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Math 564

Rewrites 2

Exercise 1.5 (Original)

Proposition. Cauchy sequences of complex numbers converge if and only if, whenever a series $\sum |a_n|$ converges, then $\sum a_n$ converges as well.

Proof. First we assume the convergence of Cauchy sequences. We will show that this implies that absolutely convergent series conditionally converge as well. Let a_n be a sequence of complex numbers such that $\sum_{n=1}^{\infty} |a_n|$ converges. This implies that the sequence of partial sums of $\sum_{n=1}^{\infty} |a_n|$ (we denote the N^th partial sum as $|A|_N$) converges and therefore must be a Cauchy sequence. In particular, this implies that given $\epsilon > 0$, there exists some N such that for $K, M \geq N$ (without lose of generality we can assume that $K > M$)

$$||A|_K - |A|_M| < \epsilon\,.$$

Observe that

$$\epsilon > ||A|_K - |A|_M| = |A|_K - |A|_M = \sum_{M+1}^{K} |a_n| \geq |\sum_{M+1}^{K} a_n| = |A_K + A_M|\,,$$

where A_M is the M^{th} partial sum of $\sum_{n=1}^{\infty} a_n$. Thus, we see that A_N is a Cauchy sequence as well and therefore it converges.

Next assume that if a series converges absolutely then, it converges conditionally. We will show that this implies the convergence of Cauchy sequences. Let a_n be a Cauchy sequence of complex numbers. Since a_n is Cauchy there exists some N_1 such that for $n > N_1$

$$|a_n - a_{N_1}| < \frac{1}{2}\,.$$

Similarly, there exists some N_2 such that for $n > N_2$

$$|a_n - a_{N_2}| < \frac{1}{2^2}\,.$$

Proceeding inductively we see that in general We can find a N_k such that for $n > N_k$

$$|a_n - a_{N_k}| < \frac{1}{2^k}\,.$$

Note that we have created an increasing sequence of indices $N_1 < N_2 < ... < N_k$. Thus we can replace some terms in the above inequalities as follows

$$|a_n - a_{N_k}| < \frac{1}{2^k}$$
$$|a_{N_k} - a_{N_{k-1}}| < \frac{1}{2^{k-1}}$$
$$.$$
$$.$$
$$.$$
$$|a_{N_2} - a_{N_1}| < \frac{1}{2}\,,$$

where $n > N_k$. Setting $n = N_{K+1}$ and combining these inequalities into sums we obtain

$$\sum_{k=1}^{K} |a_{N_{k+1}} - a_{N_k}| < \sum_{k=1}^{K} \frac{1}{2^k}\,.$$

Observe that the right-hand side of this inequality is a geometric series that converges as $K \to \infty$. Therefore the left-hand side must also be a convergent series. By our hypothesis, since $\sum_{k=1}^{K} |a_{N_{k+1}} - a_{N_k}|$ is convergent, it follows that $\sum_{k=1}^{K} a_{N_{k+1}} - a_{N_k}$ is convergent as well. Observe that this series is telescopic and therefore

$$\sum_{k=1}^{K} a_{N_{k+1}} - a_{N_k} = a_{N_{k+1}} = a_n$$

for $n > N_k$. Hence, the Cauchy sequence a_n is convergent.

$\square$

Rewrite of Exercise 1.5

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Proof. First we assume the convergence of Cauchy sequences. We will show that this implies that absolutely convergent series conditionally converge as well. Let $\{a_n\}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} |a_n|$ converges. This implies that the sequence of partial sums of $\sum_{n=1}^{\infty} |a_n|$ (we denote the $N^t h$ partial sum as $|A|_N$) converges and therefore must be a Cauchy sequence. In particular, this implies that given $\epsilon > 0$, there exists some N such that for $K, M \geq N$ (without lose of generality we can assume that $K > M$)

$$||A|_K - |A|_M| < \epsilon\,.$$

Observe that

$$\epsilon > ||A|_K - |A|_M| = |A|_K - |A|_M = \sum_{M+1}^{K} |a_n| \geq |\sum_{M+1}^{K} a_n| = |A_K + A_M|\,,$$

where A_M is the M^{th} partial sum of $\sum_{n=1}^{\infty} a_n$. Thus, we see that A_N is a Cauchy sequence as well and therefore it converges.

Next assume that if a series converges absolutely then, it converges conditionally. We will show that this implies the convergence of Cauchy sequences. Let $\{a_n\}$ be a Cauchy sequence of complex numbers. Since $\{a_n\}$ is Cauchy there exists some N_1 such that for $n > N_1$

$$|a_n - a_{N_1}| < \frac{1}{2}\,.$$

Similarly, there exists some $N_2 > N_1$ such that for $n > N_2$

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Proceeding inductively we see that in general We can find a $N_k > N_{k-1}$ such that for $n > N_k$

$$|a_n - a_{N_k}| < \frac{1}{2^k}\,.$$

Note that we have created an increasing sequence of indices $N_1 < N_2 < ... < N_k$. Thus we can replace some terms in the above inequalities as follows

$$|a_{N_{k+1}} - a_{N_k}| < \frac{1}{2^k}$$

$$|a_{N_k} - a_{N_{k-1}}| < \frac{1}{2^{k-1}}$$

$$\vdots$$

$$|a_{N_2} - a_{N_1}| < \frac{1}{2}\,,$$

Combining these inequalities into sums we obtain

$$\sum_{k=1}^{K} |a_{N_{k+1}} - a_{N_k}| < \sum_{k=1}^{K} \frac{1}{2^k}\,.$$

Observe that the right-hand side of this inequality is a geometric series that converges as $K \to \infty$. Therefore the left-hand side must also be a convergent series. By our hypothesis, since $\sum_{k=1}^{K} |a_{N_{k+1}} - a_{N_k}|$ is convergent, it follows that $\sum_{k=1}^{K} \left(a_{N_{k+1}} - a_{N_k}\right)$ is convergent as well. Observe that this series is telescopic and therefore

$$\sum_{k=1}^{K} a_{N_{k+1}} - a_{N_k} = a_{N_{K+1}} - a_{N_1}\,.$$

Thus, we have shown that $\langle a_n \rangle$ contains a convergent subsequence. This means that the Cauchy sequence a_n must be convergent as well.

□

Exercise 2.11 (Original)

Proposition. Given a projection P on a Hilbert space $\mathcal{H}$, the following holds:

1. $I - P$ is also a projection
2. $\mathcal{R}(P) = \mathcal{N}(I - P)$
3. $\mathcal{H} = \mathcal{R}(P) + \mathcal{N}(P)$

Proof. Let $z \in \mathcal{H}$, then observe that

$$[I(z) - P(z)]^2 = I(I(z)) - 2I(P(z)) + P(P(z)) = I(z) - 2P(z) + P(z) = I(z) - P(z)\,.$$

Thus the first result holds.

Next we will show that $\mathcal{R}(P) \subseteq \mathcal{N}(I-P)$ and $\mathcal{R}(P) \supseteq \mathcal{N}(I-P)$. First observe that given $z \in \mathcal{R}(P)$, there must exist some $w \in \mathcal{H}$ such that $P(w) = z$. However since P is a projection on $\mathcal{H}$, it must also hold that $P(P(w)) = P(z) = z$. Thus z is also the image under P of itself. It follows that $(I - P)(z) = z - z = 0$ and therefore $z \in \mathcal{N}(I - P)$. Now given $z \in \mathcal{N}(I - P)$ we observe that $I(z) - P(z) = z - P(z) = 0$, which implies $P(z) = z$. Hence $z \in \mathcal{R}(P)$. Consequently, we see that result two holds as well.

For the final result, it is clear that $\mathcal{R}(P) + \mathcal{N}(P) \subseteq \mathcal{H}$. Therefore we will show that $\mathcal{H} \subseteq \mathcal{R}(P) + \mathcal{N}(P)$.

I could not figure out how to make this work. Am I not understanding what is being asked? Is it not the union of the two sets?

□

Rewrite of Exercise 2.11

Proposition. Given a projection P on a Hilbert space $\mathcal{H}$, the following holds:

1. $I - P$ is also a projection
2. $\mathcal{R}(P) = \mathcal{N}(I - P)$
3. $\mathcal{H} = \mathcal{R}(P) + \mathcal{N}(P)$

Proof. Let $z \in \mathcal{H}$, then observe that

$$[I(z) - P(z)]^2 = I(I(z)) - 2I(P(z)) + P(P(z)) = I(z) - 2P(z) + P(z) = I(z) - P(z)\,.$$

Thus the first result holds.

Next we will show that $\mathcal{R}(P) \subseteq \mathcal{N}(I-P)$ and $\mathcal{R}(P) \supseteq \mathcal{N}(I-P)$. First observe that given $z \in \mathcal{R}(P)$, there must exist some $w \in \mathcal{H}$ such that $P(w) = z$. However since P is a projection on $\mathcal{H}$, it must also hold that

$P(P(w)) = P(z) = z$. Thus z is also the image under P of itself. It follows that $(I-P)(z) = z - z = 0$ and therefore $z \in \mathcal{N}(I-P)$. Now given $z \in \mathcal{N}(I-P)$ we observe that $I(z) - P(z) = z - P(z) = 0$, which implies $P(z) = z$. Hence $z \in \mathcal{R}(P)$. Consequently, we see that result two holds as well.

For the final result, it is clear that $\mathcal{R}(P) + \mathcal{N}(P) \subseteq \mathcal{H}$. Therefore we will show that $\mathcal{H} \subseteq \mathcal{R}(P) + \mathcal{N}(P)$. Given $z \in \mathcal{H}$ observe that

$$z = P(z) + z - P(z) = P(z) + I(z) - P(z) = P(z) + (I-P)(z)\,.$$

Note that $P(z) \in \mathcal{R}(P)$ and $(I-P)(z) \in \mathcal{R}(I-P)$. From the previous results we know that

$$\mathcal{R}(I-P) = \mathcal{N}(P)\,.$$

Hence, we see that for all $z \in \mathcal{H}$,

$$z \in \mathcal{R}(P) + \mathcal{N}(P)\,.$$

It follows that $\mathcal{H} = \mathcal{R}(P) + \mathcal{N}(P)$.

□

Exercise 5 from D'Angelo HW2 (Original)

Proposition:

$$\int_0^{2\pi} \cos^{2N}(\theta)d\theta = 0 \text{ for } N \in \mathbb{N}\,.$$

Proof. In order to solve this problem, we will make the following substitution

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}\,.$$

Thus, the integral can be rewritten as

$$\int_0^{2\pi} \cos^{2N}(\theta)d\theta = \int_0^{2\pi} \Big(\frac{e^{i\theta} + e^{-i\theta}}{2}\Big)^{2N} d\theta =$$

$$2^{-2N} \int_0^{2\pi} (e^{i\theta} + e^{-i\theta})^{2N} d\theta\,.$$

Using the binomial theorem we can rewrite this as

$$2^{-2N} \int_0^{2\pi} \sum_{k=0}^{2N} \binom{2N}{k} (e^{i\theta})^{2N-k} (e^{-i\theta})^k d\theta = 2^{-2N} \sum_{k=0}^{2N} \binom{2N}{k} \int_0^{2\pi} e^{i\theta(2N-2k)} d\theta =$$

$$2^{-2N} \sum_{k=0}^{2N} \binom{2N}{k} \Big[\frac{e^{i2\theta(N-k)}}{i2(N-K)}\Big]_0^{2\pi} = 2^{-2N} \sum_{k=0}^{2N} \binom{2N}{k} \frac{1-1}{i2(N-K)} = 0\,.$$

□

Rewrite of Exercise 5 from D'Angelo HW2

Proposition:

$$\int_0^{2\pi} \cos^{2N}(\theta)d\theta = 0 \text{ for } N \in \mathbb{N}\,.$$

Proof. In order to solve this problem, we will make the following substitution

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$$2^{-2N} \sum_{k=0, k\neq N}^{2N} \binom{2N}{k} \Big[\frac{e^{i2\theta(N-k)}}{i2(N-K)}\Big]_0^{2\pi} + \binom{2N}{N} \int_0^{2\pi} e^{i\theta 0} d\theta =$$

$$2^{-2N} \sum_{k=0}^{2N} \binom{2N}{k} \frac{1-1}{i2(N-K)} + \binom{2N}{N} \int_0^{2\pi} d\theta = 2\pi \binom{2N}{N}\,.$$

$\square$