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Math 564

Homework 11

Exercise 1.6

Proposition. The series $\sum_{n=0}^{\infty} \frac{\cos(kx)}{\log(k+2)}$ converges to a non-negative function.

Proof. As the first step we will use summation by parts twice on the N^{th} partial sum of the given series. Observe,

$$\sum_{n=0}^{N} \frac{\cos(kx)}{\log(k+2)} = \frac{1}{\log(N+2)} \sum_{n=0}^{N} \cos(nx) - \sum_{n=0}^{N-1} \left(\frac{1}{\log(n+3)} - \frac{1}{\log(n+2)} \right) \sum_{k=0}^{n} \cos(kx)\,.$$

Considering the right-hand term of this expression and summing it by parts again we have

$$-\sum_{n=0}^{N-1} \left(\frac{1}{\log(n+3)} - \frac{1}{\log(n+2)} \right) \sum_{k=0}^{n} \cos(kx) =$$

$$-\left(\frac{1}{\log(N+2)} - \frac{1}{\log(N+1)} \right) \sum_{n=0}^{N-1} \sum_{k=0}^{n} \cos(kx)$$
$$+ \sum_{n=0}^{N-2} \left(\frac{1}{\log(n+4)} - \frac{2}{\log(n+3)} + \frac{1}{\log(n+2)} \right) \sum_{k=0}^{n} \sum_{j=0}^{k} \cos(jx)\,.$$

Thus, the final expression is

$$\frac{1}{\log(N+2)} \sum_{n=0}^{N} \cos(nx) - \left(\frac{1}{\log(N+2)} - \frac{1}{\log(N+1)} \right) \sum_{n=0}^{N-1} \sum_{k=0}^{n} \cos(kx)$$
$$+ \sum_{n=0}^{N-2} \left(\frac{1}{\log(n+4)} - \frac{2}{\log(n+3)} + \frac{1}{\log(n+2)} \right) \sum_{k=0}^{n} \sum_{j=0}^{k} \cos(jx)\,. \tag{*}$$

Before we consider the limit of this expression we will derive a formula for $\sum_{n=0}^{N} \sum_{k=0}^{n} \cos(kx)$. Consider the following

$$\sum_{n=0}^{N} \sum_{k=0}^{n} \cos(kx) = \sum_{n=0}^{N} \sum_{k=0}^{n} \frac{e^{inx} + e^{-inx}}{2} = \sum_{n=0}^{N} \left(\frac{1}{2} + \frac{1}{2} \sum_{-n}^{n} e^{inx} \right) = \frac{N}{2} + \frac{1}{2} \sum_{n=0}^{N} \sum_{-n}^{n} e^{inx} =$$
$$\frac{N+1}{2} + \frac{1}{2} \sum_{n=1}^{N} \sum_{-n}^{n} e^{inx}\,.$$

By the formula calculated in exercise 1.44 we observe that

$$\frac{N+1}{2} + \frac{1}{2} \sum_{n=1}^{N} \sum_{-n}^{n} e^{inx} = \frac{N+1}{2} + \frac{1}{2} \frac{\sin^2(\frac{Nx}{2})}{\sin^2(\frac{x}{2})}\,.$$

Making appropriate substitutions into eq.(*) we obtain

$$\frac{1}{\log(N+2)} \sum_{n=0}^{N} \cos(nx) + \left(\frac{1}{\log(N+1)} - \frac{1}{\log(N+2)} \right) \left(\frac{N}{2} + \frac{1}{2} \frac{\sin^2(\frac{(N-1)x}{2})}{\sin^2(\frac{x}{2})} \right)$$
$$+ \sum_{n=0}^{N-2} \left(\frac{1}{\log(n+4)} - \frac{2}{\log(n+3)} + \frac{1}{\log(n+2)} \right) \left(\frac{n+1}{2} + \frac{1}{2} \frac{\sin^2(\frac{nx}{2})}{\sin^2(\frac{x}{2})} \right)\,.$$

Now we take the limit of this sum as $N \to \infty$. We will consider each of the three summands separately. First observe that since $\sum_{n=0}^{N} \cos(nx)$ is bounded and $\log(N+2)$ is an increasing function it follows that

$$\lim_{N\to\infty} \frac{1}{\log(N+2)} \sum_{n=0}^{N} \cos(nx) = 0\,.$$

Next, we consider the limit of the second summand

$$\lim_{N\to\infty} \left(\frac{1}{\log(N+1)} - \frac{1}{\log(N+2)} \right) \left(\frac{N}{2} + \frac{1}{2} \frac{\sin^2(\frac{(N-1)x}{2})}{\sin^2(\frac{x}{2})} \right) =$$

$$\lim_{N\to\infty} \left(\frac{N}{2\log(N+1)} - \frac{N}{2\log(N+2)} \right) + \lim_{N\to\infty} \left(\frac{1}{\log(N+1)} - \frac{1}{\log(N+2)} \right) \frac{1}{2} \frac{\sin^2(\frac{(N-1)x}{2})}{\sin^2(\frac{x}{2})}\,.$$

Note that first limit goes to zero (*not sure how to show this but seems really similar to exercise 1.47*), which leaves us with

$$\lim_{N\to\infty} \left(\frac{1}{\log(N+1)} - \frac{1}{\log(N+2)} \right) \frac{1}{2} \frac{\sin^2(\frac{(N-1)x}{2})}{\sin^2(\frac{x}{2})}\,.$$

Note that for each fixed value of x $\frac{\sin^2((N-1)x/2)}{\sin^2(x/2)}$ is bounded. Therefore this second limit also goes to zero. Finally, we consider the limit of the last summand

$$\lim_{N\to\infty} \sum_{n=0}^{N-2} \left(\frac{1}{\log(n+4)} - \frac{2}{\log(n+3)} + \frac{1}{\log(n+2)} \right) \left(\frac{n+1}{2} + \frac{1}{2} \frac{\sin^2(\frac{nx}{2})}{\sin^2(\frac{x}{2})} \right)\,.$$

I don't know how to proceed from here.

$\square$

Exercise 3.1

Proposition. The function $f(x) = e^{-x^2}$ is in the Schwartz space.

Proof. First we will show that e^{-x^2} is smooth. First we calculate the first few derivatives of f, observe that

$$f' = -2xe^{-x^2}$$

$$f'' = (-2 + 4x^2)e^{-x^2}$$

and

$$f''' = (12x - 8x^3)e^{-x^2}\,.$$

Thus, we see that the pattern seems to suggest that each n^{th} derivative will be equal to a polynomial of degree n times e^{-x^2}. We prove this using induction on n. Suppose that for n we have

$$f^{(n)} = P_n e^{-x^2}\,,$$

where P_n is a polynomial in x of degree n. Then observe that by the product rule of differentiation

$$f^{(n+1)} = P_n' e^{-x^2} + P_n(-2xe^{-x^2}) = (P_n' - 2xP_n)e^{-x^2}\,.$$

Since $(P_n' - 2xP_n)$ is a polynomial of degree $n+1$, it follows that the result holds for all $n \in \mathbb{N}$, by induction on n.

Next we show that f is indeed in the Schwartz space. We must show for all positive $a, n \in \mathbb{Z}$ that

$$\lim_{|x|\to\infty} = |x|^a(\frac{d}{dx})^n f(x) = 0\,.$$

Since $f(x)$ is an even function, we will consider only non-negative values of x and and the limit at positive infinity. The limit at negative infinity is symmetric. We have previously established that the n^{th} derivative of f is the product of a n^{th} degree polynomial, P_n, and e^{-x^2}. Thus, in general

$$x^a(\frac{d}{dx})^n f(x) = \frac{x^a(P_n)}{e^{x^2}}\,.$$

Here, $x^a(P_n)$ is an $(a+n)^{th}$ degree polynomial. Since the limit of the numerator and the denominator are both infinity as $x \to \infty$, we can apply L'Hospitals rule repeatedly (since $x^a(P_n)$ and e^{x^2} are both smooth). Hence, taking the $(a+n)^{th}$ derivative of both the numerator and the denominator, we see that

$$\frac{(\frac{d}{dx})^{a+n} x^a(P_n)}{(\frac{d}{dx})^{a+n} e^{x^2}} = \frac{c}{P_{a+n}e^{x^2}}$$

where P_{a+n} is a $(a+n)^{th}$ degree polynomial and c is a constant. It is easy to see that the limit of this ratio as $x \to \infty$ is equal to zero. Thus, $f(x) = e^{-x^2}$ is in the Schwartz space.

$\square$