711,1839,1853<br>2271,2299,2903,2919,3022,3046,3514,3516<br>3569,3572,3693,3757,3918,3926,4017,4018 |
| 145 | 764,956,988,1012,1255,1383,1507,1510<br>1658,1687,2335,2351,2363,2365,2711,3017<br>3194,3529,3690,3698,3733,3734,3913,3977 |
| 146 | 765,990,1263,1523,1659,1695,1895,1980<br>2399,2491,2775,2863,3019,3060,3322,3389<br>3545,3558,3694,3765,3917,3954,3990,4009 |

| | |
|---|---|
| 147 | 766,989,1271,1515,1662,1719,2022,2036<br>2427,2463,2877,2967,3004,3021,3375,3431<br>3450,3561,3691,3741,3798,3826,3915,3993 |
| 148 | 767,991,1279,1531,1663,1727,2527,2555<br>3023,3031,3577,3578,3695,3773,3887,3901<br>3919,3943,4025,4028,4054,4070,4082,4084 |
| 149 | 819,881,931,938,1310,1365,1372,1422<br>1811,1818,1873,1930,2165,2222,2277,2284<br>2673,2723,2730,2785,3157,3164,3214,3276 |
| 150 | 822,873,874,933,1334,1388,1421,1429<br>1738,1740,1745,1761,2166,2230,2412,2469<br>2649,2652,2714,2842,3155,3211,3235,3347 |
| 151 | 823,889,935,1002,1342,1404,1423,1493<br>1994,2001,2167,2238,2533,2540,2681,2746<br>3159,3215,3676,3747,3788,3809,3859,3866 |
| 152 | 826,993,1303,1484,2172,2215,3665,3722 |
| 153 | 827,954,995,1009,1311,1367,1486,1500<br>1815,1850,2173,2223,2279,2300,2727,3041<br>3196,3532,3669,3697,3726,3754,3921,3978 |
| 154 | 830,890,997,1001,1335,1431,1485,1516<br>1996,2017,2174,2231,2428,2471,2684,2874<br>3239,3351,3667,3673,3723,3738,3786,3793 |
| 155 | 831,999,1017,1018,1343,1487,1495,1532<br>2175,2239,2535,2556,3671,3705,3708,3727<br>3751,3770,3863,3898,4042,4044,4049,4065 |
| 156 | 875,1437,1742,1777,1846,2294,2653,2970,2981,3243,3411,3436 |
| 157 | 876,1445,1737,2358,2650,3219 |
| 158 | 877,878,1453,1461,1739,1741,1753,1769<br>1900,1957,2422,2486,2651,2654,2778,2870<br>2906,2924,3227,3251,3283,3382,3475,3493 |
| 159 | 879,1469,1743,1785,2550,2655,3034,3259,3539,3894,3948,4005 |
| 160 | 883,939,1373,1438,1822,1843,1875,1905<br>1934,1946,2286,2293,2677,2731,2789,2801<br>2979,2986,3165,3246,3278,3308,3413,3420 |

| | |
|---|---|
| 161 | 885,942,1397,1454,1881,1882,1884,1931<br>1939,1955,2421,2478,2675,2739,2793,2794<br>2796,2867,3166,3230,3277,3285,3301,3358 |
| 162 | 886,941,1389,1462,1747,1754,1765,1772<br>1897,1933,2414,2485,2678,2715,2777,2846<br>2908,2922,3163,3254,3275,3379,3477,3491 |
| 163 | 887,943,1405,1470,1913,1935,2003,2010<br>2542,2549,2679,2747,2809,3050,3167,3262<br>3279,3541,3813,3820,3870,3891,3932,4003 |
| 164 | 891,1003,1439,1501,1847,1854,1998,2033<br>2295,2302,2685,2983,3002,3045,3247,3415<br>3452,3564,3677,3755,3790,3825,3923,3994 |
| 165 | 892,1004,1447,1509,1993,2359,2366,2682,3223,3674,3731,3785 |
| 166 | 893,1006,1455,1525,1916,1959,1995,2009<br>2423,2494,2683,2810,2871,3052,3231,3287<br>3390,3557,3678,3763,3789,3817,3930,3987 |
| 167 | 894,1005,1463,1517,1997,2021,2025,2028<br>2430,2487,2686,2878,2938,2940,3255,3383<br>3479,3495,3675,3739,3787,3795,3801,3802 |
| 168 | 895,1007,1471,1533,1999,2041,2551,2558<br>2687,3066,3263,3543,3679,3771,3791,3833<br>3895,3902,3964,4007,4051,4058,4069,4076 |
| 169 | 947,1374,1819,1877,1962,2285,2734,2787,2929,3189,3292,3470 |
| 170 | 949,950,1390,1398,1749,1756,1763,1770<br>1898,1941,2413,2477,2718,2742,2780,2843<br>2905,2921,3187,3190,3299,3355,3467,3469 |
| 171 | 951,1406,2005,2026,2541,2750,2937,3191,3471,3804,3811,3867 |
| 172 | 955,1011,1375,1502,1823,1851,1879,1978<br>2287,2301,2735,2791,3043,3057,3197,3324<br>3534,3548,3701,3758,3925,3953,3982,4010 |
| 173 | 957,1014,1391,1526,1751,1767,1786,1788<br>2415,2493,2719,2847,3033,3036,3195,3387<br>3531,3555,3702,3766,3945,3946,3981,3989 |

| | |
|---|---|
| 174 | 958,1013,1399,1518,1914,1943,2012,2019<br>2429,2479,2743,2812,2875,3049,3198,3303<br>3359,3533,3699,3742,3797,3818,3929,3979 |
| 175 | 959,1015,1407,1534,2007,2042,2543,2557<br>2751,3065,3199,3535,3703,3774,3815,3836<br>3871,3899,3961,3983,4053,4060,4067,4074 |
| 176 | 1019,1503,1855,2303,3047,3580,3709,3759,3927,4026,4046,4081 |
| 177 | 1020,1511,2367,3706,3735,4041 |
| 178 | 1021,1022,1519,1527,2023,2044,2431,2495<br>2879,3068,3391,3559,3707,3710,3743,3767<br>3799,3834,3962,3991,4043,4045,4057,4073 |
| 179 | 1023,1535,2559,3711,3775,3903,4047,4055,4071,4089,4090,4092 |
| 180 | 1755,1773,1901,1965,2779,2910,2926,2934,3291,3507,3509,3510 |
| 181 | 1757,1771,1902,1973,2782,2907,2925,2998,3315,3446,3483,3501 |
| 182 | 1758,1774,1779,1781,1899,1910,1949,1974<br>2781,2806,2909,2923,2971,2974,2989,2997<br>3307,3318,3419,3437,3438,3443,3485,3499 |
| 183 | 1759,1775,1787,1789,1903,1981,2783,2911<br>2927,3035,3038,3062,3323,3515,3517,3547<br>3571,3574,3949,3950,3958,4013,4021,4022 |
| 184 | 1782,2973,3435 |
| 185 | 1783,1790,2038,2975,3005,3037,3439,3451,3563,3830,3947,3997 |
| 186 | 1791,3039,3579,3951,4029,4086 |
| 187 | 1883,1885,1886,1909,1947,1963,1966,1971<br>2795,2797,2798,2803,2931,2933,2990,2995<br>3293,3294,3309,3317,3422,3445,3486,3502 |
| 188 | 1887,1979,2799,3059,3325,3550,3957,4014 |
| 189 | 1907,1950,2805,2987,3310,3421 |
| 190 | 1911,1915,1951,1982,2014,2035,2807,2813<br>2991,3003,3051,3061,3311,3326,3423,3453<br>3549,3566,3822,3829,3933,3955,3998,4011 |
| 191 | 1917,1967,2011,2811,2935,3054,3295,3518,3573,3821,3934,4019 |
| 192 | 1918,1975,2013,2027,2030,2037,2814,2939<br>2941,2999,3006,3053,3319,3447,3454,3487<br>3503,3565,3805,3806,3819,3827,3931,3995 |

| | |
|---|---|
| 193 | 1919,1983,2015,2043,2815,3055,3063,3067<br>3327,3551,3581,3582,3823,3837,3935,3959<br>3965,4015,4027,4030,4062,4078,4083,4085 |
| 194 | 2029,2942,3511,3803 |
| 195 | 2031,2045,2943,3070,3519,3575,3807,3835,3966,4023,4059,4077 |
| 196 | 2039,2046,3007,3069,3455,3567,3831,3838,3963,3999,4061,4075 |
| 197 | 2047,3071,3583,3839,3967,4031,4063,4079,4087,4091,4093,4094 |
| 198 | 4095 |
| None | 0,1,2,3,4,5,6,8<br>9,10,11,12,13,16,17,18<br>19,20,24,25,26,27,32,33<br>34,36,37,40,41,44,45,48<br>64,65,66,67,68,70,72,73<br>74,75,80,81,82,83,88,89<br>90,91,96,98,128,129,130,131<br>132,136,137,138,139,144,145,146<br>147,148,152,153,154,155,160,176<br>192,193,194,195,200,201,202,203<br>208,209,210,211,216,217,218,219<br>256,257,258,260,262,264,265,272<br>288,304,320,322,324,326,384,400<br>416,432,512,513,514,516,517,518<br>520,521,524,525,528,532,544,545<br>548,549,552,553,556,557,576,578<br>580,582,640,644,656,660,768,770<br>772,774,832,834,836,838,1024,1025<br>1026,1028,1029,1032,1033,1036,1037,1040<br>1056,1057,1058,1060,1061,1064,1065,1068<br>1069,1072,1088,1090,1120,1122,1152,1168<br>1184,1200,1280,1296,1312,1328,1408,1424<br>1440,1456,1536,1537,1540,1541,1544,1545<br>1548,1549,1568,1569,1572,1573,1576,1577<br>1580,1581,2048,2049,2050,2052,2054,2056<br>2057,2064,2080,2096,2112,2114,2116,2118<br>2176,2192,2208,2224,2304,2305,2306,2308<br>2310,2312,2313,2320,2336,2352,2368,2370<br>2372,2374,2432,2448,2464,2480,2560,2562<br>2564,2566,2624,2626,2628,2630,2816,2818<br>2820,2822,2880,2882,2884,2886,3072,3088<br>3104,3120,3200,3216,3232,3248,3328,3344 |