Appendix A: Score Derivation

If we let ξ_i be the i^{th} component of $V_u + \int_T F(w_T, u)\ dt$, then we have

$$\begin{aligned}
S(u) &= \frac{d}{dM_u} A\left(\xi\right) \\
&= \sum_i \frac{\partial A}{\partial v_i} \frac{d\xi_i}{dM_u} \\
&= \sum_i \frac{\partial A}{\partial v_i} \frac{d}{dM_u} \left(v_{i,u} + \int_T f_i(w_T, u)\ dt \right) \\
&= \sum_i \frac{\partial A}{\partial v_i} \left(\int_T \frac{\partial}{\partial M_u} f_{i,u}(w_T)\ dt \right)
\end{aligned}$$

...by Leibeitz' Rule, since the time frame limits of integration do not not depend on M_u. Because of the associativity of maps, we can write the composition of F with w_T as a function of t:

$$S(t) = \sum_i \frac{\partial A}{\partial v_i} \left(\int_T \frac{\partial}{\partial M_u} (f_{i,u} \circ w_T)(t)\ dt \right)$$

Because of T is discrete, w_T is really just an n-vector over $\mathcal{D}$, and so our score function becomes

$$\begin{aligned}
S(u) &= \sum_i \frac{\partial A}{\partial v_i} \sum_{t_j} \frac{\partial}{\partial M_u} f_{i,u}\Big(D_{t_j}, D_{t_{j-1}}, \ldots, D_{t_{j-n+1}} \Big) \\
&= \sum_i \frac{\partial A}{\partial v_i} \sum_{t_j} \sum_m \boxed{\frac{\partial f_{i,u}}{\partial x_m}}\ \frac{dx_m}{dM_u} \Big(\{x_m\} \Big)
\end{aligned}$$

...where $\{x_m\}$ is the expanded representation of the D_i variables, and x_m is the m^{th} real argument to f. This is still pretty ugly, but fortunately, we can make sense of these quantities. The total amount of money given is the sum of the specific x_m variables that align with basis elements of $\mathcal{M}$.

$$M_u = \sum_{x_l \in \mathscr{B}\mathcal{M}} x_l$$

$$\Rightarrow \qquad \frac{dx_m}{dM_u} = \begin{cases} 1 - \sum\limits_{m \neq l, x_l \in \mathscr{B}\mathcal{M}} x_l & \text{if } x_m \in \mathscr{B}\mathcal{M} \\ 0 & \text{otherwise} \end{cases}$$

The boxed quantity is a number we can recover from the weights of our trained function f. The number to its right is even simpler; x_m is either M_u or an independent variable, and so if it simplifies to δ_{mk}

In the case that A is just the linear function $A(V) = \sum_i a_i v_i$, we can write the score completely in terms of known variables:

$$S(u) = \sum_i a_i \sum_{t_j, m} \psi \tag{1}$$