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Math 564

Homework 12

Exercise 2.31

Proposition. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof. We start by computing the Fourier series of $f(x) = (\pi - x)^2$ on $(0, 2\pi)$. Observe that for $n \neq 0$

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} (x - \pi)^2 e^{-inx} dx\,.$$

We can then make the following change of variable $y = x - \pi$. Then

$$\frac{1}{2\pi} \int_{\pi}^{\pi} y^2 e^{-iny} e^{-in\pi} dy = \frac{(-1)^n}{2\pi} \int_{\pi}^{\pi} y^2 e^{-iny} dy\,.$$

Then using integration by parts twice we have

$$\frac{(-1)^n}{2\pi} \int_{\pi}^{\pi} y^2 e^{-iny} dy = \frac{-1^n}{2\pi} \left(\frac{-1}{in} y^2 e^{-iny} \Big|_{-\pi}^{\pi} + \frac{2}{in} \int_{-\pi}^{\pi} y e^{-iny} dy \right) = \frac{-1^n}{2\pi} \left(\frac{2}{in} \int_{-\pi}^{\pi} y e^{-iny} dy \right) =$$

$$\frac{-1^n}{2\pi} \frac{2}{in} \left(\frac{-1}{in} y e^{-iny} \Big|_{-\pi}^{\pi} + \frac{1}{in} \int_{-\pi}^{\pi} e^{-iny} dy \right) =$$

$$\frac{-1^n}{2\pi} \frac{2}{n^2} \left(\pi(-1)^n + \pi(-1)^n \right) = \frac{-1^n}{2\pi} \frac{4\pi(-1)^n}{n^2} = \frac{2}{n^2}\,.$$

For $n = 0$ we have

$$\hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} (x - \pi)^2 e^{-i0x} dx = \frac{1}{2\pi} \int_0^{2\pi} (x - \pi)^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} y^2 dy = \frac{1}{2\pi} \left(\frac{\pi^3}{3} + \frac{\pi^3}{3} \right) = \frac{\pi^2}{3}\,.$$

Hence, it follows that

$$f(x) = \sum_{\infty}^{\infty} c_n e^{inx} = \frac{\pi^2}{3} + \sum_{n \neq 0} \frac{2}{n^2} e^{inx}\,.$$

Next, note that $f(0) = \pi^2$. Therefore,

$$\pi^2 = \frac{\pi^2}{3} + 2 \sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}\,.$$

Solving for $\sum_{n=1}^{\infty} \frac{1}{n^2}$ we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

□

Exercise 2.32

Proposition. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{-\pi^2}{12}$.

Proof. We start by computing the Fourier series of $f(x) = x^2$ on $(-\pi, \pi)$. Observe that for $n \neq 0$

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx \,.$$

Then using integration by parts twice we have

$$\frac{1}{2\pi} \int_{\pi}^{\pi} x^2 e^{-inx} dx = \frac{1}{2\pi} \left(\frac{-1}{in} x^2 e^{-inx} \Big|_{-\pi}^{\pi} + \frac{2}{in} \int_{-\pi}^{\pi} x e^{-inx} dx \right) = \frac{1}{2\pi} \left(\frac{2}{in} \int_{-\pi}^{\pi} x^{-inx} dx \right) =$$

$$\frac{1}{2\pi} \frac{2}{in} \left(\frac{-1}{in} x e^{-inx} \Big|_{-\pi}^{\pi} + \frac{1}{in} \int_{-\pi}^{\pi} e^{-inx} dx \right) =$$

$$\frac{1}{2\pi} \frac{2}{n^2} \left(\pi(-1)^n + \pi(-1)^n \right) = \frac{1}{2\pi} \frac{4\pi(-1)^n}{n^2} = \frac{2(-1)^n}{n^2} \,.$$

For $n = 0$ we have

$$\hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} x^2 e^{-i0x} dx = \frac{1}{2\pi} \int_0^{2\pi} x^2 dx = \frac{1}{2\pi} \left(\frac{\pi^3}{3} + \frac{\pi^3}{3} \right) = \frac{\pi^2}{3} \,.$$

Hence, it follows that

$$f(x) = \sum_{\infty}^{\infty} c_n e^{inx} = \frac{\pi^2}{3} + \sum_{n \neq 0} \frac{2(-1)^n}{n^2} e^{inx} \,.$$

Next, note that $f(0) = 0$. Therefore,

$$0 = \frac{\pi^2}{3} + 2 \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \,.$$

Solving for $\sum_{n=1}^{\infty} \frac{1}{n^2}$ we conclude that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12} \,.$$

$\square$

Exercise 2.33

Proposition. Given $\epsilon > 0$, there exists a $C_\epsilon > 0$ such that

$$|\langle x, y \rangle| \leq \epsilon \|x\|^2 + C_\epsilon \|y\|^2 \,.$$

Proof. By the Cauchy-Schwartz inequality we have

$$|\langle x, y \rangle| \leq \|x\| \|y\| = \frac{\sqrt{2\epsilon}}{\sqrt{2\epsilon}} \|x\| \|y\| = \|\sqrt{2\epsilon} x\| \| \frac{1}{\sqrt{2\epsilon}} y \| \,.$$

Then applying the AGM inequality we conclude that

$$\|\sqrt{2\epsilon} x\| \| \frac{1}{\sqrt{2\epsilon}} y \| \leq \frac{\|\sqrt{2\epsilon} x\|^2 + \|\frac{1}{\sqrt{2\epsilon}} y\|^2}{2} = \epsilon \|x\|^2 + \frac{1}{4\epsilon} \|y\|^2 \,.$$

Thus, we have

$$|\langle x, y \rangle| \leq \epsilon \|x\|^2 + \frac{1}{4\epsilon} \|y\|^2 \,,$$

which is the desired result with $C_\epsilon = \frac{1}{4\epsilon}$.

$\square$

Exercise 2.34

Proposition. Second inequality from 2.9.

Proof. First assume that

$$\|Lf\|^2 \leq \epsilon\|f\|^2 + \|T_\epsilon f\|^2\,,$$

for $\epsilon > 0$ and T_ϵ a compact operator. Then observe that

$$\epsilon\|f\|^2 + \|T_\epsilon f\|^2 \leq \epsilon\|f\|^2 + \|T_\epsilon f\|^2 + 2\sqrt{\epsilon}\|f\|\|T_\epsilon f\| = \left(\sqrt{\epsilon}\|f\| + \|T_\epsilon f\|\right)^2\,.$$

By taking square roots it then follows that

$$\|Lf\| \leq \sqrt{\epsilon}\|f\| + \|T_\epsilon f\|\,.$$

By the first inequality in Proposition 2.9, it follows that L must be compact.

Next assume that L is compact. Then for $\epsilon > 0$ there exists some compact operator K_ϵ such that

$$\|Lf\| \leq \epsilon\|f\| + \|K_\epsilon f\|\,.$$

Observe then that

$$\|Lf\|^2 \leq (\epsilon\|f\| + \|K_\epsilon f\|)^2 = \epsilon^2\|f\|^2 + 2\epsilon\|f\|\|K_\epsilon f\| + \|K_\epsilon f\|^2 \leq\,.$$

$$\epsilon^2\|f\|^2 + \epsilon^2\|f\|^2 + \|K_\epsilon f\|^2 + \|K_\epsilon f\|^2 = 2\epsilon^2\|f\|^2 + \|\sqrt{2}K_\epsilon f\|^2\,.$$

Hence, we have found the desired inequality to prove the result

$$\|Lf\|^2 \leq 2\epsilon^2\|f\|^2 + \|\sqrt{2}K_\epsilon f\|^2\,.$$

□

Exercise 2.35

Proposition. Assume $L \in \mathcal{L}(\mathcal{H})$. Then if L is compact, L^* is as well. Furthermore, L is compact if and only if L^*L is compact.

Proof. First suppose that L is compact. Then LL^* is compact as well, by Proposition 2.10. Now observe that

$$\|L^*f\| = |\langle L^*f, L^*f\rangle| = |\langle f, LL^*f\rangle|\,.$$

Then given an $\epsilon > 0$, by exercise 2.33, it follows that

$$|\langle f, LL^*f\rangle| \leq \epsilon\|f\|^2 + \frac{1}{4\epsilon}\|LL^*f\|^2 = \epsilon\|f\|^2 + \|\frac{1}{2\sqrt{\epsilon}}LL^*f\|^2\,.$$

Since LL^* is compact, it follows by Proposition 2.9 that L^* is compact as well.

First assume that L^*L is compact. Observe that

$$\|Lf\| = |\langle Lf, Lf\rangle| = |\langle f, L^*Lf\rangle|\,.$$

Then given an $\epsilon > 0$, by exercise 2.33, it follows that

$$|\langle f, L^*Lf\rangle| \leq \epsilon\|f\|^2 + \frac{1}{4\epsilon}\|L^*Lf\|^2 = \epsilon\|f\|^2 + \|\frac{1}{2\sqrt{\epsilon}}L^*Lf\|^2\,.$$

For this we can conclude that since L^*L is compact, L must be compact by Proposition 2.9.

□

Exercise 3.2

Proposition. Proposition 3.1 and 3.2 from the book.

Proof. Let $f, g \in \mathcal{S}$.

$\mathcal{S}$ is closed under differentiation.

Since $f \in \mathcal{S}$ is follows that for all $0 \leq a, b \in \mathbb{Z}$

$$\lim_{|x|\to\infty} |x|^a \left(\frac{d}{dx}\right)^b f(x) = 0\,.$$

If we let $h(x) = \frac{d}{dx} f(x)$, then since $0 \leq b+1 \in \mathbb{Z}$ we can conclude that

$$\lim_{|x|\to\infty} |x|^a \left(\frac{d}{dx}\right)^b h(x) = \lim_{|x|\to\infty} |x|^a \left(\frac{d}{dx}\right)^{b+1} f(x) = 0\,.$$

Hence, $h(x) \in \mathcal{S}$ and $\mathcal{S}$ is closed under differentiation.

$\mathcal{S}$ is closed under multiplication.

Since $f, g \in \mathcal{S}$ is follows that for all $0 \leq a, b \in \mathbb{Z}$

$$\lim_{|x|\to\infty} |x|^a \left(\frac{d}{dx}\right)^b f(x) = 0\,.$$

$$\lim_{|x|\to\infty} |x|^a \left(\frac{d}{dx}\right)^b g(x) = 0\,,$$

and in particular

$$\lim_{|x|\to\infty} g(x) = 0\,.$$

Then since f and g are infinity differentiable we see that for all $0 \leq a, b \in \mathbb{Z}$

$$0 = 0 \cdot 0 = \left(\lim_{|x|\to\infty} g(x)\right)\left(\lim_{|x|\to\infty} |x|^a \left(\frac{d}{dx}\right)^b f(x)\right) = \lim_{|x|\to\infty} |x|^a \left(\frac{d}{dx}\right)^b f(x)g(x)\,.$$

Thus, we see that $f(x)g(x) \in \mathcal{S}$ and $\mathcal{S}$ is closed under multiplication.

$\mathcal{F}$ is linear.

Observe that

$$\mathcal{F}(\alpha f + \beta g)(\xi) = \frac{1}{\sqrt{2\pi}} \int [\alpha f(x) + \beta g(x)] e^{-ix\xi} dx =$$

$$\frac{1}{\sqrt{2\pi}} \int \alpha f(x) e^{-ix\xi} dx + \frac{1}{\sqrt{2\pi}} \int \beta g(x) e^{-ix\xi} dx =$$

$$\alpha \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\xi} dx + \beta \frac{1}{\sqrt{2\pi}} \int g(x) e^{-ix\xi} dx = \alpha \mathcal{F}(f(\xi) + \beta \mathcal{F}(g)(\xi)\,.$$

Thus, $\mathcal{F}$ is linear.

Bounded : $\|\hat{f}\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|f\|_L^1$.

Observe that

$$|\mathcal{F}(f)(\xi)| = |\frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\xi} dx| \leq \frac{1}{\sqrt{2\pi}} \int |f(x) e^{-ix\xi}| dx =$$

$$\frac{1}{\sqrt{2\pi}} \int |f(x)| dx = \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}\,.$$

Conjugate : $\hat{\overline{f}}(\xi) = \overline{\hat{f}(-\xi)}$

Observe that

$$\hat{\overline{f}}(\xi) = \frac{1}{\sqrt{2\pi}} \int \overline{f}(x) e^{-ix\xi} dx = \frac{1}{\sqrt{2\pi}} \int \overline{f(x) e^{ix\xi}} dx =$$

$$\overline{\frac{1}{\sqrt{2\pi}} \int f(x) e^{ix\xi} dx} = \overline{\hat{f}(-\xi)}\,.$$

If $f_h(x) = f(x+h)$, then $\hat{f}_h(\xi) = e^{ih\xi}\hat{f}(\xi)$.

Note that

$$\hat{f}_h(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)_h e^{-ix\xi}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x+h)e^{-ix\xi}dx\,.$$

If we then make the change of variable for $y = x+h$ we obtain

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x+h)e^{-ix\xi}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i(y-h)\xi}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-iy\xi}e^{ih\xi}dx =$$

$$e^{ih\xi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-iy\xi}dx = e^{ih\xi}\hat{f}(\xi)\,.$$

The following identities hold: $D_{i\xi}\mathcal{F} = \mathcal{F}M_x$ and $\mathcal{F}D_x = M_{i\xi}\mathcal{F}$.

First observe that

$$D_{i\xi}\mathcal{F} = \frac{d}{d\xi}\hat{f}(\xi) = \frac{d}{d\xi}\frac{1}{\sqrt{2\pi}}\int f(x)e^{-ix\xi}dx = \frac{1}{\sqrt{2\pi}}\int f(x)(-ix)e^{-ix\xi}dx =$$

$$-i\mathcal{F}(M_x f)\,.$$

Next we note that

$$\hat{f}'(\xi) = \frac{1}{\sqrt{2\pi}}\int f(x)'e^{-ix\xi}dx = \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-ix\xi}\Big|_{-\infty}^{\infty} + i\xi\int f(x)e^{-ix\xi}dx\right]\,.$$

Since $f \in \mathcal{S}$ it follows that the first part of the sum goes to zero. Hence

$$\frac{1}{\sqrt{2\pi}}\left[f(x)e^{-ix\xi}\Big|_{-\infty}^{\infty} + i\xi\int f(x)e^{-ix\xi}dx\right] = \frac{1}{\sqrt{2\pi}}i\xi\int f(x)e^{-ix\xi}dx = i\xi\hat{f}(\xi)\,.$$

$\square$