Probabilistic Dependency Graphs and Inconsistency

How to model, measure, and mitigate internal conflict

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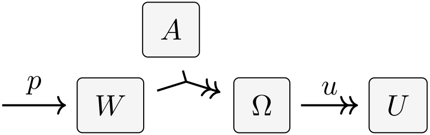
The standard way of modeling an agent with uncertainty:

- a probability distribution $p : \Delta W$ over worlds W,
- a utility function $u : \Omega \to \mathbb{R}$, some actions A.



Such agents cannot have internal conflict;

by construction, they have consistent beliefs and desires.

I like this slide

⟨ INCOMPLETE ⟩

Why build systems that can be inconsistent, if inconsistency is bad?

Why entertain the possibility of being wrong, if being wrong is bad?
Elsewhere, computer scientists take great care to model inconsistency:

- assertions and test cases: Text
-
- losses for training neural networks `(totally separate from the probabilistic model)`

I think you're agonizing too much over this. Why not just say "Because sometimes do have inconsistent beliefs. We also want to model the dynamics of making them consistent, and capture the extent to which they're inconsistent.

Yet Another Probabilistic Graphical Model

I don't think that I would remove this but modify it to say something like: not only do we want to model inconsistency, but we want to do so using a probabilistic graphical model.
Then go back to the current slides and say "In doing so, we get much more."

$\langle$ `remove slide, once intro is finished` $\rangle$

We introduce *probabilistic dependency graphs* (PDGs), a new class of graphical models for representing uncertainty.

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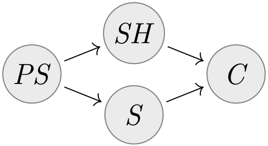
Two aspects of Bayesian Networks (BNs)

Qualitative BN, $\mathcal{G}$

an independence relation on variables

- $X \perp\!\!\!\perp_{\mathcal{G}} Y \mid \mathbf{Pa}(X)$, for all non-descendents Y of X



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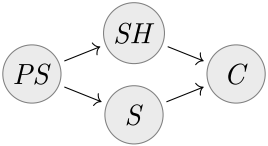
(Quantitative) BN, $\mathcal{B} = (\mathcal{G}, \mathbf{p})$

a qualitative BN ($\mathcal{G}$) and a cpd $p_X(X \mid \mathbf{Pa}(X))$ for each variable X.

- Defines a joint distribution $\Pr_{\mathcal{B}}$ with the independencies $\perp\!\!\!\perp_{\mathcal{G}}$.



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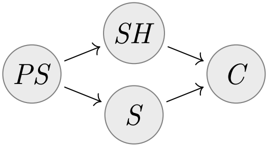

Outline for Section 1

Why is the outline coming after you've introduced BN's? What the high-level story?

Simple Example: Floomps and Guns

Grok thinks it likely (.95) that guns are illegal,
but that floomps (local slang) are legal (.90).

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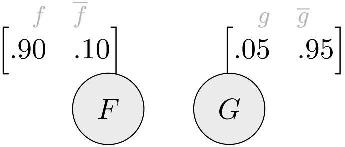

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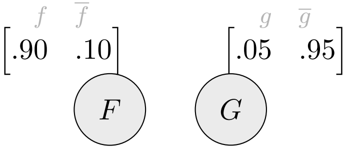
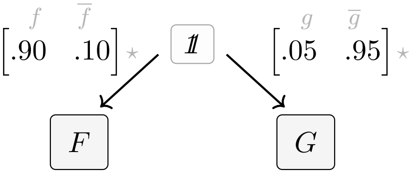


PDG



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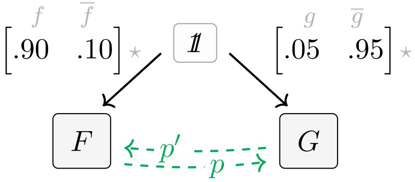

- The cpds of a PDG are attached to edges, not nodes.
- PDGs can incorporate arbitrary new probabilistic information.

Grok learns that Floomps and Guns have the same legal status (92%)

$$p(G \mid F) = \begin{array}{cc} & \begin{array}{cc} g & \bar{g} \end{array} \\ \begin{array}{c} f \\ \bar{f} \end{array} & \begin{bmatrix} .92 & .08 \\ .08 & .92 \end{bmatrix} \end{array} = \big(\, p'(F \mid G) \,\big)^{\mathsf{T}}$$

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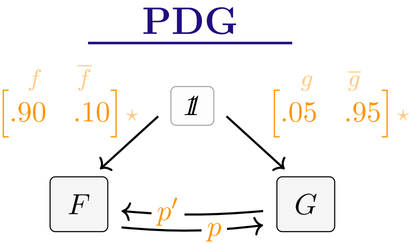

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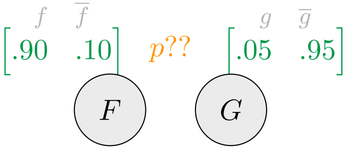

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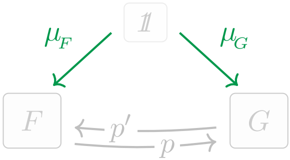

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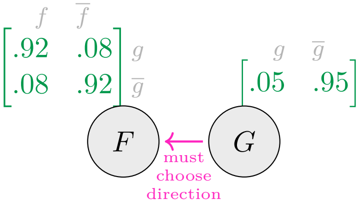

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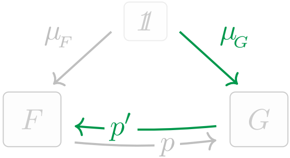

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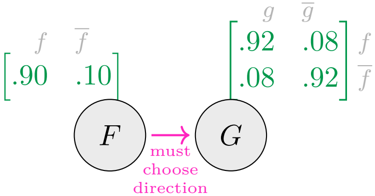

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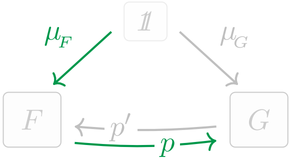

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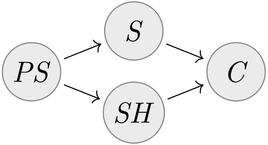

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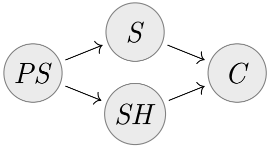

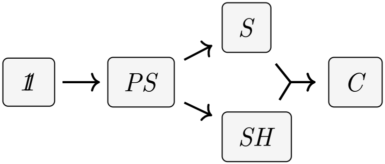

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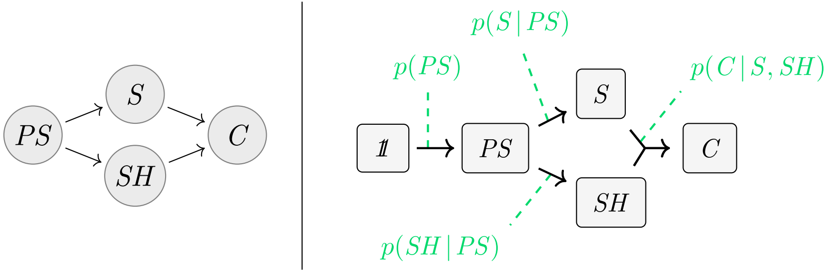

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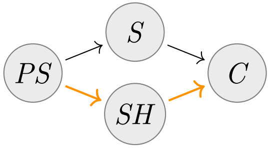

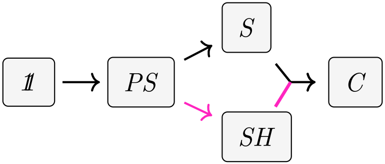

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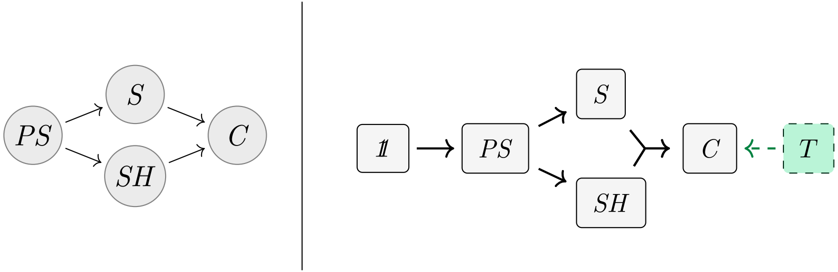

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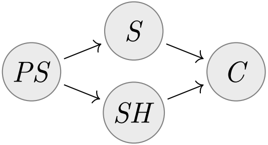

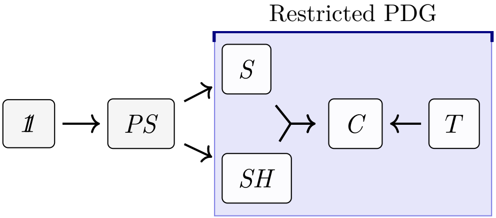

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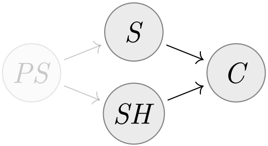

Must now give distributions on SH and S, or distinguish them as "observed" (a *conditional* BN).

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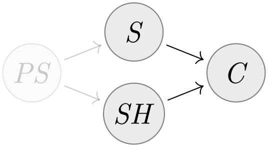

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In a qualitative BN: *removing* data results in *new* knowledge: $A \perp\!\!\!\perp C$.

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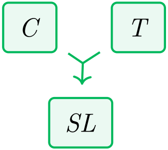

Grok wants to be supreme leader (SL).

- She notices that those who use tanning beds have more power, unless they get cancer

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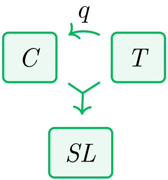

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- ...but mom says $q(C \mid T) = \begin{array}{cc} \begin{matrix} c & \bar{c} \end{matrix} & \\ \begin{bmatrix} .15 & .85 \\ .02 & .98 \end{bmatrix} & \begin{matrix} t \\ \bar{t} \end{matrix} \end{array}.$

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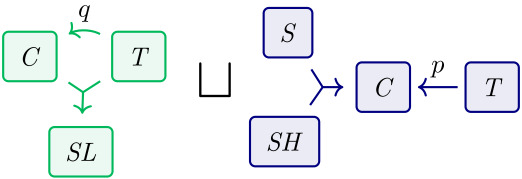

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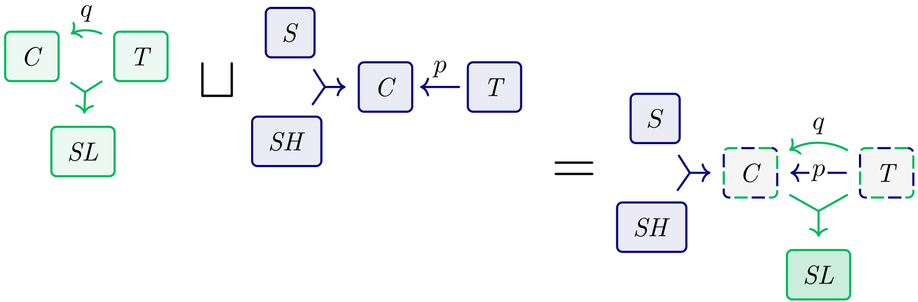

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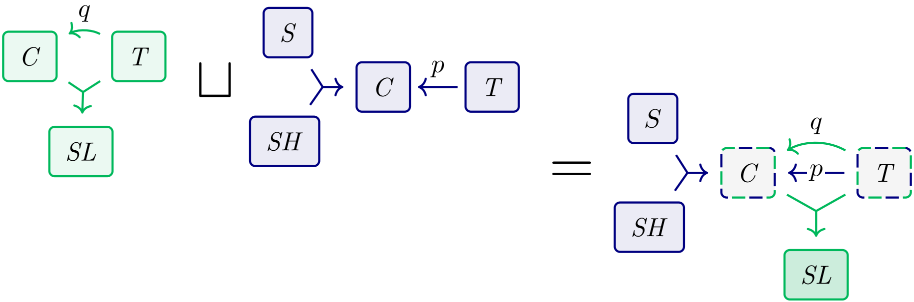

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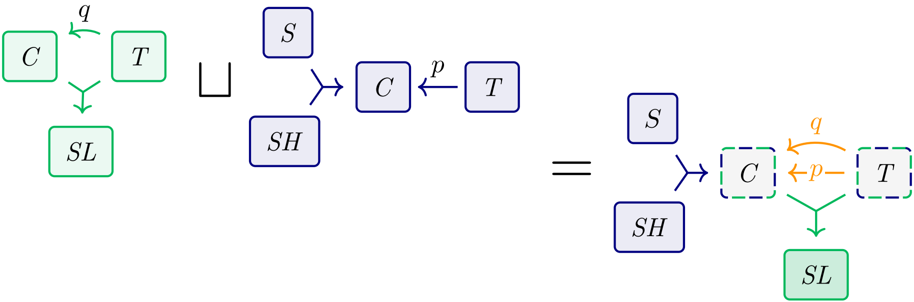

- Arbitrary PDGs may be combined without loss of information
- They may have parallel edges which directly conflict.

Outline for Section 2

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$$\mathcal{V}(\mathcal{m}) := \prod_{X \in \mathcal{N}} \mathcal{V}(X)$$

is the set of possible joint variable settings.

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write "p!" for the limit $(\beta_p \to \infty)$ of high confidence in a cpd p.

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Needless to say, I think that all the following black slides are a *terrible* idea.
This is not where you want to spend your time. It's a major distraction.

Definition (PDG)

$\mathcal{N}$: **Set** (node set)

$\mathcal{V} : \mathcal{N} \to$ **Set** (node values)

$\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \times \mathit{Label}$ (edge set)

For $X \xrightarrow{L} Y \in \mathcal{E}$,

$\mathbf{p}_L : \mathcal{V}(X) \to \Delta\mathcal{V}(Y)$ (edge cpd)

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Let **Mark** be the category of measurable spaces and Markov kernels.

Definition (PDG)

$\mathcal{N}$: **Set** (node set)

$\mathcal{V} : \mathcal{N} \to$ **Set** (node values)

$\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \times \mathit{Label}$ (edge set)

For $X \xrightarrow{L} Y \in \mathcal{E}$,

$\mathbf{p}_L : \mathcal{V}(X) \to \Delta\mathcal{V}(Y)$ (edge cpd)

$\alpha_L : \mathbb{R}$ (functional determination)

$\beta_L : \mathbb{R}$ (cpd confidence)

- $(\mathcal{N}, \mathcal{V})$ is a set of variables
- $(\mathcal{N}, \mathcal{E})$ is a multigraph
- $(\mathcal{N}, \mathcal{E}, \alpha)$, the qualitative data, forms a weighted multigraph.
- We call $(\mathcal{N}, \mathcal{E}, \mathcal{V}, \mathbf{p})$ an *unweighted* PDG
  - and give it semantics as though $\alpha_L = \beta_L = 1$.

Let **Mark** be the category of measurable spaces and Markov kernels.

Equivalent Categorical Definition

An unweighted PDG is a functor $\langle \mathbf{p}, \mathcal{V} \rangle : \mathit{Paths}(\mathcal{N}, \mathcal{E}) \to \mathbf{Mark}$.
So a PDG is a *diagram* in **Mark**, in the usual mathematical sense.

What do you do with diagrams? Take limits / colimits.

$$\cdots \qquad X_1 \twoheadrightarrow X_2 \leftarrow X_3 \qquad \cdots$$

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$$\begin{tikzcd}
 & & L \arrow[dl] \arrow[d] \arrow[dr] & & \\
\cdots & X_1 \arrow[r, two heads] & X_2 & X_3 \arrow[l] & \cdots
\end{tikzcd}$$

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$$
\begin{tikzcd}
 & & V \arrow[dl, "\forall"'] \arrow[d] \arrow[dr] & L \arrow[dll] \arrow[dl] \arrow[d] & \\
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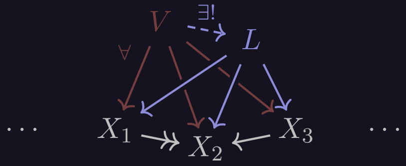

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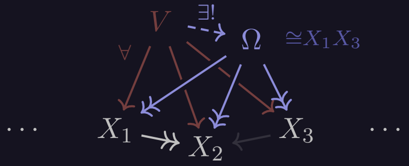

For the deterministic sub-PDG $m_{\det} \subseteq m$:

$$\lim m_{\det} = \left(\begin{array}{c}\text{natural}\\ \text{sample space}\end{array} \Omega, \quad \begin{array}{c}\text{random}\\ \text{variables}\end{array} \left\{ \tilde{X} : \Omega \to \mathcal{V}(X) \right\}_{X \in \mathcal{N}} \right)$$

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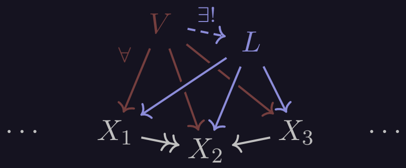

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In general: $$\lim m = \Big(\mathrm{Verts}(\underbrace{\mathbb{L}m}_{\substack{\text{Locally Consistent Polytope} \\ \text{(possible states of the Sum-Product algorithm)}}}), \quad \{ \text{variable marginals} \} \Big)$$

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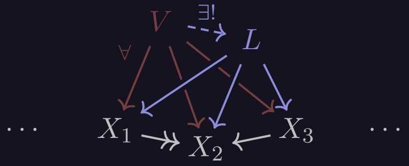

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For a BN $\mathcal{B}$: $$\lim \mathcal{m}_{\mathcal{B}} = \left(\mathbb{1}, \ \left\{ \Pr_{\mathcal{B}}(X) \right\}_{X \in \mathcal{N}} \right)$$

Outline for Section 3

Semantics of PDGs

$\{\!\{m\}\!\} \subseteq \Delta\mathcal{V}(m)$

The set of joint distributions consistent with m;

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THE SCORING FUNCTION

⟨ either finish overhaul or reinstate old slides ⟩

$$[\![\mathcal{M}]\!]_\gamma(\mu) := \mathit{Inc}_{\mathcal{M}}(\mu) + \gamma\ \mathit{IDef}_{\mathcal{M}}(\mu)$$

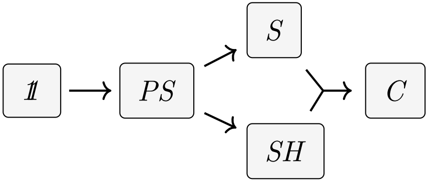
tradeoff parameter $\gamma \geq 0$

⟨ Emphasize entropy balance ⟩

Outline for Section 4

Capturing Bayesian Networks

For a BN $\mathcal{B}$ with N nodes and a vector $\beta \in \mathbb{R}^N$, let $\boldsymbol{m}_{\mathcal{B},\beta}$ be the PDG corresponding to $\mathcal{B}$, with $\alpha = \mathbf{1}$, and the given vector β of confidences.



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Theorem (*BNs are PDGs*)

If $\mathcal{B}$ is a BN and $\Pr_{\mathcal{B}}$ is the distribution it specifies, then for all $\gamma > 0$ and all vectors β,

$$[\![\boldsymbol{m}_{\mathcal{B},\beta}]\!]^*_\gamma = \{\Pr_{\mathcal{B}}\}, \quad \textit{and thus} \quad [\![\boldsymbol{m}_{\mathcal{B},\beta}]\!]^* = \Pr_{\mathcal{B}}\,.$$

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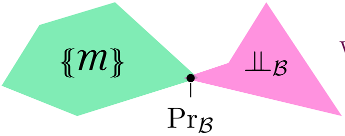
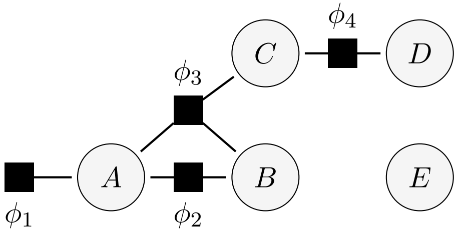
space of distributions consistent with $\boldsymbol{m}_{\mathcal{B}}$ (which minimize *Inc*)



space of distributions with independencies of $\mathcal{B}$ (which can be shown to minimize *IDef*)

$\langle$ maximum entropy result for BNs $\rangle$

Factor Graphs



You need to add some intuition here.

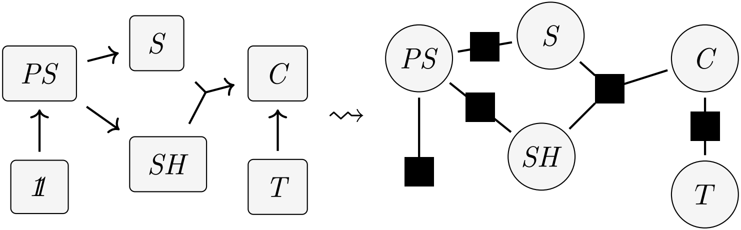
Definition

A *factor graph* Φ is a set of random variables $\mathcal{X} = \{X_i\}$ and *factors* $\{\phi_J : \mathcal{V}(X_J) \to \mathbb{R}_{\geq 0}\}_{J \in \mathcal{J}}$, where $X_J \subseteq \mathcal{X}$; define

$$\Pr_\Phi(\vec{x}) = \frac{1}{Z_\Phi} \prod_{J \in \mathcal{J}} \phi_J(\vec{x}_J),$$

where Z_Φ is the normalization constant.

PDGs as Factor Graphs



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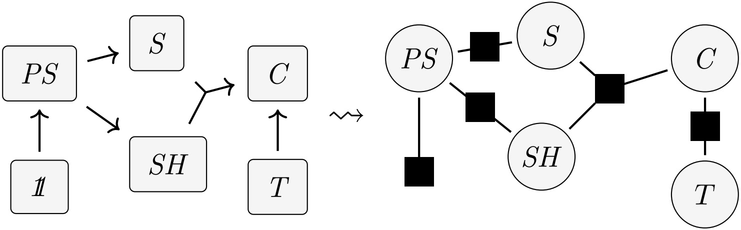
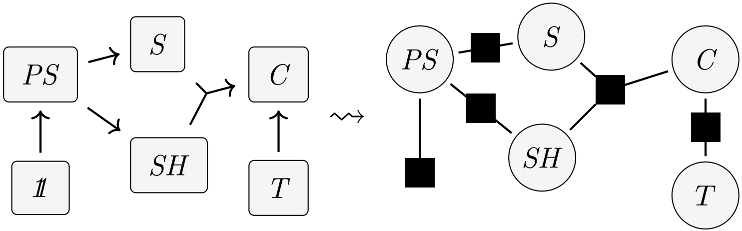


The cpds of a PDG are essentially factors. Are the semantics different?

PDGs as Factor Graphs



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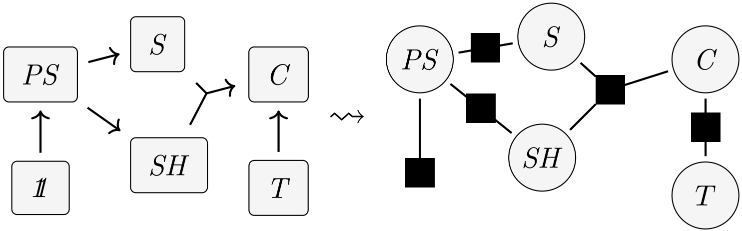

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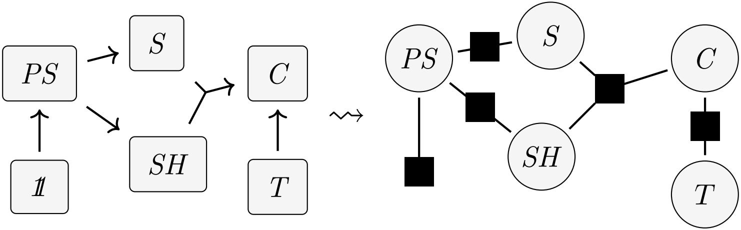

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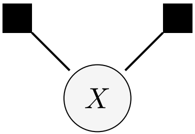
Theorem

For all unweighted PDGs $\mathcal{M}$ *and non-negative vectors* $\mathbf{v}$ *over the edges of* $\mathcal{M}$, *and all* $\gamma > 0$, *we have that* $[\![(\mathcal{M}, \mathbf{v}, \gamma\mathbf{v})]\!]_\gamma = \gamma\, \mathit{GFE}_{(\Phi_{\mathcal{M}}, \mathbf{v})}$; *consequently,* $[\![(\mathcal{M}, \mathbf{v}, \gamma\mathbf{v})]\!]_\gamma^* = \{\Pr_{(\Phi_{\mathcal{M}}, \mathbf{v})}\}$.

An Important Difference between PDGs and Factor Graphs

$$m := \begin{tikzcd} \mathbb{1} \arrow[d, bend right, "q"'] \arrow[d, bend left, "p"] \\ X \end{tikzcd}$$



$$=: \Phi$$

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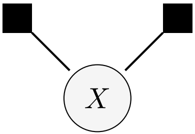

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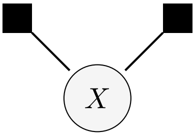

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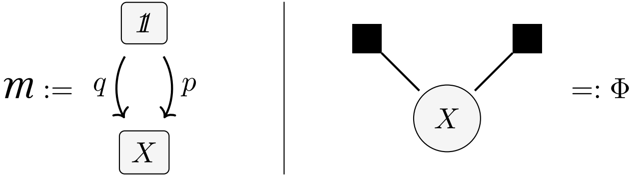

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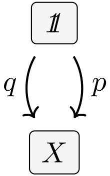
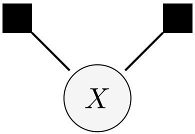
$\mathcal{m} :=$ 



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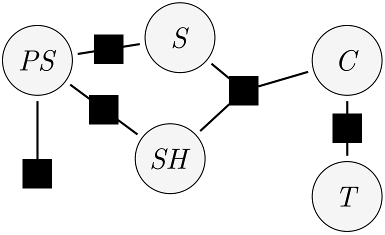
- Individual factors have *no probabilistic meaning*,
- a factor graph can fail to normalize, in which case it has no global semantics either.

Factor Graphs as PDGs



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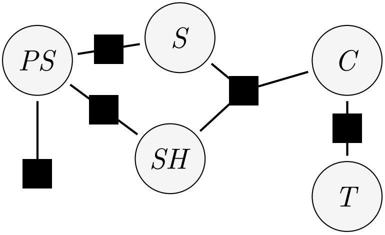

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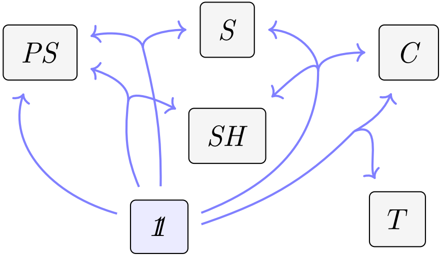

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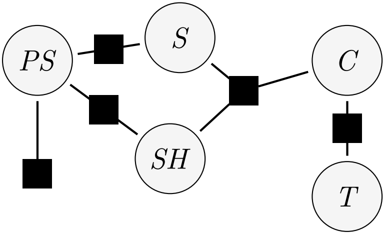

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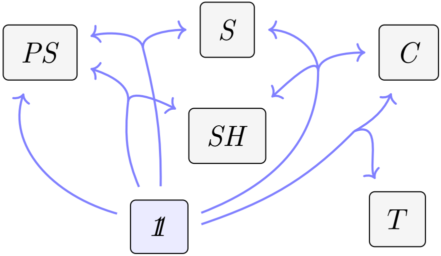

Theorem

$\Pr_\Phi = [\![\mathcal{M}_\Phi]\!]^*_1$ *for all factor graphs* Φ.

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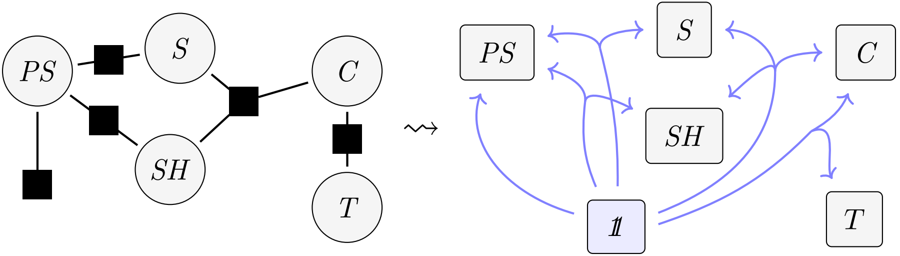

Theorem

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Theorem

For all weighted factor graphs $\Psi = (\Phi, \theta)$ *and all* $\gamma > 0$, *we have that* $\mathit{GFE}_\Psi = {}^1\!/_\gamma [\![\mathcal{m}_{\Psi,\gamma}]\!]_\gamma + C$ *for some constant* C, *so* $\Pr_\Psi$ *is the unique element of* $[\![\mathcal{m}_{\Psi,\gamma}]\!]_\gamma^*$.

⟨ `Add theorem:` $\log Z_\Phi = \langle\!\langle \mathcal{n}_\Phi \rangle\!\rangle$ ⟩

I suspect that you'll lose your audience with this

Letting $x^{\mathbf{w}}$ and $y^{\mathbf{w}}$ denote the values of X and Y, respectively, in $\mathbf{w} \in \mathcal{V}(\mathcal{M})$, we have

$$[\![\mathcal{M}]\!](\mu) = \mathop{\mathbb{E}}_{\mathbf{w}\sim\mu} \Bigg\{ \sum_{X \xrightarrow{L} Y} \Bigg[\overbrace{\beta_L \log \frac{1}{\mathbf{p}_L(y^{\mathbf{w}} | x^{\mathbf{w}})}}^{\text{log likelihood / cross entropy}} + \underbrace{(\alpha_L \gamma - \beta_L) \log \frac{1}{\mu(y^{\mathbf{w}} | x^{\mathbf{w}})}}_{\text{local regularization } (\beta_L > \alpha_L \gamma)} \Bigg] - \underbrace{\gamma \log \frac{1}{\mu(\mathbf{w})}}_{\text{global regularization}} \Bigg\}.$$

Outline for Section 5

Inference via Inconsistency Reduction

Conditioning as inconsistency resolution.
To condition on $Y = y$, in $\mathcal{M}$, simply add the edge $\mathbb{1} \xrightarrow{\delta_y} Y$ to get $\mathcal{M}_{Y=y}$.
Then $[\![\mathcal{M}_{Y=y}]\!]^* = [\![\mathcal{M}]\!]^* \mid (Y = y)$.

Add some words for what this is saying (e.g., remind the reader of what [[\M]]* is).Ar

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Querying $\Pr(Y \mid X)$ **in a PDG** $\mathcal{M}$**.**

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Text

⟨ Convexity Result ⟩

⟨ Hardness Result ⟩

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- so oracle access to inconsistency yields fast inference by gradient descent.

Make this a sub-bullet

⟨ Convexity Result ⟩

⟨ Hardness Result ⟩

Outline for Section 6

Inconsistency: The Universal Loss

- Fruitful to cast AI as optimization. But what to optimize?

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  - and a possibly-inconsistent model comes with a natural objective.

This needs more intuition. What's the model doing? Why might it be inconsistent?

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Surprising Result

Most standard objectives arise naturally as the inconsistency of the obvious PDG describing the situation.

obvious -> natural

Inconsistency: The Universal Loss

- Fruitful to cast AI as optimization. But what to optimize?
  - `Cross Entropy`, `Square Loss`, `Accuracy`, ...
  - choice is made by instinct, tradition, or pragmatics, which makes results difficult to motivate, and vulnerable to overfitting.
- Choice of *model* admits more principled discussion.
  - and a possibly-inconsistent model comes with a natural objective.

Surprising Result

Most standard objectives arise naturally as the inconsistency of the obvious PDG describing the situation.

Pedagogical Bonus

An intuitive visual language for reasoning about ineuqalities

This seems misplaced. “Inequalities” comes out of the blue.

Surprise as Inconsistency

Proposition

Consider a distribution over X with mass function $p(X)$. The surprise (or information content) $\mathrm{I}_p(x) := -\log p(X{=}x)$ at seeing a sample x is the inconsistency of the pdg containing p and the event $X = x$, i.e.,

$$\mathrm{I}_p(x) = \log \frac{1}{p(X{=}x)} = \left\langle\!\!\left\langle \xrightarrow{\;p\;} \boxed{X} \overset{X=x}{\twoheadleftarrow\!\!-\!\!-} \right\rangle\!\!\right\rangle.$$

Where did the << >> notation come from?

Surprise as Inconsistency

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- PDG semantics just so happen to give the standard meaure of compatibility between a sample and distribution.

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- PDG semantics just so happen to give the standard meaure of compatibility between a sample and distribution.
- Known as "surprise", a particular kind of internal conflict.

Variations: Surprise as Inconsistency

Proposition (marginal information as inconsistency)

If $p(X, Z)$ is a joint distribution, the (marginal) information of the (partial) observation $X = x$ is given by

$$\mathrm{I}_p(x) = \log \frac{1}{p(x)} = \left\langle\!\!\left\langle\; Z \overset{p}{\longleftarrow\!\!\longrightarrow} X \overset{x}{\twoheadleftarrow} \;\right\rangle\!\!\right\rangle.$$

While you may be used to all these funny symbols, the listener won't be. You have to be careful not to go overboard here.

Variations: Surprise as Inconsistency

Proposition (marginal information as inconsistency)

If $p(X, Z)$ is a joint distribution, the (marginal) information of the (partial) observation $X = x$ is given by

$$\mathrm{I}_p(x) = \log \frac{1}{p(x)} = \left\langle\!\!\left\langle\; \boxed{Z} \overset{p}{\swarrow\!\searrow} \boxed{X} \overset{x}{\twoheadleftarrow} \;\right\rangle\!\!\right\rangle.$$

Proposition (cross entropy as inconsistency)

Given a dataset $\underline{\mathbf{x}}$, the cross entropy $\mathrm{CE}(p, \underline{\mathbf{x}}) := -\frac{1}{|\underline{\mathbf{x}}|} \sum_{x \in \underline{\mathbf{x}}} \log p(x)$ is the inconsistency of the PDG containing p and the data distribution $\Pr_{\underline{\mathbf{x}}}$, plus the entropy of the data distribution (constant in p). That is,

I still object to the !, and I assure you the listener will have no clue of what it means.

$$\mathrm{CE}(p; \underline{\mathbf{x}}) = \left\langle\!\!\left\langle\; \boxed{Z} \overset{p}{\swarrow\!\searrow} \boxed{X} \xleftarrow{\Pr_{\underline{\mathbf{x}}}!} \;\right\rangle\!\!\right\rangle + \mathrm{H}(\Pr_{\underline{\mathbf{x}}}).$$

You're now officiallly overboard.

Proposition (Accuracy as Inconsistency)

Consider a predictor $h : X \to Y$ for true labels $f : X \to Y$, and a distribution $D(X)$. The inconsistency of believing all three is

What is accuracy?

$$\left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "D^{(\beta)}"] & \boxed{X} \arrow[r, bend left, "h"] \arrow[r, bend right, "f"'] & \boxed{Y} \end{tikzcd} \right\rangle\!\!\right\rangle = -\beta \log\Big(\mathrm{accuracy}_{f,D}(h)\Big) = \beta\ \mathrm{I}_D[f = h].$$

Proposition (Mean Square Error as Inconsistency)

$$\left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "D!"] & X \arrow[r, bend left, "\mathcal{N}(f(x),1)"] \arrow[r, bend right, "\mathcal{N}(g(x),1)"'] & Y \end{tikzcd} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \begin{tikzcd} & & \mu_f \arrow[dr, "\mathcal{N}_1"] & \\ {} \arrow[r, "D!"] & X \arrow[ur, two heads, "f"] \arrow[dr, two heads, "h"'] & & Y \\ & & \mu_h \arrow[ur, "\mathcal{N}_1"'] & \end{tikzcd} \right\rangle\!\!\right\rangle$$

$$= \mathbb{E}_D \Big(f(X) - h(X) \Big)^2 =: \mathrm{MSE}(f, h)$$

where $\mathcal{N}_1 = \mathcal{N}(-, 1)$ is the normal distribution with unit variance, and mean equal to its argument.

Proposition (Regularizers as priors)

Suppose you believe $Y \sim f_\theta(Y)$,

That is,

$$\left\langle\!\!\left\langle \quad \boxed{\Theta} \xrightarrow{f} \boxed{Y} \quad \right\rangle\!\!\right\rangle$$

What does this have to do with belief?
I'm lost.

Proposition (Regularizers as priors)

Suppose you believe $Y \sim f_\theta(Y)$, have a prior $p(\theta)$,

That is,

$$\left\langle\!\!\left\langle \xrightarrow{p^{(\beta)}} \boxed{\Theta} \xrightarrow{f} \boxed{Y} \right\rangle\!\!\right\rangle$$

Proposition (Regularizers as priors)

Suppose you believe $Y \sim f_\theta(Y)$, have a prior $p(\theta)$, and have an empirical distribution $D(Y)$ which you trust.

That is,

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Suppose you believe $Y \sim f_\theta(Y)$, have a prior $p(\theta)$, and have an empirical distribution $D(Y)$ which you trust. Then the inconsistency of also believing $\Theta = \theta_0$ is

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$$\left\langle\!\!\left\langle \begin{array}{c} \xrightarrow{p^{(\beta)}} \\ \xrightarrow[\theta_0]{} \end{array} \boxed{\Theta} \xrightarrow{f} \boxed{Y} \xleftarrow{D!} \right\rangle\!\!\right\rangle =$$

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Suppose you believe $Y \sim f_\theta(Y)$, have a prior $p(\theta)$, and have an empirical distribution $D(Y)$ which you trust.Then the inconsistency of also believing $\Theta = \theta_0$ is the regularized-cross entropy loss, and controlled by the strength β_p of the prior. That is,

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Using a (discretized) unit gaussian as a prior, $p(\theta) = \frac{1}{k}\exp(-\frac{1}{2}\theta^2)$ for a normaization constant k, the RHS becomes

$$\underbrace{\mathop{\mathbb{E}}_{D}\left[\log \frac{1}{f(Y \mid \theta_0)}\right]}_{\substack{\text{Cross entropy loss of } f_\theta \text{ w.r.t. } D \\ \text{(data-fit cost of } \theta_0\text{)}}} \quad + \quad \underbrace{\frac{\beta}{2}\theta_0^2}_{\substack{\ell_2 \text{ regularizer} \\ \text{(complexity cost of } \theta_0\text{)}}} \quad \underbrace{+\beta \log k - \mathrm{H}(D)}_{\text{constant in } f \text{ and } \theta_0}.$$

Variational Auto-Encoders, Take 1

- Structure consists of two networks:

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  $e(Z \mid X)$: encodes X's in a (small) latent space Z;

Encodes in what way? What's a latent space. I'm lost.

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  - Want to minimize "reconstruction error" for each x

$$\mathrm{Rec}(x) = - \mathop{\mathbb{E}}_{z \sim e|x} \log d(x \mid z)$$

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  - you also have a prior $p(Z)$, which you want to match.
  - Together, maximize $\mathrm{ELBO}_{p,e,d}(x) :=$
    $$-\boldsymbol{D}(e(Z|x) \,\|\, p(Z)) - \mathrm{Rec}(x) = \mathop{\mathbb{E}}_{z \sim e|x}\left[\log \frac{p(z)d(x \mid z)}{e(z \mid x)}\right]$$

What does this expression represent? Why should you want to maximize it? I continue to be lost.

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Note: not a graphical model (desipte shoehorning attempts)

I have no idea why I should think it's a graphical model or what this note is referring to.

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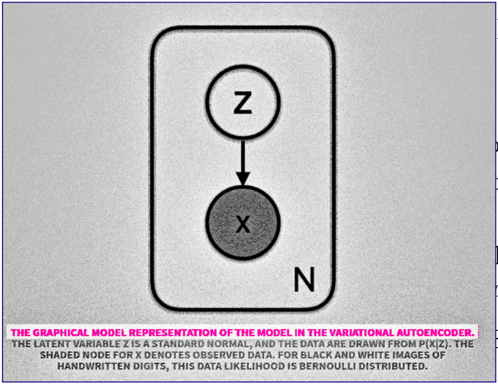

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$x) :=$

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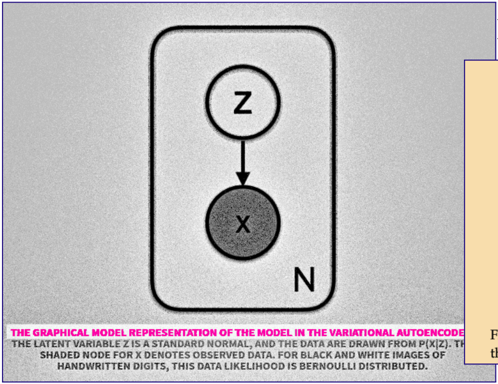

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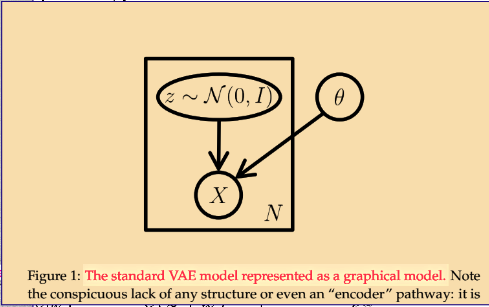

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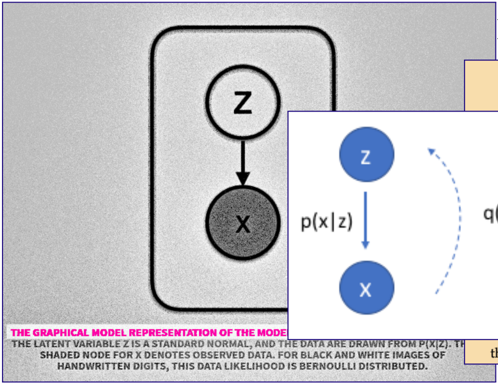

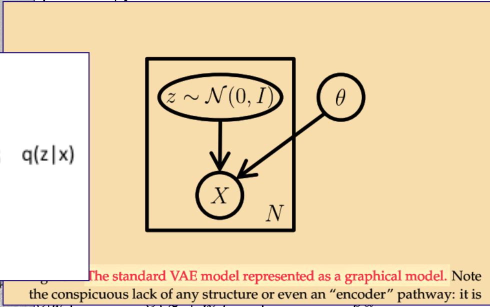

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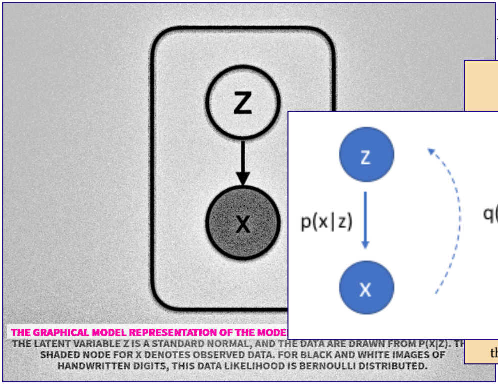

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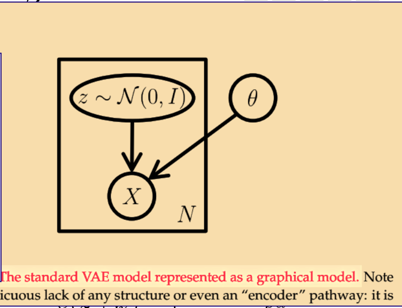

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- $e(Z \mid X)$ has same target as $p(Z)$, so BN doesn't work

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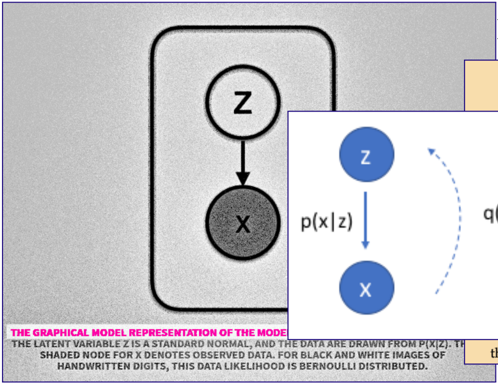

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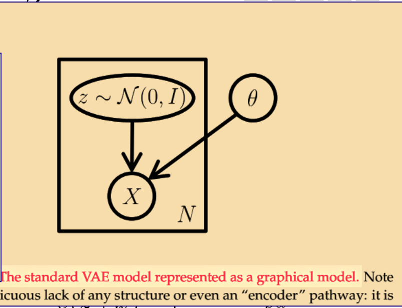

Note: not a graphical model (desipte shoehorning attempts)

- $e(Z \mid X)$ has same target as $p(Z)$, so BN doesn't work
- The heart of the VaE is not its structure, but its objective.

Variational Auto-Encoders, Take 2

- Structure:

$$\left\langle\!\!\left\langle \quad \boxed{Z} \quad \boxed{X} \quad \right\rangle\!\!\right\rangle = \mathrm{ELBO}_{p,e,d}(x)$$

Variational Auto-Encoders, Take 2

- Structure:

  $e(Z \mid X)$: encodes X in a latent space Z;

$$\left\langle\!\!\left\langle\ \begin{tikzcd} Z & X \arrow[l, bend left, "e"] \end{tikzcd}\ \right\rangle\!\!\right\rangle = \mathrm{ELBO}_{p,e,d}(x)$$

Variational Auto-Encoders, Take 2

- Structure:

  $e(Z \mid X)$: encodes X in a latent space Z;
  $d(X \mid Z)$: generate samples of X from Z.

$$\left\langle\!\!\left\langle \begin{tikzcd} Z \arrow[r, bend left, "d"] & X \arrow[l, bend left, "e"] \end{tikzcd} \right\rangle\!\!\right\rangle = \mathrm{ELBO}_{p,e,d}(x)$$

Variational Auto-Encoders, Take 2

- Structure:

  $e(Z \mid X)$: encodes X in a latent space Z;
  $d(X \mid Z)$: generate samples of X from Z.
  $p(Z)$: a prior over from Z.

$$\left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "p"] & Z \arrow[r, bend left, "d"] & X \arrow[l, bend left, "e"] \end{tikzcd} \right\rangle\!\!\right\rangle = \mathrm{ELBO}_{p,e,d}(x)$$

Variational Auto-Encoders, Take 2

- Structure:
  $e(Z \mid X)$: encodes X in a latent space Z;
  $d(X \mid Z)$: generate samples of X from Z.
  $p(Z)$: a prior over from Z.
- want to do a gradient step for a specific x.

I'm totally lost.

$$\left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "p"] & \boxed{Z} \arrow[r, bend left, "d"] & \boxed{X} \arrow[l, bend left, "e!"] & {} \arrow[l, twoheadrightarrow, "x"'] \end{tikzcd} \right\rangle\!\!\right\rangle = \mathrm{ELBO}_{p,e,d}(x)$$

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  $e(Z \mid X)$: encodes X in a latent space Z;
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  $p(Z)$: a prior over from Z.
- want to do a gradient step for a specific x.

Objective function is free:

$$\left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "p"] & \boxed{Z} \arrow[r, bend left, "d"] & \boxed{X} \arrow[l, bend left, "e!"] & {} \arrow[l, two heads, "x"'] \end{tikzcd} \right\rangle\!\!\right\rangle = \mathrm{ELBO}_{p,e,d}(x)$$

A Very Useful Fact

Believing more things can't make you any less inconsistent.

Lemma (monotonicity of inconsistency)

For all pdgs $\mathcal{m}$, $\mathcal{m}'$, *and all* $\gamma > 0$,

1. $\langle\!\langle \mathcal{m} \sqcup \mathcal{m}' \rangle\!\rangle_\gamma \geq \langle\!\langle \mathcal{m} \rangle\!\rangle_\gamma$.
2. *If* $\mathcal{m}$ *and* $\mathcal{m}'$ *have respective confidence vectors* β *and* β'*, and* $\beta \succeq \beta'$ *(that is,* $\beta_L \geq \beta'_L$ *for all* $L \in \mathcal{E}$*), then* $\langle\!\langle \mathcal{m} \rangle\!\rangle_\gamma \geq \langle\!\langle \mathcal{m}' \rangle\!\rangle_\gamma$.

Visual Proof: The Varitaional Bound

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$$\left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "p"] & Z \arrow[r, bend left, "d"] & X \arrow[l, bend left, "e!"] & {} \arrow[l, two heads, "x"'] \end{tikzcd} \right\rangle\!\!\right\rangle = -\,\mathrm{ELBO}_{p,e,d}(x).$$

Visual Proof: The Varitaional Bound

$$-\log \Pr_{p,d}(X{=}x) \;=\; \left\langle\!\!\left\langle \xrightarrow{p} \boxed{Z} \xrightarrow{d} \boxed{X} \overset{x}{\twoheadleftarrow} \right\rangle\!\!\right\rangle \qquad \left\langle\!\!\left\langle \xrightarrow{p} \boxed{Z} \underset{e!}{\overset{d}{\rightleftarrows}} \boxed{X} \overset{x}{\twoheadleftarrow} \right\rangle\!\!\right\rangle \;=\; -\,\mathrm{ELBO}_{p,e,d}(x).$$

Visual Proof: The Varitaional Bound

There's no way you'll have the time for this.

$$-\log \Pr_{p,d}(X{=}x) \;=\; \left\langle\!\!\left\langle \xrightarrow{p} \boxed{Z} \xrightarrow{d} \boxed{X} \overset{x}{\twoheadleftarrow} \right\rangle\!\!\right\rangle \leq \left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "p"] & \boxed{Z} \arrow[r, bend left, "d"] & \boxed{X} \arrow[l, bend left, "e!"] & {} \arrow[l, two heads, "x"'] \end{tikzcd} \right\rangle\!\!\right\rangle \;=\; -\mathrm{ELBO}_{p,e,d}(x).$$

Divergences and Inconsistency

Lemma (Divergences and PDGs)

The PDG divergence $\boldsymbol{D}^{\mathrm{PDG}}_{(r,s)}(p,q)$*, the inconsistency of a PDG containing* $p(X)$ *with confidence* r *and* $q(X)$ *with confidence* s*, is given by*

$$\boldsymbol{D}^{\mathrm{PDG}}_{(r,s)}(p,q) := \left\langle\!\!\left\langle \xrightarrow[(\beta:r)]{p} \boxed{X} \xleftarrow[(\beta:s)]{q} \right\rangle\!\!\right\rangle = -(r+s)\log\sum_x \Big(p(x)^r q(x)^s\Big)^{\frac{1}{r+s}}.$$

Visual Proof: Data Processing Inequality

$$\boldsymbol{D}\Big(p \,\Big\|\, q\Big) \geq \boldsymbol{D}\Big(f \circ p \,\Big\|\, f \circ q\Big)$$

$$\left\langle\!\!\left\langle \xrightarrow{\;p\;} \boxed{X} \xleftarrow{\;q\;} \right\rangle\!\!\right\rangle$$

$$\left\langle\!\!\left\langle \xrightarrow{f\circ p} \boxed{X} \xleftarrow{f\circ q} \right\rangle\!\!\right\rangle$$

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$$\left\langle\!\!\left\langle \xrightarrow{\;f \circ p\;} \boxed{X} \xleftarrow{\;f \circ q\;} \right\rangle\!\!\right\rangle$$

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$$= \left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "p"] & X_1 \arrow[rr, equal] \arrow[dr, "f!"'] & & X_2 \arrow[dl, "f!"] & {} \arrow[l, "q"'] \\ & & Y & & \end{tikzcd} \right\rangle\!\!\right\rangle$$

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Visual Proof: Data Processing Inequality

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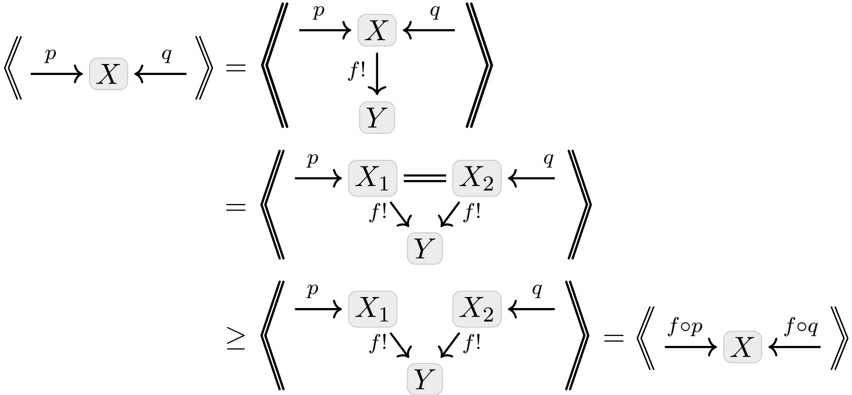
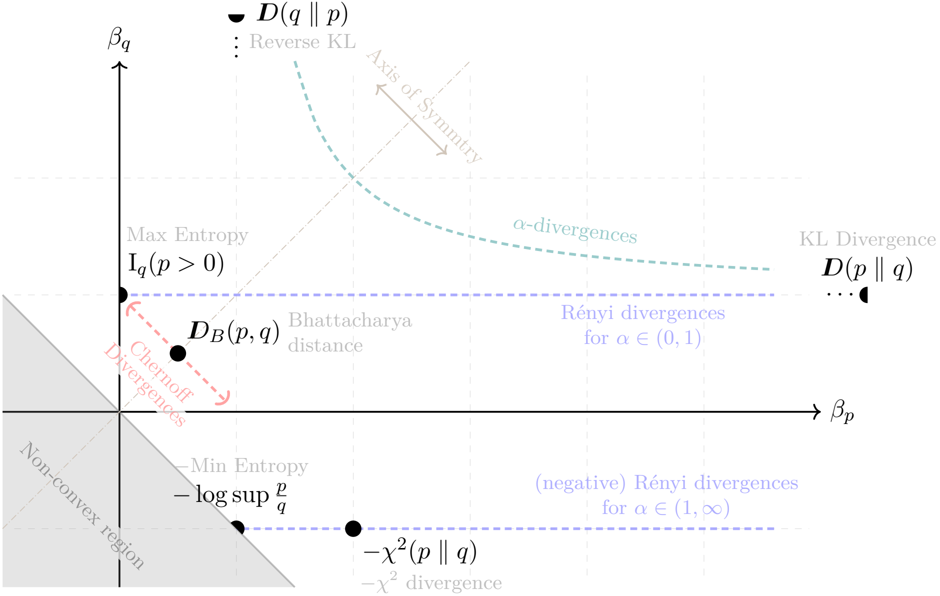


Divergences as Inconsistencies



Outline for Section 7

$\langle$ Schema Picture $\rangle$
$\langle$ Add Universal Relation Theorem $\rangle$

Outline for Section 8

Open Problems and Future Work

$\langle$ INCOMPLETE $\rangle$

- - Trace Semantics
  - Composition

Summary

$\langle$ `update with second half` $\rangle$

PDGs...

- capture inconsistency, including conflicting information from multiple sources with varying reliability.

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$\langle$ `update with second half` $\rangle$

PDGs...

- capture inconsistency, including conflicting information from multiple sources with varying reliability.
- are especially modular; to combine info from two sources, simply take a PDG union. This incorporates new data (edge cpds) and concepts (nodes) without affecting previous information.

Summary

$\langle$ `update with second half` $\rangle$

PDGs...

- capture inconsistency, including conflicting information from multiple sources with varying reliability.
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Summary

⟨ `update with second half` ⟩

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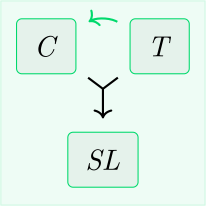
But there is much more to be done!

$\langle$ `return to initial slide, but with more conflicts` $\rangle$

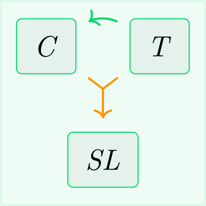
Outline for Section 9

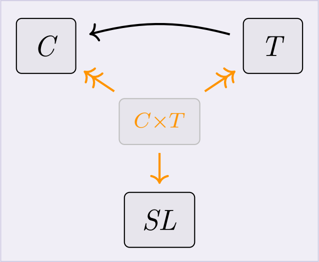
Hyper-graphs? Or merely graphs?



main definition

Hyper-graphs? Or merely graphs?





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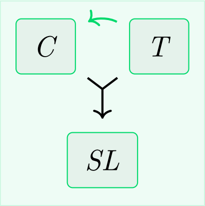

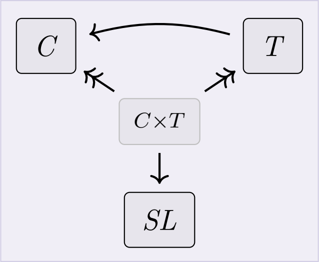

- This widget expands state space, but graphs are simpler.

main definition

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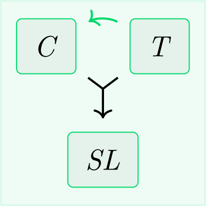

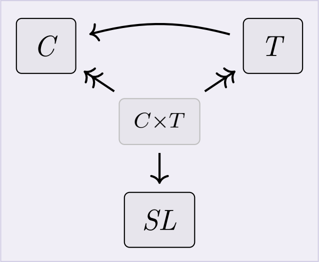

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joint distributions $\leftrightarrows$ expanded joint distributions satisfying coherence constraints

main definition

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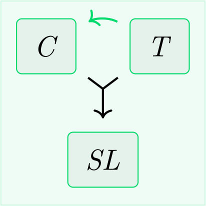

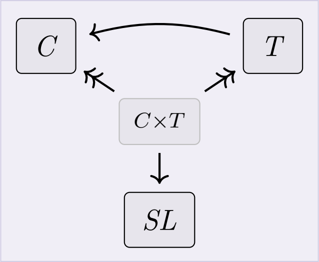

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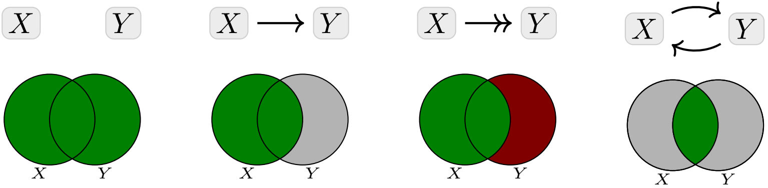
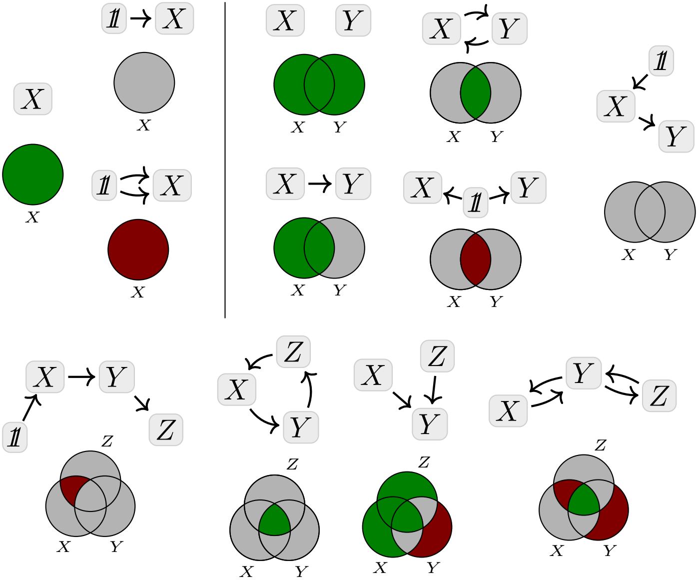
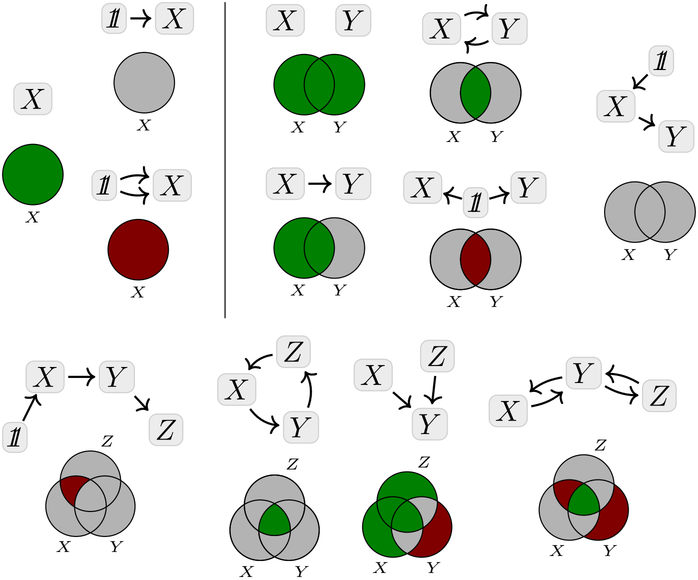
(working directly with hypergraphs is also possible)

main definition

Illustrations of *IDef*




$\mathbb{1}$
X
Y
Z



X
1 → X
X
1 → X
X
X Y
X Y
X → Y
X Y
X Y
X Y
X ← 1 → Y
X Y
1
X
Y
X Y
X → Y
1
Z
Z
X
Y
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X Y
X
Z
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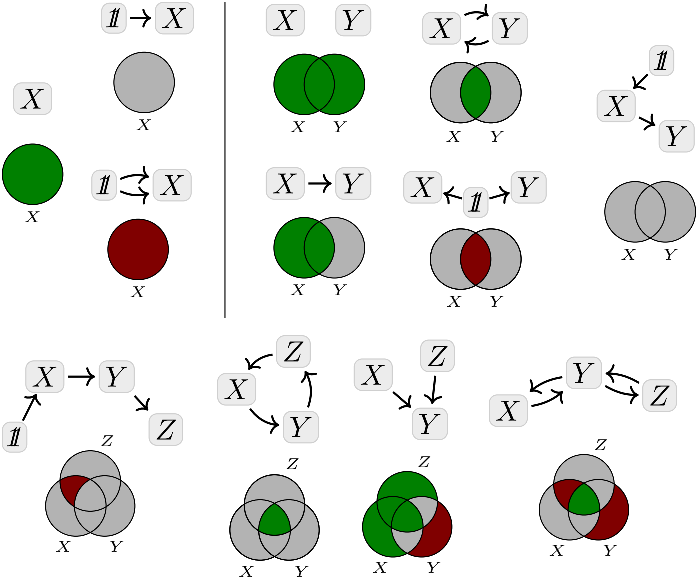
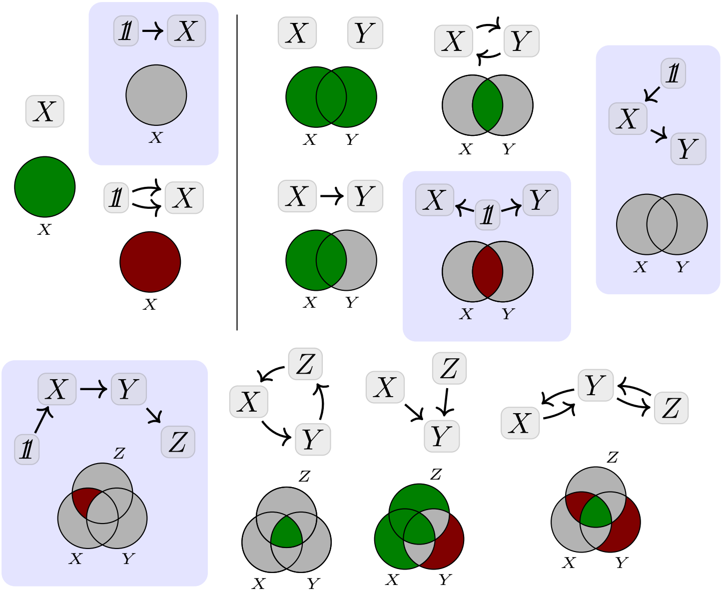
X Y
Y
X
Z
Z
X Y





X
1 → X
X
1 ⇉ X
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X Y
X Y
X ⇄ Y
X Y
X → Y
X Y
X ← 1 → Y
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1 → X → Y
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1 → X → Y → Z
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X Y