Probabilistic Dependency Graphs and Inconsistency

How to model, measure, and mitigate internal conflict

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Department of Computer Science

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Outline for Section 1

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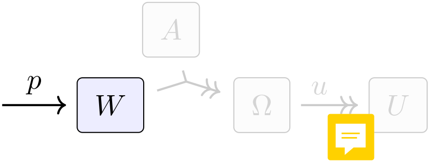

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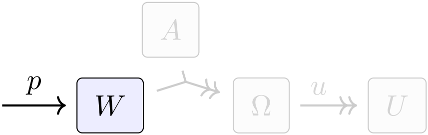

Such agents cannot have internal conflict;

by construction, they have consistent beliefs and desires.

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Freedom from perfect consistency frees up **a lot** of computation, but demands the ability to recognize and adress internal conflict.

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In doing so, we get much more . . .

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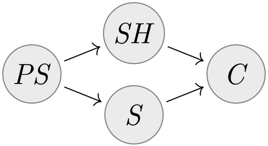
Qualitative BN, $\mathcal{G}$

an independence relation on variables

- $X \perp\!\!\!\perp_{\mathcal{G}} Y \mid \mathbf{Pa}(X)$, for all non-descendents Y of X



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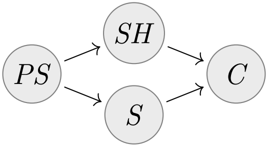
- Defines a joint distribution $\Pr_{\mathcal{B}}$ with the independencies $\perp\!\!\!\perp_{\mathcal{G}}$.



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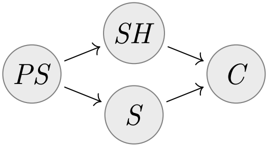

Outline for Section 2

1. Introduction
2. Modeling Examples
   - A Simple Example: What are Floomps?
   - Differences from BNs
   - PDG Union and Restriction
3. Syntax
   - Formal Definitions of PDGs
4. Semantics
5. Capturing other Graphical Models
   - Bayesian Networks
   - Factor Graphs
6. Inference
7. Inconsistency as Loss
   - Motivation
   - Standard Metrics as Inconsistency
   - Variational AutoEncoders
   - Inconsistency and Statistical Divergences
8. Other Aspects of PDGs
   - Category Theory
   - Databases
   - Other Ongoing Work

Simple Example: Floomps and Guns

Grok thinks it likely (.95) that guns are illegal,
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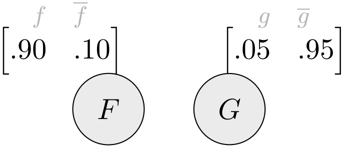

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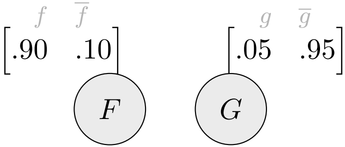

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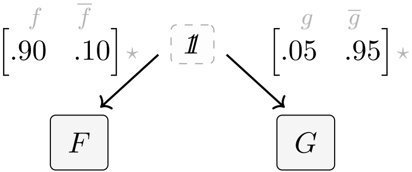

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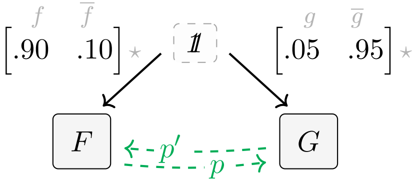

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- PDGs can incorporate arbitrary new probabilistic information.

Grok learns that Floomps and Guns have the same legal status (92%)

$$p(G\,|\,F) = \begin{array}{cc} & \begin{array}{cc} g & \bar{g} \end{array} \\ & \begin{bmatrix} .92 & .08 \\ .08 & .92 \end{bmatrix} \begin{array}{c} f \\ \bar{f} \end{array} \end{array} = \Big(\, p'(F\,|\,G) \,\Big)^{\mathsf{T}}$$

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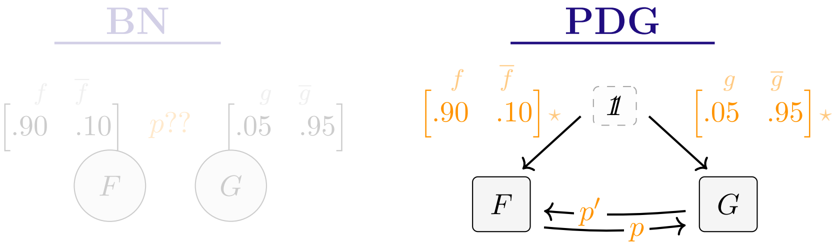

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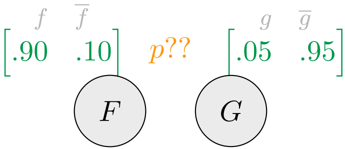

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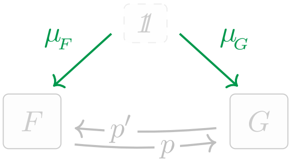

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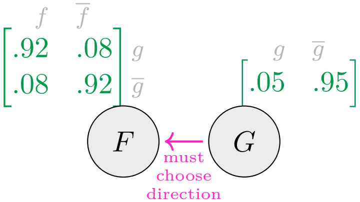

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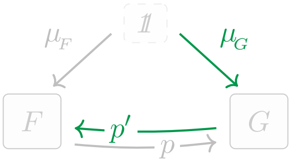

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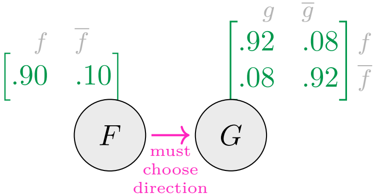

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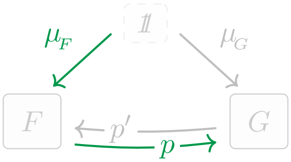

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Bayesian Networks as PDGs

$$\begin{tikzcd}[row sep=small] & S \arrow[dr] & \\ PS \arrow[ur] \arrow[dr] & & C \\ & SH \arrow[ur] & \end{tikzcd}$$

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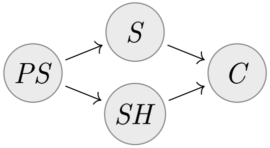

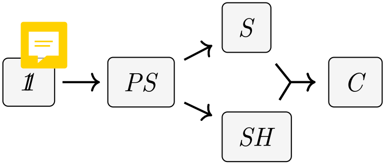

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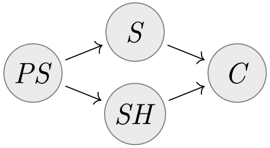

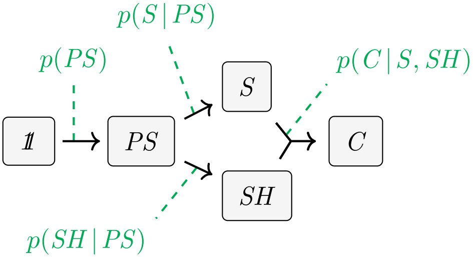

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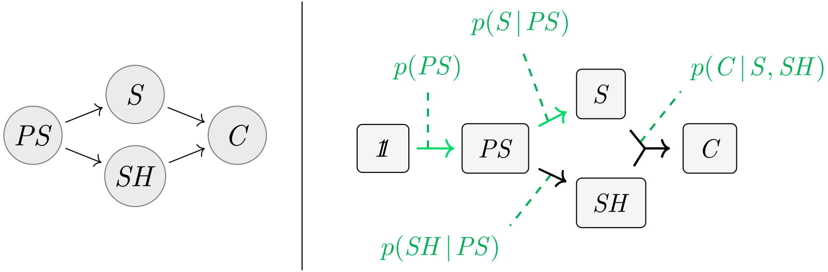

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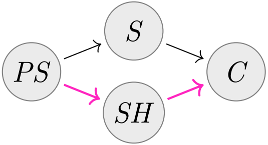

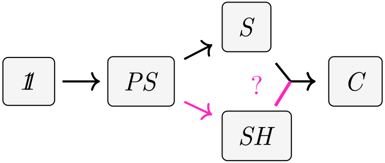

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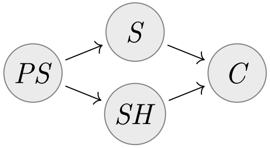

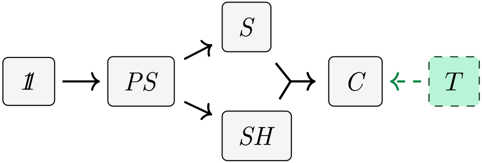

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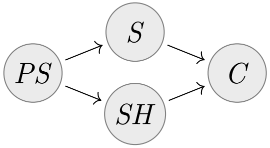

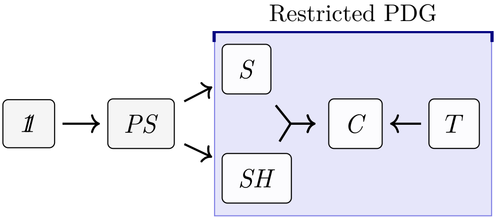

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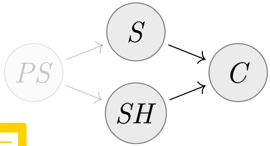

Must now give distributions on SH and S, or distinguish them as "observed" (a *conditional* BN).

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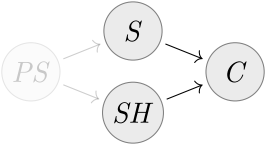

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In a qualitative BN: *removing* data results in *new* knowledge: $A \perp\!\!\!\perp C$.

$$A \to B \to C$$

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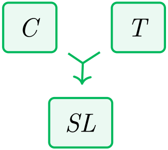

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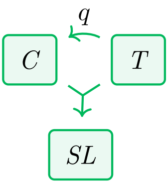

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- ...but mom says $q(C \mid T) = \begin{array}{cc} \begin{array}{cc} c & \bar{c} \end{array} & \\ \begin{bmatrix} .15 & .85 \\ .02 & .98 \end{bmatrix} & \begin{array}{c} t \\ \bar{t} \end{array} \end{array}.$

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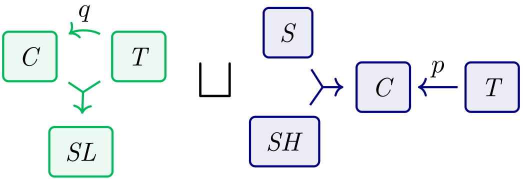

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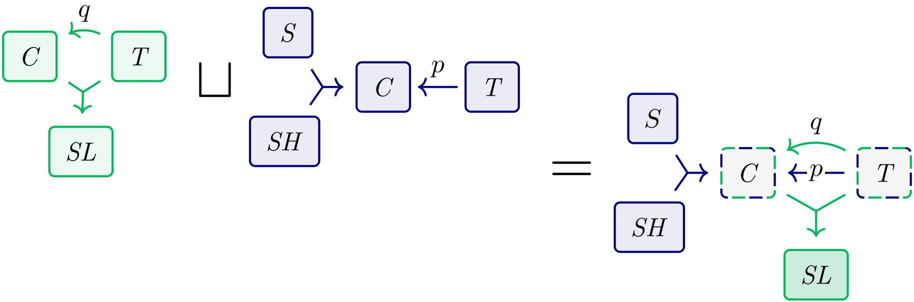

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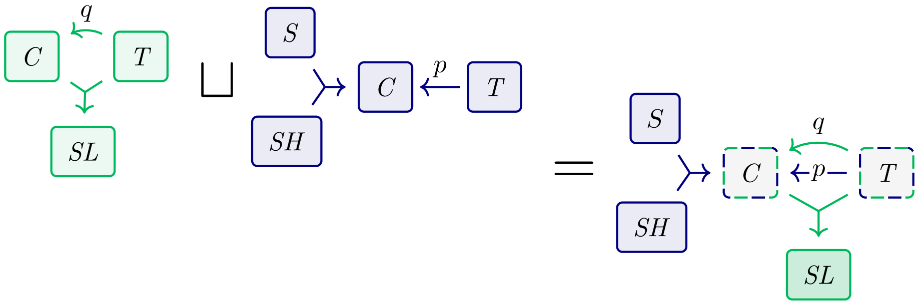

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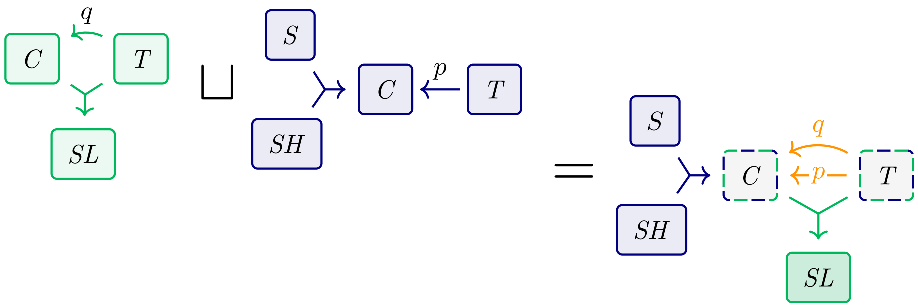

- Arbitrary PDGs may be combined without loss of information
- They may have parallel edges which directly conflict.

Outline for Section 3

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$$\boxed{\mathcal{V}(\mathcal{m}) := \prod_{X \in \mathcal{N}} \mathcal{V}(X) \qquad \text{is the set of possible joint variable settings.}}$$

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- $\mathbf{p}_L$ a cpd $\mathbf{p}_L(Y \mid X)$;
- α_L a confidence in the functional dependence $X \to Y$;
- β_L a confidence in the reliability of $\mathbf{p}_L$.

Outline for Section 4

Semantics of PDGs

$\{\!\{m\}\!\} \subseteq \Delta\mathcal{V}(m)$

The set of joint distributions consistent with m;

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The set of joint distributions consistent with $\mathcal{m}$;

$[\![\mathcal{m}]\!]_\gamma : \Delta\mathcal{V}(\mathcal{m}) \to \mathbb{R}$

A function (parameterized by $\gamma > 0$) that scores distributions by compatibility with $\mathcal{m}$;

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The set of joint distributions consistent with m;

$[\![m]\!]_\gamma : \Delta\mathcal{V}(m) \to \mathbb{R}$

A function (parameterized by $\gamma > 0$) that scores distributions by compatibility with m;

$[\![m]\!]^*_\gamma \subseteq \Delta\mathcal{V}(m)$

The distribution(s) most compatible with m
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...

trace semantics

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The Scoring Function

$$[\![\mathcal{M}]\!]_\gamma(\mu) := \mathit{Inc}_{\mathcal{M}}(\mu) + \gamma\, \mathit{IDef}_{\mathcal{M}}(\mu)$$

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Intuition: Measure μ's violation of $\mathcal{M}$'s cpds.

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$$\boldsymbol{D}(\mu \parallel \nu) = \sum_{w \in \operatorname{Supp} \mu} \mu(w) \log \frac{\mu(w)}{\nu(w)}$$

the relative entropy (KL Divergence) from ν to μ.

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$$= \mathbb{E}_\mu \sum_{X \xrightarrow{L} Y} \beta_L \Big(\underbrace{\mathrm{I}_{\mathbf{p}_L}}_{\substack{\text{code length,} \\ \text{optimized for } \mathbf{p}_L \\ \text{to communicate} \\ Y \text{ given } X}} - \underbrace{\mathrm{I}_\mu}_{\substack{\text{code length,} \\ \text{optimized for } \mu \\ \text{to communicate} \\ Y \text{ given } X}} \Big)$$

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$$[\![\mathcal{M}]\!]_\gamma(\mu) := \mathit{Inc}_{\mathcal{M}}(\mu) + \gamma\ \mathit{IDef}_{\mathcal{M}}(\mu)$$

Intuition: each edge $X \xrightarrow{L} Y$ indicates that Y is determined (perhaps noisily) by X alone.

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So a μ with $\overbrace{\text{uncertainty in } Y \text{ after } X \text{ is known}}^{\textit{measured by } \mathrm{H}(Y \mid X)}$ (beyond $\underbrace{\text{pure noise}}_{\mathrm{H}(\mu)}$) is qualitatively worse.

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The *information deficit* of a distribution μ with respect to $\mathcal{m}$ is

$$\mathit{IDef}_{\mathcal{m}}(\mu) := \sum_{X \xrightarrow{L} Y} \alpha_L \, \mathrm{H}_\mu(Y \,|\, X) - \mathrm{H}(\mu).$$

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The *information deficit* of a distribution μ with respect to $\mathcal{m}$ is

(b) # bits required to separately determine each target, knowing the source

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Information Diagrams

The Scoring Function

$$[\![\mathcal{m}]\!]_\gamma(\mu) := \mathit{Inc}_{\mathcal{m}}(\mu) + \underbrace{\gamma}_{\text{tradeoff parameter } \gamma \geq 0} \mathit{IDef}_{\mathcal{m}}(\mu)$$

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- A BN strictly enforces the qualitative picture (large γ)

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- A BN strictly enforces the qualitative picture (large γ)
- we are interested in the quantitative limit (small γ)

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Properties of the Optimal Distribution

Proposition (uniqueness for small γ)

1. *If* $0 < \gamma \leq \min_L \beta_L^{m}$, *then* $[\![m]\!]_\gamma^*$ *is a singleton.*
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More Semantics Properties

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Proposition (uniqueness for small γ)

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2. $\lim\limits_{\gamma \to 0} \llbracket \mathcal{m} \rrbracket^*_\gamma$ *exists and is a singleton.*

This lets us define $\llbracket \mathcal{m} \rrbracket^* :=$ unique element $\left(\lim\limits_{\gamma \to 0} \llbracket \mathcal{m} \rrbracket^*_\gamma \right)$.

More Semantics Properties

Properties of Inconsistency

$$\langle\!\langle \mathcal{m} \rangle\!\rangle_\gamma := \inf_\mu [\![\mathcal{m}]\!]_\gamma$$

Nice properties for minimization:

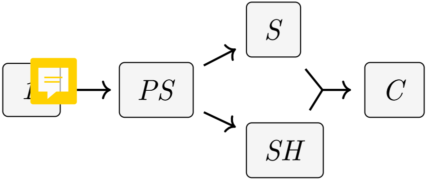
- The function $\gamma \mapsto \langle\!\langle \mathcal{m} \rangle\!\rangle_\gamma$ is continuous for all γ
- The function $p \mapsto \langle\!\langle \mathcal{m} \sqcup p \rangle\!\rangle_\gamma$ is smooth and strictly convex on its interior.

More Semantics Properties

Outline for Section 5

More Semantics Properties

Capturing Bayesian Networks



More Semantics Properties

Capturing Bayesian Networks

For a BN $\mathcal{B}$ with N nodes and a vector $\beta \in \mathbb{R}^N$, let $\boldsymbol{m}_{\mathcal{B},\beta}$ be the PDG corresponding to $\mathcal{B}$, with $\alpha = \mathbf{1}$, and the given vector β of confidences.

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Theorem (BNs are PDGs)

If $\mathcal{B}$ is a BN and $\Pr_{\mathcal{B}}$ is the distribution it specifies, then for all $\gamma > 0$ and all vectors β,

$$[\![\boldsymbol{m}_{\mathcal{B},\beta}]\!]^*_\gamma = \{\Pr_{\mathcal{B}}\}, \quad \textit{and thus} \quad [\![\boldsymbol{m}_{\mathcal{B},\beta}]\!]^* = \Pr_{\mathcal{B}}\,.$$

▸ Corolary: BNs as Maximum Entropy Distributions

Capturing Bayesian Networks

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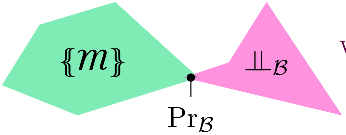
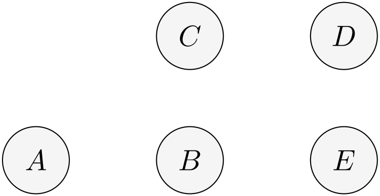
space of distributions consistent with $\mathcal{m}_{\mathcal{B}}$ (which minimize *Inc*)



space of distributions with independencies of $\mathcal{B}$ (which can be shown to minimize *IDef*)

▸ Corolary: BNs as Maximum Entropy Distributions

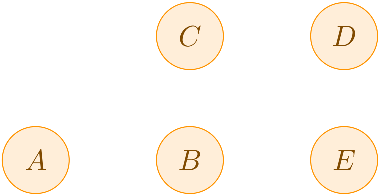
Factor Graphs



Definition

A *factor graph* Φ is

Factor Graphs



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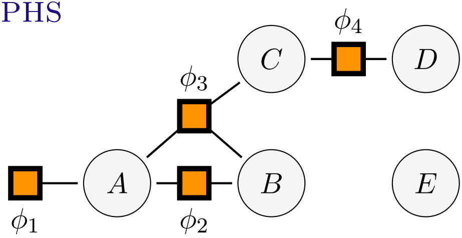
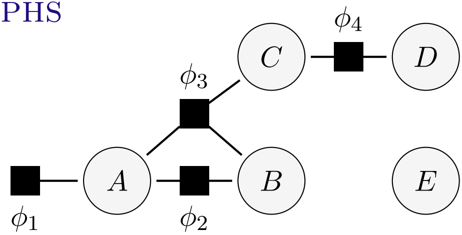
Factor Graphs



Definition

A *factor graph* Φ is a set of variables $\mathcal{X} = \{X_i\}$, and *factors* $\{\phi_J : \mathcal{V}(X_J) \to \mathbb{R}_{\geq 0}\}_{J \in \mathcal{J}}$, with $X_J \subseteq \mathcal{X}$;

Factor Graphs



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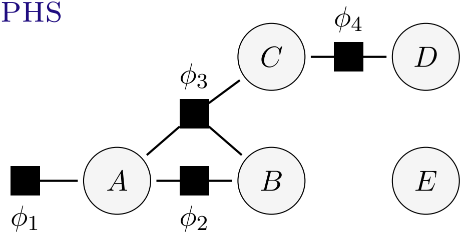
$$\Pr_\Phi(\vec{x}) := \frac{1}{Z_\Phi} \prod_{J \in \mathcal{J}} \phi_J(\vec{x}_J),$$

where Z_Φ is the normalization constant.

Factor Graphs



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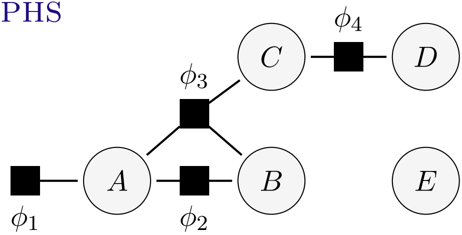
Φ defines a standard scoring function

$$\mathit{VFE}_\Phi(\mu) := \mathop{\mathbb{E}}_{\mu}\Big[- \sum_{J \in \mathcal{J}} \log \phi_J(X_J) \Big] - \mathrm{H}(\mu)$$

Factor Graphs



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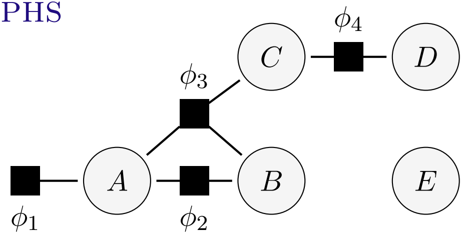
Factor Graphs



Definition

A weighted *factor graph* $\Psi = (\Phi, \theta)$ is a set of variables $\mathcal{X} = \{X_i\}$, *factors* $\{\phi_J : \mathcal{V}(X_J) \to \mathbb{R}_{\geq 0}\}_{J \in \mathcal{J}}$, and weights $(\theta_J)_{J \in \mathcal{J}}$ with $X_J \subseteq \mathcal{X}$; Ψ defines a distribution

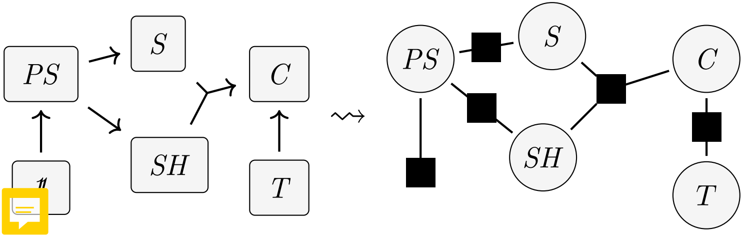
$$\Pr_\Psi(\vec{x}) := \frac{1}{Z_\Phi} \prod_{J \in \mathcal{J}} \phi_J(\vec{x}_J)^{\theta_J},$$

where Z_Ψ is the normalization constant.

Ψ defines a standard scoring function

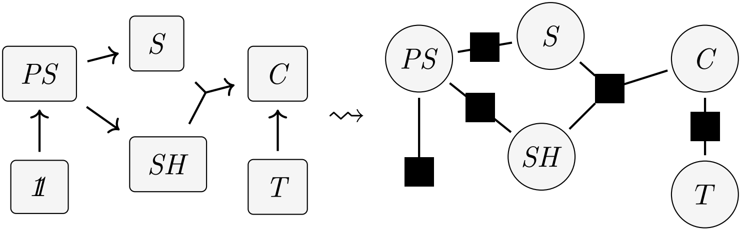
$$\mathit{VFE}_\Phi(\mu) := \mathop{\mathbb{E}}_{\mu}\Big[-\sum_{J \in \mathcal{J}} \theta_J \log \phi_J(X_J) \Big] - \mathrm{H}(\mu)$$

PDGs as Factor Graphs



▸ Precise Theorem Statement

PDGs as Factor Graphs



The cpds of a PDG are essentially factors. Are the semantics different?

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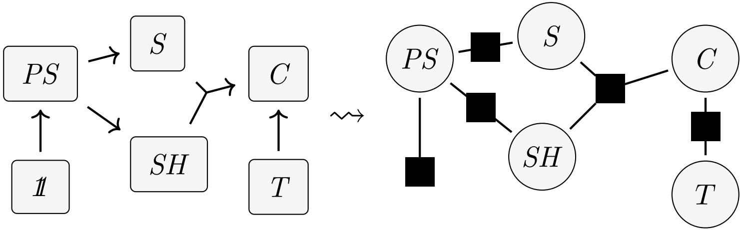

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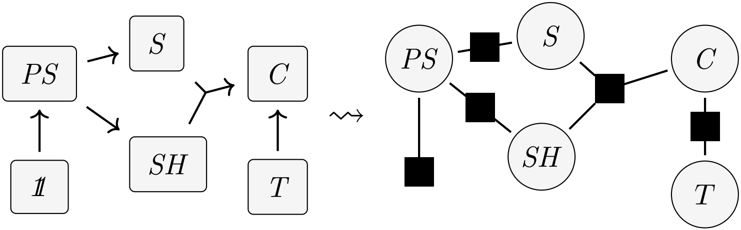

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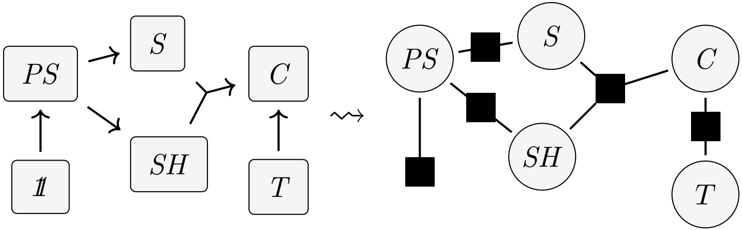
Theorem

$[\![m]\!]_1^* = \Pr_{\Phi_m}$ *for all unweighted PDGs* m.

▸ Precise Theorem Statement

PDGs as Factor Graphs



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Theorem

$[\![\mathcal{n}]\!]_1^* = \Pr_{\Phi_n}$ *for all unweighted PDGs* $\mathcal{n}$.

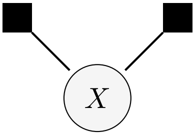
Theorem

If $\boldsymbol{\beta} = \gamma\boldsymbol{\alpha}$, *then* $[\![\mathcal{m}]\!]_\gamma^* = \Pr_{(\Phi_m, \boldsymbol{\beta})}$.

▸ Precise Theorem Statement

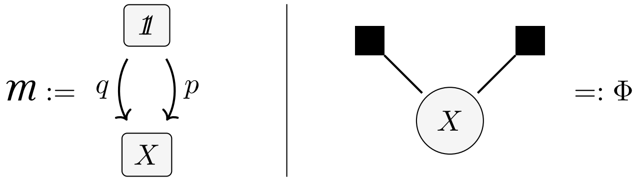
An Important Difference between PDGs and Factor Graphs

$$\mathcal{m} := \begin{tikzcd} \boxed{\mathbb{1}} \arrow[d, bend right, "q"'] \arrow[d, bend left, "p"] \\ \boxed{X} \end{tikzcd}$$



$=: \Phi$

An Important Difference between PDGs and Factor Graphs



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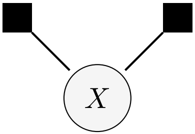

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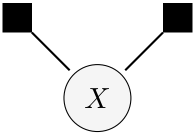
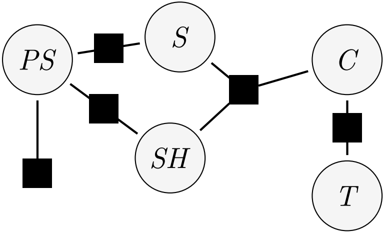


$=: \Phi$

- If $p = q$, then $[\![\mathcal{m}]\!]^* = p = q \ldots$
- $\ldots$ but $\Pr_\Phi \propto p^2$
- Individual factors have *no probabilistic meaning.*

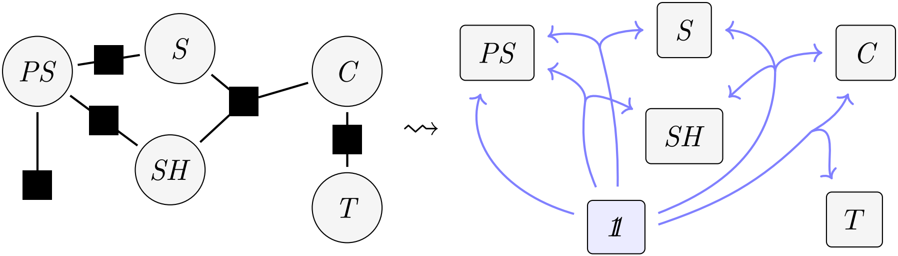
Factor Graphs as PDGs



$\rightsquigarrow$

▸ Full Theorem

Factor Graphs as PDGs



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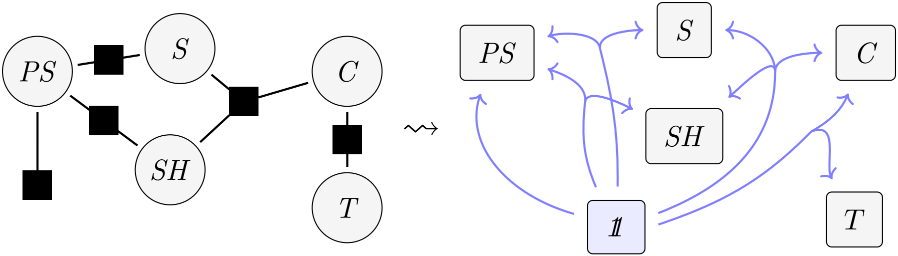

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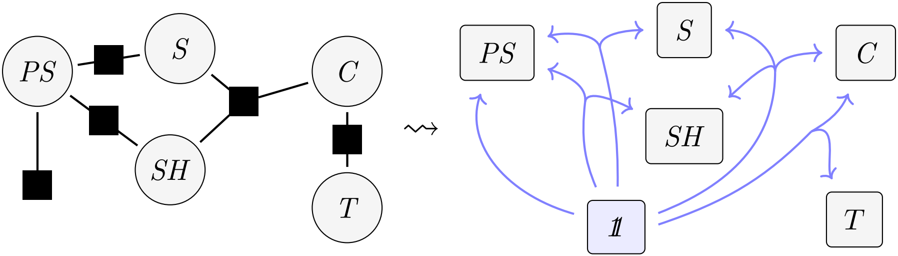

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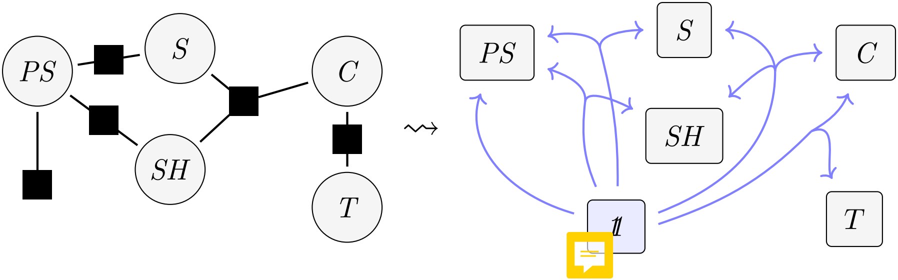

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Also: $\log Z_\Phi = \langle\!\langle \mathcal{n}_\Phi \rangle\!\rangle_1$.

PDG Semantics, Compared to that of Factor Graphs

$$[\![m]\!](\mu) = \mathbb{E}_\mu \sum_{X \xrightarrow{L} Y} \Bigg[\overbrace{\beta_L \log \frac{1}{\mathbf{p}_L(Y|X)}}^{\text{log likelihood / cross entropy}} + \underbrace{(\alpha_L \gamma - \beta_L) \log \frac{1}{\mu(Y|X)}}_{\text{local regularization (if } \beta_L > \alpha_L\gamma)} \Bigg] - \overbrace{\gamma\, \mathrm{H}(\mu)}^{\text{global regularization}}.$$

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And recall that

$$VFE_\Psi(\mu) := \mathop{\mathbb{E}}_{\mu} \left[\sum_{J \in \mathcal{J}} \theta_J \log \frac{1}{\phi_J(X_J)} \right] - \mathrm{H}(\mu)$$

Outline for Section 6

Inference via Inconsistency Reduction

Identify the event $Y{=}y$ with the cpd $\mathbb{1} \xrightarrow{\delta_y} Y$.

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Bonus

An visual language for reasoning about relationships between objective functions.

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Consider a distribution $p(X)$.

▸ For entire dataset

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Surprise is the inconsistency of simultaneously believing p and $X = x$. That is,

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- "surprise": a particular kind of internal conflict.

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Variations: Surprise as Inconsistency

Proposition (marginal information as inconsistency)

If $p(X,Z)$ is a joint distribution, the (marginal) information of the (partial) observation $X = x$ is given by

$$\mathrm{I}_p(x) = \log \frac{1}{p(x)} = \left\langle\!\!\left\langle\; \boxed{Z} \xleftarrow{\;p\;} \cdot \xrightarrow{\;p\;} \boxed{X} \twoheadleftarrow^{x} \;\right\rangle\!\!\right\rangle.$$

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Proposition (supervised setting: conditional cross entropy)

The inconsistency of the PDG containing $f(Y \mid X)$ and a high-confidence empirical distribution $\Pr_{\underline{\mathbf{xy}}}$ of samples $\underline{\mathbf{xy}} = \{(x_i, y_i)\}$ is equal to the cross entropy (plus $\mathrm{H}(Y \mid X)$, a constant that depends only on the data $\Pr_{\underline{\mathbf{xy}}}$). That is,

$$\left\langle\!\!\left\langle \begin{tikzcd} & \arrow[dl] \arrow[dr] \arrow[from=u, "\Pr_{\underline{\mathbf{xy}}}"', "(\beta:\infty)"] & \\ \boxed{X} \arrow[rr, "f"'] & & \boxed{Y} \end{tikzcd} \right\rangle\!\!\right\rangle = \frac{1}{|\underline{\mathbf{xy}}|} \sum_{(x,y) \in \underline{\mathbf{xy}}} \left[\log \frac{1}{f(y \mid x)} \right] \quad - \mathrm{H}_{\Pr_{\underline{\mathbf{xy}}}}(Y \mid X).$$

Proposition (Accuracy as Inconsistency)

Consider a predictor $h : X \to Y$ for true labels $f : X \to Y$, and a distribution $D(X)$. The inconsistency of believing all three is

$$\left\langle\!\!\left\langle \xrightarrow[(\beta)]{D} \boxed{X} \overset{h}{\underset{f}{\rightrightarrows}} \boxed{Y} \right\rangle\!\!\right\rangle = -\beta \log \Big(\text{accuracy}_{f,D}(h) \Big) = \beta \, \mathrm{I}_D[f = h].$$

Proposition (Mean Square Error as Inconsistency)

$$\left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "D", "(\beta:\infty)"'] & X \arrow[r, bend left, "\mathcal{N}(f(x),1)"] \arrow[r, bend right, "\mathcal{N}(g(x),1)"'] & Y \end{tikzcd} \right\rangle\!\!\right\rangle = \mathbb{E}_D \Big(f(X) - h(X)\Big)^2 =: \mathrm{MSE}(f,h)$$

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Suppose you believe $Y \sim f_\theta(Y)$,

That is,

$$\left\langle\!\!\left\langle \quad \boxed{\Theta} \xrightarrow{f} \boxed{Y} \quad \right\rangle\!\!\right\rangle$$

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Using a (discretized) unit gaussian as a prior, $p(\theta) = \frac{1}{k}\exp(-\frac{1}{2}\theta^2)$ for a normaization constant k, the RHS becomes

$$\underbrace{\mathbb{E}_D\left[\log \frac{1}{f(Y \mid \theta_0)}\right]}_{\substack{\text{Cross entropy loss of } f_\theta \text{ w.r.t. } D \\ \text{(data-fit cost of } \theta_0)}} + \underbrace{\frac{\beta}{2}\theta_0^2}_{\substack{\ell_2 \text{ regularizer} \\ \text{(complexity cost of } \theta_0)}} \quad \underbrace{+\beta \log k - \mathrm{H}(D)}_{\text{constant in } f \text{ and } \theta_0}.$$

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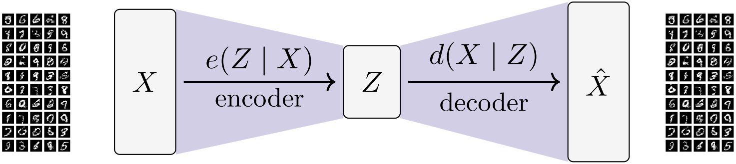
- Structure consists of two neural networks (cpds):



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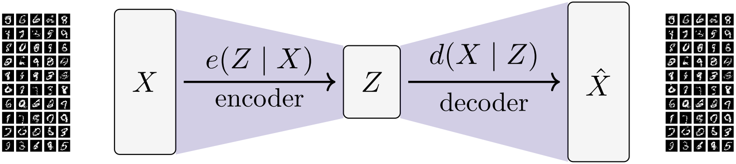

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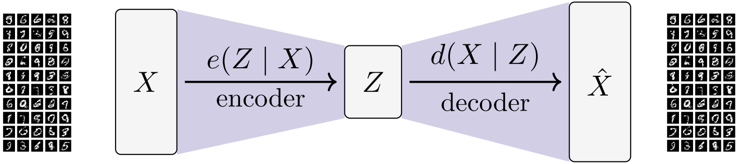

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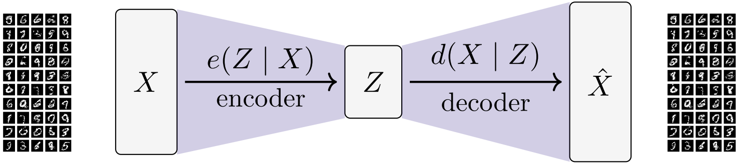

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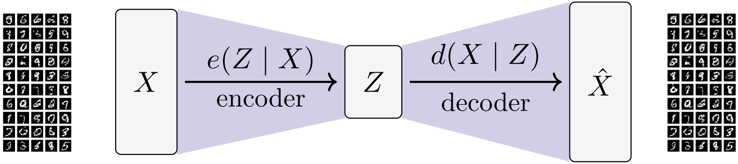

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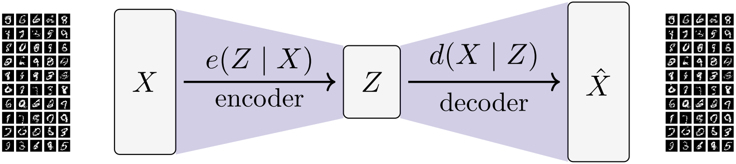

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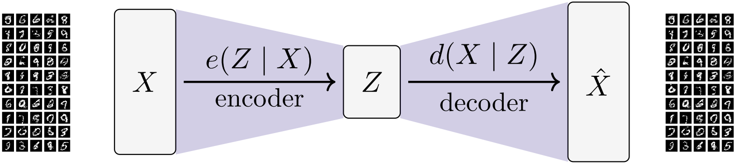

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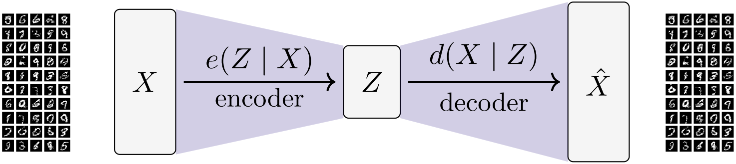

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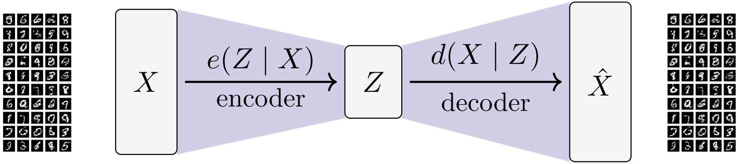

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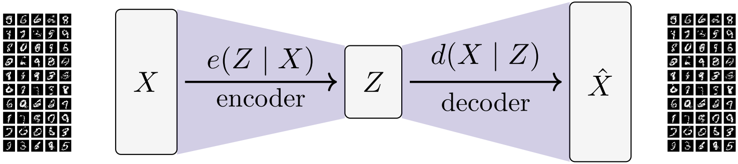
    $\mathrm{ELBO}_{p,e,d}(x) :=$

    $$\underbrace{-\boldsymbol{D}\big(e(Z|x) \,\|\, p(Z)\big)}_{\text{divergence from prior}} - \mathrm{Rec}(x) = \mathop{\mathbb{E}}_{z\sim e|x}\left[\log \frac{p(z)d(x \mid z)}{e(z \mid x)}\right] \leq \overbrace{\log \Pr_{pd}(x)}^{\text{"evidence"}}$$

Variational Auto-Encoders, Take 1

- Structure consists of two neural networks (cpds):



- Objective:
  - For each x, want to minimize $\mathrm{Rec}(x) := \overbrace{-\mathop{\mathbb{E}}_{z\sim e|x} \log d(x \mid z)}^{\text{“reconstruction error”}}$
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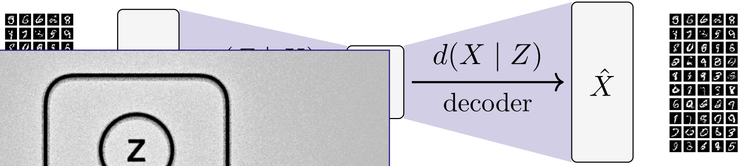

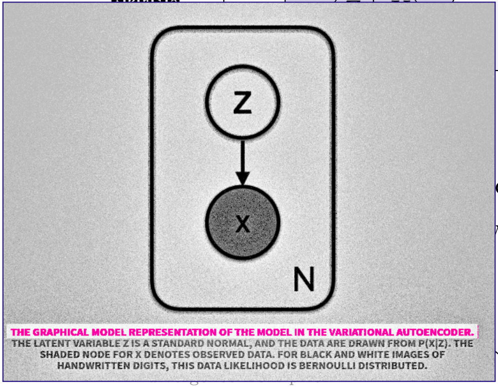

THE GRAPHICAL MODEL REPRESENTATION OF THE MODEL IN THE VARIATIONAL AUTOENCODER. THE LATENT VARIABLE Z IS A STANDARD NORMAL, AND THE DATA ARE DRAWN FROM P(X|Z). THE SHADED NODE FOR X DENOTES OBSERVED DATA. FOR BLACK AND WHITE IMAGES OF HANDWRITTEN DIGITS, THIS DATA LIKELIHOOD IS BERNOULLI DISTRIBUTED.

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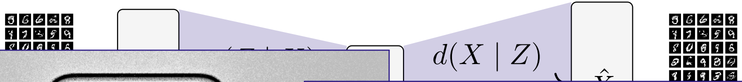

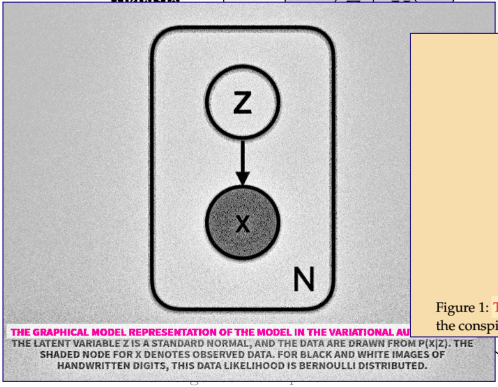

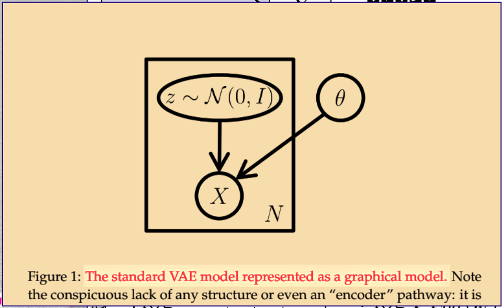

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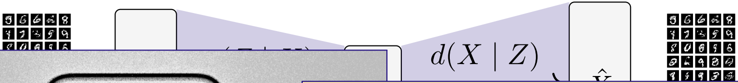

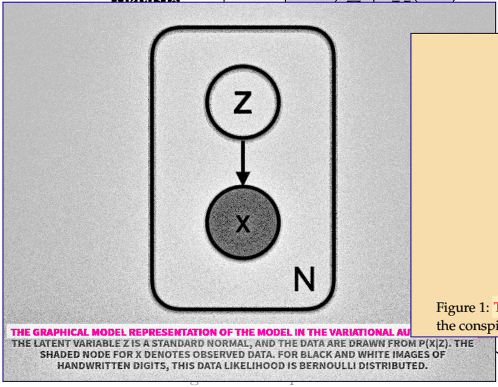

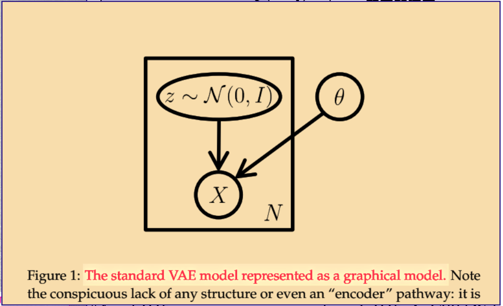

$\sim e|x \Big[\quad e(z \mid x) \quad \Big]$

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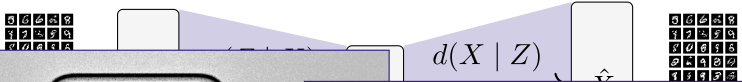

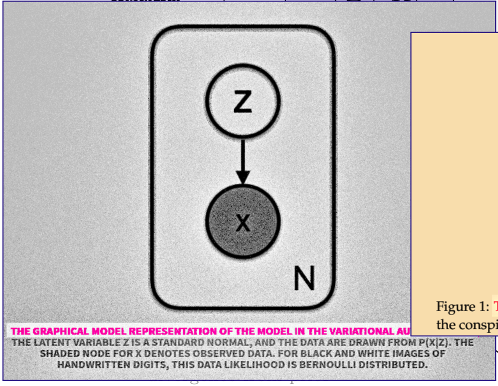

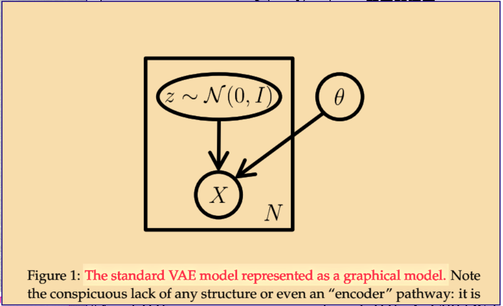

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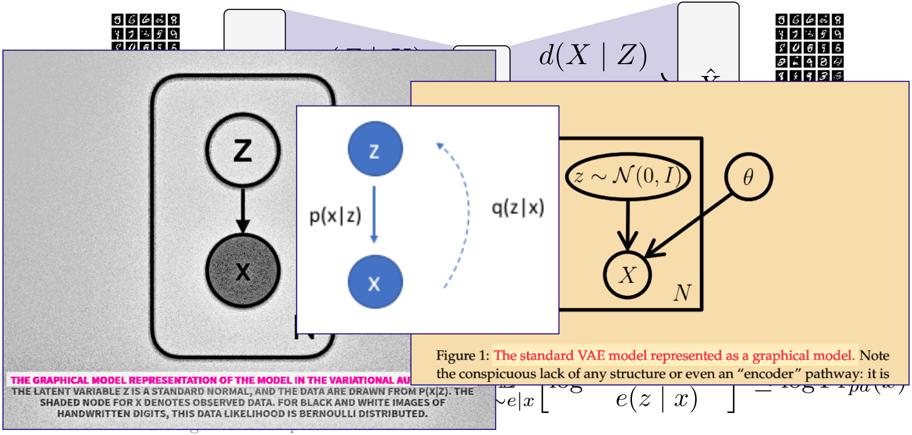

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Variational Auto-Encoders, Take 2

- Structure:

$$Z \qquad X$$

Variational Auto-Encoders, Take 2

- Structure:
  $e(Z \mid X)$: encoder

$$
\begin{tikzcd}
Z & X \arrow[l, bend left, "e"]
\end{tikzcd}
$$

Variational Auto-Encoders, Take 2

- Structure:

  $e(Z \mid X)$: encoder
  $d(X \mid Z)$: decoder

$$\begin{tikzcd} Z \arrow[r, bend left, "d"] & X \arrow[l, bend left, "e"] \end{tikzcd}$$

Variational Auto-Encoders, Take 2

- Structure:

  $e(Z \mid X)$: encoder
  $d(X \mid Z)$: decoder
  $p(Z)$: prior

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\begin{tikzcd}
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$$\begin{tikzcd} \arrow[r, "p"] & Z \arrow[r, bend left, "d"] & X \arrow[l, bend left, "e"] & \arrow[l, two heads, "x"'] \end{tikzcd}$$

Variational Auto-Encoders, Take 2

- Structure:

  $e(Z \mid X)$: encoder
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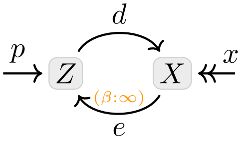
- observe a sample x
  - and trust encoding



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$$\left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "p"] & Z \arrow[rr, bend left, "d"] & & X \arrow[ll, bend left, "e", "(\infty)"'] & {} \arrow[l, two heads, "x"'] \end{tikzcd} \right\rangle\!\!\right\rangle = \mathrm{ELBO}_{p,e,d}(x)$$

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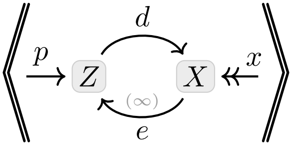

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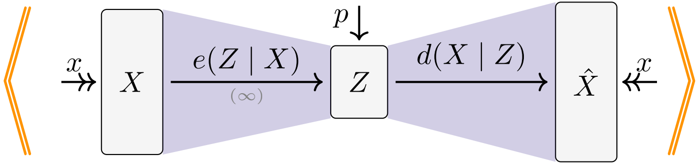

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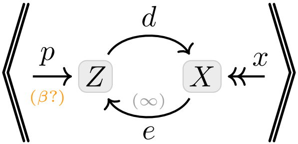

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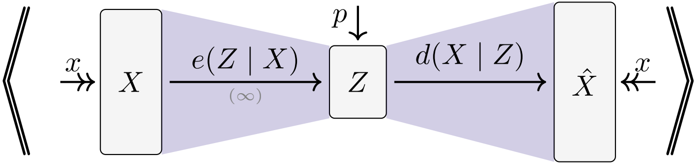

A Very Useful Fact

Believing more things can't make you any less inconsistent.

Lemma (monotonicity of inconsistency)

For all pdgs $\mathcal{M}$, $\mathcal{M}'$, and all $\gamma > 0$,

1. $\langle\!\langle \mathcal{M} \sqcup \mathcal{M}' \rangle\!\rangle_\gamma \geq \langle\!\langle \mathcal{M} \rangle\!\rangle_\gamma$.
2. *If $\mathcal{M}$ and $\mathcal{M}'$ have respective confidence vectors β and β', and $\beta \succeq \beta'$ (that is, $\beta_L \geq \beta'_L$ for all $L \in \mathcal{E}$), then $\langle\!\langle \mathcal{M} \rangle\!\rangle_\gamma \geq \langle\!\langle \mathcal{M}' \rangle\!\rangle_\gamma$.*

Visual Proof: The Varitaional Bound

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$$\left\langle\!\!\left\langle \begin{tikzcd} {} \arrow[r, "p"] & Z \arrow[r, bend left, "d"] & X \arrow[l, bend left, "e!"] & {} \arrow[l, two heads, "x"'] \end{tikzcd} \right\rangle\!\!\right\rangle = -\mathrm{ELBO}_{p,e,d}(x).$$

Visual Proof: The Varitaional Bound

$$-\log \Pr_{p,d}(X{=}x) \;=\;$$

$$\left\langle\!\!\left\langle \xrightarrow{p} \boxed{Z} \xrightarrow{d} \boxed{X} \overset{x}{\twoheadleftarrow} \right\rangle\!\!\right\rangle \qquad \left\langle\!\!\left\langle \xrightarrow{p} \boxed{Z} \underset{e!}{\overset{d}{\rightleftarrows}} \boxed{X} \overset{x}{\twoheadleftarrow} \right\rangle\!\!\right\rangle$$

$$= \;\; -\,\mathrm{ELBO}_{p,e,d}(x).$$

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\left\langle\!\!\left\langle \xrightarrow{p} \boxed{Z} \xrightarrow{d} \boxed{X} \overset{x}{\twoheadleftarrow} \right\rangle\!\!\right\rangle
\;{\color{green}\leq}\;
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Inconsistency as a Divergence

You believe both $p(X)$ and $q(X)$.

$$\xrightarrow{p} \boxed{X} \xleftarrow{q}$$

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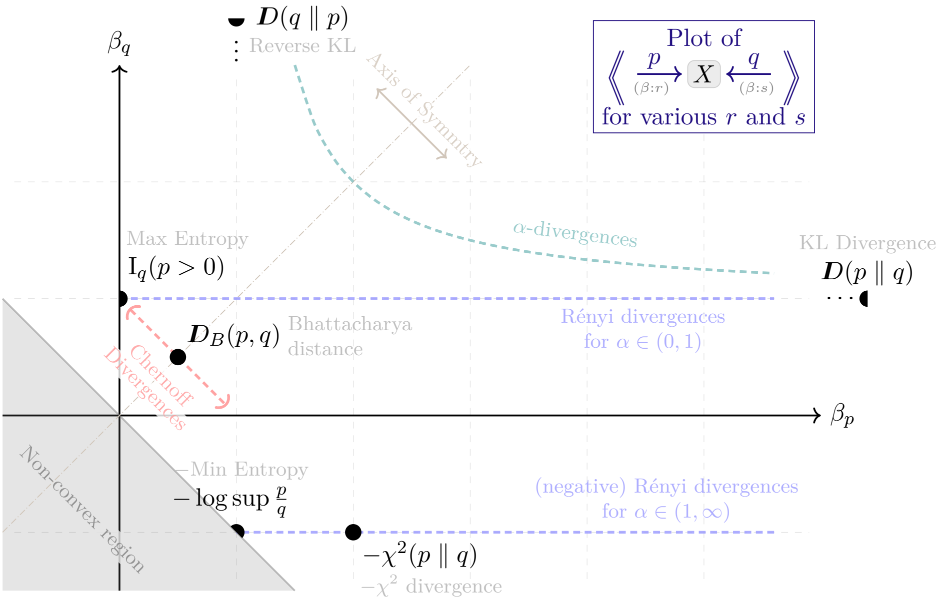
Lemma

$$\boldsymbol{D}^{\mathrm{PDG}}_{(r,s)}(p,q) = -(r+s)\log \sum_x \Big(p(x)^r q(x)^s \Big)^{\frac{1}{r+s}}.$$

Divergences as Inconsistencies



Visual Proof: Data Processing Inequality

$$\boldsymbol{D}^{\mathrm{PDG}}_{(\beta,\zeta)}\Big(p \,\Big\|\, q\Big) \geq \boldsymbol{D}^{\mathrm{PDG}}_{(\beta,\zeta)}\Big(f\circ p \,\Big\|\, f \circ q\Big)$$

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$$= \left\langle\!\!\left\langle \begin{tikzcd} & & \boxed{Y} & & \\ \arrow[r, "p", "(\beta)"'] & \boxed{X_1} \arrow[ur, bend left, "f", "(\beta)"'] \arrow[rr, equal] & & \boxed{X_2} \arrow[ul, bend right, "f"', "(\zeta)"] & \arrow[l, "q"', "(\zeta)"] \end{tikzcd} \right\rangle\!\!\right\rangle$$

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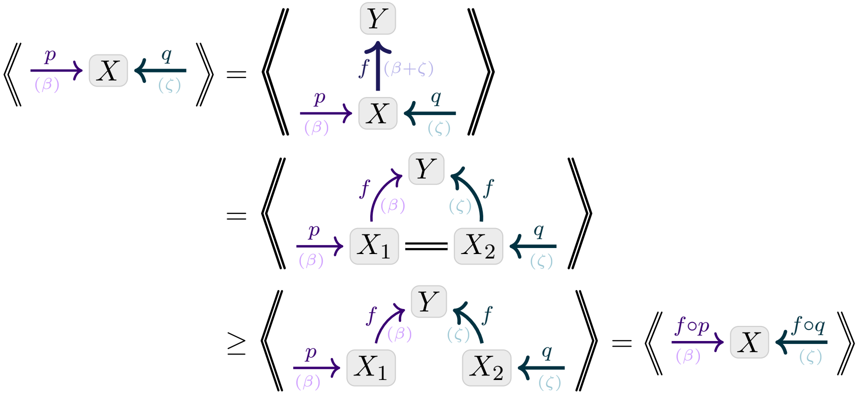

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But there is much more to be done!

Outline for Section 8

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

$$\langle\!\langle m \rangle\!\rangle = \mathrm{I}[kfa = hga] = \qquad - \log \#\{a : kfa = hga\}$$

▸ More Category Theory

Database

$\mathbf{R}_1$

A	B	C
a_1	b_1	c_1
a_2	b_2	c_2

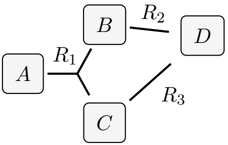
$\mathbf{R}_2$

B	D
b_2	d_1
b_3	d_2
b_4	d_3

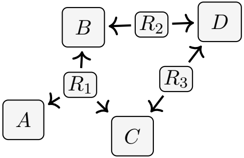
$\mathbf{R}_3$

C	D
c_2	d_1
c_1	d_3

Relational Schema



Row-PDG



Proposition

If $\mathcal{D}$ is a database and μ is a joint distribution over $\mathcal{M}_{\mathcal{D}}$, then $\mu \in \{\!\{\mathcal{M}_{\mathcal{D}}\}\!\}$ iff $Supp(\mu)$ is a universal relation for $\mathcal{D}$.

Corollary

$\mathcal{M}_{\mathcal{D}}$ is consistent iff $\mathcal{D}$ is join consistent.

Open Problems and Future Work

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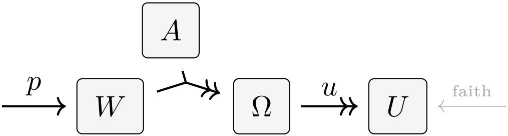
  - Extend semantics to score other objects, not just joint distributions.
  - Regarding PDGs as probabilistic automata.
  - Open Question: Do PDGs capture Dependency Networks? $*$

A Different Picture of Agency

$$\begin{tikzcd} & & A \arrow[d, two heads] & \\ {} \arrow[r, "p"] & W \arrow[r, two heads] & \Omega \arrow[r, two heads, "u"] & U \end{tikzcd}$$

A Different Picture of Agency



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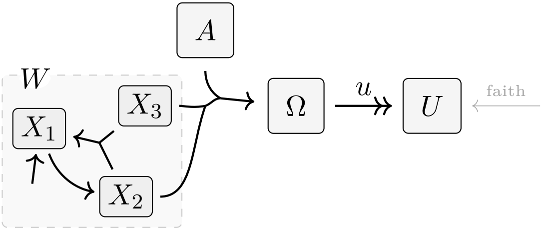

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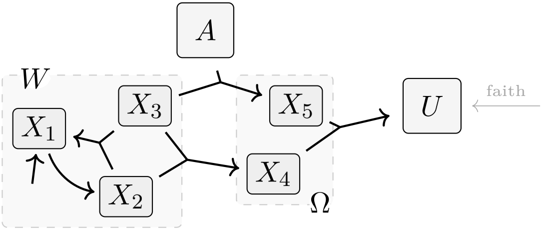

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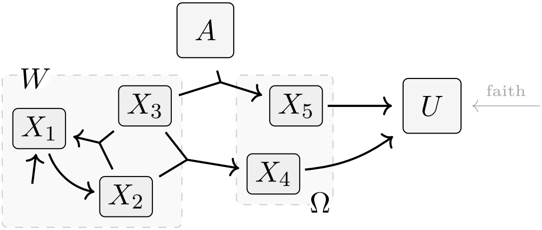

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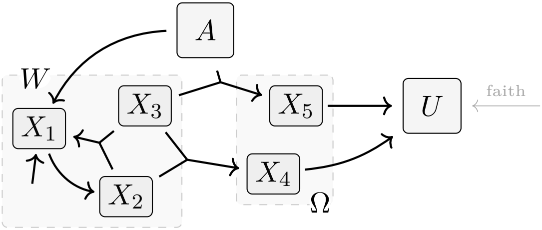

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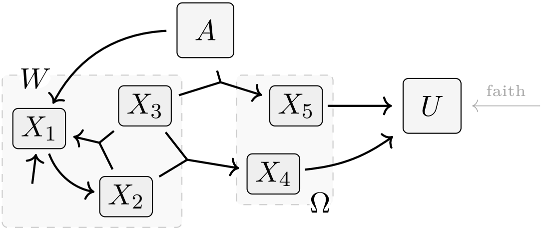

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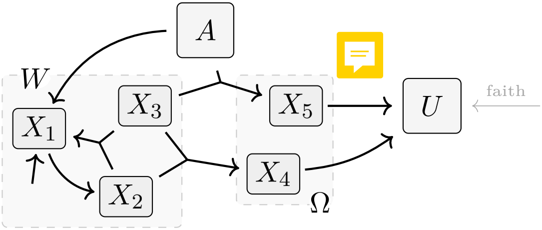

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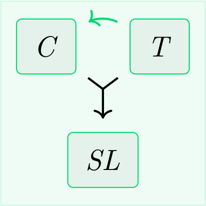
Python library available at `https://orichardson.github.io/pdg/`

Outline for Section 9

Hyper-graphs? Or merely graphs?



main definition

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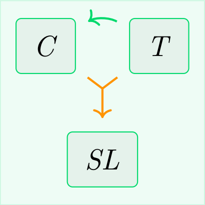

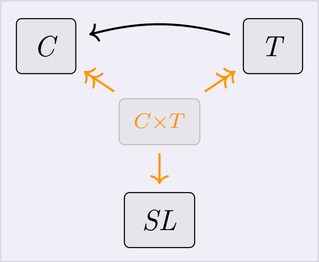

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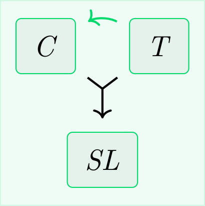

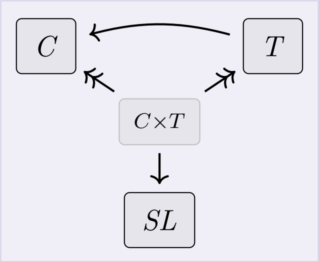

- This widget expands state space, but graphs are simpler.

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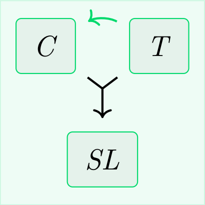

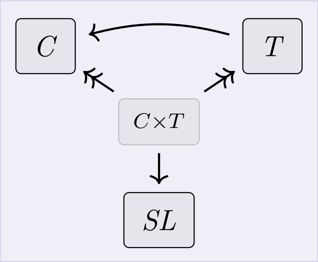

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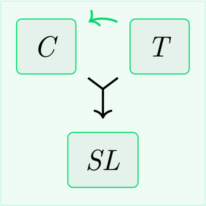

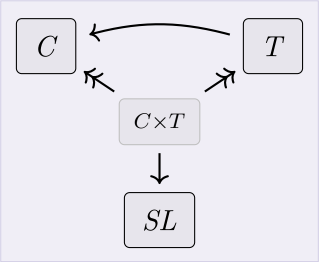

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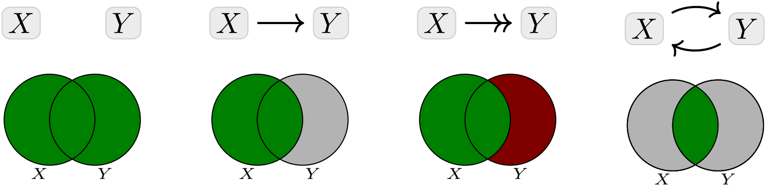
(working directly with hypergraphs is also possible)

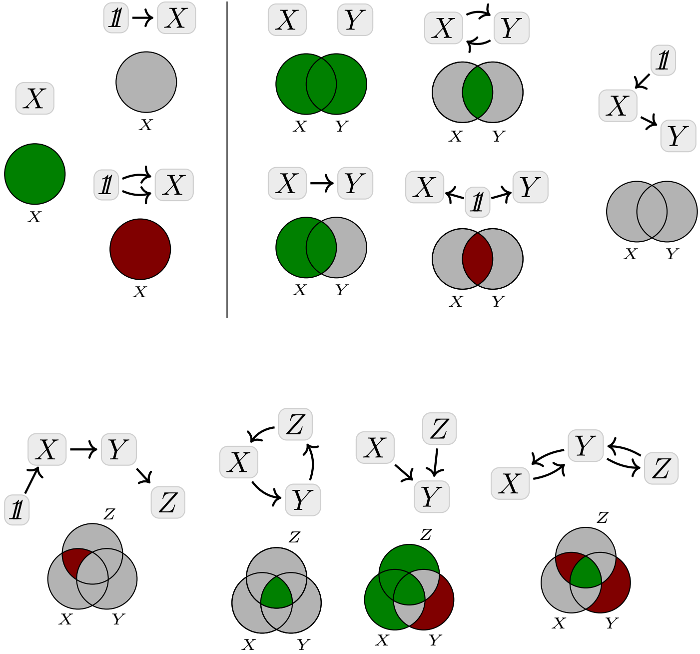
main definition

Outline for Section 10

Illustrations of *IDef*



◂ back to semantics



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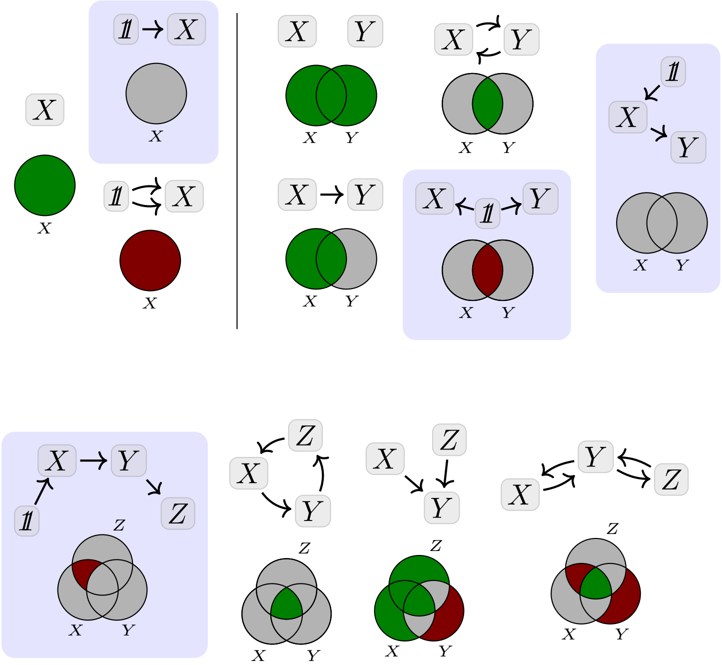

◂ back to semantics

Outline for Section 11

Relationships Between Semantics

Proposition (*the set of consistent distributions is the zero set of the scoring function*)

$\{\!\{ m \}\!\} = \{ \mu : [\![m]\!]_0(\mu) = 0 \}$.

Proposition (*If there there are distributions consistent with m, the best distribution is one of them.*)

$[\![m]\!]^* \in [\![m]\!]_0^*$, *so if m is consistent, then* $[\![m]\!]^* \in \{\!\{ m \}\!\}$.

back to semantics properties

$$[\![m]\!]_\gamma(\mu) = \mathop{\mathbb{E}}_{\mu} \log \prod_{X \xrightarrow{L} Y} \left(\frac{\mu(Y \mid X)}{\mathbf{p}_L(Y \mid X)}\right)^{\beta_L} \left(\frac{\mu(\mathcal{N})}{\prod_{X \xrightarrow{L} Y} \mu(Y \mid X)^{\alpha_L}}\right)^{\gamma}$$

Outline for Section 12

Bayesian Networks: Maximum Entropy?

Common distributions tend to maximize entropy subject to natural constraints.

distribution	constraints
Gaussian $\mathcal{N}(\mu, \sigma^2)$	mean μ, variance σ^2
Exponential $\mathrm{Exp}(\lambda)$	positive support, mean λ
Factor graphs	moment matching.
$\dots$	$\dots$

back to BN theorem

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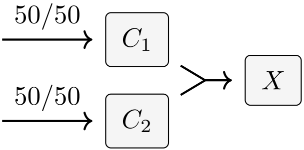

$$\Pr(X \mid C_1, C_2) = \begin{cases} \mathbb{1}[X = x_0] & \text{if } C_1 = C_2 \\ \mathrm{Unif}(X) & \text{if } C_1 \neq C_2 \end{cases}$$

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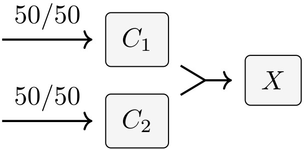

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Corollary

Among the distributions in $\{\!\{\mathcal{B}\}\!\}$, $\Pr_{\mathcal{B}}$ *has the maximum entropy, beyond the entropy of the given cpds.*

IDef says maximize: $\quad \mathrm{H}(\mu) - \sum_{X \in \mathcal{N}} \mathrm{H}_\mu(X \mid \mathbf{Pa}\, X)$

back to BN theorem

Full Factor Graph Results

Theorem (PDGs are WFGs)

For all unweighted PDGs $\mathcal{N}$ *and non-negative vectors* $\mathbf{v}$ *over the edges of* $\mathcal{N}$*, and all* $\gamma > 0$*, we have that* $[\![(\mathcal{N}, \mathbf{v}, \gamma\mathbf{v})]\!]_\gamma = \gamma\, \mathit{VFE}_{(\Phi_{\mathcal{N}}, \mathbf{v})}$*; consequently,* $[\![(\mathcal{N}, \mathbf{v}, \gamma\mathbf{v})]\!]^*_\gamma = \{\Pr_{(\Phi_{\mathcal{N}}, \mathbf{v})}\}$.

Theorem (WFGs are PDGs)

For all weighted factor graphs $\Psi = (\Phi, \theta)$ *and all* $\gamma > 0$*, we have that* $\mathit{VFE}_\Psi = {}^1\!/\!_\gamma [\![\mathcal{M}_{\Psi,\gamma}]\!]_\gamma + C$ *for some constant* C*, so* $\Pr_\Psi$ *is the unique element of* $[\![\mathcal{M}_{\Psi,\gamma}]\!]^*_\gamma$.

◂ PDG→WFG ◂ WFG→PDG

Outline for Section 13

Outline for Section 14

Definition (PDG)

$\mathcal{N}$: **Set** (node set)

$\mathcal{V} : \mathcal{N} \to$ **Set** (node values)

$\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \times \mathit{Label}$ (edge set)

For $X \xrightarrow{L} Y \in \mathcal{E}$,

$\mathbf{p}_{L} : \mathcal{V}(X) \to \Delta\mathcal{V}(Y)$ (edge cpd)

$\alpha_{L} : \mathbb{R}$ (functional determination)

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Equivalent Categorical Definition

An unweighted PDG is a functor $\langle \mathbf{p}, \mathcal{V} \rangle : \mathit{Paths}(\mathcal{N}, \mathcal{E}) \to \mathbf{Mark}$.
So a PDG is a *diagram* in **Mark**, in the usual mathematical sense.

What do you do with diagrams? Take limits / colimits.

$$\cdots \qquad X_1 \twoheadrightarrow X_2 \leftarrow X_3 \qquad \cdots$$

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$$
\begin{tikzcd}
 & & L \arrow[dl] \arrow[d] \arrow[dr] & & \\
\cdots & X_1 \arrow[r, two heads] & X_2 & X_3 \arrow[l] & \cdots
\end{tikzcd}
$$

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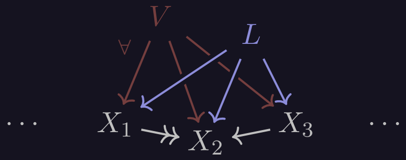

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For the deterministic sub-PDG $\mathcal{m}_{\text{det}} \subseteq \mathcal{m}$:

$$\lim \mathcal{m}_{\text{det}} = \left(\begin{matrix}\text{natural}\\ \text{sample space}\end{matrix}\ \Omega,\quad \begin{matrix}\text{random}\\ \text{variables}\end{matrix}\ \left\{ \tilde{X} : \Omega \to \mathcal{V}(X) \right\}_{X \in \mathcal{N}} \right)$$

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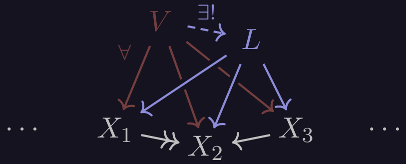

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For a BN $\mathcal{B}$: $$\lim \mathcal{m}_{\mathcal{B}} = \left(\mathbb{1},\ \left\{ \Pr_{\mathcal{B}}(X) \right\}_{X \in \mathcal{N}} \right)$$