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The transitive closure, convergence of powers and adjoint of generalized fuzzy matrices

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Abstract

Generalized fuzzy matrices are considered as matrices over a special type of semiring which is called an incline, and their transitive closure, powers, determinant and adjoint matrices are studied. An expression for the transitive closure of a matrix A as a sum of its powers and some sufficient conditions for powers of a matrix to converge are given. If the incline is commutative, a sufficient condition for nilpotency of a matrix is obtained, namely the determinants of the principal submatrices of the matrix are all equal to zero element. In addition, it is proved that A^{n-1} is equal to the adjoint matrix of A if the matrix A satisfies $A \geq I_n$.

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1. Introduction

With max–min operations the fuzzy algebra $[0, 1]$ and its matrix theory are considered by many authors, see e.g. [2,5,6,8,9,11]. Some results are extended to generalized fuzzy matrices, matrices over some sort of special semiring. For example, determinant theory, powers and nilpotent conditions of matrices over a distributive lattice are considered by Zhang [12,13] and Tan [10], and the transitivity of matrices over path algebra (i.e. additively idempotent semiring) is discussed by Hashimoto [4]. There is a special type of semiring, called incline [1,7], which is a special case of path algebra but more extensive than distributive lattice (see Golan [3], there incline is called simple semiring). In this paper generalized fuzzy matrices, matrices over an incline, are considered and some results about the transitive closure, determinant, adjoint matrices, convergence of powers and conditions for nilpotency are obtained.

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2. Preliminaries

A semiring is an algebraic structure $(R, +, \cdot)$ such that $(R, +)$ is an Abelian monoid (identity 0), (R, \cdot) is a monoid (identity 1), \cdot distributes over $+$ from either side, $r0 = 0r = 0$ for all $r \in R$ and $0 \neq 1$. Usually the semiring $(R, +, \cdot)$ is denoted by R briefly.

A semiring R is said to be an incline if $r + 1 = 1$ for all $r \in R$ (It is permissive in [7] that an incline does not contain the identity, but with the identity the definition in [7] is consistent with ours). A semiring R is additively idempotent if $r + r = r$ for all $r \in R$. An incline R is definitely an additively idempotent semiring since $r + r = r(1 + 1) = r \cdot 1 = r$. An incline R is commutative if $ab = ba$ for all $a, b \in R$.

Example 1. Any distributive lattice with the least element 0 and greatest element 1, such as fuzzy algebra $([0, 1], \vee, \wedge)$, is an incline. Here $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$.

Example 2. $([0, 1], \max, \times)$ is an incline, but not a distributive lattice. Here $a \times b$ is ordinary multiplication.

Example 3. If R is an additively idempotent semiring, the subsemiring $\{a \in R \mid a + 1 = 1\}$ is an incline.

For an incline R , defining $a \leq b$ ($a, b \in R$) if and only if $a + b = b$, it is easy to check that \leq is a partial order relation over R and satisfies the following:

Proposition 1 (Cao et al. [1] and Golan [3]). *Let R be an incline and $a, b, c \in R$, then*

- (i) $0 \leq a \leq 1$,
- (ii) if $a \leq b$, then $a + c \leq b + c$, $ac \leq bc$ and $ca \leq cb$,
- (iii) $a \leq a + b$, and $a + b$ is the least upper bound of a and b . In other words, if there is an element c satisfying $a \leq c$ and $b \leq c$, then $a + b \leq c$,
- (iv) $ab \leq a$, $ab \leq b$. In other words, ab is a lower bound of a and b ,
- (v) $acb \leq ab$,
- (vi) $a + b = 0$ if and only if $a = b = 0$,
- (vii) $ab = 1$ if and only if $a = b = 1$.

The set of square matrices of order n over an incline R is denoted by $M_n(R)$. The zero matrix O_n and the identity matrix I_n of order n are defined as if R were a field. For matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $M_n(R)$, define the operations $A + B = (a_{ij} + b_{ij})$ and $AB = (\sum_{k=1}^n a_{ik}b_{kj})$. Thus $M_n(R)$ is an additively idempotent semiring under matrix addition and multiplication with additive identity O_n and multiplicative identity I_n but no longer an incline.

The partial order relation \leq over $M_n(R)$ is defined as

$$A \leq B \quad \text{if and only if} \quad a_{ij} \leq b_{ij} \quad \text{for all } i, j.$$

That is, $A \leq B$ if and only if $A + B = B$. $A < B$ denotes $A \leq B$ and $A \neq B$.

Let R be an incline and $A \in M_n(R)$. The (i, j) -entry of square matrix A^m is denoted by $a_{ij}^{(m)}$, and obviously

$$a_{ij}^{(m)} = \sum_{1 \leq j_1, j_2, \dots, j_{m-1} \leq n} a_{ij_1} a_{j_1 j_2} \dots a_{j_{m-1} j}. \quad (1)$$

Matrix A is said to be nilpotent if $A^k = O_n$ for some $k \in \mathbb{Z}_+$, where \mathbb{Z}_+ denotes the set of all positive integers. A is idempotent if $A^2 = A$.

Definition 1. Let R be an incline and $A, B \in M_n(R)$. Matrix A is said to be transitive, if $A^2 \leq A$. Matrix B is said to be transitive closure of matrix A , if B is transitive, $A \leq B$ and $B \leq C$ for any transitive matrix C satisfying $A \leq C$. The transitive closure of matrix A is denoted by $t(A)$.

3. Results

Lemma 1. Let R be an incline and $A \in M_n(R)$. If $m \geq n$, then

$$A^m \leq \sum_{k=0}^{n-1} A^k \quad (\text{here } A^0 = I_n).$$

As a result, $A^{m+1} \leq \sum_{k=1}^n A^k$.

Proof. Let $B = \sum_{k=0}^{n-1} A^k$. Firstly, $a_{ii}^{(m)} \leq 1 = b_{ii}$ by Proposition 1(i). In the case of $i \neq j$, we consider an arbitrary summand of the right-hand side of equality (1), $a_{ij_1} a_{j_1 j_2} \dots a_{j_{m-1} j}$. Since $\{i, j_1, j_2, \dots, j_{m-1}, j\} \subset \{1, 2, \dots, n\}$ and $m+1 > n$, there are r, s such that $j_r = j_s$ ($0 \leq r < s \leq m, j_0 = i, j_m = j$). Deleting $a_{j_r j_{r+1}} a_{j_{r+1} j_{r+2}} \dots a_{j_{s-1} j_s}$ from the summand $a_{ij_1} a_{j_1 j_2} \dots a_{j_{m-1} j}$, we obtain

$$a_{ij_1} a_{j_1 j_2} \dots a_{j_{m-1} j} \leq a_{ij_1} \dots a_{j_{r-1} j_r} a_{j_s j_{s+1}} \dots a_{j_{m-1} j}$$

by Proposition 1(iv) and 1(v). If the number $r+m-s+2$ of the subscripts in the right-hand side of the inequality is still more than n , the same deleting method is used. Therefore, there is a positive integer $t \leq n-1$, such that

$$a_{ij_1} a_{j_1 j_2} \dots a_{j_{m-1} j} \leq a_{il_1} a_{l_1 l_2} \dots a_{l_{t-1} j}.$$

By equality (1) and Proposition 1(iii), we have

$$a_{ij_1} a_{j_1 j_2} \dots a_{j_{m-1} j} \leq \sum_{l_1, \dots, l_{t-1}} a_{il_1} a_{l_1 l_2} \dots a_{l_{t-1} j} = a_{ij}^{(t)} \leq \sum_{k=1}^{n-1} a_{ij}^{(k)}.$$

It follows from equality (1) and Proposition 1(iii) again that

$$a_{ij}^{(m)} \leq \sum_{k=1}^{n-1} a_{ij}^{(k)} = b_{ij}.$$

This completes the proof. \square

Theorem 1. Let R be an incline and $A \in M_n(R)$. Then the transitive closure of matrix A is given by

$$t(A) = \sum_{k=1}^n A^k.$$

Proof. Let $B = \sum_{k=1}^n A^k$. Obviously, $A \leq B$. Since $M_n(R)$ is additively idempotent, we have

$$B^2 = \sum_{k=2}^{2n} A^k \leq B + \sum_{k=n+1}^{2n} A^k.$$

By Lemma 1, $A^k \leq \sum_{l=1}^n A^l = B$ as $k > n$. Hence $B^2 \leq B$.

If there is a matrix C such that $A \leq C$ and $C^2 \leq C$, then $A^2 \leq AC \leq C^2$, and by induction we have $A^k \leq C^k \leq C$ for all positive integers k . Hence $B \leq C$. By the definition of transitive closure, we obtain

$$B = t(A) = \sum_{k=1}^n A^k. \quad \square$$

Definition 2. Let R be an incline and $A \in M_n(R)$. A is said to be power-convergent if $A^k = A^{k+1}$ for some positive integer k . If A is power-convergent the least positive integer k such that $A^k = A^{k+1}$ is called the index of A and denoted by $k(A)$. A is said to be row diagonally dominant if $a_{ii} \geq a_{ij}$ ($1 \leq i, j \leq n$). A is column diagonally dominant if $a_{ii} \geq a_{ji}$ ($1 \leq i, j \leq n$).

Theorem 2. Let R be an incline and $A \in M_n(R)$.

- (i) If $A \leq A^2$, then A is power-convergent and $a_{ij}^{(n-1)} = a_{ij}^{(n)}$ ($i \neq j$), $k(A) \leq n$.
- (ii) If $A \leq A^2$ and A is row (or column) diagonally dominant, then A is power-convergent and $k(A) \leq n - 1$.

Proof. (i) The result is obtained by Lemma 1 and the assumption.

(ii) Suppose A is row diagonally dominant. We have

$$\begin{aligned} a_{ii}^{(k)} &= \sum_{j_1, j_2, \dots, j_{k-1}} a_{ij_1} a_{j_1 j_2} \dots a_{j_{k-1} i} \\ &= \sum_{j_1} a_{ij_1} \left(\sum_{j_2, \dots, j_{k-1}} a_{j_1 j_2} \dots a_{j_{k-1} i} \right) \\ &\leq \sum_{j_1} a_{ij_1} = a_{ii} \end{aligned}$$

by Proposition 1(ii), 1(iv), and the definition of row diagonally dominant matrix. On the other hand, $a_{ii}^{(k)} \geq a_{ii}$ ($k \geq 1$) by the assumption. Therefore, $a_{ii}^{(k)} = a_{ii}$ ($k \geq 1$). This equality can be

obtained likewise for a column diagonally dominant matrix. Considering the result in (i), we have $A^{n-1} = A^n$. \square

Corollary 1. *Let R be an incline, $A \in M_n(R)$ and $A \geq I_n$. Then A is power-convergent and $k(A) \leq n - 1$.*

Under the condition of Theorem 2(i) it is possible that $A^{n-1} < A^n$.

Example 4. Let R be fuzzy algebra $([0, 1], \vee, \wedge)$ in Example 1. Take the matrix over R

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.1 & 0.1 & 0.3 \\ 0.3 & 0.2 & 0.3 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}.$$

Thus $A < A^2 < A^3$.

Thomason [11] initiated the study of convergence of powers of a max–min fuzzy matrix and Li [8] gave the result of Theorem 2(i) for the fuzzy matrix. Tan [10] pointed out that A is power-convergent and $k(A) \leq n$ if A is an $n \times n$ matrix over a distributive lattice L and satisfies $A \leq A^2$. Also Tan [10] proved that A is power-convergent if A is over L and satisfies $A^2 \leq A$. But over an incline $A^2 \leq A$ cannot guarantee A is power-convergent. For example, take 1×1 matrix $A = (0.8)$ over the incline R in Example 2. We have $A > A^2 > A^3 > \dots$. Meanwhile this example indicates that in general the following assertions are untenable for a square matrix A over an incline R .

(a) $A^k = A^{k+d}$ for some positive integers k and d .

(b) Row(or column) diagonal dominance implies $A \leq A^2$.

But these are true for a square matrix over a distributive lattice L [10]. In fact, for (b), if A is row diagonally dominant, then $a_{ij}^{(2)} = \sum_k a_{ik} a_{kj} \geq a_{ii} a_{ij} = a_{ij}$. Therefore, in Theorem 2(ii) if R is a distributive lattice the condition $A \leq A^2$ can be removed and we obtain the result over a distributive lattice L [10].

Definition 3. Let R be an incline and $A \in M_n(R)$. The determinant $|A|$ (or permanent) of matrix A is defined as follows:

$$|A| = \sum_{\sigma \in S_n} a_{1\sigma(1)} \dots a_{n\sigma(n)},$$

where S_n denotes the symmetric group of all permutations of the indices $\{1, 2, \dots, n\}$.

Definition 4. Let R be an incline and $A \in M_n(R)$ ($n \geq 2$). Matrix B is said to be adjoint matrix of matrix A if $b_{ij} = |A_{ji}|$ ($1 \leq i, j \leq n$), where A_{ji} is the matrix of order $n-1$ formed by deleting row j and column i from A . The adjoint matrix of matrix A is denoted by $\text{adj}(A)$.

In the following we consider a commutative incline.

Proposition 2. Let R be a commutative incline and $A, B \in M_n(R)$, then

- (i) $|A'| = |A|$, where A' denotes the transpose of A .
- (ii) $|rA| = r^n |A|$, where $r \in R$ and $rA = (ra_{ij})$.
- (iii) $|E_{ij}A| = |AE_{ij}| = |A|$, where E_{ij} is the matrix obtained from the identity matrix I_n by interchanging row i and row j .
- (iv) $|A| = \sum_{j=1}^n a_{ij} |A_{ij}|$, $i \in \{1, 2, \dots, n\}$.

Proof. (i)–(iii) follow from the commutativity of R and the definition of determinant. (iv) We can rewrite $|A|$ as

$$\begin{aligned} |A| &= \sum_{j=1}^n \sum_{\sigma \in S_n, \sigma(i)=j} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \\ &= \sum_{j=1}^n a_{ij} \sum_{\pi \in S_{n_i n_j}} a_{1\pi(1)} \cdots a_{i-1\pi(i-1)} a_{i+1\pi(i+1)} \cdots a_{n\pi(n)}, \end{aligned}$$

where $n_i = \{1, 2, \dots, n\} \setminus \{i\}$ and $S_{n_i n_j}$ is the set of all bijections from the set n_i to the set n_j . By the definition of determinant we see that

$$|A_{ij}| = \sum_{\pi \in S_{n_i n_j}} a_{1\pi(1)} \cdots a_{i-1\pi(i-1)} a_{i+1\pi(i+1)} \cdots a_{n\pi(n)}.$$

Thus (iv) is obtained. \square

Let A be a matrix over an incline R . Let $A(p \Rightarrow q)$ be the matrix obtained from A by replacing row q of A by row p of A .

Proposition 3. Let R be a commutative incline and $A \in M_n(R)$, then

- (i) $|AB| \geq |A||B|$.
- (ii) $|A \text{adj}(A)| \geq |A|^n + |A||\text{adj}(A)|$.
- (iii) $|\text{adj}(A)A| \geq |A|^n + |A||\text{adj}(A)|$.

Proof. (i)

$$\begin{aligned} |AB| &= \sum_{\sigma \in S_n} \left(\sum_{k=1}^n a_{1k} b_{k\sigma(1)} \sum_{k=1}^n a_{2k} b_{k\sigma(2)} \cdots \sum_{k=1}^n a_{nk} b_{k\sigma(n)} \right) \\ &= \sum_{k_1, k_2, \dots, k_n} \left(\sum_{\sigma \in S_n} a_{1k_1} a_{2k_2} \cdots a_{nk_n} b_{k_1\sigma(1)} b_{k_2\sigma(2)} \cdots b_{k_n\sigma(n)} \right) \end{aligned}$$

$$\begin{aligned} &\geq \sum_{\pi \in S_n} \left(a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} \sum_{\sigma \in S_n} b_{\pi(1)\sigma(1)} b_{\pi(2)\sigma(2)} \cdots b_{\pi(n)\sigma(n)} \right) \\ &= \sum_{\pi \in S_n} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} |B| = |A| |B|. \end{aligned}$$

(ii) Let $B = A \operatorname{adj}(A)$. Then $b_{ij} = \sum_{k=1}^n a_{ik} |A_{jk}| = |A(i \Rightarrow j)|$ by Proposition 2(iv). Thus

$$|A \operatorname{adj}(A)| = \sum_{\sigma \in S_n} |A(1 \Rightarrow \sigma(1))| |A(2 \Rightarrow \sigma(2))| \cdots |A(n \Rightarrow \sigma(n))|.$$

Therefore $|A \operatorname{adj}(A)| \geq |A(1 \Rightarrow 1)| |A(2 \Rightarrow 2)| \cdots |A(n \Rightarrow n)| = |A|^n$ and $|A \operatorname{adj}(A)| \geq |A| |\operatorname{adj}(A)|$ by (i). Thus we have $|A \operatorname{adj}(A)| \geq |A|^n + |A| |\operatorname{adj}(A)|$ by Proposition 1(iii). (iii) Since $\operatorname{adj}(A') = (\operatorname{adj}(A))'$ we have $|\operatorname{adj}(A)A| = |A' \operatorname{adj}(A')|$ by Proposition 2(i). Thus (iii) can be proved by (ii) and Proposition 2(i). \square

Zhang [12] proved Proposition 3(i) for matrices over a distributive lattice. Kim et al. [5] give

$$|A \operatorname{adj}(A)| = |\operatorname{adj}(A)A| = |A| \quad (2)$$

for max–min fuzzy matrices. Whether equality (2) is hold over an incline is not known.

For a matrix A over $M_n(R)$, let H be the $m \times m$ matrix obtained from A by deleting row q_1 , row q_2, \dots , row q_{n-m} and column q_1 , column q_2, \dots , column q_{n-m} (where $1 \leq q_1 < q_2 < \cdots < q_{n-m} \leq n$ and $1 \leq m \leq n$). The matrix H is called a principal submatrix of order m of matrix A and if $\{p_1, p_2, \dots, p_m\} \cup \{q_1, q_2, \dots, q_{n-m}\} = \{1, 2, \dots, n\}$ the principal submatrix H is denoted by $A(p_1, p_2, \dots, p_m)$. The principal submatrix of order n of the matrix A is A itself.

For a matrix A over a distributive lattice Zhang [13] proved that A is nilpotent if and only if the determinant of every principal submatrix of A is equal to 0. For a matrix over an incline we give

Theorem 3. Let R be a commutative incline and $A \in M_n(R)$.

- (i) If the determinants of the principal submatrices of A are all equal to the zero element 0 of the incline R , then A is nilpotent and $A^n = O_n$.
- (ii) If the incline R satisfies the condition

$$(D) \quad r^l \neq 0 \text{ for all } l \in \mathbb{Z}_+ \text{ and non-zero elements } r \in R$$

and for some $h \in \mathbb{Z}_+$ and any $k \in \{1, 2, \dots, n\}$ the main diagonal elements of matrix A^{hk} are all 0, then the determinants of the principal submatrices of A are all equal to 0.

Proof. (i) We examine $a_{ij_1} a_{j_1 j_2} \cdots a_{j_{n-1} j}$. There are s, t such that $j_s = j_t$ ($0 \leq s < t \leq n, j_0 = i, j_n = j$). Let $K = \{(p, q) \mid j_p = j_q, p < q\}$. Then K is not empty. Let $(k, l) \in K$ such that $l - k = \min\{q - p \mid (p, q) \in K\}$. By Proposition 1(iv) and (v), $a_{ij_1} a_{j_1 j_2} \cdots a_{j_{n-1} j} \leq a_{j_k j_{k+1}} a_{j_{k+1} j_{k+2}} \cdots a_{j_{l-1} j_l}$. The right-hand side of the inequality is equal to a summand of the expansion of determinant $|A(j_k, j_{k+1}, \dots, j_{l-1})|$ because R is commutative. By the assumption of the Theorem and Proposition 1(vi), the right-hand side of the inequality is equal to 0. Thus the left-hand side is also 0. Therefore, we obtain $a_{ij}^{(n)} = \sum_{j_1, \dots, j_{n-1}} a_{ij_1} a_{j_1 j_2} \cdots a_{j_{n-1} j} = 0$ again by Proposition 1(vi).

(ii) Let $A(p_1, p_2, \dots, p_m)$ be an $m \times m$ principal submatrix of A ($1 \leq m \leq n$). We consider $a_{p_1\sigma(p_1)} a_{p_2\sigma(p_2)} \dots a_{p_m\sigma(p_m)}$, where σ is an arbitrary permutation of indices $\{p_1, p_2, \dots, p_m\}$. Let $(p'_1, p'_2, \dots, p'_d)$ be a cycle of σ ($1 \leq d \leq m$). Then we can see that

$$\begin{aligned} (a_{p_1\sigma(p_1)} a_{p_2\sigma(p_2)} \dots a_{p_m\sigma(p_m)})^h &\leq (a_{p'_1 p'_2} a_{p'_2 p'_3} \dots a_{p'_{d-1} p'_d} a_{p'_d p'_1})^h \\ &\leq \sum_{j_1, j_2, \dots, j_{hd-1}} a_{p'_1 j_1} a_{j_1 j_2} \dots a_{j_{hd-1} p'_1} \\ &= a_{p'_1 p'_1}^{(hd)} = 0. \end{aligned}$$

We have $(a_{p_1\sigma(p_1)} a_{p_2\sigma(p_2)} \dots a_{p_m\sigma(p_m)})^h = 0$ by Proposition 1(i). Then $a_{p_1\sigma(p_1)} a_{p_2\sigma(p_2)} \dots a_{p_m\sigma(p_m)} = 0$ by the condition (D). Therefore, $|A(p_1, p_2, \dots, p_m)| = 0$. This completes the proof of Theorem 3. \square

In Theorem 3, if R is a distributive lattice the condition (D) is naturally satisfied and so the result over a distributive lattice is obtained. And in this case, the condition of Theorem 3(ii) is weaker than nilpotency of A .

For a max–min $n \times n$ fuzzy matrix A Thomason [11] proved that $\text{adj}(A) = A^{n-1}$ if A satisfies $a_{ii} \geq a_{jk}$ for all $i, j, k \leq n$. We note that if there is not the condition $a_{11} = a_{22} = \dots = a_{nn}$ which is implied by $a_{ii} \geq a_{jk}$ for all $i, j, k \leq n$ the result cannot be guaranteed. For example,

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

Although over a general incline it is possible that a matrix A under the same condition does not converge we can consider the relation between $\text{adj}(A)$ and A^{n-1} .

Such as preceding statement any of $A \leq A^2$ and $a_{ii} \geq a_{jk}$ ($1 \leq i, j, k \leq n$) cannot imply the other for a matrix A over an incline (Example 4). But we can obtain

Theorem 4. Let R be a commutative incline and $A \in M_n(R)$ ($n \geq 2$).

- (i) If $A \leq A^2$, then $|A_{ji}| \leq a_{ij}^{(n-1)}$ ($i \neq j$).
- (ii) If $a_{ii} \geq a_{jk}$ ($1 \leq i, j, k \leq n$), then $|A_{ji}| \leq a_{ij}^{(n-1)}$ ($i \neq j$) and $|A_{ii}| = a_{ii}^{(n-1)}$ ($1 \leq i \leq n$).

Proof. We note that $|A_{ji}|$ can be obtained from $|A|$ by replacing a_{ji} by 1 and all other row- j entries a_{jk} ($k \neq i$) by 0. Thus, we have

$$|A_{ji}| = \sum_{\sigma \in S_n, \sigma(j)=i} a_{1\sigma(1)} \dots a_{j-1, \sigma(j-1)} a_{j+1, \sigma(j+1)} \dots a_{n\sigma(n)}. \quad (3)$$

Since $\sigma(j) = i$, there is a cycle $(i, \sigma(i), \dots, \sigma^t(i), j)$ ($0 \leq t \leq n-2$) in the decomposition to disjoint cycles of permutation σ . The cycle becomes (i, j) in the case of $t = 0$ and $\sigma^t(i) = \sigma(\sigma^{t-1}(i))$ as $t \geq 2$.

(i) By Proposition 1(iii)–(v), commutativity of R and the assumption of the theorem, we obtain

$$\begin{aligned} a_{1\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n\sigma(n)} &\leq a_{i\sigma(i)} a_{\sigma(i)\sigma^2(i)} \cdots a_{\sigma^t(i)j} \\ &\leq \sum_{1 \leq j_1, \dots, j_t \leq n} a_{ij_1} a_{j_1 j_2} \cdots a_{j_t j} = a_{ij}^{(t+1)} \leq a_{ij}^{(n-1)} \quad (i \neq j). \end{aligned}$$

Since the above holds for every $\sigma \in S_n$ satisfying $\sigma(j) = i (i \neq j)$, we have $|A_{ji}| \leq a_{ij}^{(n-1)} (i \neq j)$ by equality (3) and Proposition 1(iii).

(ii) As $i \neq j$, we consider the general term $a_{1\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n\sigma(n)}$ in equality (3). We have

$$a_{1\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n\sigma(n)} \leq a_{i\sigma(i)} a_{\sigma(i)\sigma^2(i)} \cdots a_{\sigma^t(i)j} a_{jj}^{n-2-t}.$$

by the assumption and commutativity. Therefore,

$$|A_{ji}| \leq \sum_{1 \leq j_1, j_2, \dots, j_{n-2} \leq n} a_{ij_1} a_{j_1 j_2} \cdots a_{j_{n-2} j} = a_{ij}^{(n-1)} \quad (i \neq j).$$

As $i = j$, by the assumption we can see that $|A_{ii}| = a_{11}^{n-1} = a_{ii}^{(n-1)}$. The proof is completed. \square

Under the condition of Theorem 4(i), there is no definite relation between $|A_{ii}|$ and $a_{ii}^{(n-1)}$ (Example 4). Compared with Thomamon's result for fuzzy matrices, the condition and the conclusion of Theorem 4(i) are all weak.

Over an incline whether the equality $A^{n-1} = \text{adj}(A)$ is still hold for a matrix A satisfying $a_{ii} \geq a_{jk} (1 \leq i, j, k \leq n)$ is not known. But we can prove

Theorem 5. Let R be a commutative incline, $A \in M_n(R) (n \geq 2)$ and $A \geq I_n$, then $A^{n-1} = \text{adj}(A)$.

Proof. By Theorem 4(ii) it suffices to prove that $a_{ij}^{(n-1)} \leq |A_{ji}| (i \neq j)$. We consider an arbitrary summand $a_{ij_1} a_{j_1 j_2} \cdots a_{j_{n-2} j}$ of the expansion of $a_{ij}^{(n-1)} (i \neq j)$ according to equality (1). If the subscripts $i, j_1, \dots, j_{n-2}, j$ are pairwise different, then this summand is equal to some term in expansion (3) of $|A_{ji}|$ due to the commutativity of R . If there are two identical numbers in the subscripts $i, j_1, \dots, j_{n-2}, j$, we apply the deleting method used in the proof of Lemma 1. The method can be applied repeatedly until the subscripts left are pairwise different. Therefore, by Proposition 1(iv) and (v),

$$a_{ij_1} a_{j_1 j_2} \cdots a_{j_{n-2} j} \leq a_{il_1} a_{l_1 l_2} \cdots a_{l_m j} \quad (0 \leq m < n - 2),$$

where $i, l_1, l_2, \dots, l_m, j$ are pairwise different. The right-hand side of the above inequality becomes a_{ij} in the case $m = 0$. Let

$$\{p_1, \dots, p_{n-m-2}\} = \{1, 2, \dots, n\} \setminus \{i, l_1, l_2, \dots, l_m, j\}.$$

Since $a_{kk} = 1$ ($k = 1, 2, \dots, n$), we have

$$a_{ij_1} a_{j_1 j_2} \dots a_{j_{n-2} j} \leq a_{i l_1} a_{l_1 l_2} \dots a_{l_{m-1} j} a_{p_1 p_1} a_{p_2 p_2} \dots a_{p_{n-m-2} p_{n-m-2}}.$$

Again, the right-hand side is equal to some term in expansion (3) of $|A_{ji}|$.

Thus, we get $a_{ij_1} a_{j_1 j_2} \dots a_{j_{n-2} j} \leq |A_{ji}|$ by Proposition 1(iii). That is, each term in the expansion of $a_{ij}^{(n-1)}$ according to equality (1) is not more than $|A_{ji}|$. Therefore, $a_{ij}^{(n-1)} \leq |A_{ji}|$ by Proposition 1(iii). The proof is completed. \square

Corollary 2. Let R be a commutative incline, $A \in M_n(R)$ ($n \geq 2$) and $A \geq I_n$. Then

- (i) $\text{adj}(A)$ is idempotent and $\text{adj}(A) \geq I_n$.
- (ii) A is idempotent if and only if $\text{adj}(A) = A$.
- (iii) $\text{adj}(\text{adj}(A)) = \text{adj}(A)$.
- (iv) $A \text{adj}(A) = \text{adj}(A)A = \text{adj}(A)$.

Proof. (i) and (ii) follow from Corollary 1 and Theorem 5. (iii) is yielded from the results (i) and (ii). We prove (iv) in the following. Let $B = A \text{adj}(A)$. Then $b_{ij} = \sum_{k=1}^n a_{ik} |A_{jk}| \geq a_{ii} |A_{ji}| = |A_{ji}|$. That is, $A \text{adj}(A) \geq \text{adj}(A)$. Conversely, by Corollary 2(i), Theorem 5 and the assumption $A \geq I_n$ we see that $A \text{adj}(A) \leq A^{n-1} \text{adj}(A) = (\text{adj}(A))^2 = \text{adj}(A)$. We get $A \text{adj}(A) = \text{adj}(A)$. The proof of equality $\text{adj}(A)A = \text{adj}(A)$ is similar. \square

Ragab and Emam [9] gave the results in Corollary 2 for a max–min fuzzy matrix.

Finally, for a matrix over a distributive lattice we give

Theorem 6. Let L be a distributive lattice and $A \in M_n(L)$ ($n \geq 2$). If the entries of A satisfy $a_{ii} \geq a_{jk}$ ($1 \leq i, j, k \leq n$), then $A^{n-1} = \text{adj}(A)$.

Proof. Since a distributive lattice is a commutative incline, using Theorem 4(ii) we only need to show $a_{ij}^{(n-1)} \leq |A_{ji}|$ ($i \neq j$). Since $a \leq b$ ($a, b \in L$) implies $ab = a$ we can obtain $a_{ij}^{(n-1)} \leq |A_{ji}|$ ($i \neq j$) under the condition $a_{ii} \geq a_{jk}$ ($1 \leq i, j, k \leq n$) using the similar method as in the proof of Theorem 5.

4. Conclusion

An incline is a more general algebraic structure than a distributive lattice. This paper studied the transitive closure, convergence of powers, determinant and adjoint of a square matrix over an incline. Some results about a matrix over max–min fuzzy algebra or distributive lattice are proved to be tenable over an incline. The relation between $\text{adj}(A)$ and A^{n-1} is discussed in Theorem 4(i) under weak condition compared with that about max–min fuzzy matrix. This paper illustrate some problems about fuzzy matrices or lattice matrices can be discussed on an incline.

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