Nathan Sponberg

Homework 5

Exercise 1.38

We will calculate the Fourier series for the function

$$\cos^{2N}(x)\,.$$

Using the exponential notation for $\cos(x)$ we obtain

$$\cos^{2N}(x) = \frac{(e^{ix}+e^{-ix})^{2N}}{2^{2N}}\,.$$

Then, by the binomial theorem, we can expand the numerator as follows

$$\frac{(e^{ix}+e^{-ix})^{2N}}{2^{2N}} = \frac{\sum_{k=0}^{2N}\binom{2N}{k}e^{ix(2N-k)}e^{-ixk}}{2^{2N}} = \frac{1}{2^{2N}}\sum_{k=0}^{2N}\binom{2N}{k}e^{ix(2N-2k)}\,.$$

We note that this final expression is a Fourier series. Thus, we are done.

Exercise 1.41

We will calculate the Fourier series for the function

$$f(x) = \begin{cases} -1 & : -\pi < x < 0 \\ 1 & : 0 < x < \pi \end{cases}$$

First, we calculate the Fourier coefficients. Note that since $f(x)$ is an odd function

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0\,.$$

Now consider the case for which $n \neq 0$. We have

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} f(x)\cos(nx)dx + \int_{-\pi}^{\pi} f(x)\sin(nx)dx\Big]\,.$$

Then since $f(x)$ is odd and $\cos(x)$ is even, the left integral must be equal to zero. Thus, we have

$$\frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} f(x)\cos(nx)dx + \int_{-\pi}^{\pi} f(x)\sin(nx)dx\Big] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx =$$

$$\frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx = \frac{i}{\pi}\frac{-\cos(nx)}{n}\Big|_0^{\pi} = \frac{i(-1)^{n+1}+i}{n\pi}\,.$$

Observe that this final expression is equal to zero for $n = 2k$ and is equal to $2i/n\pi$ for $n = 2k+1$. Thus, the Fourier series for $f(x)$ is

$$\sum_{k<0}\frac{2i}{(2k+1)\pi}e^{i(2k+1)x} + \sum_{k>0}\frac{2i}{(2k+1)\pi}e^{i(2k+1)x} = \sum_{k>0}\frac{2i}{-(2k+1)\pi}e^{-i(2k+1)x} + \sum_{k>0}\frac{2i}{(2k+1)\pi}e^{i(2k+1)x} =$$

$$\sum_{k>0}\frac{2i}{(2k+1)\pi}e^{i(2k+1)x} - \frac{2i}{(2k+1)\pi}e^{-i(2k+1)x} = \sum_{k>0}\frac{2i}{(2k+1)\pi}(e^{i(2k+1)x} - e^{-i(2k+1)x}) =$$

$$\sum_{k>0}\frac{2i}{(2k+1)\pi}2i\sin[(2k+1)x] = \sum_{k>0}\frac{-4}{2k\pi+\pi}\sin(2kx+x)\,.$$

Exercise 1.42

We will calculate the Fourier series for the function

$$f(x) = e^a x\,.$$

The Fourier coefficients are

$$\hat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} e^{ax}dx = \frac{e^{2a\pi}-1}{2a\pi}\,,$$

and

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{ax}e^{-inx}dx = \frac{1}{2\pi}\frac{e^{(a-in)x}}{a-in}\Big|_0^{2\pi} = \frac{1}{2\pi}\frac{e^{(a-in)2\pi}-1}{a-in} = \frac{1}{2\pi}\frac{e^{2a\pi}-1}{a-in}\,.$$

Now we can calculate the Fourier series for $f(x)$. Observe that

$$\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{inx} = \frac{e^{2a\pi}-1}{2\pi}\left[\frac{1}{a}+\sum_{n<0}\frac{e^{inx}}{a-in}+\sum_{n>0}\frac{e^{inx}}{a-in}\right] =$$

$$\frac{e^{2a\pi}-1}{2\pi}\left[\frac{1}{a}+\sum_{n>0}\frac{e^{-inx}}{a+in}+\sum_{n>0}\frac{e^{inx}}{a-in}\right] = \frac{e^{2a\pi}-1}{2\pi}\left[\frac{1}{a}+\sum_{n>0}\frac{(a-in)e^{-inx}+(a+in)e^{inx}}{a^2+n^2}\right] =$$

$$\frac{e^{2a\pi}-1}{2\pi}\left[\frac{1}{a}+\sum_{n>0}\frac{2a\cos(nx)+2ini\sin(nx)}{a^2+n^2}\right] = \frac{e^{2a\pi}-1}{\pi}\left[\frac{1}{2a}+\sum_{n>0}\frac{a\cos(nx)-n\sin(nx)}{a^2+n^2}\right].$$

Exercise 1.43

We will calculate the Fourier series for the function

$$f(x) = \sinh(x) = \frac{e^x-e^{-x}}{2}\,.$$

The Fourier coefficients are as follows

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sinh(x)dx = 0$$

and

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^x-e^{-x}}{2}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{(1-in)x}-e^{(-1-in)x}}{2}dx =$$

$$\frac{1}{4\pi}\Big[\frac{e^{(1-in)x}}{1-in}+\frac{e^{(-1-in)x}}{1+in}\Big]_{-\pi}^{\pi} = \frac{(-1)^n}{4\pi}\Big[\frac{e^{\pi}-e^{-\pi}}{1-in}+\frac{e^{-\pi}-e^{\pi}}{1+in}\Big] =$$

$$\frac{(-1)^n}{2\pi}\Big[\frac{\sinh(\pi)}{1-in}-\frac{\sinh(\pi)}{1+in}\Big] = \frac{(-1)^n}{2\pi}\Big[\frac{(1+in)\sinh(\pi)-(1-in)\sinh(\pi)}{1+n^2}\Big] =$$

$$\frac{(-1)^n}{\pi}\Big[\frac{in\sinh(\pi)}{1+n^2}\Big]\,.$$

Now we can proceed to calculating the Fourier series. Observe that

$$\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{inx} = \frac{\sinh(\pi)}{\pi}\Big[\sum_{n<0}\frac{(-1)^n in}{1+n^2}e^{inx}+\sum_{n>0}\frac{(-1)^n in}{1+n^2}e^{inx}\Big] =$$

$$\frac{\sinh(\pi)}{\pi}\Big[\sum_{n>0}\frac{(-1)^{n+1}in}{1+n^2}e^{-inx}+\sum_{n>0}\frac{(-1)^{n}in}{1+n^2}e^{inx}\Big]=\frac{\sinh(\pi)}{\pi}\Big[\sum_{n>0}\frac{(-1)^{n}in}{1+n^2}e^{inx}-\frac{(-1)^{n}in}{1+n^2}e^{-inx}\Big]=$$

$$\frac{\sinh(\pi)}{\pi}\Big[\sum_{n>0}\frac{(-1)^{n}in}{1+n^2}(e^{inx}-e^{-inx})\Big]=\frac{\sinh(\pi)}{\pi}\Big[\sum_{n>0}\frac{(-1)^{n+1}2n\sin(nx)}{1+n^2}\Big].$$