Measuring Hierarchy with Multiple Supervisors

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26 May 2025

Abstract

This paper presents a robust framework for measuring hierarchical structures in modern organizations, where multiple supervisors and non-traditional arrangements, such as matrix structures, are common. Building on prior research, we develop a hierarchy measurement approach using directed acyclic graphs, which accommodate these complex configurations. Our approach is grounded in three core axioms—Anonymity, Subordination Removal, and the Replication Principle—which uniquely define a hierarchical pre-order for ranking organizational structures. Through a structured process of subordination removal, we provide a formal characterization of hierarchical dominance, yielding a consistent and practical hierarchy metric. This framework facilitates systematic comparisons across diverse organizational hierarchies.

Keywords: hierarchy measurement, hierarchical pre-order, hierarchical index, directed acyclic graph.
JEL classifications: D23, L22.

1. Introduction

As documented in the economic literature, organizational hierarchical structures are not merely circumstantial; they are an essential determinant of income distribution, economic power concentration, and systemic inequality. Therefore, understanding and systematically measuring hierarchical structures is crucial for quantifying the power dynamics shaping resource distribution and individual and collective economic outcomes.

The correlation between hierarchical rank and income is remarkably consistent across diverse organizations. As evidenced by case studies, income scales strongly with hierarchical rank within firms, exerting a more significant influence on earnings than any other measurable factor. This relationship follows a predictable pattern, where income increases exponentially with rank, forming the foundation for power-law distributions observed among top incomes across the economy (Fix, 2018, 2021).

Research also suggests that societies with higher energy consumption tend to develop larger, more hierarchical institutions (Fix, 2017). Hierarchical complexity rises with energy, concentrating resources at the top. By examining the "energy-hierarchy-inequality" relationship, Fix (2019) traces the origins of inequality, revealing how hierarchical structures emerged alongside increasing energy consumption and technological advancement.

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From a theoretical perspective, Simon (1957) and Lydall (1959) pioneered models demonstrating how hierarchical structures generate power-law income distributions, which characterize economic inequality. In a stylized framework with a fixed ratio between the number of supervisors in each grade and the number of immediate subordinates, they showed that when two exponential trends—income growth with hierarchical rank and decreasing positions at higher ranks—intersect, a power-law distribution naturally emerges.

As an initial step toward systematically exploring the theoretical connections between hierarchy and the determinants of power and resource distribution, Carbonell-Nicolau (2025) established a foundational framework for measuring hierarchy, specifically tailored to tree-like hierarchical structures. The present paper builds on this methodology, extending the analysis to encompass more complex hierarchical systems. Our work broadens the scope to include hierarchies where individuals may have multiple immediate supervisors, a characteristic prevalent in modern hierarchies such as matrix organizational structures.

Matrix organizations, widely adopted across various industries, including IT, consulting, aerospace, and defense, are characterized by employees reporting to both functional and project managers. According to Gallup (2017), 84% of U.S. employees work in organizations utilizing matrixed arrangements to varying degrees.

We utilize *directed acyclic graphs* to represent these complex hierarchical frameworks. These graphs, where nodes represent individuals within the hierarchy and directed edges denote supervisor-subordinate relationships, are versatile objects encompassing various organizational structures. They can accommodate multiple direct reports and supervisors while ensuring acyclicity to prevent circular reporting chains.

Directed acyclic graphs have gained prominence across disciplines for their ability to model ordered relationships with clear, unidirectional reporting structures. While these mathematical structures effectively capture formal authority flows in organizations, they may oversimplify certain organizational dynamics. For instance, when a subordinate's expertise influences superiors through feedback mechanisms, the actual influence patterns may exhibit cyclical characteristics absent from the formal structure.

These informal communication channels align with what Simon (1981) termed "informal organizations," which fall outside the scope of this paper. Simon observed that social interactions naturally evolve into robust hierarchical structures. These informal frameworks are more challenging to identify than the well-defined organigrams in formal organizational charts. Nonetheless, Krackhardt (1994) introduced formal metrics to quantify the degree to which informal organizations resemble well-structured hierarchies (see also Everett and Krackhardt, 2012).

We develop an axiomatic framework rooted in three intuitive postulates for comparing hierarchical structures: Anonymity, Subordination Removal, and the Replication Principle. Anonymity and the Replication Principle were introduced in Carbonell-Nicolau (2025). Anonymity ensures label-independence, meaning that renaming individuals in a hierarchy leaves its structural properties unchanged. The Replication Principle maintains consistency across scales by asserting that creating "organizational clones" (exact replicas of a hierarchy) preserves its hierarchical character.

The notion of Subordination Removal introduced in Carbonell-Nicolau (2025) requires adaptation for directed acyclic graphs, which allow for multiple paths connecting nodes. In our framework, removing a subordination relation means eliminating a direct edge between a node j and a direct subordinate i with the following provisos:

- If j has no superiors, no additional edges are added.
- If j has superiors, then every immediate supervisor j' of j is connected to i (bypassing j) only if there is no alternative path (besides the path through j) from j' to i.

With these fundamental axioms established, we define a core hierarchical pre-order, denoted as $\succcurlyeq_H$, based on preserving supervisory relationships across hierarchies. This binary relation extends the pre-order introduced in Carbonell-Nicolau (2025) for hierarchical trees.

For n-person hierarchies, we say that h is at least as hierarchical as h' under $\succcurlyeq_H$ if there exists a bijection ϕ from the nodes of h to the nodes of h' that preserves supervisory relationships as follows: for every i in h, every immediate supervisor j' of $\phi(i)$ in h' must correspond (via ϕ^{-1}) to a (direct or indirect) supervisor $\phi^{-1}(j')$ of i in h. Furthermore, the path from $\phi^{-1}(j')$ to i must be direct if there exists, in h', an indirect alternative path between j' and $\phi(i)$ besides the direct edge from j' to $\phi(i)$.

The hierarchical pre-order $\succcurlyeq_H$ directly corresponds to the notion of subordination relation. Specifically, strict $\succ_H$-dominance equates to successive removals of subordination relations, as formally proven in Theorem 1. This finding establishes an operational framework for structural simplification through subordination removal, conceptualized by means of a hierarchical dominance concept based on mapping direct edges in the transformed hierarchy to possibly indirect paths in the parent hierarchy.

Theorem 1 constitutes the cornerstone of our central characterization: for n-person hierarchies, $\succcurlyeq_H$-consistent hierarchical pre-orders are precisely those that satisfy the Anonymity and Subordination Removal principles (Theorem 2).

The analysis of partial completions of $\succcurlyeq_H$ parallels that in Carbonell-Nicolau (2025), yielding two measures ranked by increasing order of completion: an incomplete measure comparing the number of (direct or indirect) supervisors across pairs of nodes in two different hierarchies linked through bijective mappings, and a complete index based on the average number of supervisors in each hierarchy.

As in Carbonell-Nicolau (2025), the Replication Principle allows us to generalize our main characterization to hierarchies of varying sizes. This axiom provides the basis for an extension of $\succcurlyeq_H$ suitable for comparing hierarchies with different sizes. This extension emerges as the only hierarchical pre-order that embodies the three essential axioms of hierarchy measurement: Anonymity, Subordination Removal, and Replication Principle (Theorem 3).

A substantial body of literature on hierarchy measurement spans multiple disciplines, offering diverse hierarchical indices that comprehensively rank pairs of hierarchies. These measures range from intuitive indicators—such as the count of organizational layers from top executive to lowest employee or the "span of control" quantifying the average number of direct reports per manager—to more sophisticated statistical approaches that assess node asymmetries, directional relationships, and top-down reachability patterns in hierarchies and broader network structures (see, e.g., Trusina et al., 2004; Luo and Magee, 2011). A notable example among these advanced measures is the "global reaching centrality" concept (Mones et al., 2012), which evaluates individuals' positional centrality within hierarchies based on their capacity to reach other nodes. As demonstrated in Carbonell-Nicolau (2025), this centrality-based approach diverges fundamentally from the axiomatic framework we develop in this paper.

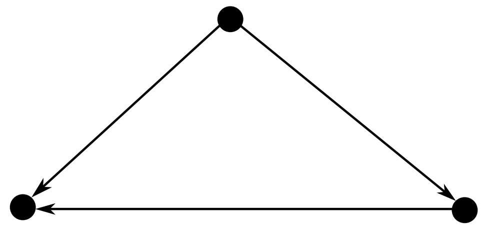

Figure 1: A hierarchy.

Corominas-Murtra et al. (2013) map directed graphs (not necessarily acyclic) onto a concise set of quantitative dimensions—Treeness, Feedforwardness, and Orderability—each capturing distinct aspects of hierarchical structure. However, unlike our framework—which differentiates among acyclic structures—their methodology assigns identical hierarchical degrees to all acyclic graphs, treating them as equivalent. In contrast, Czégel and Palla (2015) developed an approach with discriminatory power among acyclic graphs by employing a simulation technique. Their method reverses traditional information flow patterns by simulating random walks from employees upward to managers, thereby tracing information to its origins. Within this framework, nodes accumulating numerous walkers occupy superior positions in the hierarchy, while those attracting fewer walkers reside in subordinate positions. Their hierarchy score quantifies the distributional inequality of these walkers—greater concentration at upper echelons signifies a more pronounced hierarchical structure.

Our approach fundamentally diverges from these algorithmic methods by anchoring our measures in precisely articulated theoretical principles. We identify a core incomplete pre-order as the singular measure that aligns with a set of fundamental axioms designed to rank hierarchical structures unambiguously. This core measure serves as the foundation for partial completions that preserve these essential properties. Importantly, while enhancing a hierarchical measure's completeness allows for ranking more hierarchies, this expanded capacity comes with a consequential trade-off: it involves a departure from the basic axioms that characterize the core measure. This shift entails making judgments in inherently more ambiguous or less clearly delineated scenarios, introducing a tension between comprehensive applicability and theoretical coherence.

The paper is structured as follows. Section 2 introduces the foundational concept of directed acyclic graphs, providing the necessary background for understanding hierarchical structures. Section 3 establishes the axiomatic framework for measuring hierarchies and presents the main results for fixed-size hierarchies. The extension of this framework to hierarchies of varying sizes is detailed in Section 4. Finally, Section 5 concludes the paper, summarizing the main contributions and outlining directions for future research.

2. Hierarchies

We define a *hierarchy* as a *directed acyclic graph*, consisting of a set of nodes connected by directed edges with the property that no directed cycles exist within the graph. This acyclicity ensures that following any path of directed edges will never lead from a node back to itself.

In our context, nodes represent individuals, while directed edges represent subordination relations. Each edge points from a supervisor to their direct subordinate, clearly indicating the direction of authority. Individuals may have direct and indirect supervisors, the latter being supervisors of their supervisors.

Note that the acyclicity property of the directed graph implies a strict hierarchical constraint: if an individual i has k direct or indirect supervisors, then any supervisor of i must necessarily have fewer than k supervisors.

Indeed, let S_i denote the set of all supervisors (both direct and indirect) of individual i, and suppose that $j \in S_i$, meaning j is a supervisor of i. By the transitivity of the supervisory relation, any supervisor of j must also be a supervisor of i, thus the set of supervisors of j, denoted S_j, is contained in S_i: $S_j \subseteq S_i$. Furthermore, $j \in S_i$ but $j \notin S_j$ since j cannot be a supervisor of itself—this follows directly from the acyclicity of the directed graph forming the hierarchy. Consequently, S_j is a proper subset of S_i (i.e., $S_j \subsetneq S_i$), which implies that individual j has strictly fewer supervisors than individual i.

Figure 1 illustrates a simple hierarchy comprising three individuals connected by three subordination relations.

Note that this directed acyclic graph formulation provides sufficient flexibility to represent complex reporting structures where an individual may have multiple direct supervisors. This accommodates matrix organizational structures and other non-traditional hierarchical arrangements while maintaining the system's essential acyclicity property.

This approach differs from the hierarchy definition presented in Carbonell-Nicolau (2025), which constrained each individual to at most one supervisor and explicitly partitioned nodes into distinct ranks or levels. Our present conceptualization of hierarchy is more fundamental, as it relies solely on subordination relationships between individuals without imposing predefined rank structures. Nevertheless, rank analysis remains entirely compatible with our framework, which can be naturally extended to incorporate formal hierarchical levels when required.

For each individual i in a hierarchy h, we define the *sub-hierarchy* $h(i)$ as the directed acyclic graph formed by the node i together with all its direct and indirect subordinates (successors), along with all directed edges connecting these nodes in the original hierarchy.

This sub-hierarchy $h(i)$ preserves the structural relationships of the original hierarchy while focusing on the organizational subset under individual i's influence, and it constitutes a proper hierarchy in its own right.

Notationally, a hierarchy comprising $k \in \{1, 2, \ldots\}$ independent hierarchies $h_1, \ldots, h_k$ is denoted by $(h_1, \ldots, h_k)$. Here, *independence* means that these hierarchies represent k directed acyclic graphs that are pairwise disconnected—that is, no directed edge exists between any node in hierarchy h_i and any node in hierarchy h_j for $i \neq j$.

Finally, we define the *size* of a hierarchy as the cardinality of its node set—that is, the total number of individuals represented within the hierarchical structure.

3. Hierarchical pre-orders

We initially restrict our analysis to hierarchies of a fixed size n, deferring the examination of variable-sized hierarchies to a subsequent section.

Let $\mathscr{H}_n$ denote the set of all possible hierarchies containing exactly n individuals, representing the complete collection of n-person organizational structures that satisfy our directed acyclic graph definition.

A hierarchical pre-order $\succcurlyeq$ is defined as a reflexive and transitive binary relation on the set $\mathscr{H}_n$. These pre-orders represent potentially incomplete comparative assessments between pairs of hierarchies, where the notation $h \succcurlyeq h'$ carries the interpretation that "hierarchy h exhibits at least as much hierarchical structure as hierarchy h'."

Definition 1. A *hierarchical pre-order* on $\mathscr{H}_n$ is a reflexive and transitive binary relation on $\mathscr{H}_n$.

The symmetric and asymmetric parts of a hierarchical pre-order $\succcurlyeq$ are denoted by $\sim$ and $\succ$, respectively.

The following property of a hierarchy, called Anonymity, was introduced in Carbonell-Nicolau (2025).

A hierarchy h' is said to be a *relabeling* of another hierarchy h if h' is obtained from h by relabeling the individuals in h.

Formally, this can be stated as follows: there is a bijection mapping the individuals in h to those in h' preserving immediate subordination relations: for every nodes i and j in h, j is an immediate supervisor of i in h if and only if $\phi(j)$ is an immediate supervisor of $\phi(i)$ in h'.

Anonymity (A). A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}_n$ satisfies A if for any two hierarchies h and h' in $\mathscr{H}_n$, $h \sim h'$ whenever h' is a relabeling of h.

This axiom emphasizes that the essential structure of a hierarchy remains invariant under relabeling.

The concept of subordination relation removal, originally introduced by Carbonell-Nicolau (2025), is extended in this work to align with our more flexible definition of a hierarchy.

Definition 2. We say that hierarchy h' is obtained from hierarchy h by *removing a subordination relation* if there exist a subordinate i in h and an immediate supervisor j of i satisfying one of the following conditions:

- If j has no supervisors in h, then h' is the hierarchy obtained by removing the directed edge from j to i while preserving all other relationships. Specifically:
  - The sub-hierarchy rooted at node i, $h(i)$, is no longer under j's direct supervision.
  - Individual i loses exactly one direct supervisor (namely, j).
  - The structure of sub-hierarchy $h(i)$ remains unchanged.
  - All other supervisory relationships in h are preserved in h'.
- If j has at least one supervisor in h, then h' is the hierarchy obtained by:
  - Removing the directed edge from j to i.
  - For each direct supervisor j' of j in h, either
    - adding a new directed edge from j' to i if no (direct or indirect) path from j' to i exists in h besides the path through j; or

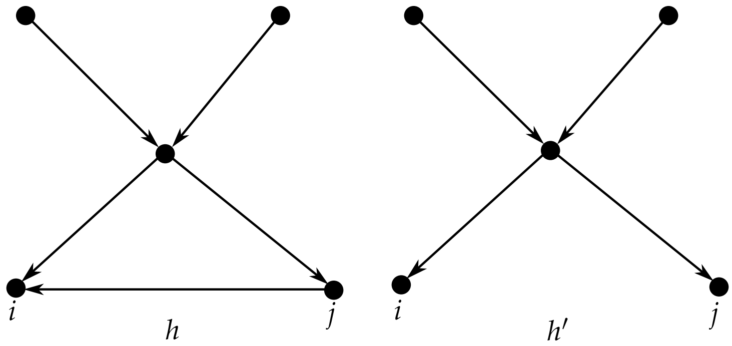

Figure 2: Subordination removal.

  * making no change if there already exists an alternative path (besides the path through j) from j' to i in h.
- Preserving the structure of sub-hierarchy $h(i)$.
- Preserving all other supervisory relationships in h.

This definition formalizes the process of subordination relation removal in two scenarios:

- When the supervisor j has no superiors, the link between j and i is simply removed.
- When j has superiors, then, for each immediate superior j' of i, i is directly connected to j', bypassing j, only if there is no alternative path (besides the path through j) from j' to i in h.

To illustrate this definition, consider the hierarchies from Figure 2, where individual i has multiple direct supervisors, including j. (We are focusing on the direct subordination relation between i and j.) In this instance, h' is obtained from h by removing the subordination relation between i and j, which simply involves removing the link between i and j, since i is already under the direct supervision of j's direct supervisor in h.

Figure 3 presents a similar example where h' is also obtained from h by removing the subordination relation between i and j. However, in this case, not only is the link between i and j removed, but a directed edge between j's immediate supervisor and i must be added in h', as that link is not present in h.

In Figure 4 we remove the subordination relation between i and j' in h by deleting the directed edge from j' and i. In this case, no direct edges are added from the supervisors of j' to i since these supervisors are already indirectly connected to i.

We maintain the principle, originally introduced by Carbonell-Nicolau (2025), that the removal of a subordination relation results in a less hierarchical structure.

Subordination Removal (SR). A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}_n$ satisfies SR if for any two hierarchies h and h' in $\mathscr{H}_n$, $h \succ h'$ whenever h' is obtained from h by removing a subordination relation.

We now extend and adapt the definition of the hierarchical pre-order $\succcurlyeq_H$ originally proposed by Carbonell-Nicolau (2025) to accommodate our generalized hierarchical framework.

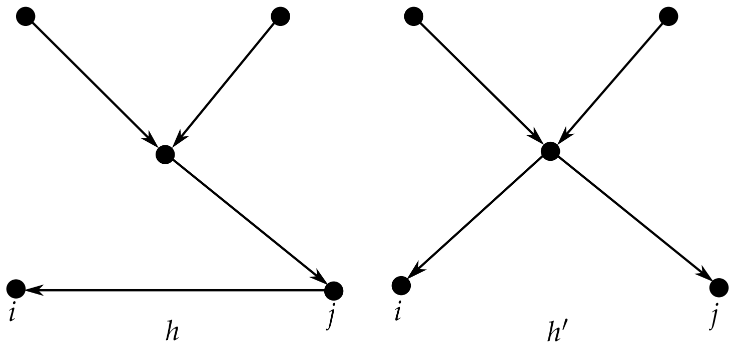

Figure 3: Subordination removal.

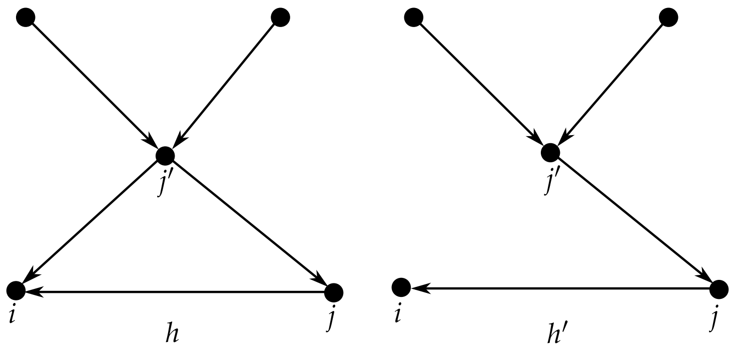

Figure 4: Subordination removal.

We define the binary relation $\succcurlyeq_H$ on $\mathscr{H}_n$ as follows.

For any two hierarchies h and h' in $\mathscr{H}_n$, $h \succcurlyeq_H h'$ if and only if there exists a bijection ϕ from the set of individuals in h to the set of individuals in h' satisfying the following condition:

- For every individual i in h such that $\phi(i)$ has at least one supervisor in h', any immediate supervisor j' of $\phi(i)$ in h' must satisfy one of the following:
  (a) If there exists an alternative indirect path from j' to $\phi(i)$ in h' besides the direct edge from j' to $\phi(i)$, then $\phi^{-1}(j')$ must be a direct supervisor of i in h.
  (b) Otherwise, $\phi^{-1}(j')$ must be either a direct or indirect supervisor of i in h.

This definition can be illustrated using the hierarchies depicted in Figure 3, where we apply the natural (identity) mapping ϕ between the nodes of h and h'. Under this bijection, individual i in h (as highlighted in the figure) maps to individual i in h', and the direct supervisor of i in h' corresponds to an indirect supervisor of i in h, demonstrating the preservation of supervisory relationships but with potentially greater hierarchical distance in h. In this case, the link between j's immediate supervisor in h and i need not be direct, since there is no alternative indirect path between these nodes in h'. This indirect rank-preservation property extends to all other nodes, as the two hierarchies maintain identical structures when node i and its associated edges are removed from both. Hence, $h \succcurlyeq_H h'$.

For further illustration, consider the hierarchies depicted in Figure 4. In this example, the identity mapping $\phi(i) = i$ between the nodes in hierarchies h and h' preserves the supervisory relations as required by the relation $\succcurlyeq_H$. Specifically, for each node i in h where i has at least one supervisor in h', every immediate supervisor of i in h' corresponds to either a direct or indirect supervisor of i in h. Moreover, since hierarchy h' contains no alternative indirect paths between nodes (i.e., there is at most one path between any two nodes), condition (a) in our definition does not apply. Therefore, we can conclude that $h \succcurlyeq_H h'$.

The binary relation $\succcurlyeq_H$ is a hierarchical pre-order satisfying Anonymity and Subordination Removal.

Lemma 1. *The binary relation $\succcurlyeq_H$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies A and SR.*

The proof of Lemma 1 is relegated to Appendix A.2.

For $h_1, h_L \in \mathscr{H}_n$, h_1 is obtained from h_L by *successive removals of subordination relations* if there are finitely many hierarchies $h_2, \dots, h_{L-1}$ in $\mathscr{H}_n$ such that h_l is obtained from h_{l+1} by removing a subordination relation, for each $l \in \{1, \dots, L-1\}$.

The following fundamental theorem establishes that the strict dominance relation $h \succ_H h'$ precisely characterizes hierarchies that can be obtained through successive removals of subordination relations. This result forms the cornerstone of our structural analysis of hierarchy transformations.

Theorem 1. *For hierarchies $h, h' \in \mathscr{H}_n$, $h \succ_H h'$ if and only if h' can be obtained from some relabeling of h by successive removals of subordination relations.*

The complete proof of Theorem 1 appears in Appendix A.3. Here, we provide an intuitive sketch of the main ideas.

The theorem characterizes hierarchical dominance in terms of structural transformations, establishing that one hierarchy dominates another precisely when the dominated hierarchy can be created by flattening the dominating hierarchy through a series of edge removals.

The proof addresses two directions:

Sufficiency. We show that if h' can be obtained from a relabeling of h by successive subordination removals, then $h \succ_H h'$.

Starting with a relabeling of h (call it $\hat{h}$), we trace the sequence of subordination removals that transforms $\hat{h}$ into h'. Since each removal creates a less hierarchical structure (Lemma 5), we establish a chain

$$\hat{h} \succ_H h_1 \succ_H h_2 \succ_H \cdots \succ_H h_L = h'$$

for some sequence of intermediate hierarchies. By transitivity of $\succ_H$, we have $\hat{h} \succ_H h'$. Since $\hat{h} \sim_H h$ (as $\hat{h}$ is merely a relabeling of h), we conclude that $h \succ_H h'$.

Necessity. We prove that if $h \succ_H h'$, then h' can be obtained from some relabeling of h by successive subordination removals.

The dominance relation $h \succ_H h'$ implies $h \succcurlyeq_H h'$, which by definition yields a bijective mapping ϕ from nodes in h to nodes in h' that preserves supervisory relationships. Our strategy is to iteratively transform h into h' through a sequence of edge removals while maintaining dominance over h' at each step.

The iterative process proceeds as follows:

1. *Strategic edge selection*. We identify an edge $e = (j \rightarrow i)$ in hierarchy h that satisfies at least one of the following conditions:

   - The edge e in h does not correspond to a direct supervisory relationship in h' when mapped via ϕ. In other words, $\phi(j)$ is not a direct supervisor of $\phi(i)$ in h'.
   - The edge e is part of a longer supervisory path in h that, when mapped through ϕ, corresponds to a direct supervisory relationship in h'. This indicates that h' has a flatter, more compressed command structure compared to h.

   When both types of edges exist in h, we prioritize the second type as their removal more efficiently transforms h toward the structure of h'. If multiple edges of the second type qualify, we select one where the subordinate node has maximum depth in the hierarchy. This carefully ordered selection ensures that after each edge removal, the resulting hierarchy maintains dominance over h' under the same bijection ϕ while systematically converging toward h''s structure.

2. *Subordination removal*. We remove the selected edge, adjusting connections as necessary according to the definition of subordination removal. This creates a new hierarchy h^*.

3. *Dominance verification*. We demonstrate that $h^* \succcurlyeq_H h'$. If $h^* \sim_H h'$, then by Lemma 4, h^* is a relabeling of h' and the proof is complete. Otherwise, we iterate the process with h^* replacing h.

After a finite number of iterations, we transform h into a structure that is equivalent to h' up to relabeling. This establishes that the hierarchical dominance relation $h \succ_H h'$

corresponds precisely to a sequence of subordination removals, providing an operational interpretation of hierarchical dominance through structural simplification.

The next phase of our analysis establishes a complete description of hierarchical pre-orders that are compatible with the pre-order $\succcurlyeq_H$. Specifically, we fully characterize all partial or complete extensions of $\succcurlyeq_H$ that preserve its fundamental ordering properties while potentially introducing additional comparability between hierarchies.

Theorem 1 constitutes the essence of our extension analysis: this result guarantees that the characterization of hierarchical pre-orders consistent with $\succcurlyeq_H$ based on axioms A and SR parallels the structural framework developed in Carbonell-Nicolau (2025).

A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}_n$ is *$\succcurlyeq_H$-consistent* if it satisfies the following conditions for all hierarchies $h, h' \in \mathscr{H}_n$:

- $h \succ_H h' \Rightarrow h \succ h'$.
- $h \sim_H h' \Rightarrow h \sim h'$.

Therefore, a $\succcurlyeq_H$-consistent hierarchical pre-order preserves all strict comparisons and equivalences of $\succcurlyeq_H$, and can only introduce new comparability between hierarchies that are incomparable under $\succcurlyeq_H$. Equivalently, $\succcurlyeq_H$-consistent hierarchical pre-orders are precisely the partial completions of $\succcurlyeq_H$.

The following theorem extends the characterization of $\succcurlyeq_H$-consistent hierarchical pre-orders established in Theorem 2 of Carbonell-Nicolau (2025) to our more general hierarchical framework. Having proven Theorem 1 for hierarchies based on directed acyclic graphs—which encompass the tree-like structures considered in the prior work—we can now demonstrate the equivalence between axioms A and SR and the property of $\succcurlyeq_H$-consistency as in the proof of Theorem 2 in Carbonell-Nicolau (2025). We provide the proof below for thoroughness.

Theorem 2. *A hierarchical pre-order on $\mathscr{H}_n$ satisfies A and SR if and only if it is $\succcurlyeq_H$-consistent.*

Proof. (*Sufficiency.*) Suppose that $\succcurlyeq$ is $\succcurlyeq_H$-consistent. Since $\succcurlyeq_H$ satisfies axioms A and SR (Lemma 1), and $\succcurlyeq$ is $\succcurlyeq_H$-consistent, it follows immediately that $\succcurlyeq$ also satisfies A and SR.

(*Necessity.*) Suppose that $\succcurlyeq$ is a hierarchical pre-order in $\mathscr{H}_n$ satisfying axioms A and SR. We must establish that $\succcurlyeq$ is $\succcurlyeq_H$-consistent, which requires proving that the following two conditions hold for all hierarchies $h, h' \in \mathscr{H}_n$:

- $h \succ_H h' \Rightarrow h \succ h'$
- $h \sim_H h' \Rightarrow h \sim h'$

First, consider the case where $h \sim_H h'$. By Lemma 4 (established in Appendix A.1), we know that h is a relabeling of h'. Since $\succcurlyeq$ satisfies axiom A, we can conclude that $h \sim h'$.

Next, consider the case where $h \succ_H h'$. According to Theorem 1, there exists a relabeling of h, denoted by h^*, such that h' can be obtained from h^* by a finite sequence of subordination relation removals. Therefore, because $\succcurlyeq$ satisfies axiom SR, we have

$$h^* \succ h_L \succ \cdots \succ h_1 \succ h'$$

for some finite sequence $h_1, \ldots, h_L$ of hierarchies in $\mathscr{H}_n$. Since h^* is a relabeling of h and $\succcurlyeq$ satisfies axiom A, we obtain

$$h \sim h^* \succ h_L \succ \cdots \succ h_1 \succ h'. \tag{1}$$

By the reflexivity and transitivity of $\succcurlyeq$, equation (1) implies that $h \succ h'$ (Sen, 2017, Lemma 1*a, p. 56), which completes the proof. ∎

As an illustrative example of a $\succcurlyeq_H$-consistent hierarchical pre-order on $\mathscr{H}_n$, let us consider the pre-order $\succcurlyeq_s$ introduced in Carbonell-Nicolau (2025) defined as follows: $h \succcurlyeq_s h'$ if and only if there exists a bijection ϕ from the set of individuals in h to the set of individuals in h' such that, for each individual i in h, the number of (direct or indirect) supervisors of i in h is greater than or equal to the number of (direct or indirect) supervisors of $\phi(i)$ in h'.

Following standard notation, we denote the symmetric and asymmetric parts of $\succcurlyeq_s$ by $\sim_s$ and $\succ_s$, respectively.

Similar to $\succcurlyeq_H$, the pre-order $\succcurlyeq_s$ also satisfies the axioms of A and SR, as formalized in the following proposition.

Proposition 1. *The binary relation $\succcurlyeq_s$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies axioms A and SR.*

The proof of Proposition 1 appears in Appendix A.4. By combining Proposition 1 with Theorem 2, we can conclude that $\succcurlyeq_s$ is $\succcurlyeq_H$-consistent. This means that $\succcurlyeq_s$ agrees with $\succcurlyeq_H$ for all pairs of hierarchies in $\mathscr{H}_n$ that are comparable under $\succcurlyeq_H$. However, the converse assertion does not hold, as demonstrated within the more restricted framework of Carbonell-Nicolau (2025).

A *hierarchical index* on $\mathscr{H}_n$ is a function $I : \mathscr{H}_n \to \mathbb{R}$ that assigns a "hierarchical degree" $I(h)$ to every hierarchy $h \in \mathscr{H}_n$.

Each index I induces a complete hierarchical order $\succcurlyeq_I$ on $\mathscr{H}_n$, defined as:

$$h \succcurlyeq_I h' \Leftrightarrow I(h) \geq I(h').$$

This relation $\succcurlyeq_I$ is a well-defined hierarchical order, satisfying reflexivity and transitivity on $\mathscr{H}_n$.

For any hierarchy $h \in \mathscr{H}_n$, let $s_h(i)$ denote the number of supervisors of individual i in h. We define the supervisor index I_s as follows:

$$I_s(h) = \frac{1}{n} \sum_{i \in h} s_h(i), \tag{2}$$

which represents the average number of supervisors per individual in hierarchy h.

We observe that $\succcurlyeq_s \subseteq \succcurlyeq_{I_s}$, meaning that whenever h is at least as hierarchical as h' under $\succcurlyeq_s$, it is also at least as hierarchical under $\succcurlyeq_{I_s}$. Furthermore, $\succcurlyeq_{I_s}$ is $\succcurlyeq_s$-consistent, as can be readily verified.

The converse, however, does not hold: $\succcurlyeq_s$ is not $\succcurlyeq_{I_s}$-consistent, as demonstrated in the restricted setting of Carbonell-Nicolau (2025).

We have established that $\succcurlyeq_{I_s}$ is $\succcurlyeq_s$-consistent and that both $\succcurlyeq_{I_s}$ and $\succcurlyeq_s$ are $\succcurlyeq_H$-consistent. Since $\succcurlyeq_{I_s}$ is a complete order on $\mathscr{H}_n$, it serves as a completion of both $\succcurlyeq_H$ and $\succcurlyeq_s$.

Given that $\succcurlyeq_{I_s}$ is reflexive, transitive, and $\succcurlyeq_H$-consistent, Theorem 2 implies that $\succcurlyeq_{I_s}$ satisfies axioms A and SR.

Proposition 2. *The hierarchical order $\succcurlyeq_{I_s}$ defined on $\mathscr{H}_n$ is complete, reflexive, and transitive, and satisfies axioms A and SR.*

4. Hierarchies of varying sizes

The extension of our analysis to hierarchies of varying sizes can be accomplished through the Replication Principle, following the methodology established in Carbonell-Nicolau (2025). This principle allows us to compare hierarchical structures across different organizational scales while preserving the essential properties of the hierarchical orders. The formal procedure involves creating comparable replicated structures and then applying our previously established results to these standardized hierarchies.

The set of all hierarchies of any size is defined as the union

$$\mathscr{H} = \bigcup_n \mathscr{H}_n,$$

where $\mathscr{H}_n$ is the set of all n-person hierarchies.

Definition 3. A hierarchical pre-order on $\mathscr{H}$ is a reflexive and transitive binary relation on $\mathscr{H}$.

A *replication* of a hierarchy $h \in \mathscr{H}$ is a hierarchy in $\mathscr{H}$ of the form $(h, \dots, h)$. By convention, h is a replication of itself.

The Replication Principle asserts that replicating a hierarchy does not change its hierarchical degree.

Replication Principle (RP). A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies RP if for any two hierarchies h and h' in $\mathscr{H}$, $h' \sim h$ whenever h' is a replication of h.

The hierarchical pre-order $\succcurlyeq_H$ on $\mathscr{H}_n$ from Section 3 can be extended to the domain $\mathscr{H}$ as follows: for $h, h' \in \mathscr{H}$, $h' \succcurlyeq_H h$ if and only if there exist equally-sized replications h_r and h'_r of h and h', respectively, such that $h'_r \succcurlyeq_H h_r$.

It is easy to verify, using Lemma 1, that the extension of $\succcurlyeq_H$ to $\mathscr{H}$ is reflexive and transitive and satisfies A and SR. Moreover, this extension also satisfies RP. Indeed, if $h' = (h, \dots, h)$ is a replication of h, then $h' \sim_H h$ because $(h, \dots, h) \sim_H (h, \dots, h)$.

Lemma 2. *The hierarchical pre-order $\succcurlyeq_H$ defined on $\mathscr{H}$ is reflexive and transitive and satisfies A, SR, and RP.*

For hierarchical pre-orders $\succcurlyeq$ defined on th extended domain $\mathscr{H}$, $\succcurlyeq$-consistency is defined as in the previous section.

A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ is *$\succcurlyeq_H$-consistent* if the following two conditions are satisfied for every pair h, h' in $\mathscr{H}$:

- $h \succ_H h' \Rightarrow h \succ h'$.
- $h \sim_H h' \Rightarrow h \sim h'$.

The following result extends Theorem 2 to the domain $\mathscr{H}$.

Theorem 3. *A hierarchical pre-order on $\mathscr{H}$ satisfies A, SR, and RP if and only if it is $\succcurlyeq_H$-consistent.*

Proof. (*Sufficiency.*) Suppose that $\succcurlyeq$ is $\succcurlyeq_H$-consistent. Since $\succcurlyeq_H$ satisfies axioms A, SR, and RP by Lemma 2, and $\succcurlyeq$ is $\succcurlyeq_H$-consistent, it follows that $\succcurlyeq$ also satisfies these three axioms.

(*Necessity.*) Suppose that $\succcurlyeq$ is a hierarchical pre-order on $\mathscr{H}$ satisfying axioms A, SR, and RP. We need to show that $\succcurlyeq$ is $\succcurlyeq_H$-consistent, which requires proving that for every pair of hierarchies $h, h' \in \mathscr{H}$:

(a) $h \succ_H h' \Rightarrow h \succ h'$.

(b) $h \sim_H h' \Rightarrow h \sim h'$.

Fix arbitrary hierarchies $h \in \mathscr{H}_m$ and $h' \in \mathscr{H}_n$. Let h_r and h'_r be n-times and m-times replications of h and h', respectively. Then h_r and h'_r are both hierarchies in $\mathscr{H}_{mn}$.

Suppose that $h \sim_H h'$. Since $\succcurlyeq_H$ satisfies RP by Lemma 2, we have:

$$h_r \sim_H h \sim_H h' \sim_H h'_r. \tag{3}$$

By Lemma 2, $\succcurlyeq_H$ is reflexive and transitive, which implies that $\sim_H$ is transitive (Sen, 2017, Lemma 1*a, p. 56). Therefore, equation (3) yields $h_r \sim_H h'_r$.

Since $h_r, h'_r \in \mathscr{H}_{mn}$ and $h_r \sim_H h'_r$, we can apply Lemma 4 from Appendix A.1 to conclude that h_r is a relabeling of h'_r.

Given that $\succcurlyeq$ satisfies axioms A and RP, we have:

$$h \sim h_r \sim h'_r \sim h'.$$

By transitivity of $\sim$, we conclude that $h \sim h'$, which establishes (b).

Now suppose that $h \succ_H h'$. Let h_r and h'_r be n-times and m-times replications of h and h', respectively. Since $\succcurlyeq_H$ satisfies RP by Lemma 2, we have:

$$h_r \sim_H h \succ_H h' \sim_H h'_r. \tag{4}$$

Because $\succcurlyeq_H$ is reflexive and transitive (Lemma 2), equation (4) implies $h_r \succ_H h'_r$ (Sen, 2017, Lemma 1*a, p. 56).

Given that $h_r, h'_r \in \mathscr{H}_{mn}$ and $h_r \succ_H h'_r$, we can apply Theorem 1 to conclude that h'_r can be obtained from some relabeling of h_r, denoted by h^*_r, by successive removals of subordination relations.

Since $\succcurlyeq$ satisfies axiom SR, there exists a finite sequence of hierarchies $h_1, \ldots, h_L$ in $\mathscr{H}_{mn}$ such that:

$$h^*_r \succ h_L \succ \cdots \succ h_1 \succ h'_r.$$

Because h^*_r is a relabeling of h_r and $\succcurlyeq$ satisfies axiom A, we have:

$$h_r \sim h^*_r \succ h_L \succ \cdots \succ h_1 \succ h'_r. \tag{5}$$

Since $\succcurlyeq$ is reflexive and transitive, equation (5) implies $h_r \succ h'_r$ (Sen, 2017, Lemma 1*a, p. 56).

Finally, because $\succcurlyeq$ satisfies axiom RP, and h_r and h'_r are replications of h and h' respectively, we have:

$$h \sim h_r \succ h'_r \sim h',$$

which implies $h \succ h'$, establishing (a). ∎

The heirarchical pre-order $\succcurlyeq_s$ introduced in Section 3 can be extended to $\mathscr{H}$ as follows: for $h, h' \in \mathscr{H}$, $h' \succcurlyeq_s h$ if and only if there exist equally-sized replications h_r and h'_r of h and h', respectively, such that $h'_r \succcurlyeq_s h_r$.

The definition of the extension $\succcurlyeq_s$, together with Proposition 1, implies that $\succcurlyeq_s$ is reflexive and transitive, and satisfies axioms A and SR. Additionally, it can be straightforwardly verified that $\succcurlyeq_s$ satisfies RP.

Proposition 3. *The hierarchical pre-order $\succcurlyeq_s$ defined on $\mathscr{H}$ is reflexive and transitive and satisfies A, SR, and RP.*

By Proposition 3 and Theorem 3, $\succcurlyeq_s$ is $\succcurlyeq_H$-consistent.

As demonstrated in Carbonell-Nicolau (2025), certain hierarchical structures remain incomparable under the $\succcurlyeq_H$ relation, whereas $\succcurlyeq_s$ induces a consistent ordering.

The hierarchical index I_s defined in equation (2) can be naturally extended to the broader domain $\mathscr{H}$. For any hierarchy $h \in \mathscr{H}$, we define:

$$I_s(h) = \frac{1}{n} \sum_{i \in h} s_h(i),$$

where $s_h(i)$ denotes the number of supervisors of individual i in hierarchy h. This index represents the average number of supervisors per individual within the hierarchy h.

The hierarchical order $\succcurlyeq_{I_s}$ induced by I_s on $\mathscr{H}$ is defined as

$$h \succcurlyeq_{I_s} h' \Leftrightarrow I_s(h) \geq I_s(h').$$

As shown in Carbonell-Nicolau (2025), $\succcurlyeq_{I_s}$ is $\succcurlyeq_s$-consistent, while $\succcurlyeq_s$ is not $\succcurlyeq_{I_s}$-consistent.

We have established that $\succcurlyeq_{I_s}$ is $\succcurlyeq_s$-consistent and that both $\succcurlyeq_{I_s}$ and $\succcurlyeq_s$ are $\succcurlyeq_H$-consistent. Consequently, $\succcurlyeq_{I_s}$, being a complete order on $\mathscr{H}$, serves as a completion of both $\succcurlyeq_H$ and $\succcurlyeq_s$.

Given that $\succcurlyeq_{I_s}$ is reflexive, transitive, and $\succcurlyeq_H$-consistent, Theorem 3 implies that $\succcurlyeq_{I_s}$ satisfies axioms A, SR, and RP.

Proposition 4. *The hierarchical order $\succcurlyeq_{I_s}$ defined on $\mathscr{H}$ is complete, reflexive, and transitive, and satisfies axioms A, SR, and RP.*

5. Concluding remarks

Reframing hierarchy as a primary driver of systemic inequality and economic power concentration shifts our understanding of these phenomena—from abstract economic forces to tangible organizational structures that can be quantified and potentially reshaped.

We propose an axiomatic approach to measuring hierarchy in directed acyclic graphs, rooted in three core principles: Anonymity, Subordination Removal, and the Replication Principle. Together, these axioms uniquely determine a foundational hierarchical pre-order based on the consistency of supervisory relationships across pairs of hierarchies. This pre-order establishes a minimal consistency threshold, enabling the development of more detailed rankings, such as those derived from the average number of supervisors.

The link between this general framework and earnings inequality remains an area for future exploration. By integrating our approach with the well-established field of inequality measurement, we can investigate how hierarchical structures influence income distribution—particularly in contexts where upward mobility in earnings aligns with advancement through organizational hierarchies. This perspective opens a compelling path for further research.

A. Appendix

A.1. Preliminary results

Lemma 3. *For $h, h' \in \mathscr{H}_n$, $h \sim_H h'$ implies implies the existence of a bijection from the nodes in h to those in h' such that for every individual i in h with $\phi(i)$ non-root in h' and for every immediate supervisor j' of $\phi(i)$ in h', one of the following holds:*

(a) If there is an alternative indirect path from j' to $\phi(i)$ in h', then $\phi^{-1}(j')$ is a direct supervisor of i in h.

(b) Otherwise, $\phi^{-1}(j')$ is a supervisor (direct or indirect) of i in h, implying there is a chain of one or more direct edges from $\phi^{-1}(j')$ to i.

Moreover:

- *For each individual i in h with k (direct or indirect) supervisors, $\phi(i)$ has k (direct or indirect) supervisors in h'.*
- *Letting S_i (respectively, $S_{\phi(i)}$) denote the set of all supervisors of i (respectively, $\phi(i)$) in h (respectively, h'), we have*

$$\phi(S_i) = \{\phi(\iota) : \iota \in S_i\} = S_{\phi(i)}. \tag{6}$$

Proof. Fix $h, h' \in \mathscr{H}_n$ and suppose that $h \sim_H h'$. Then $h \succcurlyeq_H h'$ and $h' \succcurlyeq_H h$. Hence, the following two conditions are satisfied:

- There exists a bijection ϕ from the set of individuals in h to the set of individuals in h' such that for every individual i in h with $\phi(i)$ non-root in h' and for every immediate supervisor j' of $\phi(i)$ in h', one of the following holds:
  (a) If there is an alternative indirect path from j' to $\phi(i)$ in h', then $\phi^{-1}(j')$ is a direct supervisor of i in h.
  (b) Otherwise, $\phi^{-1}(j')$ is a supervisor (direct or indirect) of i in h, implying there is a chain of one or more direct edges from $\phi^{-1}(j')$ to i.
- There exists a bijection ϕ' from the set of individuals in h' to the set of individuals in h such that for every individual i in h' with $\phi(i)$ non-root in h and for every immediate supervisor j of $\phi(i)$ in h, one of the following holds:
  - If there is an alternative indirect path from j to $\phi'(i)$ in h, then $\phi'^{-1}(j)$ is a direct supervisor of i in h'.
  - Otherwise, $\phi'^{-1}(j)$ is a supervisor (direct or indirect) of i in h', implying there is a chain of one or more direct edges from $\phi'^{-1}(j)$ to i.

Note that it suffices to prove the equality in (6) and show that for each individual i in h with k supervisors in h, $\phi(i)$ has k supervisors in h'. We first prove the last assertion.

Let I_0 (respectively, I'_0) be the set of all individuals in h (respectively, h') with no supervisors. Then

$$\phi(I_0) = \{\phi(i) : i \in I_0\} \subseteq I'_0.$$

Indeed, $j \in \phi(I_0) \setminus I'_0$ implies that there exist an unsupervised $i \in I_0$ linking to $\phi(i) = j$ in h', where j has at least one immediate supervisor, j'. But then j' links (via ϕ^{-1}) to a non-supervisor of i in h, which contradicts (a) or (b). Therefore, $\phi(I_0) \setminus I'_0 = \emptyset$, which implies that $\phi(I_0) \subseteq I'_0$.

Similarly, we can show that $\phi'(I'_0) \subseteq I_0$.

Note that the two containments $\phi(I_0) \subseteq I'_0$ and $\phi'(I'_0) \subseteq I_0$ imply that h and h' have the same number of individuals with no supervisors. Indeed, if there were more unsupervised individuals in h', then $\phi(I_0)$ would be a strict subset of I'_0, and, since both I_0 and $\phi(I_0)$ have the same cardinality, I_0 would be a smaller set than I'_0, contradicting the containment $\phi'(I'_0) \subseteq I_0$. A similar contradiction can be obtained under the assumption that there are more unsupervised individuals in h.

Given that I_0 and I'_0 have the same cardinality, and given that $\phi(I_0) \subseteq I'_0$ and $\phi'(I'_0) \subseteq I_0$, it follows that $\phi(I_0) = I'_0$ and $\phi'(I'_0) = I_0$.

Next, let I_l (respectively, I'_l) be the set of all individuals in h (respectively, h') with exactly l supervisors. Suppose that the equalities

$$\phi(I_l) = I'_l \quad \text{and} \quad \phi'(I'_l) = I_l, \quad l \in \{0, \ldots, k\}, \tag{7}$$

have been established for some $k \in \{0, 1, \ldots\}$.

It suffices to show that $\phi(I_{k+1}) = I'_{k+1}$ and $\phi'(I'_{k+1}) = I_{k+1}$.

First, we show that $\phi(I_{k+1}) \subseteq I'_{k+1}$.

If $j \in \phi(I_{k+1}) \setminus I'_{k+1}$, since $\phi(I_l) = I'_l$ for each $l \in \{0, \ldots, k\}$, we see that $j \in \phi(I_{k+1}) \setminus (\bigcup_{l=0}^{k+1} I'_l)$. Consequently, there exists $\iota \in I_{k+1}$ such that $\phi(\iota) = j$ has $\kappa' > k+1$ supervisors in h'. By (a) and (b), any one of these supervisors must link (via ϕ^{-1}) to a supervisor of ι in h. By bijectivity of ϕ, each supervisor of j maps to a distinct supervisor of ι in h, forcing ι to have $\kappa' > k+1$ supervisors—contradicting $\iota \in I_{k+1}$. Thus, the assumption that $j \in \phi(I_{k+1}) \setminus I'_{k+1}$ cannot hold, and we conclude $\phi(I_{k+1}) \subseteq I'_{k+1}$.

Similarly, we can show that $\phi'(I'_{k+1}) \subseteq I_{k+1}$.

Now the two inclusions $\phi(I_{k+1}) \subseteq I'_{k+1}$ and $\phi'(I'_{k+1}) \subseteq I_{k+1}$ imply

$$\phi(I_{k+1}) = I'_{k+1} \quad \text{and} \quad \phi'(I'_{k+1}) = I_{k+1},$$

as we sought.

It remains to prove (6). This equality follows from the fact that

$$\phi^{-1}(S_{\phi(i)}) = S_i. \tag{8}$$

To see that (8) holds, we first establish the containment $\phi^{-1}(S_{\phi(i)}) \subseteq S_i$.

Suppose that $i' \in S_{\phi(i)}$. Then there is path linking i' and $\phi(i)$ in h' through a finite set of nodes $i_1, \ldots, i_m$ in h', where

- i_1 is an immediate supervisor of $\phi(i)$;
- i_{l+1} is an immediate supervisor of i_l for all $l \in \{1, \ldots, m-1\}$; and
- i' is an immediate supervisor of i_m.

Now, applying (a)-(b) to this path, starting at $\phi(i)$, we see that

- $\phi^{-1}(i_1)$ is a supervisor of i in h;
- $\phi^{-1}(i_{l+1})$ is a supervisor of $\phi^{-1}(i_l)$ in h for each $l \in \{1, \ldots, m-1\}$; and

- $\phi^{-1}(i')$ is a supervisor of $\phi^{-1}(i_m)$ in h.

Consequently, $\phi^{-1}(i')$, which is a member of $\phi^{-1}(S_{\phi(i)})$, is a supervisor of i in h, i.e., $\phi^{-1}(i') \in S_i$.

We have seen that $\phi^{-1}(S_{\phi(i)}) \subseteq S_i$. If this inclusion were strict, $S_{\phi(i)}$ would have less elements than S_i, which contradicts the proven fact that ϕ preserves supervisor count. Hence, (8) holds. ■

Lemma 4. *For $h, h' \in \mathscr{H}_n$, $h \sim_H h'$ implies that h is a relabeling of h'.*

Proof. Choose $h, h' \in \mathscr{H}_n$ and suppose that $h \sim_H h'$. By Lemma 3, we have:

- There exists a bijection ϕ from the set of individuals in h to the set of individuals in h' satisfying the following condition:

  I For every individual i in h such that $\phi(i)$ has at least one supervisor in h', any immediate supervisor j' of $\phi(i)$ in h' must satisfy one of the following:

  I.i If there exists an alternative indirect path from j' to $\phi(i)$ in h' besides the direct edge from j' to $\phi(i)$, then $\phi^{-1}(j')$ must be a direct supervisor of i in h.

  I.ii Otherwise, $\phi^{-1}(j')$ must be either a direct or indirect supervisor of i in h.

- There exists a bijection ϕ' from the set of individuals in h' to the set of individuals in h satisfying the following condition:

  A For every individual i in h' such that $\phi'(i)$ has at least one supervisor in h, any immediate supervisor j' of $\phi'(i)$ in h must satisfy one of the following:

  A.i If there exists an alternative indirect path from j' to $\phi'(i)$ in h besides the direct edge from j' to $\phi'(i)$, then $\phi'^{-1}(j')$ must be a direct supervisor of i in h'.

  A.ii Otherwise, $\phi'^{-1}(j')$ must be either a direct or indirect supervisor of i in h'.

  Moreover:

(i) For each individual i in h with k (direct or indirect) supervisors, $\phi(i)$ has k (direct or indirect) supervisors in h'.

(ii) For each individual i in h' with k (direct or indirect) supervisors, $\phi'(i)$ has k (direct or indirect) supervisors in h.

(iii) Let S_i and $S'_{i'}$ denote the set of all supervisors of i and i' in h and h', respectively. Then, for $i \in h$ and $i' \in h'$,

$$\phi(S_i) = \{\phi(\iota) : \iota \in S_i\} = S'_{\phi(i)} \quad \text{and} \quad \phi'(S'_{i'}) = \{\phi'(\iota) : \iota \in S'_{i'}\} = S_{\phi'(i')}. \tag{9}$$

Let I_l (respectively, I'_l) denote the set of individuals in h (respectively, h') with exactly l supervisors in h (respectively, h').

We must show that h is a relabeling of h'.

To this end, we show that the immediate supervisor relationship is preserved under the mapping ϕ. Specifically:

$$j \text{ is an immediate supervisor of } i \text{ in } h \Leftrightarrow \phi(j) \text{ is an immediate supervisor of } \phi(i) \text{ in } h'.$$

We will establish this bidirectional relationship by induction.

Let us define the depth of a hierarchy $\tilde{h}$ as the maximum number of supervisors that any individual in $\tilde{h}$ can have. By (i)-(ii), h and h' have the same depth, which we denote by k.

We proceed by induction on this depth k. For the base case, when $k = 0$, all nodes in both hierarchies have no supervisors. Since there are no supervisory relationships to preserve, we can take $\phi^{-1} = \phi'$, establishing that h must be a relabeling of h' under this common bijection.

For the inductive step, assume that for any depth $k < \kappa$, we have proven the following: if two hierarchies are related by $\sim_H$ with corresponding bijections ϕ and ϕ' and have depth k, then they must be relabelings of each other under these bijections, and furthermore, ϕ and ϕ' can be chosen to satisfy $\phi^{-1} = \phi'$.

We now prove that this result extends to the case of depth $k = \kappa$.

Suppose that h and h' have depth κ. Note that (i)-(ii) give $\phi(I_\kappa) \subseteq I'_\kappa$ and $\phi'(I'_\kappa) \subseteq I_\kappa$, and, since ϕ and ϕ' are bijections, these containments imply $\phi(I_\kappa) = I'_\kappa$ and $\phi'(I'_\kappa) = I_\kappa$.

Now, let us define $h \setminus I_\kappa$ and $h' \setminus I'_\kappa$ as the hierarchies obtained from h and h' by removing the nodes in I_κ and I'_κ, respectively, along with all edges incident to these nodes.

The depth of these sub-hierarchies $h \setminus I_\kappa$ and $h' \setminus I'_\kappa$ is $\kappa - 1$, since we have removed all nodes with κ supervisors. Moreover, we can show that

$$h \setminus I_\kappa \sim_H h' \setminus I'_\kappa$$

by using the restrictions of the mappings ϕ and ϕ' to the nodes in $I \setminus I_\kappa$ and $I \setminus I'_\kappa$, respectively, where I represents the set of nodes in h and h':

- Any individual i in $h \setminus I_\kappa$ such that $\phi(i)$ has at least one supervisor in $h' \setminus I'_\kappa$ exists also in h. Moreover, any immediate supervisor j' of $\phi(i)$ in $h' \setminus I'_\kappa$ exists also in h'.
  - If there exists an alternative indirect path from j' to $\phi(i)$ in $h' \setminus I'_\kappa$ besides the direct edge from j' to $\phi(i)$, the same path exists also in h'. Hence, $\phi^{-1}(j')$ is a direct supervisor of i in h, and hence also a direct supervisor of i in $h \setminus I_\kappa$.
  - If there is no alternative indirect path from j' to $\phi(i)$ in $h' \setminus I'_\kappa$ besides the direct edge from j' to $\phi(i)$, that path does not exist in h' either. In this case, $\phi^{-1}(j')$ is a direct or indirect supervisor of i in h and hence a supervisor of i in $h \setminus I_\kappa$.

Thus, $h \setminus I_\kappa \succcurlyeq_H h' \setminus I'_\kappa$, and we can similarly show that $h' \setminus I'_\kappa \succcurlyeq_H h \setminus I_\kappa$.

Hence, by the induction hypothesis, we may conclude that:

(a) The restrictions $\phi|_{I \setminus I_\kappa}$ and $\phi'|_{I \setminus I'_\kappa}$ can be chosen so that $\phi^{-1}|_{I \setminus I_\kappa} = \phi'|_{I \setminus I'_\kappa}$.

(b) $h \setminus I_\kappa$ is a relabeling of $h' \setminus I'_\kappa$ under this common bijection.

We now show that h is a relabeling of h' under a bijection that simultaneously satisfies the properties required of both ϕ and ϕ'.

Given that conditions (a) and (b) have been established, it suffices to prove that for every node $j \in I \setminus I_\kappa$, the number of edges from j to nodes in I_κ exactly equals the number of edges from $\phi(j) = \phi'^{-1}(j)$ to nodes in I'_κ.

This edge-count equivalence is crucial because it ensures that the two hierarchies are structurally isomorphic. Specifically, there must exist an extension of the restrictions $\phi|_{I\setminus I_\kappa}$ and $\phi'|_{I\setminus I'_\kappa}$ to a full bijection across all nodes in h and h' such that all supervisory relationships are preserved. Furthermore, any such extension will maintain the essential properties previously established for ϕ and ϕ' for the entire hierarchies h and h'.

Fix $j \in I \setminus I_\kappa$. Then (a) implies

$$\phi(j) = \phi'^{-1}(j).$$

Suppose that there are ℓ edges in h of the form

$$j \longrightarrow i, \quad i \in I_\kappa.$$

We will show that there are, in h', ℓ edges of the form

$$\phi(j) = \phi'^{-1}(j) \longrightarrow i, \quad i \in I'_\kappa.$$

First, we show that

$$j \longrightarrow i, \ i \in I_\kappa \Rightarrow \phi(j) \longrightarrow \phi'^{-1}(i). \tag{10}$$

Proceeding by contradiction, suppose that $i \in I_\kappa$, $j \longrightarrow i$, and $\phi(j) \nrightarrow \phi'^{-1}(i)$.

Since $j \longrightarrow i$ in h, property A implies that $\phi(j) = \phi'^{-1}(j)$ is a supervisor of $\phi'^{-1}(i)$ in h'.

However, since $\phi(j) \nrightarrow \phi'^{-1}(i)$ in h' (i.e., there is no direct supervisory relationship), $\phi(j) = \phi'^{-1}(j)$ must be an indirect supervisor of $\phi'^{-1}(i)$ in h'. This means there exists at least one intermediate node ι' in h' forming a supervision path between $\phi(j) = \phi'^{-1}(j)$ and $\phi'^{-1}(i)$ in h':

$$\phi(j) = \phi'^{-1}(j) \rightsquigarrow \iota' \rightsquigarrow \phi'^{-1}(i), \tag{11}$$

where $a \rightsquigarrow b$ denotes a (possibly indirect) top-down supervision relation between nodes a and b.

Note that because $i \in I_\kappa$, (ii) implies $\phi'^{-1}(i) \in I'_\kappa$. Therefore, ι', being a supervisor of $\phi'^{-1}(i)$ in h', must have less than κ supervisors. Consequently, $\iota' \notin I'_\kappa$, so (a) implies

$$\phi'(\iota') = \phi^{-1}(\iota').$$

Moreover, this equality, together with (11) and (a)-(b) yields

$$j \rightsquigarrow \phi'(\iota') = \phi^{-1}(\iota'). \tag{12}$$

Furthermore, because $\iota' \in S'_{\phi'^{-1}(i)}$ (i.e., ι' is a supervisor of $\phi'^{-1}(i)$ in h') by (11), the second equality in (9) implies

$$\phi'(\iota') = \phi^{-1}(\iota') \in S_i,$$

i.e., $\phi'(\iota') = \phi^{-1}(\iota')$ is a supervisor of i in h:

$$\phi'(\iota') = \phi^{-1}(\iota') \rightsquigarrow i.$$

Combining this relation with (12) yields the following indirect path connecting j and i in h:

$$j \rightsquigarrow \phi'(\iota') = \phi^{-1}(\iota') \rightsquigarrow i. \tag{13}$$

Note that $\phi'(\iota') = \phi^{-1}(\iota')$ is distinct from j and i. Indeed, (11) implies that ι' has more supervisors than $\phi(j) = \phi'^{-1}(j)$ and less supervisors than $\phi'^{-1}(i)$ in h', and so (ii) implies that $\phi'(\iota') = \phi^{-1}(\iota')$ has more supervisors than j and less supervisors than i in h.

Thus, (13) represents an indirect path from j to i in h in addition to the direct edge $j \longrightarrow i$. Consequently, A.i implies the existence of a direct edge $\phi(j) \longrightarrow \phi'^{-1}(i)$ in h', contradicting the assumed $\phi(j) \nrightarrow \phi'^{-1}(i)$.

We conclude that (10) holds, implying that the number of edges of the form

$$\phi(j) = \phi'^{-1}(j) \longrightarrow \hat{\jmath}, \quad \hat{\jmath} \in I'_\kappa,$$

in h' is greater than or equal to the number of edges of the form $j \longrightarrow \hat{\jmath}$, where $\hat{\jmath} \in I_\kappa$, in h.

Next, we show that

$$\phi(j) = \phi'^{-1}(j) \longrightarrow i, \ i \in I'_\kappa \Rightarrow j \longrightarrow \phi^{-1}(i). \tag{14}$$

Proceeding by contradiction, suppose that $i \in I'_\kappa$, $\phi(j) = \phi'^{-1}(j) \longrightarrow i$, and $j \nrightarrow \phi^{-1}(i)$.

Since $\phi(j) = \phi'^{-1}(j) \longrightarrow i$ in h', property I implies that j is a supervisor of $\phi^{-1}(i)$ in h.

However, since $j \nrightarrow \phi^{-1}(i)$ in h (i.e., there is no direct supervisory relationship), j must be an indirect supervisor of $\phi^{-1}(i)$ in h. This means there exists at least one intermediate node ι in h forming a supervision path between j and $\phi^{-1}(i)$ in h:

$$j \rightsquigarrow \iota \rightsquigarrow \phi^{-1}(i). \tag{15}$$

Note that because $i \in I'_\kappa$, (i) implies $\phi^{-1}(i) \in I_\kappa$, so ι, being a supervisor of $\phi^{-1}(i)$ in h, must have less than κ supervisors, implying $\iota \notin I_\kappa$. Consequently, (a) implies

$$\phi'(\iota) = \phi'^{-1}(\iota).$$

Moreover, this equality, together with (15) and (a)-(b) yields

$$\phi(j) = \phi'^{-1}(j) \rightsquigarrow \phi'(\iota) = \phi'^{-1}(\iota). \tag{16}$$

Furthermore, because $\iota \in S_{\phi^{-1}(i)}$ (i.e., ι is a supervisor of $\phi^{-1}(i)$ in h) by (15), the first equality in (9) implies

$$\phi'(\iota) = \phi'^{-1}(\iota) \in S'_i,$$

i.e., $\phi'(\iota) = \phi'^{-1}(\iota)$ is a supervisor of i in h':

$$\phi'(\iota) = \phi'^{-1}(\iota) \rightsquigarrow i.$$

Combining this relation with (16) yields the following indirect path connecting $\phi(j) = \phi'^{-1}(j)$ and i in h':

$$\phi(j) = \phi'^{-1}(j) \rightsquigarrow \phi'(\iota) = \phi'^{-1}(\iota) \rightsquigarrow i. \tag{17}$$

Note that $\phi'(\iota) = \phi'^{-1}(\iota)$ is distinct from $\phi(j) = \phi'^{-1}(j)$ and i. Indeed, (15) implies that ι has more supervisors than j and less supervisors than $\phi^{-1}(i)$ in h, and so (i) implies that $\phi'(\iota) = \phi'^{-1}(\iota)$ has more supervisors than $\phi(j) = \phi'^{-1}(j)$ and less supervisors than i in h'.

Thus, (17) is an indirect path from $\phi(j) = \phi'^{-1}(j)$ to i in h' in addition to the direct edge $\phi(j) = \phi'^{-1}(j) \rightarrow i$. Consequently, I.i implies the existence of a direct edge $j \rightarrow \phi^{-1}(i)$ in h, a contradiction.

We have shown that equation (14) holds, which implies that the number of edges of the form

$$\phi(j) = \phi'^{-1}(j) \rightarrow \hat{\jmath}, \quad \hat{\jmath} \in I'_\kappa,$$

in h' is less than or equal to the number of edges of the form $j \rightarrow \hat{\jmath}$, where $\hat{\jmath} \in I_\kappa$, in h.

Combining this inequality with our previous finding that the number of edges of the form

$$\phi(j) = \phi'^{-1}(j) \rightarrow \hat{\jmath}, \quad \hat{\jmath} \in I'_\kappa,$$

in h' is greater than or equal to the number of edges of the form $j \rightarrow \hat{\jmath}$, where $\hat{\jmath} \in I_\kappa$, in h, we conclude that these two quantities must be equal.

Therefore, each of the inequalities must hold with equality, yielding the desired result. ■

Lemma 5. *For $h, h' \in \mathscr{H}_n$, if h' can be obtained from h by removing a subordination relation, then $h \succ_H h'$.*

Proof. Because h' can be obtained from h by removing a subordination relation, there exist a subordinate i in h and an immediate supervisor j of i satisfying one of the following conditions:

- If j has no supervisors in h, then h' is the hierarchy obtained by removing the directed edge from j to i while preserving all other relationships.
- If j has at least one supervisor in h, then h' is the hierarchy obtained by:
  - Removing the directed edge from j to i.
  - For each direct supervisor j' of j in h, either
    * adding a new directed edge from j' to i if no (direct or indirect) path from j' to i exists in h besides the path through j; or
    * making no change if there already exists an alternative path (besides the path through j) from j' to i in h.

We begin by proving $h \succcurlyeq_H h'$. This relation holds if there exists a bijection ϕ from the individuals in h to those in h' such that, for every individual i in h where $\phi(i)$ has at least one supervisor in h', and for every immediate supervisor j' of $\phi(i)$ in h', one of the following conditions is met:

- If there exists an alternative indirect path from j' to $\phi(i)$ in h' besides the direct edge $j' \rightarrow \phi(i)$, then $\phi^{-1}(j')$ must be a direct supervisor of i in h.
- Otherwise, if no such alternative path exists, $\phi^{-1}(j')$ must be either a direct or indirect supervisor of i in h.

Since h and h' share the same set of individuals, we use the identity map $\phi(i) = i$, which is a bijection. Suppose h' is obtained by removing the edge $j \to i$ from h, as indicated above. We verify the conditions by considering an individual ι in h such that $\phi(\iota) = \iota$ has at least one supervisor in h'.

Consider j' as an immediate supervisor of ι in h'. We analyze how j' relates to ι in h, depending on ι's position relative to the removed edge $j \to i$. If ι is not in the sub-hierarchy $h(i)$—the set of i and its direct and indirect subordinates in h—the supervisory structure above ι remains unchanged. Here, j' is an immediate supervisor of ι in h' if and only if it is in h, and alternative paths from j' to ι are the same in both hierarchies. Thus, $\phi^{-1}(j') = j'$ satisfies the condition: if alternative paths exist, j' is a direct supervisor in h; if not, it is at least a direct supervisor.

If ι is in $h(i)$ but not equal to i, meaning ι is a subordinate of i, the removal of $j \to i$ affects the structure above i, not below it. Edges to ι from its supervisors within h remain intact in h', so j''s role and alternative paths are preserved, satisfying the condition with $\phi^{-1}(j') = j'$.

Now, if $\iota = i$, the removal of $j \to i$ directly impacts i's supervisors. Let j' be an immediate supervisor of $\phi(\iota) = i$ in h'. We consider j''s role in h.

- If $j' \neq j$ and $j' \to i$ exists in h, this edge persists in h', and alternative paths are unchanged, so $\phi^{-1}(j') = j'$ is a direct supervisor in h, meeting the condition.
- If $j' = j$, in h, $j \to i$ exists, but in h', it is removed.
  - If j has no supervisors in h, i loses j as a supervisor in h', and j cannot be an immediate supervisor of i in h', making this case irrelevant.
  - If j has a supervisor j' (so $j' \to j$ in h), and the only path from j' to i in h is $j' \to j \to i$, then in h', removing $j \to i$ may add $j' \to i$. No alternative path from j' to i exists in h', as any such path would imply a path in h not via j, contradicting the condition for adding the edge. Thus, the second condition applies: $\phi^{-1}(j') = j'$ must be a direct or indirect supervisor of i in h. Since $j' \to j \to i$ in h, j' is an indirect supervisor, which suffices.

In all scenarios, the identity map ϕ satisfies the required conditions, so $h \succcurlyeq_H h'$.

Next, we show $h' \not\succcurlyeq_H h$ to confirm the strictness of the relation. Suppose, for contradiction, that $h' \succcurlyeq_H h$. Since $h \succcurlyeq_H h'$, this would imply $h \sim_H h'$, meaning h' is a relabeling of h by Lemma 4, with a bijection preserving immediate supervision relations. In h, j is an immediate supervisor of i due to $j \to i$, but in h', this edge is removed. If j has no supervisors, i loses j as a direct supervisor in h'; if j has supervisors, new edges like $j' \to i$ may appear, but j is no longer a direct supervisor. In both cases, i's immediate supervisors differ between h and h', so no bijection preserves all supervision relations from h' to h, contradicting $h' \sim_H h$. Thus, $h' \not\succcurlyeq_H h$.

Since $h \succcurlyeq_H h'$ and $h' \not\succcurlyeq_H h$, we conclude that $h \succ_H h'$. ■

A.2. Proof of Lemma 1

Lemma 1. *The binary relation $\succcurlyeq_H$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies A and SR.*

Proof. Reflexivity follows immediately from the definition of $\succcurlyeq_H$.

To see that $\succcurlyeq_H$ is transitive, suppose that

$$h \succcurlyeq_H h' \succcurlyeq_H h'', \quad \text{for } h, h', h'' \in \mathscr{H}_n. \tag{18}$$

Let $I_{\tilde{h}}$ represent the set of individuals in $\tilde{h} \in \mathscr{H}_n$. The relations in (18) imply the existence of bijections $\phi : I_h \to I_{h'}$ and $\phi' : I_{h'} \to I_{h''}$ satisfying the following:

1 For every individual i in h such that $\phi(i)$ has at least one supervisor in h', any immediate supervisor j' of $\phi(i)$ in h' must satisfy one of the following:

  1.a If there exists an alternative indirect path from j' to $\phi(i)$ in h' besides the direct edge from j' to $\phi(i)$, then $\phi^{-1}(j')$ must be a direct supervisor of i in h.

  1.b Otherwise, $\phi^{-1}(j')$ must be either a direct or indirect supervisor of i in h.

2 For every individual i in h' such that $\phi'(i)$ has at least one supervisor in h'', any immediate supervisor j' of $\phi'(i)$ in h'' must satisfy one of the following:

  2.a If there exists an alternative indirect path from j' to $\phi'(i)$ in h'' besides the direct edge from j' to $\phi'(i)$, then $\phi'^{-1}(j')$ must be a direct supervisor of i in h'.

  2.b Otherwise, $\phi'^{-1}(j')$ must be either a direct or indirect supervisor of i in h'.

Since ϕ and ϕ' are bijections, the composition $\phi^* = \phi' \circ \phi : I_h \to I_{h''}$ is also a bijection (see, e.g., Blyth, 1975, Theorem 5.10, p. 37). Thus, it suffices to show the following:

3 For every individual i in h such that $\phi^*(i)$ has at least one supervisor in h'', any immediate supervisor j' of $\phi^*(i)$ in h'' must satisfy one of the following:

  3.a If there exists an alternative indirect path from j' to $\phi^*(i)$ in h'' besides the direct edge from j' to $\phi^*(i)$, then $\phi^{*-1}(j')$ must be a direct supervisor of i in h.

  3.b Otherwise, $\phi^{*-1}(j')$ must be either a direct or indirect supervisor of i in h.

Fix individual i in h such that $\phi^*(i)$ has at least one supervisor in h''. Take any immediate supervisor j' of $\phi^*(i)$ in h''. By 2, there exists a (possibly indirect) path connecting $\phi'^{-1}(j')$ and $\phi(i)$ in h':

$$\phi'^{-1}(j') \to i_1 \to \cdots \to i_\ell \to \phi(i).$$

Now successively apply 1 to this chain to obtain the existence of the following (possibly indirect) paths connecting nodes in h:

- $\phi^{-1}(i_\ell) \rightsquigarrow i$;
- $\phi^{-1}(i_{\ell-1}) \rightsquigarrow \phi^{-1}(i_\ell)$;

  $\vdots$

- $\phi^{-1}(i_1) \rightsquigarrow \phi^{-1}(i_2)$; and
- $\phi^{-1}(\phi'^{-1}(j')) \rightsquigarrow \phi^{-1}(i_1)$.

Combining these chains gives

$$\phi^{-1}(\phi'^{-1}(j')) \rightsquigarrow i = \phi^{*-1}(j') \rightsquigarrow i. \tag{19}$$

This (possibly indirect) path connecting $\phi^{*-1}(j')$ and i in h must be direct, as required by 3.a, if there exists, in h'', an alternative indirect path from j' to $\phi^*(i)$, $j' \rightsquigarrow \phi^*(i)$, besides the direct edge $j' \to \phi^*(i)$. Indeed, the existence of such alternative path in h'' implies:

- The existence of a direct edge $\phi'^{-1}(j') \to \phi(i)$ in h', by 2.a.
- The existence of an indirect path $\phi'^{-1}(j') \rightsquigarrow \phi(i)$ besides the direct edge $\phi'^{-1}(j') \to \phi(i)$, by successive application of 2 to the path $j' \rightsquigarrow \phi^*(i)$.

Consequently, 1.a implies that the path in (19) must be a direct edge, $\phi^{*-1}(j') \to i$, if there exists, in h'', an alternative indirect path from j' to $\phi^*(i)$, $j' \rightsquigarrow \phi^*(i)$, besides the direct edge $j' \to \phi^*(i)$.

We conclude that 3 holds, implying that $\succcurlyeq_H$ is transitive.

By Lemma 5, $\succcurlyeq_H$ satisfies SR.

To see that $\succcurlyeq_H$ satisfies A, let h' be a relabeling of h. Then there exists a bijection $\phi : I_h \to I_{h'}$ with the following property:

$$\forall i, j \in h,\ j \to i \Leftrightarrow \phi(j) \to \phi(i).$$

This condition immediately gives $h \succcurlyeq_H h'$ and $h' \succcurlyeq_H h$, implying $h \sim_H h'$. ■

A.3. Proof of Theorem 1

Theorem 1. *For hierarchies $h, h' \in \mathscr{H}_n$, $h \succ_H h'$ if and only if h' can be obtained from some relabeling of h by successive removals of subordination relations.*

Proof. (*Sufficiency*.) Suppose that h' can be obtained from some relabeling of h, denoted by $\hat{h}$, by successive removals of subordination relations. Then, by applying Lemma 5, we have

$$\hat{h} \succ_H h_L \succ_H \cdots \succ_H h_1 \succ_H h'$$

for some finite sequence of hierarchies $h_1, \ldots, h_L$ in $\mathscr{H}_n$.

By the reflexivity and transitivity properties of $\succcurlyeq_H$ (Lemma 1), it follows that $\hat{h} \succ_H h'$ (Sen, 2017, Lemma 1*a, p. 56).

Moreover, since $\hat{h}$ is a relabeling of h, Lemma 1 gives us $h \sim_H \hat{h}$.

Consequently, we have

$$h \sim_H \hat{h} \succ_H h',$$

which implies that $h \succ_H h'$ by the reflexivity and transitivity of the relation $\succcurlyeq_H$ (Sen, 2017, Lemma 1*a, p. 56).

(*Necessity*.) Suppose that $h \succ_H h'$. We must show that h' can be obtained from some relabeling of h by successive removals of subordination relations.

Since $h \succ_H h'$, we have $h \succcurlyeq_H h'$, so there exists a bijection ϕ from the set of individuals in h to the set of individuals in h' satisfying the following condition:

1 For every individual i in h such that $\phi(i)$ has at least one supervisor in h', any immediate supervisor j' of $\phi(i)$ in h' must satisfy one of the following:

1.1 If there exists an alternative indirect path from j' to $\phi(i)$ in h' besides the direct edge from j' to $\phi(i)$, then $\phi^{-1}(j')$ must be a direct supervisor of i in h.

1.2 Otherwise, $\phi^{-1}(j')$ must be either a direct or indirect supervisor of i in h.

We structure the remainder of the proof as a series of claims and their logical implications. To maintain clarity in the main argument, we first present all claims and their interconnections, then provide complete proofs for each claim in sequence at the end of this section.

Claim 1. *Either there exists and edge $\jmath^* \to \iota^*$ in h such that $\phi(\jmath^*) \nrightarrow \phi(\iota^*)$ in h' or there exists an edge $\phi(\tilde{\jmath}) \to \phi(\tilde{\imath})$ in h' such that $\tilde{\jmath} \rightsquigarrow \iota \to \tilde{\imath}$ in h.*

We define the depth of a node in a hierarchy as the number of supervisors in its reporting chain, that is, the cardinality of its set of direct and indirect supervisors.

If there exists an edge $\phi(\tilde{\jmath}) \to \phi(\tilde{\imath})$ in h' such that $\tilde{\jmath} \rightsquigarrow \iota \to \tilde{\imath}$ in h, identify the edge $\iota \to \tilde{\imath}$ for which $\tilde{\imath}$ attains maximum depth among all such edges, and apply the subordination removal procedure to this edge. Otherwise, apply the same procedure to the edge $\jmath^* \to \iota^*$ from h as specified in Claim 1. Denote the resulting hierarchy as h^*.

Claim 2. *$h \succ_H h^* \succcurlyeq_H h'$, where the relation $h^* \succcurlyeq_H h'$ can be established via ϕ.*

If $h' \succcurlyeq_H h^*$, then $h^* \sim_H h'$. Consequently, by Lemma 4, we can conclude that h^* is a relabeling of h'. This completes the proof of the result. Indeed, we have shown that h' is equivalent (up to relabeling) to a hierarchy h^* that can be obtained from h through the removal of a subordination relation.

Now suppose that $h' \not\succcurlyeq_H h^*$. Then, by Claim 2, we have $h^* \succ_H h'$.

In this case, Claim 1 applies again with h^* replacing h. The proof of this statement exactly mimics the proof of Claim 1.

We now iterate this argument: if edges of the form $\iota \to \tilde{\imath}$ exist in h^*, select the one for which $\tilde{\imath}$ attains maximum depth among all such edges and apply subordination removal to this edge; otherwise, apply the procedure to the edge $\jmath^* \to \iota^*$. This operation yields a new hierarchy that assumes the role of h^* in Claim 2. By repeating this process a finite number of times, we ultimately obtain a hierarchy h° that is a relabeling of h', constructed from h through successive removals of subordination relations.

After sufficiently many such removals, the resulting hierarchy h° accounts for all edges $\phi(j) \to \phi(i)$ in h' that correspond (via ϕ^{-1}) to indirect paths between j and i in h, as well as all edges $j \to i$ in h for which $\phi(j) \nrightarrow \phi(i)$ in h'. At this point, we have $h^\circ \sim_H h'$, which by Lemma 4 implies that h° is a relabeling of h', thereby establishing our desired conclusion.

We now proceed to prove the sequence of claims stated above. For clarity and convenience, each claim will be restated immediately before its corresponding proof.

Claim 1. *Either there exists and edge $\jmath^* \to \iota^*$ in h such that $\phi(\jmath^*) \nrightarrow \phi(\iota^*)$ in h' or there exists an edge $\phi(\tilde{\jmath}) \to \phi(\tilde{\imath})$ in h' such that $\tilde{\jmath} \rightsquigarrow \iota \to \tilde{\imath}$ in h.*

Proof of Claim 1. Suppose that there is no edge $\jmath^* \to \iota^*$ in h such that $\phi(\jmath^*) \nrightarrow \phi(\iota^*)$ in h'. We must show that there exists an edge $\phi(\tilde{\jmath}) \to \phi(\tilde{\imath})$ in h' such that $\tilde{\jmath} \rightsquigarrow \iota \to \tilde{\imath}$ in h.

First, observe that h must contain at least one edge $j \to i$. If h were edgeless, then by the relation $h \succcurlyeq_H h'$, we would necessarily have that h' contains no edges either. This would imply $h \sim_H h'$, directly contradicting our assumption of strict dominance $h \succ_H h'$.

Note that the properties of ϕ in 1 imply that for every edge $\phi(j) \to \phi(i)$ in h', we have a corresponding path $j \rightsquigarrow i$ in h. Seeking a contradiction, assume that these paths are all direct edges: for every edge $\phi(j) \to \phi(i)$ in h', we have a direct edge $j \to i$ in h. We symbollically represent this implication as follows:

$$(\phi(j) \to \phi(i)) \in h' \Rightarrow (j \to i) \in h. \tag{20}$$

Since there are no edges $j \to i$ in h where $\phi(j) \nrightarrow \phi(i)$ in h', (20) implies:

$$\forall i, j \in h,\; j \to i \Leftrightarrow \phi(j) \to \phi(i).$$

This bidirectional correspondence implies $h \sim_H h'$, which contradicts our hypothesis that $h \succ_H h'$, completing our proof. □

Claim 2. *$h \succ_H h^* \succcurlyeq_H h'$, where the relation $h^* \succcurlyeq_H h'$ can be established via ϕ.*

Proof of Claim 2. Since h^* results from removing a subordination relation in h, Lemma 5 gives $h \succ_H h^*$.

The proof that $h^* \succcurlyeq_H h'$ (via ϕ) is divided into two cases. We must prove the following:

2 For every individual i in h^* such that $\phi(i)$ has at least one supervisor in h', any immediate supervisor j' of $\phi(i)$ in h' must satisfy one of the following:

  2.1 If there exists an alternative indirect path from j' to $\phi(i)$ in h' besides the direct edge from j' to $\phi(i)$, then $\phi^{-1}(j')$ must be a direct supervisor of i in h^*.

  2.2 Otherwise, $\phi^{-1}(j')$ must be either a direct or indirect supervisor of i in h^*.

Case 1. h^* results from applying subordination removal to the edge $\iota \to \tilde{\iota}$ specified in Claim 1, where $\tilde{\iota}$ has maximum depth.

Condition 2 requires that direct edges connecting nodes in h' correspond to possibly indirect paths connecting their pre-images in h^*. The subordination removal that transforms h into h^* consists of removing the edge $\iota \to \tilde{\iota}$ (and potentially adding edges linking $\tilde{\iota}$ to ι's immediate supervisors in h). Since $h \succcurlyeq_H h'$ implies that condition 1 holds, the only edges $j' \to \phi(i)$ in h' that might violate 2 are those where a pre-image path

$$\phi^{-1}(j') \rightsquigarrow i$$

in h intersects with the removed edge $\iota \to \tilde{\iota}$.

Therefore, it suffices to verify condition 2 for each node i in h^* where $\phi(i)$ has at least one supervisor in h', and for every immediate supervisor j' of $\phi(i)$ in h' such that $\phi^{-1}(j') \rightsquigarrow i$ in h and this path intersects with the edge $\iota \to \tilde{\iota}$ from h.

For such an intersection to exist, $\phi^{-1}(j')$ must be a (direct or indirect) supervisor of ι in h. Indeed, we cannot have $\phi^{-1}(j') = \iota$ unless i is a (direct or indirect) subordinate of $\tilde{\iota}$ in h, which would give us $j' \to \phi(i)$ in h' and

$$\phi^{-1}(j') = \iota \to \tilde{\iota} \rightsquigarrow i,$$

implying that i has greater depth than $\tilde{\iota}$, contradicting our selection of $\tilde{\iota}$ as having maximum depth.

Consequently, the path $\phi^{-1}(j') \rightsquigarrow i$ must traverse through a direct supervisor of ι in h. Furthermore, i cannot be a (direct or indirect) subordinate of $\tilde{\imath}$, as this would contradict $\tilde{\imath}$'s maximum depth. Nor can i be a supervisor of $\tilde{\imath}$, as this would preclude any intersection with the removed edge.

Hence, $i = \tilde{\imath}$ and the path $\phi^{-1}(j') \rightsquigarrow i$ necessarily passes through a direct supervisor of ι in h. In this situation, the removal of edge $\iota \to \tilde{\imath}$ does not prevent condition 2 from being satisfied:

- If there exists an alternative indirect path from j' to $\phi(i)$ in h' besides the direct edge $j' \to \phi(i)$, then by condition 1.1, $\phi^{-1}(j')$ must be a direct supervisor of i in h, i.e., $\phi^{-1}(j') \to i$ in h. This direct edge persists in h^* and does not intersect with the removed edge $\iota \to \tilde{\imath}$. This establishes condition 2.1.

- If no alternative indirect path exists from j' to $\phi(i)$ in h' beyond the direct edge $j' \to \phi(i)$, then $\phi^{-1}(j') \rightsquigarrow i = \tilde{\imath}$ in h^*. This is because the path $\phi^{-1}(j') \rightsquigarrow i = \tilde{\imath}$ in h traverses through an immediate supervisor of ι in h which, after subordination removal, must connect (directly or indirectly) to $i = \tilde{\imath}$ in h^*. By definition of subordination removal, a direct edge from any immediate supervisor of ι in h to $i = \tilde{\imath}$ must be added if these nodes are not already connected in h. This establishes condition 2.2.

Case 2. h^* results from applying subordination removal to an edge $\jmath^* \to \iota^*$ in h such that $\phi(\jmath^*) \nrightarrow \phi(\iota^*)$ in h'.

This case applies when h^* contains no edges of the form $\iota \to \tilde{\imath}$ as specified in Claim 1. As in the previous case, it suffices to verify condition 2 for each node i in h^* where $\phi(i)$ has at least one supervisor in h', and for every immediate supervisor j' of $\phi(i)$ in h' such that $\phi^{-1}(j') \rightsquigarrow i$ in h and this path intersects with the edge $\jmath^* \to \iota^*$ from h.

First, observe that $\phi^{-1}(j') = \jmath^*$ requires $\phi(\jmath^*) = j'$. Since $\phi(\jmath^*) \nrightarrow \phi(\iota^*)$ in h', in this scenario an intersection between $\phi^{-1}(j') \rightsquigarrow i$ and $\jmath^* \to \iota^*$ can only occur if i is a (direct or indirect) subordinate of ι^* in h. This would produce $\phi(\jmath^*) \to \phi(i)$ in h' and $\jmath^* \to \iota^* \rightsquigarrow i$, contradicting the premise that h^* has no edges of the form $\iota \to \tilde{\imath}$ as specified in Claim 1.

Therefore, the intersection between $\phi^{-1}(j') \rightsquigarrow i$ and $\jmath^* \to \iota^*$ can only occur if $\phi^{-1}(j')$ is a (direct or indirect) supervisor of $\jmath^*$ in h and $i = \iota^*$. We consider two subcases:

- If there exists an alternative indirect path from j' to $\phi(i)$ in h' besides the direct edge $j' \to \phi(i)$, then by condition 1.1, $\phi^{-1}(j') \to i$ in h. This direct edge persists in h^* and does not intersect with the removed edge $\jmath^* \to \iota^*$, thereby establishing condition 2.1.

- If no alternative indirect path exists from j' to $\phi(i)$ in h' beyond the direct edge $j' \to \phi(i)$, then $\phi^{-1}(j') \rightsquigarrow i = \iota^*$ in h^*. Indeed, $\phi^{-1}(j')$ is a (direct or indirect) supervisor of $\jmath^*$ in h, so the path $\phi^{-1}(j') \rightsquigarrow i = \iota^*$ in h passes through an immediate supervisor of $\jmath^*$ in h which, after subordination removal, must connect (directly or indirectly) to $i = \iota^*$ in h^*. This establishes condition 2.2. □

The proof of Theorem 1 is now complete. ■

A.4. Proof of Proposition 1

Proposition 1. *The binary relation $\succcurlyeq_s$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies axioms* A *and* SR.

Proof. Reflexivity follows immediately from the definition of $\succcurlyeq_s$.

To see that $\succcurlyeq_s$ is transitive, suppose that

$$h \succcurlyeq_s h' \succcurlyeq_s h'', \quad \text{for } h, h', h'' \in \mathscr{H}_n.$$

Then, letting $I_{\hat{h}}$ represent the set of individuals in hierarchy $\hat{h}$, there exist bijections $\phi : I_h \to I_{h'}$ and $\phi' : I_{h'} \to I_{h''}$ satisfying the following:

- For each individual i in h, the number of supervisors of i in h, $s_h(i)$, is greater than or equal to the number of supervisors of $\phi(i)$ in h', $s_{h'}(\phi(i))$.
- For each individual i in h', the number of supervisors of i in h', $s_{h'}(i)$, is greater than or equal to the number of supervisors of $\phi'(i)$ in h'', $s_{h''}(\phi'(i))$.

Since ϕ and ϕ' are bijections, the composition $\phi^* = \phi' \circ \phi$ is also a bijection (see, e.g., Blyth, 1975, Theorem 5.10, p. 37). Moreover, for each individual i in h, we have

$$s_h(i) \geq s_{h'}(\phi(i)) \geq s_{h''}(\phi'(\phi(i))).$$

Consequently, for each individual i in h,

$$s_h(i) \geq s_{h''}([\phi' \circ \phi](i)) = s_{h''}(\phi^*(i)),$$

implying that $h \succcurlyeq_s h''$.

To see that $\succcurlyeq_s$ satisfies A, let h' be a relabeling of h. Then there exists a bijection $\phi : I_h \to I_{h'}$ with the following property:

$$\forall i, j \in h, \ j \to i \Leftrightarrow \phi(j) \to \phi(i).$$

This condition immediately gives $h \succcurlyeq_s h'$ and $h' \succcurlyeq_s h$, implying $h \sim_s h'$.

It remains to show that $\succcurlyeq_s$ satisfies SR. Suppose that h' can be obtained from h by removing a subordination relation. Then there exist a subordinate i in h and an immediate supervisor j of i satisfying one of the following conditions:

(i) If j has no supervisors in h, then h' is the hierarchy obtained by removing the directed edge from j to i while preserving all other relationships.

(ii) If j has at least one supervisor in h, then h' is the hierarchy obtained by:

  - Removing the directed edge from j to i.
  - For each direct supervisor j' of j in h, either
    - adding a new directed edge from j' to i if no (direct or indirect) path from j' to i exists in h besides the path through j; or
    - making no change if there already exists an alternative path (besides the path through j) from j' to i in h.

We must show that $h \succ_s h'$.

Recall that obtaining h' from h via subordination removal involves eliminating an edge $\jmath \to \iota$ from h and potentially adding edges between ι and $\jmath$'s direct supervisors. These new edges are added precisely when removing $\jmath \to \iota$ would otherwise break the chain of command between ι and these supervisors.

This operation has a clear effect on the supervisory structure: each node in h' has at most the same number of supervisors as its counterpart in h, with at least one node having strictly fewer supervisors in h' than in h. Therefore, the average number of supervisors in h must exceed that in h'.

To establish $h \succ_s h'$, it suffices to demonstrate that $h \succcurlyeq_s h'$. This is because if both $h \succcurlyeq_s h'$ and $h' \succcurlyeq_s h$ were true, we would have $h \sim_s h'$, which would imply equal average numbers of supervisors in both hierarchies—contradicting our earlier conclusion.

Thus, we only need to prove that $h \succcurlyeq_s h'$.

This relation holds if there exists a bijection φ from the individuals in h to those in h' such that

$$s_h(\iota) \geq s_{h'}(\varphi(\iota)) \quad \text{for each } \iota \in h. \tag{21}$$

Let us take the identity map, where $\varphi(\iota) = \iota$ for each $\iota \in h$. Recall that the subordination removal in question eliminates the edge $j \to i$ from h, as specified in items (i)-(ii).

We consider two cases:

- If j has no supervisors in h, then h' is obtained by simply removing the directed edge from j to i while preserving all other relationships. In this scenario, only i and its subordinates experience a reduction in supervisor count, while all other nodes maintain the same number of supervisors. Thus, inequality (21) holds for all nodes.

- If j has at least one supervisor in h, then the removal of edge $j \to i$ causes i to lose j as a supervisor in h'. Although new edges may be added between i and j's supervisors, these supervisors were already indirectly supervising i in h (through j). Therefore, i experiences a net decrease in its supervisor count. Similarly, all subordinates of i lose j as a supervisor without gaining any new supervisors. For all other nodes, the supervisor count remains unchanged between h and h'.

In both cases, inequality (21) is satisfied for all nodes, establishing that $h \succcurlyeq_s h'$. ■

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