On Progressive Tax Systems with Heterogeneous Preferences

Oriol Carbonell-Nicolau*

9 August 2024

Abstract

The properties of progressive income tax systems vis-à-vis standard measures of inequality and polarization have been studied elsewhere, both for economies with exogenous and endogenous income. In the case of endogenous income, preferences are assumed to be identical across consumers. This paper relaxes the preference homogeneity assumption. Using the relative Lorenz inequality order and the relative Foster-Wolfson bipolarization order, we show that income tax systems are inequality and bipolarization reducing—no matter what the economy's initial conditions are—only if they are progressive; and we identify conditions on heterogeneous consumer preferences under which progressive tax systems are inequality and bipolarization reducing.

Keywords: Progressive income taxation, income inequality, income bipolarization, endogenous income.
JEL classifications: D31, D63, D71.

1. Introduction

Normatively, progressive income tax systems can be viewed as essential mechanisms for the reduction of "market-driven" income inequality. The theoretical literature on the foundations of progressive taxation goes back to the seminal result on the equivalence between tax progressivity—in the sense of increasing average tax rates on income—and the inequality reducing property (see Jakobsson, 1976; Fellman, 1976; Kakwani, 1977).

This result, which is couched in terms of exogenous income, has been extended in several directions (see, e.g., Hemming and Keen, 1983; Eichhorn et al., 1984; Liu, 1985; Formby et al., 1986; Thon, 1987; Latham, 1988; Thistle, 1988; Moyes, 1988, 1989, 1994; Le Breton et al., 1996; Ebert and Moyes, 2000; Ju and Moreno-Ternero, 2008; Zoli, 2018; Kakwani and Son, 2021; Carbonell-Nicolau, 2019, 2024). For the most part, these extensions maintain the exogenous income framework. The case of endogenous income presents some subtleties. In the exogenous case, where gross incomes are fixed, the mapping from gross incomes to net incomes fully determines the redistributive effect of an income tax. By contrast, the endogenous case introduces the additional effect of taxation on gross incomes,

*Department of Economics, Rutgers University, 75 Hamilton St., New Brunswick, NJ 08901. E-mail: carbonell-nicolau@rutgers.edu.

which may counter the equalizing transition from gross incomes to net incomes, depending on the magnitude and distributional incidence of the elasticity of gross income with respect to nontaxed income. In fact, there are consumer preferences for which progressive tax schedules increase inequality (see Allingham, 1979; Ebert and Moyes, 2003, 2007).

An extension of the Jakobsson-Fellman-Kakwani result to the case of endogenous income is provided in Carbonell-Nicolau and Llavador (2018, 2021b). Using the relative Lorenz inequality order, these works show that inequality reducing tax schedules—i.e., tax schedules effecting a more equal post-tax income distribution for every wage/ability distribution—are necessarily progressive, in the sense of increasing marginal tax rates on income; and identify necessary and sufficient conditions on consumer preferences for various sets of progressive tax schedules to be inequality reducing.

A further paper, Carbonell-Nicolau and Llavador (2021a), establishes the equivalence between the inequality reducing property and an analogous property formulated in terms of a different metric, the Foster-Wolfson relative bipolarization order (Foster and Wolfson, 2010; Wang and Tsui, 2000; Chakravarty, 2009, 2015), which has been used in the literature as a measure of the size of the middle class (see, e.g., Foster and Wolfson, 2010; Wolfson, 1994). Thus, tax schedules are inequality reducing if and only if they are bipolarization reducing, an equivalence that extends the scope of the Jakobsson-Fellman-Kakwani result and its variants.

The analysis of behavioral labor responses to income taxation is based on the standard Mirrlees model (Mirrlees, 1971), which assumes away any differences in preferences among consumers. This paper relaxes the preference homogeneity assumption. The framework of analysis is the Mirrlees model, augmented with a second source of heterogeneity, namely preference heterogeneity, in addition to the standard variation in wages/abilities. Thus, an economy consists of a distribution of wages/abilities and a distribution of preferences.

The introduction of heterogeneous preferences may further exacerbate the distributional distortions of taxation on gross incomes intrinsic to the endogenous income framework. In fact, the effect of taxation on the distribution of gross incomes, being richer under two sources of variation in individual attributes—wages and preferences—across the population, may counteract with added intensity the direct distributional effect of a tax on net incomes. Thus, conditions on preference profiles are needed that effectively resolve these potential trade-offs in favor of a net reduction in inequality and bipolarization.

We formulate a single crossing condition on model primitives that extends the standard agent monotonicity condition (Mirrlees, 1971; Seade, 1982) to the case of heterogeneous preferences. Under this condition, an extension of the results in Carbonell-Nicolau and Llavador (2018, 2021b) can be proven for families of utility vectors that are "sufficiently rich." Specifically, if $\boldsymbol{u} = (u_1, \ldots, u_n)$ is a distribution of utility functions describing the preferences of n individuals, the distribution $\boldsymbol{u}' = (u_1', \ldots, u_n')$ is called a *simple transformation* of $\boldsymbol{u}$ if $\boldsymbol{u}'$ takes the form $(u_i, \ldots, u_i, u_{i+1}, \ldots, u_{i+1})$ for some i, i.e., if the first (resp., last) i (resp, $n - i$) individuals are endowed with the utility function u_i (resp., u_{i+1}). A set of preference profiles is *closed under simple transformations* if it contains the simple transformations of its elements.

For families of preference profiles that are closed under simple transformations, we first show that inequality reducing tax schedules—i.e., tax schedules effecting a more equal post-tax income distribution for every economy—are necessarily progressive. We then establish the equivalence between inequality reducing and bipolarization reducing tax schedules in economies with heterogeneous preferences, and identify necessary and sufficient conditions on the distribution of preferences for a progressive tax schedule to be inequality and bipolarization reducing.

The main results are illustrated by means of a simple example where the individuals' preferences are represented by a family of quasilinear utility functions.

2. Characterizing income tax progressivity

We extend the model in Carbonell-Nicolau and Llavador (2018, 2021a,b) by allowing for heterogeneity of preferences across individuals.

Individual preferences are described by means of a utility function. All utility functions share the following properties. First, they are assumed to be real-valued functions defined on consumption-labor pairs (x, l) in the product set $\mathbb{R}_+ \times [0, L]$, where $0 < L < \infty$. For an individual endowed with a utility function u, $u(x, l)$ represents the individual's utility from x units of consumption and l units of labor. Throughout the sequel, all utility functions u are assumed to satisfy the following conditions:

(i) u is continuous.

(ii) $u(\cdot, l)$ strictly increasing in x for each $l \in [0, L)$ and $u(x, \cdot)$ strictly decreasing in l for each $x > 0$.

(iii) u is strictly quasiconcave on $\mathbb{R}_{++} \times [0, L)$ and twice continuously differentiable on $\mathbb{R}_{++} \times (0, L)$.

(iv) For each $x > 0$,

$$\liminf_{l \uparrow L} MRS(x, l) = \infty \quad \text{and} \quad \limsup_{l \downarrow 0} MRS(x, l) < \infty, \tag{1}$$

where, for $(x, l) \in \mathbb{R}_{++} \times (0, L)$, $MRS(x, l)$ represents the marginal rate of substitution of labor for consumption, i.e.,

$$MRS(x, l) = -\frac{\partial u(x, l)}{\partial l} \Bigg/ \frac{\partial u(x, l)}{\partial x}.$$

(v) For each $a > 0$, there exists $l > 0$ such that $u(al, l) > u(0, 0)$.

The first condition in (iv) requires the marginal rate of substitution of x for l to diverge, for fixed $x > 0$, as leisure vanishes. The second condition in (iv) is a technical condition ensuring that indifference curves do not become arbitrarily steep as $l \downarrow 0$. The last condition, (v), implies that, in the absence of taxes, an individual whose wage rate is $a > 0$ always consumes a positive amount.

The set of all utility functions satisfying the conditions (i)-(v) is denoted by $\mathcal{U}$.

A *tax schedule* is a continuous and nondecreasing map $T : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

- $T(y) \le y$ for each $y \in \mathbb{R}_+$.
- The map $y \mapsto y - T(y)$ is nondecreasing (i.e., T is order-preserving).

For every pre-tax income level $y \in \mathbb{R}_+$, $T(y)$ represents the associated tax liability ($T(y)$ being a subsidy if $T(y) < 0$).

A tax schedule T is *piecewise linear* if $\mathbb{R}_+$ can be partitioned into finitely many intervals $I_1, \ldots, I_K$ satisfying the following: for each k, there exist $\beta \in \mathbb{R}$ and $t \in [0,1)$ such that $T(y) = \beta + ty$ for all $y \in I_k$.[1]

The set of all piecewise linear tax schedules is denoted by $\mathscr{T}$.

An individual of ability $a > 0$ whose utility function is $u \in \mathscr{U}$ and who chooses $l \in [0, L]$ units of labor and faces a tax schedule $T \in \mathscr{T}$, affords $al - T(al)$ units of consumption and derives a utility of $u(al - T(al), l)$. Thus, the individual's problem is

$$\max_{l \in [0,L]} u(al - T(al), l). \tag{2}$$

A solution to (2) is denoted by $l^u(a, T)$.[2] It expresses the utility maximizing units of labor as a function of the "wage rate" a and the tax schedule T. Corresponding *pre-tax* and *post-tax income functions* are denoted by

$$y^u(a, T) = al^u(a, T) \quad \text{and} \quad x^u(a, T) = al^u(a, T) - T(al^u(a, T)),$$

respectively.[3,4] In the special case when T is identically zero (no taxation), we write $y^u(a, T) = y^u(a, 0)$ and $x^u(a, T) = x^u(a, 0)$. Note that $x^u(a, 0) = y^u(a, 0)$.

In the special case when T is a fixed subsidy, i.e., $T(y) = -b$ for all y and some $b \geq 0$, (2) has a unique solution (by the quasiconcavity of u on $\mathbb{R}_{++} \times [0, L)$), denoted by $l^u(a, b)$, with associated pre-tax and post-tax incomes $y^u(a, b)$ and $x^u(a, b)$, respectively.

We now define the class of utility vectors for which our main characterizations of progressive income tax systems are valid.

Throughout the sequel, the number of individuals is fixed at n.

We consider groups of individuals of size n and describe their preferences by means of utility functions in $\mathscr{U}$. Thus, a vector $(u_1, \ldots, u_n) \in \mathscr{U}^n$ of utility functions lists the individual preferences for each member of the group, where u_i represents individual i's utility function ($i \in \{1, \ldots, n\}$).

An *wage rate distribution*, also referred to as an *ability distribution*, is a vector $(a_1, \ldots, a_n) \in \mathbb{R}^n_{++}$, with its coordinates arranged in increasing order, i.e., $a_1 \leq \cdots \leq a_n$.

Note that any a-individual's utility function $u(x, l)$ defined on income-labor pairs can be reformulated in terms of net income-gross income pairs, (x, y), via the equation $y = al$, which relates before-tax income, y, to the number of hours worked, l: $u(x, y/a)$. The marginal rate of substitution of x for y,

$$\eta^a_u(x, y) = -(1/a) \cdot \frac{\partial u(x, y/a)}{\partial l} \bigg/ \frac{\partial u(x, y/a)}{\partial x}, \tag{3}$$

expresses the individual's required compensation, in terms of net income, for a one-unit marginal increase in the quantity of gross income.

[1]Note that 100% marginal tax rates are ruled out by assumption.

[2]Note that (2) has a solution because the objective function is continuous and the feasible set is compact.

[3]A solution to (2) exists, but need not be unique, and so pre-tax and post-tax solution functions are not uniquely defined.

[4]Since marginal tax rates are less than unity for the tax schedules in $\mathscr{T}$, condition (v) ensures that income levels $x^u(a, T)$ are positive.

Let $\mathbb{U}$ be the set of all utility vectors $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathscr{U}^n$ satisfying the following conditions:

(a) For each i,

$$\eta^{a}_{u_i}(x, y) \geq \eta^{a'}_{u_i}(x, y), \quad \text{for all } (x, y) \in \mathbb{R}_{++} \times (0, aL) \text{ and } a' \geq a > y/L.$$

(b) For each $i < n$ and $a > 0$,

$$\eta^{a}_{u_i}(x, y) \geq \eta^{a}_{u_{i+1}}(x, y), \quad \text{for all } (x, y) \in \mathbb{R}_{++} \times (0, aL).$$

The first condition, (a), states that the individuals' required compensation, in terms of net income, for an extra dollar of gross income, decreases with the wage rate.

The second condition, (b), compares consumption bundles across utility functions. It states that the required compensation, in terms of net income, for an extra dollar of gross income decreases for higher-order coordinates in the preference vector $(u_1, \ldots, u_n)$.

The above conditions extend the standard agent monotonicity condition introduced in Mirrlees (1971) (see also Seade (1982)) to vectors of heterogeneous preferences. Indeed, in the case when the utility function is common across individuals, (a) and (b) reduce to condition (a), which is the original single crossing condition in Mirrlees (1971) (see Mirrlees, 1971, p. 182). Note that (b) is also a single crossing condition akin to the Mirrlees condition.

The conditions (a) and (b) place constraints on the order of incomes across individuals. Given an ability distribution $(a_1, \ldots, a_n)$ with $a_1 \leq \cdots \leq a_n$, u_i represents the utility function of individual i, and so higher-order coordinates in the vector $\boldsymbol{u} = (u_1, \ldots, u_n)$ describe the preferences of higher-ability individuals. A higher-order coordinate in $\boldsymbol{u}$, together with higher ability, implies higher consumption.

Lemma 1. *Suppose that $T \in \mathscr{T}$. For each wage rate distribution $0 < a_1 \leq \cdots \leq a_n$ and every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}$,*

$$x^{u_1}(a_1, T) \leq \cdots \leq x^{u_n}(a_n, T).\text{[5]} \tag{4}$$

The proof of Lemma 1 is relegated to Appendix A.1.

[5] Recall that a solution to (2) exists, but need not be unique, and so the solutions functions $x^u(\cdot)$ are not uniquely defined. This fact introduces a technical subtlety in cases when, for some $i < n$, both the problems

$$\max_{l \in [0,L]} u_i(al - T(al), l) \quad \text{and} \quad \max_{l \in [0,L]} u_{i+1}(al - T(al), l)$$

happen to have multiple solutions for some a. In fact, in these particular cases, solutions $x^{u_i}(a, T)$ and $x^{u_{i+1}}(a, T)$ can be selected that violate the order in (4). To avoid these "pathologies," we shall impose a certain consistency in the choice of selections from the solution correspondence: for those points a for which said correspondence is multi-valued, the inequality "$x^{u_i}(a, T) \leq x^{u_{i+1}}(a, T)$" should be read to mean that for every solution function x^{u_i}, there exists a solution function $x^{u_{i+1}}$ such that

$$x^{u_i}(a, T) \leq x^{u_{i+1}}(a, T).$$

It is worth noting that the existence of multiple solutions is not a concern for those tax schedules in $\mathscr{T}$ that are marginal-rate progressive—i.e., convex—since, for marginal-rate progressive T, the solution functions $x^{u_i}(a, T)$ are uniquely defined. As per Theorem 1, inequality reducing tax schedules can only be found within the subset of marginal-rate progressive tax schedules.

Given a utility vector $(u_1, \ldots, u_n) \in \mathscr{U}^n$, an ability distribution $0 < a_1 \leq \cdots \leq a_n$, and post-tax income functions $x^{u_1}, \ldots, x^{u_n}$, a tax schedule T in $\mathscr{T}$ gives rise to a post-tax income distribution

$$(x^{u_1}(a_1, T), \ldots, x^{u_n}(a_n, T)).$$

Similarly,

$$(x^{u_1}(a_1, 0), \ldots, x^{u_n}(a_n, 0))$$

represents the income distribution in the absence of taxes.

Given two distributions $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ with positive total income, we say that $\boldsymbol{x}$ *Lorenz dominates* $\boldsymbol{y}$, a dominance relation denoted by "$\boldsymbol{x} \succcurlyeq_L \boldsymbol{y}$," if

$$\frac{\sum_{i=1}^{l} x_{[i]}}{\sum_{i=1}^{n} x_{[i]}} \geq \frac{\sum_{i=1}^{l} y_{[i]}}{\sum_{i=1}^{n} y_{[i]}}, \quad \text{for all } l \in \{1, \ldots, n\},$$

where $(x_{[1]}, \ldots, x_{[n]})$ (resp., $(y_{[1]}, \ldots, y_{[n]})$) is a rearrangement of the coordinates in $\boldsymbol{x}$ (resp., $\boldsymbol{y}$) in increasing order: $x_{[1]} \leq \cdots \leq x_{[n]}$ and $y_{[1]} \leq \cdots \leq y_{[n]}$.

A tax schedule $T \in \mathscr{T}$ is said to be *inequality reducing with respect to* $\mathbb{U}' \subseteq \mathbb{U}$, or $\mathbb{U}'$-ir, if

$$(x^{u_1}(a_1, T), \ldots, x^{u_n}(a_n, T)) \succcurlyeq_L (x^{u_1}(a_1, 0), \ldots, x^{u_n}(a_n, 0))$$

for each ability distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of income functions $(x^{u_1}, \ldots, x^{u_n})$, and every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$.

The subset of all $\mathbb{U}'$-ir tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{\mathbb{U}'\text{-ir}}$.

We now define the families of utility vectors that will be used in the formulation of our main results. To this end, we first define the wage elasticity of income for a utility function at wage rate a and non-wage income b.

For $(a, b) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$, the *wage elasticity of income* for u at (a, b) is defined by

$$\zeta^u(a, b) := \frac{\partial(a l^u(a, b) + b)}{\partial a} \cdot \frac{a}{a l^u(a, b) + b}.$$ [6,7]

Recall that the set of all piecewise linear tax schedules is denoted by $\mathscr{T}$.

A *marginal-rate progressive tax schedule* is a convex tax schedule in $\mathscr{T}$, which exhibits marginal tax rates that increase with income. The set of marginal-rate progressive tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{\text{m-prog}}$.

For every (piecewise linear) income tax schedule T in $\mathscr{T}_{\text{m-prog}}$, there exist

$$0 = e_0 < e_1 < \cdots < e_M = \infty$$

and intervals

$$I_1 = [e_0, e_1], \ldots, I_M = [e_{M-1}, e_M),$$

satisfying the following: for each m, there exist $b_m \geq 0$ and $t_m \in [0, 1)$ such that $T(y) = -b_m + t_m y$ for all $y \in I_m$, and

$$b_1 < \cdots < b_M \quad \text{and} \quad t_1 < \cdots < t_M,$$

[6]The condition (v) guarantees that $a l^u(a, b) + b$ is positive.

[7]For each $b \geq 0$, the derivative of the map $a \mapsto l^u(a, b)$ exists for all but at most one $a > 0$. See Carbonell-Nicolau and Llavador (2018, footnote 15).

where the last inequalities follow from the convexity of T.

Note that the extension of

$$-b_m + t_m y$$

to the entire domain $\mathbb{R}_+$, which is denoted by $T_m(y)$, is itself an income tax schedule in $\mathcal{T}_{\text{m-prog}}$. Thus, there are M many such linear extensions in $\mathcal{T}_{\text{m-prog}}$.

More generally, the set of all the linear extensions obtained from $T \in \mathcal{T}_{\text{m-prog}}$ in this manner is contained in $\mathcal{T}_{\text{m-prog}}$, and the cardinality of this set is equal to the number of tax brackets in T.

Using the above terminology, and given $T \in \mathcal{T}_{\text{m-prog}}$, define the class $\mathbb{U}_T$ of all utility vectors $(u_1, \ldots, u_n)$ in $\mathbb{U}$ satisfying the following conditions:

(I) For each $i < n$,

$$\frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} \geq \frac{x^{u_{i+1}}(a, T)}{x^{u_{i+1}}(a, 0)}, \quad \text{for all } a > 0.$$

(II) For each i,

$$\zeta^{u_i}((1 - t_m)a, b_m) \leq \zeta^{u_i}(a, 0)$$

whenever $a > 0$ and $m \in \{1, \ldots, M\}$ satisfy

$$y^{u_i}((1 - t_m)a, b_m) \in [e_{m-1}, e_m]$$

(i.e., individual i's gross income under the linear tax T_m lies in the m-th tax bracket for T).

Condition (I) states that the net income $x^u(a, T)$ as a fraction of $x^u(a, 0)$ decreases as the order rank for the vector of utility functions $(u_1, \ldots, u_n)$ increases.

Condition (II) states that, for a fixed utility function u and ability level a, the combined effect of the tax subsidy b and the proportional tax t decreases the wage elasticity of income.

Conditions (I) and (II) can be given the following interpretation. Condition (I) can be shown to be equivalent to the statement that T is inequality reducing in an economy where all individuals have the same wage rate, a, but differ in their preferences. Condition (II) is equivalent to the statement that T is inequality reducing in an economy where all individuals have the same preferences, while the distribution of wage rates is not degenerate. Since there are two sources of heterogeneity, i.e., there is simultaneous variation in preferences and wage rates, both conditions are needed.

It is worth pointing out that condition (II) is expressed in terms of linear tax schedules, while our main results offer characterizations of inequality reducing, nonlinear income tax schedules. Roughly speaking, this is a consequence of the fact that, in the absence of heterogeneous preferences, a full characterization of the classes of inequality reducing, nonlinear income tax schedules can be provided solely in terms of behavioral responses to linear taxation. This, unfortunately, is not the case under heterogeneous preferences. In fact, condition (I) is generally needed. However, since T is a piecewise linear tax schedule, the net income functions $x^{u_i}(a, T)$ and $x^{u_{i+1}}(a, T)$ from condition (I) take a relatively simple form, as demonstrated by the following proposition.

Proposition 1. *Given $u \in \mathcal{U}$, $T \in \mathcal{T}_{\text{m-prog}}$, and $a > 0$, there exists $m \in \{1, \ldots, M\}$ such that one and only one of the following two conditions holds.*

1. $x^u(a, T) = x^u((1 - t_m)a, b_m)$.

2. $x^u(a,T) = (1-t_m)e_m + b_m$ *and*

$$x^u((1-t_{m+1})a, b_{m+1}) < (1-t_m)e_m + b_m < x^u((1-t_m)a, b_m).$$

The proof of Proposition 1 is given in Appendix A.2.

The main results of the paper are stated for classes of preferences profiles that are "sufficiently" rich in the following sense.

Given $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{U}$, $\boldsymbol{u}' = (u'_1, \ldots, u'_n)$ is called a *simple transformation of* $\boldsymbol{u}$ if there exists $\iota \in \{0, 1, \ldots, n\}$ such that

$$\begin{aligned} u'_j &= u_\iota, && \text{for each } j \le \iota, \\ u'_j &= u_{\iota+1}, && \text{for each } j \ge \iota + 1. \end{aligned}$$

A subset $\mathbb{U}' \subseteq \mathbb{U}$ is *closed under simple transformations* if $\boldsymbol{u} \in \mathbb{U}'$ implies that $\boldsymbol{u}' \in \mathbb{U}'$ for every simple transformation $\boldsymbol{u}'$ of $\boldsymbol{u}$. In words, $\mathbb{U}'$ is closed under simple transformations if it contains the simple transformations of all of its members.

Given $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and given $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{U}'$ and $i \in \{1, \ldots, n\}$, $(u_i, \ldots, u_i)$ is also an element of $\mathbb{U}'$. To see this, note first that

$$(u_1, \ldots, u_1) \quad \text{and} \quad (u_n, \ldots, u_n)$$

are simple transformations of $\boldsymbol{u}$ (take $\iota = 0$ and $\iota = n$, respectively, in the definition of a simple transformation). Next, note that, for $i < n$,

$$\boldsymbol{u}' = (u_i, \ldots, u_i, u_{i+1}, \ldots, u_{i+1})$$

is a simple transformation of $\boldsymbol{u}$, and so $\boldsymbol{u}' \in \mathbb{U}'$. Now observe that

$$\boldsymbol{u}'' = (u_i, \ldots, u_i)$$

is a simple transformation of $\boldsymbol{u}'$, implying that $\boldsymbol{u}'' \in \mathbb{U}'$.

The following is the first main result of this paper. Its proof is relegated to Appendix A.3.

Theorem 1. *For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} \Leftrightarrow \left[T \in \mathscr{T}_{\text{m-prog}} \quad \textit{and} \quad \mathbb{U}' \subseteq \mathbb{U}_T\right].$$

This result characterizes income tax schedules that are inequality reducing with respect to a universe of preference vectors $\mathbb{U}' \subseteq \mathbb{U}$: a tax schedule $T \in \mathscr{T}$ is inequality reducing with respect to $\mathbb{U}'$ if and only if T is marginal-rate progressive (i.e., convex) and $\mathbb{U}'$ is contained in $\mathbb{U}_T$.

Theorem 1 can be extended to a second characterization of progressivity in terms of inequality *and* bipolarization reducing tax schedules.

The Foster-Wolfson bipolarization order (Foster and Wolfson, 2010; Wang and Tsui, 2000; Chakravarty, 2009, 2015) is a measure of the degree of income polarization between two income groups, taking median income as the demarcation point.

For two income distributions $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ with the same median income, m, we write $\boldsymbol{y} \succcurlyeq_{FW} \boldsymbol{x}$ to indicate that $\boldsymbol{y}$ is *more bipolarized than* $\boldsymbol{x}$, if

$$\sum_{k \le i < \frac{n+1}{2}} (m - x_i) \le \sum_{k \le i < \frac{n+1}{2}} (m - y_i), \quad \forall k : 1 \le k < \frac{n+1}{2},$$
$$\sum_{\frac{n+1}{2} < i \le k} (x_i - m) \le \sum_{\frac{n+1}{2} < i \le k} (y_i - m), \quad \forall k : \frac{n+1}{2} < k \le n.$$

The Foster-Wolfson bipolarization order compares income distributions on the basis of an aggregate measure of the deviation of income levels from median income, with lower aggregate deviations corresponding to less bipolarized distributions.

Assuming that proportional changes in income do not alter the degree of bipolarization, $\succcurlyeq_{\mathrm{FW}}$ can be extended to pairs of income distributions with different median incomes as follows.

Let $m(\boldsymbol{x})$ (resp., $m(\boldsymbol{y})$) denote the median income of $\boldsymbol{x}$ (resp., $\boldsymbol{y}$), and suppose that $m(\boldsymbol{x}) > 0$ and $m(\boldsymbol{y}) > 0$. Then the transformation

$$\boldsymbol{y}' = \frac{m(\boldsymbol{x})}{m(\boldsymbol{y})}(y_1, \ldots, y_n)$$

of $\boldsymbol{y}$ has the same median as $\boldsymbol{x}$ and we write

$$\boldsymbol{y} \succcurlyeq_{\mathrm{FW}} \boldsymbol{x} \Leftrightarrow \boldsymbol{y}' \succcurlyeq_{\mathrm{FW}} \boldsymbol{x}.$$

A tax schedule $T \in \mathscr{T}$ is said to be *bipolarization reducing with respect to* $\mathbb{U}' \subseteq \mathbb{U}$, or $\mathbb{U}'$-bpr, if

$$(x^{u_1}(a_1, 0), \ldots, x^{u_n}(a_n, 0)) \succcurlyeq_{\mathrm{FW}} (x^{u_1}(a_1, T), \ldots, x^{u_n}(a_n, T))$$

for each ability distribution $0 < a_1 \le \cdots \le a_n$, each vector of income functions $(x^{u_1}, \ldots, x^{u_n})$, and every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$.

The subset of all $\mathbb{U}'$-bpr tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

The equivalence between the inequality reducing property and its counterpart formulation in terms of the Foster-Wolfson dominance relation was first established in Carbonell-Nicolau and Llavador (2021a, Theorem 4) for economies with homogeneous preferences. The following result extends the equivalence to economies with heterogeneous preferences. The proof is given in Appendix A.4.

Theorem 2. *If* $\mathbb{U}' \subseteq \mathbb{U}$ *is closed under simple transformations, then* $\mathscr{T}_{\mathbb{U}'\text{-ir}} = \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

Theorem 2 states that if a domain of utility vectors $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations, then a tax schedule is inequality reducing with respect to $\mathbb{U}'$ if and only if it is bipolarization reducing with respect to $\mathbb{U}'$.

Theorem 1 and Theorem 2 combined immediately give the following result.

Corollary 1. *For* $\mathbb{U}' \subseteq \mathbb{U}$, *where* $\mathbb{U}'$ *is closed under simple transformations, and* $T \in \mathscr{T}$,

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} = \mathscr{T}_{\mathbb{U}'\text{-bpr}} \Leftrightarrow \left[T \in \mathscr{T}_{\text{m-prog}} \quad \textit{and} \quad \mathbb{U}' \subseteq \mathbb{U}_T\right].$$

This result states that a tax schedule $T \in \mathscr{T}$ is inequality and bipolarization reducing with respect to $\mathbb{U}'$ if and only if T is marginal-rate progressive (i.e., convex) and $\mathbb{U}'$ is contained in $\mathbb{U}_T$.

3. An example

Consider the quasilinear utility function

$$u_{(\alpha,\beta)}(x,l) = x - \alpha l^{\beta},$$

where $\alpha > 0$ and $\beta > 1$.

It is easy to verify that $u_{(\alpha,\beta)} \in \mathscr{U}$ (i.e., that the conditions (i)-(v) hold for the utility function $u_{(\alpha,\beta)}$).

To begin, we assume that α is common across individuals, while allowing β to vary. In this context, we can identify domains of utility vectors for which no tax schedule is inequality and/or bipolarization reducing.

Since

$$\eta^{a}_{u_{(\alpha_i,\beta_i)}}(x,y) = \frac{\alpha_i\beta_i}{a}\left(\frac{y}{a}\right)^{\beta_i-1},$$

we see that $\eta^{a}_{u_{(\alpha_i,\beta_i)}}(x,y)$ is nonincreasing in a for every (x,y). Moreover,

$$\begin{aligned}&\eta^{a}_{u_{(\alpha_i,\beta_i)}}(x,y) \geq \eta^{a}_{u_{(\alpha_{i+1},\beta_{i+1})}}(x,y)\\ &\qquad\Leftrightarrow \ln\alpha_i + \ln\beta_i + (\beta_i - 1)\ln(y/a) \geq \ln\alpha_{i+1} + \ln\beta_{i+1} + (\beta_{i+1}-1)\ln(y/a),\end{aligned}$$

Consequently, for fixed α, a vector of utilities

$$(u_{(\alpha,\beta_1)},\ldots,u_{(\alpha,\beta_n)})$$

belongs to $\mathbb{U}$ if and only if

$$\beta_1 \geq \cdots \geq \beta_n. \tag{5}$$

Now fix any $T \in \mathscr{T}_{\text{m-prog}}$, and let $\mathbb{U}'$ be the set of all preference profiles

$$(u_{(\alpha,\beta_1)},\ldots,u_{(\alpha,\beta_n)})$$

satisfying (5). It is readily verified that $\mathbb{U}'$ is closed under simple transformations. Hence, by Corollary 1, T is inequality/bipolarization reducing if and only if $\mathbb{U}' \subseteq \mathbb{U}_T$.

We claim that $\mathbb{U}' \nsubseteq \mathbb{U}_T$, implying that T is not inequality/bipolarization reducing. To see that $\mathbb{U}' \nsubseteq \mathbb{U}_T$, choose

$$(u_{(\alpha,\beta_1)},\ldots,u_{(\alpha,\beta_n)}) \in \mathbb{U}'$$

with $\beta_1 > \beta_2$ and fix $a > 0$ such that $y^{u_{(\alpha,\beta_1)}}(a,T) \in (0,e_1)$.[8] Because

$$0 < y^{u_{(\alpha,\beta_1)}}(a',T) \leq y^{u_{(\alpha,\beta_1)}}(a,T), \quad \text{for all } 0 < a' < a$$

[8]Such an a exists by Lemma 3 in Appendix A.3.

(where the inequality follows from Lemma 1), a can be chosen close enough to 0 to ensure that $a/\alpha < 1$. In addition, because the map

$$\beta \mapsto y^{u_{(\alpha,\beta)}}(a,T)$$

is continuous, β_2 can be taken close enough to β_1 to ensure that $y^{u_{(\alpha,\beta_2)}}(a,T) \in (0,e_1)$. Consequently,

$$\frac{x^{u_{(\alpha,\beta_1)}}(a,T)}{x^{u_{(\alpha,\beta_1)}}(a,0)} = \frac{(1-t_1)a\left(\frac{(1-t_1)a}{\alpha}\right)^{1/(\beta_1-1)}+b_1}{a\left(\frac{a}{\alpha}\right)^{1/(\beta_1-1)}} < \frac{(1-t_1)a\left(\frac{(1-t_1)a}{\alpha}\right)^{1/(\beta_2-1)}+b_1}{a\left(\frac{a}{\alpha}\right)^{1/(\beta_2-1)}} = \frac{x^{u_{(\alpha,\beta_2)}}(a,T)}{x^{u_{(\alpha,\beta_2)}}(a,0)}.[9]$$

Hence, $\mathbb{U}' \not\subseteq \mathbb{U}_T$.

Next, we fix β and allow α to vary, and identify domains of utility vectors for which T is inequality and/or bipolarization reducing.

First, note that for fixed β, a vector of utilities

$$(u_{(\alpha,\beta_1)},\ldots,u_{(\alpha,\beta_n)})$$

belongs to $\mathbb{U}$ if and only if

$$\alpha_1 \geq \cdots \geq \alpha_n. \tag{6}$$

Let $\mathbb{U}'$ be the set of all preference profiles

$$(u_{(\alpha,\beta_1)},\ldots,u_{(\alpha,\beta_n)})$$

satisfying (6). Since $\mathbb{U}'$ is closed under simple transformations, and since $\mathbb{U}' \subseteq \mathbb{U}_T$, Corollary 1 implies that T is inequality/bipolarization reducing.

To see that $\mathbb{U}' \subseteq \mathbb{U}_T$, note first that

$$\zeta^{u_{(\alpha,\beta)}}(a,b) = \frac{a^{\frac{\beta}{\beta-1}}\beta}{(\beta-1)\left(b(\alpha\beta)^{\frac{1}{\beta-1}}+a^{\frac{\beta}{\beta-1}}\right)},$$

and so $\zeta^{u_{(\alpha,\beta)}}(a,b)$ is decreasing in b and nondecreasing in a.[10] Consequently,

$$\zeta^{u_{(\alpha,\beta)}}((1-t_m)a,b_m) \leq \zeta^{u_{(\alpha,\beta)}}(a,0)$$

[9] The inequality follows from the fact that

$$\frac{\partial\left(\frac{(1-t_1)a\left(\frac{(1-t_1)a}{\alpha}\right)^{1/(\beta-1)}+b_1}{a\left(\frac{a}{\alpha}\right)^{1/(\beta-1)}}\right)}{\partial\beta} = \frac{b(\alpha a)^{1/(\beta-1)}\ln(a/\alpha)}{a^{(\beta+1)/(\beta-1)}(\beta-1)^2} < 0,$$

which holds by virtue of the inequality $a/\alpha < 1$.

[10] To see that $\zeta^{u_{(\alpha,\beta)}}(a,b)$ is nondecreasing in a, note that

$$\frac{\partial\zeta^{u_{(\alpha,\beta)}}(a,b)}{\partial a} = \frac{ba^{\frac{1}{\beta-1}}\beta^{\frac{2\beta-1}{\beta-1}}\alpha^{\frac{1}{\beta-1}}}{(\beta-1)^2\left(b(\alpha\beta)^{\frac{1}{\beta-1}}+a^{\frac{\beta}{\beta-1}}\right)^2} \geq 0.$$

whenever $a > 0$ and $m \in \{1, \ldots, M\}$ satisfy

$$y^{u_{(\alpha,\beta)}}((1-t_m)a, b_m) \in [e_{m-1}, e_m].$$

It remains to show that, for each $i < n$,

$$\frac{x^{u_{(\alpha_i,\beta)}}(a,T)}{x^{u_{(\alpha_i,\beta)}}(a,0)} \geq \frac{x^{u_{(\alpha_{i+1},\beta)}}(a,T)}{x^{u_{(\alpha_{i+1},\beta)}}(a,0)}, \quad \text{for all } a > 0.$$

Fix $i < n$ and $a > 0$. For each $m \in \{1, \ldots, M-1\}$, define $\overline{\alpha}(m)$ and $\underline{\alpha}(m)$ by

$$x^{u_{(\overline{\alpha}(m),\beta)}}((1-t_m)a, b_m) = (1-t_m)a\left(\frac{(1-t_m)a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)} + b_m = (1-t_m)e_m + b_m$$

and

$$x^{u_{(\underline{\alpha}(m),\beta)}}((1-t_{m+1})a, b_{m+1}) = (1-t_{m+1})a\left(\frac{(1-t_{m+1})a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)} + b_{m+1} = (1-t_{m+1})e_m + b_{m+1}.$$

Note that $\overline{\alpha}(m) > \underline{\alpha}(m)$ for each $m \leq M-1$ and $\underline{\alpha}(m) > \overline{\alpha}(m+1)$ for $m < M-1$. Indeed, because

$$\eta^a_{u_{(\overline{\alpha}(m),\beta)}}((1-t_m)e_m + b_m, e_m) = \frac{\overline{\alpha}(m)\beta}{a}\left(\frac{e_m}{a}\right)^{\beta-1} = 1 - t_m,$$

$$\eta^a_{u_{(\underline{\alpha}(m),\beta)}}((1-t_{m+1})e_m + b_{m+1}, e_m) = \frac{\underline{\alpha}(m)\beta}{a}\left(\frac{e_m}{a}\right)^{\beta-1} = 1 - t_{m+1},$$

and

$$\eta^a_{u_{(\overline{\alpha}(m+1),\beta)}}((1-t_{m+1})e_{m+1} + b_{m+1}, e_{m+1}) = \frac{\overline{\alpha}(m+1)\beta}{a}\left(\frac{e_{m+1}}{a}\right)^{\beta-1} = 1 - t_{m+1},$$

and since $e_1 < \cdots < e_M$ and (by the convexity of T)

$$t_1 < \cdots < t_M,$$

we have $\overline{\alpha}(m) > \underline{\alpha}(m)$ and $\underline{\alpha}(m) > \overline{\alpha}(m+1)$.

Note also that

$$x^{u_{(\alpha,\beta)}}(a, T_m) = x^{u_{(\alpha,\beta)}}((1-t_m)a, b_m), \quad \text{for each } m \in \{1, \ldots, M\},$$

where, recall, $T_m(y) = -b_m + t_m y$ for every y. This follows from the fact that both

$$l^{u_{(\alpha,\beta)}}(a, T_m) \quad \text{and} \quad l^{u_{(\alpha,\beta)}}((1-t_m)a, b_m)$$

solve the problem

$$\max_{l \in [0,L]} u_{(\alpha,\beta)}((1-t_m)al + b_m, l).$$

By Proposition 1 and Lemma 1, there are four cases to consider.

1. $m' \geq m$,

$$x^{u_{(\alpha_i,\beta)}}(a,T) = x^{u_{(\alpha_i,\beta)}}((1-t_m)a, b_m), \quad \text{and} \quad x^{u_{(\alpha_{i+1},\beta)}}(a,T) = x^{u_{(\alpha_{i+1},\beta)}}((1-t_{m'})a, b_{m'}).$$

2. $m' \geq m$,

$$x^{u_{(\alpha_i,\beta)}}(a,T) = x^{u_{(\alpha_i,\beta)}}((1-t_m)a, b_m), \quad \text{and} \quad x^{u_{(\alpha_{i+1},\beta)}}(a,T) = (1-t_{m'})e_{m'} + b_{m'}.$$

3. $m' \geq m$,

$$x^{u_{(\alpha_i,\beta)}}(a,T) = (1-t_m)e_m + b_m, \quad \text{and} \quad x^{u_{(\alpha_{i+1},\beta)}}(a,T) = x^{u_{(\alpha_{i+1},\beta)}}((1-t_{m'})a, b_{m'}).$$

4. $m' \geq m$,

$$x^{u_{(\alpha_i,\beta)}}(a,T) = (1-t_m)e_m + b_m, \quad \text{and} \quad x^{u_{(\alpha_{i+1},\beta)}}(a,T) = (1-t_{m'})e_{m'} + b_{m'}.$$

We consider only the first case, since the other three cases can be handled similarly. In the first case, we have, by (6) and Lemma 1,

$$\begin{aligned}
&\alpha_i \geq \overline{\alpha}(m) > \underline{\alpha}(m) \geq \alpha_{i+1} && \text{if } m' = m+1, \\
&\alpha_i \geq \overline{\alpha}(m) > \underline{\alpha}(m) > \cdots > \overline{\alpha}(m'-1) > \underline{\alpha}(m'-1) \geq \alpha_{i+1} && \text{if } m' > m+1.
\end{aligned}$$

If $m = m'$, then

$$\begin{aligned}
\frac{x^{u_{(\alpha_i,\beta)}}(a,T)}{x^{u_{(\alpha_i,\beta)}}(a,0)} &= \frac{x^{u_{(\alpha_i,\beta)}}((1-t_m)a, b_m)}{x^{u_{(\alpha_i,\beta)}}(a,0)} \\
&= \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\alpha_i}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\alpha_i}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\alpha_{i+1}}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\alpha_{i+1}}\right)^{1/(\beta-1)}} \\
&= \frac{x^{u_{(\alpha_{i+1},\beta)}}(a,T)}{x^{u_{(\alpha_{i+1},\beta)}}(a,0)}.
\end{aligned}$$

[11]

If $m' = m+1$, then

$$\begin{aligned}
\frac{x^{u_{(\alpha_i,\beta)}}(a,T)}{x^{u_{(\alpha_i,\beta)}}(a,0)} &= \frac{x^{u_{(\alpha_i,\beta)}}((1-t_m)a, b_m)}{x^{u_{(\alpha_i,\beta)}}(a,0)} \\
&= \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\alpha_i}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\alpha_i}\right)^{1/(\beta-1)}}
\end{aligned}$$

[11]It is a tedious but straightforward task to show that

$$\frac{\partial\left(\frac{(1-t_m)a\left(\frac{(1-t_m)a}{\alpha}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\alpha}\right)^{1/(\beta-1)}}\right)}{\partial\alpha} \geq 0.$$

$$
\begin{aligned}
&\geq \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&= \frac{(1-t_m)e_m + b_m}{a\left(\frac{a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_{m+1})e_m + b_{m+1}}{a\left(\frac{a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&= \frac{(1-t_{m+1})a\left(\frac{(1-t_{m+1})a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)} + b_{m+1}}{a\left(\frac{a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_{m+1})a\left(\frac{(1-t_{m+1})a}{\alpha_{i+1}}\right)^{1/(\beta-1)} + b_{m+1}}{a\left(\frac{a}{\alpha_{i+1}}\right)^{1/(\beta-1)}} \\
&= \frac{x^{u_{(\alpha_{i+1},\beta)}}(a,T)}{x^{u_{(\alpha_{i+1},\beta)}}(a,0)}.
\end{aligned}
$$

If $m' > m+1$, then

$$
\begin{aligned}
\frac{x^{u_{(\alpha_i,\beta)}}(a,T)}{x^{u_{(\alpha_i,\beta)}}(a,0)} &= \frac{x^{u_{(\alpha_i,\beta)}}((1-t_m)a, b_m)}{x^{u_{(\alpha_i,\beta)}}(a,0)} \\
&= \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\alpha_i}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\alpha_i}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&= \frac{(1-t_m)e_m + b_m}{a\left(\frac{a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_{m+1})e_m + b_{m+1}}{a\left(\frac{a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&= \frac{(1-t_{m+1})a\left(\frac{(1-t_{m+1})a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)} + b_{m+1}}{a\left(\frac{a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_{m+1})a\left(\frac{(1-t_{m+1})a}{\overline{\alpha}(m+1)}\right)^{1/(\beta-1)} + b_{m+1}}{a\left(\frac{a}{\overline{\alpha}(m+1)}\right)^{1/(\beta-1)}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1-t_{m+1})e_{m+1} + b_{m+1}}{a\left(\frac{a}{\overline{\alpha}(m+1)}\right)^{1/(\beta-1)}} \\
&\vdots \\
&\geq \frac{(1-t_{m'})a\left(\frac{(1-t_{m'})a}{\underline{\alpha}(m'-1)}\right)^{1/(\beta-1)} + b_{m'}}{a\left(\frac{a}{\underline{\alpha}(m'-1)}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_{m'})a\left(\frac{(1-t_{m'})a}{\alpha_{i+1}}\right)^{1/(\beta-1)} + b_{m'}}{a\left(\frac{a}{\alpha_{i+1}}\right)^{1/(\beta-1)}} \\
&= \frac{x^{u_{(\alpha_{i+1},\beta)}}(a,T)}{x^{u_{(\alpha_{i+1},\beta)}}(a,0)}.
\end{aligned}
$$

4. Concluding remarks

We have studied inequality and bipolarization reducing income tax schedules in economies with endogenous income and heterogeneous preferences. We have introduced a single crossing condition on vectors of utilities—a condition akin to the standard agent monotonicity condition of Mirrlees (1971)—ensuring that income increases with the wage rate. This property allows us to provide a full characterization of inequality and bipolarization reducing income tax schedules in terms of taxpayer preference profiles and the structure of the tax code.

A recurrent criticism for this paper's line of inquiry points to the fact that income as a measure of "welfare" disregards the welfare effects derived from the "consumption" of leisure. There are, however, good reasons not to adopt the classical welfare metric—namely utility, rather than income—and study, instead, the distributional effects of income taxes on vectors of "felicity" indices. In fact, most (if not all) measures of inequality and polarization—being "cardinal" in nature—are "incompatible" with a meaningful evaluation of utility distributions, in the sense that said measures are not, in general, invariant to order-preserving utility transformations, which do not alter consumer behavior in the neoclassical framework.

Alternatively, one might consider broader measures of "consumption," including the "value" of leisure. This approach requires, however, "comparable" metrics for the "value" of leisure across individuals.[12] For example, one might consider using the "opportunity cost" of leisure. But this is, in general, a "censored" variable, since there is no observable wage rate for those individuals who do not work.

[12] In the special case of quasilinear preferences (see, e.g., Section 3), the consumers' "value" of leisure can be measured using the same monetary units as long as individual preferences are known.

A. Proofs

A.1. Proof of Lemma 1

Lemma 1. *Suppose that $T \in \mathscr{T}$. For each wage rate distribution $0 < a_1 \leq \cdots \leq a_n$ and every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}$,*

$$x^{u_1}(a_1, T) \leq \cdots \leq x^{u_n}(a_n, T).^{13} \tag{4}$$

Proof. Given a tax schedule $T \in \mathscr{T}$, a wage rate distribution $0 < a_1 \leq \cdots \leq a_n$, and a vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}$, the inequality

$$x^{u_i}(a_i, T) \leq x^{u_{i+1}}(a_{i+1}, T), \quad \text{for } i < n,$$

follows from the two inequalities

$$x^{u_i}(a_i, T) \leq x^{u_i}(a_{i+1}, T) \quad \text{and} \quad x^{u_i}(a_{i+1}, T) \leq x^{u_{i+1}}(a_{i+1}, T).$$

The first inequality is a consequence of the Mirrlees single crossing condition, (a) (see Mirrlees, 1971, Theorem 1).

Similarly, condition (b) implies the second inequality. This is a consequence of the fact that the indifference curve for the map

$$(x, y) \in \mathbb{R}^2_{++} \mapsto u_{i+1}(x, y/a_{i+1}) \tag{7}$$

going through the bundle $(x^{u_i}(a_{i+1}, T), y^{u_i}(a_{i+1}, T))$ must lie (weakly) above the indifference curve for the map

$$(x, y) \in \mathbb{R}^2_{++} \mapsto u_i(x, y/a_{i+1}) \tag{8}$$

going though the same bundle for all points (x, y) satisfying $y \leq y^{u_i}(a_{i+1}, T)$, implying that the problem

$$\begin{aligned} &\max_{(x,y) \in \mathbb{R}_+ \times [0, a_{i+1} L]} u_{i+1}(x, y/a_{i+1}) \\ &\text{s.t. } x \leq y - T(y) \end{aligned}$$

has a solution (x^*, y^*) such that

$$y^* \geq y^{u_i}(a_{i+1}, T) \quad \text{and} \quad x^* \geq x^{u_i}(a_{i+1}, T).^{14}$$

To see this, suppose that, on the contrary, there exists a point $y' < y^{u_i}(a_{i+1}, T)$ such that

$$u_i(x', y'/a_{i+1}) = \overline{u}_i, \quad u_{i+1}(x'', y'/a_{i+1}) = \overline{u}_{i+1}, \quad \text{and } x' > x'',$$

[13] When multiple solution functions x^{u_i} exist, "$x^{u_i}(a_i, T) \leq x^{u_{i+1}}(a_{i+1}, T)$" means that for every solution function x^{u_i}, there exists a solution function $x^{u_{i+1}}$ such that $x^{u_i}(a, T) \leq x^{u_{i+1}}(a, T)$. See Footnote 5.

[14] If $y^* < y^{u_i}(a_{i+1}, T)$ for all solutions (x^*, y^*), then there exists a feasible bundle (x°, y°) with $y^\circ < y^{u_i}(a_{i+1}, T)$ strictly above the first indifference curve, and hence strictly above the second indifference curve, implying that an individual whose utility function is u_i and whose wage rate is a_{i+1} prefers (x°, y°) over $(x^{u_i}(a_{i+1}, T), y^{u_i}(a_{i+1}, T))$, a contradiction.

where $u_i(x, y/a_{i+1}) = \overline{u}_i$ (resp., $u_{i+1}(x, y/a_{i+1}) = \overline{u}_{i+1}$) represents the indifference curve for the map in (8) (resp., (7)) going through the bundle $(x^{u_i}(a_{i+1}, T), y^{u_i}(a_{i+1}, T))$.

Note that, because

$$\eta_{u_i}^{a_{i+1}}(x, y) \geq \eta_{u_{i+1}}^{a_{i+1}}(x, y), \quad \text{for all } (x, y) \in \mathbb{R}_{++} \times (0, aL),$$

the indifference curve $u_i(x, y/a_{i+1}) = \overline{u}_i$ must lie (weakly) above the curve $u_{i+1}(x, y/a_{i+1}) = \overline{u}_{i+1}$ for all $y \geq y'$. In addition, both curves intersect at $y^{u_i}(a_{i+1}, T)$.

Now let $u_i(x, y/a_{i+1}) = \hat{u}_i$ represent the equation of the indifference curve for the map

$$(x, y) \in \mathbb{R}^2_{++} \mapsto u_i(x, y/a_{i+1})$$

going through the point (x'', y'). Since $x'' < x'$, we have $\hat{u}_i < \overline{u}_i$.

Thus, we have three indifference curves,

$$u_i(x, y/a_{i+1}) = \overline{u}_i, \quad u_i(x, y/a_{i+1}) = \hat{u}_i, \quad \text{and } u_{i+1}(x, y/a_{i+1}) = \overline{u}_i,$$

satisfying the following conditions:

- $\overline{u}_i > \hat{u}_i$;
- the indifference curve $u_i(x, y/a_{i+1}) = \overline{u}_i$ must lie (weakly) above the curve $u_{i+1}(x, y/a_{i+1}) = \overline{u}_{i+1}$ for all $y \geq y'$;
- the indifference curves $u_i(x, y/a_{i+1}) = \overline{u}_i$ and $u_{i+1}(x, y/a_{i+1}) = \overline{u}_i$ intersect at $y^{u_i}(a_{i+1}, T)$;
- the indifference curves $u_i(x, y/a_{i+1}) = \hat{u}_i$ and $u_{i+1}(x, y/a_{i+1}) = \overline{u}_i$ intersect at y'; and
- because
$$\eta_{u_i}^{a_{i+1}}(x, y) \geq \eta_{u_{i+1}}^{a_{i+1}}(x, y), \quad \text{for all } (x, y) \in \mathbb{R}_{++} \times (0, aL),$$
the indifference curve $u_i(x, y/a_{i+1}) = \hat{u}_i$ must lie (weakly) above the curve $u_{i+1}(x, y/a_{i+1}) = \overline{u}_{i+1}$ for all $y \geq y'$.

Consequently, the indifference curves $u_i(x, y/a_{i+1}) = \hat{u}_i$ and $u_i(x, y/a_{i+1}) = \overline{u}_i$ must intersect, which contradicts the inequality $\overline{u}_i > \hat{u}_i$. ■

A.2. Proof of Proposition 1

Proposition 1. *Given $u \in \mathscr{U}$, $T \in \mathscr{T}_{\text{m-prog}}$, and $a > 0$, there exists $m \in \{1, \ldots, M\}$ such that one and only one of the following two conditions holds.*

1. $x^u(a, T) = x^u((1 - t_m)a, b_m)$.
2. $x^u(a, T) = (1 - t_m)e_m + b_m$ *and*
$$x^u((1 - t_{m+1})a, b_{m+1}) < (1 - t_m)e_m + b_m < x^u((1 - t_m)a, b_m).$$

Proof. Pick $u \in \mathscr{U}$, $T \in \mathscr{T}_{\text{m-prog}}$, and $a > 0$. By the condition (v),

$$y^u(a, T) > 0 = e_0.$$

If

$$e_{m-1} < y^u(a,T) < e_m, \quad \text{some } m \in \{1,\ldots,M\},$$

then

$$\eta_a^u(y^u(a,T) - T(y^u(a,T)), y^a(a,T)) = 1 - t_m, \tag{9}$$

and so

$$y^u(a,T) = y^u(a,T_m)$$

and

$$x^u(a,T) = x^u(a,T_m) = x^u((1-t_m)a, b_m). \tag{10}$$

If

$$y^u(a,T) = e_m, \quad \text{some } m \in \{1,\ldots,M\}, \tag{11}$$

then

$$1 - t_{m+1} \le \eta_a^u(y^u(a,T) - T(y^u(a,T)), y^a(a,T)) \le 1 - t_m.^{15}$$

If (11) and (9) hold, then (10) also holds.

If (11) holds and

$$\eta_a^u(y^u(a,T) - T(y^u(a,T)), y^a(a,T)) = 1 - t_{m+1},$$

then

$$x^u(a,T) = x^u(a,T_{m+1}) = x^u((1-t_{m+1})a, b_{m+1}).$$

If (11) holds and

$$1 - t_{m+1} < \eta_a^u(y^u(a,T) - T(y^u(a,T)), y^a(a,T)) < 1 - t_m,$$

then

$$y^u(a,T_{m+1}) < e_m < y^u(a,T_m),$$

whence

$$x^u((1-t_{m+1})a, b_{m+1}) = x^u(a,T_{m+1}) < (1-t_m)e_m + b_m < x^u(a,T_m) = x^u((1-t_m)a, b_m). \quad \blacksquare$$

A.3. Proof of Theorem 1

To begin, we state and prove a series of intermediate results.

Lemma 2. *Given $u \in \mathcal{U}$, $(x,y) \in \mathbb{R}^2_{++}$, and $\gamma \in (0,\infty)$, there exists $a > 0$ satisfying*

$$a > y/L \quad \text{and} \quad \eta_u^a(x,y) = \gamma.$$

[15] Indeed, given that

$$t_1 < \cdots < t_M,$$

$$1 - t_m < \eta_a^u(y^u(a,T) - T(y^u(a,T)), y^a(a,T))$$

implies $y^u(a,T) < e_m$ and

$$\eta_a^u(y^u(a,T) - T(y^u(a,T)), y^a(a,T)) < 1 - t_{m+1}$$

implies $y^u(a,T) > e_m$.

Proof. The statement follows from the Intermediate Value Theorem, since the map

$$a \mapsto \eta_u^a(x, y)$$

is continuous and, by (1),

$$\liminf_{a \downarrow y/L} \eta_u^a(x, y) = \liminf_{a \downarrow y/L} \frac{1}{a} MRS(x, y/a) = \infty,$$

and

$$\limsup_{a \to \infty} \eta_u^a(x, y) = \limsup_{a \to \infty} \frac{1}{a} MRS(x, y/a) = 0. \qquad \blacksquare$$

Lemma 3. *Given $u \in \mathscr{U}$, a linear $T \in \mathscr{T}$, and $e > 0$, there exists $a > 0$ such that $y^u(a, T) = e$.*

Proof. Choose $u \in \mathscr{U}$, a linear $T \in \mathscr{T}$, and $e > 0$. Let $t \in [0, 1)$ represent the marginal tax rate for the linear tax T. Note that, for $a > 0$ with $a > e/L$, the condition

$$\eta_u^a(e - T(e), e/a) = 1 - t$$

is sufficient for e to solve the problem

$$\max_{y \in [0, aL]} u(y - T(y), y/a).$$

By Lemma 2, there exists $a > 0$ such that

$$a > e/L \quad \text{and} \quad \eta_u^a(e - T(e), e/a) = 1 - t,$$

implying that $y^u(a, T) = e$. $\blacksquare$

Lemma 4. *Given $u \in \mathscr{U}$ and a linear $T \in \mathscr{T}$, the map $a \mapsto y^u(a, T)$ is continuous.*

Proof. Fix $u \in \mathscr{U}$ and a linear $T \in \mathscr{T}$. Note that $y^u(a, T)$ is the unique solution to the problem

$$\max_{y \in [0, aL]} u(y - T(y), y/a),$$

and so the map $a \mapsto y^u(a, T)$ is continuous by the Maximum Theorem. $\blacksquare$

Lemma 5. *Suppose that $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations. Then a tax schedule $T \in \mathscr{T}$ is $\mathbb{U}'$-ir if and only if*

$$\frac{x^{u_1}(a_1, T)}{x^{u_1}(a_1, 0)} \geq \cdots \geq \frac{x^{u_n}(a_n, T)}{x^{u_n}(a_n, 0)} \tag{12}$$

for each wage rate distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$, and every vector of income functions $(x^{u_1}, \ldots, x^{u_n})$.

Proof. Suppose that (12) holds for each wage rate distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$, and every vector of income functions $(x^{u_1}, \ldots, x^{u_n})$. We must show that T is $\mathbb{U}'$-ir, i.e., that

$$(x^{u_1}(a_1, T), \ldots, x^{u_n}(a_n, T)) \succcurlyeq_L (x^{u_1}(a_1, 0), \ldots, x^{u_n}(a_n, 0))$$

for each wage rate distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of income functions $(x^{u_1}, \ldots, x^{u_n})$, and every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$. But this follows from Marshall et al. (1967, Theorem 2.4), since $x^{u_1}(a_1, 0) > 0$ (by condition (v)), and

$$x^{u_1}(a_1, T) \leq \cdots \leq x^{u_n}(a_n, T) \quad \text{and} \quad x^{u_1}(a_1, 0) \leq \cdots \leq x^{u_n}(a_n, 0)$$

(by Lemma 1).

We now prove the contrapositive of the converse assertion. Suppose that, for some $i < n$,

$$\frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} < \frac{x^{u_{i+1}}(a', T)}{x^{u_{i+1}}(a', 0)}, \quad \text{for some } a' \geq a > 0,$$

and some vectors $(x^{u_1}, \ldots, x^{u_n})$ and $(u_1, \ldots, u_n) \in \mathbb{U}'$. We must show that T is not $\mathbb{U}'$-ir.

For the wage distribution

$$(a_1^*, \ldots, a_n^*), \quad \text{where } a_j^* = a \text{ for } j \leq i \text{ and } a_j^* = a' \text{ for } j > i,$$

and the preference profile

$$(u_1^*, \ldots, u_i^*, u_{i+1}^*, \ldots, u_n^*) = (u_i, \ldots, u_i, u_{i+1}, \ldots, u_{i+1})$$

(which, being a simple transformation of $(u_1, \ldots, u_n)$, is an element of $\mathbb{U}'$), we have

$$\frac{x^{u_1^*}(a_1^*, 0)}{x^{u_1^*}(a_1^*, T)} = \cdots = \frac{x^{u_i^*}(a_i^*, 0)}{x^{u_i^*}(a_i^*, T)} = \frac{x^{u_i}(a, 0)}{x^{u_i}(a, T)}$$
$$> \frac{x^{u_{i+1}}(a', 0)}{x^{u_{i+1}}(a', T)} = \frac{x^{u_{i+1}^*}(a_{i+1}^*, 0)}{x^{u_{i+1}^*}(a_{i+1}^*, T)} = \cdots = \frac{x^{u_n^*}(a_n^*, 0)}{x^{u_n^*}(a_n^*, T)}. \tag{13}$$

Applying Theorem 2.4 in Marshall et al. (1967), one obtains

$$(x^{u_1^*}(a_1^*, 0), \ldots, x^{u_n^*}(a_n^*, 0)) \succcurlyeq_L (x^{u_1^*}(a_1^*, T), \ldots, x^{u_n^*}(a_n^*, T)). \tag{14}$$

If

$$(x^{u_1^*}(a_1^*, 0), \ldots, x^{u_n^*}(a_n^*, 0)) \succ_L (x^{u_1^*}(a_1^*, T), \ldots, x^{u_n^*}(a_n^*, T)), \tag{15}$$

then T is not $\mathbb{U}'$-ir and the proof is complete.

To see that (15) holds, consider the following two (exhaustive) cases:

Case 1.

$$\frac{x^{u_1^*}(a_1^*, 0)}{\sum_\iota x^{u_\iota^*}(a_\iota^*, 0)} > \frac{x^{u_1^*}(a_1^*, T)}{\sum_\iota x^{u_\iota^*}(a_\iota^*, T)}.$$

In this case, (14) implies (15).

Case 2.

$$\frac{x^{u_1^*}(a_1^*, 0)}{\sum_\iota x^{u_\iota^*}(a_\iota^*, 0)} = \frac{x^{u_1^*}(a_1^*, T)}{\sum_\iota x^{u_\iota^*}(a_\iota^*, T)}.$$

In this case, since

$$x^{u_1^*}(a_1^*, 0) = \cdots = x^{u_i^*}(a_i^*, 0) \quad \text{and} \quad x^{u_1^*}(a_1^*, T) = \cdots = x^{u_i^*}(a_i^*, T),$$

it follows that

$$\frac{\sum_{\iota=1}^{i} x^{u_\iota^*}(a_\iota^*,0)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,0)} = \frac{i x^{u_1^*}(a_1^*,0)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,0)} = \frac{i x^{u_1^*}(a_1^*,T)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,T)} = \frac{\sum_{\iota=1}^{i} x^{u_\iota^*}(a_\iota^*,T)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,T)} \tag{16}$$

and

$$\frac{x^{u_i^*}(a_i^*,0)}{\sum_\iota x^{u_\iota^*}(a_\iota^*,0)} = \frac{x^{u_i^*}(a_i^*,T)}{\sum_\iota x^{u_\iota^*}(a_\iota^*,T)}.$$

Note that the last equality, together with the inequality in (13), implies that

$$\frac{x^{u_{i+1}^*}(a_{i+1}^*,T)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,T)} > \frac{x^{u_{i+1}^*}(a_{i+1}^*,0)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,0)}.$$

This inequality, together with (16), implies that

$$\frac{\sum_{\iota=1}^{i+1} x^{u_\iota^*}(a_\iota^*,0)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,0)} < \frac{\sum_{\iota=1}^{i+1} x^{u_\iota^*}(a_\iota^*,T)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,T)},$$

which contradicts (14).

Hence, (15) holds and the proof is complete. ■

Lemma 6. *For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} \Rightarrow T \in \mathscr{T}_{\text{m-prog}}.$$

Proof. For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$, suppose that $T \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$. Because $\mathbb{U}'$ is closed under simple transformations, $(u_1, \ldots, u_n) \in \mathbb{U}'$ implies that $(u_1, \ldots, u_1) \in \mathbb{U}'$, and so T is inequality reducing with respect to $\{(u_1, \ldots, u_1)\}$ whenever $(u_1, \ldots, u_n) \in \mathbb{U}'$. Applying Theorem 1 in Carbonell-Nicolau and Llavador (2018) gives $T \in \mathscr{T}_{\text{m-prog}}$. ■

Lemma 7. *For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} \Rightarrow \mathbb{U}' \subseteq \mathbb{U}_T.$$

Proof. Suppose that $\mathbb{U}' \nsubseteq \mathbb{U}_T$. Then there exists $(u_1, \ldots, u_n)$ in $\mathbb{U}' \setminus \mathbb{U}_T$. It will be shown that $T \notin \mathscr{T}_{\mathbb{U}'\text{-ir}}$. Note that, by Lemma 5, it suffices to show that there exist a wage distribution

$$0 < \alpha_1 \leq \cdots \leq \alpha_n$$

and a preference vector

$$(v_1, \ldots, v_n) \in \mathbb{U}'$$

such that

$$\frac{x^{v_i}(\alpha_i,T)}{x^{v_i}(\alpha_i,0)} < \frac{x^{v_{i+1}}(\alpha_{i+1},T)}{x^{v_{i+1}}(\alpha_{i+1},0)}, \quad \text{some } i < n.$$

Since $(u_1, \ldots, u_n) \in \mathbb{U}' \setminus \mathbb{U}_T$, either

$$\frac{x^{u_i}(a,T)}{x^{u_i}(a,0)} < \frac{x^{u_{i+1}}(a,T)}{x^{u_{i+1}}(a,0)}, \quad \text{some } i \text{ and } a, \tag{17}$$

or

$$\zeta^{u_i}((1-t_m)a, b_m) > \zeta^{u_i}(a,0), \text{ some } i, a, m \text{ such that } y^{u_i}((1-t_m)a, b_m) \in [e_{m-1}, e_m]. \quad (18)$$

Suppose first that (17) holds. Define the wage distribution

$$(\alpha_1, \ldots, \alpha_n) = (a, \ldots, a)$$

and the vector of utility functions

$$v = (v_1, \ldots, v_n) = (u_i, \ldots, u_i, u_{i+1}, \ldots, u_{i+1}).$$

Note that, because $(u_1, \ldots, u_n) \in \mathbb{U}'$, and since $\mathbb{U}'$ is closed under simple transformations, we have $v \in \mathbb{U}'$.

The above definitions, together with (17), yield

$$\frac{x^{v_i}(\alpha_i, T)}{x^{v_i}(\alpha_i, 0)} = \frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} < \frac{x^{u_{i+1}}(a, T)}{x^{u_{i+1}}(a, 0)} = \frac{x^{v_{i+1}}(\alpha_{i+1}, T)}{x^{v_{i+1}}(\alpha_{i+1}, 0)},$$

as we sought.

Next, suppose that (18) holds. Then the map

$$\beta \mapsto \frac{x^{u_i}(\beta, T_m)}{x^{u_i}(\beta, 0)} \quad (19)$$

is strictly increasing at a.[16]

Suppose first that

$$y^{u_i}((1-t_m)a, b_m) \in [e_{m-1}, e_m).$$

Since the map in (19) is strictly increasing at a, for any $a' > a$ close enough to a, we have

$$\frac{x^{u_i}(a, T_m)}{x^{u_i}(a, 0)} < \frac{x^{u_i}(a', T_m)}{x^{u_i}(a', 0)}. \quad (20)$$

By Lemma 1,

$$y^{u_i}(a', T_m) \geq y^{u_i}(a, T_m),$$

and so, by continuity of the map $\beta \mapsto y^{u_i}(\beta, T_m)$ (Lemma 4), we have, for $a' > a$ close enough to a,

$$e_{m-1} \leq y^{u_i}(a, T_m) \leq y^{u_i}(a', T_m) < e_m, \quad (21)$$

where the first and last inequalities follow from (18). Note that (21) implies that

$$y^{u_i}(a, T_m) = y^{u_i}(a, T) \quad \text{and} \quad y^{u_i}(a', T_m) = y^{u_i}(a', T),$$

whence

$$x^{u_i}(a, T_m) = x^{u_i}(a, T) \quad \text{and} \quad x^{u_i}(a', T_m) = x^{u_i}(a', T).$$

Hence, (20) yields

$$\frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} < \frac{x^{u_i}(a', T)}{x^{u_i}(a', 0)}. \quad (22)$$

[16]Indeed, $\zeta^{u_i}((1-t_m)a, b_m) \leq \zeta^{u_i}(a,0)$ if and only if the map $\beta \mapsto \frac{x^{u_i}(\beta, T_m)}{x^{u_i}(\beta, 0)}$ is nonincreasing at point a.

Define the wage distribution

$$(\alpha_1, \dots, \alpha_n) = (a, a', \dots, a')$$

and the vector of utility functions

$$\boldsymbol{v} = (v_1, \dots, v_n) = (u_i, \dots, u_i).$$

Note that, because $(u_1, \dots, u_n) \in \mathbb{U}'$, and since $\mathbb{U}'$ is closed under simple transformations, we have $\boldsymbol{v} \in \mathbb{U}'$.

The above definitions, together with (22), yield

$$\frac{x^{v_1}(\alpha_1, T)}{x^{v_1}(\alpha_1, 0)} = \frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} < \frac{x^{u_i}(a', T)}{x^{u_i}(a', 0)} = \frac{x^{v_2}(\alpha_2, T)}{x^{v_2}(\alpha_2, 0)},$$

as we sought.

It remains to consider the case when

$$y^{u_i}((1 - t_m)a, b_m) = e_m.$$

Since the map in (19) is strictly increasing at a, for any $a' < a$ close enough to a, we have

$$\frac{x^{u_i}(a, T_m)}{x^{u_i}(a, 0)} > \frac{x^{u_i}(a', T_m)}{x^{u_i}(a', 0)}. \tag{23}$$

By Lemma 1,

$$y^{u_i}(a', T_m) \leq y^{u_i}(a, T_m),$$

and so, by continuity of the map $\beta \mapsto y^{u_i}(\beta, T_m)$ (Lemma 4), we have, for $a' < a$ close enough to a,

$$e_m = y^{u_i}(a, T_m) \geq y^{u_i}(a', T_m) > e_{m-1}. \tag{24}$$

Note that (24) implies that

$$y^{u_i}(a, T_m) = y^{u_i}(a, T) \quad \text{and} \quad y^{u_i}(a', T_m) = y^{u_i}(a', T),$$

whence

$$x^{u_i}(a, T_m) = x^{u_i}(a, T) \quad \text{and} \quad x^{u_i}(a', T_m) = x^{u_i}(a', T).$$

Hence, (23) yields

$$\frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} > \frac{x^{u_i}(a', T)}{x^{u_i}(a', 0)}. \tag{25}$$

Define the wage distribution

$$(\alpha_1, \dots, \alpha_n) = (a', a, \dots, a)$$

and the vector of utility functions

$$\boldsymbol{v} = (v_1, \dots, v_n) = (u_i, \dots, u_i).$$

Note that, because $(u_1, \ldots, u_n) \in \mathbb{U}'$, and since $\mathbb{U}'$ is closed under simple transformations, we have $v \in \mathbb{U}'$. Moreover,

$$\frac{x^{v_1}(\alpha_1, T)}{x^{v_1}(\alpha_1, 0)} = \frac{x^{u_i}(a', T)}{x^{u_i}(a', 0)} < \frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} = \frac{x^{v_2}(\alpha_2, T)}{x^{v_2}(\alpha_2, 0)},$$

where the inequality follows from (25). ■

Lemma 8. *For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} \Leftarrow \left[T \in \mathscr{T}_{\text{m-prog}} \quad \textit{and} \quad \mathbb{U}' \subseteq \mathbb{U}_T\right].$$

Proof. Suppose that $T \in \mathscr{T}_{\text{m-prog}}$ and $\mathbb{U}' \subseteq \mathbb{U}_T$. By Lemma 5, it suffices to show that

$$\frac{x^{u_1}(a_1, T)}{x^{u_1}(a_1, 0)} \geq \cdots \geq \frac{x^{u_n}(a_n, T)}{x^{u_n}(a_n, 0)} \tag{26}$$

for each ability distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$, and every vector of income functions $(x^{u_1}, \ldots, x^{u_n})$.

Choose $0 < a_1 \leq \cdots \leq a_n$, $(u_1, \ldots, u_n) \in \mathbb{U}'$, and $(x^{u_1}, \ldots, x^{u_n})$.

The proof proceeds by induction on the number of brackets for T.

Suppose first that T is linear. By condition (II), we have, for each i,

$$\frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} \geq \frac{x^{u_i}(a', T)}{x^{u_i}(a', 0)}, \quad \text{whenever } a' \geq a.$$

By condition (I), we have for each $i < n$,

$$\frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} \geq \frac{x^{u_{i+1}}(a, T)}{x^{u_{i+1}}(a, 0)}, \quad \text{for all } a > 0. \tag{27}$$

Hence, for each $i < n$,

$$\frac{x^{u_i}(a_i, T)}{x^{u_i}(a_i, 0)} \geq \frac{x^{u_i}(a_{i+1}, T)}{x^{u_i}(a_{i+1}, 0)} \geq \frac{x^{u_{i+1}}(a_{i+1}, T)}{x^{u_{i+1}}(a_{i+1}, 0)},$$

implying (26), as we sought.

Now suppose that the lemma has been proven for any m-bracket tax schedule, where $m \in \{1, \ldots, M-1\}$, for some $M > 1$. It will be shown that the lemma is also true for an M-bracket tax schedule.

Suppose that T is an M-bracket tax schedule. Because T is piecewise linear in $\mathscr{T}_{\text{m-prog}}$, there exist

$$0 = e_0 < e_1 < \cdots < e_M = \infty$$

and intervals

$$I_1 = [e_0, e_1], \ldots, I_M = [e_{M-1}, e_M),$$

satisfying the following: for each m, there exist $b_m \geq 0$ and $t_m \in [0, 1)$ such that $T(y) = -b_m + t_m y$ for all $y \in I_m$, and

$$b_1 < \cdots < b_M \quad \text{and} \quad t_1 < \cdots < t_M.$$

For $m \in \{1, \ldots, M\}$, let
$$T_m(y) = -b_m + t_m y.$$

Because $(u_1, \ldots, u_n) \in \mathbb{U}$, we have

$$x^{u_1}(a_1, T_1) = b_1 + (1 - t_1)y^{u_1}(a_1, T_1) \le \cdots \le x^{u_n}(a_n, T_1) = b_1 + (1 - t_1)y^{u_n}(a_n, T_1).$$

Let i_1 be the largest i for which

$$x^{u_i}(a_i, T_1) \le b_1 + (1 - t_1)e_1.$$

Then
$$y^{u_1}(a_1, T_1) \le \cdots \le y^{u_{i_1}}(a_{i_1}, T_1) \le e_1,$$

and so
$$x^{u_i}(a_i, T_1) = x^{u_i}(a_i, T), \quad \text{for each } i \in \{1, \ldots, i_1\},$$

since T and T_1 coincide on $[e_0, e_1]$. Because T_1 is linear, the induction hypothesis implies that $T_1 \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$, and so Lemma 5 implies that

$$\frac{x^{u_1}(a_1, T)}{x^{u_1}(a_1, 0)} = \frac{x^{u_1}(a_1, T_1)}{x^{u_1}(a_1, 0)} \ge \cdots \ge \frac{x^{u_{i_1}}(a_{i_1}, T_1)}{x^{u_{i_1}}(a_{i_1}, 0)} = \frac{x^{u_{i_1}}(a_{i_1}, T)}{x^{u_{i_1}}(a_{i_1}, 0)}. \tag{28}$$

Next, let T^* be defined as follows:

$$T^*(y) = \begin{cases} T(y) & \text{if } y \ge e_1, \\ T_2(y) & \text{if } y < e_1. \end{cases}$$

It is easy to see that T is an $M - 1$-bracket tax schedule in $\mathscr{T}_{\text{m-prog}}$. Consequently, the induction hypothesis gives $T^* \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$, and so Lemma 5 implies that

$$\frac{x^{u_1}(a_1, T^*)}{x^{u_1}(a_1, 0)} \ge \cdots \ge \frac{x^{u_n}(a_n, T^*)}{x^{u_n}(a_n, 0)}. \tag{29}$$

Now let i_2 be the smallest $i \ge i_1$ for which

$$x^{u_i}(a_i, T) \ge b_2 + (1 - t_2)e_1.$$

Then Lemma 1 implies that

$$e_1 \le y^{u_{i_2}}(a_{i_2}, T) \le \cdots \le y^{u_n}(a_n, T).$$

and so
$$x^{u_i}(a_i, T^*) = x^{u_i}(a_i, T), \quad \text{for each } i \in \{i_2, \ldots, n\},$$

since T^* and T coincide on $[e_1, \infty)$. Consequently, (29) gives

$$\frac{x^{u_{i_2}}(a_{i_2}, T)}{x^{u_{i_2}}(a_{i_2}, 0)} \ge \cdots \ge \frac{x^{u_n}(a_n, T)}{x^{u_n}(a_n, 0)}. \tag{30}$$

Note that the definition of i_1 and i_2 entails

$$i_1 \le i_2 \le i_1 + 1,$$

and so, in light of (28) and (30), the proof will be complete if we show that

$$\frac{x^{u_{i_1}}(a_{i_1}, T)}{x^{u_{i_1}}(a_{i_1}, 0)} \ge \frac{x^{u_{i_2}}(a_{i_2}, T)}{x^{u_{i_2}}(a_{i_2}, 0)}. \tag{31}$$

Using (27), we see that

$$\frac{x^{u_{i_1}}(a_{i_1}, T)}{x^{u_{i_1}}(a_{i_1}, 0)} \ge \frac{x^{u_{i_2}}(a_{i_1}, T)}{x^{u_{i_2}}(a_{i_1}, 0)}. \tag{32}$$

Consequently, it suffices to show that

$$\frac{x^{u_{i_2}}(a_{i_1}, T)}{x^{u_{i_2}}(a_{i_1}, 0)} \ge \frac{x^{u_{i_2}}(a_{i_2}, T)}{x^{u_{i_2}}(a_{i_2}, 0)}, \tag{33}$$

since this inequality, combined with (32), gives (31).

Since $a_{i_1} \le a_{i_2}$, we have

$$y^{u_{i_2}}(a_{i_1}, T) \le y^{u_{i_2}}(a_{i_2}, T)$$

(see Lemma 1).

Let m be the bracket for the gross income $y^{u_{i_2}}(a_{i_2}, T)$, i.e.,

$$y^{u_{i_2}}(a_{i_2}, T) \in [e_{m-1}, e_m].$$

Suppose first that there is no m' such that

$$y^{u_{i_2}}(a_{i_1}, T) \le e_{m'} \le y^{u_{i_2}}(a_{i_2}, T).$$

In this case,

$$e_{m-1} < y^{u_{i_2}}(a_{i_1}, T) = y^{u_{i_2}}(a_{i_1}, T_m) \le y^{u_{i_2}}(a_{i_2}, T_m) = y^{u_{i_2}}(a_{i_2}, T) < e_m.$$

Since T_m is linear, we know that $T_m \in \mathcal{T}_{\mathrm{U'\text{-}ir}}$, and so Lemma 5 implies (33).[17]

Now suppose that there are exactly k thresholds

$$y^{u_{i_2}}(a_{i_1}, T) \le e_{m_1} < \cdots < e_{m_k} \le y^{u_{i_2}}(a_{i_2}, T) \tag{34}$$

between $y^{u_{i_2}}(a_{i_1}, T)$ and $y^{u_{i_2}}(a_{i_2}, T)$, for some $k \in \{1, \ldots, M-1\}$. Suppose further that (33) has been proven when the number of thresholds between $y^{u_{i_2}}(a_{i_1}, T)$ and $y^{u_{i_2}}(a_{i_2}, T)$ is less than k. It will be shown that (33) holds.

First, we show that there exist $\alpha \le \alpha'$ such that

$$y^{u_{i_2}}(\alpha, T_{m_1}) = e_{m_1} = y^{u_{i_2}}(\alpha', T_{m_1+1}). \tag{35}$$

[17] This can be seen by applying Lemma 5 to the wage rate distribution $(a_{i_1}, a_{i_2}, \ldots, a_{i_2})$ and the preference vector $(u_{i_2}, \ldots, u_{i_2})$.

The existence of α and α' satisfying (35) follows from Lemma 3. To see that $\alpha \leq \alpha'$, it suffices to observe that

$$\eta_{u_{i_2}}^{\alpha}(x^{u_{i_2}}(\alpha, T_{m_1}), y^{u_{i_2}}(\alpha, T_{m_1})) = 1 - t_{m_1} > 1 - t_{m_1+1} = \eta_{u_{i_2}}^{\alpha'}(x^{u_{i_2}}(\alpha', T_{m_1+1}), y^{u_{i_2}}(\alpha', T_{m_1+1})),$$

implying that

$$y^{u_{i_2}}(\alpha, T_{m_1}) < y^{u_{i_2}}(\alpha', T_{m_1}),$$

whence $\alpha \leq \alpha'$ (by Lemma 1).

Next, observe that

$$y^{u_{i_2}}(\beta, T) = e_{m_1}, \quad \text{for all } \beta \in [\alpha, \alpha']. \tag{36}$$

Indeed, (35) implies that

$$y^{u_{i_2}}(\alpha, T) = e_{m_1} = y^{u_{i_2}}(\alpha', T),$$

and so Lemma 1 implies (36).

Note that, since $x^{u_{i_2}}(\alpha, 0) \leq x^{u_{i_2}}(\alpha', 0)$ (by Lemma 1), (36) implies that

$$\frac{x^{u_{i_2}}(\alpha, T)}{x^{u_{i_2}}(\alpha, 0)} = \frac{(1 - t_{m_1})e_{m_1} + b_{m_1}}{x^{u_{i_2}}(\alpha, 0)} \geq \frac{(1 - t_{m_1})e_{m_1} + b_{m_1}}{x^{u_{i_2}}(\alpha', 0)} = \frac{x^{u_{i_2}}(\alpha', T)}{x^{u_{i_2}}(\alpha', 0)}. \tag{37}$$

We are now ready to prove (33). First, consider the case when

$$a_{i_1} \leq \alpha \leq \alpha' \leq a_{i_2}. \tag{38}$$

Since $a_{i_1} \leq \alpha$, we have

$$e_{m_1 - 1} < y^{u_{i_2}}(a_{i_1}, T) \leq y^{u_{i_2}}(\alpha, T) = e_{m_1},$$

implying that

$$y^{u_{i_2}}(a_{i_1}, T) = y^{u_{i_2}}(a_{i_1}, T_{m_1}) \quad \text{and} \quad y^{u_{i_2}}(\alpha, T) = y^{u_{i_2}}(\alpha, T_{m_1}).$$

Since T_{m_1} is linear, we know that $T_{m_1} \in \mathscr{T}_{\mathrm{U'\text{-}ir}}$, and so Lemma 5 implies that

$$\frac{x^{u_{i_2}}(a_{i_1}, T)}{x^{u_{i_2}}(a_{i_1}, 0)} \geq \frac{x^{u_{i_2}}(\alpha, T)}{x^{u_{i_2}}(\alpha, 0)}. \tag{39}$$

Combining this inequality with (37) gives

$$\frac{x^{u_{i_2}}(a_{i_1}, T)}{x^{u_{i_2}}(a_{i_1}, 0)} \geq \frac{x^{u_{i_2}}(\alpha', T)}{x^{u_{i_2}}(\alpha', 0)}. \tag{40}$$

If $\alpha' = a_{i_2}$, we see that (33) holds.

It remains to consider the case when $\alpha' < a_{i_2}$. If

$$x^{u_{i_2}}(\alpha', T) = x^{u_{i_2}}(a_{i_2}, T),$$

then

$$\frac{x^{u_{i_2}}(a_{i_1}, T)}{x^{u_{i_2}}(a_{i_1}, 0)} \geq \frac{x^{u_{i_2}}(\alpha', T)}{x^{u_{i_2}}(\alpha', 0)} = \frac{x^{u_{i_2}}(a_{i_2}, T)}{x^{u_{i_2}}(\alpha', 0)} \geq \frac{x^{u_{i_2}}(a_{i_2}, T)}{x^{u_{i_2}}(a_{i_2}, 0)},$$

where the first inequality uses (40) and the last inequality follows from the inequality

$$x^{u_{i_2}}(\alpha', 0) \leq x^{u_{i_2}}(a_{i_2}, 0),$$

which is implied by Lemma 1. Thus, (33) holds.

Now suppose that

$$x^{u_{i_2}}(\alpha', T) < x^{u_{i_2}}(a_{i_2}, T).$$

By Lemma 3, there exists $\beta \in (\alpha', a_{i_2})$ close enough to α' such that

$$y^{u_{i_2}}(\alpha', T) = e_{m_1} < y^{u_{i_2}}(\beta, T) < e_{m_1+1}. \tag{41}$$

Note that

$$y^{u_{i_2}}(\alpha', T) = y^{u_{i_2}}(\alpha', T_{m_1+1}) \quad \text{and} \quad y^{u_{i_2}}(\beta, T) = y^{u_{i_2}}(\beta, T_{m_1+1}).$$

Since T_{m_1+1} is linear, we obtain

$$\frac{x^{u_{i_2}}(\alpha', T)}{x^{u_{i_2}}(\alpha', 0)} \geq \frac{x^{u_{i_2}}(\beta, T)}{x^{u_{i_2}}(\beta, 0)}.$$

Combining this inequality with (40) gives

$$\frac{x^{u_{i_2}}(a_{i_1}, T)}{x^{u_{i_2}}(a_{i_1}, 0)} \geq \frac{x^{u_{i_2}}(\beta, T)}{x^{u_{i_2}}(\beta, 0)}. \tag{42}$$

Next, note that (41) and (34) imply

$$e_{m_1} < y^{u_{i_2}}(\beta, T) \leq e_{m_1+1} < \cdots < e_{m_k} \leq y^{u_{i_2}}(a_{i_2}, T),$$

i.e., the number of thresholds between $y^{u_{i_2}}(\beta, T)$ and $y^{u_{i_2}}(a_{i_2}, T)$ is less than k. Consequently,

$$\frac{x^{u_{i_2}}(\beta, T)}{x^{u_{i_2}}(\beta, 0)} \geq \frac{x^{u_{i_2}}(a_{i_2}, T)}{x^{u_{i_2}}(a_{i_2}, 0)}.$$

This inequality, together with (42), gives (33), as we sought.

Next, consider the case when

$$a_{i_1} \leq a_{i_2} \leq \alpha.$$

Then

$$e_{m_1-1} < y^{u_{i_2}}(a_{i_1}, T) \leq y^{u_{i_2}}(a_{i_2}, T) \leq y^{u_{i_2}}(\alpha, T) = e_{m_1},$$

implying that

$$y^{u_{i_2}}(a_{i_1}, T) = y^{u_{i_2}}(a_{i_1}, T_{m_1}) \quad \text{and} \quad y^{u_{i_2}}(a_{i_2}, T) = y^{u_{i_2}}(a_{i_2}, T_{m_1}).$$

Since T_{m_1} is linear, we obtain (33).

It remains to consider the case when

$$a_{i_1} \leq \alpha \leq a_{i_2} \leq \alpha'.$$

This implies (39), as in the first case (38). To see that (33) holds, note that

$$\frac{x^{u_{i_2}}(a_{i_1},T)}{x^{u_{i_2}}(a_{i_1},0)} \geq \frac{x^{u_{i_2}}(\alpha,T)}{x^{u_{i_2}}(\alpha,0)} = \frac{x^{u_{i_2}}(a_{i_2},T)}{x^{u_{i_2}}(\alpha,0)} \geq \frac{x^{u_{i_2}}(a_{i_2},T)}{x^{u_{i_2}}(a_{i_2},0)},$$

where the first inequality uses (39), the equality follows from (36), and the last inequality is a consequence of the inequality $x^{u_{i_2}}(\alpha,0) \leq x^{u_{i_2}}(a_{i_2},0)$ (which follows from Lemma 1). ■

We are now ready to prove Theorem 1.

Theorem 1. *For* $\mathbb{U}' \subseteq \mathbb{U}$*, where* $\mathbb{U}'$ *is closed under simple transformations, and* $T \in \mathscr{T}$*,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} \Leftrightarrow \left[T \in \mathscr{T}_{\text{m-prog}} \quad \textit{and} \quad \mathbb{U}' \subseteq \mathbb{U}_T\right].$$

Proof. The equivalence is an immediate consequence of Lemma 6, Lemma 7, and Lemma 8. ■

A.4. Proof of Theorem 2

Theorem 2. *If* $\mathbb{U}' \subseteq \mathbb{U}$ *is closed under simple transformations, then* $\mathscr{T}_{\mathbb{U}'\text{-ir}} = \mathscr{T}_{\mathbb{U}'\text{-bpr}}$*.*

Proof. Suppose that $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations. First, we prove the containment $\mathscr{T}_{\mathbb{U}'\text{-ir}} \subseteq \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

Pick $T \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$, $0 < a_1 \leq \cdots \leq a_n$, $(u_1,\ldots,u_n) \in \mathbb{U}'$, and a vector of income functions $(x^{u_1},\ldots,x^{u_n})$.

First, suppose that n is odd. Let a_m denote the median ability level. For $i < m$, we have

$$\begin{aligned}
&\frac{1}{x^{u_m}(a_m,T)}(x^{u_m}(a_m,T) - x^{u_i}(a_i,T)) \\
&\qquad = \frac{1}{x^{u_m}(a_m,0)}\left[x^{u_m}(a_m,0) - \left(\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \cdot \frac{x^{u_m}(a_m,0)}{x^{u_m}(a_m,T)}\right)x^{u_i}(a_i,0)\right] \\
&\qquad \leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_m}(a_m,0) - x^{u_i}(a_i,0)),
\end{aligned}$$

where the last inequality follows from the inequality

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \geq \frac{x^{u_m}(a_m,T)}{x^{u_m}(a_m,0)},$$

which holds because $a_i \leq a_m$ and T is $\mathbb{U}'$-ir (see Lemma 5).

Similarly, for $i > m$, we have

$$\begin{aligned}
&\frac{1}{x^{u_m}(a_m,T)}(x^{u_i}(a_i,T) - x^{u_m}(a_m,T)) \\
&\qquad = \frac{1}{x^{u_m}(a_m,0)}\left[\left(\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \cdot \frac{x^{u_m}(a_m,0)}{x^{u_m}(a_m,T)}\right)x^{u_i}(a_i,0) - x^{u_m}(a_m,0)\right] \\
&\qquad \leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_i}(a_i,0) - x^{u_m}(a_m,0)),
\end{aligned}$$

where the last inequality follows from the inequality

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \leq \frac{x^{u_m}(a_m,T)}{x^{u_m}(a_m,0)},$$

which holds because $a_m \leq a_i$ and T is $\mathbb{U}'$-ir (see Lemma 5).

Because

$$\frac{1}{x^{u_m}(a_m,T)}(x^{u_m}(a_m,T) - x^{u_i}(a_i,T)) \leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_m}(a_m,0) - x^{u_i}(a_i,0)), \quad \text{for } i < n,$$
$$\frac{1}{x^{u_m}(a_m,T)}(x^{u_i}(a_i,T) - x^{u_m}(a_m,T)) \leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_i}(a_i,0) - x^{u_m}(a_m,0)), \quad \text{for } i > n,$$

we see that

$$\frac{x^{u_m}(a_m,T)}{x^{u_m}(a_m,0)}(x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0)) \succcurlyeq_{FW} (x^{u_1}(a_1,T),\ldots,x^{u_n}(a_n,T)),$$

whence

$$(x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0)) \succcurlyeq_{FW} (x^{u_1}(a_1,T),\ldots,x^{u_n}(a_n,T)).$$

Consequently, $T \in \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

Next, suppose that n is even. Let

$$m = m(x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0)) = \frac{x^{u_{n/2}}(a_{n/2},0) + x^{u_{(n/2)+1}}(a_{(n/2)+1},0)}{2}$$

and

$$m' = m(x^{u_1}(a_1,T),\ldots,x^{u_n}(a_n,T)) = \frac{x^{u_{n/2}}(a_{n/2},T) + x^{u_{(n/2)+1}}(a_{(n/2)+1},T)}{2}.$$

For $i \leq n/2$, we have

$$\begin{aligned}\frac{1}{m'}(m' - x^{u_i}(a_i,T)) &= \frac{1}{m}\left[m - \left(\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)}\frac{m}{m'}\right)x^{u_i}(a_i,0)\right]\\ &\leq \frac{1}{m}(m - x^{u_i}(a_i,0)),\end{aligned}$$

where the last inequality follows from the inequalities

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \geq \frac{x^{u_{n/2}}(a_{n/2},T)}{x^{u_{n/2}}(a_{n/2},0)} \geq \frac{m'}{m};$$

the first inequality holds because $a_i \leq a_{n/2}$ and T is $\mathbb{U}'$-ir (see Lemma 5); the second inequality is expressible as

$$\frac{x^{u_{n/2}}(a_{n/2},T)}{x^{u_{n/2}}(a_{n/2},0)} \geq \frac{x^{u_{(n/2)+1}}(a_{(n/2)+1},T)}{x^{u_{(n/2)+1}}(a_{(n/2)+1},0)},$$

which holds because $a_{n/2} \leq a_{(n/2)+1}$ and T is $\mathbb{U}'$-ir (see Lemma 5).

Because

$$\frac{1}{m'}(m' - x^{u_i}(a_i, T)) \leq \frac{1}{m}(m - x^{u_i}(a_i, 0)), \quad \text{for } i \leq n/2,$$
$$\frac{1}{m'}(x^{u_i}(a_i, T) - m') \leq \frac{1}{m}(x^{u_i}(a_i, 0) - m), \quad \text{for } i \geq (n/2) + 1,$$

we have

$$(x^{u_1}(a_1, 0), \dots, x^{u_n}(a_n, 0)) \succcurlyeq_{FW} (x^{u_1}(a_1, T), \dots, x^{u_n}(a_n, T)).$$

Consequently, $T \in \mathcal{T}_{\mathbb{U}'\text{-bpr}}$.

It remains to prove the containment $\mathcal{T}_{\mathbb{U}'\text{-ir}} \supseteq \mathcal{T}_{\mathbb{U}'\text{-bpr}}$.

Choose $T \in \mathcal{T}_{\mathbb{U}'\text{-bpr}}$, $0 < a_1 \leq \cdots \leq a_n$, $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathbb{U}'$, and a vector of income functions $(x^{u_1}, \dots, x^{u_n})$.

First, suppose that n is odd. Pick $i < n$ and a_i, and define an ability distribution $0 < a'_1 \leq \cdots \leq a'_n$ satisfying

$$a'_j = a_i \text{ for each } j \leq i \quad \text{and} \quad a'_j = a_{i+1} \text{ for each } j \geq i + 1.$$

Note that

$$a'_{m-1} = a_i \leq a'_m \leq a_{i+1} = a'_{m+1},$$

where a'_m represents the median ability level. Moreover, either $a'_m = a_i$ or $a'_m = a_{i+1}$. Suppose that $a'_m = a_{i+1}$ (the other case can be handled similarly).

Because $\mathbb{U}'$ is closed under simple transformations, the utility vector $\boldsymbol{u}' = (u'_1, \dots, u'_n)$, where

$$\begin{aligned} u'_j &= u_i, && \text{for each } j \leq i, \\ u'_j &= u_{i+1}, && \text{for each } j \geq i + 1, \end{aligned}$$

is a member of $\mathbb{U}'$.

Because T is $\mathbb{U}'$-bpr,

$$\frac{1}{x^{u'_m}(a'_m, T)} \sum_{j=i}^{m} (x^{u'_m}(a'_m, T) - x^{u'_j}(a'_j, T)) \leq \frac{1}{x^{u'_m}(a'_m, 0)} \sum_{j=i}^{m} (x^{u'_m}(a'_m, 0) - x^{u'_j}(a'_j, 0)). \tag{43}$$

Since $a'_m = a_{i+1}$, $a'_j = a_{i+1}$, and $u'_j = u_{i+1}$ for $j \geq i + 1$, we have

$$\sum_{j=i+1}^{m} (x^{u'_m}(a'_m, T) - x^{u'_j}(a'_j, T)) = 0 \quad \text{and} \quad \sum_{j=i+1}^{m} (x^{u'_m}(a'_m, 0) - x^{u'_j}(a'_j, 0) = 0.$$

Consequently, (43) can be expressed as

$$\frac{1}{x^{u'_m}(a'_m, T)}(x^{u'_m}(a'_m, T) - x^{u'_i}(a'_i, T)) \leq \frac{1}{x^{u'_m}(a'_m, 0)}(x^{u'_m}(a'_m, 0) - x^{u'_i}(a'_i, 0)),$$

whence

$$\frac{x^{u'_i}(a'_i, T)}{x^{u'_i}(a'_i, 0)} \geq \frac{x^{u'_m}(a'_m, T)}{x^{u'_m}(a'_m, 0)}.$$

Now since $u'_i = u_i$, $u'_m = u_{i+1}$, $a'_i = a_i$, and $a'_m = a_{i+1}$, it follows that

$$\frac{x^{u_i}(a_i, T)}{x^{u_i}(a_i, 0)} \geq \frac{x^{u_{i+1}}(a_{i+1}, T)}{x^{u_{i+1}}(a_{i+1}, 0)}.$$

Since $i < n$ was arbitrary, we see that

$$\frac{x^{u_1}(a_1, T)}{x^{u_1}(a_1, 0)} \geq \cdots \geq \frac{x^{u_n}(a_n, T)}{x^{u_n}(a_n, 0)}.$$

Since $0 < a_1 \leq \cdots \leq a_n$, $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{U}'$, and $(x^{u_1}, \ldots, x^{u_n})$ were arbitrary, Lemma 5 implies that $T \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$.

Next, suppose that n is even. Pick $i < n$ and a_i, and define an ability distribution $0 < a'_1 \leq \cdots \leq a'_n$ satisfying

$$a'_j = a_i \text{ for each } j \leq i \quad \text{and} \quad a'_j = a_{i+1} \text{ for each } j \geq i + 1.$$

Because $\mathbb{U}'$ is closed under simple transformations, the utility vector $\boldsymbol{u}' = (u'_1, \ldots, u'_n)$, where

$$\begin{aligned} u'_j &= u_i, && \text{for each } j \leq i, \\ u'_j &= u_{i+1}, && \text{for each } j \geq i + 1, \end{aligned}$$

is a member of $\mathbb{U}'$.

Note that the income distributions

$$(x^{u'_1}(a'_1, T), \ldots, x^{u'_n}(a'_n, T)) \quad \text{and} \quad (x^{u'_1}(a'_1, 0), \ldots, x^{u'_n}(a'_n, 0))$$

satisfy

$$x^{u'_1}(a'_1, T) = \cdots = x^{u'_i}(a'_i, T) \leq m' \leq x^{u'_{i+1}}(a'_{i+1}, T) = \cdots = x^{u'_n}(a'_n, T), \tag{44}$$

where m' represents the median income for $(x^{u'_1}(a'_1, T), \ldots, x^{u'_n}(a'_n, T))$, and

$$x^{u'_1}(a'_1, 0) = \cdots = x^{u'_i}(a'_i, 0) \leq m \leq x^{u'_{i+1}}(a'_{i+1}, 0) = \cdots = x^{u'_n}(a'_n, 0), \tag{45}$$

where m represents the median income for $(x^{u'_1}(a'_1, 0), \ldots, x^{u'_n}(a'_n, 0))$.

As in the previous case, it suffices to show that

$$\frac{x^{u_i}(a_i, T)}{x^{u_i}(a_i, 0)} \geq \frac{x^{u_{i+1}}(a_{i+1}, T)}{x^{u_{i+1}}(a_{i+1}, 0)}.$$

If

$$x^{u'_1}(a'_1, T) = \cdots = x^{u'_i}(a'_i, T) = m' = x^{u'_{i+1}}(a'_{i+1}, T) = \cdots = x^{u'_n}(a'_n, T), \tag{46}$$

then

$$x^{u'_1}(a'_1, 0) = \cdots = x^{u'_i}(a'_i, 0) = m = x^{u'_{i+1}}(a'_{i+1}, 0) = \cdots = x^{u'_n}(a'_n, 0). \tag{47}$$

Indeed, $x^{u'_i}(a'_i,0) < x^{u'_{i+1}}(a'_{i+1},0)$ implies that $x^{u'_i}(a'_i,T) < x^{u'_{i+1}}(a'_{i+1},T)$, since marginal tax rates are less than unity. Under (46)-(47), we have

$$\frac{x^{u'_i}(a'_i,T)}{x^{u'_i}(a'_i,0)} = \frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} = \frac{m'}{m} = \frac{x^{u_{i+1}}(a_{i+1},T)}{x^{u_{i+1}}(a_{i+1},0)} = \frac{x^{u'_{i+1}}(a'_{i+1},T)}{x^{u'_{i+1}}(a'_{i+1},0}.$$

If $x^{u'_i}(a'_i,T) < x^{u'_{i+1}}(a'_{i+1},T)$ then $x^{u'_i}(a'_i,0) < x^{u'_{i+1}}(a'_{i+1},0)$. Thus, if $m' = x^{u'_i}(a'_i,T)$ (resp., $m' = x^{u'_{i+1}}(a'_{i+1},T)$), then $m = x^{u'_i}(a'_i,0)$ (resp., $m = x^{u'_{i+1}}(a'_{i+1},0)$). We consider the case when $m' = x^{u'_i}(a'_i,T)$ and $m = x^{u'_i}(a'_i,0)$ and omit the other case, which can be handled similarly.

Suppose that $m' = x^{u'_i}(a'_i,T)$ and $m = x^{u'_i}(a'_i,0)$. Because T is $\mathbb{U}'$-bpr,

$$\frac{1}{m'} \sum_{\frac{n+1}{2}<j\leq i+1} (x^{u'_j}(a'_j,T) - m') \leq \frac{1}{m} \sum_{\frac{n+1}{2}<j\leq i+1} (x^{u'_j}(a'_j,0) - m). \tag{48}$$

Given (44)-(45), and since $m' = x^{u'_i}(a'_i,T)$ and $m = x^{u'_i}(a'_i,0)$, we see that (48) can be expressed as

$$\frac{1}{m'}(x^{u'_{i+1}}(a'_{i+1},T) - m') \leq \frac{1}{m}(x^{u'_{i+1}}(a'_{i+1},0) - m),$$

whence

$$\frac{x^{u'_i}(a'_i,T)}{x^{u'_i}(a'_i,0)} = \frac{m'}{m} \geq \frac{x^{u'_{i+1}}(a'_{i+1},T)}{x^{u'_{i+1}}(a'_{i+1},0)}.$$

Now since $u'_i = u_i$ and $u'_{i+1} = u_{i+1}$, it follows that

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \geq \frac{x^{u_{i+1}}(a_{i+1},T)}{x^{u_{i+1}}(a_{i+1},0)},$$

as we sought. ■

References

Allingham, M. (1979) "Inequality and progressive taxation: An example," *Journal of Public Economics*, Vol. 11, pp. 273–274, DOI: http://dx.doi.org/10.1016/0047-2727(79)90009-4.

Carbonell-Nicolau, Oriol (2019) "On the distributional effects of commodity taxation," *Journal of Public Economic Theory*, Vol. 21, pp. 687–707, DOI: http://dx.doi.org/10.1111/jpet.12370.

——— (2024) "Inequality and Bipolarization Reducing Mixed Taxation," mimeograph.

Carbonell-Nicolau, Oriol and Humberto Llavador (2018) "Inequality reducing properties of progressive income tax schedules: The case of endogenous income," *Theoretical Economics*, Vol. 13, pp. 39–60, DOI: http://dx.doi.org/10.3982/TE2533.

——— (2021a) "Inequality, Bipolarization, and Tax Progressivity," *American Economic Journal: Microeconomics*, Vol. 13, pp. 492–513, DOI: http://dx.doi.org/10.1257/mic.20190111.

——— (2021b) "Elasticity determinants of inequality-reducing income taxation," *The Journal of Economic Inequality*, Vol. 19, pp. 163–183, DOI: http://dx.doi.org/10.1007/s10888-020-09461-8.

Chakravarty, S. R. (2009) *Inequality, Polarization and Poverty*: Springer Netherlands, pp.1–178, DOI: http://dx.doi.org/10.1007/978-0-387-79253-8.

——— (2015) *Inequality, Polarization and Conflict: An Analytical Study*, Economic Studies in Inequality, Social Exclusion and Well-Being: Springer, DOI: http://dx.doi.org/10.1007/978-81-322-2166-1.

Ebert, U. and P. Moyes (2000) "Consistent income tax structures when households are heterogeneous," *Journal of Economic Theory*, Vol. 90, pp. 116–150, DOI: http://dx.doi.org/10.1006/jeth.1999.2582.

——— (2003) "The difficulty of income redistribution with labour supply," *Economics Bulletin*, Vol. 8, pp. 1–9.

——— (2007) "Income taxation with labor responses," *Journal of Public Economic Theory*, Vol. 9, pp. 653–682, DOI: http://dx.doi.org/10.1111/j.1467-9779.2007.00324.x.

Eichhorn, W., H. Funke, and W. F. Richter (1984) "Tax progression and inequality of income distribution," *Journal of Mathematical Economics*, Vol. 13, pp. 127–131, DOI: http://dx.doi.org/10.1016/0304-4068(84)90012-0.

Fellman, J. (1976) "The effect of transformations of Lorenz curves," *Econometrica*, Vol. 44, pp. 823–824, DOI: http://dx.doi.org/10.2307/1913450.

Formby, John P., W. James Smith, and David Sykes (1986) "Income Redistribution and Local Tax Progressivity: A Reconsideration," *The Canadian Journal of Economics*, Vol. 19, pp. 808–811.

Foster, J. E. and M. C. Wolfson (2010) "Polarization and the decline of the middle class: Canada and the U.S.," *Journal of Economic Inequality*, Vol. 8, pp. 247–273, DOI: http://dx.doi.org/10.1007/s10888-009-9122-7.

Hemming, R. and M. J. Keen (1983) "Single-crossing conditions in comparisons of tax progressivity," *Journal of Public Economics*, Vol. 20, pp. 373–380, DOI: http://dx.doi.org/10.1016/0047-2727(83)90032-4.

Jakobsson, U. (1976) "On the measurement of the degree of progression," *Journal of Public Economics*, Vol. 5, pp. 161–168, DOI: http://dx.doi.org/10.1016/0047-2727(76)90066-9.

Ju, B.-G. and J. D. Moreno-Ternero (2008) "On the equivalence between progressive taxation and inequality reduction," *Social Choice and Welfare*, Vol. 30, pp. 561–569, DOI: http://dx.doi.org/10.1007/s00355-007-0254-z.

Kakwani, N. C. (1977) “Applications of Lorenz curves in economic analysis,” *Econometrica*, Vol. 45, pp. 719–728, DOI: http://dx.doi.org/10.2307/1911684.

Kakwani, Nanak and Hyun Hwa Son (2021) “Normative measures of tax progressivity: An international comparison,” *Journal of Economic Inequality*, Vol. 19, pp. 185–212.

Latham, R. (1988) “Lorenz-dominating income tax functions,” *International Economic Review*, Vol. 29, pp. 185–198, DOI: http://dx.doi.org/10.2307/2526818.

Le Breton, M., P. Moyes, and A. Trannoy (1996) “Inequality reducing properties of composite taxation,” *Journal of Economic Theory*, Vol. 69, pp. 71–103, DOI: http://dx.doi.org/10.1006/jeth.1996.0038.

Liu, P.-W. (1985) “Lorenz domination and global tax progressivity,” *Canadian Journal of Economics*, Vol. 18, pp. 395–399, DOI: http://dx.doi.org/10.2307/135143.

Marshall, Albert W., Ingram Olkin, and Frank Proschan (1967) “Monotonicity of Ratios and Meand and Other Applications of Majorization,” in Oven Shisa ed. *Inequalities: Proceedings*, Vol. II, New York: Academic Press, pp. 177–190.

Mirrlees, J. A. (1971) “Exploration in the theory of optimum income taxation,” *Review of Economic Studies*, Vol. 38, pp. 175–208, DOI: http://dx.doi.org/10.2307/2296779.

Moyes, P. (1988) “A note on minimally progressive taxation and absolute income inequality,” *Social Choice and Welfare*, Vol. 5, pp. 227–234, DOI: http://dx.doi.org/10.1007/BF00735763.

——— (1989) “Some classes of functions that preserve the inequality and welfare orderings of income distributions,” *Journal of Economic Theory*, Vol. 49, pp. 347–359, DOI: http://dx.doi.org/https://doi.org/10.1016/0022-0531(89)90087-2.

——— (1994) “Inequality reducing and inequality preserving transformations of incomes: Symmetric and individualistic transformations,” *Journal of Economic Theory*, Vol. 63, pp. 271–298, DOI: http://dx.doi.org/10.1006/jeth.1994.1043.

Seade, J. (1982) “On the sign of the optimum marginal income tax,” *Review of Economic Studies*, Vol. 49, pp. 637–643, DOI: http://dx.doi.org/10.2307/2297292.

Thistle, P. D. (1988) “Uniform progressivity, residual progression, and single-crossing,” *Journal of Public Economics*, Vol. 37, pp. 121–126, DOI: http://dx.doi.org/10.1016/0047-2727(88)90009-6.

Thon, D. (1987) “Redistributive properties of progressive taxation,” *Mathematical Social Sciences*, Vol. 14, pp. 185–191, DOI: http://dx.doi.org/10.1016/0165-4896(87)90021-7.

Wang, You-Qiang and Kai-Yuen Tsui (2000) “Polarization orderings and new classes of polarization indices,” *Journal of Public Economic Theory*, Vol. 2, pp. 349–363, DOI: http://dx.doi.org/10.1111/1097-3923.00042.

Wolfson, Michael C. (1994) “When Inequalities Diverge,” *American Economic Review: Papers and Proceedings*, Vol. 353, pp. 353–358.

Zoli, Claudio (2018) “A note on progressive taxation and inequality equivalence,” in J.A. Bishop and J.G. Rodríguez eds. *Inequality, Taxation and Intergenerational Transmission (Research on Economic Inequality, Vol. 26)*: Emerald Publishing Limited, pp. 15–33.