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Math 564

Homework 7

Exercise 2.11

Proposition. Given a projection P on a Hilbert space $\mathcal{H}$, the following holds:

1. $I - P$ is also a projection
2. $\mathcal{R}(P) = \mathcal{N}(I - P)$
3. $\mathcal{H} = \mathcal{R}(P) + \mathcal{N}(P)$

Proof. Let $z \in \mathcal{H}$, then observe that

$$[I(z) - P(z)]^2 = I(I(z)) - 2I(P(z)) + P(P(z)) = I(z) - 2P(z) + P(z) = I(z) - P(z)\,.$$

Thus the first result holds.

Next we will show that $\mathcal{R}(P) \subseteq \mathcal{N}(I-P)$ and $\mathcal{R}(P) \supseteq \mathcal{N}(I-P)$. First observe that given $z \in \mathcal{R}(P)$, there must exist some $w \in \mathcal{H}$ such that $P(w) = z$. However since P is a projection on $\mathcal{H}$, it must also hold that $P(P(w)) = P(z) = z$. Thus z is also the image under P of itself. It follows that $(I - P)(z) = z - z = 0$ and therefore $z \in \mathcal{N}(I - P)$. Now given $z \in \mathcal{N}(I - P)$ we observe that $I(z) - P(z) = z - P(z) = 0$, which implies $P(z) = z$. Hence $z \in \mathcal{R}(P)$. Consequently, we see that result two holds as well.

For the final result, it is clear that $\mathcal{R}(P) + \mathcal{N}(P) \subseteq \mathcal{H}$. Therefore we will show that $\mathcal{H} \subseteq \mathcal{R}(P) + \mathcal{N}(P)$.

I could not figure out how to make this work. Am I not understanding what is being asked? Is it not the union of the two sets?

□

Exercise 2.12

Proposition. For a fixed w in a Hilbert space $\mathcal{H}$, linear operator $P(v)$ on $\mathcal{H}$ given by

$$P(v) = \frac{\langle v, w\rangle}{\|w\|^2} w$$

is a projection on $\mathcal{H}$.

Proof. Observe that

$$P(P(v)) = P\Big(\frac{\langle v, w\rangle}{\|w\|^2} w\Big) = \frac{\langle \frac{\langle v,w\rangle}{\|w\|^2} w, w\rangle}{\|w\|^2} w\,.$$

Since the inner product of a Hilbert space $\mathcal{H}$ is a complex scalar we can factor out $\langle v, w\rangle/\|w\|^2$. Thus, we see that

$$\frac{\langle \frac{\langle v,w\rangle}{\|w\|^2} w, w\rangle}{\|w\|^2} w = \frac{\langle v, w\rangle}{\|w\|^2}\frac{\langle w, w\rangle}{\|w\|^2} w = \frac{\langle v, w\rangle}{\|w\|^2} w = P(v)\,.$$

□

Exercise 2.13

Proposition. Let $\mathcal{H} = L^2[-1, 1]$ and V_e and V_o be the subspaces of even and odd functions, respectively, in $\mathcal{H}$. Then V_e is orthogonal to V_o.

Proof. The inner product in $L^2[-1,1]$ is defined as

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx\,.$$

If f is an arbitrary function in V_e and g is an arbitrary function in V_o, then we observe that fg must itself be an odd function. Hence, it follows that for all $f \in V_e$ and all $g \in V_o$

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx = 0\,.$$

Therefore, V_e and V_o are orthogonal subspaces of $\mathcal{H}$.

□

Exercise 2.16

First note that the inequality

$$0 \leq \|z - \sum_{n=1}^{N} \langle z, z_n\rangle z_n\|^2$$

must be true since the squared norm in a Hilbert space will always return a non-negative real number. Next, observe that

$$\|z - \sum_{n=1}^{N} \langle z, z_n\rangle z_n\|^2 = \Big\langle z - \sum_{n=1}^{N} \langle z, z_n\rangle z_n, z - \sum_{n=1}^{N} \langle z, z_n\rangle z_n \Big\rangle =$$

$$\Big\langle z, z\Big\rangle - \Big\langle z, \sum_{n=1}^{N} \langle z, z_n\rangle z_n \Big\rangle - \Big\langle \sum_{n=1}^{N} \langle z, z_n\rangle z_n, z\Big\rangle + \Big\langle \sum_{n=1}^{N} \langle z, z_n\rangle z_n, \sum_{n=1}^{N} \langle z, z_n\rangle z_n \Big\rangle =$$

$$\|z\|^2 - \sum_{n=1}^{N} \overline{\langle z, z_n\rangle} \langle z, z_n\rangle - \sum_{n=1}^{N} \langle z, z_n\rangle \overline{\langle z, z_n\rangle} + \|\sum_{n=1}^{N} \langle z, z_n\rangle z_n\|^2 =$$

$$\|z\|^2 - 2\sum_{n=1}^{N} |\langle z, z_n\rangle|^2 + \|z_n\|^2 |\sum_{n=1}^{N} \langle z, z_n\rangle|^2 = \|z\|^2 - \sum_{n=1}^{N} |\langle z, z_n\rangle|^2\,.$$

Thus, we have

$$0 \leq \|z - \sum_{n=1}^{N} \langle z, z_n\rangle z_n\|^2 = \|z\|^2 - \sum_{n=1}^{N} |\langle z, z_n\rangle|^2\,.$$

Exercise 2.17

Proposition. For $\mathcal{H} = L^2[0,1]$, the orthogonal projection of x^2 onto $\text{span}(\{1, x\})$ is $x-1/6$. For $\mathcal{H} = L^2[-1,1]$, the orthogonal projection of x^2 onto $\text{span}(\{1, x\})$ is $1/3$.

Proof. We will find the minimum of the expression $\|x^2 - xa - b\|$ which will give us the orthogonal projection onto the spanning set. First we consider the case of $L^2[0,1]$. Observe that

$$\|x^2 - xa - b\| = \int_0^1 (x^2 - xa - b)^2 = \int_0^1 x^4 - 2ax^3 - 2bx^2 + a^2x^2 + 2abx + b^2 dx =$$

$$\frac{1}{5} - \frac{1}{2}a - \frac{2}{3}b + \frac{1}{3}a^2 + ab + b^2\,.$$

We then take partial derivatives of this function in terms of a and b. This yields

$$\frac{\partial}{\partial a} = -\frac{1}{2} + \frac{2}{3}a + b$$

and

$$\frac{\partial}{\partial b} = -\frac{2}{3} + a + 2b\,.$$

Setting these equal to zero we obtain

$$b = \frac{1}{2} - \frac{2}{3}a$$

and

$$a = \frac{2}{3} - 2b\,.$$

From this it follows that $a = 1$ and $b = -1/6$. Next we consider the case of $L^2[-1,1]$. Observe that

$$\|x^2 - xa - b\| = \int_{-1}^{1} (x^2 - xa - b)^2 = \int_{-1}^{1} x^4 - 2ax^3 - 2bx^2 + a^2x^2 + 2abx + b^2 dx =$$

$$\frac{2}{5} - \frac{4}{3}b + \frac{2}{3}a^2 + 2b^2\,.$$

We then take partial derivatives of this function in terms of a and b. This yields

$$\frac{\partial}{\partial a} = \frac{4}{3}a$$

and

$$\frac{\partial}{\partial b} = -\frac{4}{3} + 4b\,.$$

Setting these equal to zero we obtain

$$b = \frac{1}{3}$$

and

$$a = 0\,.$$

From this it follows that $a = 0$ and $b = 1/3$.

$\square$