Nathan Sponberg
Math 564

**Homework 7**

**Exercise 2.2**

**Proposition.** The Cauchy-Schwarz inequality $|\langle x, y\rangle| \leq \|x\|\|y\|$ holds in $\mathbb{R}^n$.

*Proof.* We will prove the Cauhcy-Schwarz inequality using induction. In particular, we will show the following for $\mathbb{R}^n$

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 = \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \Big(\sum_{i=1}^{n} x_i y_i\Big)^2 = \sum_{1\leq j<k\leq n}^{n} (x_j y_k - x_k y_j)^2$$

for all $n \in \mathbb{N}$. Note that since the right most expression is a sum of square terms, it must be non-negative. We proceed with a proof by induction.

**Basis Step.** We start with the case of $n = 2$ since the $n = 1$ case is too trivial for the proof. Observe that

$$\sum_{i=1}^{2} x_i^2 \sum_{i=1}^{2} y_i^2 - \Big(\sum_{i=1}^{2} x_i y_i\Big)^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1y_1 + x_2y_2)^2 =$$

$$(x_1y_1)^2 + (x_2y_1)^2 + (x_1y_2)^2 + (x_2y_2)^2 - (x_1y_1)^2 - 2x_1y_1x_2y_2 - (x_2y_2)^2 =$$

$$(x_1y_2)^2 - 2x_1y_1x_2y_2 + (x_2y_1)^2 = (x_1y_2 - x_2y_1)^2 .$$

Thus the result holds for $n = 2$.

**Induction Step.** Assume that for $n$ the result holds. Then we consider the case of $n + 1$. We examine the following expression

$$\sum_{i=1}^{n+1} x_i^2 \sum_{i=1}^{n+1} y_i^2 - \Big(\sum_{i=1}^{n+1} x_i y_i\Big)^2 .$$

Extracting the $n + 1^{st}$ term from each summation we can expand as follows

$$\sum_{i=1}^{n+1} x_i^2 \sum_{i=1}^{n+1} y_i^2 - \Big(\sum_{i=1}^{n+1} x_i y_i\Big)^2 =$$

$$\Big(x_{n+1}^2 + \sum_{i=1}^{n} x_i^2\Big)\Big(y_{n+1}^2 + \sum_{i=1}^{n} y_i^2\Big) - \Big(x_n y_{n+1} + \sum_{i=1}^{n} x_i y_i\Big)^2 =$$

$$x_{n+1}^2 y_{n+1}^2 + y_{n+1}^2 \sum_{i=1}^{n} x_i^2 + x_{n+1}^2 \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$$

$$-x_{n+1}^2 y_{n+1}^2 - 2x_{n+1}y_{n+1} \sum_{i=1}^{n} x_i y_i - \Big(\sum_{i=1}^{n} x_i y_i\Big)^2 .$$

We observe that by our induction hypothesis $\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \Big(\sum_{i=1}^{n} x_i y_i\Big)^2 = \sum_{1\leq j<k\leq n}^{n}(x_j y_k - x_k y_j)^2$. Hence we consider the following terms from the previous expression

$$x_{n+1}^2 y_{n+1}^2 + y_{n+1}^2 \sum_{i=1}^{n} x_i^2 + x_{n+1}^2 \sum_{i=1}^{n} y_i^2 - x_{n+1}^2 y_{n+1}^2 - 2x_{n+1}y_{n+1} \sum_{i=1}^{n} x_i y_i =$$

$$y_{n+1}^2 \sum_{i=1}^{n} x_i^2 - 2x_{n+1}y_{n+1} \sum_{i=1}^{n} x_i y_i + x_{n+1}^2 \sum_{i=1}^{n} y_i^2 = .$$

$$\sum_{j=1}^{n}(x_j y_{n+1})^2 - 2x_j y_{n+1} x_{n+1} y_j + (x_{n+1} y_j)^2 = \sum_{j=1}^{n}(x_j y_{n+1} - x_{n+1} y_j)^2$$

Thus, we have

$$\sum_{i=1}^{n+1} x_i^2 \sum_{i=1}^{n+1} y_i^2 - \Big(\sum_{i=1}^{n+1} x_i y_i\Big)^2 = \sum_{1\le j<k\le n} (x_j y_k - x_k y_j)^2 + \sum_{j=1}^{n}(x_j y_{n+1} - x_{n+1} y_j)^2 =$$

$$\sum_{1\le j<k\le n}^{n+1} (x_j y_k - x_k y_j)^2 \,.$$

Consequently, by induction on $n$, the result holds for all $n \in \mathbb{N}$.

$\square$

**Exercise 2.3**

**Proposition.** The Cauchy-Schwarz inequality $|\langle x, y\rangle| \le \|x\|\|y\|$ holds in $\mathbb{R}^n$.

*Proof.* We use the method of Lagrange multipliers for this proof. First note that The inequality

$$\sum_{i=1}^{n} x_i y_i \le \Big(\sum_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n} y_i^2\Big)^{\frac{1}{2}}$$

is equivalent to

$$\sum_{i=1}^{n} x_i \frac{y_i}{\|y\|} \le \Big(\sum_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}} .$$

Hence we will consider the expression $\sum_{i=1}^{n} x_i z_i$ (which we refer to as [*]) where $z_i = \frac{y_i}{\|y\|}$. We then use Lagrange multipliers to maximize this function. First note that we treat $\vec{x}$ as fixed and maximize this function with respect to $\vec{y}$. We use $\sum_{i=1}^{n} z_i^2 = 1$ as a constraint since

$$\|z\| = \Big(\sum_{i=1}^{n} z_i^2\Big)^{\frac{1}{2}} = 1 = \sum_{i=1}^{n} z_i^2$$

(due to $\vec{z}$ being normalized). The Lagrangian of this system is

$$\Lambda(\vec{z}, \lambda) = \sum_{i=1}^{n} x_i z_i + \lambda\Big(\sum_{i=1}^{n} z_i^2 - 1\Big) .$$

Computing the gradient of this in terms of the $z_i$ and $\lambda$ gives us

$$\nabla_{\vec{z},\lambda}\Lambda(\vec{z}, \lambda) = \Big(\vec{x} + 2\lambda\vec{z}, \sum_{i=1}^{n} z_i^2 - 1\Big) = \Big(\vec{x} + 2\lambda\vec{z}, 0\Big) = \vec{x} + 2\lambda\vec{z} .$$

Now setting this equal to zero we observe that

$$\vec{x} = -2\lambda\vec{z} .$$

Since we have two vectors that are equal, we know that their norms are equal as well. Thus,

$$\frac{\|x\|}{2} = |\lambda|\|z\| = |\lambda| .$$

Making this substitution for $\lambda$ into the expression $\nabla \sum_{i=1}^{n} x_i z_i = -\lambda \nabla \sum_{i=1}^{n} z_i^2$ yields the following

$$\vec{x} = \frac{\|x\|}{2} 2\vec{z} = \|x\| \vec{z}$$

which implies

$$\vec{z} = \frac{\vec{x}}{\|x\|} .$$

Therefore, this is the value of $\vec{z}$ that gives us a maximum for the function [*]. This implies that

$$\sum_{i=1}^{n} x_i z_i \leq \sum_{i=1}^{n} x_i \frac{x_i}{\|x\|} = \frac{\sum_{i=1}^{n} x_i^2}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} = \|x\| .$$

This gives us the equivalent inequality that we wanted. Thus, the result holds.

$\square$