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Math 564

Homework 1

Exercise 1.2

Proposition. The infinite series $\sum\limits_{n=2}^{\infty} \frac{\sin(nx)}{\log(n)}$ converges.

Proof. We will use Corollary 1.1 to show that the series converges. First we will re-index the series to start at $n = 1$,

$$\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log(n)} = \sum_{n=1}^{\infty} \frac{\sin[(n+1)x]}{\log(n+1)}\,.$$

Then set $a_n = 1/\log(n)$ and $b_n = \sin[(n+1)x]$. Observe that $a_n \to 0$ as $n \to \infty$. Next consider the series $\sum |a_{n+1} - a_n|$. Note that $a_{n+1} < a_n$ for all $n \in \mathbb{N}$, thus

$$\sum_{n=1}^{N} \Big| \frac{1}{\log(n+2)} - \frac{1}{\log(n+1)} \Big| = \sum_{n=1}^{N} \frac{1}{\log(n+1)} - \frac{1}{\log(n+2)} =$$

$$\frac{1}{\log(2)} - \frac{1}{\log(3)} + \frac{1}{\log(3)} - \frac{1}{\log(4)} + \cdots - \frac{1}{\log(N+1)} + \frac{1}{\log(N+1)} - \frac{1}{\log(N+2)} =$$

$$\frac{1}{\log(2)} - \frac{1}{N+2}\,.$$

Note that as $N \to \infty$, $1/\log(N+2) \to 0$ and the above series converges to $1/\log(2)$.
As the final step, we show that the partial sums of $\sum_{n=1}^{N} b_n$ (which we denote as B_N) are bounded. Recall the following identity

$$\sin[(n+1)x] = \frac{e^{i(n+1)x} - e^{-i(n+1)x}}{2i}\,.$$

From this it follows that

$$\sum_{n=1}^{N} \sin[(n+1)x] = \sum_{n=1}^{N} \frac{e^{i(n+1)x} - e^{-i(n+1)x}}{2i} = \frac{1}{2i}\Big[\sum_{n=1}^{N} e^{i(n+1)x} - \sum_{n=1}^{N} e^{-i(n+1)x}\Big]$$

Consider the term $\sum_{n=1}^{N} e^{i(n+1)x}$. We note that since $e^{ix} \neq 1$, this is a finite geometric series. Hence we have

$$\sum_{n=1}^{N} e^{i(n+1)x} = e^{ix}\frac{1 - e^{i(N+1)x}}{1 - e^{ix}}\,.$$

Observe that

$$\Big| e^{ix}\frac{1 - e^{i(N+1)x}}{1 - e^{ix}} \Big| \leq \frac{|1 - e^{i(N+1)x}|}{|1 - e^{ix}|} \leq \frac{2}{|1 - e^{ix}|}\,.$$

Note that the right most term in this inequality is not dependant on N and thus $\sum_{n=1}^{N} e^{i(n+1)x}$ is bounded. Similarly, $-\sum_{n=1}^{N} e^{-i(n+1)x}$ is also bounded and therefore the partial sums B_N, are bounded. Consequently, by Corollary 1.1, $\sum\limits_{n=2}^{\infty} \frac{\sin(nx)}{\log(n)}$ must converge.

□

Proposition. For $\alpha > 0$, the infinite series $\sum\limits_{n=1}^{\infty} \frac{\sin(nx)}{n^{\alpha}}$ converges.

Proof. Again we use Corollary 1.1 to prove convergence. Note that if we again set $b_n = \sin(nx)$, then the partial sums B_N are bounded (as shown in the text).
Next we set $a_n = 1/n^\alpha$. Since $\alpha > 0$ it is clear that $a_n \to 0$ as $n \to \infty$. As a final step, consider the series $\sum |a_{n+1} - a_n|$. We note that $1/n^\alpha > 1/n^\alpha$ for all $n \in \mathbb{N}$. It follows that

$$\Big|\sum_{n=1}^{N} \frac{1}{(n+1)^\alpha} - \frac{1}{n^\alpha}\Big| = \sum_{n=1}^{N} \frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha}\,,$$

which is a telescopic series and therefore converges as $N \to \infty$. Thus, by Corollary 1.1 $\sum\limits_{n=1}^{\infty} \frac{\sin(nx)}{n^\alpha}$ converges.

□

Exercise 1.4

Proposition. Let w be a primitive third root of unity. Then $\sum\limits_{n=1}^{\infty} \frac{w^{n-1}}{n^{1/3}}$ converges, but $\sum\limits_{n=1}^{\infty} \left(\frac{w^{n-1}}{n^{1/3}}\right)^3 = \sum\limits_{n=1}^{\infty} \frac{1}{n}$, which diverges.

Proof. We use Corollary 1.1 in order to prove our hypothesis. Set $a_n = 1/n^{1/3}$. It is clear that $a_n \to 0$ as $n \to \infty$. Considering the series $\sum |a_{n+1} - a_n|$, we note that it can be rewritten as

$$\sum_{n=1}^{N} \frac{1}{(n)^{1/3}} - \frac{1}{(n+1)^{1/3}}\,.$$

This is a telescopic series and since $a_n \to 0$, this series must converge.
Next we set $b_n = w^{n-1}$ and consider the partial sums given by $\sum_{n=0}^{N-1} w^n$. Since $w \neq 1$, each of these partial sums is a finite geometric series. In particular, we note that for any $N = 3k$ where $k \in \mathbb{Z}$ we have

$$\sum_{n=0}^{N-1} w^n = \frac{1 - w^N}{1 - w} = \frac{1 - (w^3)^k}{1 - w} = 0\,.$$

Thus, we see that every third partial sum is zero and therefore the partial sums of the b_n terms must be bounded. Consequently, by Corollary 1.1, $\sum\limits_{n=1}^{\infty} \frac{w^{n-1}}{n^{1/3}}$ converges.

□

Exercise 1.5

Proposition. Cauchy sequences of complex numbers converge if and only if, whenever a series $\sum |a_n|$ converges, then $\sum a_n$ converges as well.

Proof. First we assume the convergence of Cauchy sequences. We will show that this implies that absolutely convergent series conditionally converge as well. Let a_n be a sequence of complex numbers such that $\sum_{n=1}^{\infty} |a_n|$ converges. This implies that the sequence of partial sums of $\sum_{n=1}^{\infty} |a_n|$ (we denote the $N^t h$ partial sum as $|A|_N$) converges and therefore must be a Cauchy sequence. In particular, this implies that given $\epsilon > 0$, there exists some N such that for $K, M \geq N$ (without lose of generality we can assume that $K > M$)

$$||A|_K - |A|_M| < \epsilon\,.$$

Observe that

$$\epsilon > ||A|_K - |A|_M| = |A|_K - |A|_M = \sum_{M+1}^{K} |a_n| \geq |\sum_{M+1}^{K} a_n| = |A_K + A_M|\,,$$

where A_M is the M^{th} partial sum of $\sum_{n=1}^{\infty} a_n$. Thus, we see that A_N is a Cauchy sequence as well and therefore it converges.

Next assume that if a series converges absolutely then, it converges conditionally. We will show that this implies the convergence of Cauchy sequences. Let a_n be a Cauchy sequence of complex numbers. Since a_n is Cauchy there exists some N_1 such that for $n > N_1$

$$|a_n - a_{N_1}| < \frac{1}{2}.$$

Similarly, there exists some N_2 such that for $n > N_2$

$$|a_n - a_{N_2}| < \frac{1}{2^2}.$$

Proceeding inductively we see that in general We can find a N_k such that for $n > N_k$

$$|a_n - a_{N_k}| < \frac{1}{2^k}.$$

Note that we have created an increasing sequence of indices $N_1 < N_2 < ... < N_k$. Thus we can replace some terms in the above inequalities as follows

$$\begin{gathered} |a_n - a_{N_k}| < \frac{1}{2^k} \\ |a_{N_k} - a_{N_{k-1}}| < \frac{1}{2^{k-1}} \\ . \\ . \\ . \\ |a_{N_2} - a_{N_1}| < \frac{1}{2}, \end{gathered}$$

where $n > N_k$. Setting $n = N_{K+1}$ and combining these inequalities into sums we obtain

$$\sum_{k=1}^{K} |a_{N_{k+1}} - a_{N_k}| < \sum_{k=1}^{K} \frac{1}{2^k}.$$

Observe that the right-hand side of this inequality is a geometric series that converges as $K \to \infty$. Therefore the left-hand side must also be a convergent series. By our hypothesis, since $\sum_{k=1}^{K} |a_{N_{k+1}} - a_{N_k}|$ is convergent, it follows that $\sum_{k=1}^{K} a_{N_{k+1}} - a_{N_k}$ is convergent as well. Observe that this series is telescopic and therefore

$$\sum_{k=1}^{K} a_{N_{k+1}} - a_{N_k} = a_{N_{k+1}} = a_n$$

for $n > N_k$. Hence, the Cauchy sequence a_n is convergent.

$\square$

Exercise 1.7

First we rewrite $f(\theta)$ as follows

$$f(\theta) = 1 + a\cos(\theta) = \frac{a}{2}e^{-i\theta} + 1 + \frac{a}{2}e^{i\theta}.$$

Using the Riesz-Fejer Theorem we can derive $q(z)$ to be

$$z(\frac{a}{2}z^{-1} + 1 + \frac{a}{2}z) = \frac{a}{2} + z + \frac{a}{2}z^2.$$

Using the quadratic formula, we see that the roots of this polynomial are

$$z = \frac{-1 \pm \sqrt{1-a^2}}{a}.$$

Setting $\xi = \frac{-1+\sqrt{1-a^2}}{a}$ we can now factor $q(z)$ as follows

$$q(z) = \frac{a}{2}(z-\xi)(z-\bar{\xi}^{-1})\,.$$

Recall that on the unit circle $z = 1/\bar{z}$ and consequently

$$z - \bar{\xi}^{-1} = \frac{1}{\bar{z}} - \frac{1}{\bar{\xi}} = \frac{\bar{\xi}-\bar{z}}{\bar{z}\bar{\xi}}\,.$$

Thus, we see that

$$q(z) = \frac{-a}{2\bar{z}\bar{\xi}}(z-\xi)(\bar{z}-\bar{\xi})\,.$$

Taking the modulus of $q(z)$ we see that

$$|q(z)| = \frac{|a|}{2|\bar{\xi}|}(z-\xi)(\bar{z}-\bar{\xi})\,.$$

Finally, by the Riesz-Fejer Theorem we know that $|p(z)|^2 = |q(z)|$. Thus

$$|p(z)| = \sqrt{|q(z)|} = \sqrt{\frac{|a|}{2|\bar{\xi}|}}|(z-\xi)| = \sqrt{\frac{|a|}{2\frac{|-1+\sqrt{1-a^2}|}{|a|}}} \cdot |(z - \frac{-1+\sqrt{1-a^2}}{a})| =$$

$$\sqrt{\frac{a^2}{2|-1+\sqrt{1-a^2}|}} \cdot |(z - \frac{-1+\sqrt{1-a^2}}{a})| = \frac{a}{(2|-1+\sqrt{1-a^2}|)^{1/2}} \cdot |(z - \frac{-1+\sqrt{1-a^2}}{a})|$$