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Math 564

Homework 2

Exercise 1.13

Proposition: The Hermitian symmetric polynomial $R(t,\bar{t}) = a + bt + \bar{b}\bar{t} + c|t|^2$ has a minimum at the point $(-\bar{b}/c, -b/c)$, where $a \in \mathbb{R}$, $B \in \mathbb{C}$ and $c > 0$.

Proof. Set $t = x + iy$ and $b = k + iz$ for $x, y, k, z \in \mathbb{R}$. Then we can rewrite the Hermitian symmetric polynomial in terms of the two real variables x and y. Observe that

$$a + bt + \bar{b}\bar{t} + c|t|^2 = a + (k + iz)(x + iy) + (k - iz)(x - iy) + c(x^2 + y^2) =$$
$$a + kx + izx + iky - zy + kx - izx - iky - zy + cx^2 + cy^2 =$$
$$a + 2kx - 2zy + cx^2 + cy^2 = a + (cx^2 + 2kx) + (cy^2 - 2zy)\,.$$

Taking the partial derivatives of this polynomial (which we call $f(x, y)$) we see that

$$f_x = 2cx + 2k$$

and

$$f_y = 2cy - 2z\,.$$

These derivatives are zero at the points $x = -k/c$ and $y = z/c$ respectively. Next, we note that $f_{xx} = 2c = f_{yy}$ and $f_{xy} = 0$. Thus, the function f, has a minimum at the point $(-k/c, z/c)$. This means that the $R(t,\bar{t})$ has a minimum at the point $(-\bar{b}/c, -b/c)$.

□

Exercise 1.17

Proposition: The following series

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}\,,$$

converges for any square matrix M, of complex numbers.

Proof. We consider a general matrix form of the given series. Note that

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(M^n)_{11}}{n!} & \cdots & \sum_{n=0}^{\infty} \frac{(M^n)_{1m}}{n!} \\ \vdots & \ddots & \vdots \\ \sum_{n=0}^{\infty} \frac{(M^n)_{m1}}{n!} & \cdots & \sum_{n=0}^{\infty} \frac{(M^n)_{mm}}{n!} \end{pmatrix} = \left(\sum_{n=0}^{\infty} \frac{(M^n)_{jk}}{n!} \right)_{1 \le j,k \le m}\,,$$

where $(M^n)_{jk}$ represents the entry from the j^{th} row and k^{th} column of the matrix M^n. We then set $A = \max\{|M_{jk}|\}$. Next, we prove by induction on n that

$$|(M^n)_{jk}| \le m^{n-1}A^n \text{ for } 1 \le j, k \le m\,.(*)$$

By definition $|M_{jk}| \le A$ for $1 \le j, k \le m$. In order to illustrate the inductive argument, we will also consider the case of $|(M^2)_{jk}|$. Setting R_j and C_k to the j^{th} row vector and the k^{th} column vector of M respectively, we observe that

$$|(M^2)_{jk}| \le |R_j \cdot C_k|\,.$$

Note that the modulus of the product, of any two entries in R_j and C_k must be bounded by A^2. The dot product of the two vectors in $\mathbb{C}^m$ gives us a sum of m items. Hence, it follows that

$$|(M^2)_{jk}| \le |R_j \cdot C_k| \le mA^2\,.$$

We now proceed inductively on n. Assume that for a given n the result from (*) holds. Next, consider the case of $n+1$, i.e., $|(M^{n+1})_{jk}|$. Setting $(R^n)_j$ equal to the j^{th} column vector of M^n, we observe that

$$|(M^{n+1})_{jk}| = |(R^n)_j \cdot C_k| \, .$$

Consider the vector $(R^n)_j$; we see that it is equal to

$$\Big((R^{n-1})_j \cdot C_1, (R^{n-1})_j \cdot C_2, ..., (R^{n-1})_j \cdot C_m\Big) \, .$$

We note that by our induction hypothesis, the modulus of each of the m entries in this vector is less than or equal to $m^{n-1}A^n$. The modulus of each of the entries in C_k is less than or equal to A. Thus, it follows that

$$|(M^{n+1})_{jk}| = |(R^n)_j \cdot C_k| \leq m^n A^n + 1 \, ,$$

and by induction on n we see that the result holds for all $n \in \mathbb{N}$. As our final step we note that the above result indicates that

$$\left(\textstyle\sum_{n=0}^{\infty} \frac{|(M^n)_{jk}|}{n!}\right)_{1\leq j,k\leq m} \leq \left(\textstyle\sum_{n=0}^{\infty} \frac{m^{n-1}A^n}{n!}\right)_{1\leq j,k\leq m} \, .$$

Observe that

$$\sum_{n=0}^{\infty} \frac{m^{n-1}A^n}{n!} = \frac{1}{m}\sum_{n=0}^{\infty} \frac{m^n A^n}{n!} = m^{-1}e^{mA} \, .$$

Thus, each entry in the matrix $\left(\textstyle\sum_{n=0}^{\infty} \frac{|(M^n)_{jk}|}{n!}\right)_{1\leq j,k\leq m}$ converges and therefore e^M must converge.

□

Exercise 1.18

Proposition: If B is an invertible matrix, then for each positive integer k

$$(BMB^{-1})^k = BM^kB^{-1} \, .$$

Proof. We will prove this by induction on k. For $k = 2$ we see that

$$(BMB^{-1})^2 = BMB^{-1}BMB^{-1} = BM^2B^{-1} \, .$$

Now we assume that the result holds for n and we will show that the same is true for $n+1$. Observe that

$$(BMB^{-1})^n + 1 = (BMB^{-1})^n BMB^{-1} = BM^nB^{-1}BMB^{-1} = BM^{n+1}B^{-1} \, .$$

Thus, the result holds for all $n \in \mathbb{N}$.

□

Exercise 1.19

Proposition: If B is invertible, then $Be^MB^{-1} = e^{BMB^{-1}}$.

Proof. Observe that

$$Be^MB^{-1} = B\Big[\sum_{n=0}^{\infty} \frac{M^n}{n!}\Big]B^{-1} = \sum_{n=0}^{\infty} \frac{BM^nB^{-1}}{n!} = \sum_{n=0}^{\infty} \frac{(BMB^{-1})^n}{n!} = e^{BMB^{-1}} \, .$$

□

Exercise 1.22

Proposition: For $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $e^{At} = \begin{pmatrix} e^{\lambda t} & 1 + te^{\lambda t} \\ 0 & e^{\lambda}t \end{pmatrix}$.

Proof. First we will show by induction on n that $A^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$. For $n = 2$, it is easy to see that

$$A^2 = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} .$$

Assume that this result holds for a given n. Then we consider the case for $n + 1$. Observe that

$$A^{n+1} = A^n A = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix} .$$

Thus, we see that the result holds for all $n \in \mathbb{N}$. Now we can examine e^{At}. We know that

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} .$$

Using the above result for A^n, this can be rewritten as

$$\sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} & \sum_{n=0}^{\infty} \frac{n\lambda^{n-1}t^n}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} & \sum_{n=0}^{\infty} \frac{\lambda^{n-1}t^n}{(n-1)!} \\ 0 & \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \end{pmatrix} = .$$

$$\begin{pmatrix} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} & 1 + t\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 1 + te^{\lambda t} \\ 0 & e^{\lambda}t \end{pmatrix} .$$

□

Exercise 1.24

As an initial step we would like to find the general solution to the homogeneous differential equation $(D^2 + m^2)y = 0$. Observe that we can factor the differential operator term so that we obtain

$$(D + im)(D - im) = 0 .$$

Then by inspection, and application of corollary 1.4, we obtain the general solution

$$y_0 = c_1 e^{-imx} + c_2 e^{imx} .$$

We now attempt to generate a particular solution to the inhomogeneous differential equation $(D^2 + m^2) = e^x$ using the method outlined in 4.1. First, we start with the expression $(D - im)g_1 = e^x$ and solve for g_1. We assume a solution of the form $g_1 = c(x)e^{imx}$. Then it follows that

$$(D - im)g_1 = c'(x)e^{imx} + c(x)ime^{imx} - c(x)ime^{imx} = c'(x)e^{imx} = e^x .$$

Solving for $c(x)$ we obtain

$$c(x) = \int_a^x e^t e^{-imt} dt$$

and substituting this into $g_1 = c(x)e^{imx}$ we get

$$g_1 = e^{imx} \int_a^x e^{t(1-im)} dt = \frac{-1}{1 - im} e^{imx} [e^{x(1-im)} - \lim_{a \to \infty} e^{a(1-im)}] = -(1 - im)^{-1} e^x .$$

We now consider the expression $(D + im)g_2 = g_1 = -(im)^{-1}e^x$ and we solve for g_2. By a similar process as above we obtain

$$g_2 = -(1 + im)^{-1} e^{-imx} \int_a^x e^{t(1+im)} dt = \frac{-e^x}{(1 + im)(1 - im)} = \frac{-e^x}{1 + m^2} .$$

Thus, the general solution to the inhomogeneous differential equation $(D^2 + m^2)y = e^x$ is

$$y = c_1 e^{-imx} + c_2 e^{imx} - (1 + m^2)^{-1} e^x .$$