HARMONIC MORPHISMS AND HYPERELLIPTIC GRAPHS

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ABSTRACT. We study harmonic morphisms of graphs as a natural discrete analogue of holomorphic maps between Riemann surfaces. We formulate a graph-theoretic analogue of the classical Riemann-Hurwitz formula, study the functorial maps on Jacobians and harmonic 1-forms induced by a harmonic morphism, and present a discrete analogue of the canonical map from a Riemann surface to projective space. We also discuss several equivalent formulations of the notion of a hyperelliptic graph, all motivated by the classical theory of Riemann surfaces. As an application of our results, we show that for a 2-edge-connected graph G which is not a cycle, there is at most one involution ι on G for which the quotient G/ι is a tree. We also show that the number of spanning trees in a graph G is even if and only if G admits a non-constant harmonic morphism to the graph B_2 consisting of 2 vertices connected by 2 edges. Finally, we use the Riemann-Hurwitz formula and our results on hyperelliptic graphs to classify all hyperelliptic graphs having no Weierstrass points.

1. INTRODUCTION

1.1. **Notation and terminology.** Throughout this paper, a *Riemann surface* will mean a compact, connected one-dimensional complex manifold, and (unless otherwise specified) a *graph* will mean a finite, connected multigraph without loop edges. A graph with no multiple edges is called *simple*. We will denote by $V(G)$ and $E(G)$, respectively, the set of vertices and edges of G. For a vertex $x \in V(G)$ and an edge $e \in E(G)$, we write $x \in e$ if e is incident to x.

We denote by $g(G) := |E(G)| - |V(G)| + 1$ the *genus* of G; this is the dimension of the cycle space of G. (In graph theory, the term "genus" is traditionally used for a different concept, namely, the smallest genus of any surface in which the graph can be embedded, and the integer $g = g(G)$ is called the "cyclomatic number" of G. We call g the genus of G in order to highlight the analogy with Riemann surfaces.)

For $k \geq 2$, a graph G is called *k-edge-connected* if $G - W$ is connected for every set W of at most $k - 1$ edges of G. (By convention, we consider the trivial graph having one vertex and no edges to be k-edge-connected for all k.) Alternatively, define a *cut* to be the set of all edges connecting a vertex in V_1 to a vertex in V_2 for some partition of $V(G)$ into disjoint subsets V_1 and V_2. Then G is k-edge-connected if and only if every non-empty cut has size at least k.

A *bridge* is an edge of G whose deletion increases the number of connected components of G. A (connected) graph is 2-edge-connected if and only if it contains no bridge.

Finally, if $A \subseteq V(G)$, we denote by $\chi_A : V(G) \to \{0, 1\}$ the characteristic function of A, and for $x \in A$ we let $\text{outdeg}_A(x)$ denote the number of edges $e = xy$ in $E(G)$ with $y \notin A$.

1.2. **Motivation and discussion of main results.** In [BN], the authors investigated some new analogies between graphs and Riemann surfaces, formulating the notion of a *linear system* on a graph and proving a graph-theoretic analogue of the classical Riemann-Roch theorem. The theory of linear systems on graphs has applications to understanding the *Jacobian of a finite graph*, a group which is analogous to the Jacobian of a Riemann surface, and which has appeared in many

different guises throughout the literature (e.g., as the "Picard group" in [BdlHN97], the "critical group" in [Big97], the "sandpile group" in [Dha90], and the "group of components" in [Lor91]).

The present paper can be viewed as a natural sequel to [BN]. In classical algebraic geometry, one is usually interested not just in Riemann surfaces themselves, but also in the holomorphic maps between them. Thus, we are naturally led to ask: what is the "correct" graph-theoretic analogue of a holomorphic map between Riemann surfaces? In other words, is there a category consisting of graphs, together with certain maps between them, which closely mirrors the category of Riemann surfaces, together with the holomorphic maps between them? In this paper, we hope to convince the reader that the notion of a *harmonic morphism of graphs*, introduced by Urakawa in [Ura00], has essentially all of the desired features.

Actually, since we want to allow graphs with multiple edges, we need to slightly modify the definition of a harmonic morphism from [Ura00], since Urakawa assumes that all of his graphs are simple. Recall that a holomorphic map $\phi : X \to X'$ between Riemann surfaces is one which locally pulls back holomorphic functions on X' to holomorphic functions on X. Although the notion of a holomorphic function does not make sense in the context of graphs, there is a natural notion of a *harmonic function* (see (1.7) below for the definition). Urakawa defines a harmonic morphism $\phi : V(G) \to V(G')$ between simple graphs G, G' to be a function which locally pulls back harmonic functions on G' to harmonic functions on G, i.e., a function such that for every $x \in V(G)$ and every function $f : V(G') \to \mathbb{R}$ which is harmonic at $\phi(x)$, the composition $f \circ \phi$ is harmonic at x. What makes this a workable and useful notion is Theorem 2.5 from [Ura00], which asserts that a function $\phi : V(G) \to V(G')$ between simple graphs is a harmonic morphism if and only if ϕ is *horizontally conformal*, meaning that:

(HC1) For all adjacent vertices $x, y \in V(G)$, we have either $\phi(x) = \phi(y)$ or $\phi(x)$ is adjacent to $\phi(y)$, and

(HC2) For all $x \in V(G)$, the quantity

$$|\{y \in V(G) \mid \phi(y) = y' \text{ and } y \text{ is adjacent to } x\}|$$

is the same for all $y' \in V(G')$ adjacent to $x' := \phi(x)$.

However, the equivalence between harmonic morphisms and horizontally conformal maps fails for graphs which are not simple (c.f. Remark 2.10 below). Because of this, we take an analogue of (HC1) and (HC2) as our definition of a harmonic morphism between multigraphs (see Definition 2.1 for a precise definition). A harmonic morphism in this sense does indeed pull back harmonic functions to harmonic functions, but the converse does not always hold.

One of the key features of defining harmonic morphisms in terms of horizontal conformality is that given a harmonic morphism $\phi : G \to G'$, it is possible to assign a well-defined *multiplicity* $m_\phi(x)$ to each vertex $x \in V(G)$ (analogous to the *ramification index* $e_\phi(x)$ at $x \in X$ for a non-constant holomorphic map $\phi : X \to X'$ between Riemann surfaces) in such a way that the sum $\deg(\phi)$ of the multiplicities at all vertices mapping to a given vertex $x' \in V(G')$ is independent of x'. We define the *degree* of ϕ to be this number.

Harmonic morphisms between graphs enjoy numerous properties analogous to classical properties from algebraic geometry. For example, if $\phi : X \to X'$ is a non-constant holomorphic map of degree $\deg(\phi)$ between Riemann surfaces having genus g and g', respectively, then:

(RS1) ϕ is surjective and $g \geq g'$.

(RS2) The *Riemann-Hurwitz* formula $2g - 2 = \deg(\phi)(2g' - 2) + \sum_{x \in X} (e_\phi(x) - 1)$ holds.

(RS3) ϕ induces functorial maps $\phi_* : \mathrm{Jac}(X) \to \mathrm{Jac}(X')$ and $\phi^* : \mathrm{Jac}(X') \to \mathrm{Jac}(X)$ between the Jacobians of X and X'.

(RS4) ϕ induces functorial maps $\phi_* : \Omega^1(X) \to \Omega^1(X')$ and $\phi^* : \Omega^1(X') \to \Omega^1(X)$ between the complex vector spaces $\Omega^1(X)$ and $\Omega^1(X')$ of holomorphic 1-forms on X and X', respectively.

(RS5) If D is a divisor on X, then $\dim |\phi_*(D)| \geq \dim |D|$, where $|D|$ denotes the complete linear system associated to D. In particular, if X is hyperelliptic and $g(X') \geq 2$, then X' is hyperelliptic as well.

We will prove graph-theoretic analogues of all of these classical facts. We will also describe some situations in which the naive analogue of certain classical facts does not hold. For example, in algebraic geometry the map $\phi^* : \mathrm{Jac}(X') \to \mathrm{Jac}(X)$ is sometimes injective and sometimes not; more precisely, it is known that ϕ^* fails to be injective if and only if ϕ has a nontrivial unramified abelian subcover. However, the analogous map $\phi^* : \mathrm{Jac}(G') \to \mathrm{Jac}(G)$ in the graph-theoretic context turns out to always be injective; this appears to be a rather subtle fact with some useful applications.

As a basic testing ground for our "dictionary" between graphs and Riemann surfaces, we consider in detail the graph-theoretic analogue of a *hyperelliptic* Riemann surface. This is particularly interesting because classically, there are many different equivalent characterizations of what it means for a Riemann surface X of genus at least 2 to be hyperelliptic. As just a sample, we mention the following:

(H1) There exists a divisor D of degree 2 on X for which $r(D) := \dim |D|$ is equal to 1.

(H2) There exists an involution ι for which X/ι has genus 0.

(H3) There is a degree 2 holomorphic map $\phi : X \to \mathbb{P}^1$.

(H4) There is an automorphism ι of X for which $\iota^* : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$ is multiplication by -1.

(H5) There is an automorphism ι of X for which $\iota^* : \Omega^1(X) \to \Omega^1(X)$ is multiplication by -1.

(H6) The symmetric square $S_{x_0}^{(2)} : \mathrm{Div}_+^2(X) \to \mathrm{Jac}(X)$ of the Abel-Jacobi map (relative to some base point $x_0 \in X$) is not injective.

(H7) The canonical map $\psi_X : X \to \mathbb{P}(\Omega^1(X))$ is not injective.

When any of these equivalent conditions are satisfied, there is a unique automorphism ι satisfying (H2), (H4), and (H5), called the *hyperelliptic involution*.

For a 2-edge-connected graph G of genus at least 2, we take the analogue of (H1) to be the definition of what it means for G to be hyperelliptic. (This definition was already introduced in [Bak07].) We then prove that the graph-theoretic analogues of conditions (H1)-(H5) above are all equivalent to one another, and that the hyperelliptic involution ι on a graph satisfying any of these conditions is unique. However, in the graph-theoretic context it turns out that (H1) $\Rightarrow$ (H6) $\Leftrightarrow$ (H7), so that hyperelliptic graphs satisfy the analogues of conditions (H6) and (H7), but there are non-hyperelliptic 2-edge-connected graphs G of genus at least 2 which also satisfy these conditions. In fact, we will see that the graph-theoretic analogues of conditions (H6) and (H7) are equivalent to the condition that G is not 3-edge-connected.

As an application of our results, and to illustrate another difference with the theory of Riemann surfaces, we conclude our paper with a discussion of Weierstrass points on hyperelliptic graphs. (The notion of a Weierstrass points on graphs was introduced in [Bak07]; see §5 for a definition.) Classically, a hyperelliptic Riemann surface of genus $g \geq 2$ possesses exactly $2g + 2$ Weierstrass points, namely, the fixed points of the hyperelliptic involution, and every Riemann surface of genus at least 2 has Weierstrass points. The situation for graphs is less orderly, as there are infinite families of graphs having no Weierstrass points at all. Using our rather precise knowledge about

the structure of hyperelliptic graphs, we give a classification of all hyperelliptic graphs having no Weierstrass points. We leave as an open problem whether or not there exist further (non-hyperelliptic) examples of Weierstrass-pointless graphs.

Occasionally, our foundational results on harmonic morphisms and hyperelliptic graphs lead to applications to more traditional-sounding graph-theoretic questions. For example, as a consequence of our study of hyperelliptic graphs, we will show that for a 2-edge-connected graph G of genus at least 2, there is at most one involution ι on G whose quotient is a tree. We also show that the number κ_G of spanning trees in a graph G is even if and only if G admits a non-constant degree 2 harmonic morphism to the graph B_2 consisting of 2 vertices connected by 2 edges.

Although in this paper we view our graph-theoretic results as "analogous" to classical results from algebraic geometry, there is in fact a closer connection between the two worlds than one might at first imagine. One such connection arises from the specialization of divisors on an arithmetic surface, and is explored in [Bak07]. We expect that the ideas in the present paper will help spur further interactions between graph theory, on the one hand, and arithmetic, algebraic, and tropical geometry on the other.

It would be interesting to prove analogues of the results in the present paper for metric graphs, and more generally for tropical curves, but we have not attempted to do so here. It would also be interesting to generalize some of our results to higher dimensions. At least in the context of Riemannian polyhedra (which are higher-dimensional analogues of metric graphs), there is already a rich literature concerning the notion of a harmonic morphism (see, e.g., [EF01]). However, it appears that the questions being addressed in [EF01] and the references therein are of a somewhat different flavor than the ones which we study here.

We have endeavored to make this paper as self-contained as possible. Therefore, we summarize in §1.3 below all of the facts from [BN] which we will be using. We have also rewritten certain proofs from [Ura00], because our notation differs somewhat from Urakawa's, and because we work in the somewhat more general setting of multigraphs. A good reference for many of the facts about Riemann surfaces which we will be discussing in this paper is [Mir95], and a basic graph theory reference is [Bol98].

1.3. **Background material from** [BN]**.** Following [BN], we denote by $\mathrm{Div}(G)$ the free abelian group on $V(G)$. We refer to elements of $\mathrm{Div}(G)$ as *divisors* on G. We can write each element $D \in \mathrm{Div}(G)$ uniquely as $D = \sum_{x \in V(G)} D(x)(x)$ with $D(x) \in \mathbb{Z}$. We say that D is *effective*, and write $D \geq 0$, if $D(x) \geq 0$ for all $x \in V(G)$. For $D \in \mathrm{Div}(G)$, we define the *degree* of D by the formula $\deg(D) = \sum_{x \in V(G)} D(x)$. We denote by $\mathrm{Div}^0(G)$ the subgroup of $\mathrm{Div}(G)$ consisting of divisors of degree zero. Finally, we denote by $\mathrm{Div}^k_+(G) = \{E \in \mathrm{Div}(G) \ : \ E \geq 0, \deg(E) = k\}$ the set of effective divisors of degree k on G.

Let $C^0(G, \mathbb{Z})$ be the group of $\mathbb{Z}$-valued functions on $V(G)$. For $f \in C^0(G, \mathbb{Z})$, we define the *divisor* of f by the formula

$$\mathrm{div}(f) = \sum_{x \in V(G)} \sum_{e = xy \in E(G)} (f(x) - f(y))\, (x).$$

The divisor of f can be naturally identified with the graph-theoretic Laplacian of f. Divisors of the form $\mathrm{div}(f)$ for some $f \in C^0(G, \mathbb{Z})$ are called *principal*; we denote by $\mathrm{Prin}(G)$ the group of principal divisors on G. It is easy to see that every principal divisor has degree zero, so that $\mathrm{Prin}(G)$ is a subgroup of $\mathrm{Div}^0(G)$.

The Jacobian of G, denoted $\mathrm{Jac}(G)$, is defined to be the quotient group

$$\mathrm{Jac}(G) = \mathrm{Div}^0(G)/\mathrm{Prin}(G).$$

One can show using Kirchhoff's Matrix-Tree Theorem (c.f. [Big97, §14]) that $\mathrm{Jac}(G)$ is a finite abelian group of order κ_G, where κ_G is the number of spanning trees in G.

We define an equivalence relation $\sim_G$ on $\mathrm{Div}(G)$ by writing $D \sim_G D'$ if and only if $D - D' \in \mathrm{Prin}(G)$, and set

$$|D| = \{E \in \mathrm{Div}(G) \ : \ E \geq 0 \text{ and } E \sim_G D\}.$$

We refer to $|D|$ as the *(complete) linear system* associated to D, and when $D \sim D'$ we call the divisors D and D' *linearly equivalent*. We will usually just write $D \sim D'$, rather than $D \sim_G D'$, when the graph G is understood.

For later use, we note the following simple fact about the linear equivalence relation on G:

Lemma 1.1. *We have $(x) \sim (y)$ for all $x, y \in V(G)$ if and only if G is a tree.*

Proof. This follows from the fact that $|\mathrm{Jac}(G)| = \kappa_G$, together with the observation that the group $\mathrm{Div}^0(G)$ is generated by the divisors of the form $(x) - (y)$ with $x, y \in V(G)$. $\qquad\square$

Given a divisor D on G, define $r(D) = -1$ if $|D| = \emptyset$, and otherwise set

$$r(D) = \max\{k \in \mathbb{Z} \ : \ |D - E| \neq \emptyset \ \forall \ E \in \mathrm{Div}_+^k(G)\}.$$

Note that $r(D)$ depends only on the linear equivalence class of D, and therefore is an invariant of the linear system $|D|$. When we wish to emphasize the underlying graph G, we will sometimes write $r_G(D)$ instead of $r(D)$.

For later use, we recall from [BN, Lemma 2.1] the following simple lemma:

Lemma 1.2. *For all $D, D' \in \mathrm{Div}(G)$ such that $r(D), r(D') \geq 0$, we have $r(D+D') \geq r(D)+r(D')$.*

We define the *canonical divisor* on G to be

$$K_G = \sum_{x \in V(G)} (\deg(x) - 2)(x).$$

We have $\deg(K_G) = 2g - 2$, where $g = |E(G)| - |V(G)| + 1$ is the *genus* of G (or, in more traditional language, *cyclomatic number* of G).

The following result is proved in [BN, Theorem 1.12]:

Theorem 1.3 (Riemann-Roch for graphs). *Let D be a divisor on a graph G. Then*

$$r(D) - r(K_G - D) = \deg(D) + 1 - g.$$

As a consequence of Lemma 1.2 and Theorem 1.3, we have the following graph-theoretic analogue of a classical result known as Clifford's theorem (see [BN, Corollary 3.5] for a proof):

Corollary 1.4 (Clifford's Theorem for graphs). *Let D be a divisor on a graph G for which $|D| \neq \emptyset$ and $|K_G - D| \neq \emptyset$. Then*

$$r(D) \leq \frac{1}{2}\deg(D) \ .$$

The next result (Theorem 3.3 from [BN]) is very useful for computing $r(D)$ in specific examples, and also plays an important role in the proof of Theorem 1.3. For each linear ordering $<$ on $V(G)$, we define a corresponding divisor $\nu \in \mathrm{Div}(G)$ of degree $g - 1$ by the formula

$$\nu = \sum_{x \in V(G)} (|\{e = xy \in E(G) \ : \ y < x\}| - 1)(x).$$

Theorem 1.5. *For every $D \in \mathrm{Div}(G)$, exactly one of the following holds:*

    (1) $r(D) \geq 0$; *or*
    (2) $r(\nu - D) \geq 0$ *for some divisor ν associated to a linear ordering $<$ of $V(G)$.*

Finally, we recall some facts from [BN] and [BdlHN97] about the graph-theoretic analogue of the Abel-Jacobi map from a Riemann surface to its Jacobian.

For a fixed base point $x_0 \in V(G)$, we define the *Abel-Jacobi map* $S_{x_0} : G \to \mathrm{Jac}(G)$ by the formula

$$(1.6) \qquad\qquad\qquad S_{x_0}(x) = [(x) - (x_0)] \ .$$

The map S_{x_0} can be characterized by the following universal property (see §3 of [BdlHN97]). A map $\varphi : G \to A$ from $V(G)$ to an abelian group A is called *harmonic* if for each $x \in V(G)$, we have

$$(1.7) \qquad\qquad\qquad \deg(x) \cdot \varphi(x) = \sum_{e = xy \in E(G)} \varphi(y) \ .$$

Then S_{x_0} is universal among all harmonic maps from G to abelian groups sending x_0 to 0, in the following precise sense:

Lemma 1.8. *If $\varphi : G \to A$ is any map sending $x_0 \in V(G)$ to 0, then there is a unique group homomorphism $\psi : \mathrm{Jac}(G) \to A$ such that $\varphi = \psi \circ S_{x_0}$.*

We also define, for each integer $k \geq 1$, a map $S_{x_0}^{(k)} : \mathrm{Div}_+^k(G) \to \mathrm{Jac}(G)$ by

$$S_{x_0}^{(k)}((x_1) + \cdots + (x_k)) = S_{x_0}(x_1) + S_{x_0}(x_2) + \cdots + S_{x_0}(x_k) \ .$$

The following result is proved in [BN, Theorem 1.8]:

Theorem 1.9. *The map $S_{x_0}^{(k)}$ is injective if and only if G is $(k + 1)$-edge-connected.*

2. Harmonic morphisms

2.1. **Definition and basic properties of harmonic morphisms.** Harmonic morphisms between simple graphs were defined and studied in [Ura00]. Here, we reproduce some definitions from [Ura00], but with several variations due to the fact that we allow our graphs to have multiple edges.

Let G, G' be graphs. A function $\phi : V(G) \cup E(G) \to V(G') \cup E(G')$ is said to be a *morphism* from G to G' if $\phi(V(G)) \subseteq V(G')$, and for every $x \in V(G)$ and $e \in E(G)$ such that $x \in e$, either $\phi(e) \in E(G')$ and $\phi(x) \in \phi(e)$, or $\phi(e) = \phi(x)$. We write $\phi : G \to G'$ for brevity. If $\phi(E(G)) \subseteq E(G')$ then we say that ϕ is a *homomorphism*. A bijective homomorphism is called an *isomorphism*, and an isomorphism $\phi : G \to G$ is called an *automorphism*.

We now come to the key definition in this paper.

Definition. A morphism $\phi : G \to G'$ is said to be *harmonic* (or *horizontally conformal*) if for all $x \in V(G), y \in V(G')$ such that $y = \phi(x)$, the quantity $|\{e \in E(G)|x \in e, \ \phi(e) = e'\}|$ is the same for all edges $e' \in E(G')$ such that $y \in e'$.

Remark 2.1. One can check directly from the definition that the composition of two harmonic morphisms is again harmonic. Therefore the set of all graphs, together with the harmonic morphisms between them, forms a category.

Let $\phi : G \to G'$ be a morphism and let $x \in V(G)$. Define the *vertical multiplicity* of ϕ at x by

$$v_\phi(x) = |\{e \in E(G) \,|\phi(e) = \phi(x)\}|.$$

This is simply the number of *vertical edges* incident to x, where an edge e is called *vertical* if $\phi(e) \in V(G)$ (and is called *horizontal* otherwise).

If ϕ is harmonic and $|V(G')| > 1$, we define the *horizontal multiplicity* of ϕ at x by

$$m_\phi(x) = |\{e \in E(G)|x \in e, \ \phi(e) = e'\}|$$

for any edge $e' \in E(G)$ such that $\phi(x) \in e'$. By the definition of a harmonic morphism, $m_\phi(x)$ is independent of the choice of e'. When $|V(G')| = 1$, we define $m_\phi(x)$ to be 0 for all $x \in V(G)$.

If $\deg(x)$ denotes the degree of a vertex x, we have the following basic formula relating the horizontal and vertical multiplicities:

$$(2.2) \qquad\qquad \deg(x) = \deg(\phi(x))m_\phi(x) + v_\phi(x).$$

We say that a harmonic morphism $\phi : G \to G'$ is *non-degenerate* if $m_\phi(x) \geq 1$ for every $x \in V(G)$. (The motivation for this definition comes from Theorem 5.12 below.)

If $|V(G')| > 1$, we define the *degree* of a harmonic morphism $\phi : G \to G'$ by the formula

$$(2.3) \qquad\qquad \deg(\phi) := |\{e \in E(G) \mid \phi(e) = e'\}|$$

for any edge $e' \in E(G')$. (When $|V(G')| = 1$, we define $\deg(\phi)$ to be 0.) By the following lemma (c.f. [Ura00, Lemma 2.12]), the right-hand side of (2.3) does not depend on the choice of e' (and therefore $\deg(\phi)$ is well-defined):

Lemma 2.4. *The quantity* $|\{e \in E(G) \mid \phi(e) = e'\}|$ *is independent of the choice of* $e' \in E(G')$.

Proof. Let $y \in V(G')$, and suppose there are two edges $e', e'' \in E(G')$ incident to y. Since ϕ is horizontally conformal, for each $x \in V(G)$ with $\phi(x) = y$ we have

$$|\{e \in E(G) \mid x \in e, \ \phi(e) = e'\}| = |\{\tilde{e} \in E(G) \mid x \in \tilde{e}, \ \phi(\tilde{e}) = e''\}|.$$

Therefore

$$
\begin{aligned}
|\{e \in E(G) \mid \phi(e) = e'\}| &= \sum_{x \in \phi^{-1}(y)} |\{e \in E(G) \mid x \in e, \ \phi(e) = e'\}| \\
(2.5) \qquad &= \sum_{x \in \phi^{-1}(y)} |\{\tilde{e} \in E(G) \mid x \in \tilde{e}, \ \phi(\tilde{e}) = e''\}| \\
&= |\{\tilde{e} \in E(G) \mid \phi(\tilde{e}) = e''\}|.
\end{aligned}
$$

Now suppose e', e'' are arbitrary edges of G'. Since G is connected, the result follows by applying (2.5) to each pair of consecutive edges in any path connecting e' and e''. $\qquad\square$

According to the next result, the degree of a harmonic morphism $\phi : G \to G'$ is just the number of preimages under ϕ of any vertex of G', counting multiplicities:

Lemma 2.6. *For any vertex $y \in G'$, we have*

$$\deg(\phi) = \sum_{\substack{x \in V(G) \\ \phi(x)=y}} m_\phi(x).$$

Proof. Choose an edge $e' \in E(G')$ with $y \in e'$. Then

$$\sum_{x \in \phi^{-1}(y)} m_\phi(x) = \sum_{x \in \phi^{-1}(y)} \sum_{e \in \phi^{-1}(e'),\, x \in e} 1$$
$$= |\phi^{-1}(e')| = \deg(\phi).$$

$\square$

As with morphisms of Riemann surfaces in algebraic geometry, a harmonic morphism of graphs must be either constant or surjective. More generally, one has the following:

Lemma 2.7. *Let $\phi : G \to G'$ be a harmonic morphism with $|V(G')| > 1$. Then $\deg(\phi) = 0$ if and only if ϕ is constant, and $\deg(\phi) > 0$ if and only if ϕ is surjective.*

Proof. If ϕ is constant, then clearly $\deg(\phi) = 0$. Moreover, it follows from Lemmas 2.4 and 2.6 that ϕ is surjective if and only if $\deg(\phi) > 0$. So it remains only to show that if $\deg(\phi) = 0$, then ϕ is constant. For this, suppose we have $\phi(x) = y$. Since $m_\phi(x) = 0$, it follows that $\phi(e) = y$ for every edge e with $x \in e$. Thus $\phi(x') = y$ for every neighbor x' of x. As G is connected, it follows that every vertex and every edge of G is mapped under ϕ to y. $\square$

2.2. **Harmonic morphisms and harmonic maps to abelian groups.** Recall that given a graph G and an abelian group A, a function $f : V(G) \to A$ is said to be *harmonic* at $x \in V(G)$ if

$$\sum_{e=xy \in E(G)} (f(x) - f(y)) = 0.$$

A morphism $\phi : G \to G'$ is said to be *A-harmonic* if for any $y = \phi(x)$ and any function $f : V(G') \to A$ harmonic at y, the function $f \circ \phi$ is harmonic at x.

Proposition 2.8. *Let G and G' be graphs, and let $\phi : G \to G'$ be a harmonic morphism. Then ϕ is A-harmonic for every abelian group A.*

Proof. (c.f. Lemma 2.11 of [Ura00]) Let $x \in V(G)$, $y \in V(G')$ be such that $y = \phi(x)$, and let $f : V(G') \to A$ be harmonic at y, i. e.

$$\sum_{e=zy \in E(G')} f(z) = \deg(y)f(y).$$

Then we have

$$\sum_{e=zx\in E(G)} f(\phi(z)) = \sum_{\substack{e=zx\in E(G) \\ \phi(e)=y}} f(\phi(z)) + \sum_{e'=z'y\in E(G')} \left(\sum_{\substack{e=zx\in E(G) \\ \phi(e)=e'}} f(\phi(z)) \right)$$

$$= v_\phi(x)f(y) + \sum_{e'=z'y\in E(G')} m_\phi(x)f(z')$$

$$= v_\phi(x)f(y) + m_\phi(x)\deg(y)f(y)$$

$$= (v_\phi(x) + m_\phi(x)\deg(\phi(x)))f(\phi(x))$$

$$= \deg(x)f(\phi(x)) \qquad\qquad \text{(by (2.2)),}$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

If G' is a simple graph (i.e., without multiple edges), then the converse of Proposition 2.8 also holds:

Proposition 2.9. *If G' is a simple graph, then for a morphism $\phi : G \to G'$, the following are equivalent:*

   (1) *ϕ is harmonic (i.e., horizontally conformal).*
   (2) *ϕ is A-harmonic for every abelian group A.*
   (3) *ϕ is $\mathbb{R}$-harmonic.*

Proof. It follows from Proposition 2.8 that (1) implies (2), and it is immediate that (2) implies (3). It remains to show that (3) implies (1), which we do following [Ura00, Lemma 2.7].

For a vertex $x \in V(G)$ and an edge $e' \in E(G')$ such that $\phi(x) \in e'$, let

$$k_x(e') = |\{e \in E(G) | x \in e, \ \phi(e) = e'\}|.$$

We need to prove that $k_x(e')$ is independent of the choice of e'. Let $\phi(x) = y$, let $e' = yz$, and define a function $f_{e'} : V(G') \to \mathbb{R}$ as follows. Let $f_{e'}(z) = 1$, let $f_{e'}(y) = 1/\deg(y)$, and let $f_{e'}(w) = 0$ for $w \in V(G')\backslash\{y, z\}$. Then $f_{e'}$ is harmonic at y, so by (3), $f_{e'} \circ \phi$ is harmonic at x. It follows that

$$\frac{\deg(x)}{\deg(y)} = \deg(x)f_{e'}(\phi(x)) = \sum_{e=xw\in E(G)} f_{e'}(\phi(w))$$

$$= \sum_{\substack{e=xw\in E(G) \\ \phi(w)=y}} f_{e'}(y) + \sum_{\substack{e=xw\in E(G) \\ \phi(w)=z}} f_{e'}(z)$$

$$= \frac{v_\phi(x)}{\deg(y)} + k_x(e') \qquad\qquad \text{(since G' is simple).}$$

Therefore $k_x(e') = (\deg(x) - v_\phi(x))/\deg(\phi(x))$ is independent of the choice of e', as desired. $\quad\square$

Remark 2.10. If G' is not simple, then the converse of Proposition 2.9 may fail, as one sees easily by taking G to be the graph with 2 vertices x, y connected by a single edge e, G' to be the graph with 2 vertices x', y' connected by two edges e'_1, e'_2, and $\phi : G \to G'$ to be the morphism which sends x, y to x', y', respectively, and e to e'_1.

2.3. **The Riemann-Hurwitz formula for graphs.** Let $\phi : G \to G'$ be a harmonic morphism. We define the push-forward homomorphism $\phi_* : \mathrm{Div}(G) \to \mathrm{Div}(G')$ by

$$(2.11) \qquad\qquad \phi_*(D) = \sum_{x \in V(G)} D(x)(\phi(x)).$$

Similarly, we define the pullback homomorphism $\phi^* : \mathrm{Div}(G') \to \mathrm{Div}(G)$ by

$$(2.12) \qquad\qquad \phi^*(D') = \sum_{y \in V(G')} \sum_{\substack{x \in V(G) \\ \phi(x)=y}} m_\phi(x) D'(y)(x).$$

Lemma 2.13. *If $\phi : G \to G'$ is a harmonic morphism and $D' \in \mathrm{Div}(G')$, then $\deg(\phi^*(D')) = \deg(\phi) \cdot \deg(D')$.*

Proof. This follows from Lemma 2.6 and the definition of ϕ^*. $\qquad\qquad\square$

A basic fact about harmonic morphisms of graphs is that one has the following analogue of the classical Riemann-Hurwitz formula from algebraic geometry:

Theorem 2.14 (Riemann-Hurwitz for graphs). *Let G, G' be graphs, and let $\phi : G \to G'$ be a harmonic morphism. Then:*

   (1) *The canonical divisors on G and G' are related by the formula*

$$(2.15) \qquad\qquad K_G = \phi^* K_{G'} + R_G,$$

      where

$$R_G = 2 \sum_{x \in V(G)} (m_\phi(x) - 1)(x) + \sum_{x \in V(G)} v_\phi(x)(x).$$

   (2) *If G, G' have genus g and g', respectively, then*

$$(2.16) \qquad\qquad 2g - 2 = \deg(\phi)(2g' - 2) + \sum_{x \in V(G)} \left(2(m_\phi(x) - 1) + v_\phi(x) \right).$$

   (3) *If ϕ is non-constant, then $2g - 2 \geq \deg(\phi)(2g' - 2)$ and $g \geq g'$.*

Proof. By definition, we have $(\phi^* K_{G'})(x) = m_\phi(x)(\deg(\phi(x)) - 2)$. On the other hand, by (2.2) we have

$$K_G(x) = \deg(x) - 2 = \deg(\phi(x)) m_\phi(x) + v_\phi(x) - 2$$
$$= (\phi^* K_{G'})(x) + 2 m_\phi(x) + v_\phi(x) - 2 = (\phi^* K_{G'} + R_G)(x)$$

for every $x \in V(G)$, which proves (1). Part (2) follows immediately from Lemma 2.13 upon computing the degrees of the divisors on both sides of (2.15). In order to verify (3), we claim that if ϕ is non-constant then $\deg(R_G) \geq 0$. This is clear if G has no *vertical leaves* (i.e., degree 1 vertices x having $m_\phi(x) = 0$). On the other hand, suppose x is a vertical leaf, and let $e = xy$ be the unique edge with $x \in e$. Then if $\overline{G}$ is the graph obtained by contracting e to y, the induced map $\overline{G} \to G'$ is still harmonic and non-constant, and $\deg(R_{\overline{G}}) = \deg(R_G)$. Continuing in this way, we can reduce our claim to the already established case where G has no vertical leaves. $\qquad\square$

Remark 2.17. In the classical Riemann-Hurwitz formula from algebraic geometry, for a non-constant holomorphic map $\phi : X \to X'$ between Riemann surfaces of genus g and g', respectively, one has

$$2g - 2 = \deg(\phi)(2g' - 2) + \sum_{x \in X} \left(e_\phi(x) - 1 \right),$$

where $e_\phi(x)$ denotes the ramification index of ϕ at x. Note that there is no analogue in algebraic geometry of the "vertical multiplicities" $v_\phi(x)$, and there is an extra factor of 2 in the right-hand side of (2.16). Also, note that for Riemann surfaces one has a *linear equivalence* $K_X \sim \phi^* K_{X'} + R_X$ (which is all that can be expected, since there are just canonical *divisor classes* on X and X', not canonical divisors), but in (2.15) we have an actual equality of divisors.

3. Examples

In this section, we give some examples of harmonic and non-harmonic morphisms.

Example 3.1 (A harmonic morphism). The morphism shown in Figure 1 is harmonic, with horizontal and vertical multiplicities $m_\phi(x)$ and $v_\phi(x)$, respectively, labeled next to the corresponding vertices.
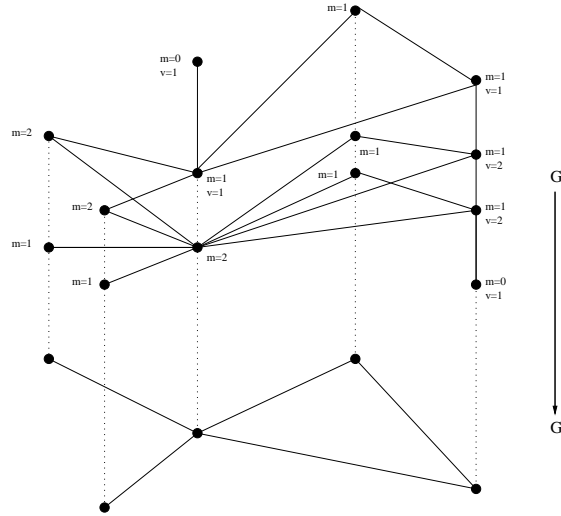

FIGURE 1. A harmonic morphism $\phi : G \to G'$ of degree 3.

Example 3.2 (Harmonic morphisms to trees). Every graph G admits a non-constant harmonic morphism to a tree. More precisely, suppose $|V(G)| \geq 2$ and let $x \in V(G)$ be a vertex of degree $k \geq 1$. Let T be the graph consisting of two vertices a, b connected by a single edge e', and let ϕ be the morphism sending x to a and every $y \in V(G)\backslash\{x\}$ to b, and sending an edge $e \in E(G)$ to e' if $x \in e$, and to b otherwise. Then ϕ is a harmonic morphism of degree k.

Example 3.3 (Automorphisms). A graph automorphism $\alpha : G \to G$ is a non-degenerate harmonic morphism of degree 1.

Example 3.4 (Coverings). A morphism $\phi : G \to G'$ is a *covering* of degree $d \geq 1$ if $\deg(x) = \deg(\phi(x))$ for every $x \in V(G)$ and $\phi^{-1}(e')$ consists of d disjoint edges for every edge $e' \in E(G')$. A covering is a harmonic morphism; more precisely, a covering morphism is the same thing as a harmonic morphism for which $m_\phi(x) = 1$ and $v_\phi(x) = 0$ for all $x \in V(G)$.

Example 3.5 (Collapsing). Let $p \in V(G)$ be a cut vertex, so that G can be partitioned into two subsets G_1 and G_2 which intersect only at p. We define the *collapsing* of G relative to G_1 to be the graph G' obtained by contracting all vertices and edges in G_1 to $\{p\}$. Let $\phi : G \to G'$ be the

morphism which sends G_1 to p and is the identity on G_2. Then if $|V(G_2)| > 1$, it is easy to see that ϕ is a harmonic morphism of degree 1 (c.f. [Ura00, Proposition 4.2]).

Example 3.6 (Contracting bridges is *not* harmonic). Let $e \in E(G)$ be a bridge, and let $\overline{G}$ be the graph obtained by contracting e. Then there is an evident *contraction morphism* $\rho : G \to \overline{G}$ which is surjective on both vertices and edges. However, ρ is not in general a harmonic morphism, as in Figure 2.
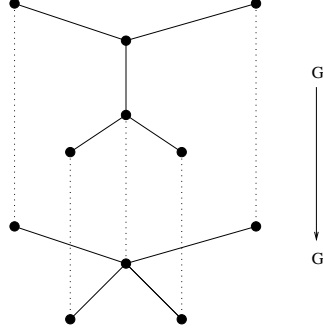

FIGURE 2. A non-harmonic morphism $\rho : G \to G'$.

4. Functorial properties

In this section, we discuss how harmonic morphisms between graphs induce different kinds of functorial maps between divisor groups, Jacobians, and harmonic 1-forms.

4.1. **Induced maps on Jacobians.** In §2.3, we introduced homomorphisms $\phi_* : \mathrm{Div}(G) \to \mathrm{Div}(G')$ and $\phi^* : \mathrm{Div}(G') \to \mathrm{Div}(G)$ associated to a harmonic morphism $\phi : G \to G'$. These homomorphisms are related by the following simple formula:

Lemma 4.1. *Let $\phi : G \to G'$ be a harmonic morphism, and let $D' \in \mathrm{Div}(G')$. Then $\phi_*(\phi^*(D')) = \deg(\phi)D'$.*

Proof. This follows from Lemma 2.6 and the definitions of ϕ_* and ϕ^*. $\qquad\qquad\square$

Suppose $\phi : G \to G'$ is a harmonic morphism and that $f : V(G) \to A$ and $f' : V(G') \to A$ are functions, where A is an abelian group. We define $\phi_* f : V(G') \to A$ by

$$\phi_* f(y) := \sum_{\substack{x \in V(G) \\ \phi(x)=y}} m_\phi(x) f(x)$$

and $\phi^* f' : V(G) \to A$ by

$$\phi^* f' := f' \circ \phi.$$

Proposition 4.2. *Let $\phi : G \to G'$ be a harmonic morphism, let $f : V(G) \to \mathbb{Z}$ and $f' \in V(G') \to \mathbb{Z}$. Then*

$$(4.3) \qquad\qquad\qquad\qquad \phi_*(\mathrm{div}(f)) = \mathrm{div}(\phi_* f)$$

and

$$(4.4) \qquad\qquad\qquad\qquad \phi^*(\mathrm{div}(f')) = \mathrm{div}(\phi^* f').$$

Proof. We start by proving (4.3). We have

$$\mathrm{div}(f) = \sum_{e=xy\in E(G)} (f(x) - f(y))((x) - (y)).$$

By the linearity of ϕ_*, we have

(4.5) $$\phi_*(\mathrm{div}(f)) = \sum_{e=xy\in E(G)} (f(x) - f(y))(\phi(x) - \phi(y)).$$

By the definition of $\phi_* f$, we have
(4.6)

$$\mathrm{div}(\phi_* f) = \sum_{e'=x'y'\in E(G')} \left(\sum_{x\in V(G),\, \phi(x)=x'} m_\phi(x)f(x) - \sum_{y\in V(G),\, \phi(y)=y'} m_\phi(y)f(y) \right) ((x') - (y')).$$

Note that terms in (4.5) corresponding to edges in $\phi^{-1}(V(G'))$ are zero. Therefore, to derive (4.3) from (4.5) and (4.6), it suffices to prove that

$$\sum_{e=xy\in\phi^{-1}(e')} (f(x) - f(y)) = \sum_{x\in V(G),\, \phi(x)=x'} m_\phi(x)f(x) - \sum_{y\in V(G),\, \phi(y)=y'} m_\phi(y)f(y)$$

for every edge $e' = x'y' \in E(G')$. This last identity holds by the definition of m_ϕ.

We now prove (4.4). Let $D' := \mathrm{div}(f')$. We have $D'(y) = \deg(y)f'(y) - \sum_{e=zy\in E(G')} f'(z)$ for every $y \in V(G')$, so by the definition of ϕ^*, we have

(4.7) $$(\phi^* D')(x) = m_\phi(x)D'(\phi(x)) = m_\phi(x)\deg(\phi(x))f'(\phi(x)) - m_\phi(x) \sum_{e=z\phi(x)\in E(G')} f'(z)$$

for every $x \in V(G)$. We now consider $\mathrm{div}(\phi^* f')(x)$. We have

$$\mathrm{div}(\phi^* f')(x) = \mathrm{div}(f' \circ \phi)(x) = \deg(x)f'(\phi(x)) - \sum_{e=xy\in E(G)} f'(\phi(y)).$$

By (2.2), we have

$$\deg(x)f'(\phi(x)) = m_\phi(x)\deg(\phi(x))f'(\phi(x)) + \sum_{e=xy\in E(G),\, \phi(y)=\phi(x)} f'(\phi(y)).$$

Therefore

(4.8) $$\mathrm{div}(\phi^* f')(x) = m_\phi(x)\deg(\phi(x))f'(\phi(x)) - \sum_{e=xy\in E(G),\, \phi(y)\neq\phi(x)} f'(\phi(y)).$$

Moreover, for every edge $e' = z\phi(x) \in E(G')$ we have

$$\sum_{e=xy,\, \phi(e)=e'} f'(\phi(y)) = m_\phi(x)f'(z),$$

and therefore

$$\sum_{e=xy\in E(G),\, \phi(y)\neq\phi(x)} f'(\phi(y)) = m_\phi(x) \sum_{e'=z\phi(x)\in E(G')} f'(z).$$

Thus (4.4) follows from (4.7) and (4.8). □

In particular:

Corollary 4.9. *If* $\phi : G \to G'$ *is a harmonic morphism, then* $\phi_*(\mathrm{Prin}(G)) \subseteq \mathrm{Prin}(G')$ *and* $\phi^*(\mathrm{Prin}(G')) \subseteq \mathrm{Prin}(G)$.

As a consequence of Corollary 4.9, we see that ϕ induces group homomorphisms (which we continue to denote by ϕ_*, ϕ^*)

$$\phi_* : \mathrm{Jac}(G) \to \mathrm{Jac}(G'),\ \phi^* : \mathrm{Jac}(G') \to \mathrm{Jac}(G).$$

It is straightforward to check that if $\psi : G \to G'$ and $\phi : G' \to G''$ are harmonic morphisms and $D \in \mathrm{Div}(G), D'' \in \mathrm{Div}(G'')$, then $\phi \circ \psi : G \to G''$ is harmonic, and we have $(\phi \circ \psi)_*(D) = \phi_*(\psi_*(D))$ and $(\phi \circ \psi)^*(D'') = \psi^*(\phi^*(D''))$. Therefore we obtain two different functors from the category of graphs (together with harmonic morphisms between them) to the category of abelian groups: a covariant "Albanese" functor $(G \mapsto \mathrm{Jac}(G), \phi \mapsto \phi_*)$ and a contravariant "Picard" functor $(G \mapsto \mathrm{Jac}(G), \phi \mapsto \phi^*)$. (This terminology comes from the corresponding notions in algebraic geometry.)

Remark 4.10. A more conceptual definition of the Albanese functor ϕ_* is as follows. Choose a base vertex $x_0 \in G$, and let $S = S_{x_0} : G \to \mathrm{Jac}(G)$ and $S' = S_{\phi(x_0)} : G' \to \mathrm{Jac}(G')$ denote the corresponding Abel-Jacobi maps. Since $S' : G' \to \mathrm{Jac}(G')$ is a harmonic function, it follows from Proposition 2.8 that the pullback $S' \circ \phi$ is a harmonic map from G to $\mathrm{Jac}(G')$. As $S' \circ \phi$ sends x_0 to 0, it follows from Lemma 1.8 that there is a unique homomorphism $\psi : \mathrm{Jac}(G) \to \mathrm{Jac}(G')$ such that $S' \circ \phi = \psi \circ S$. From the uniqueness of ψ, it follows easily that $\psi = \phi_*$.

In particular, a harmonic morphism $\phi : G \to G'$ gives rise to a commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\ \ \phi\ \ } & G' \\
{\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S'} \\
\mathrm{Jac}(G) & \xrightarrow{\ \ \phi_*\ \ } & \mathrm{Jac}(G')
\end{array}
$$

As an application of Corollary 4.9, we have the following result:

Corollary 4.11. *If $\phi : G \to G'$ is a non-constant harmonic morphism, then for every $D \in \mathrm{Div}(G)$ we have $r_{G'}(\phi_*(D)) \geq r_G(D)$.*

Proof. By Lemma 2.7, ϕ is surjective on vertices. Let $D' := \phi_*(D)$, and let k be a nonnegative integer. For every effective divisor $E' \in \mathrm{Div}(G')$ of degree k, we can choose $E \in \mathrm{Div}(G)$ such that $\phi_*(E) = E'$. If $r_G(D) \geq k$, then by definition $D - E = F + P$ with F effective and P principal. Applying ϕ_* and using the fact that $\phi_*(P) \in \mathrm{Prin}(G')$, we see that $D' - E'$ is equivalent to the effective divisor $\phi_*(F)$, and therefore $r_{G'}(D') \geq k$ as well. □

We now investigate some useful general properties of the induced maps ϕ_* and ϕ^* on Jacobians. In the classical algebraic geometry setting, ϕ_* is always surjective but ϕ^* is sometimes injective and sometimes not. More precisely, $\phi^* : \mathrm{Jac}(X') \to \mathrm{Jac}(X)$ is injective if and only if $\phi : X \to X'$ has a nontrivial unramified abelian subcover. The situation for graphs is simpler, since as we will now show, ϕ_* is always surjective and ϕ^* is always injective. The surjectivity of ϕ_* is easy:

Lemma 4.12. *Let $\phi : G \to G'$ be a non-constant harmonic morphism. Then $\phi_* : \mathrm{Jac}(G) \to \mathrm{Jac}(G')$ is surjective.*

Proof. It follows from Lemma 2.7 and the linearity of ϕ_* that ϕ_* is a surjective map from $\mathrm{Div}(G)$ to $\mathrm{Div}(G')$, which implies surjectivity on the level of Jacobians. □

The injectivity of ϕ^* is much more subtle (as one would expect, since the analogous statement for Riemann surfaces is false):

Theorem 4.13. *Let $\phi : G \to G'$ be a non-constant harmonic morphism. Then $\phi^* : \mathrm{Jac}(G') \to \mathrm{Jac}(G)$ is injective.*

Proof. We first set the following notation. For a function $f : V(G) \to \mathbb{Z}$, let $\max(f) = \max_{x \in V(G)} f(x)$, let $\min(f) = \min_{x \in V(G)} f(x)$, and let $s(f) = \max(f) - \min(f)$. Let $M(f) = \{x \in V(G) \,|\, f(x) = \max(f)\}$, and let $m(f) = \{x \in V(G) \,|\, f(x) = \min(f)\}$.

It suffices to show that $D' \in \mathrm{Prin}(G')$ for every $D' \in \mathrm{Div}(G')$ such that $\phi^*(D') \in \mathrm{Prin}(G)$. Suppose for the sake of contradiction that there exists a divisor $D' \in \mathrm{Div}(G') \smallsetminus \mathrm{Prin}(G')$ such that $\phi^*(D') = \mathrm{div}(f)$ for some $f : V(G) \to \mathbb{Z}$. Choose such a D' for which $s(f)$ is minimized, and subject to this condition such that $|M(f)|$ is minimized. Let $D := \phi^*(D') = \mathrm{div}(f)$. Clearly $s(f) \neq 0$, as otherwise $D = 0$, and therefore $D' = 0 \in \mathrm{Prin}(G')$, a contradiction. Therefore there exists a vertex $x_0 \in M(f)$ with a neighbor in $V(G) \smallsetminus M(f)$. For every $x \in M(f)$, one has

$$D(x) = \mathrm{div}(f)(x) \geq |\{e \in E(G) \mid e = xy, y \in V(G) \smallsetminus M(f)\}|.$$

It follows that $D(x) \geq 0$ for every $x \in M(f)$, and that $D(x_0) > 0$. Similarly, for every $x \in m(f)$ one has either $D(x) < 0$, or else $D(x) = 0$ and all the neighbors of x belong to $m(f)$. Let $X = \phi^{-1}(\phi(x_0)) \cap m(f)$. Since $D(x_0) > 0$, we have $D'(\phi(x_0)) > 0$ as well, so by the definition of ϕ^* it follows that $D(x) > 0$ for every $x \in \phi^{-1}(\phi(x_0))$ with $m_\phi(x) > 0$. Therefore X consists entirely of vertices $x \in \phi^{-1}(\phi(x_0))$ with $m_\phi(x) = 0$ and $D(x) = 0$. But then all the neighbors of vertices in X belong to $\phi^{-1}(\phi(x_0))$, and thus by the above must belong to X. Since G is connected, it follows that X is empty, i.e.,

$$(4.14) \qquad\qquad \phi^{-1}(\phi(x_0)) \cap m(f) = \emptyset.$$

Let $\chi : V(G') \to \mathbb{Z}$ be the characteristic function of $\{\phi(x_0)\}$, and let $D'' = D' - \mathrm{div}(\chi)$. We claim that D'' contradicts the choice of D'. Clearly, $D'' \in \mathrm{Div}(G') \smallsetminus \mathrm{Prin}(G')$. By Proposition 4.2, we have

$$\phi^*(D'') = \phi^*(D') - \phi^*(\mathrm{div}(\chi)) = \mathrm{div}(f) - \mathrm{div}(\phi^*\chi) = \mathrm{div}(f - \chi \circ \phi).$$

Let $D^\star = \phi^*(D'')$ and let $f^\star = f - \chi \circ \phi$. We have $f^\star(x) = f(x) - 1$ for every $x \in \phi^{-1}(\phi(x_0))$, and $f^\star(x) = f(x)$ otherwise. By (4.14), we have $\min(f) = \min(f^\star)$, and clearly $\max(f) \geq \max(f^\star)$. Therefore $s(f) \geq s(f^\star)$. Moreover, either $s(f) > s(f^\star)$ or $\max(f) = \max(f^\star)$. In the second case, we have $M(f^\star) \subseteq M(f) \smallsetminus \{x_0\}$, and thus $|M(f)| > |M(f^\star)|$. It follows that D'' contradicts the choice of D', as claimed. $\square$

4.2. **Eulerian cuts and harmonic morphisms.** Let $\kappa_G = |\mathrm{Jac}(G)|$ denote the number of spanning trees in a graph G. From either Lemma 4.12 or Theorem 4.13, we immediately deduce the following corollary:

Corollary 4.15. *If there exists a non-constant harmonic morphism from G to G', then $\kappa_{G'}$ divides κ_G.*

Define an *Eulerian cut* in a graph G to be a non-empty cut which is also an even subgraph of G; equivalently, an Eulerian cut is a partition of $V(G)$ into non-empty disjoint subsets X and X' in such a way that there are an even number of edges connecting each vertex in X (resp. X') to vertices in X' (resp. X). According to a theorem of Chen [Che71] (see also [Big97, Proposition 35.2]), G has an Eulerian cut if and only if κ_G is even. From Corollary 4.15, it therefore follows that if G' has an Eulerian cut and there exists a non-constant harmonic morphism from G to G', then G has an Eulerian cut as well. We can strengthen this observation using the following result, which characterizes the existence of Eulerian cuts in terms of non-constant harmonic maps from G to the graph B_2 consisting of 2 vertices connected by 2 edges:

Theorem 4.16. *Let G be a graph. Then the following are equivalent:*

(1) *G has an Eulerian cut.*
(2) *There is a non-constant harmonic morphism from G to B_2.*
(3) *κ_G is even.*

Proof. Although the equivalence $(1) \Leftrightarrow (3)$ is just Chen's theorem, for the reader's convenience we will provide a self-contained proof of this result. Our proof of $(3) \Rightarrow (1)$ is borrowed from the unpublished manuscript [Epp96].

$(\mathbf{1}) \Rightarrow (\mathbf{2})$: Suppose that G admits an Eulerian cut S. We claim that there exists a partition of S into non-empty disjoint subsets $S_1, S_2 \subseteq E(G)$ such that

$$|\{e \in S_1 \mid x \in e\}| = |\{e \in S_2 \mid x \in e\}|$$

for every $x \in V(G)$. Indeed, it is well-known (see [Bol98, §I.1,Theorem 1]) that the edge set of any Eulerian graph can be decomposed into edge-disjoint cycles. Since the graph $G[S]$ with vertex set $V(G)$ and edge set S is Eulerian and bipartite, it follows that $G[S]$ decomposes into edge-disjoint *even* cycles. It is trivial to obtain the required partition for an even cycle. By composing the resulting partitions of even cycles, one then obtains the required partition (S_1, S_2) of S.

We now construct a non-constant harmonic morphism $\phi : G \to B_2$ as follows. Let the vertices of B_2 be labeled x and y, and let the edges of B_2 be labeled e_1 and e_2. Let $X \subseteq V(G)$ be one of the sides of S. For $z \in V(G)$, let $\phi(z) = x$ if $x \in X$, and let $\phi(z) = y$ otherwise. For $i \in \{1, 2\}$ and $e \in S_i$, let $\phi(e) = e_i$. Finally, if $e = z_1 z_2 \in E(G) \smallsetminus S$, we set $\phi(e) = \phi(z_1) = \phi(z_2)$. It follows from the definition of S_1 and S_2 that ϕ is a non-constant harmonic morphism.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$: We have $\kappa_{B_2} = 2$. Therefore, if G admits a non-constant harmonic morphism to B_2, then κ_G is even by Corollary 4.15.

$(\mathbf{3}) \Rightarrow (\mathbf{1})$: (c.f. [Epp96, Proof of Theorem 6]) Let $\Lambda(G) = H^1(G, \mathbb{Z}) \subset H^1(G, \mathbb{R})$ denote the *lattice of integral flows* on G, and let $\Lambda^{\#}(G)$ be the lattice dual to $\Lambda(G)$ under the standard Euclidean inner product $\langle\,,\,\rangle$ on $C^1(G, \mathbb{R}) \supseteq H^1(G, \mathbb{R})$. Explicitly, we have

$$\Lambda^{\#} = \{\omega \in H^1(G, \mathbb{R}) \mid \langle \omega, \omega' \rangle \in \mathbb{Z} \text{ for all } \omega' \in H^1(G, \mathbb{Z})\}.$$

By [BdlHN97] (see also [Big97, §29]), there is a canonical isomorphism $\mathrm{Jac}(G) \cong \Lambda^{\#}(G)/\Lambda(G)$.

Suppose that κ_G is even. Then $\mathrm{Jac}(G)$ has an element of order 2, so there is a flow $\omega \in \Lambda^{\#}(G)$ such that $\omega \notin \Lambda(G)$ but $2\omega \in \Lambda(G)$. Thus the value of ω along each edge of G is a half-integer, and the set S of edges along which ω is non-integral is non-empty. Since $\delta(\omega) = 0$, it follows that every vertex in S has even degree. So it suffices to prove that S is a cut. To see this, choose a vertex $x \in V(G)$, and partition $V(G)$ into disjoint subsets A and B as follows: a vertex $y \in V(G)$ belongs to A (resp. B) iff it can be connected to x by a path containing an *odd* (resp. *even*) number of edges in S. Since G is connected, $A \cup B = V(G)$. Furthermore, we have $A \cap B = \emptyset$, because otherwise there would be a cycle C in G containing an odd number of edges of S, and therefore $\langle \omega, \chi_C \rangle \notin \mathbb{Z}$, contradicting the fact that $\omega \in \Lambda^{\#}(G)$. Finally, to see that S is indeed a cut, note that each edge $e \in S$ connects a vertex in A to a vertex in B (since e itself is a path with one edge in S), and an edge $e' \notin S$ cannot connect a vertex in A to a vertex in B (since e' is a path with no edges in S). Thus S is precisely the cut consisting of all edges connecting A to B. $\square$

Remark 4.17. Here is a more direct argument for proving $(1) \Rightarrow (3)$ which makes use of Theorem 1.5. Let S be an Eulerian cut in G separating the subsets $X, Y \subset V(G)$. It is easy to see that there exists an ordering $x_1, \ldots, x_k$ of X such that for every $i \in \{1, \ldots, k\}$ either $\mathrm{outdeg}_X(x_i) > 0$ or $x_j x_i \in E(G)$ for some $j < i$. Similarly, there exists an ordering $y_1, \ldots, y_\ell$ of Y such that for every $i \in \{1, \ldots, l\}$ either $\mathrm{outdeg}_Y(y_i) > 0$ or $y_j y_i \in E(G)$ for some $j < i$. Define a divisor

$D \in \mathrm{Div}^0(G)$ by setting $D(x) := \frac{1}{2} \mathrm{outdeg}_X(x)$ for $x \in X$, and $D(y) := -\frac{1}{2} \mathrm{outdeg}_Y(y)$ for $y \in Y$. Then $2D = \mathrm{div}(\chi_X) \sim 0$. However, using Theorem 1.5, we see that D itself is not equivalent to 0, since $D \leq \nu$, where ν is the divisor associated to the linear order $y_1 < \cdots < y_\ell < x_1 < \cdots < x_k$ on $V(G)$. Thus D corresponds to an element of order 2 in $\mathrm{Jac}(G)$, and in particular $\kappa_G = |\mathrm{Jac}(G)|$ is even.

4.3. Induced maps on harmonic 1-forms.

We now turn to a discussion of harmonic 1-forms and the maps induced on them by a harmonic morphism.

We begin with some notation and terminology. Let $\vec{E}(G)$ denote the set of directed edges of G. For $e \in \vec{E}(G)$, we let $o(e), t(e)$ denote the *origin* and *terminus* of e, respectively. We denote by $\overline{e}$ the directed edge representing the same undirected edge as e, but with the opposite orientation. From the definition of a morphism of graphs, it follows easily that a morphism $\phi : G \to G'$ induces a natural map from $\vec{E}(G)$ to $\vec{E}(G') \cup V(G')$.

Let A be an abelian group, and let $C^1(G, A)$ denote the space of 1-cochains on G with values in A, i.e., functions $\omega : \vec{E}(G) \to A$ with the property that $\omega(e) = -\omega(\overline{e})$ for all $e \in \vec{E}(G)$. As usual, we also let $C^0(G, A)$ denote the space of all functions $f : V(G) \to A$. We define the *coboundary operator* $\delta : C^1(G, A) \to C^0(G, A)$ by the formula

$$(4.18) \qquad \delta(\omega)(x) := \sum_{\substack{e \in \vec{E}(G) \\ t(e) = x}} \omega(e).$$

An *A-flow* (or simply a *flow* if $A = \mathbb{R}$) on G is a 1-cochain $\omega \in C^1(G, A)$ such that $\delta(\omega) = 0$. We denote by $H^1(G, A)$ the space of A-flows on G. When $A = \mathbb{R}$, we will also refer to $\mathbf{H}^1(G) := H^1(G, \mathbb{R})$ as the space of *harmonic 1-forms* on G; it is analogous to the space $\Omega^1(X)$ of holomorphic 1-forms on a Riemann surface X. For example, it is well-known that $\dim_{\mathbb{R}} \mathbf{H}^1(G) = g$ (just as $\dim_{\mathbb{C}} \Omega^1(X) = g$ in the Riemann surface case).

Suppose $\phi : G \to G'$ is a harmonic morphism and that $\omega \in C^1(G, A), \omega' \in C^1(G', A)$ are 1-cochains. We define the *pullback* $\phi^* \omega' \in C^1(G, A)$ by

$$(\phi^* \omega')(e) := \begin{cases} \omega'(\phi(e)) & \text{if } \phi(e) \in \vec{E}(G') \\ 0 & \text{otherwise} \end{cases}$$

and the *push-forward* (or *trace*) $\phi_* \omega \in C^1(G', A)$ by

$$\phi_* \omega(e') := \sum_{\substack{e \in \vec{E}(G) \\ \phi(e) = e'}} \omega(e).$$

Proposition 4.19. *Let $\phi : G \to G'$ be a harmonic morphism and let $\omega \in \mathbf{H}^1(G), \omega' \in \mathbf{H}^1(G')$ be harmonic 1-forms. Then:*

(1) $\phi^* \omega' \in \mathbf{H}^1(G)$.
(2) $\phi_* \omega \in \mathbf{H}^1(G')$.

Proof. To establish (1), we follow [Ura00, Proof of Theorem 2.13]. For every $x \in V(G)$, we have

$$\sum_{\substack{e \in \vec{E}(G), t(e) = x \\ \phi(e) \in \vec{E}(G')}} \omega'(\phi(e)) = \sum_{\substack{e' \in \vec{E}(G') \\ t(e') = \phi(x)}} \sum_{\substack{e \in \vec{E}(G), x \in e \\ \phi(e) = e'}} \omega'(\phi(e)) = m_\phi(x) \sum_{\substack{e' \in \vec{E}(G') \\ t(e') = \phi(x)}} \omega'(e').$$

Since $\delta(\omega') = 0$ and $(\phi^*\omega')(e) = 0$ for all vertical edges $e \in \vec{E}(G)$, for all $x \in V(G)$ we have

$$
\delta(\phi^*\omega')(x) = \sum_{\substack{e \in \vec{E}(G) \\ t(e)=x}} (\phi^*\omega')(e) = \sum_{\substack{e \in \vec{E}(G), t(e)=x \\ \phi(e) \in \vec{E}(G')}} \omega'(\phi(e))
$$

$$
= m_\phi(x) \sum_{\substack{e' \in \vec{E}(G') \\ t(e')=\phi(x)}} \omega'(e') = m_\phi(x)\delta(\omega')(\phi(x))
$$

$$
= 0,
$$

which proves (1).

For (2), note that for every $y \in V(G')$, we have

(4.20)
$$
\sum_{\substack{e' \in \vec{E}(G') \\ t(e')=y}} \sum_{\substack{e \in \vec{E}(G) \\ \phi(e)=e'}} \omega(e) = \sum_{\substack{x \in V(G) \\ \phi(x)=y}} \sum_{\substack{e \in \vec{E}(G) \\ t(e)=x}} \omega(e),
$$

since each vertical edge in $E(G)$ gets counted twice in the sum on the right-hand side of (4.20), once with each orientation, and therefore the net contribution to the sum from such an edge is zero. Therefore

$$
\delta(\phi_*\omega)(y) = \sum_{\substack{e' \in \vec{E}(G') \\ t(e)=y}} (\phi_*\omega)(e') = \sum_{\substack{e' \in \vec{E}(G') \\ t(e')=y}} \sum_{\substack{e \in \vec{E}(G) \\ \phi(e)=e'}} \omega(e)
$$

$$
= \sum_{\substack{x \in V(G) \\ \phi(x)=y}} \sum_{\substack{e \in \vec{E}(G) \\ t(e)=x}} \omega(e) \qquad \text{by (4.20)}
$$

$$
= \sum_{\substack{x \in V(G) \\ \phi(x)=y}} \delta(\omega)(x) = 0,
$$

proving (2). $\square$

As a consequence of Proposition 4.19, we see that ϕ induces linear transformations (which we continue to denote by ϕ^*, ϕ_*)

$$
\phi^* : \mathbf{H}^1(G') \to \mathbf{H}^1(G), \ \phi_* : \mathbf{H}^1(G) \to \mathbf{H}^1(G').
$$

It is straightforward to check that the association $(G', \phi) \mapsto (\mathbf{H}^1(G'), \phi^*)$ (resp. $(G, \phi) \mapsto (\mathbf{H}^1(G'), \phi_*)$) is a contravariant (resp. covariant) functor from the category of graphs (together with harmonic morphisms between them) to the category of vector spaces.

It follows easily from the definitions that

(4.21)
$$
\phi_*\phi^*(\omega') = \deg(\phi)\omega'
$$

for all $\omega' \in \mathbf{H}^1(G')$ (compare with Lemma 4.1). As a consequence, we obtain the following result, which provides another way to see that if ϕ is a non-constant harmonic morphism from a graph of genus g to a graph of genus g', then $g' \leq g$ (c.f. Theorem 2.14):

Corollary 4.22. *If $\phi : G \to G'$ is a non-constant harmonic morphism, then $\phi^* : \mathbf{H}^1(G') \to \mathbf{H}^1(G)$ is injective and $\phi_* : \mathbf{H}^1(G) \to \mathbf{H}^1(G')$ is surjective.*

Proof. Both the injectivity of ϕ^* and the surjectivity of ϕ_* follow easily from (4.21). However, one can also prove the injectivity of ϕ^* directly (c.f. [Ura00, Proof of Theorem 2.13]): if $\phi^*(\omega') = 0$, then $\omega'(\phi(e)) = 0$ for all horizontal edges $e \in E(G)$, and since ϕ maps the set of horizontal edges of G surjectively onto $E(G')$, it follows that $\omega' = 0$. $\square$

By functoriality, an automorphism α of a graph G induces an automorphism α^* of the vector space $\mathbf{H}^1(G)$. For later use, we note the following property of the corresponding map $\mathrm{Aut}(G) \to \mathrm{Aut}(\mathbf{H}^1(G))$:

Proposition 4.23. *If G is a 2-edge-connected graph of genus at least 2, then the natural map from* $\mathrm{Aut}(G)$ *to* $\mathrm{Aut}(\mathbf{H}^1(G))$ *is injective.*

Proof. Let $\beta, \beta' \in \mathrm{Aut}(G)$. By considering the automorphism $\alpha := \beta'\beta^{-1}$, it suffices to prove that if α^* is the identity map on $\mathbf{H}^1(G)$, then α is the identity map on G. So suppose $\alpha^* = \mathrm{Id}$. Then every directed cycle in G is mapped onto itself. Let C be an (undirected) simple cycle in G (i.e., a cycle with no repeated vertices), let $x \in C$ be a vertex of degree at least 3, and let $x' = \alpha(x)$. Let $e \in C$ be the directed edge with $o(e) = x$, let $e' = \alpha(e) \in C$, and let $e'' \in \vec{E}(G)$ be a directed edge with $o(e'') = x$ and $e'' \notin C$. Since G is 2-edge-connected, e'' belongs to a simple cycle C'', and we can choose C'' so that either $V(C) \cap V(C'') = \{x\}$, or else so that $E(C) \cap E(C'')$ is a path in C containing e.

Case I: $V(C) \cap V(C'') = \{x\}$.

In this case, $x' \in V(C) \cap V(C'') = \{x\}$ so $\alpha(x) = x$. But then $\alpha(e) = e$, since α^* preserves *directed* cycles of G. From this it follows easily that α is the identity map on C.

Case II: $E(C) \cap E(C'')$ is a path in C containing e.

In this case, we must also have $e' \in C''$. Suppose $e' \neq e$. Then as $\alpha(e'') \notin C$, the cycle C'' can be directed so that it consists of the unique path in C from x to x' followed by the unique path in $C'' \backslash C$ from x' to x. But then α restricted to C'' is orientation-reversing, a contradiction. We conclude that $\alpha(e) = e$, and hence α is the identity map on C in this case as well.

It follows that the restriction of α to every simple cycle C of G is the identity map. Since G is 2-edge-connected, this implies that α is the identity map on all of G. $\square$

Remark 4.24. Proposition 4.23 is the analogue of the fact from algebraic geometry that if X is a Riemann surface of genus at least 2, then the natural map from $\mathrm{Aut}(X)$ to $\mathrm{Aut}(\Omega^1(X))$ is injective.

As a consequence of Proposition 4.23, we obtain the following non-trivial restriction on the automorphism group of a 2-edge-connected graph of genus at least 2:

Corollary 4.25. *If G is a 2-edge-connected graph of genus $g \geq 2$, then the group $\mathrm{Aut}(G)$ is isomorphic to a subgroup of the group $\mathrm{GL}(g, \mathbb{Z})$ of invertible $g \times g$ matrices with coefficients in $\mathbb{Z}$.*

Proof. Since $\mathrm{Aut}(G)$ acts faithfully on the g-dimensional vector space $\mathbf{H}^1(G, \mathbb{R})$ and preserves the lattice $\mathbf{H}^1(G, \mathbb{Z})$, the result follows. $\square$

Remark 4.26. By a theorem of Minkowski, the torsion group $\mathrm{GL}(n, \mathbb{Z})_{\mathrm{tors}}$ of $\mathrm{GL}(n, \mathbb{Z})$ is finite for all $n \geq 1$, and every prime divisor p of $|\mathrm{GL}(n, \mathbb{Z})_{\mathrm{tors}}|$ satisfies $p \leq n + 1$. In particular, Corollary 4.25 implies that if a 2-edge-connected graph G of genus $g \geq 2$ has an automorphism of prime order p then $p \leq g + 1$. This bound is sharp, since the graph B_{n+1} consisting of 2 vertices joined by $n + 1$ edges has genus n and $|\mathrm{Aut}(B_{n+1})| = 2(n + 1)!$.

5. Hyperelliptic graphs

5.1. **Definition and basic properties.** We say that a graph G is *hyperelliptic* if there exists a divisor $D \in \mathrm{Div}(G)$ such that $\deg(D) = 2$ and $r(D) = 1$. By Riemann-Roch for graphs, if G is

hyperelliptic then $g(G) \geq 2$, and by Clifford's theorem for graphs, if $g(G) \geq 2$ and $\deg(D) = 2$, then $r(D) = 1$ if and only if $r(D) \geq 1$.

Example 5.1. Every graph of genus 2 is hyperelliptic. Indeed, if $g(G) = 2$, then by Riemann-Roch for graphs, the canonical divisor K_G has $\deg(K_G) = 2$ and $r(K_G) = 1$.

Example 5.2. Let the graph $G = B(l_1, l_2, \ldots, l_n)$ consist of two vertices x and y and $n \geq 3$ internally disjoint paths joining x to y with lengths $l_1, l_2, \ldots, l_n$. Then G is hyperelliptic. More specifically, we claim that $r((x)+(y)) = 1$. To prove this, it suffices to show that $|(x)+(y)-(z)| \neq \emptyset$ for every $z \in V(G)$. Consider one of the paths joining x and y, and let $x, z_1, z_2, \ldots, z_{l-1}, y$ be the vertices of this path in order. Then $(x) + (y) \sim (z_i) + (z_{l-i})$, and therefore $|(x) + (y) - (z_i)| \neq \emptyset$ for every $1 \leq i \leq l - 1$. Thus $r((x) + (y)) = 1$, and our claim follows.

Although the graph $G = B(1, 1, \ldots, 1)$ has edge connectivity equal to $|E(G)|$, which can be arbitrarily large, the following result shows that every other hyperelliptic graph has edge connectivity at most 2:

Lemma 5.3. *If G is a hyperelliptic graph, then either $|V(G)| = 2$ (so that G is isomorphic to a graph of the form $B(1, 1, \ldots, 1)$) or G has edge connectivity at most 2.*

Proof. Let $D = (x) + (x')$ be an effective divisor of degree 2 on G with $r(D) = 1$. If $|V(G)| > 2$, choose a vertex $y \in V(G)$ with $y \notin \{x, x'\}$. Since $r(D) = 1$, there exists $y' \in V(G)$ such that $(x) + (x') \sim (y) + (y')$, and therefore the map $S^{(2)} : \mathrm{Div}_+^2(G) \to \mathrm{Jac}(G)$ is not injective. By Theorem 1.9, it follows that G is not 3-edge-connected. $\square$

A classical result from algebraic geometry asserts that if X is a hyperelliptic Riemann surface and $\phi : X \to X'$ is a non-constant holomorphic map with $g(X') \geq 2$, then X' is also hyperelliptic. Using Corollary 4.11, we obtain the following analogous result for graphs:

Corollary 5.4. *If G is hyperelliptic and $\phi : G \to G'$ is a non-constant harmonic morphism onto a graph G' with $g(G') \geq 2$, then G' is hyperelliptic as well.*

As in classical algebraic geometry, we can also show in the graph-theoretic setting that there is at most one complete linear system $|D|$ of degree 2 on a graph G for which $r(D) = 1$:

Proposition 5.5. *If D, D' are degree 2 divisors on G with $r(D) = r(D') = 1$, then $D \sim D'$.*

Proof. We may assume that $g := g(G) \geq 2$. Consider the divisor $E := D + (g-2)D'$ of degree $2g-2$ on G. By Lemma 1.2, we have $r(E) \geq g-1$. By Riemann-Roch for graphs, we have $r(K_G - E) \geq 0$; since $\deg(K_G - E) = 0$, it follows that $K_G \sim E$. Applying the same reasoning to $E' := (g-1)D'$, we see that $K_G \sim E'$, and therefore $D \sim D'$ as desired. $\square$

5.2. Hyperelliptic graphs, involutions, and harmonic morphisms.
Our next goal is to obtain a graph-theoretic analogue of the well-known result from algebraic geometry that the following are equivalent for a Riemann surface X of genus at least 2: (i) X is hyperelliptic; (ii) X admits a non-constant holomorphic map of degree 2 onto the Riemann sphere; and (iii) there is an involution $\iota : X \to X$ whose quotient is isomorphic to the Riemann sphere. We begin by discussing quotients in the category of graphs (together with morphisms between them).

Let H be a finite group acting on a graph G, i.e., suppose we are given a homomorphism $H \to \mathrm{Aut}(G)$. We write $h \cdot x$ for the action of an element $h \in H$ on an element x of $V(G) \cup E(G)$. We define the quotient graph G/H, together with a canonical morphism $\pi_H : G \to G/H$, as follows.

For $x, y \in V(G) \cup E(G)$, let $x \sim_H y$ if there exists an element $h \in H$ such that $h \cdot x = y$. Then $\sim_H$ is an equivalence relation on $V(G) \cup E(G)$. The *quotient graph G/H* is constructed as follows. The vertices of G/H are the equivalence classes of $V(G)$ with respect to $\sim_H$. The edges of G/H correspond to those equivalence classes of $E(G)$ with respect to $\sim_H$ which consist of edges whose ends are inequivalent. It is readily verified that G/H is a graph in our sense of the word (i.e., a connected multigraph with no loop edges). The *quotient morphism $\pi_H : G \to G/H$* maps every vertex of G to its equivalence class, every edge of G whose ends are inequivalent to the edge of G/H corresponding to its equivalence class, and every edge of G with equivalent ends to the equivalence class of its ends. It is straightforward to check that π_H is a surjective morphism of graphs (though not necessarily a harmonic morphism), and by construction we have $\pi_H(h \cdot x) = \pi_H(x)$ for all $h \in H$ and all $x \in V(G) \cup E(G)$. In fact, the morphism $\pi_H : G \to G/H$ has the following universal property: if $\pi' : G \to G/H$ is any morphism of graphs for which $\pi'(h \cdot x) = \pi'(x)$ for all $h \in H$ and all $x \in V(G) \cup E(G)$, then there exists a unique morphism $\psi : G/H \to G'$ such that $\pi' = \psi \circ \pi_H$. This universal property uniquely characterizes G/H up to isomorphism.

If $H = \langle \phi \rangle$ is a cyclic subgroup of $\mathrm{Aut}(G)$, we will often write G/ϕ instead of G/H and $\phi^\sim$ instead of π_ϕ.

An automorphism ι of a graph G is called an *involution* if $\iota \circ \iota$ is the identity automorphism. We say that an involution ι is *mixing* if for every edge $e = xy \in E(G)$ such that $\iota(e) = e$ we have $\iota(x) = y$. Equivalently, ι is mixing if and only if it does not fix any directed edge of G. The following lemma shows that if $|V(G)| > 2$, there is a one-to-one correspondence between mixing involutions of G and non-degenerate harmonic morphisms of degree two from G to a graph G'.

Lemma 5.6. *Let G, G' be graphs, and let $\phi : G \to G'$ be a non-degenerate harmonic morphism of degree 2. Then there is a mixing involution ι of G for which $\phi = \iota^\sim$. Conversely, let $|V(G)| > 2$ and let $\iota : G \to G$ be a mixing involution. Then $\iota^\sim$ is a non-degenerate harmonic morphism of degree two.*

Proof. Let $\phi : G \to G'$ be a non-degenerate harmonic morphism of degree 2. For $x \in V(G)$, if there exists $y \neq x$ such that $\phi(y) = \phi(x)$ then we define $\iota(x) = y$. Otherwise, we define $\iota(x) = x$. For every $e \in E(G)$ such that $\phi(e) \in E(G')$, there is a unique edge $e' \in E(G)$ such that $e' \neq e$ and $\phi(e') = \phi(e)$, and we define $\iota(e) = e'$. Define $\iota(e) = e$ for every $e \in E(G)$ such that $\phi(e) \in V(G')$.

If $x \in V(G)$, $e \in E(G)$, $x \in e$ and $\phi(e) \in E(G')$ then either $\iota(x) \in \iota(e)$, or $x \in \iota(e)$. In the second case, $m_\phi(x) = 2$ and therefore by non-degeneracy of ϕ we have $x = \iota(x)$. It follows easily from this that ι is a morphism. Clearly $\iota \circ \iota$ is the identity map. In particular, ι is a bijection. Therefore ι is an involution, and ι is mixing by definition. Finally, it is easy to see that $\phi = \iota^\sim$.

Now suppose $|V(G)| > 2$, and let $\iota : G \to G$ be a mixing involution of G. Denote G/ι by G'. Note that $|V(G')| \geq |V(G)|/2 > 1$. Consider a vertex $x \in V(G)$, let $y = \iota^\sim(x)$, and consider an edge $e' = yy' \in E(G')$. Then there exists an edge $e = xx'$ in G such that $\iota^\sim(e) = e'$, and $(\iota^\sim)^{-1}(e') = \{e, \iota(e)\}$. Therefore $|\{d \in E(G) | x \in d, \ \iota^\sim(d) = e'\}| = 1$ if $x \neq \iota(x)$ and $|\{d \in E(G) | x \in d, \ \iota^\sim(d) = e'\}| = 2$ otherwise. It follows that $m_{\iota^\sim}(x)$ is well defined and positive, and that $\sum_{\iota^\sim(y)=z} m_{\iota^\sim}(y) = 2$ for every $z \in V(G')$. Therefore, $\iota^\sim$ is a non-degenerate harmonic morphism of degree two, as claimed. $\square$

The following result will be used to reduce the study of general hyperelliptic graphs to the special case of graphs which are 2-edge-connected.

Lemma 5.7. *Let G be a graph, let $\overline{G}$ be the graph obtained by contracting every bridge of G, and let $\rho : G \to \overline{G}$ be the natural surjective morphism. Then for every divisor $D \in \mathrm{Div}(G)$, we have $D \sim_G 0$ if and only if $\rho_*(D) \sim_{\overline{G}} 0$, where $\rho_*(D)$ is defined as in (2.11).*

Remark 5.8. Note that the morphism $\rho : G \to \overline{G}$ is not necessarily harmonic, c.f. Example 3.6.

Proof of Lemma 5.7. It suffices by induction to prove the result with $\overline{G}$ replaced by the graph obtained by contracting a *single* bridge e. We begin with some notation. Let x_1, x_2 be the endpoints of e, and let $\overline{x} = \rho(x_1) = \rho(x_2)$. Let G_1, G_2 be the connected components of $G - e$ containing x_1 and x_2, respectively, and for $i = 1, 2$, let $\overline{G}_i = \rho(G_i)$, so that $\overline{G} = \overline{G}_1 \cup \overline{G}_2$ and $\overline{G}_1 \cap \overline{G}_2 = \{\overline{x}\}$. Note that $(x_1) \sim (x_2)$ on G; this follows from the observation that $(x_1) - (x_2) = \mathrm{div}(\chi_{G_1})$.

Let $D \in \mathrm{Div}(G)$. Suppose first that D is a principal divisor on G; we want to show that $\rho_*(D)$ is a principal divisor on $\overline{G}$. It suffices by linearity to consider the case where $D = \mathrm{div}(\chi_y)$ for some $y \in V(G)$. If $y \notin \{x_1, x_2\}$, then $\rho_*(D) = \mathrm{div}(\chi_{\rho(y)})$. Otherwise, we have $\rho_*(\mathrm{div}(\chi_{x_1})) = \mathrm{div}(\chi_{V(\overline{G}_2)})$ and $\rho_*(\mathrm{div}(\chi_{x_2})) = \mathrm{div}(\chi_{V(\overline{G}_1)})$. This proves that $\rho_*(D)$ is principal.

In the other direction, suppose that $\rho_*(D)$ is principal; we want to show that D itself is principal. By linearity, it suffices to consider the case where $\rho_*(D) = \mathrm{div}(\chi_z)$ for some $z \in V(\overline{G})$. If $z \neq \overline{x}$, then $\rho^{-1}(z)$ consists of a single element, and $D = \mathrm{div}(\chi_{\rho^{-1}(z)})$. If $z = \overline{x}$, then $\rho^{-1}(z) = \{x_1, x_2\}$ and using the fact that $(x_1) \sim (x_2)$ it is easy to see that $D \sim \mathrm{div}(\chi_{\{x_1, x_2\}})$. This proves that D is principal. $\qquad\square$

Remark 5.9. As alluded to in [BN, Remark 4.8], one can use Lemma 5.7 to obtain an alternate proof of Corollary 4.7 from [BN] which does not make use of circuit theory.

Corollary 5.10. *Let G be a graph, let $\overline{G}$ be the graph obtained by contracting every bridge of G, and let $\rho : G \to \overline{G}$ be the natural surjective morphism. Then for every divisor $D \in \mathrm{Div}(G)$, we have* $r_G(D) = r_{\overline{G}}(\rho_*(D))$.

Proof. Let $k \geq 0$ be an integer, and let $D \in \mathrm{Div}(G)$. Suppose $r(D) \geq k$, and let $\overline{D} = \rho_*(D)$. Then for every effective divisor $E \in \mathrm{Div}(G)$ of degree k, there exists an effective divisor $E' \in \mathrm{Div}(G)$ such that $D - E \sim E'$, and thus $\overline{D} - \rho_*(D) \sim \rho_*(E')$ by Lemma 5.7. Since $\rho_* : \mathrm{Div}(G) \to \mathrm{Div}(\overline{G})$ is surjective and preserves degrees and effectivity, it follows that $r(\overline{D}) \geq k$.

Conversely, suppose $r(\rho_*(D)) \geq k$. Then for every effective divisor $E \in \mathrm{Div}(G)$ of degree k, there exists an effective divisor $E' \in \mathrm{Div}(G)$ such that $\rho_*(D) - \rho_*(E) \sim \rho_*(E')$. By Lemma 5.7, it follows that $D - E \sim E'$, and thus $r(D) \geq k$ as desired. $\qquad\square$

Corollary 5.11. *Let G be a graph, and let $\overline{G}$ be the graph obtained by contracting every bridge of G. Then G is hyperelliptic if and only if $\overline{G}$ is hyperelliptic.*

Proof. This follows immediately from Corollary 5.10 and the surjectivity of $\rho_* : \mathrm{Div}(G) \to \mathrm{Div}(\overline{G})$. $\qquad\square$

Because of Corollary 5.11, when studying hyperelliptic graphs there is no loss of generality if we restrict our attention to graphs which are 2-edge-connected. And it turns out that for 2-edge-connected graphs, there are several equivalent characterizations of what it means to be hyperelliptic:

Theorem 5.12. *For a 2-edge-connected graph G of genus $g \geq 2$, the following conditions are equivalent:*

(1) *G is hyperelliptic.*
(2) *There exists an involution $\iota : G \to G$ such that G/ι is a tree.*
(3) *There exists a non-degenerate degree two harmonic morphism ϕ from G to a tree, or $|V(G)| = 2$.*

Proof. If $|V(G)| = 2$ then it is easily verified that conditions (1), (2) and (3) all hold. Therefore in what follows we assume $|V(G)| > 2$.

$(1) \Rightarrow (2)$. Let D be a divisor of degree 2 on G with $r(D) = 1$. For every $x \in V(G)$, we have $|D - (x)| \neq \emptyset$ and $\deg(D - (x)) = 1$. Since G is 2-edge-connected, there exists a unique $y \in V(G)$ such that $D - (x) \sim (y)$. Define $\iota(x) = y$.

Our next goal is to define ι on $E(G)$. Consider an edge $e = xy \in E(G)$. If $\iota(x) = y$, we define $\iota(e) = e$. If $\iota(x) \neq y$, then let $D_1 = (x) + (\iota(x))$ and let $D_2 = (y) + (\iota(y))$. By the definition of ι, we have $D_1 \sim D \sim D_2$. Therefore, there exists a non-constant function $f : V(G) \to \mathbb{Z}$ such that $D_1 - D_2 = \operatorname{div}(f)$. Let $M(f)$ be the set of all the vertices $z \in V(G)$ for which $f(z)$ is maximal. For every vertex $z \in M(f)$, we have

$$D_1(z) \geq (\operatorname{div}(f))(z) = \sum_{e'=zz' \in E(G)} (f(z) - f(z')) \geq |\{e' = zz' \in E(G) \mid z' \in V(G) \smallsetminus M(f)\}|.$$

Therefore $\deg(D_1) \geq |\delta(M(f))|$, where for $X \subseteq V(G)$ we denote by $\delta(X)$ the set of all edges of G having exactly one end in X. On the other hand, $|\delta(M(f))| \geq 2$ by the 2-edge connectivity of G. It follows that $|\delta(M(f))| = 2$, and that $x, \iota(x) \in M(f)$. Analogously, we can conclude that f is minimized on y and $\iota(y)$, and therefore that $y, \iota(y) \in V(G) \smallsetminus M(f)$. It follows that $e \in \delta(M(f))$. Define $\iota(e)$ to be the unique edge e^* such that $\delta(M(f)) = \{e, e^*\}$. Let x' be the end of e^* in $M(f)$. By the argument above we have $D_1 = (x') + (x)$. Therefore $x' = \iota(x)$. By the symmetry between x and y, we conclude that e^* joins $\iota(x)$ and $\iota(y)$. Therefore ι is an automorphism, and clearly $\iota \circ \iota$ is the identity.

By Lemma 5.6, we know that $\phi = \iota^\sim$ is a harmonic morphism. For every $x, y \in V(G/\iota)$ we have

$$\phi^*((x)) = (x) + (\iota(x)) \sim D \sim (y) + (\iota(y)) = \phi^*((y)).$$

Therefore, by Theorem 4.13, we have $(x) \sim (y)$ for all $x, y \in V(G)$. It follows from Lemma 1.1 that G/ι is a tree, as desired.

$(2) \Leftrightarrow (3)$. Consider an involution ι satisfying (2). For every edge $e = xy \in E(G)$ such that $x \neq \iota(y)$, the set of edges $\{e, \iota(e)\}$ is the preimage of an edge of G/ι, and therefore forms a cut in G. It follows that $e \neq \iota(e)$, and therefore ι is mixing. The equivalence of (2) and (3) now follows from Lemma 5.6.

$(3) \Rightarrow (1)$. Let $\phi : G \to T$ be a non-degenerate harmonic morphism of degree two, where T is a tree. Let $y_0 \in V(T)$ be chosen arbitrarily and let $D := \phi^*((y_0))$. Then D is an effective divisor of degree 2 on G. We claim that $r(D) = 1$. Clearly, $r(D) \leq 1$. Therefore, it suffices to show that $|D - (x)| \neq \emptyset$ for every $x \in V(G)$. Note that $(y) \sim (y')$ for every pair of vertices $y, y' \in V(T)$. Therefore $(\phi(x)) \sim (y_0)$, and by Proposition 4.2 we have $D \sim \phi^*((\phi(x))) \geq m_\phi(x)(x)$. By since ϕ is non-degenerate, we have $m_\phi(x) > 0$, and therefore $\phi^*((\phi(x))) = (x) + (x')$ for some $x' \in V(G)$, which implies that $|D - (x)| \neq \emptyset$ as desired. $\qquad \square$

Remark 5.13. One can use Theorem 5.12 to give an alternate proof of Lemma 5.3 which does not make use of Theorem 1.9. Indeed, if G is 2-edge-connected and $|V(G)| > 2$, then by Theorem 5.12 there is a non-degenerate harmonic morphism ϕ of degree 2 from G to a tree T with $|E(T)| > 0$. If $e' \in E(T)$ and $e, \iota(e)$ are the distinct edges of G mapping to e' under ϕ, then it is easy to see that $G - \{e, \iota(e)\}$ is disconnected. Thus G is not 3-edge-connected.

It is worth stating explicitly the following fact which was established during the course of our proof of Theorem 5.12:

Corollary 5.14. *If G is a 2-edge-connected hyperelliptic graph, then for any involution ι for which G/ι is a tree, we have $(x) + (\iota(x)) \sim (y) + (\iota(y))$ for all $x, y \in V(G)$. In particular, $r((x) + (\iota(x))) = 1$ for all $x \in V(G)$.*

From Corollary 5.14 and Proposition 5.5, we obtain the following graph-theoretic result whose statement does not involve harmonic morphisms at all:

Corollary 5.15. *If G is a 2-edge-connected graph of genus at least 2, then there is at most one involution ι of G whose quotient is a tree.*

Proof. By Corollary 5.14, if ι is such an involution then $r((x) + (\iota(x))) = 1$ for all $x \in V(G)$. So if ι and ι' are two such involutions, then $(x) + (\iota(x)) \sim (x) + (\iota'(x))$ for all $x \in V(G)$ by Proposition 5.5. Thus $(\iota(x)) \sim (\iota'(x))$ for all $x \in V(G)$. Since G is 2-edge-connected, it follows from Theorem 1.9 that $\iota(x) = \iota'(x)$ for all $x \in V(G)$, i.e., $\iota = \iota'$. $\square$

If G is a 2-edge-connected hyperelliptic graph, we call the unique involution ι whose quotient is a tree the *hyperelliptic involution* on G.

Remark 5.16. Corollary 5.15 is the graph-theoretic analogue of the fact that the hyperelliptic involution on a hyperelliptic Riemann surface is unique. We will give another proof of Corollary 5.15 in Remark 5.20 below.

Remark 5.17. It follows from the proofs of Theorem 5.12 and Corollary 5.15 that if G is a 2-edge-connected hyperelliptic graph and $r((x) + (y)) = 1$ for some $x, y \in V(G)$, then $y = \iota(x)$.

As a consequence of the uniqueness of the hyperelliptic involution, we obtain the following corollary:

Corollary 5.18. *If G is a 2-edge-connected hyperelliptic graph with hyperelliptic involution ι, then ι belongs to the center of the group $\mathrm{Aut}(G)$.*

Proof. Let $\tau \in \mathrm{Aut}(G)$, and consider the automorphism $\iota' := \tau^{-1} \iota \tau$. It is easy to check that ι' is an involution, and that τ induces an isomorphism from G/ι' to G/ι, so that G/ι' is a tree. By Corollary 5.15, we have $\iota' = \iota$, and therefore ι and τ commute, as desired. $\square$

5.3. **Equivalent characterizations of the hyperelliptic involution.** For a Riemann surface X of genus at least 2 and $\iota : X \to X$ an automorphism, the following are equivalent: (i) X is hyperelliptic with hyperelliptic involution ι; (ii) $\iota_* : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$ is multiplication by -1; (iii) $\iota^* : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$ is multiplication by -1; (iv) $\iota_* : \Omega^1(X) \to \Omega^1(X)$ is multiplication by -1; and (v) $\iota^* : \Omega^1(X) \to \Omega^1(X)$ is multiplication by -1. We now show that a similar characterization holds for 2-edge-connected graphs with genus at least 2.

Theorem 5.19. *Let G be a 2-edge-connected graph of genus $g \geq 2$, and let $\iota \in \mathrm{Aut}(G)$. Then the following are equivalent:*

   (1) *G is hyperelliptic with hyperelliptic involution ι.*
   (2) *$\iota_* : \mathrm{Jac}(G) \to \mathrm{Jac}(G)$ is multiplication by -1.*
   (3) *$\iota^* : \mathrm{Jac}(G) \to \mathrm{Jac}(G)$ is multiplication by -1.*
   (4) *$\iota_* : \mathbf{H}^1(G) \to \mathbf{H}^1(G)$ is multiplication by -1.*
   (5) *$\iota^* : \mathbf{H}^1(G) \to \mathbf{H}^1(G)$ is multiplication by -1.*

Proof. Since ι is a harmonic morphism of degree 1 from G to itself, $\iota_* \circ \iota^*$ is the identity map on both $\mathrm{Jac}(G)$ and $\mathbf{H}^1(G)$. It follows easily that $(2) \Leftrightarrow (3)$ and $(4) \Leftrightarrow (5)$. So it suffices to prove that $(1) \Leftrightarrow (2)$ and $(1) \Leftrightarrow (5)$.

$(\mathbf{1}) \Rightarrow (\mathbf{2})$. If G is hyperelliptic with hyperelliptic involution ι, then by Corollary 5.14, for every $x, y \in V(G)$, we have $(x) + (\iota(x)) \sim (y) + (\iota(y))$. Thus $(x) - (y) \sim (\iota(y)) - (\iota(x)) = \iota_*((y) - (x))$. Since the group $\mathrm{Div}^0(G)$ is generated by divisors of the form $(x) - (y)$, it follows that $\iota_* \equiv -1$ on $\mathrm{Jac}(G)$.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$. If $\iota_* \equiv -1$ on $\mathrm{Jac}(G)$, then $(x)+(\iota(x)) \sim (y)+(\iota(y))$ for all $x, y \in V(G)$. In particular, for any $x \in V(G)$, we have $r((x) + (\iota(x))) = 1$. Thus G is hyperelliptic. By Remark 5.17, ι is the hyperelliptic involution on G.

$(\mathbf{1}) \Rightarrow (\mathbf{5})$. Suppose G is hyperelliptic with hyperelliptic involution ι, and let $\pi : G \to T$ be the corresponding quotient map from G to a tree T. If $|V(G)| = 2$, it is clear that (5) holds, so we may assume that $|V(T)| > 1$. Let $e' \in \vec{E}(T)$ be a directed edge of T. Since π is a harmonic morphism of degree 2, there are two distinct directed edges $e, \iota(e)$ of G mapping onto e'. Let $\omega \in \mathbf{H}^1(G)$. Since T is a tree, we have $\mathbf{H}^1(T) = 0$, and therefore $(\pi_*\omega)(e') = 0$. On the other hand, by definition we have

$$(\pi_*\omega)(e') = \omega(e) + \omega(\iota(e)) = \omega(e) + (\iota^*\omega)(e).$$

Since π is surjective on oriented edges, it follows that $(\iota^*\omega)(e) = -\omega(e)$ for all $e \in \vec{E}(G)$ such that $\pi(e) \in \vec{E}(T)$. But for $e \in \vec{E}(G)$ with $\pi(e) \in V(T)$, we have $\iota(e) = \bar{e}$, and thus $(\iota^*\omega)(e) = -\omega(e)$ for such edges as well. It follows that $(\iota^*\omega)(e) = -\omega(e)$ for all $e \in \vec{E}(G)$, as desired.

$(\mathbf{5}) \Rightarrow (\mathbf{1})$. Suppose $\iota^* \equiv -1$ on $\mathbf{H}^1(G)$. Then $(\iota^2)^*$ is the identity map on $\mathbf{H}^1(G)$, so ι is an involution by Proposition 4.23. If $\iota(e) = e$ for some directed edge e, then letting ω be the characteristic function of any simple cycle containing e, we have

$$\omega(e) = \omega(\iota(e)) = (\iota^*\omega)(e) = -\omega(e),$$

so that $\omega(e) = 0$, a contradiction. Therefore ι is mixing. If $|V(G)| = 2$, it is easy to verify directly that (1) holds. So we may assume without loss of generality that $|V(G)| > 2$. By Lemma 5.6, we know that $\pi := \iota^\sim : G \to G' := G/\iota$ is a non-degenerate harmonic morphism of degree 2. It remains to show that G' is a tree. Since $\pi \circ \iota = \pi$, we have $\iota^*(\pi^*(\omega')) = \pi^*(\omega')$ for every $\omega' \in \mathbf{H}^1(G')$ by functoriality. Since $\iota^* \equiv -1$ on $\mathbf{H}^1(G)$, we conclude that $\pi^*(\omega') = -\pi^*(\omega')$, and therefore $\pi^*(\omega') = 0$, for every $\omega' \in \mathbf{H}^1(G')$. But $\pi^* : \mathbf{H}^1(G') \to \mathbf{H}^1(G)$ is injective, so it follows that $\mathbf{H}^1(G') = 0$, i.e., G' is a tree. $\square$

Remark 5.20. Combining Proposition 4.23 with the proof of $(1) \Rightarrow (5)$ in Theorem 5.19 yields another proof of Corollary 5.15 (i.e., of the uniqueness of the hyperelliptic involution).

As an application of Theorem 5.19, we establish a special case of [Bak07, Conjecture 3.14]. To state the result, given a graph G and a positive integer k, we define $\sigma_k(G)$ to be the graph obtained by replacing each edge of G by a path consisting of k edges.

Corollary 5.21. *Let G be a graph, and let k be a positive integer. Then G is hyperelliptic if and only if $\sigma_k(G)$ is hyperelliptic.*

Proof. By Corollary 5.11, we may assume without loss of generality that G (and therefore $\sigma_k(G)$ as well) is a 2-edge-connected graph of genus at least 2. If G is hyperelliptic, then by Theorem 5.19 there is an automorphism ι of G which acts as -1 on $\mathbf{H}^1(G)$. Identifying $V(G)$ with a subset of $V(\sigma_k(G))$ in the obvious way induces an isomorphism between $\mathbf{H}^1(G)$ and $\mathbf{H}^1(\sigma_k(G))$, and it is easy to see that ι can be extended to an automorphism of $\sigma_k(G)$ which acts as -1 on $\mathbf{H}^1(\sigma_k(G))$. Therefore $\sigma_k(G)$ is hyperelliptic. Conversely, suppose that $\sigma_k(G)$ is hyperelliptic. Then there is an automorphism ι' of $G' := \sigma_k(G)$ which acts as -1 on $\mathbf{H}^1(G')$. By an argument similar to the proof of Proposition 4.23, it follows that ι' induces an automorphism ι of G which acts as -1 on $\mathbf{H}^1(G)$ (the key point is that every cycle in G' contains a vertex of degree at least 3, which must belong to $V(G)$, and which must be sent by ι to another such vertex). Therefore G is hyperelliptic as well. $\square$

5.4. **The canonical map and 3-edge-connectivity.** We now turn to a discussion of a graph-theoretic analogue of the "canonical map" from a Riemann surface to projective space. In algebraic geometry, the following are equivalent for a Riemann surface X of genus at least 2: (i) X is not hyperelliptic; (ii) the symmetric square $S^{(2)} : \mathrm{Div}_+^2(X) \to \mathrm{Jac}(X)$ of the Abel-Jacobi map is injective; and (iii) the canonical map $\psi_X : X \to \mathbb{P}(\Omega^1(X))$ is injective. We have already seen that the analogues of (i) and (ii) are not equivalent for 2-edge-connected graphs of genus at least 2; indeed, by Theorem 1.9, $S^{(2)} : \mathrm{Div}_+^2(G) \to \mathrm{Jac}(G)$ is injective if and only if G is 3-edge-connected, and this is a strictly weaker condition than G being non-hyperelliptic (if $|V(G)| > 2$). We now define a graph-theoretic version ψ_G of the canonical map, and show that the analogues of conditions (ii) and (iii) for graphs are equivalent. In other words, we will show that ψ_G is injective if and only if G is 3-edge-connected.

Let G be a 2-edge-connected graph, and let $\mathbf{H}^1(G)$ be the space of harmonic 1-forms on G, as defined in §4. We write $\mathbb{P}(\mathbf{H}^1(G))$ for the projective space consisting of all hyperplanes (linear subspaces of codimension 1) in $\mathbf{H}^1(G)$. We define the *canonical map* $\psi_G : E(G) \to \mathbb{P}(\mathbf{H}^1(G))$ by sending an edge $e \in E(G)$ to the hyperplane $W(e) := \{\omega \in \mathbf{H}^1(G) \;:\; \omega(e) = 0\}$. Note that the condition $\omega(e) = 0$ is independent of the orientation of e, so it makes sense to ask whether or not ω vanishes on an undirected edge. Also, the fact that G is 2-edge-connected guarantees that $W(e) \neq \mathbf{H}^1(G)$, so $W(e)$ is indeed a hyperplane.

Our main observation about the canonical map is the following proposition:

Proposition 5.22. *Let G be a 2-edge-connected graph. Then the following are equivalent:*
  (1) *The canonical map $\psi_G : E(G) \to \mathbb{P}(\mathbf{H}^1(G))$ is injective.*
  (2) *The map $S^{(2)} : \mathrm{Div}_+^2(G) \to \mathrm{Jac}(G)$ is injective.*
  (3) *G is 3-edge-connected.*

Proof. We already know by Theorem 1.9 that $(2) \Leftrightarrow (3)$, so it suffices to prove that $(1) \Leftrightarrow (3)$. Suppose first that G is 3-edge-connected, and let $e_1, e_2 \in E(G)$. Since $G - \{e_1, e_2\}$ is connected, there is a cycle C containing e_1 but not e_2. The characteristic function χ_C of C is then a flow belonging to $W(e_2)$ but not $W(e_1)$, from which it follows that ψ_G is injective.

Conversely, suppose G is not 3-edge-connected. Then there exist edges $e_1, e_2 \in E(G)$ such that $G - \{e_1, e_2\}$ is disconnected. It follows that any flow $\omega \in \mathbf{H}^1(G)$ which is non-zero on e_1 must also be non-zero on e_2. Thus $W(e_1) = W(e_2)$, and ψ_G is not injective. $\qquad\square$

Remark 5.23. One can define an analogue $\psi_{G,A}$ of the canonical map for flows with values in an arbitrary abelian group A, and certain graph-theoretic assertions about A-flows translate nicely into statements about $\psi_{G,A}$. For example, for $A = \mathbb{Z}/5\mathbb{Z}$, Tutte's famous 5-flow conjecture (c.f. [Bol98, §X.4, p. 348]) is equivalent to the assertion that if G is a 2-edge-connected graph, then the image of $\psi_{G,\mathbb{Z}/5\mathbb{Z}} : E(G) \to \mathbb{P}(H^1(G, \mathbb{Z}/5\mathbb{Z}))$ is contained in an affine subspace (i.e., there exists a hyperplane in $\mathbb{P}(H^1(G, \mathbb{Z}/5\mathbb{Z}))$ disjoint from $\psi_{G,\mathbb{Z}/5\mathbb{Z}}(E(G))$).

5.5. **Hyperelliptic graphs without Weierstrass points.** We conclude by using Theorem 5.12 and the Riemann-Hurwitz formula for graphs to give a complete characterization of all hyperelliptic graphs having no Weierstrass points. (Graphs with no Weierstrass points are quite interesting from the point of view of arithmetic geometry, c.f. [Bak07, Corollary 4.10].)

Recall from [Bak07] that, by analogy with the theory of Riemann surfaces, a vertex $x \in V(G)$ is called a *Weierstrass point* if $r(g(x)) \geq 1$. An example is given in [Bak07] of a family of graphs of genus at least 2 with no Weierstrass points, namely the family $B_n = B(1, 1, \ldots, 1)$ consisting of

two vertices joined by $n \geq 3$ edges. This is in contrast to the classical situation, in which every Riemann surface of genus at least 2 has Weierstrass points. (It is also proved in in [Bak07] that every *metric graph* of genus at least 2 does have Weierstrass points.)

Remark 5.24. On a hyperelliptic Riemann surface X, the Weierstrass points are precisely the fixed points of the hyperelliptic involution. For a 2-edge-connected graph G, it is easy to see that a fixed point of the hyperelliptic involution is a Weierstrass point, and if $g(G) = 2$ then the converse also holds. However, if $g(G) \geq 3$ then the converse does not always hold, as the following example shows. Let G be the hyperelliptic graph $B(3,3,3,3)$ of genus 3, and let $x, y \in V(G)$ be the internal vertices of one of the edges of G. Then it is not hard to verify that x and y are Weierstrass points. Since $\iota(x) = y$, we see that these points are not fixed by the hyperelliptic involution ι on G.

It turns out that apart from a few exceptions, hyperelliptic graphs almost always have Weierstrass points. One exception is the family of graphs B_n mentioned above. Another is the family of graphs $B(l_1, l_2, l_3)$, where l_1, l_2, l_3 are odd positive integers (c.f. Example 5.2). A third exception is the family of graphs $\Phi(l)$ described in the next paragraph.

For every integer $l \geq 1$, let the graph $\Phi(l)$ consist of two disjoint paths $P = [x_0, x_1, \ldots, x_l]$ and $Q = [y_0, y_2, \ldots, y_l]$ of length l, together with two pairs of parallel edges joining x_0 to y_0 and x_l to y_l, respectively. It is easy to verify that for the unique involution $\iota : G \to G$ sending x_i to y_i, the quotient graph $\Phi(l)/\iota$ is isomorphic to a path of length l. Thus $\Phi(l)$ is hyperelliptic for all l.

Remark 5.25. It follows from Corollary 5.10 that $x \in V(G)$ is a Weierstrass point if and only if $\rho(x) \in V(\overline{G})$ is a Weierstrass point, where $\overline{G}$ is the 2-edge-connected graph obtained by contracting every bridge of G. So without loss of generality, when studying Weierstrass points on graphs it suffices to consider graphs which are 2-edge-connected.

Theorem 5.26. *The following are the only 2-edge-connected hyperelliptic graphs with no Weierstrass points:*

(1) *The graph B_n for some integer $n \geq 3$.*
(2) *The graph $B(l_1, l_2, l_3)$ for some odd integers $l_1, l_2, l_3 \geq 1$.*
(3) *The graph $\Phi(l)$ for some integer $l \geq 1$.*

Proof. Let G be a 2-edge-connected hyperelliptic graph with no Weierstrass points. If $|V(G)| = 2$, then G is isomorphic to B_n for some $n \geq 3$, so without loss of generality, we may assume that $|V(G)| > 2$. By Theorem 5.12, there exists a non-degenerate degree 2 harmonic morphism $\phi : G \to T$ for some tree T with $|V(T)| > 1$. Note that for every $t \in V(T)$ we have $r(\phi^*((t))) = 1$. If $m_\phi(x) = 2$ for some $x \in V(G)$, then x is a Weierstrass point, as $r(g(x)) \geq r(2(x)) = r(\phi^*(\phi(x))) = 1$. Therefore, we may assume without loss of generality that $m_\phi(x) = 1$ for every $x \in V(G)$, so that every $t \in V(T)$ has exactly two preimages under ϕ. By the Riemann-Hurwitz formula for graphs (Theorem 2.14), we have $\sum_{x \in V(G)} v_\phi(x) = 2g + 2$.

Consider a vertex $t \in V(T)$ with $\deg(t) = 1$. Let $\phi^{-1}(t) = \{x, x'\}$, and let x'' be the unique neighbor of x in $V(G) \smallsetminus \phi^{-1}(t)$ (which is well-defined since $m_\phi(x) = 1$). It is easy to see that there are $v_\phi(x) + 1$ edges incident to x, namely the $v_\phi(x)$ vertical edges connecting x to x' and the horizontal edge connecting x to x''. Also, since G is 2-edge-connected, we have $\deg(x) \geq 2$, so by (2.2) we know that $v_\phi(x) \geq 1$. It follows that

$$(v_\phi(x) + 2)(x) \sim (x) + v_\phi(x)(x') + (x'') \geq (x) + (x') = \phi^*(t).$$

Therefore $r((v_\phi(x) + 2)(x)) \geq 1$, and x is a Weierstrass point of G if $v_\phi(x) \leq g - 2$. Thus we may assume without loss of generality that $v_\phi(x) \geq g - 1$ for every $x \in V(G)$ such that $\deg(\phi(x)) = 1$. Let $k := |\{t \in V(T) \mid \deg(t) = 1\}| \geq 2$. We have $2g + 2 = \sum_{x \in V(G)} v_\phi(x) \geq 2k(g - 1) \geq 4(g - 1)$.

It follows that either $g = 2$ and $k \leq 3$, or else $g = 3$ and $k = 2$. In the latter case, T is a path and $v_\phi(x) = 0$ for every $x \in V(G)$ such that $\deg(\phi(x)) > 1$.

If $g = 2$, it is easy to see that G must be isomorphic to the graph $B(l_1, l_2, l_3)$ for some integers $l_1, l_2, l_3 \geq 1$, and if l_i is even for some $i \in \{1, 2, 3\}$ then the middle vertex of the path of length l_i is a Weierstrass point by Example 5.2. If $g = 3$, then by the above we have $v_\phi(x) = 2$ for every $x \in V(G)$ such that $\deg(\phi(x)) = 1$. It follows easily that G is isomorphic to the graph $\Phi(|E(T)|)$.

It remains to show that if G is one of the graphs in (1), (2) or (3), then it has no Weierstrass points. We start by considering the case when G satisfies (1), i.e., $|V(G)| = 2$ and $|E(G)| = n \geq 3$. Let $V(G) = \{x, y\}$. By Theorem 1.5 we have $r((n-1)(x) - (y)) = -1$. Therefore $r(g(x)) = r((n-1)(x)) \leq r((n-1)(x) - (y)) + 1 = 0$. It follows that x is not a Weierstrass point, and by symmetry neither is y.

Now suppose that G is of the form (2), so that $g = 2$. As each l_i is odd, the hyperelliptic involution ι on G has no fixed points (since, in the notation of Example 5.2, we have $\iota(x) = y$ and $\iota(z_i) = z_{l-i}$). Therefore G has no Weierstrass points by Remark 5.24.

Finally, suppose that G is of the form (3), i.e., that G is isomorphic to the graph $\Phi(l)$ for some integer $l \geq 1$. Let the vertices of G be labeled as in the definition of $\Phi(l)$. By symmetry, it suffices to prove that $r(3(x_i)) = 0$ for every integer i such that $0 \leq i \leq l$. Suppose first that $l \leq 3i \leq 2l$. Then $3(x_i) \sim (x_0) + (x_l) + (x_{3i-l})$. Consider the following linear order $<$ on $V(G)$:

$$y_0 < \cdots < y_l < x_0 < \cdots < x_{3i-l-1} < x_l < \cdots < x_{3i-l+1} < x_{3i-l}.$$

The divisor associated to this order is equal to $(x_0) + (x_l) + (x_{3i-l}) - (y_0)$. It follows that $r(3(x_i)) = r((x_0) + (x_l) + (x_{3i-l})) = 0$ in this case. Suppose now that $3i < l$. Then we have

$$3(x_i) \sim (x_{3i}) + 2(x_0) \sim (x_{3i+1}) + 2(y_0) \sim (x_l) + (y_0) + (y_{l-3i-1}).$$

This time, we consider the following linear order $<$ on $V(G)$:

$$y_l < x_l < \cdots < x_0 < y_0 < \cdots < y_{l-3i-2} < y_{l-1} < \cdots < y_{l-3i} < y_{l-3i-1}.$$

The divisor associated to this order is equal to $(x_l) + (y_0) + (y_{l-3i-1}) - (y_l)$. Again it follows that $r(3(x_i)) = 0$. The last remaining case, where $3i > 2l$, follows by symmetry from the case $3i < l$. $\square$

Remark 5.27. It would be interesting to characterize *all* 2-edge-connected graphs G having no Weierstrass points. We do not at present know of any examples in which G is not hyperelliptic.

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