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Inequality and bipolarization-reducing mixed taxation ☆

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ABSTRACT

Progressive income and commodity tax structures have been examined independently in terms of their ability to reduce income inequality and bipolarization. Rather than focusing on income and commodity taxes in isolation, this paper studies mixed tax systems, which subject both income and consumption to taxation. It provides necessary and sufficient conditions on the structure of these systems that ensure a reduction in income inequality and bipolarization for both exogenous and endogenous income scenarios. Commodity taxation is "superfluous" in the case of exogenous income, as any post-tax income distribution achievable through a mixed tax system can be replicated by income taxation alone. In contrast, when income is endogenous, there are cases where relying solely on income taxation is ineffective, while mixed tax structures have equalizing and depolarizing potential.

1. Introduction

The analysis of progressive income tax structures and their impact on income inequality reduction traces back to the seminal works of Jakobsson (1976), Fellman (1976), and Kakwani (1977). These authors established the equivalence between increasing average tax rates on income — known as average-rate income tax progressivity — and an income tax schedule's consistent ability to reduce income inequaliThisfoundationalconcepthassincety. This foundational concept has since been extended in several directions (see, e.g., Hemming and Keen, 1983; Eichhorn et al., 1984; Liu, 1985; Formby et al., 1986; Thon, 1987; Latham, 1988; Thistle, 1988; Moyes, 1988, 1989, 1994; Le Breton et al., 1996; Ebert and Moyes, 2000; Ju and Moreno-Ternero, 2008).[1,2]

While the Jakobsson–Fellman–Kakwani result is framed in terms of endowment economies with exogenous income, extensions of their analysis to the case of endogenous income are provided in Carbonell-Nicolau and Llavador (2018, 2021a). A related study by Carbonell-Nicolau and Llavador (2021b) adopts an alternative evaluation criterion for the distributional effects of income tax policies, focusing on their ability to reduce income bipolarization, as measured by a relative metric proposed in Foster and Wolfson (2010). This study establishes the general equivalence between inequality and bipolarization-reducing income tax schedules. Additionally, the effects of commodity taxation — as opposed to income taxation — on income inequality have been studied in Carbonell-Nicolau (2019).

The present paper examines mixed tax systems — i.e., tax systems that subject both income and consumption to taxation — and their effectiveness in reducing income inequality and bipolarization. The analysis begins with the case of exogenous income. Each mixed tax system establishes a mapping from an initial distribution of endowment incomes to a corresponding post-tax income distribution. If the latter distribution Lorenz dominates (in the relative sense) the former distribution for every possible initial distribution, the underlying tax system is classified as *inequality-reducing*.

The first main result of the paper (Theorem 1) states that a mixed tax system is inequality-reducing if and only if net income increases with pre-tax income and the tax system is jointly average-rate progressive, meaning it exhibits increasing average tax rates on income. This result is followed by a discussion of its interpretation and implications for the tax treatment of luxuries and necessities.

The Foster–Wolfson bipolarization order, in its relative form (Foster and Wolfson, 2010; Chakravarty, 2009, 2015), is a well-accepted measure of an income distribution's degree of polarization between two income groups separated by the distribution's median income. A mixed tax system is considered *bipolarization-reducing* if it results in a less

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[1] Additional foundations for income tax progressivity emerge from the principle of equal sacrifice (Young, 1987, 1988, 1990), as shown in Mitra and Ok (1996, 1997). See also D'Antoni (1999).

[2] Alternative normative rationales for income redistribution have been proposed, based on concepts of incentive compatibility (or non-manipulability) and "solidarity" principles. These approaches have been used to provide axiomatic justifications for linear income taxation, as demonstrated in the works of Ju et al. (2007), Casajus (2015a,b), Yokote and Casajus (2017).

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bipolarized post-tax income distribution, as measured by the Foster–Wolfson order, regardless of the pre-tax income distribution to which the tax system is applied. This alternative criterion for evaluating mixed tax systems can also be characterized in terms of joint average-rate progressivity: a mixed tax system is bipolarization-reducing if and only if net income increases with pre-tax income and the tax system is jointly average-rate progressive (Theorem 2).

The case of endogenous income is examined in Section 2.2. In this scenario, individuals choose their labor supply and preferred consumption bundles based on their wage rate and any applicable taxes on consumption and income. A given wage rate distribution and mixed tax system lead to an income distribution determined by individual labor and consumption choices.

The definitions of inequality (respectively, bipolarization) reducing tax systems remain similar to the case of exogenous income. However, in the endogenous case, income distributions from a taxless environment are compared with those resulting from mixed taxation. Specifically, a mixed tax system is considered inequality (respectively, bipolarization) reducing if it produces a more equal (respectively, less bipolarized) income distribution, relative to the taxless distribution, for any distribution of wage rates.

The following characterizations of inequality (respectively, bipolarization) reducing tax systems are proven for the case of endogenous income. First, a mixed tax system is inequality-reducing only if net income increases with the wage rate and the income tax is marginal-rate progressive (i.e., it exhibits increasing marginal tax rates on income) (Theorem 4). Second, families of inequality-reducing mixed tax systems can be characterized by a condition on the wage elasticity of net income (Theorem 5), which is amenable to interpretation in terms of the tax treatment of luxuries and necessities. Finally, the equivalence between inequality and bipolarization-reducing tax systems is established in Theorem 6.

Mixed tax systems have been studied in the literature on optimal taxation, which focuses on tax policies that maximize welfare in the utilitarian, Benthamite sense. Specifically, combined tax policies that subject both income and consumption to taxation have been examined in Atkinson and Stiglitz (1976), which establishes a "redundance theorem" asserting that (under some assumptions) "the optimal tax system can rely solely on income taxation". A comparable result can be demonstrated in our framework for the case of exogenous income. In particular, if the consumption of inferior goods is not taxed, any post-tax income distribution derived from an inequality-reducing (respectively, a bipolarization-reducing) mixed tax system is attainable via pure income taxation. This finding, however, does not hold in the case of endogenous income: as demonstrated by an example, there are cases when *only* mixed tax systems are effective in achieving the goal of inequality and bipolarization reduction.

2. Distributional properties of mixed taxation

This section develops notions of progressivity for mixed tax systems and characterizes them through measures of income inequality and bipolarization. Our analysis proceeds in two stages. First, in Section 2.1, we examine the case where income is exogenous — unresponsive to income taxation — while allowing consumption to respond to commodity taxation. Then, in Section 2.2, we extend the analysis to consider endogenous income responses.

2.1. Exogenous income

Individual preferences are represented by a utility function $u : \mathbb{R}^K_+ \to \mathbb{R}$ defined on commodity bundles $\boldsymbol{x} = (x_1, \dots, x_K) \in \mathbb{R}^K_+$, where x_k denotes the quantity of traded good $k \in \{1, \dots, K\}$.

The utility function u is assumed continuous and nondecreasing, strictly increasing on $\mathbb{R}^K_{++}$, and strictly quasiconcave on $\mathbb{R}^K_{++}$.

An *income tax schedule* is a continuous and nondecreasing map $T : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

- $T(y) \le y$ for each $y \in \mathbb{R}_+$.
- The map $y \mapsto y - T(y)$ is nondecreasing (i.e., T is order-preserving).

Here, $T(y) > 0$ (respectively, $T(y) < 0$) represents the tax paid (respectively, a subsidy received) by an individual whose income is y.

A *commodity tax profile* is a vector $\boldsymbol{\tau} = (\tau_1, \dots, \tau_K)$ of tax rates, one for each traded good. For each good k, $\tau_k > 0$ (respectively, $\tau_k < 0$) represents the tax liability (respectively, subsidy) paid (respectively, received) per unit of good k consumed.

A *mixed tax system* is a tuple $(T, \boldsymbol{\tau})$, where T is an income tax schedule and $\boldsymbol{\tau}$ is a commodity tax profile.

Given an income tax schedule T, a commodity tax profile $\boldsymbol{\tau} = (\tau_1, \dots, \tau_K)$, and a commodity price vector $\boldsymbol{p} = (p_1, \dots, p_K)$ such that $p_k > 0$ and $p_k + \tau_k > 0$ for each k, an individual whose income is y solves the following problem:

$$\max_{(x_1,\dots,x_K)\in\mathbb{R}^K_+} u(x_1, \dots, x_K)$$
$$\text{s.t. } (p_1 + \tau_1)x_1 + \cdots + (p_K + \tau_K)x_K \le y - T(y).$$

The properties of u entail that this problem has a unique solution, denoted by

$$x_1(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)), \dots, x_K(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)),$$

where each $x_k(\boldsymbol{p}', y')$ represents the individual's Marshallian demand function for good k corresponding to net price vector $\boldsymbol{p}'$ and net income y'. These demand functions, in conjunction with the mixed tax system $(T, \boldsymbol{\tau})$, determine *net income* (income after all tax payments):

$$z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) = y - T(y) - \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)) = \sum_{k=1}^{K} p_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)). \quad (1)$$

An *income distribution* is a vector $\boldsymbol{y} = (y_1, \dots, y_n)$, where y_i represents individual i's income level. The population size, n, takes values in the set of natural numbers. Let $(y_{[1]}, \dots, y_{[n]})$ be a rearrangement of the coordinates in $\boldsymbol{y}$ such that

$$y_{[1]} \le \cdots \le y_{[n]}.$$

Throughout the sequel, we restrict attention to income distributions whose median income is positive.

In this paper, inequality is measured by means of the relative Lorenz order, defined as follows. Given two income distributions $\boldsymbol{y} = (y_1, \dots, y_n)$ and $\boldsymbol{y}' = (y'_1, \dots, y'_n)$, $\boldsymbol{y}'$ is said to *Lorenz dominate* $\boldsymbol{y}$, denoted by "$\boldsymbol{y}' \succcurlyeq_L \boldsymbol{y}$", if

$$\frac{\sum_{i=1}^{l} y'_{[i]}}{\sum_{i=1}^{n} y'_{[i]}} \ge \frac{\sum_{i=1}^{l} y_{[i]}}{\sum_{i=1}^{n} y_{[i]}}, \quad \text{for all } l \in \{1, \dots, n\}.$$

The interpretation of the dominance relation "$\boldsymbol{y}' \succcurlyeq_L \boldsymbol{y}$" is that "$\boldsymbol{y}'$ is at least as equal as $\boldsymbol{y}$".

Given a price vector $\boldsymbol{p}$ and a pre-tax income distribution $(y_1, \dots, y_n)$, a mixed tax system $(T, \boldsymbol{\tau})$ yields a post-tax income distribution

$$(z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)).$$

Given a price vector $\boldsymbol{p}$, a mixed tax system $(T, \boldsymbol{\tau})$ is *inequality-reducing* if

$$(z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)) \succcurlyeq_L (y_1, \dots, y_n)$$

for every pre-tax income distribution $(y_1, \dots, y_n)$ and every population size n.

An income tax schedule T is *inequality-reducing* if the mixed tax system $(T, 0)$, where the commodity tax profile is identically zero, is inequality-reducing. Similarly, a commodity tax profile $\boldsymbol{\tau}$ is *inequality-reducing* if the mixed tax system $(0, \boldsymbol{\tau})$, where the income tax schedule is identically zero, is inequality-reducing.

An income tax schedule is *average-rate progressive* if, for $y > 0$, the average tax rate $T(y)/y$ is nondecreasing in y.

Given a price vector $\boldsymbol{p}$, a commodity tax profile $\boldsymbol{\tau} = (\tau_1, \dots, \tau_K)$ is *average-rate progressive* if, for $y > 0$, the average tax rate

$$\frac{1}{y}\left(\sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}+\boldsymbol{\tau}, y)\right)$$

is nondecreasing in y.

A mixed tax system $(T, \boldsymbol{\tau})$ is *separately average-rate progressive* if T (respectively, $\boldsymbol{\tau}$) is average-rate progressive; and *jointly average-rate progressive* if, for $y > 0$, the average tax rate,

$$\frac{1}{y}\left(T(y) + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))\right),$$

is nondecreasing in y.

Joint and separate average-rate progressivity are logically nested in the following sense. Separate average-rate progressivity implies joint average-rate progressivity, but the converse is not true.[3]

Our first main result is a characterization of inequality-reducing mixed tax systems in terms of jointly progressive mixed taxation.

Theorem 1. *Given a price vector* $\boldsymbol{p}$*, a mixed tax system* $(T, \boldsymbol{\tau})$ *is inequality-reducing if and only if the net income function* $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ *is nondecreasing in the pre-tax income level* y *and* $(T, \boldsymbol{\tau})$ *is jointly average-rate progressive.*

The proof of Theorem 1 is relegated to Appendix A.

In general, $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ need not be nondecreasing in the pre-tax income level y. Indeed, it is easy to see, using the identity in (1), that if good k is inferior, and if p_k is large enough relative to the (gross) prices of the other goods ($p_{k'}$, $k' \neq k$), then the map $y \mapsto z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ may well be decreasing in y. Partial differentiation of the identity in (1) with respect to y gives

$$\begin{aligned}\frac{\partial z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)}{\partial y} &= (1 - T'(y))\left(1 - \sum_{k=1}^{K} \tau_k \cdot \partial_2 x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))\right) \\ &= (1 - T'(y)) \sum_{k=1}^{K} p_k \cdot \partial_2 x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y)),\end{aligned}$$

where, for every good k, $\partial_2 x_k(\boldsymbol{p}', y')$ denotes the partial derivative of the Marshallian demand function with respect to its second variable, income, evaluated at the price vector $\boldsymbol{p}'$ and income level y'.[4] If good k is inferior, so that $\partial_2 x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y)) < 0$ for each $\boldsymbol{p}+\boldsymbol{\tau}$ and $y - T(y)$, and if p_k is large enough, relative to the prices of the other goods, then $\frac{\partial z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)}{\partial y} < 0$.[5]

The following are immediate corollaries of Theorem 1.

- An income tax schedule is inequality-reducing if and only if it is average-rate progressive. This is the classic Jakobsson–Fellman–Kakwani result (Jakobsson, 1976; Fellman, 1976; Kakwani, 1977).
- A commodity tax profile is inequality-reducing if and only if it is average-rate progressive and net income is nondecreasing with pre-tax income.

Inequality-reducing (hence average-rate progressive) commodity tax profiles have been characterized in terms of the tax treatment of luxury (respectively, necessary) commodities (Carbonell-Nicolau, 2019).

A *luxury* (respectively, *necessary*) *commodity* is a commodity for which the proportion of total income spent on it rises (respectively, declines) with income.

Assuming differentiable demand functions, a luxury commodity k can be formally defined in terms of the following condition:[6]

$$\frac{\partial(p_k x_k(\boldsymbol{p}, y)/y)}{\partial y} > 0, \quad \text{for every } (\boldsymbol{p}, y). \tag{2}$$

(Recall that $x_k(\boldsymbol{p}, y)$ denotes the standard Marshallian demand function for good k.)

A commodity k is a necessity if

$$\frac{\partial(p_k x_k(\boldsymbol{p}, y)/y)}{\partial y} < 0, \quad \text{for every } (\boldsymbol{p}, y). \tag{3}$$

Conditions (2) and (3) can be expressed as follows:

$$\frac{\partial x_k(\boldsymbol{p}, y)}{\partial y} > \frac{x_k(\boldsymbol{p}, y)}{y}, \quad \text{for every } (\boldsymbol{p}, y), \tag{4}$$

and

$$\frac{\partial x_k(\boldsymbol{p}, y)}{\partial y} < \frac{x_k(\boldsymbol{p}, y)}{y}, \quad \text{for every } (\boldsymbol{p}, y). \tag{5}$$

These conditions lend themselves to interpretation. For example, multiplying both sides of (4) by p_k, we see that good k is a luxury if the marginal propensity to spend on good k (i.e., the fraction of an extra dollar spent on good k) exceeds the average propensity to spend on good k (i.e., the current fraction of total income spent on good k). Condition (5) can be understood in a similar way.

It is easy to see that luxury goods are necessarily normal goods (in the sense that their demand increases with income), but the converse does not generally hold. Similarly, inferior goods are necessities, but necessities need not be inferior.

A reformulation of average-rate progressivity illuminates the link between Theorem 1 and the tax treatment of luxuries and necessities.

To begin, consider commodity tax profiles separately. Assuming differentiability of the demand functions, a commodity tax profile is average-rate progressive if, for $y > 0$,

$$\frac{\partial}{\partial y}\left(\frac{1}{y}\left(\sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}+\boldsymbol{\tau}, y)\right)\right) \geq 0,$$

which is expressible as

$$\sum_{k=1}^{K}\left(\partial_2 x_k(\boldsymbol{p}+\boldsymbol{\tau}, y) - \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau}, y)}{y}\right)\tau_k \geq 0, \tag{6}$$

where, for every good k, $\partial_2 x_k(\boldsymbol{p}', y')$ denotes the partial derivative of the Marshallian demand function with respect to its second variable, income, evaluated at the price vector $\boldsymbol{p}'$ and income level y'.

Recall that a good is a luxury if the bracketed term is positive on its domain and a necessity if it is negative on its domain. Thus, a commodity tax profile is average-rate progressive if it taxes luxuries and/or subsidizes necessities.

Similarly, joint average-rate progressivity can be expressed as

$$\frac{\partial}{\partial y}\left(\frac{1}{y}\left(T(y) + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))\right)\right) \geq 0, \quad y > 0.$$

Under differentiability of T and the demand functions $x_k(\boldsymbol{p}', y')$ ($k \in \{1, \dots, K\}$), this inequality can be written as

$$\begin{aligned}&T'(y) + (1 - T'(y))\left(\sum_{k=1}^{K} \tau_k \partial_2 x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))\right) \\ &\qquad \geq \frac{T(y)}{y} + \sum_{k=1}^{K} \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))}{y}, \quad y > 0.\end{aligned} \tag{7}$$

[3] See Appendix F for a proof of this assertion.

[4] The use of this notation is intended to resolve ambiguities between the perturbed variable (income) and its current level, y'.

[5] Note that, in addition, if k is an inferior good, then τ_k needs to be small enough relative to the sizes of those commodity tax rates τ_l for which l is a normal good.

[6] Differentiability is obviously not necessary for the definition of a luxury commodity.

The left-hand side represents the total fraction of an extra dollar (at the income level y) paid as tax—the sum of the income tax fraction on that extra dollar, $T'(y)$, plus the fraction of the extra dollar, net of income taxes, levied as consumption tax,

$$(1 - T'(y))\left(\sum_{k=1}^{K} \tau_k \partial_2 x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y))\right).$$

The right-hand side of (7) represents the total fraction of every dollar (at the current income level, y) paid as tax—the sum of the share of income tax per dollar, $T(y)/y$, plus the consumption tax share per dollar of income,

$$\sum_{k=1}^{K} \tau_k \cdot \frac{x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y))}{y}.$$

Based on this interpretation of joint average-rate progressivity and Theorem 1, the inequality-reducing effects of an average-rate progressive income tax schedule, along with commodity taxes on luxuries and subsidies on necessities, should be intuitively appealing.

A fundamentally different metric commonly used to evaluate income distributions is the Foster–Wolfson bipolarization order (Foster and Wolfson, 2010; Chakravarty, 2009, 2015), a measure of the degree of polarization between two income groups, taking median income as the demarcation point.

For two income distributions $\boldsymbol{y} = (y_1, \dots, y_n)$ and $\boldsymbol{y}' = (y'_1, \dots, y'_n)$ with the same median income, m, we write $\boldsymbol{y}' \succcurlyeq_{FW} \boldsymbol{y}$ to indicate that $\boldsymbol{y}'$ is *at least as bipolarized as* $\boldsymbol{y}$ if

$$\sum_{k \le i < \frac{n+1}{2}} (m - y_{[i]}) \le \sum_{k \le i < \frac{n+1}{2}} (m - y'_{[i]}), \quad \forall k : 1 \le k < \frac{n+1}{2},$$

$$\sum_{\frac{n+1}{2} < i \le k} (y_{[i]} - m) \le \sum_{\frac{n+1}{2} < i \le k} (y'_{[i]} - m), \quad \forall k : \frac{n+1}{2} < k \le n.$$

This order evaluates pairs of income distributions on the basis of an "average deviation" between individual income levels and median income, with lower average deviations corresponding to less bipolarized income distributions.

Assuming that proportional changes in income do not alter the "degree" of bipolarization, $\succcurlyeq_{FW}$ can be extended to pairs of income distributions with different median incomes.

Let $m(\boldsymbol{y})$ (respectively, $m(\boldsymbol{y}')$) denote the median income of $\boldsymbol{y}$ (respectively, $\boldsymbol{y}'$), and suppose that $m(\boldsymbol{y}) > 0$ and $m(\boldsymbol{y}') > 0$. Then the transformation

$$\boldsymbol{y}'' = \frac{m(\boldsymbol{y})}{m(\boldsymbol{y}')}(y'_1, \dots, y'_n)$$

of $\boldsymbol{y}'$ has the same median as $\boldsymbol{y}$.

The binary relation $\succcurlyeq_{FW}$ can now be extended over pairs $(\boldsymbol{y}, \boldsymbol{y}')$ as follows:

$$\boldsymbol{y}' \succcurlyeq_{FW} \boldsymbol{y} \Leftrightarrow \boldsymbol{y}'' \succcurlyeq_{FW} \boldsymbol{y}.$$

Given a price vector $\boldsymbol{p}$, a mixed tax system $(T, \boldsymbol{\tau})$ is *bipolarization-reducing* if

$$(y_1, \dots, y_n) \succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n))$$

for every pre-tax income distribution $(y_1, \dots, y_n)$ and every population size n.

Carbonell-Nicolau and Llavador (2021b) established the equivalence between inequality-reducing and bipolarization-reducing income tax schedules. We now establish the equivalence between inequality-reducing and bipolarization-reducing mixed tax systems.

Theorem 2. *Given a price vector $\boldsymbol{p}$, a mixed tax system $(T, \boldsymbol{\tau})$ is bipolarization-reducing if and only if the net income function $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in the pre-tax income level y and $(T, \boldsymbol{\tau})$ is jointly average-rate progressive.*

The proof of Theorem 2 is given in Appendix B.

Combining Theorem 1 and Theorem 2 immediately gives the following result.

Theorem 3. *Given a price vector $\boldsymbol{p}$, a mixed tax system is inequality-reducing if and only if it is bipolarization-reducing.*

As a special case of Theorem 3, we obtain the following result: given a price vector $\boldsymbol{p}$, a commodity tax profile is inequality-reducing if and only if it is bipolarization-reducing.

We now investigate whether mixed taxation is "redundant" for reducing inequality and bipolarization.

First, we illustrate that income taxation is sometimes necessary because there are certain preferences under which commodity taxation has no equalizing potential.

Consider a two-good example with $u(x_1, x_2) = x_1 x_2$. The demand functions are given by

$$x_1(\boldsymbol{p}, y) = \frac{y}{2p_1} \quad \text{and} \quad x_2(\boldsymbol{p}, y) = \frac{y}{2p_2},$$

and so

$$\frac{\partial x_k(\boldsymbol{p}, y)}{\partial y} = \frac{x_k(\boldsymbol{p}, y)}{y}, \quad k \in \{1, 2\}.$$

Thus, both goods are neither luxuries nor necessities. Consequently, commodity taxation is neutral — it neither increases nor decreases inequality — implying that any strictly equalizing tax system requires income taxation.

Next, we establish conditions under which commodity taxation becomes "superfluous:" any post-tax income distribution achievable through a mixed tax system can be replicated through income taxation alone. This finding parallels the Atkinson–Stiglitz theorem in optimal taxation, which demonstrates that income taxation suffices for implementing optimal tax policies when leisure and consumption are separable (Atkinson and Stiglitz, 1976).[7]

Suppose that $(T, \boldsymbol{\tau})$ is a mixed tax system such that $\boldsymbol{\tau}$ does not tax any inferior good. Take a price vector $\boldsymbol{p}$ and suppose that the net income function $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in the pre-tax income level y. Then there exists an income tax schedule T^* satisfying the following: given an income distribution $(y_1, \dots, y_n)$, both $(T, \boldsymbol{\tau})$ and T^* give rise to the same post-tax income distribution:

$$(z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)) = (z(\boldsymbol{p}, T^*, 0, y_1), \dots, z(\boldsymbol{p}, T^*, 0, y_n)).$$

To see this, note that setting

$$T^*(y) = T(y) + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y))$$

gives

$$z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) = y - T(y) - \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)) = z(\boldsymbol{p}, T^*, 0, y), \quad \text{for all } y.$$

To see that T^* is a proper income tax schedule, note first that, because $\boldsymbol{\tau}$ does not tax inferior goods, T^* is nondecreasing. Moreover, $T^*(y) \le y$ for all y (since $T(y) \le y$ and

$$y - T(y) - \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)) \ge \sum_{k=1}^{K} p_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)) \ge 0$$

for all y) and, because $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in y, the map $y \mapsto y - T^*(y)$ is nondecreasing.

Theorem 2 and Theorem 3, combined with the above observations, yield a stronger result: when a mixed tax system $(T, \boldsymbol{\tau})$ reduces inequality (or bipolarization), there exists an equivalent income tax schedule T^* that generates identical post-tax income distributions. This

[7] While Atkinson and Stiglitz's analysis assumes endogenous income, we defer our treatment of income responses to Section 2.2.

equivalence holds for any initial income distribution and any price vector $\boldsymbol{p}$, provided that: (i) no inferior goods are taxed under $\boldsymbol{\tau}$, and (ii) the net income function $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in the pre-tax income level y.

2.2. Endogenous income

In the case of endogenous income, preferences are described by means of a utility function u defined on consumption bundles and labor hours, $(\boldsymbol{x}, l) \in \mathbb{R}^K_+ \times [0, L]$, where $\boldsymbol{x} = (x_1, \dots, x_K)$ represents a bundle of K commodities, l is a measure of working hours, and $L > 0$.[8,9]

In this section, we restrict attention to piecewise linear tax schedules (see the definition of a tax schedule at the beginning of Section 2.1).

A tax schedule T is *piecewise linear* if $\mathbb{R}_+$ can be partitioned into finitely many intervals $I_1, \dots, I_M$ satisfying the following: for each m, there exist $\beta_m \in \mathbb{R}$ and $t_m \in [0, 1)$ such that $T(y) = \beta_m + t_m y$ for all $y \in I_m$.

The set of all piecewise linear tax schedules is denoted by $\mathcal{T}$.

Individuals differ in their hourly wage $a > 0$. An individual who supplies $l \in [0, L]$ units of labor and faces an income tax schedule T earns a net income of $al - T(al)$. Given a price vector $\boldsymbol{p} = (p_1, \dots, p_K)$ and a mixed tax system $(T, \boldsymbol{\tau}) = (T, \tau_1, \dots, \tau_K)$ where $T \in \mathcal{T}$ and $p_k + \tau_k > 0$ for each k, the individual's optimization problem is:

$$\begin{aligned} &\max_{(x_1,\dots,x_K,l)\in\mathbb{R}^K_+\times[0,L]} u(x_1, \dots, x_K, l) \\ &\text{s.t. } (p_1 + \tau_1)x_1 + \dots + (p_K + \tau_K)x_K \le al - T(al). \end{aligned} \tag{8}$$

Throughout the sequel, u is assumed to satisfy the following conditions:

(i) u is continuous.

(ii) $u(\cdot, l)$ is nondecreasing and strictly increasing on $\mathbb{R}^K_{++}$ for each $l \in [0, L]$ and $u(\boldsymbol{x}, \cdot)$ is strictly decreasing for each $\boldsymbol{x} \in \mathbb{R}^K_{++}$.

(iii) Given $\boldsymbol{p}$, $T \in \mathcal{T}$, $a > 0$, and $aL \ge y > 0$, let $\boldsymbol{x}(\boldsymbol{p}, T, a, y)$ denote a solution to

$$\begin{aligned} &\max_{(x_1,\dots,x_K)\in\mathbb{R}^K_+} u(x_1, \dots, x_K, y/a) \\ &\text{s.t. } p_1x_1 + \dots + p_Kx_K \le y - T(y). \end{aligned} \tag{9}$$

Because u is continuous and the feasible set is compact, a solution exists. Under the condition (iv) below, the solution is unique (see Footnote 10).

Suppose that $\boldsymbol{x}(\boldsymbol{p}, T, a, y)$ is continuous in (a, y) for each $\boldsymbol{p}$ and T. Choose a commodity k and a gross labor income level $y > 0$. The marginal rate of substitution of x_k for y for an "a-individual" is given by

$$MRS^a_k(\boldsymbol{x}, y) = -\frac{(1/a)(\partial u(\boldsymbol{x}, y/a)/\partial l)}{\partial u(\boldsymbol{x}, y/a)/\partial x_k}.$$

It represents the amount of extra good k an individual would require as compensation for an extra marginal unit of gross labor income. We assume that (a) $MRS^a_k(\boldsymbol{x}(\boldsymbol{p}, T, a, y), y)$ is well defined for each k, $\boldsymbol{p}$, $T \in \mathcal{T}$, $a > 0$, and $y > 0$ and continuous in (a, y) for each $\boldsymbol{p}$ and T; and (b) for each k, $\boldsymbol{p}$, $T \in \mathcal{T}$, and $y > 0$,

$$\lim_{a \searrow y/L} MRS^a_k(\boldsymbol{x}(\boldsymbol{p}, T, a, y), y) = \infty \text{ and } \lim_{a\to\infty} MRS^a_k(\boldsymbol{x}(\boldsymbol{p}, T, a, y), y) = 0.$$

(iv) u is quasiconcave and exhibits the following form of "strict quasiconcavity:" given $\boldsymbol{p}$, $T \in \mathcal{T}$, $a > 0$, $aL \ge y > 0$, $aL \ge y' > 0$, and solutions $\boldsymbol{x}$ and $\boldsymbol{x}'$ to the problems (9) and

$$\begin{aligned} &\max_{(x_1,\dots,x_K)\in\mathbb{R}^K_+} u(x_1, \dots, x_K, y'/a) \\ &\text{s.t. } p_1x_1 + \dots + p_Kx_K \le y' - T(y'), \end{aligned}$$

respectively, the following condition is satisfied:

$$u(\alpha(\boldsymbol{x}, y/a) + (1-\alpha)(\boldsymbol{x}', y'/a)) > \min\{u(\boldsymbol{x}, y/a), u(\boldsymbol{x}', y'/a)\}$$

for all $\alpha \in (0, 1)$ whenever $y \ne y'$ or $\boldsymbol{x} \ne \boldsymbol{x}'$.[10]

(v) For each $\boldsymbol{p}$, $T \in \mathcal{T}$, and $a > 0$, there exist $\boldsymbol{x} \in \mathbb{R}^K_{++}$ and $l > 0$ such that

$$p_1x_1 + \dots + p_Kx_K \le al - T(al)$$

and $u(\boldsymbol{x}, l) > u(\boldsymbol{0}, 0)$.

The condition (iii) states that the compensation (in terms of good k) required by an individual for an extra marginal unit of labor income at labor income level y and at a utility maximizing bundle $\boldsymbol{x}(\boldsymbol{p}, T, a, y)$ (a) tends to infinity as aL approaches y from above;[11] and (b) tends to zero as a diverges to ∞. The last condition, (v), implies that, under a mixed tax system $(T, \boldsymbol{\tau})$, and given a price vector $\boldsymbol{p}$, an individual whose wage rate is $a > 0$ always consumes a positive amount of at least one good, i.e., $x^u_k(\boldsymbol{p}, T, \boldsymbol{\tau}, a) > 0$ for at least one k, implying that $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) > 0$.

A solution to (8) is denoted by

$$(x^u_1(\boldsymbol{p}, T, \boldsymbol{\tau}, a), \dots, x^u_K(\boldsymbol{p}, T, \boldsymbol{\tau}, a), l^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)). \tag{11}$$

[12]The notation used in this section makes the dependence of a solution to (8) on the utility function, u, explicit. Keeping this dependence in mind will be convenient when characterizing the inequality-reducing properties of tax schedules in terms of conditions on u.

A corresponding *net income* function is denoted by

$$\begin{aligned} z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) &= al^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) - T(al^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)) - \sum_{k=1}^K \tau_k x^u_k(\boldsymbol{p}, T, \boldsymbol{\tau}, a) \\ &= \sum_{k=1}^K p_k x^u_k(\boldsymbol{p}, T, \boldsymbol{\tau}, a). \end{aligned} \tag{13}$$

In the absence of taxes, the solution in (11) is denoted by

$$(x^u_1(\boldsymbol{p}, 0, 0, a), \dots, x^u_K(\boldsymbol{p}, 0, 0, a), l^u(\boldsymbol{p}, 0, 0, a)).$$

We now formulate the last two conditions on the utility function u:

[8] Here, we allow u to take the value $-\infty$. The example presented in Appendix G features a utility function $u(x_1, x_2, l)$ such that $u(0, x_2, l) = -\infty$ for all x_2 and l.

[9] In this paper, preferences are assumed to be homogeneous across individuals. This assumption is standard in the literature on optimal income taxation, which builds on the seminal paper of Mirrlees (1971). Our baseline model is the Mirrlees model, augmented to allow for more than one commodity. Carbonell-Nicolau (2024) allows for preference heterogeneity but confines attention to income taxation.

[10] This condition implies that the problem (9) has a unique solution. To see this, suppose that there are two distinct solutions, $\boldsymbol{x}$ and $\boldsymbol{x}'$, to (9). Then (iv) implies that

$$u(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{x}', y/a) > u(\boldsymbol{x}, y/a) = u(\boldsymbol{x}', y/a) \tag{10}$$

for $\alpha \in (0, 1)$. But since

$$p_1x_1 + \dots + p_Kx_K \le y - T(y) \quad \text{and} \quad p_1x'_1 + \dots + p_Kx'_K \le y - T(y),$$

it follows that

$$p_1(\alpha x_1 + (1-\alpha)x'_1) + \dots + p_K(\alpha x_K + (1-\alpha)x'_K) \le y - T(y).$$

This, together with (10), contradicts that $\boldsymbol{x}$ and $\boldsymbol{x}'$ solve (9).

[11] Note that $a \searrow y/L$ implies that $l \nearrow L$.

[12] The problem (8) has at least one solution. To see this, consider first the following problem:

$$\max_{y\in[0,aL]} u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y), y/a). \tag{12}$$

Note that if this problem has a solution, y^*, then

$$(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y^*), y^*/a)$$

solves (8). Thus, it suffices to show that (12) has a solution. But (12) has a solution because the objective function is continuous in y and the feasible set is a closed interval.

(vi) Given $\boldsymbol{p}$ and $(T, \boldsymbol{\tau})$ with $T \in \mathcal{T}$, if $l^u(\boldsymbol{p}, T, \boldsymbol{\tau}, \cdot)$ has a discontinuity point at some $a > 0$, so does $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, \cdot)$.

(vii) Given $b \geq 0$ and $\boldsymbol{p}$, the map $a \mapsto al^u(\boldsymbol{p}, T, 0, a)+b$ is nondecreasing, where T is defined by $T(y) = -b$ for all $y \geq 0$.

Condition (vi) says that discontinuities in gross income with respect to a translate into similar discontinuities in net income.[13]

Condition (vii) states that, for any fixed subsidy $b \geq 0$ and in the absence of commodity taxation, net income is nondecreasing with a.[14]

The set of all utility functions satisfying the conditions (i)–(vii) is denoted by $\mathcal{U}$.

A *wage distribution* is a vector $(a_1, \dots, a_n) \in \mathbb{R}^n_{++}$, where n is the population size and a_i represents individual i's wage rate.

Given a price vector $\boldsymbol{p}$ and a wage distribution $(a_1, \dots, a_n)$, a mixed tax system $(T, \boldsymbol{\tau})$ generates an income distribution

$$(z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \dots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n)).$$

In the absence of taxation, i.e., when both T and $\boldsymbol{\tau}$ are identically zero, the resulting income distribution is

$$(z^u(\boldsymbol{p}, 0, 0, a_1), \dots, z^u(\boldsymbol{p}, 0, 0, a_n)).$$

A mixed tax system $(T, \boldsymbol{\tau})$ is *inequality-reducing with respect to* $\boldsymbol{p}$ *and* u, or $(\boldsymbol{p}, u)$*-ir*, if

$$(z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \dots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n)) \succcurlyeq_L (z^u(\boldsymbol{p}, 0, 0, a_1), \dots, z^u(\boldsymbol{p}, 0, 0, a_n))$$

for each wage distribution $(a_1, \dots, a_n)$, every post-tax income function z^u, and every population size n.

When the underlying $\boldsymbol{p}$ and u are clear from the context, we sometimes refer to $(\boldsymbol{p}, u)$-ir mixed tax systems simply as inequality-reducing mixed tax systems.

An income tax schedule is *marginal-rate progressive* if it is convex.

If $T \in \mathcal{T}$ is marginal-rate progressive, then the net income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is uniquely defined.[15]

Theorem 4. *For $T \in \mathcal{T}$, a mixed tax system $(T, \boldsymbol{\tau})$ is inequality-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$ only if the net income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a and T is marginal-rate progressive.*

The proof of Theorem 4 is relegated to Appendix C.

For any income tax schedule $T \in \mathcal{T}$, we have $T(0) = -b$ for some $b \geq 0$. Here, b can be viewed as a subsidy deducted from any tax liability. If b exceeds an individual's tax obligations, the individual receives the excess subsidy as a direct payment.

Let $\mathcal{T}_{m\text{-}prog}$ be the set of all marginal-rate progressive income tax schedules in $\mathcal{T}$. Every income tax schedule T in $\mathcal{T}_{m\text{-}prog}$ is piecewise linear, and so $\mathbb{R}_+$ can be partitioned into finitely many intervals $I_1, \dots, I_M$ satisfying the following: for each m, there exist $b \in \mathbb{R}$ and $t \in [0, 1)$ such that $T(y) = -b + ty$ for all $y \in I_m$. Note that, because T is convex, $b \geq 0$. Note also that the extension of $-b + ty$ to the entire domain $\mathbb{R}_+$ is itself an income tax schedule in $\mathcal{T}_{m\text{-}prog}$. We call each such linear income tax schedules a *linear extension* of T. Thus, there are M many linear extensions of T in $\mathcal{T}_{m\text{-}prog}$. More generally, the number of linear extensions of $T \in \mathcal{T}_{m\text{-}prog}$ is equal to the number of tax brackets in T, and the set of all linear extensions of T is contained in $\mathcal{T}_{m\text{-}prog}$.

A subset of tax schedules $\mathcal{S} \subseteq \mathcal{T}_{m\text{-}prog}$ is *closed under linear extensions* if $\mathcal{S}$ contains the linear extensions of its members.

We now characterize the set of inequality-reducing mixed tax systems $(T, \boldsymbol{\tau})$, where $T \in \mathcal{S} \subseteq \mathcal{T}_{m\text{-}prog}$ and $\mathcal{S}$ is closed under linear extensions, in terms of conditions on the utility function u.

Consider the tax schedule $T \in \mathcal{T}$ defined by $T(y) = -b$ for all $y \geq 0$ and some $b \geq 0$. For this particular tax schedule, we write

$$\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y) = \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, -b, a, y).$$

Given a price vector $\boldsymbol{p} = (p_1, \dots, p_K)$, $b \geq 0$, a commodity tax profile $\boldsymbol{\tau} = (\tau_1, \dots, \tau_K)$ with $p_k + \tau_k > 0$ for each k, and $a > 0$, the problem

$$\max_{y \in [0, aL]} u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, -b, a, y), y/a) \tag{16}$$

has a unique solution, $y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)$, which represents the gross income for an individual whose wage rate is a and who receives a subsidy of size b and faces a commodity tax profile $\boldsymbol{\tau}$.[16] When consumption is not taxed, i.e., when $\boldsymbol{\tau} = 0$, we write $y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a) = y^u(\boldsymbol{p}, -b, 0, a)$.

The corresponding after-tax income is given by

$$z^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a) = y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a) + b - \sum_{k=1}^{K} \tau_k x_k^u(\boldsymbol{p} + \boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a) + b),$$

where $x_k^u(\boldsymbol{p}', y')$ denotes the standard (Marshallian) demand function at price vector $\boldsymbol{p}'$ and income level y'. When $\boldsymbol{\tau} = 0$ and $b = 0$,

$$z^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a) = z^u(\boldsymbol{p}, 0, 0, a) = y^u(\boldsymbol{p}, 0, 0, a)$$

represents the solution to

$$\max_{y \in [0, aL]} u(\boldsymbol{x}(\boldsymbol{p}, 0, a, y), y/a).$$

Let $\mathcal{S}$ be a subset of income tax schedules in $\mathcal{T}_{m\text{-}prog}$, and let $\mathcal{S}'$ be a subset of commodity tax profiles. The set of all linear extensions of the elements of $\mathcal{S}$ is denoted by $\mathcal{L}_{\mathcal{S}}$.

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[13] By Eq. (13), condition (vi) holds if there are no inferior goods. More generally, if $l^u(\boldsymbol{p}, T, \boldsymbol{\tau}, \cdot)$ has a discontinuity at a, gross (hence net) labor income is also discontinuous at a, and so the demands for the K commodities will generally exhibit a discontinuity at a, which will generally translate into a discontinuity in $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, \cdot)$.

[14] An increase in a represents an increase in the "price" of leisure, which triggers a substitution effect in demand toward more labor income (and more goods and services) and away from leisure, and an income effect, which increases the demand for leisure (if leisure is a normal good). If the income effect does not outweigh the substitution effect, condition (vii) holds. More generally, even when the income effect dominates, condition (vii) holds if the reduction in the labor supply is "small" relative to the increase in the wage rate.

[15] To see this, note first that if the problem

$$\max_{y \in [0, aL]} u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y), y/a) \tag{14}$$

has a unique solution, then so does the problem (8), implying that the net income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is uniquely defined. Indeed, because the function $\boldsymbol{x}(\boldsymbol{p}, T, a, y)$ is uniquely defined for each $(\boldsymbol{p}, T, a, y)$ (Footnote 10), if there exist two distinct solutions, $(\boldsymbol{x}', l')$ and $(\boldsymbol{x}'', l'')$, to the problem (8), then $l' \neq l''$ (otherwise, i.e., if $l' = l''$, then $\boldsymbol{x}' = \boldsymbol{x}'' = \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, al')$) and there exist two distinct solutions, al' and al'', to (14). Thus, it suffices to show that (14) has a unique solution whenever T is convex. (The problem (14) was shown to have a solution in Footnote 12.) To see that (14) has a unique solution, suppose that there are two distinct solutions, y and y'. Then (iv) implies that

$$\begin{aligned} &u(\alpha(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y), y/a) + (1-\alpha)(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y'), y'/a)) \\ &= u(\alpha \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y) + (1-\alpha)\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y'), (\alpha y + (1-\alpha)y')/a) \\ &> u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y), y/a) = u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y'), y'/a) \end{aligned} \tag{15}$$

for $\alpha \in (0, 1)$. Because $y, y' \in [0, aL]$, we have

$$\alpha y + (1-\alpha)y' \in [0, aL].$$

In addition, because T is a convex function,

$$\begin{aligned} &\boldsymbol{p} \cdot (\alpha \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y) + (1-\alpha)\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y')) \\ &= \alpha(\boldsymbol{p} \cdot \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y)) + (1-\alpha)(\boldsymbol{p} \cdot \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y')) \\ &\leq \alpha(y - T(y)) + (1-\alpha)(y' - T(y')) \\ &= \alpha y + (1-\alpha)y' - (\alpha T(y) + (1-\alpha)T(y')) \\ &\leq \alpha y + (1-\alpha)y' - T(\alpha y + (1-\alpha)y'); \end{aligned}$$

here, for $\boldsymbol{x} = (x_1, \dots, x_K) \in \mathbb{R}^K_+$,

$$\boldsymbol{p} \cdot \boldsymbol{x} = p_1 x_1 + \cdots + p_K x_K.$$

Consequently, (15) contradicts that y and y' solve (14).

[16] For the proof of uniqueness of the solution, refer to Footnote 15.

The linear extensions in $\mathcal{L}_{\mathcal{S}}$ take the form $-b+ty$, where $b \geq 0$ represents the intercept and $t \in [0,1)$ the slope. Let

$$B(\mathcal{L}_{\mathcal{S}}) = \{b \geq 0 : -b+ty \in \mathcal{L}_{\mathcal{S}},\ \text{some } t\}$$

denote the set of all vertical-axis intercepts, and let

$$R(\mathcal{L}_{\mathcal{S}}) = \{t \in [0,1) : -b+ty \in \mathcal{L}_{\mathcal{S}},\ \text{some } b\}.$$

denote the set of all slopes (marginal tax rates) of these linear extensions.

The next result offers a characterization of subsets of inequality-reducing mixed tax systems.

Theorem 5. *Suppose that $\mathcal{S} \subseteq \mathcal{T}_{m\text{-}prog}$ is closed under linear extensions. Suppose that $\mathcal{S}'$ is a subset of commodity tax profiles. Then the mixed tax systems in $\mathcal{S} \times \mathcal{S}'$ are inequality-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$ if and only if the following two conditions are satisfied:*

(i) the net income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a for each $T \in \mathcal{L}_{\mathcal{S}} \cup \{0\}$ and $\boldsymbol{\tau} \in \mathcal{S}'$;[17] *and*

(ii) the quotient

$$\frac{z^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)}{z^u(\boldsymbol{p}, 0, 0, a)}$$

is nonincreasing in a for every $(b, t, \boldsymbol{\tau}) \in B(\mathcal{L}_{\mathcal{S}}) \times R(\mathcal{L}_{\mathcal{S}}) \times \mathcal{S}'$.

The proof of Theorem 5 is presented in Appendix D.

Let us now take a closer look at condition (ii) in Theorem 5. Assuming differentiability of z^u with respect to a, this condition can be expressed as follows:

$$\frac{\partial z^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)}{\partial a} \Big/ \frac{\partial z^u(\boldsymbol{p}, 0, 0, a)}{\partial a} \leq \frac{z^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)}{z^u(\boldsymbol{p}, 0, 0, a)},$$
$$\text{for each } a > 0 \text{ and } (b, t, \boldsymbol{\tau}) \in B(\mathcal{L}_{\mathcal{S}}) \times R(\mathcal{L}_{\mathcal{S}}) \times \mathcal{S}',$$

which states that the ratio of the marginal effects is less than the ratio of levels. This condition can be equivalently formulated in terms of elasticities:

$$\zeta^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a) \leq \zeta^u(\boldsymbol{p}, 0, 0, a),$$
$$\text{for each } a > 0 \text{ and } (b, t, \boldsymbol{\tau}) \in B(\mathcal{L}_{\mathcal{S}}) \times R(\mathcal{L}_{\mathcal{S}}) \times \mathcal{S}', \tag{17}$$

where

$$\zeta^u(\boldsymbol{p}', -b', \boldsymbol{\tau}', a') = \frac{\partial z^u(\boldsymbol{p}', -b', \boldsymbol{\tau}', a')}{\partial a} \cdot \frac{a'}{z^u(\boldsymbol{p}', -b', \boldsymbol{\tau}', a')}$$

represents the *wage elasticity of net income* at $(\boldsymbol{p}', -b', \boldsymbol{\tau}', a')$.

The right-hand side of the inequality in (17) is the elasticity of untaxed income at $\boldsymbol{p}$ and a:

$$\zeta^u(\boldsymbol{p}, 0, 0, a) = \frac{\partial y^u(\boldsymbol{p}, 0, 0, a)}{\partial a} \cdot \frac{a}{y^u(\boldsymbol{p}, 0, 0, a)}.$$

Since

$$\frac{\partial y^u(\boldsymbol{p}, 0, 0, a)}{\partial a} = l^u(\boldsymbol{p}, 0, 0, a) + a \cdot \frac{\partial l^u(\boldsymbol{p}, 0, 0, a)}{\partial a},$$

we have

$$\zeta^u(\boldsymbol{p}, 0, 0, a) = 1 + \epsilon^u(\boldsymbol{p}, 0, 0, a),$$

where

$$\epsilon^u(\boldsymbol{p}', -b', \boldsymbol{\tau}', a') = \frac{\partial l^u(\boldsymbol{p}', -b', \boldsymbol{\tau}', a')}{\partial a} \cdot \frac{a'}{l^u(\boldsymbol{p}', -b', \boldsymbol{\tau}', a')}$$

represents the *wage elasticity of the labor supply* at $(\boldsymbol{p}', -b', \boldsymbol{\tau}', a')$.

The left-hand side of the inequality in (17) is the elasticity of (directly and indirectly) taxed income at $(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)$, where the income tax consists of a fixed subsidy $b \geq 0$ and the commodity tax profile is given by $\boldsymbol{\tau}$. This elasticity can be expressed as follows:

$$\zeta^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a) = \zeta^u(\boldsymbol{p}+\boldsymbol{\tau}, -b, 0, (1-t)a) \cdot \frac{1 - \sum_{k=1}^K \tau_k \cdot \frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)+b)}{\partial y}}{1 - \sum_{k=1}^K \tau_k \cdot \frac{x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)+b)}{y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)+b}}. \tag{18}$$

[18]

The equality in (18) relates two distinct elasticities: the wage elasticity of net income versus the wage elasticity of income gross of consumption taxes but net of income taxes and subsidies.

Consumption taxes affect net income through two channels:

First, by distorting relative prices, consumption taxes — in combination with the subsidy b — influence individual labor supply decisions and thus labor income. This mechanism is captured by the elasticity term on the right-hand side of (18).

Second, consumption taxes determine an individual's tax liability through their interaction with marginal and average consumption propensities. This effect is reflected in the ratio term on the right-hand side of (18).

Note that

$$\sum_{k=1}^K \tau_k \cdot \frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)+b)}{\partial y}$$

represents the marginal consumption tax rate: the fraction of an additional dollar of income paid in consumption taxes. Consequently,

$$1 - \sum_{k=1}^K \tau_k \cdot \frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)+b)}{\partial y}$$

represents the marginal disposable income rate: the fraction of an additional dollar available for goods and services after consumption taxes.

Similarly,

$$1 - \sum_{k=1}^K \tau_k \cdot \frac{x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)+b)}{y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)+b}$$

represents the average propensity to consume goods and services: the fraction of total income remaining after consumption taxes.

For a luxury good k, we have

$$\frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)+b)}{\partial y} > \frac{x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)+b)}{y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, (1-t)a)+b}.$$

[17] Here 0 denotes the linear tax schedule T defined by $T(y) = 0$ for all y.

[18] Routine calculations give

$$\zeta^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a) = \left(\frac{\partial y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)+b}{\partial a} \cdot \frac{a}{y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)+b}\right) \cdot \frac{1 - \sum_{k=1}^K \tau_k \cdot \frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)+b)}{\partial y}}{1 - \sum_{k=1}^K \tau_k \cdot \frac{x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)+b)}{y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)+b}},$$

where

$$\frac{\partial y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)+b}{\partial a} \cdot \frac{a}{y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)+b}$$

is the wage elasticity of income (net of income taxes, but gross of consumption taxes) at $(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)$. Since

$$z^u(\boldsymbol{p}+\boldsymbol{\tau}, -b, 0, a) = y^u(\boldsymbol{p}+\boldsymbol{\tau}, -b, 0, a) + b,$$

and since

$$y^u(\boldsymbol{p}+\boldsymbol{\tau}, -b, 0, a) = y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)$$

(as inspection of the problem (16) reveals), we have

$$\zeta^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a) = \zeta^u(\boldsymbol{p}+\boldsymbol{\tau}, -b, 0, a) \cdot \frac{1 - \sum_{k=1}^K \tau_k \cdot \frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)+b)}{\partial y}}{1 - \sum_{k=1}^K \tau_k \cdot \frac{x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)+b)}{y^u(\boldsymbol{p}, -b, \boldsymbol{\tau}, a)+b}}.$$

Eq. (18) follows immediately from this expression.

Thus, taxing luxury goods ensures that the ratio of the fraction of an extra dollar spent on goods and services to the average fraction of every dollar spent on goods and services,

$$\frac{1-\sum_{k=1}^{K}\tau_k\cdot\frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p},-b,\boldsymbol{\tau},(1-t)a)+b)}{\partial y}}{1-\sum_{k=1}^{K}\tau_k\cdot\frac{x_k^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p},-b,\boldsymbol{\tau},(1-t)a)+b)}{y^u(\boldsymbol{p},-b,\boldsymbol{\tau},(1-t)a)+b}},$$

is less than one, which helps reduce the wage elasticity of net income, $\zeta^u(\boldsymbol{p},-b,\boldsymbol{\tau},(1-t)a)$ (see (18)), and therefore relaxes the constraints in (17).[19]

Luxury good taxation relaxes the constraint in (17), but its ultimate impact on the wage elasticity of net income involves three interacting forces: (i) the combination of income subsidies b and proportional tax rates t; (ii) labor supply responses to consumption tax-induced price distortions; and (iii) the direct effect of consumption taxation on marginal and average propensities to consume goods and services. These forces can act in opposing directions, making the net effect theoretically ambiguous.

We now turn to the equivalence between inequality-reducing and bipolarization-reducing mixed tax systems in the case of endogenous income.

Recall that, given a price vector $\boldsymbol{p}$ and a wage distribution $(a_1,\dots,a_n)$, a mixed tax system $(T,\boldsymbol{\tau})$ gives rise to an income distribution

$$(z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_1),\dots,z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_n)).$$

In the absence of taxation, the resulting income distribution is

$$(z^u(\boldsymbol{p},0,0,a_1),\dots,z^u(\boldsymbol{p},0,0,a_n)).$$

A mixed tax system $(T,\boldsymbol{\tau})$ is *bipolarization-reducing with respect to* $\boldsymbol{p}$ *and* u, or $(\boldsymbol{p},u)$*-bpr*, if

$$(z^u(\boldsymbol{p},0,0,a_1),\dots,z^u(\boldsymbol{p},0,0,a_n))\succcurlyeq_{FW}(z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_1),\dots,z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_n))$$

for each wage distribution $(a_1,\dots,a_n)$, every post-tax income function z^u, and every population size n.

Theorem 6. *For $T\in\mathcal{T}$, a mixed tax system $(T,\boldsymbol{\tau})$ is inequality-reducing with respect to $\boldsymbol{p}$ and $u\in\mathcal{U}$ if and only if it is bipolarization-reducing with respect to $\boldsymbol{p}$ and u.*

The proof of Theorem 6 is given in Appendix E.

We conclude this section by examining the joint effect of direct and indirect taxation on inequality and bipolarization, comparing it to the impact of direct taxation alone.

Unlike the case of exogenous income, commodity taxation is not generally "superfluous" here. In fact, a mixed tax system may reduce inequality in situations where income taxation alone fails to promote economic equality. We provide a detailed, albeit technical, example illustrating this point in Appendix G, but we will briefly highlight its main features here.

As demonstrated in Appendix G, there exist quasilinear preferences for which no income tax schedule (except for a pure subsidy) reduces inequality or bipolarization. Theorem 5 outlines necessary and sufficient conditions for a mixed tax system to be inequality-reducing, which can be applied to pure direct taxation as well. Appendix G shows that condition (ii) in Theorem 5 is violated in the case of pure income taxation (other than a pure subsidy). Equivalently, condition (17) fails when $\boldsymbol{\tau}=0$. Under pure direct taxation, there is no relative price distortion between goods and services, causing the ratio on the right-hand side of (18) to vanish.

In contrast, a tax on luxury goods, combined with a pure income subsidy, relaxes condition (17) (via the ratio on the right-hand side of (18)). This combination can produce an inequality and bipolarization-reducing mixed tax system, provided the income subsidy is sufficiently large.

[19] A similar argument can be made if necessities are subsidized.

3. Concluding remarks

We have studied mixed tax systems, i.e., tax systems subjecting both income and consumption to taxation, and their ability to reduce income inequality and bipolarization. We have identified necessary and sufficient conditions on the structure of mixed tax systems ensuring a reduction in income inequality and bipolarization, in both the cases of exogenous and endogenous income.

Commodity taxation is shown to be "redundant" for exogenous income, which means that any post-tax income distribution generated by a mixed tax system is attainable by means of a pure income tax. In contrast, there are instances where relying solely on income taxation does not yield more equal or less bipolarized endogenous income distributions, while mixed tax structures possess universal equitable and depolarizing effects.

We conclude with three comments.

First, it is worth reiterating the problems associated with studying *welfare* inequality and bipolarization-reducing tax systems as opposed to taxation aimed at reducing *income* inequality and bipolarization. These problems stem from the fact that Lorenz (respectively, Foster–Wolfson) dominance is not generally invariant to order-preserving utility transformations.

Second, the monotonicity of net income in pre-tax income or in wages is essential for characterizing inequality-reducing mixed tax systems, as shown in Theorem 1 and Theorem 5. Our analysis suggests that this condition may fail for high-priced inferior goods, as discussed following Theorem 1. A systematic investigation of how pricing and tax policies for inferior goods affect this monotonicity condition is left for future research.

Finally, while the present analysis treats exogenous and endogenous income separately, a natural extension — despite the challenges it may pose — would allow for heterogeneous sources of income at the individual level (e.g., "capital" income vs. labor income).

Appendix A. Proof of Theorem 1

The proof of Theorem 1 is based on the following result, which is well-known in the literature.

Lemma 1. *Suppose that $\boldsymbol{y}'=(y_1',\dots,y_n')$ and $\boldsymbol{y}=(y_1,\dots,y_n)$ are two income distributions with*

$$y_1\le\cdots\le y_n\quad\text{and}\quad y_1'\le\cdots\le y_n'.$$

If $y_\imath$ is the first positive income level in $\boldsymbol{y}$, then

$$\frac{y_\imath'}{y_\imath}\ge\cdots\ge\frac{y_n'}{y_n}\Rightarrow\boldsymbol{y}'\succcurlyeq_L\boldsymbol{y}.$$

Proof. The case when $\imath=1$ is proven in Marshall et al. (1967, Theorem 2.4).

Suppose that $\imath>1$.[20] We must show that

$$\frac{\sum_{i=1}^{l}y_i'}{\sum_{i=1}^{n}y_i'}\ge\frac{\sum_{i=1}^{l}y_i}{\sum_{i=1}^{n}y_i},\quad\text{for all } l\in\{1,\dots,n\}.$$

For fixed l, the inequality is equivalent to

$$\left(\sum_{i=l+1}^{n}y_i\right)\left(\sum_{i=1}^{l}y_i'\right)-\left(\sum_{i=1}^{l}y_i\right)\left(\sum_{i=l+1}^{n}y_i'\right)\ge 0.\tag{19}$$

This is clearly true if $l\in\{1,\dots,\imath-1\}$, since, in this case, $\sum_{i=1}^{l}y_i=0$.

For $l\ge\imath$, (19) can be expressed as

$$\left(\sum_{i=l+1}^{n}y_i\right)\left(\sum_{i=1}^{\imath-1}y_i'+\sum_{i=\imath}^{l}y_i'\right)-\left(\sum_{i=1}^{l}y_i\right)\left(\sum_{i=l+1}^{n}y_i'\right)\ge 0.\tag{20}$$

[20] The proof of this case can be found in the proof of Proposition 3.1 in Le Breton et al. (1996).

For the sub-distributions

$$(y'_\iota,\dots,y'_n) \quad \text{and} \quad (y_\iota,\dots,y_n),$$

we know that $(y'_\iota,\dots,y'_n) \succcurlyeq_L (y_\iota,\dots,y_n)$, and so

$$\frac{\sum_{i=\iota}^{l} y'_i}{\sum_{i=\iota}^{n} y'_i} \geq \frac{\sum_{i=\iota}^{l} y_i}{\sum_{i=\iota}^{n} y_i},$$

whence

$$\left(\sum_{i=l+1}^{n} y_i\right)\left(\sum_{i=\iota}^{l} y'_i\right) - \left(\sum_{i=\iota}^{l} y_i\right)\left(\sum_{i=l+1}^{n} y'_i\right) \geq 0,$$

implying that (20) holds, as we sought. ■

Using Lemma 1, Theorem 1 can be proven as follows.

Theorem 1. *Given a price vector $\boldsymbol{p}$, a mixed tax system $(T,\boldsymbol{\tau})$ is inequality-reducing if and only if the net income function $z(\boldsymbol{p},T,\boldsymbol{\tau},y)$ is nondecreasing in the pre-tax income level y and $(T,\boldsymbol{\tau})$ is jointly average-rate progressive.*

Proof. Suppose that the mixed tax system $(T,\boldsymbol{\tau})$ is inequality-reducing. We must show that $z(\boldsymbol{p},T,\boldsymbol{\tau},y)$ is nondecreasing in y and

$$\frac{1}{y}\left(T(y)+\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y)\right) \leq \frac{1}{y'}\left(T(y')+\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y')\right), \quad \text{for } y' > y > 0. \tag{21}$$

To begin, we assume that

$$z(\boldsymbol{p},T,\boldsymbol{\tau},y') < z(\boldsymbol{p},T,\boldsymbol{\tau},y), \quad \text{for some } y' > y \geq 0,$$

and derive a contradiction. Note that $y > 0$ (otherwise $z(\boldsymbol{p},T,\boldsymbol{\tau},y)=0$) and, for large enough n,

$$\frac{y}{(n-1)y+y'} > \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{(n-1)z(\boldsymbol{p},T,\boldsymbol{\tau},y)+z(\boldsymbol{p},T,\boldsymbol{\tau},y')}.$$

Consequently, for the income distributions

$$\boldsymbol{y}^* = (y_1^*,\dots,y_n^*) = (y,\dots,y,y')$$

and

$$\boldsymbol{z}^* = (z_1^*,\dots,z_n^*) = (z(\boldsymbol{p},T,\boldsymbol{\tau},y),\dots,z(\boldsymbol{p},T,\boldsymbol{\tau},y),z(\boldsymbol{p},T,\boldsymbol{\tau},y')),$$

we have $\boldsymbol{z}^* \not\succcurlyeq_L \boldsymbol{y}^*$, contradicting that $(T,\boldsymbol{\tau})$ is inequality-reducing.

It remains to prove (21). Fix $y' > y > 0$. Define the distribution $\boldsymbol{y}'' = (y''_1,\dots,y''_n)$ by $y''_1 = y$ and $y''_i = y'$ for $i \neq 1$. Since $(T,\boldsymbol{\tau})$ is inequality-reducing, we have

$$(z(\boldsymbol{p},T,\boldsymbol{\tau},y''_1),\dots,z(\boldsymbol{p},T,\boldsymbol{\tau},y''_n)) \succcurlyeq_L (y''_1,\dots,y''_n), \tag{22}$$

implying that

$$\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y''_1)}{\sum_{i=1}^{n} z(\boldsymbol{p},T,\boldsymbol{\tau},y''_i)} \geq \frac{y''_1}{\sum_{i=1}^{n} y''_i},$$

i.e.,

$$\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y)}{z(\boldsymbol{p},T,\boldsymbol{\tau},y)+(n-1)z(\boldsymbol{p},T,\boldsymbol{\tau},y')} \geq \frac{y}{y+(n-1)y'}. \tag{23}$$

Eq. (22) also implies

$$\frac{\sum_{i=1}^{n-1} z(\boldsymbol{p},T,\boldsymbol{\tau},y''_i)}{\sum_{i=1}^{n} z(\boldsymbol{p},T,\boldsymbol{\tau},y''_i)} \geq \frac{\sum_{i=1}^{n-1} y''_i}{\sum_{i=1}^{n} y''_i},$$

i.e.,

$$\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y)+(n-2)z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{z(\boldsymbol{p},T,\boldsymbol{\tau},y)+(n-1)z(\boldsymbol{p},T,\boldsymbol{\tau},y')} \geq \frac{y+(n-2)y'}{y+(n-1)y'},$$

whence

$$1-\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y)+(n-2)z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{z(\boldsymbol{p},T,\boldsymbol{\tau},y)+(n-1)z(\boldsymbol{p},T,\boldsymbol{\tau},y')} \leq 1-\frac{y+(n-2)y'}{y+(n-1)y'},$$

or

$$\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{z(\boldsymbol{p},T,\boldsymbol{\tau},y)+(n-1)z(\boldsymbol{p},T,\boldsymbol{\tau},y')} \leq \frac{y'}{y+(n-1)y'}.$$

Combining this inequality with (23) yields

$$\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y)}{y} \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y)+(n-1)z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{y+(n-1)y'} \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{y'},$$

or

$$\frac{1}{y}\left(T(y)+\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y)\right) \leq \frac{1}{y'}\left(T(y')+\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y')\right),$$

as we sought.

Conversely, suppose that the mixed tax system $(T,\boldsymbol{\tau})$ is jointly average-rate progressive and $z(\boldsymbol{p},T,\boldsymbol{\tau},y)$ is nondecreasing in y. Fix any income distribution $\boldsymbol{y}=(y_1,\dots,y_n)$. Without loss of generality, suppose that

$$y_1 \leq \cdots \leq y_n.$$

Let y_ι be the first positive coordinate in $\boldsymbol{y}$. Then

$$\frac{1}{y_\iota}\left(T(y_\iota)+\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y_\iota)\right) \leq \cdots \leq \frac{1}{y_n}\left(T(y_n)+\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y_n)\right).$$

Equivalently,

$$\frac{1}{y_\iota}\left(y_\iota - T(y_\iota)-\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y_\iota)\right) \geq \cdots \geq \frac{1}{y_n}\left(y_n - T(y_n)-\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y_n)\right),$$

or

$$\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_\iota)}{y_\iota} \geq \cdots \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_n)}{y_n}.$$

By Lemma 1, it follows that

$$(z(\boldsymbol{p},T,\boldsymbol{\tau},y_1),\dots,z(\boldsymbol{p},T,\boldsymbol{\tau},y_n)) \succcurlyeq_L (y_1,\dots,y_n).$$

Since $\boldsymbol{y}$ was arbitrary, $(T,\boldsymbol{\tau})$ is inequality-reducing. ■

Appendix B. Proof of Theorem 2

Theorem 2. *Given a price vector $\boldsymbol{p}$, a mixed tax system $(T,\boldsymbol{\tau})$ is bipolarization-reducing if and only if the net income function $z(\boldsymbol{p},T,\boldsymbol{\tau},y)$ is nondecreasing in the pre-tax income level y and $(T,\boldsymbol{\tau})$ is jointly average-rate progressive.*

Proof. Choose a bipolarization-reducing mixed tax system $(T,\boldsymbol{\tau})$. First, we show that $z(\boldsymbol{p},T,\boldsymbol{\tau},y)$ is nondecreasing in y. Proceeding by contradiction, suppose that

$$z(\boldsymbol{p},T,\boldsymbol{\tau},y') < z(\boldsymbol{p},T,\boldsymbol{\tau},y), \quad \text{for some } y' > y \geq 0.$$

Note that $y > 0$ (otherwise $z(\boldsymbol{p},T,\boldsymbol{\tau},y)=0$). Consider the income distributions

$$\boldsymbol{y}^* = (y_1^*,\dots,y_n^*) = (y,\dots,y,y')$$

and

$$\boldsymbol{z}^* = (z_1^*,\dots,z_n^*) = (z(\boldsymbol{p},T,\boldsymbol{\tau},y),\dots,z(\boldsymbol{p},T,\boldsymbol{\tau},y),z(\boldsymbol{p},T,\boldsymbol{\tau},y')).$$

For $n > 2$, we have $m(\boldsymbol{y}^*)=y$ and $m(\boldsymbol{z}^*)=z(\boldsymbol{p},T,\boldsymbol{\tau},y)$. Therefore,

$$\frac{1}{m(\boldsymbol{y}^*)}\sum_{1\leq i<\frac{n+1}{2}}(m(\boldsymbol{y}^*)-y^*_{[i]}) = 0 < \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y)-z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{m(\boldsymbol{z}^*)} = \frac{1}{m(\boldsymbol{z}^*)}\sum_{1\leq i<\frac{n+1}{2}}(m(\boldsymbol{z}^*)-z^*_{[i]}),$$

and so $\frac{m(z^*)}{m(y^*)}y^* \not\succcurlyeq_{FW} z^*$, whence $y^* \not\succcurlyeq_{FW} z^*$, contradicting that $(T, \boldsymbol{\tau})$ is bipolarization-reducing.

It remains to show that $(T, \boldsymbol{\tau})$ is jointly average-rate progressive. Suppose that (T, τ) is not jointly average-rate progressive. Then there exist $0 < y < y'$ such that

$$\frac{1}{y}\left(T(y) + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}, T, \boldsymbol{\tau}, y)\right) > \frac{1}{y'}\left(T(y') + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}, T, \boldsymbol{\tau}, y')\right),$$

or

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)}{y} < \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')}{y'}. \tag{24}$$

Suppose first that n is odd. Take an income distribution $\boldsymbol{y} = (y_1, \dots, y_n)$ with

$$y_1 \leq \cdots \leq y_n,$$

$y_{m-1} = y$, and $y_m = y'$, where $m = (n+1)/2$, so that y_m is the median income in $\boldsymbol{y}$. Then

$$\begin{aligned}\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m) - z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{m-1})}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} &= 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{m-1})}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} \\ &= 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')} \\ &> 1 - \frac{y}{y'} \\ &= \frac{y_m - y_{m-1}}{y_m},\end{aligned}$$

where the inequality uses (24). Consequently,

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)}{y_m}(y_1, \dots, y_n) \not\succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)),$$

whence

$$(y_1, \dots, y_n) \not\succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)),$$

a contradiction.

If n is even, set $y_{n/2} = y$ and $y_{(n/2)+1} = y'$ and note that

$$m = m(z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)) = \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2}) + z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{(n/2)+1})}{2}$$

and

$$m' = m(\boldsymbol{y}) = \frac{y_{n/2} + y_{(n/2)+1}}{2}.$$

Hence,

$$\begin{aligned}\frac{m - z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2})}{m} &= 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2})}{m} \\ &= 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)}{\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y)+z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{2}} \\ &= 1 - 2\frac{1}{1 + \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{z(\boldsymbol{p},T,\boldsymbol{\tau},y)}} \\ &> 1 - 2\frac{1}{1 + \frac{y'}{y}} \\ &= \frac{m' - y}{m'},\end{aligned}$$

where the inequality uses (24). Consequently,

$$(y_1, \dots, y_n) \not\succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)),$$

a contradiction.

Conversely, suppose that the mixed tax system $(T, \boldsymbol{\tau})$ is jointly average-rate progressive and $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in y. Choose an income distribution $\boldsymbol{y} = (y_1, \dots, y_n)$ with

$$y_1 \leq \cdots \leq y_n.$$

Suppose first that n is odd. Let y_m represent the median income in $\boldsymbol{y}$. Because $(T, \boldsymbol{\tau})$ is jointly average-rate progressive, we have

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{y_i} \geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)}{y_m} \geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_j)}{y_j}, \quad \text{for } i < m \text{ and } j > m.$$

Therefore,

$$\begin{aligned}\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m) - z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} &= 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} \\ &\leq 1 - \frac{y_i}{y_m} = \frac{y_m - y_i}{y_m}, \quad \text{for } i < m, \\ \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i) - z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} &= \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} - 1 \\ &\leq \frac{y_i}{y_m} - 1 = \frac{y_i - y_m}{y_m}, \quad \text{for } i > m.\end{aligned}$$

Consequently,

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)}{y_m}(y_1, \dots, y_n) \succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)),$$

whence

$$(y_1, \dots, y_n) \succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)).$$

Since $\boldsymbol{y}$ was arbitrary, $(T, \boldsymbol{\tau})$ is bipolarization-reducing.

Now suppose that n is even. Then the median incomes for $\boldsymbol{y}$ and

$$(z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n))$$

are given by

$$m = \frac{y_{n/2} + y_{(n/2)+1}}{2} \quad \text{and} \quad m' = \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2}) + z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{(n/2)+1})}{2},$$

respectively, and so

$$\begin{aligned}\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{y_i} \geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2})}{y_{n/2}} \geq \frac{m'}{m} &\geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{(n/2)+1})}{y_{(n/2)+1}} \\ &\geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_j)}{y_j}, \quad \text{for } i \leq n/2 \text{ and } j \geq (n/2)+1,\end{aligned}$$

where the second and third inequalities hold because

$$\begin{aligned}\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2})}{y_{n/2}} \geq \frac{m'}{m} \Leftrightarrow \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2})}{y_{n/2}} &\geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2}) + z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{(n/2)+1})}{y_{n/2} + y_{(n/2)+1}} \\ \Leftrightarrow \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2})}{y_{n/2}} &\geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{(n/2)+1})}{y_{(n/2)+1}}\end{aligned}$$

and

$$\begin{aligned}\frac{m'}{m} \geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{(n/2)+1})}{y_{(n/2)+1}} \Leftrightarrow \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2}) + z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{(n/2)+1})}{y_{n/2} + y_{(n/2)+1}} &\geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{(n/2)+1})}{y_{(n/2)+1}} \\ \Leftrightarrow \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2})}{y_{n/2}} &\geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{(n/2)+1})}{y_{(n/2)+1}}.\end{aligned}$$

Therefore,

$$\frac{m' - z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{m'} = 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{m'} \leq 1 - \frac{y_i}{m} = \frac{m - y_i}{m}, \text{ for } i \leq n/2,$$

and

$$\begin{aligned}\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i) - m'}{m'} &= \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{m'} - 1 \\ &\leq \frac{y_i}{m} - 1 = \frac{y_i - m}{m}, \text{ for } i \geq (n/2)+1.\end{aligned}$$

Consequently,

$$(y_1, \dots, y_n) \succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)).$$

Since $\boldsymbol{y}$ was arbitrary, $(T, \boldsymbol{\tau})$ is bipolarization-reducing. ■

Appendix C. Proof of Theorem 4

The following lemma will be used in the proof of Theorem 4.

Lemma 2. *Given $u \in \mathcal{U}$, p, and a mixed tax system $(T, \boldsymbol{\tau})$, suppose that $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a. Then $(T, \boldsymbol{\tau})$ is inequality-reducing if and only if*

$$\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)}{z^u(\boldsymbol{p}, 0, 0, a)} \geq \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, 0, 0, a')}, \quad \text{whenever } a' > a > 0. \tag{25}$$

Proof. The sufficiency part of the statement follows from condition (vii) and Lemma 1.

To see that (25) holds whenever $(T, \boldsymbol{\tau})$ is inequality-reducing, suppose that

$$\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)}{z^u(\boldsymbol{p}, 0, 0, a)} < \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, 0, 0, a')}, \quad \text{for some } a' > a > 0.$$

[21]Then, for the wage distribution

$$(a_1^*, \dots, a_n^*) = (a, a' \dots, a'),$$

we have

$$\frac{z^u(\boldsymbol{p}, 0, 0, a_1^*)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*)} > \frac{z^u(\boldsymbol{p}, 0, 0, a_2^*)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_2^*)} = \cdots = \frac{z^u(\boldsymbol{p}, 0, 0, a_n^*)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n^*)}. \tag{26}$$

Therefore, Lemma 1 gives

$$(z^u(\boldsymbol{p}, 0, 0, a_1^*), \dots, z^u(\boldsymbol{p}, 0, 0, a_n^*)) \succcurlyeq_L (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*), \dots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n^*)). \tag{27}$$

If

$$(z^u(\boldsymbol{p}, 0, 0, a_1^*), \dots, z^u(\boldsymbol{p}, 0, 0, a_n^*)) \succ_L (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*), \dots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n^*)), \tag{28}$$

then $(T, \boldsymbol{\tau})$ is not inequality-reducing and the proof is complete.

To see that (28) holds, note that, if

$$\frac{z^u(\boldsymbol{p}, 0, 0, a_1^*)}{\sum_i z^u(\boldsymbol{p}, 0, 0, a_i^*)} > \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*)}{\sum_i z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i^*)},$$

then, by (27), we see that (28) holds. If, on the other hand,

$$\frac{z^u(\boldsymbol{p}, 0, 0, a_1^*)}{\sum_i z^u(\boldsymbol{p}, 0, 0, a_i^*)} = \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*)}{\sum_i z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i^*)},$$

then the inequality in (26) implies that

$$\frac{z^u(\boldsymbol{p}, 0, 0, a_2^*)}{\sum_i z^u(\boldsymbol{p}, 0, 0, a_i^*)} < \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_2^*)}{\sum_i z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i^*)},$$

whence

$$\frac{z^u(\boldsymbol{p}, 0, 0, a_1^*) + z^u(\boldsymbol{p}, 0, 0, a_2^*)}{\sum_i z^u(\boldsymbol{p}, 0, 0, a_i^*)} < \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*) + z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_2^*)}{\sum_i z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i^*)},$$

which contradicts (27). ■

Theorem 4. *For $T \in \mathcal{T}$, a mixed tax system $(T, \boldsymbol{\tau})$ is inequality-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$ only if the net income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a and T is marginal-rate progressive.*

Proof. Choose $\boldsymbol{p}$, a mixed tax system $(T, \boldsymbol{\tau})$ with $T \in \mathcal{T}$, and $u \in \mathcal{U}$. Suppose that

$$z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a') < z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), \qquad \text{for } a' > a > 0.$$

By condition (vii), the map $a \mapsto z^u(\boldsymbol{p}, 0, 0, a) = al^u(\boldsymbol{p}, 0, 0, a)$ is nondecreasing.

For the income distributions

$$\boldsymbol{z} = (z_1, \dots, z_n) = (z^u(\boldsymbol{p}, 0, 0, a), \dots, z^u(\boldsymbol{p}, 0, 0, a), z^u(\boldsymbol{p}, 0, 0, a'))$$

and

$$\boldsymbol{z}' = (z_1', \dots, z_n') = (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), \dots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')),$$

we have, for large enough n,

$$\frac{z^u(\boldsymbol{p}, 0, 0, a)}{(n-1)z^u(\boldsymbol{p}, 0, 0, a) + z^u(\boldsymbol{p}, 0, 0, a')} > \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a') + (n-1)z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)},$$

implying that $\boldsymbol{z}' \not\succcurlyeq_L \boldsymbol{z}$, and so $(T, \boldsymbol{\tau})$ is not inequality-reducing.

We now assume that $(T, \boldsymbol{\tau})$ is inequality-reducing and T is not marginal-rate progressive, and derive a contradiction. By the previous argument, we know that $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a.

Since $T \in \mathcal{T}$, we can partition $\mathbb{R}_+$ into finitely many intervals $I_1, \dots, I_J$ such that T is linear on I_j for each j. Because T is nonconvex, there exist two contiguous intervals,

$$I_j = [\underline{y}, y^*] \quad \text{and} \quad I_{j'} = [y^*, \overline{y}],$$

such that $\underline{y} < y^* < \overline{y}$ and T is concave on $I_j \cup I_{j'}$. Therefore, the restriction of the map

$$y \in \mathbb{R}_+ \mapsto f(y)$$

from pre-tax income y to post-tax income $f(y) = y - T(y)$ to the set $I_j \cup I_{j'}$ can be expressed as follows:

$$f(y) = \begin{cases} \alpha + \beta y & \text{if } y \in I_j, \\ \alpha' + \beta' y & \text{if } y \in I_{j'}, \end{cases}$$

where $\alpha, \alpha' \in \mathbb{R}$, $\alpha > \alpha'$, and $\beta' > \beta > 0$. Note that $f(y) > 0$ if $y > 0$ (since marginal tax rates are less than unity).

Recall that the marginal rate of substitution of x_k for y for an "a-individual" is given by

$$MRS_k^a(\boldsymbol{x}, y) = -\frac{(1/a)(\partial u(\boldsymbol{x}, y/a)/\partial l)}{\partial u(\boldsymbol{x}, y/a)/\partial x_k}.$$

It represents the amount of extra good k an individual should receive as compensation for an extra marginal unit of gross labor income. Note that $y \leq aL$.

Recall that

$$(x_1^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), \dots, x_K^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), l^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a))$$

represents a solution to the problem

$$\max_{(x_1, \dots, x_K, l) \in \mathbb{R}_+^K \times [0, L]} u(x_1, \dots, x_K, l)$$
$$\text{s.t. } (p_1 + \tau_1)x_1 + \dots + (p_K + \tau_K)x_K \leq al - T(al).$$

An individual whose wage is $a \geq \underline{y}/L$ and whose labor supply is $l = \underline{y}/a$ earns gross (respectively, net) labor income $\underline{y}$ (respectively, $f(\underline{y})$).

Let $\boldsymbol{x}(a, y)$ solve

$$\max_{(x_1, \dots, x_K) \in \mathbb{R}_+^K} u(x_1, \dots, x_K, y/a)$$
$$\text{s.t. } (p_1 + \tau_1)x_1 + \dots + (p_K + \tau_K)x_K \leq f(y).$$

By the condition (iii),

$$\lim_{a \searrow \underline{y}/L} MRS_k^a(\boldsymbol{x}(a, \underline{y}), \underline{y}) = \infty.$$

Therefore (since the indifference curves for the utility function

$$(\boldsymbol{x}, y) \in \mathbb{R}_+^K \times [0, aL] \mapsto u(\boldsymbol{x}, y/a)$$

are convex by the quasiconcavity of u), there exists $\underline{a} > 0$ such that $al^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) \leq \underline{y} < y^*$ for all $a \leq \underline{a}$.

An individual whose wage is $a \geq \overline{y}/L$ and whose labor supply is $l = \overline{y}/a$ earns gross (respectively, net) labor income $\overline{y}$ (respectively, $f(\overline{y})$).

[21] This part of the argument follows part of the proof of Proposition 3.1 in Le Breton et al. (1996).

Because

$$\lim_{a\to\infty} MRS_k^a(\boldsymbol{x}(a,\overline{y}),\overline{y}) = 0,$$

there exists $\overline{a} > 0$ such that $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \geq \overline{y} > y^*$ for all $a \geq \overline{a}$.

Let

$$a^* = \inf\{a > 0 : al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) > y^*\}.$$

Then $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) > y^*$ for all $a > a^*$ and $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \leq y^*$ for all $a < a^*$.

We claim that

$$\sup\{al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) : a < a^*\} < y^*. \tag{29}$$

To see this, note that $\sup\{al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) : a < a^*\} = y^*$ implies that there are sequences (a_n) and (y_n) such that $a_n \nearrow a^*$, each y_n is a solution to

$$\max_{y\in[0,a_nL]} u(\boldsymbol{x}(a_n,y), y/a_n), \tag{30}$$

and $y_n \nearrow y^*$. A necessary condition for y_n to solve (30) is

$$MRS_k^{a_n}(\boldsymbol{x}(a_n,y_n),y_n) = \frac{\beta}{p_k+\tau_k} > 0. \tag{31}$$

Because $a_n \to a^*$, $y_n \to y^*$, and $MRS_k^a(\boldsymbol{x}(a,y),y)$ is continuous in (a,y), we have

$$MRS_k^{a_n}(\boldsymbol{x}(a_n,y_n),y_n) \to MRS_k^{a^*}(\boldsymbol{x}(a^*,y^*),y^*).$$

Consequently, (31) gives

$$MRS_k^{a^*}(\boldsymbol{x}(a^*,y^*),y^*) = \frac{\beta}{p_k+\tau_k}.$$

Hence,

$$(p_k+\tau_k)MRS_k^{a^*}(\boldsymbol{x}(a^*,y^*),y^*) = \beta < \beta' = \frac{df(y^*)}{dy},$$

and so there exists $\hat{y} \in I_{j'}$ such that

$$u(\boldsymbol{x}(a^*,y^*),y^*/a^*) < u(\boldsymbol{x}(a^*,\hat{y}),\hat{y}/a^*).$$

Consequently, since the map

$$(a,y) \in \mathbb{R}^2_{++} \mapsto u(\boldsymbol{x}(a,y),y/a)$$

is continuous and $a_n \to a^*$ and $y_n \to y^*$,

$$u(\boldsymbol{x}(a_n,y_n),y_n/a_n) \to u(\boldsymbol{x}(a^*,y^*),y^*/a^*) < u(\boldsymbol{x}(a^*,\hat{y}),\hat{y}/a^*).$$

Since $u(\boldsymbol{x}(a^*,y^*),y^*/a^*) < u(\boldsymbol{x}(a^*,\hat{y}),\hat{y}/a^*)$, we may choose $\varepsilon > 0$ such that

$$u(\boldsymbol{x}(a^*,y^*),y^*/a^*) + 2\varepsilon < u(\boldsymbol{x}(a^*,\hat{y}),\hat{y}/a^*).$$

Since $u(\boldsymbol{x}(a_n,y_n),y_n/a_n) \to u(\boldsymbol{x}(a^*,y^*),y^*/a^*)$, there exists N such that

$$u(\boldsymbol{x}(a_n,y_n),y_n/a_n) < u(\boldsymbol{x}(a^*,y^*),y^*/a^*) + \varepsilon, \quad \text{for all } n \geq N.$$

Moreover, since $a_n \to a^*$, there exists M such that

$$u(\boldsymbol{x}(a^*,\hat{y}),\hat{y}/a^*) - \varepsilon < u(\boldsymbol{x}(a_n,\hat{y}),\hat{y}/a_n), \quad \text{for all } n \geq M.$$

Consequently, for $n \geq \max\{M,N\}$,

$$u(\boldsymbol{x}(a_n,y_n),y_n/a_n) < u(\boldsymbol{x}(a_n,\hat{y}),\hat{y}/a_n),$$

contradicting that y_n solves (30).

Hence, (29) holds. Therefore, because $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) > y^*$ for all $a > a^*$ and $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \leq y^*$ for all $a < a^*$, we see that there exist $0 < y < y'$ such that

$$al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \leq y \quad \text{for } a < a^*,$$
$$al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \geq y' \quad \text{for } a > a^*.$$

Thus, $l^u(\boldsymbol{p},T,\boldsymbol{\tau},\cdot)$ has a discontinuity at a^*, and so, by the condition (vi), $z^u(\boldsymbol{p},T,\boldsymbol{\tau},\cdot)$ has a corresponding discontinuity at a^*. Hence, because $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a, there exist $z < z'$ such that

$$z^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \leq z \quad \text{for } a < a^*, \tag{32}$$

$$z^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \geq z' \quad \text{for } a > a^*. \tag{33}$$

Next, we show that the map $z^u(\boldsymbol{p},0,0,\cdot)$ is continuous. Note that $z^u(\boldsymbol{p},0,0,a)$ solves

$$\max_{y\in[0,aL]} u(\boldsymbol{x}(\boldsymbol{p},0,a,y), y/a).$$

By the Maximum Theorem, the correspondence

$$a > 0 \rightrightarrows \operatorname*{argmax}_{y\in[0,aL]} u(\boldsymbol{x}(\boldsymbol{p},0,a,y), y/a)$$

is upper hemicontinuous. We will show that this correspondence is, in fact, single-valued, implying that it is a continuous function, i.e., the map $z^u(\boldsymbol{p},0,0,\cdot)$ is continuous.

Suppose that there exist $a > 0$ and distinct $y' > 0$ and $y'' > 0$ in

$$\operatorname*{argmax}_{y\in[0,aL]} u(\boldsymbol{x}(\boldsymbol{p},0,a,y), y/a).$$

[22]

Fix $\alpha \in (0,1)$. By the condition (iv),

$$\begin{aligned} &u(\alpha(\boldsymbol{x}(\boldsymbol{p},0,a,y'),y'/a) + (1-\alpha)(\boldsymbol{x}(\boldsymbol{p},0,a,y''),y''/a)) \\ &= u(\alpha\boldsymbol{x}(\boldsymbol{p},0,a,y') + (1-\alpha)\boldsymbol{x}(\boldsymbol{p},0,a,y''),(\alpha y' + (1-\alpha)y'')/a) \\ &\qquad > \min\{u(\boldsymbol{x}(\boldsymbol{p},0,a,y'),y'/a), u(\boldsymbol{x}(\boldsymbol{p},0,a,y''),y''/a)\}. \end{aligned} \tag{34}$$

Because $y', y'' \in [0,aL]$, we have

$$\alpha y' + (1-\alpha)y'' \in [0,aL].$$

In addition,

$$\begin{aligned} \boldsymbol{p}\cdot(\alpha\boldsymbol{x}(\boldsymbol{p},0,a,y') + (1-\alpha)\boldsymbol{x}(\boldsymbol{p},0,a,y'')) &= \alpha(\boldsymbol{p}\cdot\boldsymbol{x}(\boldsymbol{p},0,a,y')) \\ &+ (1-\alpha)(\boldsymbol{p}\cdot\boldsymbol{x}(\boldsymbol{p},0,a,y'')) \leq \alpha y' + (1-\alpha)y''. \end{aligned}$$

[23] Therefore, (34) contradicts that y' and y'' are members of $\operatorname{argmax}_{y\in[0,aL]} u(\boldsymbol{x}(\boldsymbol{p},0,a,y),y/a)$.

Given (32)–(33) and the continuity of the map $z^u(\boldsymbol{p},0,0,\cdot)$, we see that, for $0 < a < a^* < a'$ with a and a' close enough to a^*, we have

$$\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)}{z^u(\boldsymbol{p},0,0,a)} < \frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a')}{z^u(\boldsymbol{p},0,0,a')}.$$

By Lemma 2, $(T,\boldsymbol{\tau})$ is not inequality-reducing, a contradiction. ■

Appendix D. Proof of Theorem 5

The proof of Theorem 5 is based on the following lemma.

Lemma 3. *Suppose that $\mathcal{S} \subseteq \mathcal{T}_{m\text{-}prog}$ is closed under linear extensions. Suppose that $\mathcal{S}'$ is a subset of commodity tax profiles. Then the mixed tax systems in $\mathcal{S}\times\mathcal{S}'$ are inequality-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$ if and only if the following two conditions are satisfied:*

(i) the net income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a for each $T \in \mathcal{L}_{\mathcal{S}} \cup \{0\}$ and $\boldsymbol{\tau} \in \mathcal{S}'$; and

(ii) the members of $\mathcal{L}_{\mathcal{S}} \times \mathcal{S}'$ are inequality-reducing.

Proof. Suppose that the mixed tax systems in $\mathcal{S}\times\mathcal{S}'$ are inequality-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$. Then the members of $\mathcal{L}_{\mathcal{S}}\times\mathcal{S}'$ are inequality-reducing (since $\mathcal{L}_{\mathcal{S}} \subseteq \mathcal{S}$) and, by Theorem 4, the net income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a for each $T \in \mathcal{L}_{\mathcal{S}}$ and $\boldsymbol{\tau} \in \mathcal{S}'$.

Now assume (i)–(ii). Fix $(T,\boldsymbol{\tau}) \in \mathcal{S}\times\mathcal{S}'$. We must show that $(T,\boldsymbol{\tau})$ is inequality-reducing with respect to $\boldsymbol{p}$ and u. By Lemma 2, it suffices to show that $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a and

$$\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)}{z^u(\boldsymbol{p},0,0,a)} \geq \frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a')}{z^u(\boldsymbol{p},0,0,a')}, \quad \text{whenever } a' > a > 0.$$

[22] Recall that the condition (v) guarantees that $y' > 0$ and $y'' > 0$.

[23] Here, for $\boldsymbol{x} = (x_1,\ldots,x_K) \in \mathbb{R}^K_+$, $\boldsymbol{p}\cdot\boldsymbol{x} = p_1x_1 + \cdots + p_Kx_K$.

Because T is piecewise linear in $\mathcal{T}_{m\text{-}prog}$, there exist

$$0 < e_1 < \cdots < e_M < \infty$$

and intervals

$$I_1 = [0, e_1], \ldots, I_M = [e_{M-1}, e_M], I_{M+1} = [e_M, \infty)$$

satisfying the following: for each m, there exist $b_m \geq 0$ and $t_m \in [0, 1)$ such that $T(y) = -b_m + t_m y$ for all $y \in I_m$. Moreover,

$$b_1 < \cdots < b_{M+1} \quad \text{and} \quad t_1 < \cdots < t_{M+1}.$$

For $m \in \{1, \ldots, M+1\}$, let $T_m(y) = -b_m + t_m y$. Suppose that $y_m(a)$ and $y(a)$ denote the solutions to

$$\max_{y \in [0, aL]} u(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_m, a, y), y/a)$$

and

$$\max_{y \in [0, aL]} u(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, y), y/a),$$

respectively.[24]

By the Maximum Theorem, the function $y_m(a)$ is continuous.

Next, we show that there exists $\overline{a}_1 > 0$ such that $y_1(\overline{a}_1) = e_1$. To see this, note that there exists a small enough $\alpha > 0$ such that $y_1(\alpha) < e_1$. Moreover, because

$$\lim_{a \to \infty} MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_1, a, e_1), e_1) = 0$$

for each k, there exists a large enough $\beta > 0$ such that $y_1(\beta) > e_1$. Consequently, the Intermediate Value Theorem gives $\overline{a}_1 > 0$ such that $y_1(\overline{a}_1) = e_1$.

Similarly, we can show the following:

$$\begin{aligned} &\exists \overline{a}_1 > 0 : y_1(\overline{a}_1) = e_1, \\ &\exists \underline{a}_2 > 0, \overline{a}_2 > 0 : y_2(\underline{a}_2) = e_1 \text{ and } y_2(\overline{a}_2) = e_2, \\ &\ \vdots \\ &\exists \underline{a}_M > 0, \overline{a}_M > 0 : y_M(\underline{a}_M) = e_{M-1} \text{ and } y_M(\overline{a}_M) = e_M, \\ &\exists \underline{a}_{M+1} > 0 : y_{M+1}(\underline{a}_{M+1}) = e_M. \end{aligned} \tag{35}$$

Moreover,

$$\overline{a}_1 \leq \underline{a}_2 \leq \overline{a}_2 \leq \cdots \leq \underline{a}_M \leq \overline{a}_M \leq \underline{a}_{M+1}. \tag{36}$$

We prove this in two steps.

First, we show that

$$\underline{a}_m \leq \overline{a}_m, \quad m \in \{2, \ldots, M\}. \tag{37}$$

Note that, for $m \in \{1, \ldots, M+1\}$, the map

$$a \mapsto l^u(\boldsymbol{p}+\boldsymbol{\tau}, T_m, 0, a)$$

is nondecreasing. Indeed, for $a' > a > 0$, we have

$$\begin{aligned} al^u(\boldsymbol{p}+\boldsymbol{\tau}, T_m, 0, a) &= al^u(\boldsymbol{p}+\boldsymbol{\tau}, -b_m, 0, (1-t_m)a) \\ &\leq al^u(\boldsymbol{p}+\boldsymbol{\tau}, -b_m, 0, (1-t_m)a') = al^u(\boldsymbol{p}+\boldsymbol{\tau}, T_m, 0, a'), \end{aligned}$$

where the inequality follows from condition (vii). Hence, the map $a \mapsto y_m(a)$ is nondecreasing. Consequently, for $m \in \{2, \ldots, M\}$, since

$$y_m(\underline{a}_m) = e_{m-1} < e_m = y_m(\overline{a}_m),$$

we see that $\underline{a}_m \leq \overline{a}_m$. This establishes (37).

It remains to show that

$$\overline{a}_m \leq \underline{a}_{m+1}, \quad m \in \{1, \ldots, M\}.$$

We only show that $\overline{a}_1 \leq \underline{a}_2$, since the other inequalities can be handled similarly. Proceeding by contradiction, suppose that $\overline{a}_1 > \underline{a}_2$. Since $y_2(\underline{a}_2) = e_1$, for every k we have

$$(p_k + \tau_k) MRS_k^{\underline{a}_2}(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_2, \underline{a}_2, e_1), e_1) = 1 - t_2.$$

[24] These problems have a unique solution. See Footnote 15.

Now since $\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_2, \underline{a}_2, e_1)$ solves

$$\max_{(x_1, \ldots, x_K) \in \mathbb{R}_+^K} u(x_1, \ldots, x_K, e_1/\underline{a}_2)$$

$$\text{s.t. } p_1 x_1 + \cdots + p_K x_K \leq e_1 - T_2(e_1),$$

and since $T_1(e_1) = T_2(e_1)$, we have

$$\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_2, \underline{a}_2, e_1) = \boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_1, \underline{a}_2, e_1),$$

and so

$$(p_k + \tau_k) MRS_k^{\underline{a}_2}(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_1, \underline{a}_2, e_1), e_1) = 1 - t_2 < 1 - t_1,$$

implying that $y_1(\underline{a}_2) > e_1$, a contradiction, since $y_1(a)$ is nondecreasing and $y_1(\overline{a}_1) = e_1$.

We have seen that (35) and (36) hold. Note that

$$y(a) = \begin{cases} y_1(a) & \text{if } a \in (0, \overline{a}_1], \\ y_2(a) & \text{if } a \in [\underline{a}_2, \overline{a}_2], \\ \vdots & \vdots \\ y_M(a) & \text{if } a \in [\underline{a}_M, \overline{a}_M], \\ y_{M+1}(a) & \text{if } a \geq \underline{a}_{M+1}. \end{cases} \tag{38}$$

Moreover, for $a \in [\overline{a}_1, \underline{a}_2]$ we have $y(a) = e_1$. To see this, fix $a \in [\overline{a}_1, \underline{a}_2]$. For each k we have

$$(p_k + \tau_k) MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1) \in [1 - t_2, 1 - t_1], \tag{39}$$

implying that $y(a) = e_1$. Prior to verifying the inclusion in (39), we argue that $y(a) = e_1$ is a consequence of (39).

Under (39), an increase in labor income, dy, starting at $(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1)$, requires at least

$$MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1) dy \geq \frac{1 - t_2}{p_k + \tau_k} dy$$

extra units of good k to keep utility constant, since the indifference curves for the utility function

$$(\boldsymbol{x}, y) \mapsto u(\boldsymbol{x}, y/a)$$

are convex (by quasiconcavity of u). But increasing labor income by dy only brings about an extra (net) labor income of at most $(1 - t_2)dy$ (recall that

$$1 - t_1 > \cdots > 1 - t_{M+1}),$$

which can afford at most $\frac{1-t_2}{p_k+\tau_k} dy$ extra units of good k. Thus, an increase in labor income is not welfare improving. A similar argument can be made for a reduction (rather than an increase) in labor income, taking the bundle $(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1)$ as the initial point.

To see that (39) holds, we assume that

$$(p_k + \tau_k) MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1) < 1 - t_2 \tag{40}$$

and derive a contradiction (the case when

$$(p_k + \tau_k) MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1) > 1 - t_1$$

can be handled similarly). Since

$$\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1) = \boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_2, a, e_1),$$

(40) implies that

$$(p_k + \tau_k) MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_2, a, e_1), e_1) < 1 - t_2,$$

whence $y_2(a) > e_1$, a contradiction, since $y_2(a)$ is nondecreasing, $a \leq \underline{a}_2$, and $y_2(\underline{a}_2) = e_1$.

We have seen that $y(a) = e_1$ whenever $a \in [\overline{a}_1, \underline{a}_2]$. Similarly, we can show that

$$y(a) = \begin{cases} e_1 & \text{if } a \in [\overline{a}_1, \underline{a}_2], \\ \vdots & \vdots \\ e_M & \text{if } a \in [\overline{a}_M, \underline{a}_{M+1}]. \end{cases} \tag{41}$$

We are now ready to show that $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a and

$$\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)}{z^u(\boldsymbol{p},0,0,a)} \geq \frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a')}{z^u(\boldsymbol{p},0,0,a')}, \quad \text{whenever } a' > a > 0.$$

Choose $a' > a > 0$. Suppose that $a \in (0,\overline{a}_1)$ and $a' > \underline{a}_{M+1}$ (the other cases can be handled similarly). Then

$$\begin{aligned} z^u(\boldsymbol{p},T,\boldsymbol{\tau},a') &= z^u(\boldsymbol{p},T_{M+1},\boldsymbol{\tau},a') && \text{(by (38))}\\ &\geq z^u(\boldsymbol{p},T_{M+1},\boldsymbol{\tau},\underline{a}_{M+1}) && \text{(by (i))}\\ &= z^u(\boldsymbol{p},T_M,\boldsymbol{\tau},\overline{a}_M) && \text{(by (38) and (41))}\\ &\geq z^u(\boldsymbol{p},T_M,\boldsymbol{\tau},\underline{a}_M) && \text{(by (i))}\\ &\ \vdots \\ &\geq z^u(\boldsymbol{p},T_2,\boldsymbol{\tau},\underline{a}_2)\\ &= z^u(\boldsymbol{p},T_1,\boldsymbol{\tau},\overline{a}_1) && \text{(by (38) and (41))}\\ &\geq z^u(\boldsymbol{p},T_1,\boldsymbol{\tau},a) && \text{(by (i)).} \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a')}{z^u(\boldsymbol{p},0,0,a')} &= \frac{z^u(\boldsymbol{p},T_{M+1},\boldsymbol{\tau},a')}{z^u(\boldsymbol{p},0,0,a')} && \text{(by (38))}\\ &\leq \frac{z^u(\boldsymbol{p},T_{M+1},\boldsymbol{\tau},\underline{a}_{M+1})}{z^u(\boldsymbol{p},0,0,\underline{a}_{M+1})} && \text{(by (ii) and Lemma 2)}\\ &= \frac{z^u(\boldsymbol{p},T_M,\boldsymbol{\tau},\overline{a}_M)}{z^u(\boldsymbol{p},0,0,\underline{a}_{M+1})} && \text{(by (38) and (41))}\\ &\leq \frac{z^u(\boldsymbol{p},T_M,\boldsymbol{\tau},\overline{a}_M)}{z^u(\boldsymbol{p},0,0,\overline{a}_M)} && \text{(by (i))}\\ &\ \vdots\\ &\leq \frac{z^u(\boldsymbol{p},T_2,\boldsymbol{\tau},\underline{a}_2)}{z^u(\boldsymbol{p},0,0,\underline{a}_2)}\\ &= \frac{z^u(\boldsymbol{p},T_1,\boldsymbol{\tau},\overline{a}_1)}{z^u(\boldsymbol{p},0,0,\underline{a}_2)} && \text{(by (38) and (41))}\\ &\leq \frac{z^u(\boldsymbol{p},T_1,\boldsymbol{\tau},\overline{a}_1)}{z^u(\boldsymbol{p},0,0,\overline{a}_1)} && \text{(by (i))}\\ &\leq \frac{z^u(\boldsymbol{p},T_1,\boldsymbol{\tau},a)}{z^u(\boldsymbol{p},0,0,a)} && \text{(by (ii) and Lemma 2).} \end{aligned}$$

This completes the proof. ■

Theorem 5. *Suppose that $\mathcal{S} \subseteq \mathcal{T}_{m\text{-}prog}$ is closed under linear extensions. Suppose that $\mathcal{S}'$ is a subset of commodity tax profiles. Then the mixed tax systems in $\mathcal{S} \times \mathcal{S}'$ are inequality-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$ if and only if the following two conditions are satisfied:*

(i) the net income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a for each $T \in \mathcal{L}_S \cup \{0\}$ and $\boldsymbol{\tau} \in \mathcal{S}'$;[25] and

(ii) the quotient

$$\frac{z^u(\boldsymbol{p},-b,\boldsymbol{\tau},(1-t)a)}{z^u(\boldsymbol{p},0,0,a)}$$

is nonincreasing in a for every $(b,t,\boldsymbol{\tau}) \in B(\mathcal{L}_S) \times R(\mathcal{L}_S) \times \mathcal{S}'$.

Proof. By Lemma 3, the mixed tax systems in $\mathcal{S} \times \mathcal{S}'$ are inequality-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$ if and only if the following two conditions are satisfied:

- the net income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a for each $T \in \mathcal{L}_S \cup \{0\}$ and $\boldsymbol{\tau} \in \mathcal{S}'$; and
- the members of $\mathcal{L}_S \times \mathcal{S}'$ are inequality-reducing.

Given the first bullet point and using Lemma 2, the second bullet point is expressible as follows:

$$\frac{z^u(\boldsymbol{p},-b+ty,\boldsymbol{\tau},a)}{z^u(\boldsymbol{p},0,0,a)}$$

is nonincreasing in a every $(b,t,\boldsymbol{\tau}) \in B(\mathcal{L}_S) \times R(\mathcal{L}_S) \times \mathcal{S}'$. Since

$$z^u(\boldsymbol{p},-b+ty,\boldsymbol{\tau},a) = (1-t)al^u(\boldsymbol{p},-b+ty,\boldsymbol{\tau},a) + b - \sum_{k=1}^{K} \tau_k x_k^u(\boldsymbol{p},-b+ty,\boldsymbol{\tau},a)$$

and

$$(x_1^u(\boldsymbol{p},-b+ty,\boldsymbol{\tau},a),\dots,x_K^u(\boldsymbol{p},-b+ty,\boldsymbol{\tau},a),l^u(\boldsymbol{p},-b+ty,\boldsymbol{\tau},a))$$

solves

$$\max_{(x_1,\dots,x_K,l)\in\mathbb{R}_+^K\times[0,L]} u(x_1,\dots,x_K,l)$$
$$\text{s.t. } (p_1+\tau_1)x_1+\cdots+(p_K+\tau_K)x_K \leq b+(1-t)al,$$

it follows that $z^u(\boldsymbol{p},-b+ty,\boldsymbol{\tau},a)$ can also be expressed as $z^u(\boldsymbol{p},-b,\boldsymbol{\tau},(1-t)a)$. Therefore, the second bullet point can be written as follows:

$$\frac{z^u(\boldsymbol{p},-b,\boldsymbol{\tau},(1-t)a)}{z^u(\boldsymbol{p},0,0,a)}$$

is nonincreasing in a for every $(b,t,\boldsymbol{\tau}) \in B(\mathcal{L}_S) \times R(\mathcal{L}_S) \times \mathcal{S}'$. ■

Appendix E. Proof of Theorem 6

First, we state and prove the following intermediate result.

Lemma 4. *For $T \in \mathcal{T}$, suppose that $(T,\boldsymbol{\tau})$ is bipolarization-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$. Then, the net income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a.*

Proof. Suppose that $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is not nondecreasing in a. Then, there exist $a' > a > 0$ such that

$$z^u(\boldsymbol{p},T,\boldsymbol{\tau},a') < z^u(\boldsymbol{p},T,\boldsymbol{\tau},a).$$

Consider the income distributions

$$\boldsymbol{z}^* = (z_1^*,\dots,z_n^*) = (z^u(\boldsymbol{p},0,0,a),\dots,z^u(\boldsymbol{p},0,0,a),z^u(\boldsymbol{p},0,0,a'))$$

and

$$\boldsymbol{z}^{**} = (z_1^{**},\dots,z_n^{**}) = (z^u(\boldsymbol{p},T,\boldsymbol{\tau},a),\dots,z^u(\boldsymbol{p},T,\boldsymbol{\tau},a),z^u(\boldsymbol{p},T,\boldsymbol{\tau},a')).$$

For $n > 2$, we have $m(\boldsymbol{z}^*) = z^u(\boldsymbol{p},0,0,a)$ and $m(\boldsymbol{z}^{**}) = z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$. Therefore, since

$$z^u(\boldsymbol{p},0,0,a) \leq \cdots \leq z^u(\boldsymbol{p},0,0,a) \leq z^u(\boldsymbol{p},0,0,a'),$$

where the last inequality follows from the condition (vii), we have

$$\frac{1}{m(\boldsymbol{z}^*)} \sum_{1\leq i<\frac{n+1}{2}} (m(\boldsymbol{z}^*) - z_{[i]}^*) = 0 < \frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a) - z^u(\boldsymbol{p},T,\boldsymbol{\tau},a')}{m(\boldsymbol{z}^{**})}$$
$$= \frac{1}{m(\boldsymbol{z}^{**})} \sum_{1\leq i<\frac{n+1}{2}} (m(\boldsymbol{z}^{**}) - z_{[i]}^{**}),$$

and so $\frac{m(\boldsymbol{z}^{**})}{m(\boldsymbol{z}^*)}\boldsymbol{z}^* \not\succcurlyeq_{FW} \boldsymbol{z}^{**}$, whence $\boldsymbol{z}^* \not\succcurlyeq_{FW} \boldsymbol{z}^{**}$. ■

Next, we prove Theorem 6.

Theorem 6. *For $T \in \mathcal{T}$, a mixed tax system $(T,\boldsymbol{\tau})$ is inequality-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$ if and only if it is bipolarization-reducing with respect to $\boldsymbol{p}$ and u.*

Proof. For $T \in \mathcal{T}$, suppose that a mixed tax system $(T,\boldsymbol{\tau})$ is inequality-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$. Fix a wage distribution $(a_1,\dots,a_n)$, an income function z^u, and a population size n. We must show that

$$(z^u(\boldsymbol{p},0,0,a_1),\dots,z^u(\boldsymbol{p},0,0,a_n)) \succcurlyeq_{FW} (z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_1),\dots,z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_n)). \quad (42)$$

By Theorem 4, the net income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a. By the condition (vii), the net income function $z^u(\boldsymbol{p},0,0,a)$ is nondecreasing in a.

[25] Here 0 denotes the linear tax schedule T defined by $T(y) = 0$ for all y.

We prove (42) when n is odd (the case when n is even can be handled similarly). Choose a wage distribution $(a_1,\dots,a_n)$ with

$$a_1 \le \cdots \le a_n.$$

Let $m = (n+1)/2$, so that a_m is the median wage. Because $(T,\boldsymbol{\tau})$ is inequality-reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$, and since the net income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ (respectively, $z^u(\boldsymbol{p},0,0,a)$) is nondecreasing in a, Lemma 2 implies that

$$\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_i)}{z^u(\boldsymbol{p},0,0,a_i)} \ge \frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)}{z^u(\boldsymbol{p},0,0,a_m)} \ge \frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_j)}{z^u(\boldsymbol{p},0,0,a_j)}, \quad \text{for } i<m \text{ and } j>m.$$

Hence,

$$\begin{aligned}
\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)-z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_i)}{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)} &= 1-\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_i)}{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)}\\
&\le 1-\frac{z^u(\boldsymbol{p},0,0,a_i)}{z^u(\boldsymbol{p},0,0,a_m)} = \frac{z^u(\boldsymbol{p},0,0,a_m)-z^u(\boldsymbol{p},0,0,a_i)}{z^u(\boldsymbol{p},0,0,a_m)}, \qquad \text{for } i<m,\\
\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_i)-z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)}{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)} &= \frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_i)}{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)}-1\\
&\le \frac{z^u(\boldsymbol{p},0,0,a_i)}{z^u(\boldsymbol{p},0,0,a_m)}-1 = \frac{z^u(\boldsymbol{p},0,0,a_i)-z^u(\boldsymbol{p},0,0,a_m)}{z^u(\boldsymbol{p},0,0,a_m)}, \qquad \text{for } i>m,
\end{aligned}$$

Consequently,

$$\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)}{z^u(\boldsymbol{p},0,0,a_m)}(z^u(\boldsymbol{p},0,0,a_1),\dots,z^u(\boldsymbol{p},0,0,a_n)) \succcurlyeq_{FW} (z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_1),\dots,z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_n)),$$

implying (42).

Conversely, suppose that $(T,\boldsymbol{\tau})$ is bipolarization-reducing with respect to $\boldsymbol{p}$ and u. Fix a wage distribution $(a_1,\dots,a_n)$, an income function z^u, and a population size n. We must show that

$$(z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_1),\dots,z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_n)) \succcurlyeq_L (z^u(\boldsymbol{p},0,0,a_1),\dots,z^u(\boldsymbol{p},0,0,a_n)). \tag{43}$$

By Lemma 4, the net income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a. By the condition (vii), the net income function $z^u(\boldsymbol{p},0,0,a)$ is nondecreasing in a.

We only prove (43) when n is odd (the case when n is even can be proven similarly). Proceeding by contradiction, suppose that (43) is false. Then Lemma 2 implies that there exist $a'>a>0$ such that

$$\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)}{z^u(\boldsymbol{p},0,0,a)} < \frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a')}{z^u(\boldsymbol{p},0,0,a')}. \tag{44}$$

Pick a wage distribution $(a_1,\dots,a_n)$ with

$$a_1 \le \cdots \le a_n$$

such that the median wage, a_m, where $m=(n+1)/2$, is equal to a', and $a_{m-1}=a$. Then

$$\begin{aligned}
\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)-z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_{m-1})}{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)} &= 1-\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_{m-1})}{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)}\\
&= 1-\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)}{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a')}\\
&> 1-\frac{z^u(\boldsymbol{p},0,0,a)}{z^u(\boldsymbol{p},0,0,a')}\\
&= \frac{z^u(\boldsymbol{p},0,0,a_m)-z^u(\boldsymbol{p},0,0,a_{m-1})}{z^u(\boldsymbol{p},0,0,a_m)},
\end{aligned}$$

where the inequality follows from (44). Consequently,

$$\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_m)}{z^u(\boldsymbol{p},0,0,a_m)}(z^u(\boldsymbol{p},0,0,a_1),\dots,z^u(\boldsymbol{p},0,0,a_n)) \not\succcurlyeq_{FW} (z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_1),\dots,z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_n)),$$

whence

$$(z^u(\boldsymbol{p},0,0,a_1),\dots,z^u(\boldsymbol{p},0,0,a_n)) \not\succcurlyeq_{FW} (z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_1),\dots,z^u(\boldsymbol{p},T,\boldsymbol{\tau},a_n)),$$

a contradiction. ∎

Appendix F. Joint and separate average-rate progressivity

In this section, we show that separate average-rate progressivity implies joint average-rate progressivity, but the converse is not true.

For an example of a jointly average-rate progressive tax system that is not separately average-rate progressive, consider the utility function

$$u(x_1,x_2) = 2\sqrt{x_1}+x_2,$$

whose associated (Marshallian) demand functions are given by

$$x_1(p_1,p_2,y) = \begin{cases}(p_2/p_1)^2 & \text{if } y \ge p_2^2/p_1,\\ y/p_1 & \text{if } y < p_2^2/p_1,\end{cases} \quad \text{and}$$

$$x_2(p_1,p_2,y) = \begin{cases}\frac{y}{p_2}-\frac{p_2}{p_1} & \text{if } y \ge p_2^2/p_1,\\ 0 & \text{if } y < p_2^2/p_1.\end{cases}$$

Let $(T,\boldsymbol{\tau})$ be a mixed tax system such that $\boldsymbol{\tau}=(0,\tau_2)$ and

$$T(y) = \begin{cases}\beta' y & \text{if } (1-\beta')y < (p_2+\tau_2)^2/p_1,\\ \frac{(p_2+\tau_2)^2}{(1-\beta')p_1}(\beta'-\beta)+\beta y & \text{if } (1-\beta')y \ge (p_2+\tau_2)^2/p_1,\end{cases}$$

where $1>\beta'>\beta>0$.

Because T is concave, $(T,\boldsymbol{\tau})$ fails to be separately average-rate progressive, and yet $(T,\boldsymbol{\tau})$ is jointly average-rate progressive if $\tau_2(1-\beta') \ge p_2(\beta'-\beta)$. To see this, note that

$$\begin{aligned}
&\frac{1}{y}(T(y)+\tau_2 x_2(\boldsymbol{p},T,\boldsymbol{\tau},y))\\
&= \begin{cases}\beta' & \text{if } (1-\beta')y < (p_2+\tau_2)^2/p_1,\\ \beta+\frac{\tau_2(1-\beta)}{p_2+\tau_2}+\frac{p_2+\tau_2}{(1-\beta')p_1 y}(p_2(\beta'-\beta)-\tau_2(1-\beta')) & \text{if } (1-\beta')y \ge (p_2+\tau_2)^2/p_1.\end{cases}
\end{aligned}$$

Hence, $\frac{1}{y}(T(y)+\tau_2 x_2(\boldsymbol{p},T,\boldsymbol{\tau},y))$ is nondecreasing in y if $\tau_2(1-\beta') \ge p_2(\beta'-\beta)$.

We now show that separate average-rate progressivity implies joint average-rate progressivity.

Suppose that $(T,\boldsymbol{\tau})$ is separately average-rate progressive. Assuming differentiability of T and the demand functions $x_k(\boldsymbol{p}',y')$ ($k \in \{1,\dots,K\}$), joint average-rate progressivity is expressible as

$$T'(y)+(1-T'(y))\left(\sum_{k=1}^K \tau_k \partial_2 x_k(\boldsymbol{p}+\boldsymbol{\tau},y-T(y))\right) \ge \frac{T(y)}{y}+\sum_{k=1}^K \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau},y-T(y))}{y}, \quad y>0. \tag{45}$$

Because $\boldsymbol{\tau}$ is average-rate progressive, it taxes luxuries and/or subsidizes necessities (refer to the discussion following expression (6)). Since luxury goods are normal and inferior goods are necessities, it follows that the bracketed summation on the left-hand side of expression (45) is nonnegative. Consequently, it suffices to show that

$$T'(y) \ge \frac{T(y)}{y}+\sum_{k=1}^K \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau},y-T(y))}{y}, \quad y>0,$$

i.e.,

$$T'(y)-\frac{T(y)}{y} \ge \sum_{k=1}^K \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau},y-T(y))}{y}, \quad y>0. \tag{46}$$

Note that the left-hand side of (46) is greater than or equal to one, since T is average-rate progressive. Therefore,

$$1 \ge \sum_{k=1}^K \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau},y-T(y))}{y}, \quad y>0,$$

is a sufficient condition for (46) to hold. But the last inequality is true, since

$$\sum_{k=1}^K \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau},y-T(y))}{y}$$

is the fraction of every dollar of income paid as consumption tax at $y > 0$. Formally, we have

$$y - \sum_{k=1}^{K} \tau_k \cdot x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y)) \geq y - T(y) - \sum_{k=1}^{K} \tau_k \cdot x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y)) = z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) \geq 0.$$

Appendix G. On mixed vs. pure direct taxation

Unlike in the exogenous case, commodity taxation is *not* generally "superfluous" when income is endogenous, as noted at the conclusion of Section 2.2. In fact, a mixed tax system may prove inequality-reducing in situations where income taxation alone fails to have any equalizing effect.

To illustrate this point, suppose that there are two goods (i.e., $K = 2$) and consider the following utility function:

$$u(x_1, x_2, l) = -\frac{1}{x_1} + x_2 - \frac{1}{1-l}, \tag{47}$$

where $l \in [0, 1]$.

For this utility function, we have

$$x_1^u(\boldsymbol{p}, y) = \begin{cases} \frac{y}{p_1} & \text{if } y < \sqrt{p_1 p_2}, \\ \sqrt{\frac{p_2}{p_1}} & \text{if } y \geq \sqrt{p_1 p_2}, \end{cases}$$

$$x_2^u(\boldsymbol{p}, y) = \begin{cases} 0 & \text{if } y < \sqrt{p_1 p_2}, \\ \frac{y - \sqrt{p_1 p_2}}{p_2} & \text{if } y \geq \sqrt{p_1 p_2}, \end{cases} \tag{48}$$

$$\boldsymbol{x}(\boldsymbol{p}, T, a, y) = \left(\min\left\{ \sqrt{\frac{p_2}{p_1}}, \frac{y - T(y)}{p_1} \right\}, \max\left\{ 0, \frac{y - T(y) - \sqrt{p_1 p_2}}{p_2} \right\} \right),$$

$$y^u(\boldsymbol{p}, -b, 0, a) + b = \begin{cases} b & \text{if } b \geq \sqrt{p_1 p_2},\ a < p_2, \\ a\left(1 - \sqrt{\frac{p_2}{a}}\right) + b & \text{if } b \geq \sqrt{p_1 p_2},\ a \geq p_2, \\ b & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b < \sqrt{p_1 p_2}, \\ & \quad b > \sqrt{a p_1}, \\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b < \sqrt{p_1 p_2}, \\ & \quad b \leq \sqrt{a p_1}, \\ \sqrt{p_1 p_2} & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b \geq \sqrt{p_1 p_2}, \\ & \quad b \leq \sqrt{a p_1},\ a < p_2, \\ \sqrt{p_1 p_2} & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b \geq \sqrt{p_1 p_2}, \\ & \quad b \leq \sqrt{a p_1},\ a \geq p_2, \\ & \quad a\left(1 - \sqrt{\frac{p_2}{a}}\right) + b < \sqrt{p_1 p_2}, \\ a\left(1 - \sqrt{\frac{p_2}{a}}\right) + b & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b \geq \sqrt{p_1 p_2}, \\ & \quad b \leq \sqrt{a p_1},\ a \geq p_2, \\ & \quad a\left(1 - \sqrt{\frac{p_2}{a}}\right) + b \geq \sqrt{p_1 p_2}, \end{cases} \tag{49}$$

and

$$\zeta^u(\boldsymbol{p}, -b, 0, a) = \begin{cases} 0 & \text{if } b \geq \sqrt{p_1 p_2},\ a < p_2, \\ \frac{a - \sqrt{a p_2} + \frac{1}{2}\sqrt{\frac{a}{p_2}}}{a - \sqrt{a p_2} + b} & \text{if } b \geq \sqrt{p_1 p_2},\ a \geq p_2, \\ 0 & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b < \sqrt{p_1 p_2}, \\ & \quad b > \sqrt{a p_1}, \\ \frac{a(\sqrt{a p_1} - b + \frac{a+b}{2})}{(a + \sqrt{a p_1})(a+b)} & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b < \sqrt{p_1 p_2}, \\ & \quad b \leq \sqrt{a p_1}, \\ 0 & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b \geq \sqrt{p_1 p_2}, \\ & \quad b \leq \sqrt{a p_1},\ a < p_2, \\ 0 & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b \geq \sqrt{p_1 p_2}, \\ & \quad b \leq \sqrt{a p_1},\ a \geq p_2, \\ & \quad a\left(1 - \sqrt{\frac{p_2}{a}}\right) + b < \sqrt{p_1 p_2}, \\ \frac{a - \sqrt{a p_2} + \frac{1}{2}\sqrt{\frac{a}{p_2}}}{a - \sqrt{a p_2} + b} & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b \geq \sqrt{p_1 p_2}, \\ & \quad b \leq \sqrt{a p_1},\ a \geq p_2, \\ & \quad a\left(1 - \sqrt{\frac{p_2}{a}}\right) + b \geq \sqrt{p_1 p_2}. \end{cases} \tag{50}$$

It is straightforward to verify that u satisfies conditions (i)–(vii).[26] Since

$$\frac{\partial x_1^u(\boldsymbol{p}, y)}{\partial y} = \begin{cases} \frac{1}{p_1} & \text{if } y < \sqrt{p_1 p_2}, \\ 0 & \text{if } y \geq \sqrt{p_1 p_2}, \end{cases} \quad \text{and} \quad \frac{x_1^u(\boldsymbol{p}, y)}{y} = \begin{cases} \frac{1}{p_1} & \text{if } y < \sqrt{p_1 p_2}, \\ \sqrt{\frac{1}{y}\frac{p_2}{p_1}} & \text{if } y \geq \sqrt{p_1 p_2}, \end{cases}$$

and

$$\frac{\partial x_2^u(\boldsymbol{p}, y)}{\partial y} = \begin{cases} 0 & \text{if } y < \sqrt{p_1 p_2}, \\ \frac{1}{p_2} & \text{if } y \geq \sqrt{p_1 p_2}, \end{cases} \quad \text{and} \quad \frac{x_2^u(\boldsymbol{p}, y)}{y} = \begin{cases} 0 & \text{if } y < \sqrt{p_1 p_2}, \\ \frac{1}{p_2} - \frac{1}{y}\sqrt{\frac{p_1}{p_2}} & \text{if } y \geq \sqrt{p_1 p_2}, \end{cases}$$

good 1 is a necessity and good 2 is a luxury.

In the special case when $\mathcal{S}' = \{\boldsymbol{\tau} = 0\}$ (no commodity taxes), Theorem 5 (together with (17)) implies that the following is a necessary condition for a subset $\mathcal{S} \subseteq \mathcal{T}_{m\text{-}prog}$ to be inequality-reducing with respect to $\boldsymbol{p}$ and u:

$$\zeta^u(\boldsymbol{p}, -b, 0, (1-t)a) \leq \zeta^u(\boldsymbol{p}, 0, 0, a), \quad \text{for each } a > 0 \text{ and } (b, t) \in B(\mathcal{L}_{\mathcal{S}}) \times R(\mathcal{L}_{\mathcal{S}}).$$

We will show that, for any $(b, t) \in B(\mathcal{L}_{\mathcal{S}}) \times R(\mathcal{L}_{\mathcal{S}})$ with $t > 0$, there exists $a > 0$ such that

$$\zeta^u(\boldsymbol{p}, -b, 0, (1-t)a) > \zeta^u(\boldsymbol{p}, 0, 0, a), \tag{51}$$

implying that no income tax schedule other than a pure subsidy, $T(y) = -b$, is inequality-reducing.

Choose $(b, t) \in B(\mathcal{L}_{\mathcal{S}}) \times R(\mathcal{L}_{\mathcal{S}})$. Using (50), we see that, for any sufficiently large a, (51) can be written as

$$\frac{(1-t)a - \sqrt{(1-t)a p_2} + \frac{1}{2}\sqrt{\frac{(1-t)a}{p_2}}}{(1-t)a - \sqrt{(1-t)a p_2} + b} > \frac{a - \sqrt{a p_2} + \frac{1}{2}\sqrt{\frac{a}{p_2}}}{a - \sqrt{a p_2}}.$$

Arranging terms yields

$$\frac{a}{2}\sqrt{\frac{a}{p_2}}(\sqrt{1-t} - (1-t)) > b(a - \sqrt{a p_2}) + \frac{b}{2}\sqrt{\frac{a}{p_2}}. \tag{52}$$

[26] Condition (vi) holds because no good is inferior (see Footnote 13). Condition (vii) holds because $y^u(\boldsymbol{p}, -b, 0, a) + b$ is nondecreasing in a.

Since $\sqrt{1-t}-(1-t)>0$, we see that there is a large enough a such that (52) holds.[27]

Thus, for the utility function in (47), there is no inequality-reducing income tax schedule other than a pure subsidy. We now show that there are *mixed* tax systems that are inequality-reducing.

Consider a mixed tax system $(T,\boldsymbol{\tau})$ such that $T(y)=-b$, for $b\geq 0$, and $\boldsymbol{\tau}=(\tau_1,\tau_2)=(0,\tau_2)$, where $\tau_2>0$, i.e., the commodity tax profile taxes the luxury good.

Using (17), (18), and Theorem 5, we see that $(T,\boldsymbol{\tau})$ is inequality-reducing with respect to $\boldsymbol{p}$ and u if and only if the following two conditions are satisfied:

- the net income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a for $T=0$ and $T=-b$; and
- the inequality

$$\zeta^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)\cdot\frac{1-\tau_2\cdot\frac{\partial x_2^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b)}{\partial y}}{1-\tau_2\cdot\frac{x_2^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b)}{y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b}}\leq\zeta^u(\boldsymbol{p},0,0,a)\quad(54)$$

holds for each $a>0$.

We will show that, given $\boldsymbol{p}$ and $\tau_2>0$, there exists $\underline{b}\geq 0$ such that these two conditions are satisfied if $b\geq\underline{b}$, implying that any mixed tax system $(T,\boldsymbol{\tau})$ such that $T(y)=-b$, for $b\geq\underline{b}$, and $\boldsymbol{\tau}=(\tau_1,\tau_2)=(0,\tau_2)$, is inequality-reducing with respect to $\boldsymbol{p}$ and u.

To see that the net income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a for $T=0$ and $T=-b$, note first that

$$\begin{aligned}z^u(\boldsymbol{p},-b,\boldsymbol{\tau},a)&=y^u(\boldsymbol{p},-b,\boldsymbol{\tau},a)+b-\tau_2x_2^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p},-b,\boldsymbol{\tau},a)+b)\\&=y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b-\tau_2x_2^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,a)+b).\end{aligned}$$

It suffices to show that $z^u(\boldsymbol{p},-b,\boldsymbol{\tau},a)$ is nondecreasing in a for any $b\geq 0$. Note that, by (48),

$$z^u(\boldsymbol{p},-b,\boldsymbol{\tau},a)=\begin{cases}y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b\\\qquad\text{if } y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b<\sqrt{p_1(p_2+\tau_2)},\\(y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b)(1-\frac{\tau_2}{p_2+\tau_2})+\tau_2\sqrt{\frac{p_1}{p_2+\tau_2}}\\\qquad\text{if } y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b\geq\sqrt{p_1(p_2+\tau_2)}.\end{cases}$$

Consequently, it suffices to show that $y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b$ is nondecreasing in a for any $b\geq 0$. This can be verified using (49).[28]

The Eq. (54) can be expressed as

$$\begin{aligned}&\zeta^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)\leq\zeta^u(\boldsymbol{p},0,0,a)\\&\qquad\text{if } y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b\leq\sqrt{p_1(p_2+\tau_2)},\\&\zeta^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)\cdot\frac{1-\frac{\tau_2}{p_2+\tau_2}}{1-\frac{\tau_2}{p_2+\tau_2}\left(1-\frac{\sqrt{p_1(p_2+\tau_2)}}{y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b}\right)}\leq\zeta^u(\boldsymbol{p},0,0,a)\\&\qquad\text{if } y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b>\sqrt{p_1(p_2+\tau_2)}.\end{aligned}$$

Since

$$\frac{1-\frac{\tau_2}{p_2+\tau_2}}{1-\frac{\tau_2}{p_2+\tau_2}\left(1-\frac{\sqrt{p_1(p_2+\tau_2)}}{y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b}\right)}\leq 1$$

whenever $y^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)+b>\sqrt{p_1(p_2+\tau_2)}$, it suffices to show that there exists $\underline{b}\geq 0$ such that

$$\zeta^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)\leq\zeta^u(\boldsymbol{p},0,0,a),\qquad\text{for } a>0 \text{ and } b\geq\underline{b}.\tag{55}$$

Let

$$\underline{b}=\max\left\{\frac{p_2}{1+2p_2},a^*+\frac{\sqrt{a^*}}{\sqrt{p_2+\tau_2}}+\sqrt{\frac{p_1}{p_2+\tau_2}},\sqrt{p_1(p_2+\tau_2)}\right\},\tag{56}$$

where $a^*>0$ is implicitly defined by the following equation:

$$a^*=\sqrt{a^*p_2}+\sqrt{p_1p_2}.\tag{57}$$

Note that

$$\underline{b}\geq\frac{p_2}{1+2p_2}\geq\frac{2\sqrt{ap_2}-a}{1+2p_2},\qquad\text{for all } a>0.\tag{58}$$

This inequality will be used later.

Using (50), $\zeta^u(\boldsymbol{p},0,0,a)$ can be expressed as follows:

$$\zeta^u(\boldsymbol{p},0,0,a)=\begin{cases}\frac{\sqrt{ap_1}+\frac{a}{2}}{a+\sqrt{ap_1}}&\text{if } a<\sqrt{ap_2}+\sqrt{p_1p_2},\\0&\text{if } a\geq\sqrt{ap_2}+\sqrt{p_1p_2},\ a<p_2,\\\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}}&\text{if } a\geq\sqrt{ap_2}+\sqrt{p_1p_2},\ a\geq p_2.\end{cases}\tag{59}$$

Using (50), (59), and the inequalities $b\geq\underline{b}\geq\sqrt{p_1(p_2+\tau_2)}$ (from (56)), $\zeta^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)-\zeta^u(\boldsymbol{p},0,0,a)$ can be expressed as follows:

$$\zeta^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)-\zeta^u(\boldsymbol{p},0,0,a)=$$

$$\begin{cases}-\frac{\sqrt{ap_1}+\frac{a}{2}}{a+\sqrt{ap_1}}&\text{if } a<p_2+\tau_2,\\&a<\sqrt{ap_2}+\sqrt{p_1p_2},\\\frac{a-\sqrt{a(p_2+\tau_2)}+\frac{1}{2}\sqrt{\frac{a}{p_2+\tau_2}}}{a-\sqrt{a(p_2+\tau_2)}+b}-\frac{\sqrt{ap_1}+\frac{a}{2}}{a+\sqrt{ap_1}}&\text{if } a\geq p_2+\tau_2,\\&a<\sqrt{ap_2}+\sqrt{p_1p_2},\\0&\text{if } a<p_2,\\&a\geq\sqrt{ap_2}+\sqrt{p_1p_2},\\0-\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}}&\text{if } a<p_2+\tau_2, a\geq p_2,\\&a\geq\sqrt{ap_2}+\sqrt{p_1p_2},\\\frac{a-\sqrt{a(p_2+\tau_2)}+\frac{1}{2}\sqrt{\frac{a}{p_2+\tau_2}}}{a-\sqrt{a(p_2+\tau_2)}+b}\\\qquad-\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}}&\text{if } a\geq p_2+\tau_2,\\&a\geq\sqrt{ap_2}+\sqrt{p_1p_2},\end{cases}$$

To establish (55), consider the following cases for the expression

$$\zeta^u(\boldsymbol{p}+\boldsymbol{\tau},-b,0,a)-\zeta^u(\boldsymbol{p},0,0,a)$$

given above.

In the second case, we have

$$\frac{a-\sqrt{a(p_2+\tau_2)}+\frac{1}{2}\sqrt{\frac{a}{p_2+\tau_2}}}{a-\sqrt{a(p_2+\tau_2)}+b}\leq\frac{\sqrt{ap_1}+\frac{a}{2}}{a+\sqrt{ap_1}}\tag{60}$$

[27] The inequality

$$\frac{(\alpha a^{3/2})/(2\sqrt{p_2})}{ba+\frac{ba^{1/2}}{2\sqrt{p_2}}}>1,\tag{53}$$

where $\alpha=\sqrt{1-t}-(1-t)$, is sufficient for (52) to hold. Since

$$\lim_{a\to\infty}\frac{(\alpha a^{3/2})/(2\sqrt{p_2})}{ba+\frac{ba^{1/2}}{2\sqrt{p_2}}}=\lim_{a\to\infty}\frac{(\frac{3\alpha}{2}a^{1/2})/(2\sqrt{p_2})}{b+\frac{(b/2)a^{-1/2}}{2\sqrt{p_2}}}=\infty$$

(by l'Hôpital's rule), it follows that (53) holds for large enough a.

[28] Note that both $a(1-\sqrt{p_2/a})+b$ and $\frac{a(\sqrt{ap_1}-b)}{a+\sqrt{ap_1}}+b$ are nondecreasing in a. To see that $\frac{a(\sqrt{ap_1}-b)}{a+\sqrt{ap_1}}+b$ is nondecreasing in a, note that

$$\partial\left(\frac{a(\sqrt{ap_1}-b)}{a+\sqrt{ap_1}}\right)\Big/\partial a=$$

$$\frac{(\sqrt{ap_1}-b+\frac{1}{2}\sqrt{ap_1})(a+\sqrt{ap_1})-a(\sqrt{ap_1}-b)\left(1+\frac{p_1}{2\sqrt{ap_1}}\right)}{(a+\sqrt{ap_1})^2}.$$

This expression is nonnegative if and only if

$$\frac{1}{2}\sqrt{ap_1}(\sqrt{ap_1}-b+a+\sqrt{ap_1})\geq 0.$$

The last inequality holds if $\sqrt{ap_1}-b\geq 0$.

if and only if

$$a+\sqrt{a}\left(\frac{1}{\sqrt{p_2+\tau_2}}-\sqrt{p_2+\tau_2}\right)+\sqrt{\frac{p_1}{p_2+\tau_2}}\leq\frac{2b\sqrt{p_1}}{\sqrt{a}}+b.$$

A sufficient condition for this inequality to hold is

$$a+\frac{\sqrt{a}}{\sqrt{p_2+\tau_2}}+\sqrt{\frac{p_1}{p_2+\tau_2}}\leq b,$$

which holds for every $0<a<\sqrt{ap_2}+\sqrt{p_1p_2}$ if

$$a^*+\frac{\sqrt{a^*}}{\sqrt{p_2+\tau_2}}+\sqrt{\frac{p_1}{p_2+\tau_2}}\leq b,$$

where, recall, a^* is defined by Eq. (57). Since this inequality is true (see (56)), it follows that (60) holds, and so (55) holds.

In the fifth case, we have

$$\frac{a-\sqrt{a(p_2+\tau_2)}+\frac{1}{2}\sqrt{\frac{a}{p_2+\tau_2}}}{a-\sqrt{a(p_2+\tau_2)}+b}\leq\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}} \tag{61}$$

for $b\geq\underline{b}$. To see this, note first that

$$\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}+b}\leq\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}},$$

since $b\geq 0$. Moreover, because

$$\frac{\partial}{\partial p_2}\left(\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}+b}\right)\leq 0$$

is equivalent to

$$b\geq\frac{2\sqrt{ap_2}-a}{1+2p_2},$$

and since this inequality holds for $b\geq\underline{b}$ by virtue of (58), it follows that (61) holds for $b\geq\underline{b}$. Thus, in the fifth case, (55) holds.

We conclude that, given $\boldsymbol{p}$ and $\tau_2>0$, there exists $\underline{b}\geq 0$ such that any mixed tax system $(T,\boldsymbol{\tau})$ such that $T(y)=-b$, for $b\geq\underline{b}$, and $\boldsymbol{\tau}=(\tau_1,\tau_2)=(0,\tau_2)$, is inequality-reducing with respect to $\boldsymbol{p}$ and u. By contrast, no income tax schedule (other than a pure subsidy) is inequality-reducing.

Data availability

No data was used for the research described in the article.

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