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ORIGINAL PAPER

Measuring hierarchy

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Abstract

This paper presents a novel axiomatic approach to measuring and comparing hierarchical structures. Hierarchies are fundamental across a range of disciplines—from ecology to organizational science—yet existing measures of hierarchical degree often lack systematic criteria for comparison. We introduce a mathematically rigorous framework based on a simple partial pre-order over hierarchies, denoted as $\succcurlyeq_H$, and demonstrate its equivalence to intuitively appealing axioms for hierarchy comparisons. Our analysis yields three key results. First, we establish that for fixed-size hierarchies, one hierarchy is strictly more hierarchical than another according to $\succcurlyeq_H$ if the latter can be derived from the former through a series of subordination removals. Second, we fully characterize the hierarchical pre-orders that align with $\succcurlyeq_H$ using two fundamental axioms: Anonymity and Subordination Removal. Finally, we extend our framework to varying-size hierarchies through the introduction of a Replication Principle, which enables consistent comparisons across different scales.

1 Introduction

The concept of hierarchy is essential for understanding emergent properties in complex systems across a wide range of scientific disciplines. In ecology and earth sciences, hierarchical frameworks clarify the organization of ecosystems, encompassing everything from individual organisms to vast biomes. Similarly, social scientists utilize hierarchical models to examine power dynamics, social stratification, and organizational behavior within human societies. The evolution of economic entities—from simple partnerships to multinational corporations—has been marked by increasingly sophisticated hierarchical structures.

A critical examination of these hierarchical structures reveals both their advantages and disadvantages across various contexts. In biology, hierarchies can be beneficial by reducing conflict and enhancing group coordination among individuals. For instance, social hierarchies in animal species often lead to established dominance relationships

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that minimize aggressive encounters and promote stability within groups. This stability can facilitate cooperative behaviors and improve resource allocation.

In organizational science and management, hierarchies establish clear lines of authority, accountability, and communication within institutions. This structured approach can enhance coordination, decision-making, and strategy implementation. However, such frameworks can also impede the pursuit of desirable goals. For instance, Wright (2024) argues that deeply entrenched knowledge-based hierarchies may yield negative epistemic consequences by reinforcing conservative selection biases against innovative research. This indicates that while hierarchies can streamline processes, they may also stifle creativity and adaptability.

Furthermore, it is possible that the evolution of hierarchical structures in human societies has significantly contributed to the historic development of income and wealth inequality, which has now reached unprecedented levels. Although authoritative evidence supporting this hypothesis is currently lacking, the notion aligns with some findings in the literature and remains intuitively compelling. This perspective raises critical questions about the social implications of hierarchical organization and its potential role in perpetuating disparities.

To explore the link between hierarchy and income distribution further, we note that hierarchical structures exhibit a strong correlation with earnings distributions. Some researchers have employed theoretical hierarchical models to explain observed worker compensation patterns. Empirical evidence suggests that the compensation of a firm's highest-paid official is primarily related to firm size, while other variables—particularly profit—have minimal explanatory power (Roberts 1956). Simon (1957) proposes a simple hierarchical structure to elucidate the relationship between CEO compensation and firm size. Similarly, Lydall (1959) employs a comparable hierarchical mechanism to generate a labor income distribution whose upper tail aligns with empirical power law distributions.

More recently, Fix (2018, 2019) adapted the hierarchical models of Simon (1957) and Lydall (1959) in light of new data, finding that relative income within firms scales strongly with the average number of subordinates under an individual's control.

Both Simon (1957) and Lydall (1959) postulate a specific hierarchical structure in which each supervisor has the same number of immediate subordinates—a concept often referred to as the "span of control." Under this assumption, these authors express worker compensation as a function of both the span of control and the ratio of an individual's salary to those of their immediate subordinates. As noted by Simon (1957), the span of control serves as a measure of the "steepness" of organizational hierarchies. However, empirically, the span of control varies across ranks. This variability is documented in Fix (2018, 2019), where the span of control is replaced by the average number of subordinates.

Despite these insights, measures of hierarchical "steepness" remain "informal" due to the absence of a systematic criterion for comparing hierarchies. The distributional consequences of hierarchical structures are not well understood—both theoretically, owing to a lack of a general theory for hierarchy measurement, and empirically, due to the paucity of publicly available data.

Despite the pivotal role that hierarchies play in shaping organizational structures and influencing earnings distribution, our ability to systematically analyze and compare these structures remains limited. While several measures of hierarchical degree have been developed by researchers across various disciplines, they predominantly take the form of hierarchical indices that result in a complete ranking of hierarchical structures.[1] Although these measures may seem intuitively appealing, they often lack a solid theoretical foundation.

This paper represents a first attempt to advance beyond informal measures of hierarchical steepness by introducing a systematic, mathematically grounded approach. Methodologically, our analysis is akin to the axiomatic underpinnings of inequality measurement (see, e.g., Cowell 2016; Chakravarty 2009, 2015).

We present an axiomatic theory of hierarchy measurement, which serves as a crucial precondition for systematically analyzing how organizational design shapes economic outcomes and societal structures—particularly the relationship between income distribution and hierarchical frameworks. This theoretical approach also contributes to a broader body of literature, as hierarchies are fundamental to the evolution and transformation of societies as a whole. Indeed, the transition from small-scale to large-scale societies is closely linked with increasing hierarchical complexity and per capita energy capture (see, e.g., Turchin and Gavrilets 2009; Fix 2017; Bichler and Nitzan 2020; Graeber and Wengrow 2021).

Turning to the specifics of the formal analysis, we demonstrate that a simple partial pre-order defined over the set of hierarchies is equivalent to a set of intuitively appealing axioms or principles for hierarchy comparisons. This equivalence lays the groundwork for a comprehensive characterization of hierarchy measures that align with this partial pre-order.

We consider hierarchies represented as a series of nodes connected by paths. Each node symbolizes an individual, while the paths between nodes delineate the subordination relationships among them. Our focus is on hierarchies in which each subordinate has only one immediate supervisor.

A hierarchical pre-order $\succcurlyeq$ is a reflexive and transitive binary relation over the set of all hierarchies. Because $\succcurlyeq$ need not be complete, $\succcurlyeq$ may render no judgment over some comparisons of hierarchies. When two hierarchies h and h' are comparable under $\succcurlyeq$, we write "$h \succcurlyeq h'$" to indicate that "h is at least as hierarchical as h'."

For hierarchies with the same number of individuals, we adopt two basic criteria for hierarchy comparisons. First, relabeling the individuals in a hierarchy does not alter its hierarchical structure. This property is called Anonymity. The second criterion is based on the notion of Subordination Removal. We say that a hierarchy h' is obtained from another hierarchy h by removal of a subordination relation if the sub-hierarchy $h(i)$ of h that begins at an immediate subordinate i of a supervisor j in h is moved up one level in the hierarchy so that i is no longer an immediate subordinate of j, but rather either an unsupervised individual (if j has no supervisors) or an immediate subordinate of j's immediate supervisor. The sub-hierarchy $h(i)$ remains otherwise intact, and the structure of h' is otherwise identical to that of h. The Subordination Removal postulate asserts that removing a subordination relation creates a less hierarchical structure.

[1] See, e.g., Mones et al. (2012), Corominas-Murtra et al. (2013), Krackhardt (1994), Trusina et al. (2004), Luo and Magee (2011) and Czégel and Palla (2015).

We introduce a particular hierarchical pre-order, denoted by $\succsim_H$, which is instrumental in the formulation of our results. Specifically, we say that hierarchy h is at least as hierarchical as h' under $\succsim_H$ (i.e., $h \succsim_H h'$) if there exists a bijection between the sets of individuals in both hierarchies with a particular property: for every individual i in h that corresponds to a subordinate j in h', the immediate supervisor of j in h' must correspond to a supervisor of i in h. This relationship not only establishes a one-to-one mapping between the individuals but also preserves the supervisory dynamics across the two hierarchies.

It turns out that the hierarchical pre-order $\succsim_H$ is closely related to the concept of subordination relation. Notably, a key result of this paper (Theorem 1) establishes that for any two hierarchies h and h' with the same number of individuals, h is strictly more hierarchical than h' under $\succsim_H$ (i.e., $h \succ_H h'$) if and only if h' can be derived from a relabeling of h through a series of removals of subordination relations.

A hierarchical pre-order is said to be $\succsim_H$-consistent if it agrees with $\succsim_H$ for pairs of hierarchies that are comparable under $\succsim_H$. The second primary result of this paper (Theorem 2) characterizes the class of $\succsim_H$-consistent hierarchical pre-orders in terms of two axioms: for a fixed number of individuals, n, a hierarchical pre-order on the set of all hierarchies of size n satisfies Anonymity and Subordination Removal if and only if it is $\succsim_H$-consistent.

Similar results can be obtained for extensions of hierarchical pre-orders to pairs of hierarchies of varying sizes. These results require the introduction of a third fundamental criterion alongside the existing Anonymity and Subordination Removal conditions: replicating a hierarchy—which yields two identical hierarchies, a superstructure that is itself a hierarchy in its own right—does not alter its hierarchical structure. This property is called Replication Principle.

Using the Replication Principle, the hierarchical pre-order $\succsim_H$ can be extended to pairs of hierarchies of varying sizes as follows: $h \succsim_H h'$ if there exist two equally-sized replications, h_r and h'_r, of h and h', respectively, such that $h_r \succsim_H h'_r$.

The third main result of this paper (Theorem 3) states that a hierarchical pre-order on the set of all hierarchies (of any size) satisfies Anonymity, Subordination Removal, and the Replication Principle if and only if it is $\succsim_H$-consistent.

We examine two examples of $\succsim_H$-consistent hierarchical pre-orders. The first compares hierarchies based on the number of supervisors between pairs of linked individuals. This pre-order represents a partial completion of $\succsim_H$. The second example, which provides a complete ordering, is derived from a hierarchical index that computes the average number of supervisors for each hierarchy. Despite apparent similarities, our measure differs fundamentally from the hierarchical metric in Mones et al. (2012). We provide a thorough examination of this relationship within our framework in Sect. 4.

The literature on rank mobility (see, e.g., D'Agostino and Dardanoni 2009; Bossert et al. 2016) is pertinent to the analysis presented in this paper.[2] Rank mobility examines how individuals transition across indicators of economic or social "status" over time. Since hierarchy inherently involves "rank" echelons that can be viewed as measures of

[2] For additional context, see the surveys on income mobility by Maasoumi (1998), Fields and Ok (1999), and Jäntti and Jenkins (2015).

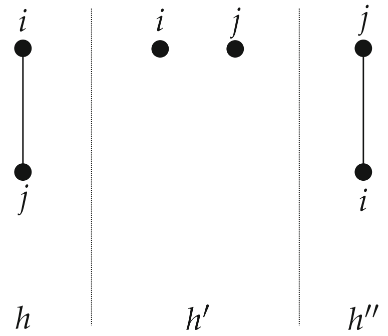


Fig. 1 Rank mobility vs. hierarchical measurement

"status," there is a potential link between measuring hierarchical degree and assessing rank mobility.

However, this connection is, at best, tenuous. In fact, a fundamental difference exists between rank mobility and the comparison of hierarchical structures, as illustrated by the following example.

Consider the three simple hierarchies depicted in Fig. 1.

In hierarchies h and h'', there are two levels, or ranks. In hierarchy h, individual j is subordinate to individual i, while in h'', j supervises i. In contrast, hierarchy h' has no subordination relations, with both i and j occupying the same rank.

Notice that the transition from h to h'' involves both individuals moving across ranks, whereas only one individual changes rank in the transition from h to h'. This indicates that the shift from h to h'' reflects a greater degree of rank mobility. In fact, rank mobility is maximal in the transition from h to h''.

Despite the difference in rank mobility, our measures of hierarchical structure classify h and h'' as *equally* hierarchical. This is because both h and h'' share the same hierarchical structure and differ only in the labeling of individuals. According to the "Anonymity" axiom, this means that both hierarchies are essentially equivalent.

Thus, while h and h'' are maximally hierarchical and h' is minimally hierarchical, rank mobility is higher—indeed, maximal—in the transition from h to h''.

This analysis demonstrates that the degree of rank mobility between hierarchies is independent of their relative hierarchical structure. Specifically, we have shown that two equally hierarchical structures (h and h'') can be connected by a transition involving maximal rank mobility, while a transition to a less hierarchical structure (h') involves lower rank mobility.

The paper is structured as follows. Section 2 introduces the concept of hierarchy along with its relevant terminology. Section 3 formally defines (potentially incomplete) hierarchical pre-orders for hierarchies of fixed size. It presents the Anonymity and Subordination Removal axioms, and introduces a specific hierarchical pre-order, denoted by $\succcurlyeq_H$, based on supervisory rank comparisons across hierarchies. This pre-order satisfies the aforementioned axioms (see Lemma 1) and is utilized in the first main result (Theorem 1) to fully characterize successive removals of subordination relations. The second main result (Theorem 2) demonstrates the equivalence between the Anonymity and Subordination Removal axioms and the $\succcurlyeq_H$-consistency of a hierarchical pre-order—its alignment with $\succcurlyeq_H$ whenever two hierarchies can be compared under this pre-order. Section 3 concludes with a discussion on two specific (partial) completions of the hierarchical pre-order $\succcurlyeq_H$.

Section 4 broadens the analysis from Sect. 3 to hierarchies of varying sizes. It introduces the Replication Principle and extends the hierarchical pre-order $\succcurlyeq_H$ to encompass hierarchies of any size, ensuring compliance with the Replication Principle, as well as the Anonymity and Subordination Removal axioms (see Lemma 2). The third main result (Theorem 3) establishes the equivalence of these three axioms with $\succcurlyeq_H$-consistent hierarchical pre-orders. Additionally, Sect. 4 explores specific (partial) completions of $\succcurlyeq_H$ within the expanded domain of hierarchies of any size, comparing them to the hierarchical index proposed by Mones et al. (2012).

Finally, Sect. 5 summarizes our key findings and explores potential avenues for future research.

2 Hierarchies

We begin with the definition of a hierarchy.

Definition 1 A ***hierarchy*** is defined as a set of $n \in \mathbb{N}$ individuals satisfying the following conditions:

- There exists a set of level-0 individuals, I_0, such that each $i \in I_0$ has either no subordinates or a set of level-1 subordinates, S_i, satisfying the following conditions:
  - $S_i \cap I_0 = \emptyset$ for each $i \in I_0$.
  - $S_i \cap S_{i'} = \emptyset$ for each $i, i' \in I_0$ with $i \neq i'$.

  In words, there is no overlap between the set of level-0 individuals, I_0, and the set of i's subordinates, S_i; and a subordinate of $i \in I_0$ in S_i cannot be, at the same time, a subordinate of another level-0 individual $i' \in I_0$.
- Suppose that the set of level-k individuals has been defined, where $k \in \{0, 1, \ldots, K-1\}$, and where K represents the total number of levels in the hierarchy. The set of level-$(k+1)$ individuals is defined as follows. Each level-$(k+1)$ subordinate, j, has either no subordinates or a set of level-$(k+2)$ subordinates, S_j, satisfying the following conditions:
  - For each level-$(k+1)$ subordinate j, the set of level-$(k+2)$ subordinates of j, S_j, contains no level-κ individuals, where $\kappa \in \{0, \ldots, k+1\}$.
  - For any two distinct level-$(k+1)$ subordinates j and j', the sets of level-$(k+2)$ subordinates S_j and $S_{j'}$ of j and j', respectively, are disjoint.

For each level-k subordinate i in a given hierarchy h, where $k \in \{1, \ldots, K\}$ (and where K denotes the total number of levels in h), there is one level-$(k-1)$ supervisor, $p_h(i)$, one level-$(k-2)$ supervisor, $p_h^2(i)$, etc. The supervisor $p_h(i)$ of i is called i's immediate supervisor. $p_h^2(i)$, $p_h^3(i)$, $\ldots$ are indirect supervisors of i, being supervisors of i's immediate supervisor, $p_h(i)$.

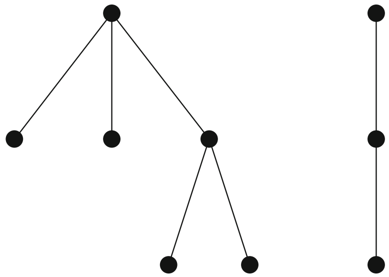


Fig. 2 A hierarchy

In the sequel, the subscript "h" in expressions like $p_h(i)$ and $p_h^2(i)$ is sometimes omitted for simplicity. These omissions should not cause confusion, as the underlying hierarchy can be easily inferred from the context.

Individuals i and j in a given hierarchy are related if there is a path linking them, i.e., if either $i = j$ or $j = p^l(i)$ for some l. If $i \neq j$, we say that i is a subordinate of j. If $i \neq j = p(i)$, we say that i is an immediate subordinate of j.

For each individual i in a hierarchy h, the ***sub-hierarchy*** containing i and all of i's subordinates constitutes a properly defined hierarchy, denoted by $h(i)$. The subordinates of i are the members of the sub-hierarchy $h(i)$ other than i.

Note that, given a hierarchy h and its set of level-0 individuals, I_0, h can be represented as a vector of sub-hierarchies $(h(i))_{i \in I_0}$. Note also that, given a hierarchy $h = (h(i))_{i \in I_0}$, $(h(i))_{i \in I}$, where $I \subseteq I_0$, is a hierarchy in its own right, sometimes also referred to as a sub-hierarchy of h.

Hierarchies can be conveniently represented graphically as a series of nodes linked by paths. Each node represents an individual in the hierarchy. Figure 2 presents a hierarchy with two level-0 individuals, four level-1 individuals, and three level-2 individuals.

It should be noted that our definition of a hierarchy is not flexible enough to account for structures where a subordinate reports to multiple immediate supervisors.

3 Hierarchical pre-orders

To begin, we consider pre-orders on hierarchies of a fixed size. Extensions of these pre-orders to hierarchies of varying sizes are studied in Sect. 4.

Let $\mathscr{H}_n$ be the set of all n-person hierarchies.

Definition 2 A ***hierarchical pre-order*** on $\mathscr{H}_n$ is a reflexive and transitive binary relation $\succcurlyeq$ on $\mathscr{H}_n$.

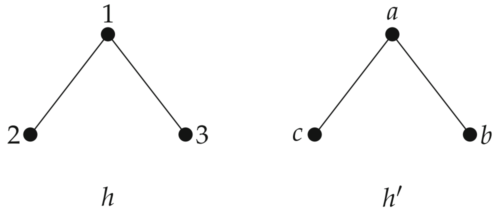
For $h, h' \in \mathscr{H}_n$, "$h \succcurlyeq h'$" means that "h is at least as hierarchical as h'."

Note that hierarchical pre-orders are not necessarily complete.

The symmetric and asymmetric parts of $\succcurlyeq$ are denoted by $\sim$ and $\succ$, respectively. For $h, h' \in \mathscr{H}_n$, the interpretation of the dominance relation "$h \succ h'$" (respectively, "$h \sim h'$") is that "h is more hierarchical than h'" (respectively, "h and h' are equally hierarchical").

We now present two basic properties of hierarchical pre-orders.

Fig. 3 Relabeling



A hierarchy h' is said to be a ***relabeling*** of another hierarchy h if h' is obtained from h by relabeling the individuals in h. For example, the two hierarchies in Fig. 3, h and h', are relabelings of each other.

Anonymity (A) A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}_n$ satisfies A if for any two hierarchies h and h' in $\mathscr{H}_n$, $h \sim h'$ whenever h' is a relabeling of h.

The Anonymity axiom asserts that, given a hierarchical pre-order, all relabelings of a given hierarchy belong to the same equivalence class. Note that relabelings do not alter the fundamental structure of a hierarchy; they simply rename the individuals involved.

We now introduce the notion of subordination relation removal.

Definition 3 We say that h' is obtained from a hierarchy h by ***removing a subordination relation*** if there exists a level-k subordinate i in h, where $k \in \{1, \ldots, K\}$ (with K denoting the total number of levels in the hierarchy h), satisfying the following conditions:

- If i's immediate supervisor in h, $p_h(i)$, is a level-0 individual, then h' is the hierarchy in which the sub-hierarchy $h(i)$ is no longer under $p_h(i)$'s supervision, i becomes a level-0 individual, and the sub-hierarchy that begins at i is $h(i)$; h' is otherwise equal to h.
- If i's immediate supervisor in h, $p_h(i)$, is a not level-0 individual, then $p_h(i)$ is an immediate subordinate of $p_h^2(i)$, i.e., $p_h(i) \in S_{p_h^2(i)}$. In this case, h' is the hierarchy in which the sub-hierarchy $h(i)$ is no longer under $p_h(i)$'s supervision, but rather under the direct supervision of $p_h^2(i)$, so that i is no longer a level-k subordinate, but rather a level-$(k-1)$ subordinate in $S_{p_h^2(i)}$, and the sub-hierarchy that begins at i is $h(i)$; h' is otherwise equal to h.

Figure 4 illustrates the previous definition. In Fig. 4, h' can be obtained from h by taking the sub-hierarchy $h(a)$ that starts at the node labeled "a" and moving it up so that the individuals in $h(a)$ are no longer subordinates of the level-0 individual in h.

Figure 5 presents another example where individual b loses one subordinate, individual c, who becomes a direct subordinate of b's supervisor, a.

The following axiom states that removing a subordination relation creates a less hierarchical structure.

Subordination Removal (SR) A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}_n$ satisfies SR if for any two hierarchies h and h' in $\mathscr{H}_n$, $h \succ h'$ whenever h' is obtained from h by removing a subordination relation.

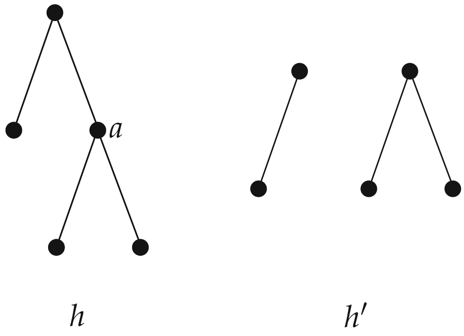


Fig. 4 Removing a subordination relation

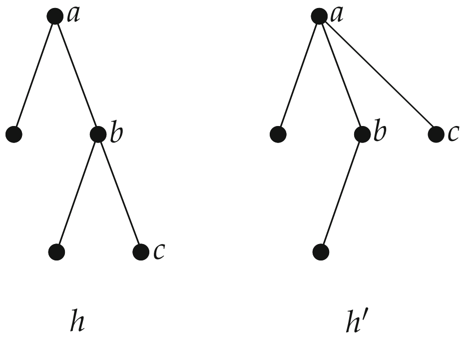


Fig. 5 Removing a subordination relation

We now define a particular hierarchical pre-order, denoted by $\succcurlyeq_H$, based on the comparison of supervisory ranks across hierarchies.

For any two hierarchies h and h' in $\mathscr{H}_n$, $h \succcurlyeq_H h'$ if and only if there exists a bijection ϕ from the set of individuals in h to the set of individuals in h' satisfying the following: for each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p_{h'}(\phi(i))$, is linked (via ϕ^{-1}) to a supervisor j of i in h, so that j is in the path from i to i's level-0 supervisor: $\phi^{-1}(p_{h'}(\phi(i))) = j = p_h^l(i)$ for some l.

As an example, consider the two hierarchies given in Fig. 6. The bijection ϕ from the set of individuals in h to the set of individuals in h' is represented by means of double-arrowed lines connecting nodes across the two hierarchies.

One can easily verify that the linked pairs across hierarchies depicted in Fig. 6 satisfy the conditions from the definition of $\succcurlyeq_H$. For example, take individual 2 in h, who is linked to individual b in h', and whose immediate supervisor, a, is linked to individual 1 in h, who is a supervisor of 2. As another example, take individual 4 (in h), who is linked to individual c (in h'), whose immediate supervisor, b, is linked to 2 (in h), a supervisor of 4. Note that 2 lies in the path from 4 to 4's level-0 supervisor, 1, in h. Similar conditions can be verified for the other nodes in h. Hence, $h \succcurlyeq_H h'$.

The symmetric and asymmetric parts of $\succcurlyeq_H$ are denoted by $\sim_H$ and $\succ_H$, respectively.

Our main results reveal an intrinsic relationship between the hierarchical pre-order $\succcurlyeq_H$ and our two basic axioms.

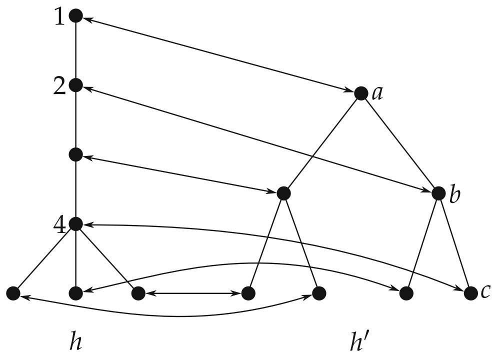


Fig. 6 $h \succcurlyeq_H h'$

We begin the analysis with a preliminary result, which asserts that $\succcurlyeq_H$ satisfies both Anonymity and Subordination Removal.

Lemma 1 *The hierarchical pre-order $\succcurlyeq_H$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies* A *and* SR.

The proof of Lemma 1 is relegated to Appendix A.2.

Before fully characterizing the hierarchical pre-order $\succcurlyeq_H$, we should note an important property: while $\succcurlyeq_H$ is reflexive and transitive, it is not complete. A simple example illustrates this incompleteness.

Consider the two hierarchies h and h' shown in Fig. 7. We will prove that no bijection ϕ exists mapping individuals from h to those in h' that satisfies the required supervisory relationship: for any individual i in h whose image $\phi(i)$ is not at level 0, the immediate supervisor of $\phi(i)$ in h' must be mapped (via ϕ^{-1}) to some supervisor of i in h.

The proof is by contradiction. Any valid bijection must map subordinates in h to subordinates in h'—if not, some level-0 individual i in h' would map to a subordinate in h, violating the supervisory requirement for i, who has no supervisor. Without loss of generality, assume i maps to i' and j maps to j' as shown in the figure. By definition of $\succcurlyeq_H$, the shared supervisor of i and j would need to map to both the supervisor of j' and the supervisor of i'—an impossibility since these are distinct individuals.

This incompleteness, while preventing comparison of certain hierarchies, aligns with our goal of characterizing hierarchical pre-orders through fundamental, intuitive axioms. Such axioms typically cannot definitively rank structures when multiple competing factors are at play. Our approach mirrors the inequality measurement literature, where the fundamental Lorenz pre-order exists alongside more complete measures like the Gini index.

By establishing this core hierarchical pre-order through basic axioms, we create a foundation for studying all pre-orders that extend it, including both partial and complete extensions. These extensions can offer additional comparisons in ambiguous cases, though analysts must evaluate whether such added comparability resolves ambiguities appropriately. We will explore specific examples of such extensions later in this work.

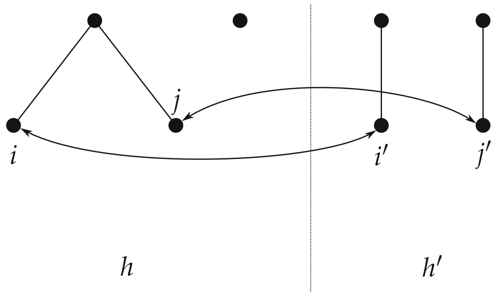


Fig. 7 Incompleteness of $\succsim_H$

The notion of successive subordination removal, introduced next, plays an important role in our first main result.

For $h_1, h_L \in \mathscr{H}_n$, h_1 is obtained from h_L by ***successive removals of subordination relations*** if there are finitely many hierarchies $h_2, \ldots, h_{L-1}$ in $\mathscr{H}_n$ such that h_l is obtained from h_{l+1} by removing a subordination relation, for each $l \in \{1, \ldots, L-1\}$.

Our first main result provides a complete characterization of successive subordination relation removals in terms of the hierarchical pre-order $\succsim_H$.

Theorem 1 *For $h, h' \in \mathscr{H}_n$, $h \succ_H h'$ if and only if h' can be obtained from some relabeling of h by successive removals of subordination relations.*

The following discussion aims to offer visual and intuitive insights into Theorem 1. The formal proof is available in Appendix A.3.

Suppose that h and h' are hierarchies in $\mathscr{H}_n$. First, note that Theorem 1 is trivially true when $n = 1$ (i.e., when both h and h' are 1-person hierarchies). Proceeding by induction, we argue that Theorem 1 is true for arbitrary $n > 1$ if we know it is true for $(n-1)$-sized hierarchies.

Consider the hierarchies h and h' depicted in Fig. 8 and suppose that $h \succ_H h'$. Then $h \succsim_H h'$, which implies that there exists a bijection ϕ mapping the individuals in h to those in h' with the following property: for every individual i in h such that $\phi(i)$ is not a level-0 individual in h', the immediate supervisor of $\phi(i)$ in h' is linked (via ϕ^{-1}) to a supervisor of i in h.

Suppose that the bijection ϕ is represented by the double-arrowed lines connecting the nodes across the hierarchies, as shown in Fig. 8. Now let $h\backslash\iota$ be the hierarchy obtained from h by removing individual ι, as shown in Fig. 9. Note that all the direct subordinates of ι in h become level-0 individuals in $h\backslash\iota$.

Similarly, let $h'\backslash\phi(\iota)$ denote the hierarchy formed by removing the individual $\phi(\iota)$ from h', as illustrated in Fig. 9. This results in the two direct subordinates of $\phi(\iota)$ in h' becoming level-0 individuals, leading to a total of exactly four level-0 individuals in $h'\backslash\phi(\iota)$.

Note that the bijection represented in Fig. 9 by the double-arrowed lines is the restriction of ϕ to the hierarchy $h\backslash\iota$. The properties of ϕ imply that for every individual i in $h\backslash\iota$ where $\phi(i)$ is not a level-0 individual in $h'\backslash\phi(\iota)$, the immediate supervisor of $\phi(i)$ in $h'\backslash\phi(\iota)$ is connected (through ϕ^{-1}) to a supervisor of i in $h\backslash\iota$.

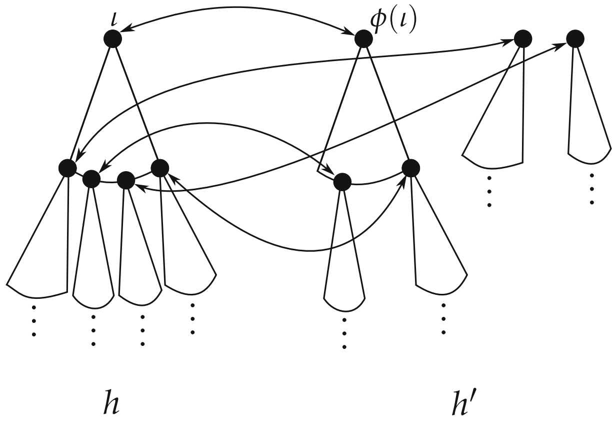


Fig. 8 Illustrating Theorem 1

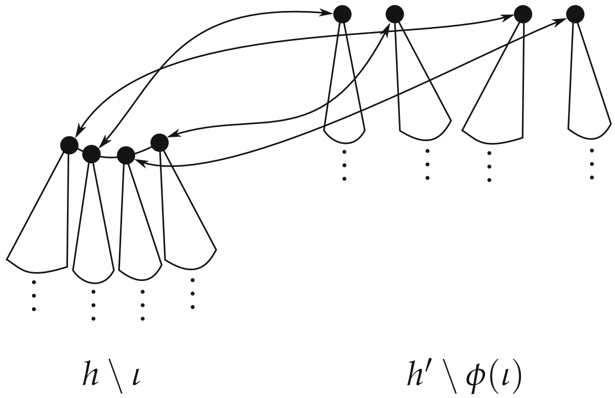


Fig. 9 Illustrating Theorem 1

Hence, $h\backslash\iota \succsim_H h'\backslash\phi(\iota)$. There are two cases to consider:

$$\text{(i) } h\backslash\iota \succ_H h'\backslash\phi(\iota) \quad \text{and} \quad \text{(ii) } h\backslash\iota \sim_H h'\backslash\phi(\iota).$$

In the first case, the induction hypothesis implies that $h'\backslash\phi(\iota)$ can be obtained from $h\backslash\iota$ by successive removals of subordination relations, since both $h\backslash\iota$ and $h'\backslash\phi(\iota)$ are $(n-1)$-person hierarchies.[3]

Note that, in the transition from $h\backslash\iota$ to $h'\backslash\phi(\iota)$, the removal of each subordination relation affects only the individuals within a single sub-hierarchy of $h\backslash\iota$. This means that the entire sequence of subordination removals from $h\backslash\iota$ to $h'\backslash\phi(\iota)$ can be decom-

[3] More precisely, $h'\backslash\phi(\iota)$ can be obtained from some relabeling of $h\backslash\iota$ by successive removals of subordination relations. While not explicitly mentioned in the remainder of this discussion, it is implicitly understood that two hierarchies are considered equivalent if they share the same structure, regardless of how their nodes are labeled.

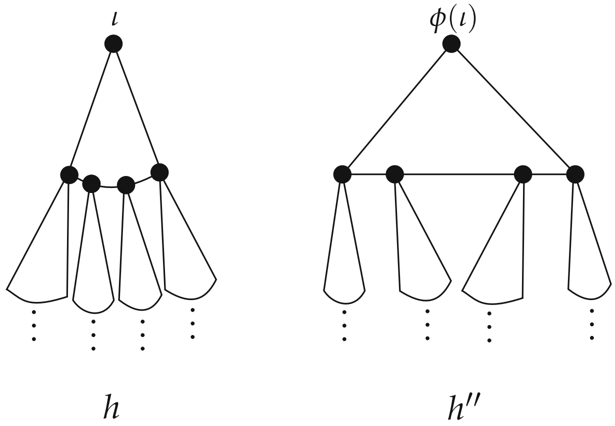


Fig. 10 Illustrating Theorem 1

posed into subsequences that transform sub-hierarchies of $h\backslash\iota$ into sub-hierarchies of $h'\backslash\phi(\iota)$.

For instance, in the specific case illustrated in Fig. 9, this indicates that each of the four sub-hierarchies in $h'\backslash\phi(\iota)$ can be derived, through a series of subordination relation removals, from exactly one of the four sub-hierarchies in $h\backslash\iota$.

Now, consider the hierarchy formed by adding the individual ι at the top of $h\backslash\iota$, as depicted in the left panel of Fig. 10. Observe that the resulting hierarchy is precisely h. Similarly, let h'' be the hierarchy created by adding the individual $\phi(\iota)$ at the top of hierarchy $h'\backslash\phi(\iota)$, as shown in the right panel of Fig. 10.

As we have noted, each of the four sub-hierarchies in $h'\backslash\phi(\iota)$ can be derived from exactly one of the four sub-hierarchies in $h\backslash\iota$ through a series of subordination relation removals. This implies that the hierarchy h'' from Fig. 10 can be obtained from h by successive removals of subordination relations. This is not yet the conclusion we seek, as our goal is to demonstrate that h' (rather than h'') can be obtained from h through successive removals of subordination relations. However, note that in our example, h' can be derived from h'' by moving two of the four level-1 sub-hierarchies in h'' up to level 0.

Since h'' (respectively, h') can be obtained from h (respectively, h'') by successive removals of subordination relations, it follows that h' can also be obtained from h by successive removals of subordination relations, as we sought.

It remains to consider the case when $h\backslash\iota \sim_H h'\backslash\phi(\iota)$. To address this case, we rely on Lemma 4 (proven in Appendix A.1). In our context, this lemma states that $h\backslash\iota \sim_H h'\backslash\phi(\iota)$ implies that $h\backslash\iota$ is a relabeling of $h'\backslash\phi(\iota)$. Consequently, the hierarchies h and h'' depicted in Fig. 10 are also relabelings of one another. Thus, since we have established that h' can be derived from h'' through successive removals of subordination relations, we conclude that h' can be obtained from h by successive removals of subordination relations, as we aimed to show.

Theorem 1 establishes the equivalence between two distinct concepts: the intuitive process of successively removing subordination relations within a hierarchy and the more formal notion embodied by the pre-order $\succsim_H$, which is defined through the comparison of supervisory ranks across different hierarchies.

A hierarchical pre-order $\succsim$ on $\mathscr{H}_n$ is $\succsim_H$***-consistent*** if the following two conditions are satisfied for every pair h, h' in $\mathscr{H}_n$:

- $h \succ_H h' \Rightarrow h \succ h'$.
- $h \sim_H h' \Rightarrow h \sim h'$.

Thus, a $\succsim_H$-consistent hierarchical pre-order can only differ from $\succsim_H$ for pairs of incomparable hierarchies under $\succsim_H$. In other words, $\succsim_H$-consistent hierarchical pre-orders are partial completions of $\succsim_H$.

The next main result states that the Anonymity and Subordination Removal axioms fully characterize $\succsim_H$-consistent hierarchical pre-orders.

Theorem 2 *A hierarchical pre-order on $\mathscr{H}_n$ satisfies A and SR if and only if it is $\succsim_H$-consistent.*

The proof of Theorem 2, which builds on Theorem 1, is sufficiently straightforward to be included in the main text. This proof references two lemmata proven in Appendix A.

Proof of Theorem 2 [*Sufficiency*.] Suppose that $\succsim$ is $\succsim_H$-consistent. Because $\succsim_H$ satisfies A and SR (Lemma 1), and since $\succsim$ is $\succsim_H$-consistent, it follows that $\succsim$ also satisfies A and SR.

[*Necessity*.] Suppose that $\succsim$ is a hierarchical pre-order on $\mathscr{H}_n$ satisfying A and SR. We must show that $\succsim$ is $\succsim_H$-consistent, i.e., that the following two conditions are satisfied for every pair h, h' in $\mathscr{H}_n$:

- $h \succ_H h' \Rightarrow h \succ h'$.
- $h \sim_H h' \Rightarrow h \sim h'$.

Suppose first that $h \sim_H h'$. Then, by applying Lemma 4 (which is stated and proven in Appendix A.1), we can conclude that h is a relabeling of h'. Because $\succsim$ satisfies A, it follows that $h \sim h'$.

Now suppose that $h \succ_H h'$. By Theorem 1, h' can be obtained from some relabeling of h, denoted by h^*, by successive removals of subordination relations. Therefore, there exist hierarchies $h_1, \ldots, h_L$ in $\mathscr{H}_n$ such that

$$h' \leftarrow_{RS} h_1 \leftarrow_{RS} \cdots \leftarrow_{RS} h_L \leftarrow_{RS} h^*,$$

where, for $\hat{h}, \overline{h} \in \mathscr{H}_n$, "$\hat{h} \leftarrow_{RS} \overline{h}$" symbolically indicates that "$\hat{h}$ can be obtained from $\overline{h}$ by removing a subordination relation."

Consequently, because $\succsim$ satisfies SR, we have

$$h^* \succ h_L \succ \cdots \succ h_1 \succ h',$$

and since h^* is a relabeling of h and $\succsim$ satisfies A, we see that

$$h \sim h^* \succ h_L \succ \cdots \succ h_1 \succ h'. \tag{1}$$

Because $\succsim$ is reflexive and transitive, (1) implies that $h \succ h'$ (Sen 2017, Lemma 1*a, p. 56), as desired. □

As an example of a $\succsim_H$-consistent hierarchical pre-order on $\mathscr{H}_n$, consider the pre-order $\succsim_s$ defined as follows: $h \succsim_s h'$ if and only if there exists a bijection ϕ from the set of individuals in h to the set of individuals in h' such that, for each i in h, the number of supervisors of i in h is greater than or equal to the number of supervisors of $\phi(i)$ in h'.

In other words, h is considered more hierarchical than h' under $\succsim_s$ if there exists a bijective correspondence between the individuals in the two hierarchies such that each individual i in h has at least as many (or more) supervisors compared to the individual in h' that corresponds to i.

The symmetric and asymmetric parts of $\succsim_s$ are denoted, as usual, by $\sim_s$ and $\succ_s$, respectively.

To illustrate, consider again the hierarchy from Fig. 6. It is easy to see that the bijection represented via double-arrowed lines connecting nodes across hierarchies has the property that each linked pair of individuals has (weakly) more supervisors in h than in h'. Thus, $h \succsim_s h'$.

Like $\succsim_H$, the pre-order $\succsim_s$ also satisfies Anonymity and Subordination Removal.

Proposition 1 *The hierarchical pre-order $\succsim_s$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies A and SR.*

The proof of Proposition 1 is provided in Appendix A.4.

By Proposition 1 and Theorem 2, $\succsim_s$ is $\succsim_H$-consistent, i.e., $\succsim_s$ agrees with $\succsim_H$ for pairs of hierarchies in $\mathscr{H}_n$ that are comparable under $\succsim_H$.

The converse assertion is false, i.e., $\succsim_H$ is not $\succsim_s$-consistent. In fact, in general, $h \succsim_s h'$ need not imply $h \succsim_H h'$. To see this, consider the two seven-person hierarchies depicted in Fig. 11. It is easy to see that, for the bijection represented in the figure, each linked pair of individuals has (weakly) more supervisors in h than in h'. Thus, $h \succsim_s h'$.

However, $h \not\succsim_H h'$. To understand this, we refer to Lemma 6 from Appendix A.1, which is restated here for the reader's convenience:

> Suppose that I_0 (respectively, I'_0) represents the set of level-0 individuals in h (respectively, h'). Then $h \succsim_H h'$ if and only if there exists a finite partition of I'_0 consisting of $\#I_0$ elements,
>
> $$\{I'_1, \ldots, I'_{\#I_0}\},$$
>
> where $\#I_0$ denotes the cardinality of I_0, such that for each $i \in I_0$, $h(i) \succsim_H (h'(\iota))_{\iota \in I'_i}$.

Note that, for the sub-hierarchy of h, $\hat{h}$, depicted in Figure 11, we cannot have $\hat{h} \succsim_H (h'(\iota))_{\iota\in I'}$ for $I' \subseteq I'_0$, since $\hat{h} \succsim_H (h'(\iota))_{\iota\in I'}$ implies that $\hat{h}$ and $(h'(\iota))_{\iota\in I'}$ have the same size, and yet all the sub-hierarchies of h' with only one level-0 supervisor have more than two individuals.

Hence, $h \not\succsim_H h'$, and Theorem 1 implies that h' cannot be obtained from some relabeling of h by successive removals of subordination relations.

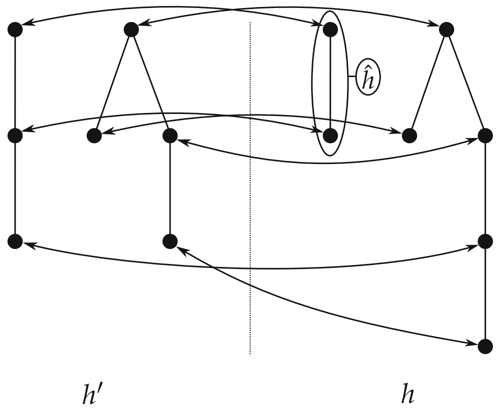


Fig. 11 $h \succ_s h'$ and $h \nsucc_H h'$

Next, we consider complete orders over the set of hierarchies $\mathscr{H}_n$ induced by hierarchical indices.

A *hierarchical index* on $\mathscr{H}_n$ is a map $I : \mathscr{H}_n \to \mathbb{R}$ that assigns a "hierarchical degree" $I(h)$ to every hierarchy $h \in \mathscr{H}_n$. The index I gives rise to a complete hierarchical order on $\mathscr{H}_n$, $\succcurlyeq_I$, defined as follows:

$$h \succcurlyeq_I h' \Leftrightarrow I(h) \geq I(h').$$

Clearly, $\succcurlyeq_I$ is a properly defined hierarchical order, i.e., a reflexive and transitive order on $\mathscr{H}_n$.

For example, given $h \in \mathscr{H}_n$, let $s_h(i)$ represent the number of supervisors of i in h and define

$$I_s(h) = \frac{1}{n} \sum_{i \in h} s_h(i), \tag{2}$$

i.e., $I_s(h)$ denotes the average number of supervisors per individual in the hierarchy h.

Clearly, $\succcurlyeq_s \subseteq \succcurlyeq_{I_s}$, i.e., h is at least as hierarchical as h' under $\succcurlyeq_{I_s}$ whenever h is at least as hierarchical as h' under $\succcurlyeq_s$. What is more, $\succcurlyeq_{I_s}$ is $\succcurlyeq_s$-consistent. Indeed, for $h, h' \in \mathscr{H}_n$, $h \succ_s h'$ implies that there exists a bijection ϕ from the set of individuals in h to the set of individuals in h' such that

$$s_h(i) \geq s_{h'}(\phi(i)), \quad \text{for each } i \in h,$$

with the inequality being strict for at least one i. Consequently,

$$I_s(h) = \frac{1}{n} \sum_{i \in h} s_h(i) > \frac{1}{n} \sum_{i \in h} s_{h'}(\phi(i)) = \frac{1}{n} \sum_{i \in h'} s_{h'}(i) = I_s(h'),$$

whence $h \succ_{I_s} h'$. Similarly, one can show that $h \sim_s h'$ implies $h \sim_{I_s} h'$.

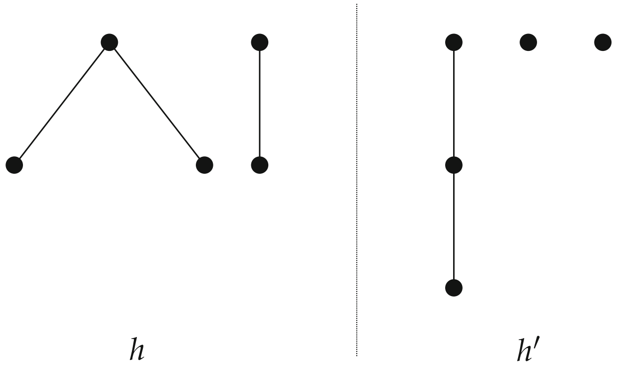


Fig. 12 $h \succsim_{I_s} h'$ and $h \not\succsim_s h'$

Conversely, $\succsim_s$ is not $\succsim_{I_s}$-consistent. Indeed, $\succsim_{I_s}$-dominance need not imply $\succsim_s$-dominance, as the following example illustrates.

Consider the hierarchies h and h' depicted in Fig. 12. It is easily verified that the average number of supervisors per individual in either h or h' is $3/5$:

$$I_s(h) = I_s(h') = 3/5.$$

Hence, $h \sim_{I_s} h'$. Nevertheless, $h \nsim_s h'$. Put differently, we have $h \succsim_{I_s} h'$ and yet $h \not\succsim_s h'$. To see that $h \not\succsim_s h'$, it suffices to note that there is one individual in h' who has two supervisors, while all individuals in h have at most one supervisor.

We have seen that $\succsim_{I_s}$ is $\succsim_s$-consistent and that $\succsim_{I_s}$ and $\succsim_s$ are $\succsim_H$-consistent. Thus, $\succsim_{I_s}$, being a complete order on $\mathscr{H}_n$, is a completion of $\succsim_H$ and $\succsim_s$.

Since $\succsim_{I_s}$ is reflexive, transitive, and $\succsim_H$-consistent, Theorem 2 implies that $\succsim_{I_s}$ satisfies A and SR.

Proposition 2 *The hierarchical order $\succsim_{I_s}$ defined on $\mathscr{H}_n$ is complete, reflexive, and transitive and satisfies* A *and* SR.

4 Comparing hierarchies of varying sizes

In this section, we expand the previous analysis to encompass hierarchies of different sizes.

Recall that $\mathscr{H}_n$ is the set of n-person hierarchies. The superset

$$\mathscr{H} = \bigcup_n \mathscr{H}_n$$

represents the set of hierarchies of any size.

Definition 4 A hierarchical pre-order on $\mathscr{H}$ is a reflexive and transitive binary relation on $\mathscr{H}$.

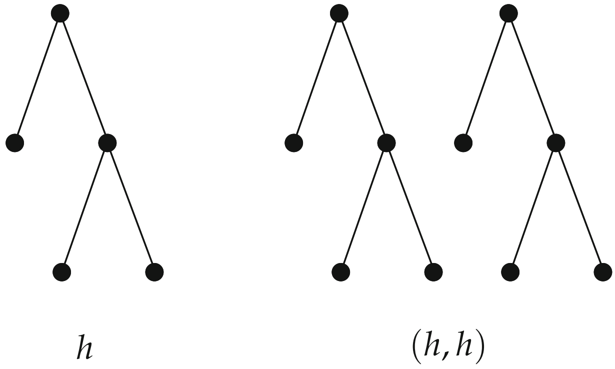


Fig. 13 (h, h) is a replication of h

A ***replication*** of a hierarchy $h \in \mathscr{H}$ is a hierarchy in $\mathscr{H}$ of the form $(h, \dots, h)$. By convention, h is a replication of itself.

For example, the ten-person hierarchy (h, h) in Fig. 13 is a replication of the five-person hierarchy h.

The Replication Principle allows one to compare hierarchies in $\mathscr{H}$ with their replications.

Replication Principle (RP) A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies RP if for any two hierarchies h and h' in $\mathscr{H}$, $h' \sim h$ whenever h' is a replication of h.

This axiom asserts that, for a hierarchical pre-order on $\mathscr{H}$, any replication of a given hierarchy h is as hierarchical as h.

Note that, according to the Replication Principle, two hierarchies in $\mathscr{H}$ can only be compared through replication if the number of individuals in one hierarchy is a multiple of the number in the other. For instance, if h is a two-person hierarchy, then any replication of h will result in a $2k$-person hierarchy, where $k \in \mathbb{N}$.

The hierarchical pre-order $\succcurlyeq_H$ on $\mathscr{H}_n$ introduced in Sect. 3 can be extended to the domain $\mathscr{H}$ as follows: for $h, h' \in \mathscr{H}$, $h' \succcurlyeq_H h$ if and only if there exists n such that h_r (respectively, h'_r) is a replication of h (respectively, h') in $\mathscr{H}_n$ and $h'_r \succcurlyeq_H h_r$.

In other words: for $h, h' \in \mathscr{H}$, h' is at least as hierarchical as h if and only if there exist equally sized replications of h' and h, denoted h'_r and h_r respectively, such that h'_r is at least as hierarchical as h_r.

Lemma 1 implies that the extension of $\succcurlyeq_H$ to $\mathscr{H}$ is reflexive and transitive and satisfies A and SR.[4] Moreover, the extension $\succcurlyeq_H$ satisfies RP. Indeed, if $h' = (h, \dots, h)$ is a replication of h, then $h' \sim_H h$ because $(h, \dots, h) \sim_H (h, \dots, h)$.

[4] Indeed, given h, h', and h'' in $\mathscr{H}$, we have

$$h \succcurlyeq_H h,$$
$$[h \succcurlyeq_H h' \;\&\; h' \succcurlyeq_H h''] \Rightarrow [h_r \succcurlyeq_H h'_r \;\&\; h'_r \succcurlyeq_H h''_r] \Rightarrow h_r \succcurlyeq_H h''_r \Rightarrow h \succcurlyeq_H h'',$$

where the relation "$h_r \succcurlyeq_H h''_r$" follows from the transitivity of $\succcurlyeq_H$ restricted to the domain $\mathscr{H}_n$ (Lemma 1).
To see that $\succcurlyeq_H$, defined on $\mathscr{H}$, satisfies A, suppose that h and h' are hierarchies in $\mathscr{H}$ such that h' is a relabeling of h. Then h and h' have the same size, n. Since the restriction of $\succcurlyeq_H$ to $\mathscr{H}_n$ satisfies A (Lemma 1), we have $h \sim_H h'$.
To see that $\succcurlyeq_H$ satisfies SR, suppose that h and h' are hierarchies in $\mathscr{H}$ and that h' can be obtained from

Lemma 2 *The hierarchical pre-order $\succcurlyeq_H$ defined on $\mathscr{H}$ is reflexive and transitive and satisfies* A, SR, *and* RP.

Given the extension $\succcurlyeq_H$ defined on $\mathscr{H}$, one can define $\succcurlyeq_H$-consistency as in the previous section.

A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ is $\succcurlyeq_H$***-consistent*** if the following two conditions are satisfied for every pair h, h' in $\mathscr{H}$:

- $h \succ_H h' \Rightarrow h \succ h'$.
- $h \sim_H h' \Rightarrow h \sim h'$.

The following result states that the axioms A, SR, and RP fully characterize $\succcurlyeq_H$-consistency for hierarchical pre-orders defined on $\mathscr{H}$. This result extends Theorem 2 to the domain $\mathscr{H}$.

Theorem 3 *A hierarchical pre-order on $\mathscr{H}$ satisfies* A, SR, *and* RP *if and only if it is $\succcurlyeq_H$-consistent.*

Similar to Theorem 2, Theorem 3 can be proven using Theorem 1 and two ancillary lemmas proven in Appendix A.

Proof of Theorem 3 [*Sufficiency*.] Suppose that $\succcurlyeq$ is $\succcurlyeq_H$-consistent. Because $\succcurlyeq_H$ satisfies A, SR, and RP (Lemma 2), and since $\succcurlyeq$ is $\succcurlyeq_H$-consistent, it follows that $\succcurlyeq$ also satisfies A, SR, and RP.

[*Necessity*.] Suppose that $\succcurlyeq$ is a hierarchical pre-order on $\mathscr{H}$ satisfying A, SR, and RP. We must show that $\succcurlyeq$ is $\succcurlyeq_H$-consistent, i.e., that the following two conditions are satisfied for every pair h, h' in $\mathscr{H}$:

(a) $h \succ_H h' \Rightarrow h \succ h'$.
(b) $h \sim_H h' \Rightarrow h \sim h'$.

Fix h and h' in $\mathscr{H}$. Suppose that $h \in \mathscr{H}_m$ and $h' \in \mathscr{H}_n$. Let h_r (respectively, h'_r) be an n-times (respectively, m-times) replication of h (respectively, h'). Then h_r and h'_r are hierarchies in $\mathscr{H}_{mn}$.

Suppose first that $h \sim_H h'$. Since $\succcurlyeq_H$ satisfies RP (Lemma 2),

$$h_r \sim_H h \sim_H h' \sim_H h'_r. \tag{3}$$

Because $\succcurlyeq_H$ is reflexive and transitive (Lemma 2), $\sim_H$ is transitive (Sen 2017, Lemma 1*a, p. 56). Consequently, (3) implies that $h_r \sim_H h'_r$.

Since $h_r, h'_r \in \mathscr{H}_{mn}$ and $h_r \sim_H h'_r$, h_r is a relabeling of h'_r (see Lemma 4 in Appendix A.1). Because $\succcurlyeq$ satisfies A and RP, it follows that

$$h \sim h_r \sim h'_r \sim h'.$$

By transitivity of $\sim$, we see that $h \sim h'$. This establishes (b).

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h by removing a subordination relation. Then h and h' have the same size, n. Since the restriction of $\succcurlyeq_H$ to $\mathscr{H}_n$ satisfies SR (Lemma 1), we have $h \succ_H h'$.

Now suppose that $h \succ_H h'$. Let h_r (respectively, h'_r) be an n-times (respectively, m-times) replication of h (respectively, h'). Since $\succsim_H$ satisfies RP (Lemma 2),

$$h_r \sim_H h \succ_H h' \sim_H h'_r. \tag{4}$$

Because $\succsim_H$ is reflexive and transitive (Lemma 2), (4) gives $h_r \succ_H h'_r$ (Sen 2017, Lemma 1*a, p. 56).

Since $h_r, h'_r \in \mathscr{H}_{mn}$ and $h_r \succ_H h'_r$, h'_r can be obtained from some relabeling of h_r, denoted by h^*_r, by successive removals of subordination relations (Theorem 1). Therefore, there exist hierarchies $h_1, \ldots, h_L$ in $\mathscr{H}_n$ such that

$$h'_r \leftarrow_{RS} h_1 \leftarrow_{RS} \cdots \leftarrow_{RS} h_L \leftarrow_{RS} h^*_r,$$

where, for $\hat{h}, \overline{h} \in \mathscr{H}_{mn}$, "$\hat{h} \leftarrow_{RS} \overline{h}$" means that "$\hat{h}$ can be obtained from $\overline{h}$ by removing a subordination relation."

Consequently, because $\succsim$ satisfies SR,

$$h^*_r \succ h_L \succ \cdots \succ h_1 \succ h'_r,$$

and since h^*_r is a relabeling of h_r and $\succsim$ satisfies A, we see that

$$h_r \sim h^*_r \succ h_L \succ \cdots \succ h_1 \succ h'_r. \tag{5}$$

Because $\succsim$ is reflexive and transitive, (5) implies that $h_r \succ h'_r$ (Sen 2017, Lemma 1*a, p. 56). Since $\succsim$ satisfies RP, and since h_r (respectively, h'_r) is a replication of h (respectively, h'), it follows that

$$h \sim h_r \succ h'_r \sim h',$$

implying that $h \succ h'$. This establishes (a). □

Recall the hierarchical pre-order $\succsim_s$ on $\mathscr{H}_n$ introduced in Sect. 3: $h \succsim_s h'$ if and only if there exists a bijection ϕ between the sets of individuals in the two hierarchies such that, for each i in h, the number of supervisors of i in h is greater than or equal to the number of supervisors of $\phi(i)$ in h'.

This pre-order can be extended to $\mathscr{H}$ as follows: for $h, h' \in \mathscr{H}$, $h' \succsim_s h$ if and only if there exists n such that h_r (respectively, h'_r) is a replication of h (respectively, h') in $\mathscr{H}_n$ and $h'_r \succsim_s h_r$.

Proposition 1 implies that the extension of $\succsim_s$ to $\mathscr{H}$ is reflexive and transitive and satisfies A and SR.[5] In addition, the extension $\succsim_s$ satisfies RP. Indeed, if $h' = (h, \ldots, h)$ is a replication of h, then $h' \sim_s h$ because $(h, \ldots, h) \sim_s (h, \ldots, h)$.

Proposition 3 *The hierarchical pre-order $\succsim_s$ defined on $\mathscr{H}$ is reflexive and transitive and satisfies* A, SR, *and* RP.

[5] This assertion can be proven using a method entirely analogous to that employed in the proof from Footnote 4, which demonstrates that the extension of $\succsim_H$ to $\mathscr{H}$ is reflexive, transitive, and satisfies the conditions A and SR.

By Proposition 3 and Theorem 3, $\succcurlyeq_s$ is $\succcurlyeq_H$-consistent.

In Sect. 3, we demonstrated that for the hierarchical pre-orders $\succcurlyeq_H$ and $\succcurlyeq_s$ defined on $\mathscr{H}_n$, $\succcurlyeq_s$-dominance need not imply $\succcurlyeq_H$-dominance.[6] This implies that the extensions $\succcurlyeq_H$ and $\succcurlyeq_s$ defined on $\mathscr{H}$ share the same property. Consequently, since $\succcurlyeq_s$, defined on $\mathscr{H}$, is $\succcurlyeq_H$-consistent, $\succcurlyeq_s$ aligns with $\succcurlyeq_H$ for pairs of hierarchies that are comparable under $\succcurlyeq_H$, but there are cases where $\succcurlyeq_H$ deems two hierarchies incomparable, whereas $\succcurlyeq_s$ still orders them.

The hierarchical index I_s defined in (2) can also be extended to $\mathscr{H}$: for $h \in \mathscr{H}$, let

$$I_s(h) = \frac{1}{n} \sum_{i \in h} s_h(i), \tag{6}$$

where $s_h(i)$ denotes the number of supervisors of i in h. For each $h \in \mathscr{H}$, $I_s(h)$ represents the average number of supervisors per individual in the hierarchy h.

The hierarchical order $\succcurlyeq_{I_s}$ induced by I_s on $\mathscr{H}$ is defined by

$$h \succcurlyeq_{I_s} h' \Leftrightarrow I_s(h) \geq I_s(h').$$

The order $\succcurlyeq_{I_s}$ agrees with $\succcurlyeq_s$ for those pairs of hierarchies $h, h' \in \mathscr{H}$ that can be $\succcurlyeq_s$-ordered. In other words, $\succcurlyeq_{I_s}$ is $\succcurlyeq_s$-consistent. To see this, suppose that $h \succ_s h'$ for $h, h' \in \mathscr{H}$. Then there exists n such that h_r (respectively, h'_r) is a replication of h (respectively, h') in $\mathscr{H}_n$ and $h_r \succ_s h'_r$. Since $h_r, h'_r \in \mathscr{H}_n$ and $h_r \succ_s h'_r$, we have $I_s(h_r) > I_s(h'_r)$ (since we know, from the previous section, that the restriction of $\succcurlyeq_{I_s}$ to $\mathscr{H}_n$ is $\succcurlyeq_s$-consistent). Next, note that if $h \in \mathscr{H}_m$ and h_r is a k-times replication of h, so that $mk = n$, then

$$I_s(h_r) = \frac{1}{n} \sum_{i \in h_r} s_{h_r}(i) = \frac{1}{mk} k \sum_{i \in h} s_h(i) = I_s(h).$$

Similarly, $I_s(h'_r) = I_s(h')$. Consequently,

$$I_s(h) = I_s(h_r) > I_s(h'_r) = I_s(h'),$$

whence $h \succ_{I_s} h'$.

In a similar vein, one can show that $h \sim_s h'$, for $h, h' \in \mathscr{H}$, implies $h \sim_{I_s} h'$. Thus, $\succcurlyeq_{I_s}$ is $\succcurlyeq_s$-consistent.

Conversely, $\succcurlyeq_s$ is not $\succcurlyeq_{I_s}$-consistent. In fact, for $h, h' \in \mathscr{H}$, $h \succcurlyeq_{I_s} h'$ need not imply $h \succcurlyeq_s h'$. To illustrate this point, refer back to Fig. 12 and its detailed explanation. Figure 12 represents two hierarchies h and h' in $\mathscr{H}$ for which $h \sim_{I_s} h'$ and yet $h \nsim_s h'$.

We have seen that $\succcurlyeq_{I_s}$ is $\succcurlyeq_s$-consistent and that $\succcurlyeq_{I_s}$ and $\succcurlyeq_s$ are $\succcurlyeq_H$-consistent. Thus, $\succcurlyeq_{I_s}$, being a complete order on $\mathscr{H}$, is a completion of $\succcurlyeq_H$ and $\succcurlyeq_s$.

Since $\succcurlyeq_{I_s}$ is reflexive, transitive, and $\succcurlyeq_H$-consistent, Theorem 3 implies that $\succcurlyeq_{I_s}$ satisfies A, SR, and RP.

6 Refer to Fig. 11 and its detailed explanation.

Proposition 4 *The hierarchical order $\succsim_{I_s}$ defined on $\mathscr{H}$ is complete, reflexive, and transitive and satisfies* A, SR, *and* RP.

Remark 1 The hierarchical index $I_s : \mathscr{H} \to \mathbb{R}$ defined in (6) can be equivalently formulated in terms of the average number of subordinates. Formally, let $I_b : \mathscr{H} \to \mathbb{R}$ be defined by

$$I_b(h) = \frac{1}{n} \sum_{i \in h} b_h(i),$$

where $b_h(i)$ denotes the number of subordinates of i in h.

We claim that

$$I_s(h) = I_b(h), \quad \text{for every } h \in \mathscr{H}. \tag{7}$$

We first establish the equivalence for hierarchies with exactly one level-0 individual. Consider the following induction argument. The equivalence is clearly true for any 1-person hierarchy. Now assume that (7) is satisfied for any $(n-1)$-person hierarchy h with exactly one level-0 individual, where $n > 1$. We will show that (7) must also hold for any n-person hierarchy with exactly one level-0 individual.

Suppose that h is an n-person hierarchy, where $n > 1$, and that h has only one level-0 individual, ι. Let $h \backslash \iota$ denote the hierarchy derived from h by removing individual ι. Then $h \backslash \iota$ is an $(n-1)$-person hierarchy and we have

$$I_s(h \backslash \iota) = \frac{1}{n-1} \sum_{i \in h \backslash \iota} s_{h \backslash \iota}(i) = \frac{1}{n-1} \sum_{i \in h \backslash \iota} b_{h \backslash \iota}(i) = I_b(h \backslash \iota)$$

by the induction hypothesis. Using this equality, we can write

$$\begin{aligned} I_s(h) &= \frac{1}{n} \sum_{i \in h} s_h(i) = \frac{1}{n} \sum_{i \in h \backslash \iota} (s_{h \backslash \iota}(i) + 1) = \frac{1}{n} \sum_{i \in h \backslash \iota} s_{h \backslash \iota}(i) + \frac{n-1}{n} \\ &= \frac{n-1}{n} \cdot I_s(h \backslash \iota) + \frac{n-1}{n} = \frac{n-1}{n} \cdot I_b(h \backslash \iota) + \frac{n-1}{n} \\ &= \frac{1}{n} \sum_{i \in h \backslash \iota} b_{h \backslash \iota}(i) + \frac{n-1}{n} \\ &= \frac{1}{n} \left(\sum_{i \in h \backslash \iota} b_{h \backslash \iota}(i) + n - 1 \right) = \frac{1}{n} \sum_{i \in h} b_h(i) = I_b(h), \end{aligned}$$

as desired.

Now let h be an arbitrary n-person hierarchy. Then h can be expressed as

$$h = (h(i))_{i \in I_0},$$

where I_0 denotes the set of all level-0 individuals in h. Let $n_{h(i)}$ represent the number of individuals in $h(i)$ for $i \in I_0$. Because

$$I_s(h(i)) = I_b(h(i)), \quad \text{for all } i \in I_0,$$

we have

$$
\begin{aligned}
I_s(h) = \frac{1}{n}\sum_{j\in h} s_h(j) &= \frac{1}{n}\sum_{i\in I_0}\sum_{j\in h(i)} s_h(j) \\
&= \frac{1}{n}\sum_{i\in I_0}[n_{h(i)} I_s(h(i))] = \frac{1}{n}\sum_{i\in I_0}[n_{h(i)} I_b(h(i))] \\
&= \frac{1}{n}\sum_{i\in I_0}\sum_{j\in h(i)} b_h(j) = \frac{1}{n}\sum_{j\in h} b_h(j) = I_b(h),
\end{aligned}
$$

which establishes the desired conclusion.

The hierarchical index $I_s : \mathscr{H} \to \mathbb{R}$ defined in (6) bears certain similarities to the "global reaching centrality" measure introduced by Mones et al. (2012). This measure is defined for unweighted directed graphs—a framework that naturally encompasses our notion of hierarchies. Indeed, any hierarchy can be mapped to a directed graph by replacing the paths linking individuals with directed edges (one-sided arrows). This mapping can be constructed in two distinct ways: either by directing arrows from each supervisor to their immediate subordinates, or conversely, from each subordinate to their immediate supervisor.

Using the subordinate-to-supervisor mapping, we can express the core concept from Mones et al. (2012) as follows. For an individual i in an n-person hierarchy h, the "local reaching centrality" $C_h(i)$ is defined as the proportion of all individuals in h, other than i, that can be reached from i via outgoing edges. More formally, $C_h(i)$ is calculated by dividing the number of individuals reachable from i through outgoing edges by the total number of individuals in h minus 1 (excluding i itself from both the numerator and denominator).

Let $\overline{C}_h$ denote the maximum local reaching centrality among all individuals in hierarchy h. The *global reaching centrality* of h, denoted by $GRC(h)$, is then defined as the average deviation of individuals' local reaching centrality from this maximum:

$$
GRC(h) = \frac{\sum_{i=1}^{n}(\overline{C}_h - C_h(i))}{n-1},
$$

where n is the number of individuals in the hierarchy.

To illustrate this concept, consider the five-person hierarchy h depicted in Fig. 14. In this representation, directed edges indicate subordination relations, with arrows pointing from subordinates to their immediate supervisors. This example allows us to demonstrate the calculation of both local reaching centralities for individual nodes and the global reaching centrality for the entire hierarchy.

In Fig. 14, local reaching centralities are displayed next to each individual's node. As can be observed, the maximum local reaching centrality $\overline{C}_h$ is $1/2$. Using these values, we can compute the global reaching centrality:

$$
GRC(h) = \frac{\frac{1}{2}-\frac{1}{2}}{4} + \frac{\frac{1}{2}-\frac{1}{4}}{4} + \frac{\frac{1}{2}-0}{4} + \frac{\frac{1}{2}-0}{4} + \frac{\frac{1}{2}-0}{4} = \frac{7}{16}.
$$

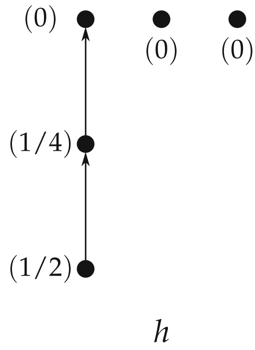

Fig. 14 Calculating GRC_h

We note that

$$C_h(i) = \frac{s_h(i)}{n-1}, \quad \text{for each individual } i \text{ in } h,$$

where $s_h(i)$ represents the number of supervisors of i in h. With this foundation, we can now explore the relationship between the individual supervision index $I_s(h)$ and the global reaching centrality index $GRC(h)$ as follows:

$$\begin{aligned} GRC(h) &= \frac{\sum_{i=1}^{n}(\overline{C}_h - C_h(i))}{n-1} = \frac{n\overline{C}_h - \sum_{i=1}^{n} C_h(i)}{n-1} = \frac{n\overline{C}_h - \frac{1}{n-1}\sum_{i=1}^{n} s_h(i)}{n-1} \\ &= \frac{n\overline{C}_h - \frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n} s_h(i)}{n-1} = \frac{n\overline{C}_h - \frac{n}{n-1} I_s(h)}{n-1} = \frac{n}{n-1}\left(\overline{C}_h - \frac{I_s(h)}{n-1}\right). \end{aligned} \tag{8}$$

The relationship between $I_s(h)$ and $GRC(h)$ described by (8) implies that for pairs of equally-sized hierarchies h and h' in $\mathscr{H}$ with identical maximum local reaching centralities, i.e., satisfying $\overline{C}_h = \overline{C}_{h'}$, we have

$$GRC(h) \geq GRC(h') \Leftrightarrow I_s(h') \geq I_s(h). \tag{9}$$

However, this equivalence no longer holds for pairs of hierarchies with different maximum local reaching centralities. In fact, the global reaching centrality index fails to be $\succcurlyeq_H$-consistent.

To see this, consider first the two three-person hierarchies represented in Fig. 15. These hierarchies have the same number of individuals and the same maximum local reaching centralities, since

$$\overline{C}_h = \overline{C}_{h'} = \frac{1}{2}.$$

Consequently, (9) holds, and so

$$I_s(h) = \frac{2}{3} > \frac{1}{3} = I_s(h')$$

implies that $GRC(h) < GRC(h')$.

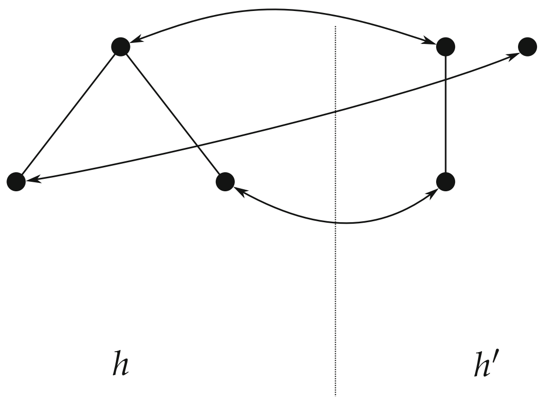

Fig. 15 $h \succ_H h'$ and $GRC(h) < GRC(h')$

On the other hand, it is easy to see that $h \succcurlyeq_H h'$ (for example, using the bijection depicted in Fig. 15) and yet $h' \not\succcurlyeq_H h$, so that $h \succ_H h'$. To see that $h' \not\succcurlyeq_H h$, consider that any bijection between the set of individuals in h' and those in h must associate a level-0 individual in h', denoted as i, with a level-1 individual in h. However, the supervisor of this level-1 individual cannot be linked to i or any supervisor of i.

Thus, we have $h \succ_H h'$ and $GRC(h) < GRC(h')$.

Now consider the two hierarchies, h and h', depicted in Fig. 16. We can establish that $h \succcurlyeq_H h'$ using the bijection illustrated in the figure. However, the reverse relation does not hold: $h' \not\succcurlyeq_H h$. To see why, consider any bijection from the individuals in h' to those in h. Such a bijection must map some level-1 individual i in h' to a subordinate in h. This subordinate in h necessarily has a supervisor that does not correspond to any supervisor of i in h', thus violating the conditions for $h' \succcurlyeq_H h$.

Note that the maximum local reaching centralities differ across hierarchies in this example:

$$\overline{C}_{h'} = \frac{1}{2} < 1 = \overline{C}_h.$$

Since

$$I_s(h') = \frac{2}{3} < 1 = I_s(h),$$

we can use (8) to compute the corresponding global reaching centralities:

$$GRC(h') = \frac{1}{4} < \frac{3}{4} = GRC(h).$$

This example, in conjunction with the previous one, yields a key insight into the relationship between hierarchical structures and centrality measures. Firstly, in the last example we observe that $h \succ_H h'$, indicating that hierarchy h is strictly more hierarchical than h' according to $\succcurlyeq_H$. Secondly, we find that the global reaching centrality measures of these hierarchies satisfy $GRC(h') < GRC(h)$. This ordering stands in contrast to our previous example, where the relationship between global reaching centrality and $\succcurlyeq_H$ was reversed. This reversal demonstrates that the global reaching

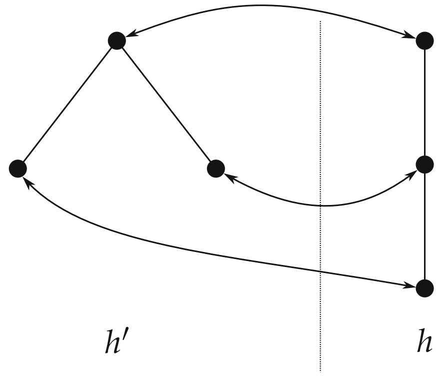

Fig. 16 $h \succ_H h'$ and $GRC(h) > GRC(h')$

centrality index is not $\succcurlyeq_H$-consistent. In other words, a more hierarchical structure as defined by $\succcurlyeq_H$ does not necessarily imply a lower global reaching centrality.

Alternatively, we can reformulate global reaching centrality by considering the downward flow of authority rather than the upward chain of command, i.e., using directed edges that flow from supervisors to their immediate subordinates, without altering the key insights derived from our previous analysis. This alternative formulation provides a different perspective on hierarchical relationships while maintaining consistency with our established findings regarding the non-monotonicity of global reaching centrality with respect to $\succcurlyeq_H$.

Indeed, this alternative formulation of global reaching centrality yields consistent results with our previous findings. Examining the example in Fig. 15, we observe that

$$h \succ_H h' \quad \text{and} \quad GRC(h) > GRC(h'),$$

indicating that the more hierarchical structure h corresponds to a higher global reaching centrality. Conversely, the example from Fig. 16 demonstrates that

$$h \succ_H h' \quad \text{and} \quad GRC(h') > GRC(h).$$

Here, despite h being more hierarchical, it exhibits a lower global reaching centrality than h'. This juxtaposition of results clearly illustrates the non-monotonic relationship between hierarchical structure and global reaching centrality, reinforcing our earlier conclusion that GRC is not $\succcurlyeq_H$-consistent, even under this alternative formulation.

5 Concluding remarks

This paper introduces a novel axiomatic framework for measuring and comparing hierarchical structures. By establishing a mathematical foundation for hierarchy comparison, our work represents a first step towards a systematic analysis of how organizational architecture influences economic outcomes and shapes societal structures.

At the core of our framework are three novel axiomatic principles that provide a basis for systematic hierarchy comparison. The Anonymity axiom establishes that the underlying hierarchical structure remains invariant under any relabeling of individuals within the hierarchy, ensuring that the hierarchical nature of an organization is determined by its structural relationships rather than by the specific identities of its members. The Subordination Removal axiom formalizes the intuitive notion that eliminating a supervisory relationship necessarily results in a less hierarchical structure, providing a crucial basis for comparing the relative "steepness" of different hierarchies. Finally, the Replication Principle extends our framework beyond the constraints of fixed-size comparisons by stipulating that replicating a hierarchy—creating multiple identical copies of its structure—preserves its hierarchical degree, thus enabling meaningful comparisons between hierarchies of different sizes while maintaining consistency with our other axioms.

We introduce a new and straightforward hierarchical pre-order, $\succcurlyeq_H$, defined through the comparison of supervisory ranks across hierarchies. We demonstrate that this pre-order is intrinsically related to our axioms. Specifically, we show that one hierarchy is strictly more hierarchical than another under $\succcurlyeq_H$ if and only if the latter can be derived from the former—up to relabeling—through a series of successive subordination removals.

We define $\succcurlyeq_H$-consistent hierarchical pre-orders as those that align with $\succcurlyeq_H$ when comparing two hierarchies that can be ranked under $\succcurlyeq_H$. We characterize all $\succcurlyeq_H$-consistent hierarchical pre-orders via the Anonymity and Subordination Removal axioms and extend our framework to accommodate hierarchies of varying sizes through the Replication Principle.

Our analysis reveals that the notion of hierarchical degree encapsulated in our basic axioms is fundamentally distinct from the essence of rank mobility measurement. Additionally, we study examples of partial and full completions of the core hierarchical pre-order $\succcurlyeq_H$, comparing them to existing hierarchical indices.

The theoretical foundations established in this work naturally point to a number of avenues for future research. The first involves expanding our framework to encompass networks with multiple immediate supervisors, as the current focus on hierarchies with single immediate supervisors may not capture all real-world organizational structures.

The second avenue leverages our framework's refined hierarchical comparison methods to explore the interplay between organizational design and socioeconomic outcomes. This research direction examines the broad implications of organizational structure for income distribution and social dynamics.

Of particular interest is the prospect of a nuanced theoretical analysis investigating the correlation between "deeper hierarchy" and established measures of income inequality, poverty, and polarization. Even in the context of "economic growth," deeply entrenched hierarchies can hinder equitable resource distribution. In such scenarios, similar levels of absolute poverty become increasingly normatively regrettable. Understanding these determinants requires a comprehensive analysis of how power dynamics and social stratification might counteract poverty alleviation efforts, drawing on existing research about poverty reduction failure.[7]

Data availability This work is theoretical in nature; therefore, it does not involve dataset analysis or generation.

Appendix A

A.1 Preliminary lemmata

Lemma 3 *For $h, h' \in \mathscr{H}_n$, $h \sim_H h'$ implies that there exists a bijection ϕ from the set of individuals in h to the set of individuals in h' satisfying the following*:

(a) For each level-k individual i in h, $\phi(i)$ is a level-k individual in h'.
(b) For each individual i in h, the number of immediate subordinates of i in h equals the number of immediate subordinates of $\phi(i)$ in h'.
(c) For each individual i in h, the set

$$\phi(I_{h(i)}) = \{\phi(i) : i \in I_{h(i)}\},$$

where $I_{h(i)}$ denotes the set of all individuals in the sub-hierarchy $h(i)$, is equal to the set of all individuals in the sub-hierarchy $h'(\phi(i))$.

Proof Since $h \succcurlyeq_H h'$, there exists, by definition, a bijection ϕ from the set of individuals in h to the set of individuals in h' satisfying the following:

(I) For each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p_{h'}(\phi(i))$, links (via ϕ^{-1}) to a supervisor j of i in h: $\phi^{-1}(p_{h'}(\phi(i))) = j = p_h^l(i)$ for some l.

Similarly, since $h' \succcurlyeq_H h$, there exists a bijection ϕ' from the set of individuals in h' to the set of individuals in h satisfying the following: for each individual i in h' such that $\phi'(i)$ is not a level-0 individual, the immediate supervisor of $\phi'(i)$ in h, $p_h(\phi'(i))$, links (via ϕ'^{-1}) to a supervisor j of i in h': $\phi'^{-1}(p_h(\phi'(i))) = j = p_{h'}^l(i)$ for some l.

Let I_0 (respectively, I_0') be the set of all level-0 individuals in h (respectively, h').

First, we show that

$$\phi(I_0) = \{\phi(i) : i \in I_0\} \subseteq I_0'.$$

To see this, note that $j \in \phi(I_0)\backslash I_0'$ implies that there exist $i \in I_0$ and a level-k individual j in h', where $k > 0$, such that $\phi(i) = j$. But then $\phi^{-1}(p_{h'}(\phi(i))) \neq p_h^l(i)$ for any l, which contradicts (I). Therefore, $\phi(I_0)\backslash I_0' = \emptyset$, which implies that $\phi(I_0) \subseteq I_0'$.

Similarly, we can show that $\phi'(I_0') \subseteq I_0$.

Next, let I_l (respectively, I_l') be the set of all level-l individuals in h (respectively, h'). Suppose that the containments $\phi(I_l) \subseteq I_l'$ and $\phi'(I_l') \subseteq I_l$ have been proven for

[7] Poverty reduction failure is studied in Kanbur and Mukherjee (2007) and Chakravarty and D'Ambrosio (2013).

each $l \in \{0, \ldots, k\}$ and some $k \geq 0$. Then the containments $\phi(I_{k+1}) \subseteq I'_{k+1}$ and $\phi'(I'_{k+1}) \subseteq I_{k+1}$ can also be proven.

To see this, note first that, for each l, the two containments $\phi(I_l) \subseteq I'_l$ and $\phi'(I'_l) \subseteq I_l$ imply that h and h' have the same number of level-l individuals. Indeed, if there were more level-l individuals in h', then $\phi(I_l)$ would be a strict subset of I'_l, and, since both I_l and $\phi(I_l)$ have the same cardinality, I_l would be a smaller set than I'_l, contradicting the containment $\phi'(I'_l) \subseteq I_l$. A similar contradiction can be obtained under the assumption that there are more level-l individuals in h.

Now, if $j \in \phi(I_{k+1}) \backslash I'_{k+1}$, since $\phi(I_l) \subseteq I'_l$ and I_l and I'_l have the same cardinality for each $l \in \{0, \ldots, k\}$, we see that $j \in \phi(I_{k+1}) \backslash (\bigcup_{l=0}^{k+1} I'_l)$. Consequently, there exist $i \in I_{k+1}$ and a level-κ' individual j in h', where $\kappa' > k + 1$, such that $\phi(i) = j$. But then $\phi^{-1}(p_{h'}(\phi(i)))$ must be a level-κ individual in h, where $\kappa \geq k + 1$. Indeed, if $\phi^{-1}(p_{h'}(\phi(i)))$ where a level-l individual in h for some $l \in \{0, \ldots, k\}$, then $\phi(I_l) \nsubseteq I'_l$ (since $p_{h'}(\phi(i)) = p_{h'}(j)$ is a level-$(\kappa' - 1)$ individual in h' and $\kappa' > k + 1$), contradicting the assumed containment $\phi(I_l) \subseteq I'_l$. Consequently, $\phi^{-1}(p_{h'}(\phi(i))) \neq p_h^{\ell}(i)$ for any ℓ, which contradicts (I). Therefore, $\phi(I_{k+1}) \backslash I'_{k+1} = \emptyset$, which implies that $\phi(I_{k+1}) \subseteq I'_{k+1}$.

Next, fix a level-k individual i in h. Since $\phi(I_k) \subseteq I'_k$, it follows that $\phi(i)$ is a level-k individual in h'. This establishes (a).

To see that (b) holds, let i be an individual in h. Suppose that i is a level-k individual. Proceeding by contradiction, suppose that the number of immediate subordinates of i in h is not equal to the number of immediate subordinates of $\phi(i)$ in h'. If he number of immediate subordinates of $\phi(i)$ is greater, then (by (a)) there exists a subordinate j of $\phi(i)$ linking (via ϕ^{-1}) to a level-$(k + 1)$ subordinate ι in h whose immediate supervisor, i^*, is not i. But then $\phi(\iota) = j$ and $\phi(i)$ is j's immediate supervisor in h', and yet $\phi(i)$ links (via ϕ^{-1}) to $i \neq i^*$, implying that i is not a supervisor of ι, which contradicts (I).

Hence, the number of immediate subordinates of i in h must be greater than or equal to the number of immediate subordinates of $\phi(i)$ in h'.

If the number of immediate subordinates of i in h is greater than the number of immediate subordinates of $\phi(i)$ in h', there exists an immediate subordinate ι of i in h such that $\phi(\iota)$'s immediate supervisor in h', $p_{h'}(\phi(\iota))$, is not $\phi(i)$. But then $\phi^{-1}(p_{h'}(\phi(\iota)))$ is a level-k individual different from i, implying that $\phi^{-1}(p_{h'}(\phi(\iota)))$ is not a supervisor of ι in h, which contradicts (I).

Thus, the number of immediate subordinates of i in h is equal to the number of immediate subordinates of $\phi(i)$ in h'. This establishes (b).

It only remains to prove (c). Fix an individual i in h, and let $I_{h(i)}$ (respectively, $I_{h'(\phi(i))}$) be the set of all individuals in the hierarchy $h(i)$ (respectively, $h'(\phi(i))$). We must show that $\phi(I_{h(i)}) = I_{h'(\phi(i))}$.

Suppose that i is a level-k individual. Note that it suffices to prove the following: Suppose that j is a level-$(k+l)$ individual in $h(i)$ for $l \geq 0$. Then $\phi(S_j) = S_{\phi(j)}$, where S_j (respectively, $S_{\phi(j)}$) represents the set of immediate subordinates of j (respectively, $\phi(j)$) in h (respectively, h').

Suppose that j is a level-$(k + l)$ individual in $h(i)$ for $l \geq 0$. Suppose that there exists $\iota \in S_j$ such that $\phi(\iota) \notin S_{\phi(j)}$. Then, since ι is a level-$(k + l + 1)$ individual in h, so that $\phi(\iota)$ is a level-$(k+l+1)$ individual in h' (by (a)), $\phi(\iota)$'s immediate supervisor in h', $p_{h'}(\phi(\iota))$, is a level-$(k+l)$ individual in h' who links (via ϕ^{-1}) to a level-$(k+l)$ individual in h, $\phi^{-1}(p_{h'}(\phi(\iota)))$. Note that $\phi^{-1}(p_{h'}(\phi(\iota)))$, being different from j (since $\phi(\iota) \notin S_{\phi(j)}$ and so $p_{h'}(\phi(\iota)) \neq \phi(j)$), is not a supervisor of ι in h. Since this contradicts (I), we see that $\phi(S_j) \subseteq S_{\phi(j)}$. But then $\phi(S_j) = S_{\phi(j)}$, since S_j and $S_{\phi(j)}$ (and hence $\phi(S_j)$ and $S_{\phi(j)}$) have the same cardinality (by (b)). □

Lemma 4 *For $h, h' \in \mathscr{H}_n$, $h \sim_H h'$ implies that h is a relabeling of h'.*

Proof It suffices to show that there exists a bijection ϕ from the set of individuals in h to the set of individuals in h' such that $h(i)$ is a relabeling of $h'(\phi(i))$ for each i in h.

Let ϕ be the bijection given by Lemma 3. Let K be the largest level for which there are level-K individuals in h. Then all the level-K individuals in h have zero subordinates. By item (a) of Lemma 3, for any level-K individual i in h, $\phi(i)$ is a level-K individual in h'; moreover, since $\phi(i)$ has zero subordinates, item (b) of Lemma 3 implies that $h(i)$ is a relabeling of $h'(\phi(i))$.

Suppose that $h(i)$ has been shown to be a relabeling of $h'(\phi(i))$ for each level-k individuals i in h, where $k \in \{K, K-1, \ldots, 1\}$. Then $h(i)$ is a relabeling of $h'(\phi(i))$ for each level-$(k-1)$ individual i in h.

To see this, fix a level-$(k-1)$ individual i in h. Let S_i (respectively, S_i') be the set of level-k subordinates of i (respectively, $\phi(i)$) in h (respectively, h'). If

$$\phi(S_i) = \{\phi(j) : j \in S_i\} = S_i'$$

were true, then, because $h(j)$ is a relabeling of $h'(\phi(j))$ for each $j \in S_i$, it would follow that $h(i)$ is a relabeling of $h'(\phi(i))$. Thus, it suffices to show that $\phi(S_i) = S_i'$.

By items (a) and (c) of Lemma 3, we know that $\phi(S_i)$ is a set of level-k individuals in h' contained in the set of all individuals in the sub-hierarchy $h'(\phi(i))$. Since S_i' is the set of all level-k individuals in $h'(\phi(i))$, item (a) of Lemma 3 gives $\phi(S_i) \subseteq S_i'$. But then $\phi(S_i) = S_i'$, since S_i and S_i' (and hence $\phi(S_i)$ and S_i') have the same cardinality (by item (b) of Lemma 3). □

Lemma 5 *For $h, h' \in \mathscr{H}_n$, if h' can be obtained from h by removing a subordination relation, then $h \succ_H h'$.*

Proof We proceed by induction on n. The statement is clearly true if $n = 1$. We now prove the statement for any $n > 1$ under the assumption that it is true for m-person hierarchies, where $m \in \{1, \ldots, n-1\}$.

Because h' can be obtained from h by removing a subordination relation, there exists a level-k subordinate i^* in h, where $k \in \{1, \ldots, K\}$ (and where K denotes the total number of levels in the hierarchy h), satisfying the following:

(i) If i^*'s immediate supervisor, $p_h(i^*)$, is a level-0 individual in h, then h' is the hierarchy in which the sub-hierarchy $h(i^*)$ is no longer under $p_h(i^*)$'s supervision, i^* becomes a level-0 individual, and the sub-hierarchy that begins at i^* is $h(i^*)$; h' is otherwise equal to h.

(ii) If i^*'s immediate supervisor in h, $p_h(i^*)$, is a not level-0 individual, then $p_h(i^*)$ is an immediate subordinate of $p_h^2(i^*)$, i.e., $p_h(i^*) \in S_{p_h^2(i^*)}$. In this case, h'

is the hierarchy in which the sub-hierarchy $h(i^*)$ is no longer under $p_h(i^*)$'s supervision, but rather under the direct supervision of $p_h^2(i^*)$, so that i^* is no longer a level-k subordinate, but rather a level-$(k-1)$ subordinate in $S_{p_h^2(i^*)}$, and the sub-hierarchy that begins at i^* is $h(i^*)$; h' is otherwise equal to h.

First, we show that $h \succsim_H h'$. To see this, let ϕ be the identity map from the set of individuals in h to the set of individuals in h'. It suffices to prove the following:

$(*)$ For each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p_{h'}(\phi(i))$, links (via ϕ^{-1}) to a supervisor j of i in h: $\phi^{-1}(p_{h'}(\phi(i))) = j = p_h^l(i)$ for some l.

Note that if the sub-hierarchy $h(i^*)$ is removed from h and the sub-hierarchy $h'(\phi(i^*)) = h'(i^*)$ is removed from h', the remaining hierarchies are identical. Therefore, for any individual i in h not in $h(i^*)$, $(*)$ holds.

Next, fix an individual i in $h(i^*)$. If $i \neq i^*$, then, since the two sub-hierarchies $h(i^*)$ and $h'(i^*)$ are identical, and since ϕ is the identity map, $(*)$ holds.

It remains to prove $(*)$ for $i = i^*$. Note that if $\phi(i^*) = i^*$ is not a level-0 individual in h', then (ii) must hold. But then the immediate supervisor of $\phi(i^*) = i^*$ in h' is $p_{h'}^2(i^*)$, which links (via ϕ^{-1}) to $p_h^2(i^*)$ in h, a supervisor of i^* in h, implying that $(*)$ holds.

Since $h \succsim_H h'$, it remains to show that $h' \not\succsim_H h$. Proceeding by contradiction, $h' \succsim_H h$ implies that $h' \sim_H h$. Consequently, h' is a relabeling of h (Lemma 4), contradicting that h' can be obtained from h by removing a subordination relation. □

Lemma 6 *Suppose that $h, h' \in \mathscr{H}_n$, and let I_0 (respectively, I_0') be the set of level-0 individuals in h (respectively, h'). The following two statements are equivalent:*

(i) $h \succsim_H h'$.
(ii) *There exists a finite partition of I_0' consisting of $\#I_0$ elements,*

$$\{I_1', \ldots, I_{\#I_0}'\},$$

where $\#I_0$ denotes the cardinality of I_0, such that for each $i \in I_0$, $h(i) \succsim_H (h'(\iota))_{\iota \in I_i'}$.

Proof Suppose that (ii) holds. Then there exists a finite partition of I_0' consisting of $\#I_0$ elements,

$$\{I_1', \ldots, I_{\#I_0}'\}, \tag{10}$$

such that for each $i \in I_0$, there exists a bijection ϕ_i from the set of individuals in $h(i)$ to the set of individuals in $(h'(\iota))_{\iota \in I_i'}$ satisfying the following: for each individual j in $h(i)$ such that $\phi_i(j)$ is not a level-0 individual,

$$\phi_i^{-1}(p_{(h'(\iota))_{\iota \in I_i'}}(\phi_i(j))) = p_{h(i)}^l(j), \quad \text{for some } l. \tag{11}$$

Define a function ϕ from the set of individuals in h to the set of individuals in h' as follows:

$$\phi(j) = \phi_i(j) \quad \text{if } j \in h(i),\ i \in I_0.$$

We claim that ϕ is a bijection. To see this, note first that if j' is an individual in h' then j' is an individual in the sub-hierarchy $h'(i')$ for some $i' \in I'_0$. Since $i' \in I'_0$, there exists $i \in I_0$ such that $i' \in I'_i$. Thus, j' is an individual in the sub-hierarchy $(h'(\iota))_{\iota \in I'_i}$, and so there exists an individual j in the sub-hierarchy $h(i)$ such that $j = \phi_i^{-1}(j')$. Because $i \in I_0$ and $j \in h(i)$, we see that

$$\phi(j) = \phi_i(j) = \phi_i(\phi_i^{-1}(j')) = j'.$$

Hence, ϕ is an onto map.

To see that ϕ is one-to-one, note that for every j in h, there is a unique $i \in I_0$ such that j is an individual in the sub-hierarchy $h(i)$, and so there is a unique element I'_i of the partition in (10) and a unique individual i' in the sub-hierarchy $(h'(\iota))_{\iota \in I'_i}$ such that $\phi(j) = i'$.

Thus, ϕ is a bijection.

Next, fix an arbitrary individual j in h such that $\phi(j)$ is not a level-0 individual. Then $j \in h(i)$ for some $i \in I_0$ and $\phi(j) = \phi_i(j)$, where ϕ_i is a bijection from the set of individuals in $h(i)$ to the set of individuals in $(h'(\iota))_{\iota \in I'_i}$ satisfying (11). Therefore, since

$$\phi^{-1}(p_{h'}(\phi(j))) = \phi^{-1}(p_{h'}(\phi_i(j))) = \phi^{-1}(p_{(h'(\iota))_{\iota \in I'_i}}(\phi_i(j))) = \phi_i^{-1}(p_{(h'(\iota))_{\iota \in I'_i}}(\phi_i(j)))$$

and

$$p_h^l(j) = p_{h(i)}^l(j),$$

we have

$$\phi^{-1}(p_{h'}(\phi_i(j))) = p_h^l(j), \quad \text{for some } l,$$

and so (i) holds.

Now suppose that (i) holds. Then, there exists a bijection ϕ from the set of individuals in h to the set of individuals in h' satisfying the following:

(∗) For each individual i in h such that $\phi(i)$ is not a level-0 individual,

$$\phi^{-1}(p_{h'}(\phi(i))) = p_h^l(i), \quad \text{for some } l.$$

For each $i \in I_0$ (respectively, $i \in I'_0$), let $I_{h(i)}$ (respectively, $I_{h'(i)}$) be the set of all individuals in $h(i)$ (respectively, $h'(i)$).

First, we show that

$$\forall i \in I'_0, \exists j \in I_0 : \phi^{-1}(I_{h'(i)}) = \{\phi^{-1}(\iota) : \iota \in I_{h'(i)}\} \subseteq I_{h(j)}. \tag{12}$$

Fix $i \in I'_0$. Then $\phi^{-1}(i)$ is an individual in h. Let j be the level-0 supervisor of $\phi^{-1}(i)$ in h. It suffices to show that $\phi^{-1}(I_{h'(i)}) \subseteq I_{h(j)}$.

Proceeding by contradiction, suppose that there exists $\iota \in \phi^{-1}(I_{h'(i)}) \backslash I_{h(j)}$. Then $\iota \in I_{h(\iota^*)}$ for some $\iota^* \in I_0 \backslash \{j\}$. Note that $\phi(\iota) \neq i$, since $\phi^{-1}(i) \neq \iota$. Since i is the only level-0 individual in $h'(i)$, and since $I_{h'(i)} \ni \phi(\iota) \neq i$, $\phi(\iota)$, an individual in the sub-hierarchy $h'(i)$, is not a level-0 individual. Therefore, by $(*)$,

$$\phi^{-1}(p_{h'}(\phi(\iota))) = p_h^l(\iota), \quad \text{for some } l,$$

implying that

$$\phi^{-1}(p_{h'}(\phi(\iota))) \in I_{h(\iota^*)}. \tag{13}$$

If $p_{h'}(\phi(\iota))$ is a level-0 individual, since $p(\phi(\iota)) \in I_{h'(i)}$, then $p(\phi(\iota)) = i$ (since i is the only level-0 individual in $h'(i)$); in this case, since $\phi^{-1}(i) \in I_{h(j)}$ and $j \neq \iota^*$, $\phi^{-1}(p_{h'}(\phi(\iota))) = \phi^{-1}(i)$ cannot be a member of $I_{h(\iota^*)}$, contradicting (13).

If $p(\phi(\iota))$ is not a level-0 individual, then, again applying $(*)$, we see that

$$\phi^{-1}(p^2_{h'}(\phi(\iota))) = p_h^l(\iota), \quad \text{for some } l,$$

implying that

$$\phi^{-1}(p^2_{h'}(\phi(\iota))) \in I_{h(\iota^*)}. \tag{14}$$

If $p^2_{h'}(\phi(\iota))$ is a level-0 individual, since $p^2_{h'}(\phi(\iota)) \in I_{h'(i)}$, then $p^2_{h'}(\phi(\iota)) = i$; in this case, since $\phi^{-1}(i) \in I_{h(j)}$ and $j \neq \iota^*$, $\phi^{-1}(p^2_{h'}(\phi(\iota))) = \phi^{-1}(i)$ cannot be a member of $I_{h(\iota^*)}$, which contradicts (14).

If $p^2(\phi(\iota))$ is not a level-0 individual, again applying $(*)$, we see that

$$\phi^{-1}(p^3_{h'}(\phi(\iota))) = p^l(\iota), \quad \text{for some } l,$$

implying that $\phi^{-1}(p^3_{h'}(\phi(\iota))) \in I_{h(\iota^*)}$. This argument can be reiterated until a contradiction is reached in finitely many steps.

This proves (12).

Next, we show that there exists a finite partition of I'_0 consisting of $\#I_0$ elements,

$$\{I'_1, \dots, I'_{\#I_0}\},$$

such that for each $i \in I_0$, we have

$$\phi(I_{h(i)}) = \{\phi(\iota) : \iota \in I_{h(i)}\} = \bigcup_{\iota \in I'_i} I_{h'(\iota)}. \tag{15}$$

To see this, note that by (12), for each $\iota \in I'_0$, there exists $j_\iota \in I_0$ such that

$$\phi^{-1}(I_{h'(\iota)}) = \{\phi^{-1}(\iota') : \iota' \in I_{h'(\iota)}\} \subseteq I_{h(j_\iota)}.$$

In addition, because ϕ is a bijection, each j_ι must be unique.

For $i \in I_0$, define

$$I'_i = \{\iota \in I'_0 : j_\iota = i\}. \tag{16}$$

First, we show that

$$\{I'_1, \dots, I'_{\#I_0}\},$$

where each I'_i is defined by (16), is a partition of I'_0. First, note that for $i, j \in I_0$ with $i \neq j$,

$$I'_i \cap I'_j = \{\iota \in I'_0 : j_\iota = i\} \cap \{\iota \in I'_0 : j_\iota = j\} = \emptyset,$$

since j_ι is uniquely defined for each $\iota \in I'_0$. Next, note that

$$\bigcup_{i \in I_0} I'_i = I'_0.$$

To see this, note that the containment $\bigcup_{i \in I_0} I'_i \subseteq I'_0$ is obvious, so we only need to show that $\bigcup_{i \in I_0} I'_i \supseteq I'_0$.

Suppose that $\iota \in I'_0$. Then $\iota \in I'_{j_\iota}$, and so $\iota \in \bigcup_{i \in I_0} I'_i$. Thus, $\bigcup_{i \in I_0} I'_i \supseteq I'_0$.

Next, we prove (15) for each $i \in I_0$.

Fix $i \in I_0$. Suppose that $j' \in \phi(I_{h(i)})$. Then there exists $j \in I_{h(i)}$ such that $j' = \phi(j)$, implying that $j' \in I_{h'(\iota)}$ for some $\iota \in I'_0$. If $\iota \notin I'_i$, then there exists $j_\iota \in I_0 \backslash \{i\}$ such that $\phi^{-1}(I_{h'(\iota)}) \subseteq I_{h(j_\iota)}$. Since $j' \in I_{h'(\iota)}$, this implies that

$$j = \phi^{-1}(j') \in I_{h(j_\iota)}.$$

But this contradicts the fact that $j \in I_{h(i)}$. Indeed, since $j_\iota \neq i$, we have

$$I_{h(j_\iota)} \cap I_{h(i)} = \emptyset.$$

Therefore, we must have $j' \in I_{h'(\iota)}$ for some $\iota \in I'_i$.

Hence, $\phi(I_{h(i)}) \subseteq \bigcup_{\iota \in I'_i} I_{h'(\iota)}$.

Conversely, if $j' \in I_{h'(\iota)}$ for some $\iota \in I'_i$, then the definition of I'_i in (16) entails that

$$\phi^{-1}(I_{h'(\iota)}) \subseteq I_{h(i)},$$

implying that $\phi^{-1}(j') \in I_{h(i)}$, and so $j' \in \phi(I_{h(i)})$.

Consequently, $\phi(I_{h(i)}) \supseteq \bigcup_{\iota \in I'_i} I_{h'(\iota)}$.

We conclude that (15) holds. Now let $\phi|_{I_{h(i)}}$ be the restriction of ϕ to $I_{h(i)}$.

By (15), $\phi|_{I_{h(i)}}$ is a bijection from $I_{h(i)}$ to $\bigcup_{\iota \in I'_i} I_{h'(\iota)}$.

By $(*)$, $\phi|_{I_{h(i)}}$ satisfies the following: for each individual j in $h(i)$ such that $\phi|_{I_{h(i)}}(j)$ is not a level-0 individual in $(h'(\iota))_{\iota \in I'_i}$,

$$\phi|_{I_{h(i)}}{}^{-1}(p_{(h'(\iota))_{\iota \in I'_i}}(\phi|_{I_{h(i)}}(j))) = p^l_{h(i)}(j), \quad \text{for some } l.$$

Consequently, $h(i) \succsim_H (h'(\iota))_{\iota \in I'_i}$. Since i was arbitrary in I_0, this establishes (ii). $\square$

Lemma 7 *Suppose that $h, h' \in \mathscr{H}_n$, and let I_0 (respectively, I'_0) be the set of level-0 individuals in h (respectively, h'). Then (i) implies (ii):*

(i) $h \succ_H h'$.

(ii) *For each $i \in I_0$, there exists $I'_i \subseteq I'_0$ such that $h(i) \succcurlyeq_H (h'(j))_{j \in I'_i}$; and there exists $i^* \in I_0$ such that $h(i^*) \succ_H (h'(j))_{j \in I'_{i^*}}$.*

Proof Suppose that $h \succ_H h'$. Then $h \succcurlyeq_H h'$, and so Lemma 6 implies that there exists a finite partition of I'_0 consisting of $\#I_0$ elements,

$$\{I'_1, \ldots, I'_{\#I_0}\},$$

where $\#I_0$ denotes the cardinality of I_0, such that for each $i \in I_0$, $h(i) \succcurlyeq_H (h'(\iota))_{\iota \in I'_i}$.

If $(h'(\iota))_{\iota \in I'_i} \succcurlyeq_H h(i)$ for each $i \in I_0$, then $h(i) \sim_H (h'(\iota))_{\iota \in I'_i}$ for each $i \in I_0$, and so by Lemma 4, $h(i)$ is a relabeling of $(h'(\iota))_{\iota \in I'_i}$ for each $i \in I_0$. But then

$$h = (h(i))_{i \in I_0} \quad \text{and} \quad h' = ((h'(\iota))_{\iota \in I'_i})_{i \in I_0}$$

are relabelings of each other, and so A implies that

$$h = (h(i))_{i \in I_0} \sim_H h' = ((h'(\iota))_{\iota \in I'_i})_{i \in I_0},$$

a contradiction (recall that $\succcurlyeq_H$ satisfies A by Proposition 1).

Therefore, there exists $i^* \in I_0$ such that $(h'(\iota))_{\iota \in I'_{i^*}} \not\succcurlyeq_H h(i^*)$, implying that $h(i^*) \succ_H (h'(\iota))_{\iota \in I'_{i^*}}$, and so (ii) holds. □

A.2 Proof of Lemma 1

Lemma 1 is restated here for the reader's convenience.

Lemma 1 *The hierarchical pre-order $\succcurlyeq_H$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies A and SR.*

Proof Reflexivity follows immediately from the definition of $\succcurlyeq_H$.

Let $I_{\widetilde{h}}$ represent the set of individuals in $\widetilde{h} \in \mathscr{H}_n$.

To see that $\succcurlyeq_H$ is transitive, suppose that

$$h \succcurlyeq_H h' \succcurlyeq_H h'', \quad \text{for } h, h', h'' \in \mathscr{H}_n.$$

Then, there exist bijections $\phi : I_h \to I_{h'}$ and $\phi' : I_{h'} \to I_{h''}$ satisfying the following:

{1} For each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p_{h'}(\phi(i))$, links (via ϕ^{-1}) to a supervisor j of i in h, i.e.,

$$\phi^{-1}(p_{h'}(\phi(i))) = j = p_h^l(i), \quad \text{for some } l.$$

{2} For each individual i in h' such that $\phi'(i)$ is not a level-0 individual, the immediate supervisor of $\phi'(i)$ in h'', $p_{h''}(\phi'(i))$, links (via ϕ'^{-1}) to a supervisor j of i in h', i.e.,

$$\phi'^{-1}(p_{h''}(\phi'(i))) = j = p_{h'}^l(i), \quad \text{for some } l.$$

Since ϕ and ϕ' are bijections, the composition $\phi^* := \phi' \circ \phi$ is also a bijection (see, e.g., Blyth 1975, Theorem 5.10, p. 37). Thus, it suffices to show the following:

($\circ$) For each individual i in h such that $\phi^*(i)$ is not a level-0 individual, the immediate supervisor of $\phi^*(i)$ in h'', $p_{h''}(\phi^*(i))$, links (via ϕ^{*-1}) to a supervisor j of i in h, i.e.,

$$\phi^{*-1}(p_{h''}(\phi^*(i))) = j = p_h^l(i), \quad \text{for some } l.$$

Fix an individual i in h such that $\phi^*(i)$ is not a level-0 individual. Proceeding by contradiction, suppose that

$$\phi^{*-1}(p_{h''}(\phi^*(i))) \neq p_h^l(i), \quad \text{for any } l. \tag{17}$$

Consider the sequence

$$i, \phi(i), p_{h'}(\phi(i)), \phi^{-1}(p_{h'}(\phi(i))), \phi^{-1}(p_{h'}(\phi(i))), p_{h'}(\phi(i)), p_{h'}^2(\phi(i)),$$
$$\phi^{-1}(p_{h'}^2(\phi(i))), \phi^{-1}(p_{h'}^2(\phi(i))), p_{h'}^2(\phi(i)), p_{h'}^3(\phi(i)), \phi^{-1}(p_{h'}^3(\phi(i))), \ldots.$$

This sequence can be subdivided into four-element cycles as follows:

$$\begin{array}{ll}
\text{Cycle 1:} & i, \phi(i), p_{h'}(\phi(i)), \phi^{-1}(p_{h'}(\phi(i))). \\
\text{Cycle 2:} & \phi^{-1}(p_{h'}(\phi(i))), p_{h'}(\phi(i)), p_{h'}^2(\phi(i)), \phi^{-1}(p_{h'}^2(\phi(i))) \\
\text{Cycle 3:} & \phi^{-1}(p_{h'}^2(\phi(i))), p_{h'}^2(\phi(i)), p_{h'}^3(\phi(i)), \phi^{-1}(p_{h'}^3(\phi(i))) \\
\text{Cycle 4:} & \phi^{-1}(p_{h'}^3(\phi(i))), p_{h'}^3(\phi(i)), p_{h'}^4(\phi(i)), \phi^{-1}(p_{h'}^4(\phi(i))) \\
\vdots & \vdots
\end{array}$$

The first and last elements of each cycle are individuals in h, while the second and third elements of each cycle are individuals in h'. Moreover, by {1}, the first and last elements of each cycle belong to the path connecting i and i's level-0 supervisor in h, i.e., if j is the first or the fourth element of a cycle, we have $j = p^l(i)$ for some l. In addition, by construction, the second and third elements of every cycle belong to the path connecting $\phi(i)$ and $\phi(i)$'s level-0 supervisor in h', i.e., if j is the second or the third element of a cycle, we have $j = p_{h'}^l(\phi(i))$ for some l.

Note that each individual in the path connecting $\phi(i)$ and $\phi(i)$'s level-0 supervisor in h' must eventually become the third element of a cycle. Hence, because

$$\phi'^{-1}(p_{h''}(\phi^*(i))) = p_{h'}^l(\phi(i)), \quad \text{for some } l \text{ (by \{2\})},$$

$\phi'^{-1}(p_{h''}(\phi^*(i)))$ is equal to the third element of some cycle ℓ. But then the fourth element of cycle ℓ, which can be expressed as

$$\phi^{-1}(\phi'^{-1}(p_{h''}(\phi^*(i)))),$$

belongs to the path connecting i and i's level-0 supervisor in h (as noted in the previous paragraph). Noting that

$$\phi^{-1}(\phi'^{-1}(p_{h''}(\phi^*(i)))) = \phi^{*-1}(p_{h''}(\phi^*(i))),$$

this contradicts our initial assumption in (17).

We conclude that ($\circ$) holds, implying that $\succsim_H$ is transitive.

By Lemma 5, $\succsim_H$ satisfies SR.

To see that $\succsim_H$ satisfies A, let h' be a relabeling of h. Then there exists a bijection $\phi : I_h \to I_{h'}$ with the following property: if each individual i in h' is assigned the label "$\phi^{-1}(i)$," then the resulting hierarchy is identical to h.

It is easy to see that, for the bijection ϕ, the following condition is satisfied: for each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p_{h'}(\phi(i))$, links (via ϕ^{-1}) to a supervisor j of i in h, i.e.,

$$\phi^{-1}(p_{h'}(\phi(i))) = j = p_h^l(i), \quad \text{for some } l.$$

Hence, $h \succsim_H h'$.

A similar condition can be verified for the bijection $\phi^{-1} : I_{h'} \to I_h$: for each individual i in h' such that $\phi^{-1}(i)$ is not a level-0 individual, the immediate supervisor of $\phi^{-1}(i)$ in h, $p_h(\phi^{-1}(i))$, links (via ϕ) to a supervisor j of i in h', i.e.,

$$\phi(p_h(\phi^{-1}(i))) = j = p_{h'}^l(i), \quad \text{for some } l.$$

Consequently, $h' \succsim_H h$ and $h \succsim_H h'$, implying that $h \sim_H h'$. □

A.3 Proof of Theorem 1

Theorem 1 *For $h, h' \in \mathscr{H}_n$, $h \succ_H h'$ if and only if h' can be obtained from some relabeling of h by successive removals of subordination relations.*

Proof [*Necessity*.] First, we prove the "only if" part of the statement under the assumption that h has only one level-0 individual.

We proceed by induction on n. The statement is clearly true if $n = 1$. We now prove the statement for any $n > 1$ under the assumption that it is true for m-person hierarchies, where $m \in \{1, \ldots, n-1\}$.

Suppose that $h \succ_H h'$. We must show that h' can be obtained from some relabeling of h by successive removals of subordination relations.

Since $h \succsim_H h'$, there exists a bijection ϕ from the set of individuals in h to the set of individuals in h' satisfying the following:

($\blacksquare$) For each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p_{h'}(\phi(i))$, links (via ϕ^{-1}) to a supervisor j of i in h, i.e.,

$$\phi^{-1}(p_{h'}(\phi(i))) = j = p_h^l(i), \quad \text{for some } l.$$

Let the (unique) level-0 individual h be denoted by ι. Then $\phi(\iota)$ is a level-0 individual in h' (otherwise $p_{h'}(\phi(\iota))$ would not link (via ϕ^{-1}) to a supervisor of ι, contradicting (■)).

If $\phi(\iota) \neq \iota$, the individuals in h can be relabeled so that $\phi(\iota) = \iota$. The resulting relabeling will be denoted again by h.

Let $h\backslash\iota$ be the hierarchy resulting from removing individual ι from h: in $h\backslash\iota$, every j in the set S_ι of all level-1 subordinates of ι becomes a level-0 individual, and the sub-hierarchy that begins at j is $h(j)$.

Let $h'\backslash\iota$ be the hierarchy resulting from removing individual ι from h':

- In $h'\backslash\iota$, every j in the set of all level-1 subordinates of ι becomes a level-0 individual, and the sub-hierarchy that begins at j is $h'(j)$.
- The structure of h' remains otherwise intact, i.e., the sub-hierarchy that begins at any level-0 i other than ι is $h'(i)$.

Let ϕ^* be the restriction of ϕ to the individuals in $h\backslash\iota$. Note that ϕ^* is a bijection between the individuals in $h\backslash\iota$ and those in $h'\backslash\iota$. Moreover, because ϕ satisfies (■) and $\phi(\iota) = \iota$, ϕ^* has the following property: for each individual i in $h\backslash\iota$ such that $\phi^*(i)$ is not a level-0 individual,

$$\phi^{*-1}(p_{h'}(\phi^*(i))) = p_h^l(i), \quad \text{for some } l.$$

Thus, we have $h\backslash\iota, h'\backslash\iota \in \mathscr{H}_{n-1}$ and $h\backslash\iota \succsim_H h'\backslash\iota$.

Suppose first that $h'\backslash\iota \succsim_H h\backslash\iota$. Then, $h\backslash\iota \sim_H h'\backslash\iota$, and Lemma 4 implies that $h\backslash\iota$ is a relabeling of $h'\backslash\iota$.

Since ι is the only level-0 individual in h, we can write

$$h = h(\iota) \quad \text{and} \quad h' = (h'(\iota), (h'(j))_{j \in I_0' \backslash \{\iota\}}),$$

where I_0' denotes the set of all level-0 individuals in h'. Now, letting S_ι (respectively, S_ι') be the set of level-1 subordinates of ι in h (respectively, h'), we can write

$$h\backslash\iota = (h(j))_{j \in S_\iota} \quad \text{and} \quad h'\backslash\iota = ((h'(j))_{j \in S_\iota'}, (h'(j))_{j \in I_0' \backslash \{\iota\}}).$$

Since $h\backslash\iota$ is a relabeling of $h'\backslash\iota$, there is no loss of generality in assuming that $h\backslash\iota$ and $h'\backslash\iota$ are identical (since the individuals in $h\backslash\iota$ can always be relabeled in such a way that $h\backslash\iota$ and $h'\backslash\iota$ are identical). Hence,

$$S_\iota = S_\iota' \cup (I_0' \backslash \{\iota\}) \quad \text{and} \quad h(j) = h'(j) \text{ for all } j \in S_\iota. \tag{18}$$

Now let $\{i_1, \ldots, i_m\}$ be an enumeration of $I_0' \backslash \{\iota\}$ and define the sequence of hierarchies $h_0, \ldots, h_m$ as follows:

- $h_0 = h$.
- h_1 is obtained from h by the removal of a subordination relation as follows: i_1 is no longer a level-1 subordinate in h under the direct supervision of ι, but rather a level-0 individual, and the sub-hierarchy that begins at i_1 is $h(i_1)$; h_1 is otherwise equal to h.

- h_2 is obtained from h_1 by removing a subordination relation as follows: i_2 is no longer a level-1 subordinate in h_1 under the direct supervision of ι, but rather a level-0 individual, and the sub-hierarchy that begins at i_2 is $h(i_2)$; h_2 is otherwise equal to h_1.

$$\vdots$$

- h_m is obtained from h_{m-1} by removing a subordination relation as follows: i_m is no longer a level-1 subordinate in h_{m-1} under the direct supervision of ι, but rather a level-0 individual, and the sub-hierarchy that begins at i_m is $h(i_m)$; h_m is otherwise equal to h_{m-1}.

Since h_ℓ is obtained from $h_{\ell-1}$ by removing a 0each $\ell \in \{1, \dots, m\}$, and since (18) and the definition of the sequence of hierarchies $h_0, \dots, h_m$ entails $h_m = h'$, we see that h' can be obtained from some relabeling of h by successive removals of subordination relations, as we sought.

Next, suppose that $h'\backslash\iota \not\approx_H h\backslash\iota$. Since $h\backslash\iota \succcurlyeq_H h'\backslash\iota$, we see that $h\backslash\iota \succ_H h'\backslash\iota$. Since $h\backslash\iota, h'\backslash\iota \in \mathscr{H}_{n-1}$ and $h\backslash\iota \succ_H h'\backslash\iota$, the induction hypothesis gives some relabeling of $h\backslash\iota$, denoted again by $h\backslash\iota$, such that

$$h'\backslash\iota \Leftarrow_{RS} h\backslash\iota;$$

here (and in the remainder of the proof), for any two hierarchies $\hat{h}$ and $\overline{h}$, "$\hat{h} \Leftarrow_{RS} \overline{h}$" means that "$\hat{h}$ can be obtained from $\overline{h}$ by successive removals of subordination relations."

Recall that h and h' can be expressed as

$$h = h(\iota) \quad \text{and} \quad h' = (h'(\iota), (h'(j))_{j \in I_0' \backslash \{\iota\}}),$$

and that $h\backslash\iota$ and $h'\backslash\iota$ are expressible as

$$h\backslash\iota = (h(j))_{j \in S_\iota} \quad \text{and} \quad h'\backslash\iota = ((h'(j))_{j \in S_\iota'}, (h'(j))_{j \in I_0' \backslash \{\iota\}}),$$

where S_ι (respectively, S_ι') represents the set of level-1 subordinates of ι in h (respectively, h').

Because every removal of a subordination relation in the transition

$$h'\backslash\iota \Leftarrow_{RS} h\backslash\iota$$

affects only the players of one and only one of the sub-hierarchies in $(h(j))_{j \in S_\iota}$, there exists a partition

$$(I_j)_{j \in S_\iota}$$

of the set $S_\iota' \cup (I_0' \backslash \{\iota\})$ such that each I_j is a subset of the set of individuals in $h(j)$ and

$$(h'(j'))_{j' \in I_j} \Leftarrow_{RS} h(j), \quad \text{for all } j \in S_\iota. \tag{19}$$

Each partition member I_j can be further partitioned into two sets: the members of I_j that are immediate subordinates of ι in h', I_j^s, and the members of I_j that are not immediate subordinates of ι in h', I_j^{ns}:

$$I_j^s = I_j \cap S'_\iota \quad \text{and} \quad I_j^{ns} = I_j \cap (I'_0 \backslash \{\iota\}).$$

Using this notation, (19) can be rewritten as

$$((h'(j'))_{j' \in I_j^s}, (h'(j'))_{j' \in I_j^{ns}}) \Leftarrow_{RS} h(j), \quad \text{for all } j \in S_\iota. \tag{20}$$

Now let h'' be the hierarchy obtained from the hierarchy

$$((h'(j'))_{j' \in I_j^s}, (h'(j'))_{j' \in I_j^{ns}})_{j \in S_\iota}$$

by adding individual ι at the top, so that ι is the only level-0 individual in h'' and the level-1 subordinates of ι are the members of

$$S'_\iota \cup (I'_0 \backslash \{\iota\}) = \bigcup_{j \in S_\iota} I_j = \left(\bigcup_{j \in S_\iota} I_j^s \right) \cup \left(\bigcup_{j \in S_\iota} I_j^{ns} \right).$$

Similarly, let h^* be the hierarchy obtained from

$$(h(j))_{j \in S_\iota}$$

by adding individual ι at the top, so that ι is the only level-0 individual in h^* and the level-1 subordinates of ι are the members of S_ι.

Note that (20) implies that

$$h'' \Leftarrow_{RS} h^*.$$

Consequently, since $h^* = h = h(\iota)$, we have

$$h'' \Leftarrow_{RS} h(\iota) = h. \tag{21}$$

Note that, by successive removals of subordination relations in h'', we can, for any level-1 subordinate j' in $\bigcup_{j \in S_\iota} I_j^{ns}$, move the sub-hierarchy $h'(j')$ up to level 0, thus obtaining the hierarchy

$$(h''', (h'(j'))_{j' \in \bigcup_{j \in S_\iota} I_j^{ns}}),$$

where h''' is a hierarchy defined as follows:

- h''' has only one level-0 individual, ι.
- The level-1 subordinates of ι are the members of $\bigcup_{j \in S_\iota} I_j^s$, and the sub-hierarchy that begins at any such level-1 subordinate j' is given by $h'(j')$.

We therefore have

$$(h''', (h'(j'))_{j' \in \bigcup_{j \in S_\iota} I_j^{ns}}) \Leftarrow_{RS} h''. \tag{22}$$

Moreover, since all the immediate subordinates of ι in h' are the same as all the immediate subordinates of ι in h''' and all the non-subordinates of ι in h' are the same as all the non-subordinates of ι in

$$(h''', (h'(j'))_{j' \in \bigcup_{j \in S_\iota} I_j^{ns}}),$$

we see that

$$h' = (h''', (h'(j'))_{j' \in \bigcup_{j \in S_\iota} I_j^{ns}}).$$

This, together with (21)–(22), gives $h' \Leftarrow_{RS} h$, as desired.

It remains to prove the "only if" part of the statement when h has more than one level-0 individual.

Suppose that $h \succ_H h'$. We must show that h' can be obtained from some relabeling of h by successive removals of subordination relations.

Let I_0 (respectively, I_0') be the set of level-0 individuals in h (respectively, h'). By Lemma 7, for each $i \in I_0$, there exists $I_i' \subseteq I_0'$ such that $h(i) \succsim_H (h'(j))_{j \in I_i'}$; and there exists $i^* \in I_0$ such that $h(i^*) \succ_H (h'(j))_{j \in I_{i^*}'}$.

Let I^* be the set of all $i \in I_0$ such that $h(i) \succ_H (h'(j))_{j \in I_i'}$. The set I^* is nonempty since $i^* \in I^*$. Note that, for each $i \in I_0 \backslash I^*$, we have $h(i) \sim_H (h'(j))_{j \in I_i'}$.

From the first part of this proof, we obtain the following:

$$(h'(j))_{j \in I_i'} \Leftarrow_{RS} h(i), \quad \text{for all } i \in I^*.$$

Therefore, since $h(i) \sim_H (h'(j))_{j \in I_i'}$ for each $i \in I_0 \backslash I^*$, and since the relation $h(i) \sim_H (h'(j))_{j \in I_i'}$ implies that $(h'(j))_{j \in I_i'}$ is a relabeling of $h(i)$ (Lemma 4), it follows that h' can be obtained from some relabeling of h by successive removals of subordination relations.

[*Sufficiency*.] Suppose that h' can be obtained from some relabeling of h, denoted by $\overline{h}$, by successive removals of subordination relations, i.e.,

$$h' \leftarrow_{RS} h_1 \leftarrow_{RS} \cdots \leftarrow_{RS} h_L \leftarrow_{RS} \overline{h}$$

for finitely many hierarchies $h_1, \ldots, h_L$; here (and in the remainder of the proof), for any two hierarchies $\hat{h}$ and $\underline{h}$, "$\hat{h} \leftarrow_{RS} \underline{h}$" means that "$\hat{h}$ can be obtained from $\underline{h}$ by removing a subordination relation." We must show that $h \succ_H h'$.

By Lemma 5,

$$\overline{h} \succ_H h_L \succ_H \cdots \succ_H h_1 \succ_H h'.$$

By reflexivity and transitivity of $\succsim_H$ (Lemma 1), it follows that $\overline{h} \succ_H h'$ (Sen 2017, Lemma 1*a, p. 56). Moreover, since $\overline{h}$ is a relabeling of h, Lemma 1 gives $h \sim_H \overline{h}$. Consequently,

$$h \sim_H \overline{h} \succ_H h',$$

implying that $h \succ_H h'$ (Sen 2017, Lemma 1*a, p. 56). □

A.4 Proof of Proposition 1

Proposition 1 *The hierarchical pre-order $\succsim_s$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies* A *and* SR.

Proof Reflexivity follows immediately from the definition of $\succsim_s$.

To see that $\succsim_s$ is transitive, suppose that

$$h \succsim_s h' \succsim_s h'', \quad \text{for } h, h', h'' \in \mathscr{H}_n.$$

Then, letting $I_{\hat{h}}$ represent the set of individuals in hierarchy $\hat{h}$, there exist bijections $\phi : I_h \to I_{h'}$ and $\phi' : I_{h'} \to I_{h''}$ satisfying the following:

- For each individual i in h, the number of supervisors of i in h, $\#_h i$, is greater than or equal to the number of supervisors of $\phi(i)$ in h', $\#_{h'}\phi(i)$.
- For each individual i in h', the number of supervisors of i in h', $\#_{h'} i$, is greater than or equal to the number of supervisors of $\phi'(i)$ in h'', $\#_{h''}\phi'(i)$.

Since ϕ and ϕ' are bijections, the composition $\phi^* := \phi' \circ \phi$ is also a bijection (see, e.g., Blyth 1975, Theorem 5.10, p. 37). Moreover, for each individual i in h, we have

$$\#_h i \geq \#_{h'}\phi(i) \geq \#_{h''}\phi'(\phi(i)).$$

Consequently, for each individual i in h,

$$\#_h i \geq \#_{h''}[\phi' \circ \phi](i) = \#_{h''}\phi^*(i),$$

implying that $h \succsim_s h''$.

To see that $\succsim_s$ satisfies A, suppose that h' is a relabeling of h. Then there exists a bijection $\phi : I_h \to I_{h'}$ with the following property: if each individual i in h' is assigned the label "$\phi^{-1}(i)$," then the resulting hierarchy is identical to h.

For the bijection ϕ, the following condition is satisfied: for each i in h, the number of supervisors of i in h is equal to the number of supervisors of $\phi(i)$ in h'.

Hence, $h \succsim_s h'$.

A similar condition can be verified for the bijection $\phi^{-1} : I_{h'} \to I_h$: for each individual i in h', the number of supervisors of i in h' is equal to the number of supervisors of $\phi^{-1}(i)$ in h.

Consequently, $h' \succsim_s h$ and $h \succsim_s h'$, implying that $h \sim_s h'$.

It remains to show that $\succsim_s$ satisfies SR. Suppose that h' can be obtained from h by removing a subordination relation. Then there exists a level-k subordinate i^* in h, where $k > 0$, satisfying the following:

(i) If i^*'s immediate supervisor in h, $p_h(i^*)$, is a level-0 individual, then h' is the hierarchy in which the sub-hierarchy $h(i^*)$ is no longer under $p_h(i^*)$'s supervision, i^* becomes a level-0 individual, and the sub-hierarchy that begins at i^* is $h(i^*)$; h' is otherwise equal to h.

(ii) If i^*'s immediate supervisor in h, $p_h(i^*)$, is a not level-0 individual, then $p_h(i^*)$ is an immediate subordinate of $p_h^2(i^*)$, i.e., $p_h(i^*) \in S_{p_h^2(i^*)}$. In this case, h' is the hierarchy in which the sub-hierarchy $h(i^*)$ is no longer under $p_h(i^*)$'s supervision, but rather under the direct supervision of individual $p_h^2(i^*)$, so that i^* is no longer a level-k subordinate, but rather a level-$(k-1)$ subordinate in $S_{p_h^2(i^*)}$, and the sub-hierarchy that begins at i^* is $h(i^*)$; h' is otherwise equal to h.

We must show that $h \succ_s h'$.

Note that the only individuals whose set of supervisors is altered as a result of the subordination removals specified in items (i) and (ii) are those in the sub-hierarchy $h(i^*)$ of h containing i^* and all of i^*'s subordinates. Moreover, after the removal of a subordination relation, the individuals in $h(i^*)$ are left with less supervisors. Consequently, if i is an individual in h not in the sub-hierarchy $h(i^*)$, the number of supervisors of i in h is equal to the number of supervisors of i in h', while if i is in $h(i^*)$, the number of supervisors of i in h is greater than the number of supervisors of i in h', implying that $h \succsim_s h'$.

It remains to show that $h' \not\succsim_s h$. Proceeding by contradiction, suppose that $h' \succsim_s h$. Then there exists a bijection $\varphi : I_{h'} \to I_h$ such that for each i in h', the number of supervisors of i in h' is greater than or equal to the number of supervisors of $\varphi(i)$ in h.

Let I^* be the (nonempty) set of all individuals in $h(i^*)$ who have the most supervisors in h among all the individuals in $h(i^*)$. Let s^* be the number of supervisors in h for the individuals in I^*. Then $s^* - 1$ is the number of supervisors in h' for the individuals in I^* (since h' can be obtained from h by removing a subordination relation and (i) and (ii) hold).

Let $\overline{s}$ be the maximum number of supervisors that an individual in h can have. Note that $\overline{s} \geq s^*$.

We claim that if $\overline{s} > s^*$, then $\varphi(I'_{\overline{s}}) = I_{\overline{s}}$.

To see this, note that if $\overline{s} > s^*$, then all the individuals in h with $\overline{s}$ supervisors are not in the sub-hierarchy $h(i^*)$, and so the number of individuals in h with $\overline{s}$ supervisors, denoted by $I_{\overline{s}}$, is equal to the number of individuals in h' with $\overline{s}$ supervisors, denoted by $I'_{\overline{s}}$.

This implies that $\varphi(I'_{\overline{s}}) \supseteq I_{\overline{s}}$. Indeed, if that were not the case, there would exist an individual $\iota \in I_{\overline{s}} \backslash \varphi(I'_{\overline{s}})$, i.e., ι would have $\overline{s}$ supervisors in h and any individual in h' with $\overline{s}$ supervisors would link, via φ, to an individual in h other than ι. But since $I_{\overline{s}} = I'_{\overline{s}}$, this would imply that for some individual in h' with less than $\overline{s}$ supervisors, ι', $\varphi(\iota') = \iota$, contradicting the fact that for each i in h', the number of supervisors of i in h' is greater than or equal to the number of supervisors of $\varphi(i)$ in h.

Since $I_{\overline{s}} = I'_{\overline{s}}$ and φ is a bijection, $\varphi(I'_{\overline{s}})$ and $I_{\overline{s}}$ have the same cardinality, and so the containment $\varphi(I'_{\overline{s}}) \supseteq I_{\overline{s}}$ implies that $\varphi(I'_{\overline{s}}) = I_{\overline{s}}$.

Similarly, for any $\ell \in \mathbb{N}$ for which $\overline{s} - \ell > s^*$, we have $\varphi(I'_{\overline{s}-\ell}) = I_{\overline{s}-\ell}$ (where $I_{\overline{s}-\ell}$ (respectively, $I'_{\overline{s}-\ell}$) represents the set of all individuals in h (respectively, h') with $\overline{s} - \ell$ supervisors).

Next, note that there exists $\ell^* \in \{0, 1, 2, \ldots\}$ such that $\overline{s} - \ell^* = s^*$, since $\overline{s} \geq s^*$. Therefore, $I_{\overline{s}-\ell^*} = I_{s^*}$. Moreover, since I_{s^*} is the set of all individuals in h with s^*

supervisors, and since I^* is the set of all individuals in the sub-hierarchy $h(i^*)$ of h with s^* supervisors, it follows that I_{s^*} contains I^*.

Next, we show that $I'_{s^*} = I_{s^*} \backslash I^*$. To see this, suppose that $i \in I'_{s^*}$. Then i is not in $h(i^*)$. Indeed, if i were in $h(i^*)$, since i has s^* supervisors in h', then i would have $s^* + 1$ supervisors in h, contradicting the fact that those individuals in $h(i^*)$ who have the most supervisors in h have s^* supervisors.

Since i is not in $h(i^*)$, we have $i \notin I^*$ (since the members of I^* are also in $h(i^*)$). Now, because h' can be obtained from h by removing a subordination relation and (i) and (ii) hold, and since the removal of a subordination relation specified in items (i) and (ii) does not affect the number of supervisors for those individuals not in $h(i^*)$, $i \in I'_{s^*}$ implies $i \in I_{s^*}$. Thus, $i \in I_{s^*} \backslash I^*$, and so $I'_{s^*} \subseteq I_{s^*} \backslash I^*$.

Conversely, suppose that $i \in I_{s^*} \backslash I^*$. Then i is not in $h(i^*)$ (since I^* is the set of all individuals in $h(i^*)$ who have s^* supervisors in h). Therefore, because the removal of a subordination relation specified in items (i) and (ii) does not affect the number of supervisors for those individuals not in $h(i^*)$, we have $i \in I'_{s^*}$. Hence, $I'_{s^*} \supseteq I_{s^*} \backslash I^*$.

Now, since $I'_{s^*} = I_{s^*} \backslash I^*$ and I^* is nonempty, it follows that the number of individuals in h with s^* supervisors exceeds the number of individuals in h' with s^* supervisors. Consequently, using the fact (proven earlier) that

$$\varphi(I'_{\bar{s}-\ell}) = I_{\bar{s}-\ell}, \quad \text{for any } \ell \in \mathbb{N} \text{ for which } \bar{s} - \ell > s^*,$$

we see that there exists some individual ι in h' with less than s^* supervisors whose corresponding individual in h, $\varphi(\iota)$, has s^* supervisors. But this contradicts the fact that the number of supervisors of ι in h' is greater than or equal to the number of supervisors of $\varphi(\iota)$ in h.

We conclude that $h' \not\succcurlyeq_s h$.

Since $h \succcurlyeq_s h'$ and $h' \not\succcurlyeq_s h$, we see that $h \succ_p h'$. □

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