Inequality and Bipolarization Reducing Mixed Taxation

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Abstract

Progressive income (resp., commodity) tax systems have been characterized in terms of their ability to reduce income inequality and bipolarization. This paper provides necessary and sufficient conditions on the structure of *mixed* tax systems—subjecting both income and consumption to taxation—ensuring a reduction in income inequality and bipolarization, both in the cases of exogenous and endogenous income. Commodity taxation is "superfluous" in the case of exogenous income, in the sense that any post-tax income distribution resulting from the implementation of a mixed tax system is attainable by means of an income tax. By contrast, under endogenous income, there are cases when relying solely on income taxes is ineffective, while mixed tax structures have equalizing and depolarizing power.

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JEL classifications: D63, D71.

1. Introduction

The analysis of progressive income tax structures and their ability to reduce income inequality goes back to the seminal works of Jakobsson (1976), Fellman (1976), and Kakwani (1977), which established the equivalence between increasing average tax rates on income—or average-rate income tax progressivity—and an income tax system's ability to invariably reduce income inequality.[1] While the Jakobsson-Fellman-Kakwani result is framed in terms of endowment economies with exogenous income, extensions of their analysis to the case of endogenous income are provided in Carbonell-Nicolau and Llavador (2018, 2021b). A further paper, Carbonell-Nicolau and Llavador (2021a), adopts an alternative evaluation criterion for the distributional effects of income tax policies, namely their ability to reduce income bipolarization, as measured by a standard relative metric proposed in Foster and Wolfson (2010), and establishes the general equivalence

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[1]Various extensions of this result can be found in Hemming and Keen (1983); Eichhorn et al. (1984); Liu (1985); Formby et al. (1986); Thon (1987); Latham (1988); Thistle (1988); Moyes (1988, 1989, 1994); Le Breton et al. (1996); Ebert and Moyes (2000); Ju and Moreno-Ternero (2008).

between inequality and bipolarization reducing income tax schedules. The effects of commodity—as opposed to income—taxation on income inequality have been studied in Carbonell-Nicolau (2019).

This paper considers *mixed tax systems*, i.e., tax systems subjecting both income and consumption to taxation, and their ability to reduce income inequality and bipolarization. The analysis begins with the case of exogenous income. Any mixed tax system determines a mapping from an initial distribution of endowment incomes to a corresponding post-tax income distribution. If the latter distribution Lorenz dominates (in the relative sense) the former distribution, for every possible initial distribution, the underlying tax system is said to be *inequality reducing*.

The first main result of the paper (Theorem 1) states that a mixed tax system is inequality reducing if and only if disposable income increases with pre-tax income and the tax system is jointly average-rate progressive (i.e., it exhibits increasing average tax rates on income). The statement of this result is followed by a discussion on its interpretation and its implications for the tax treatment of luxuries and necessities.

The Foster-Wolfson bipolarization order (Foster and Wolfson, 2010; Chakravarty, 2009, 2015) is a relatively standard measure of an income distribution's degree of polarization between two income groups delimited by the distribution's median income. A mixed tax system is said to be *bipolarization reducing* if it yields a less bipolarized post-tax income distribution, as measured by the Foster-Wolfson order, regardless of the pre-tax income distribution the tax system is applied to. This alternative evaluation criterion for mixed tax systems can be characterized in terms of joint average-rate progressivity: a mixed tax system is bipolarization reducing if and only if disposable income increases with pre-tax income and the tax system is jointly average-rate progressive (Theorem 2).

The case of endogenous income is considered in Section 2.2. Under endogenous income, individuals choose their most preferred consumption bundle and their labor supply, given their wage rate and any taxes on their consumption and income. Given a wage rate distribution and a mixed tax system, individual labor and consumption choices give rise to an income distribution. Inequality (resp., bipolarization) reducing tax systems are defined as in the case of exogenous income, except that we now compare income distributions from a taxless environment with those arising when individuals are subjected to mixed taxation: a mixed tax system is inequality (resp., bipolarization) reducing if it yields a more equal (resp., less bipolarized) income distribution, relative to the taxless distribution, for any distribution of wage rates.

The following characterizations of inequality (resp., bipolarization) reducing tax systems are proven under endogenous income. First, a mixed tax system is inequality reducing only if disposable income increases with the wage rate and the income tax is marginal-rate progressive (i.e., it exhibits increasing marginal tax rates on income) (Theorem 4). Second, families of inequality reducing mixed tax systems can be characterized in terms of a condition on the wage elasticity of disposable income (Theorem 5), which is amenable to interpretation in terms of the tax treatment of luxuries and necessities. The equivalence between inequality and bipolarization reducing tax systems is proven in Theorem 6.

Mixed tax systems have been studied in the literature on optimal taxation. In particular, combined tax policies subjecting both income and consumption to taxation and their role in attaining maximum welfare have been considered in Atkinson and Stiglitz (1976), which establishes a "redundance theorem" whereby (under some assumptions) "the optimal tax system can rely solely on income taxation." A similar result can be proven in our framework in the case of exogenous income. Specifically, if the consumption of inferior goods is not taxed, any post-tax income distribution resulting from the implementation of an inequality reducing (resp., a bipolarization reducing) mixed tax system is attainable via pure income taxation. This result, however, ceases to hold in the case of endogenous income: as illustrated by means of an example, there are cases when, unlike pure income taxes, combinations of income and consumption taxes are effective in achieving the goal of inequality and bipolarization reduction.

2. Distributional properties of mixed taxation

In this section, we consider notions of progressivity for mixed tax systems and characterize them in terms of measures of income inequality and bipolarization.

The analysis has two parts. In Section 2.1, income is treated as exogenous, i.e., unresponsive to income taxation (while consumption is responsive to commodity taxation). The analysis for the case of endogenous income is provided in Section 2.2.

2.1. Exogenous income

Individual preferences are represented by a utility function $u : \mathbb{R}^K_+ \to \mathbb{R}$ defined on commodity bundles $\boldsymbol{x} = (x_1, \ldots, x_K) \in \mathbb{R}^K_+$, where x_k denotes the quantity of traded good $k \in \{1, \ldots, K\}$.

The utility function u is assumed continuous and nondecreasing, strictly increasing on $\mathbb{R}^K_{++}$, and strictly quasiconcave on $\mathbb{R}^K_{++}$.

An *income tax schedule* is a continuous and nondecreasing map $T : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

- $T(y) \leq y$ for each $y \in \mathbb{R}_+$.
- The map $y \mapsto y - T(y)$ is nondecreasing (i.e., T is order-preserving).

Here, $T(y) > 0$ (resp., $T(y) < 0$) represents the tax paid (resp., a subsidy received) by an individual whose income is y.

A *commodity tax system* is a vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_K)$ of tax rates, one for each traded good. For each good k, $\tau_k > 0$ (resp., $\tau_k < 0$) represents the tax liability (resp., subsidy) paid (resp., received) per unit of good k consumed.

A *mixed tax system* is a tuple $(T, \boldsymbol{\tau})$, where T is an income tax schedule and $\boldsymbol{\tau}$ is a commodity tax system.

Given an income tax schedule T, a commodity tax system $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_K)$, and a commodity price vector $\boldsymbol{p} = (p_1, \ldots, p_K)$ such that $p_k > 0$ and $p_k + \tau_k > 0$ for each k, an individual whose income is y solves the following problem:

$$\begin{aligned} &\max_{(x_1,\ldots,x_K)\in\mathbb{R}^K_+} u(x_1, \ldots, x_K) \\ &\text{s.t. } (p_1 + \tau_1)x_1 + \cdots + (p_K + \tau_K)x_K \leq y - T(y). \end{aligned}$$

The properties of u entail that this problem has a unique solution, denoted by

$$x_1(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y)), \ldots, x_K(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y)),$$

where each $x_k(\boldsymbol{p}', y')$ represents the individual's Marshallian demand function for good k corresponding to net price vector $\boldsymbol{p}'$ and net income y'. These demand functions, together with the mixed tax system, give rise to disposable income

$$z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) = y - T(y) - \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y)) = \sum_{k=1}^{K} p_k x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y)). \quad (1)$$

An *income distribution* is a vector $\boldsymbol{y} = (y_1, \ldots, y_n)$, where y_i represents individual i's income level. The population size, n, takes values in the set of natural numbers. Let $(y_{[1]}, \ldots, y_{[n]})$ be a rearrangement of the coordinates in $\boldsymbol{y}$ such that

$$y_{[1]} \leq \cdots \leq y_{[n]}.$$

Throughout the sequel, we restrict attention to income distributions whose median income is positive.

In this paper, inequality is measured by means of the relative Lorenz order, defined as follows. Given two income distributions $\boldsymbol{y} = (y_1, \ldots, y_n)$ and $\boldsymbol{y}' = (y'_1, \ldots, y'_n)$, $\boldsymbol{y}'$ is said to *Lorenz dominate* $\boldsymbol{y}$, denoted by '$\boldsymbol{y}' \succcurlyeq_L \boldsymbol{y}$,' if

$$\frac{\sum_{i=1}^{l} y'_{[i]}}{\sum_{i=1}^{n} y'_{[i]}} \geq \frac{\sum_{i=1}^{l} y_{[i]}}{\sum_{i=1}^{n} y_{[i]}}, \quad \text{for all } l \in \{1, \ldots, n\}.$$

The interpretation of the dominance relation '$\boldsymbol{y}' \succcurlyeq_L \boldsymbol{y}$' is that '$\boldsymbol{y}'$ is at least as equal as $\boldsymbol{y}$.'

Given a price vector $\boldsymbol{p}$ and a pre-tax income distribution $(y_1, \ldots, y_n)$, a mixed tax system $(T, \boldsymbol{\tau})$ gives rise to a post-tax income distribution

$$(z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \ldots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)).$$

Given a price vector $\boldsymbol{p}$, a mixed tax system $(T, \boldsymbol{\tau})$ is *inequality reducing* if

$$(z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \ldots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)) \succcurlyeq_L (y_1, \ldots, y_n)$$

for every pre-tax income distribution $(y_1, \ldots, y_n)$ and every population size n.

When T (resp., $\boldsymbol{\tau}$) is identically zero, we say that the income tax schedule T (resp., the commodity tax system $\boldsymbol{\tau}$) is inequality reducing.

An income tax schedule is *average-rate progressive* if, for $y > 0$, the tax rate $T(y)/y$ is nondecreasing in y.

Given a price vector $\boldsymbol{p}$, a commodity tax system $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_K)$ is *average-rate progressive* if, for $y > 0$, the tax rate

$$\frac{1}{y}\left(\sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}+\boldsymbol{\tau}, y)\right)$$

is nondecreasing in y.

A mixed tax system $(T, \boldsymbol{\tau})$ is *separately average-rate progressive* if T (resp., $\boldsymbol{\tau}$) is average-rate progressive; and *jointly average-rate progressive* if, for $y > 0$, the average tax rate,

$$\frac{1}{y}\left(T(y) + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y))\right),$$

is nondecreasing in y.

Joint and separate average-rate progressivity are logically nested in the following sense. Separate average-rate progressivity implies joint average-rate progressivity, but the converse is not true.[2]

Our first main result is a characterization of inequality reducing mixed tax systems in terms of jointly progressive mixed taxation.

Theorem 1. *Given a price vector $\boldsymbol{p}$, a mixed tax system $(T, \boldsymbol{\tau})$ is inequality reducing if and only if the disposable income function $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in the pre-tax income level y and $(T, \boldsymbol{\tau})$ is jointly average-rate progressive.*

The proof of Theorem 1 is relegated to Appendix A.

In general, $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ need not be nondecreasing in y. Indeed, using the identity in (1) it is easy to see that if good k is inferior and if p_k is large enough relative to the (gross) prices of the other goods ($p_{k'}$, $k' \neq k$), then the map $y \mapsto z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ may well be decreasing in y.

The following are immediate corollaries of Theorem 1.

- An income tax schedule is inequality reducing if and only if it is average-rate progressive. This is the classic Jakobsson-Fellman-Kakwani result (Jakobsson, 1976; Fellman, 1976; Kakwani, 1977).
- A commodity tax system is inequality reducing if and only if it is average-rate progressive and disposable income is nondecreasing with pre-tax income.

Inequality reducing (hence average-rate progressive) commodity tax systems have been characterized in terms of the tax treatment of luxury (resp., necessary) commodities (Carbonell-Nicolau, 2019).

A luxury (resp., necessary) commodity is a commodity for which the proportion of total income spent on it rises (resp., declines) with income.

Assuming differentiable demand functions, a *luxury commodity* k can be formally defined in terms of the following condition:

$$\frac{\partial(p_k x_k(\boldsymbol{p}, y)/y)}{\partial y} > 0, \quad \text{for every } (\boldsymbol{p}, y).^{3} \tag{2}$$

(Recall that $x_k(\boldsymbol{p}, y)$ denotes the standard Marshallian demand function for good k.)

A commodity k is a *necessity* if

$$\frac{\partial(p_k x_k(\boldsymbol{p}, y)/y)}{\partial y} < 0, \quad \text{for every } (\boldsymbol{p}, y). \tag{3}$$

[2] See Appendix F for a proof of this assertion.

[3] Differentiability is obviously not necessary for this definition.

Conditions (2) and (3) can be expressed as follows:

$$\frac{\partial x_k(\boldsymbol{p},y)}{\partial y} > \frac{x_k(\boldsymbol{p},y)}{y}, \quad \text{for every } (\boldsymbol{p},y), \tag{4}$$

and

$$\frac{\partial x_k(\boldsymbol{p},y)}{\partial y} < \frac{x_k(\boldsymbol{p},y)}{y}, \quad \text{for every } (\boldsymbol{p},y). \tag{5}$$

These conditions are amenable to interpretation. For example, multiplying both sides of (4) by p_k, we see that good k is a luxury if the marginal propensity to spend on good k (i.e., the fraction of an extra dollar spent on good k) exceeds the average propensity to spend on good k (i.e., the current fraction of total income spent on good k). Condition (5) can be given a similar interpretation.

It is easy to see that luxury goods are normal (in the sense that their demand increases with income), but the converse does not generally hold. Similarly, inferior goods are necessities, but necessities need not be inferior.

A reformulation of average-rate progressivity illuminates the link between Theorem 1 and the tax treatment of luxuries and necessities.

To begin, consider commodity tax systems separately. Under differentiability of the demand functions, a commodity tax system is average-rate progressive if, for $y > 0$,

$$\frac{\partial}{\partial y}\left(\frac{1}{y}\left(\sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}+\boldsymbol{\tau},y)\right)\right) \geq 0,$$

which is expressible as

$$\sum_{k=1}^{K}\left(\partial_2 x_k(\boldsymbol{p}+\boldsymbol{\tau},y) - \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau},y)}{y}\right)\tau_k \geq 0, \tag{6}$$

where, for every good k, $\partial_2 x_k(\boldsymbol{p}',y')$ denotes the partial derivative of the Marshallian demand function with respect to its second variable, income, evaluated at the price vector $\boldsymbol{p}'$ and income level y'.[4]

Recall that a good is a luxury if the bracketed term is positive on its domain and a necessity if it is negative on its domain. Thus, a commodity tax system is average-rate progressive if it taxes luxuries and/or subsidizes necessities.

Similarly, joint average-rate progressivity can be expressed as

$$\frac{\partial}{\partial y}\left(\frac{1}{y}\left(T(y) + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}+\boldsymbol{\tau},y-T(y))\right)\right) \geq 0, \quad y > 0.$$

[4]The use of this notation is intended to resolve ambiguities between the perturbed variable (income) and its current level, y'.

Under differentiability of T and the demand functions $x_k(\boldsymbol{p}', y')$ ($k \in \{1, \ldots, K\}$), this inequality can be expressed as

$$T'(y) + (1 - T'(y)) \left(\sum_{k=1}^{K} \tau_k \partial_2 x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)) \right) \\ \geq \frac{T(y)}{y} + \sum_{k=1}^{K} \tau_k \cdot \frac{x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y))}{y}, \quad y > 0. \quad (7)$$

The left-hand side represents the total fraction of an extra dollar (at the income level y) paid as tax—the sum of the fraction of an extra dollar paid as income tax, $T'(y)$, plus the fraction of an extra dollar, net of income taxes, paid as consumption tax,

$$(1 - T'(y)) \left(\sum_{k=1}^{K} \tau_k \partial_2 x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)) \right).$$

The right-hand side of (7) represents the total fraction of every dollar (at the current income level, y) paid as tax—the sum of the fraction of every dollar paid as income tax, $T(y)/y$, plus the fraction of every dollar paid as consumption tax,

$$\sum_{k=1}^{K} \tau_k \frac{x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y))}{y}.$$

A fundamentally different metric commonly used to evaluate income distributions is the Foster-Wolfson bipolarization order (Foster and Wolfson, 2010; Chakravarty, 2009, 2015), a measure of the degree of polarization between two income groups, taking median income as the demarcation point.

For two income distributions $\boldsymbol{y} = (y_1, \ldots, y_n)$ and $\boldsymbol{y}' = (y'_1, \ldots, y'_n)$ with the same median income, m, we write $\boldsymbol{y}' \succcurlyeq_{\mathrm{FW}} \boldsymbol{y}$ to indicate that $\boldsymbol{y}'$ is *more bipolarized than* $\boldsymbol{y}$, if

$$\sum_{k \leq i < \frac{n+1}{2}} (m - y_{[i]}) \leq \sum_{k \leq i < \frac{n+1}{2}} (m - y'_{[i]}), \quad \forall k : 1 \leq k < \frac{n+1}{2},$$
$$\sum_{\frac{n+1}{2} < i \leq k} (y_{[i]} - m) \leq \sum_{\frac{n+1}{2} < i \leq k} (y'_{[i]} - m), \quad \forall k : \frac{n+1}{2} < k \leq n.$$

This order evaluates pairs of income distributions on the basis of an 'average deviation' between individual income levels and median income, with lower average deviations corresponding to less bipolarized income distributions.

Assuming that proportional changes in income do not alter the 'degree' of bipolarization, $\succcurlyeq_{FW}$ can be extended to pairs of income distributions with different median incomes.

Let $m(\boldsymbol{y})$ (resp., $m(\boldsymbol{y}')$) denote the median income of $\boldsymbol{y}$ (resp., $\boldsymbol{y}'$), and suppose that $m(\boldsymbol{y}) > 0$ and $m(\boldsymbol{y}') > 0$. Then the transformation

$$\boldsymbol{y}'' = \frac{m(\boldsymbol{y})}{m(\boldsymbol{y}')} (y'_1, \ldots, y'_n)$$

of $\boldsymbol{y}'$ has the same median as $\boldsymbol{y}$. We now make the following definition:

$$\boldsymbol{y}' \succcurlyeq_{FW} \boldsymbol{y} \Leftrightarrow \boldsymbol{y}'' \succcurlyeq_{FW} \boldsymbol{y}.$$

Given a price vector $\boldsymbol{p}$, a mixed tax system $(T, \boldsymbol{\tau})$ is *bipolarization reducing* if

$$(y_1, \dots, y_n) \succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n))$$

for every pre-tax income distribution $(y_1, \dots, y_n)$ and every population size n.

Carbonell-Nicolau and Llavador (2021a) establishes the equivalence between inequality reducing and bipolarization reducing income tax schedules. We now prove the equivalence between inequality reducing and bipolarization reducing mixed tax systems (resp., commodity tax systems).

Theorem 2. *Given a price vector $\boldsymbol{p}$, a mixed tax system $(T, \boldsymbol{\tau})$ is bipolarization reducing if and only if the disposable income function $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in the pre-tax income level y and $(T, \boldsymbol{\tau})$ is jointly average-rate progressive.*

The proof of Theorem 2 is given in Appendix B.

Combining Theorem 1 and Theorem 2 immediately gives the following result.

Theorem 3. *Given a price vector $\boldsymbol{p}$, a mixed tax system is inequality reducing if and only if it is bipolarization reducing.*

As a special case of Theorem 3, we obtain the following result: given a price vector $\boldsymbol{p}$, a commodity tax system is inequality reducing if and only if it is bipolarization reducing.

We now investigate whether mixed taxation is "redundant" in achieving the goal of inequality and bipolarization reduction. First, we illustrate that income taxation is sometimes necessary, since there exist preferences for which commodity taxation has no equalizing power.

Revisit the previous two-good example with $u(x_1, x_2) = x_1 x_2$. We have already seen that, for this utility function, commodity taxation is neutral—it neither increases nor decreases inequality—because both goods are neither luxuries nor necessities. Thus, in this case, any equalizing tax system requires income taxation.

Next, we show that, under some conditions, commodity taxation is superfluous, in the sense that any post-tax income distribution resulting from the implementation of a mixed tax system is attainable via income taxation. This result is reminiscent of the Atkinson-Stiglitz result from the literature on optimal taxation (see Atkinson and Stiglitz, 1976), whereby "the optimal tax system can rely solely on income taxation" (under separability between leisure and consumption).[5]

Suppose that $(T, \boldsymbol{\tau})$ is a mixed tax system such that $\boldsymbol{\tau}$ does not tax any inferior good. Take a price vector $\boldsymbol{p}$ and suppose that the disposable income function $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in the pre-tax income level y. Then there exists an income tax schedule T^* satisfying the following: given an income distribution $(y_1, \dots, y_n)$, both $(T, \boldsymbol{\tau})$ and T^* give rise to the same post-tax income distribution:

$$(z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)) = (z(\boldsymbol{p}, T^*, 0, y_1), \dots, z(\boldsymbol{p}, T^*, 0, y_n)).$$

[5] Atkinson and Stiglitz (1976) treat income as endogenous. The analysis of endogenous income in our framework is presented in Section 2.2.

To see this, note that setting

$$T^*(y) = T(y) + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y))$$

gives

$$z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) = y - T(y) - \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)) = z(\boldsymbol{p}, T^*, 0, y), \quad \text{for all } y.$$

To see that T^* is an income tax schedule, note first that, because $\boldsymbol{\tau}$ does not tax inferior goods, T^* is nondecreasing. Moreover, $T^*(y) \leq y$ for all y (since $T(y) \leq y$ and

$$y - T(y) - \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)) \geq \sum_{k=1}^{K} p_k x_k(\boldsymbol{p} + \boldsymbol{\tau}, y - T(y)) \geq 0$$

for all y) and, since $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in y, the map $y \mapsto y - T^*(y)$ is nondecreasing.

Note that the previous observations, together with Theorem 2 and Theorem 3, imply that, for every inequality reducing (resp., bipolarization reducing) mixed tax system $(T, \boldsymbol{\tau})$ such that $\boldsymbol{\tau}$ does not tax any inferior good, and for every price vector $\boldsymbol{p}$, there exists an income tax schedule T^* satisfying the following: given an initial income distribution, both $(T, \boldsymbol{\tau})$ and T^* give rise to the same post-tax income distribution.

2.2. Endogenous income

In the case of endogenous income, preferences are described by means of a utility function u defined on consumption bundles and labor hours, $(\boldsymbol{x}, l) \in \mathbb{R}_+^K \times [0, L)$, where $\boldsymbol{x} = (x_1, \ldots, x_K)$ represents a bundle of K commodities, l is a measure of working hours, and $0 < L \leq \infty$.[6]

In this section, we restrict attention to piecewise linear tax schedules (see the definition of a tax schedule at the beginning of Section 2.1).

A tax schedule T is *piecewise linear* if $\mathbb{R}_+$ can be partitioned into finitely many intervals $I_1, \ldots, I_M$ satisfying the following: for each m, there exist $\beta \in \mathbb{R}$ and $t \in [0, 1)$ such that $T(y) = \beta + ty$ for all $y \in I_m$.

The set of all piecewise linear tax schedules is denoted by $\mathscr{T}$.

Individuals differ in the wage that they receive for their labor supply. The disposable income available to buy goods and services for an individual who receives $\$a > 0$ per hour worked and who supplies $l \in [0, L)$ units of labor and is subjected to income taxation under a tax schedule T is given by $al - T(al)$. Thus, given a price vector $\boldsymbol{p} = (p_1, \ldots, p_K)$ and a mixed tax system $(T, \boldsymbol{\tau}) = (T, \tau_1, \ldots, \tau_K)$ such that $T \in \mathscr{T}$ and $p_k + \tau_k > 0$ for each

[6]Here, we allow u to take the value $-\infty$. The ?? presented later in this section features a utility function $u(x_1, x_2, l)$ such that $u(0, x_2, l) = -\infty$ for all x_2 and l.

k, the individual's problem is

$$\begin{aligned} &\max_{(x_1,\dots,x_K,l)\in\mathbb{R}^K_+\times[0,L)} \quad u(x_1,\dots,x_K,l) \\ &\text{s.t. } (p_1+\tau_1)x_1+\cdots+(p_K+\tau_K)x_K \le al - T(al). \end{aligned} \tag{8}$$

Throughout the sequel, u is assumed to satisfy the following conditions:

(i) u is continuous.

(ii) $u(\cdot,l)$ is nondecreasing and strictly increasing on $\mathbb{R}^K_{++}$ for each $l\in[0,L)$ and $u(\boldsymbol{x},\cdot)$ is strictly decreasing for each $\boldsymbol{x}\in\mathbb{R}^K_{++}$.

(iii) Given $\boldsymbol{p}$, $T\in\mathscr{T}$, $a>0$, and $aL>y>0$, let $\boldsymbol{x}(\boldsymbol{p},T,a,y)$ denote a solution to

$$\begin{aligned} &\max_{(x_1,\dots,x_K)\in\mathbb{R}^K_+} \quad u(x_1,\dots,x_K,y/a) \\ &\text{s.t. } p_1x_1+\cdots+p_Kx_K \le y - T(y).^{7} \end{aligned} \tag{9}$$

Suppose that $\boldsymbol{x}(\boldsymbol{p},T,a,y)$ is continuous in (a,y) for each $\boldsymbol{p}$ and T.

Choose a commodity k and a gross labor income level $y>0$. The marginal rate of substitution of x_k for y for an 'a-individual' is given by

$$MRS^a_k(\boldsymbol{x},y) = -\frac{(1/a)(\partial u(\boldsymbol{x},y/a)/\partial l)}{\partial u(\boldsymbol{x},y/a)/\partial x_k}.$$

It represents the amount of extra good k an individual would require as compensation for an extra unit of gross labor income.

We assume that (a) $MRS^a_k(\boldsymbol{x}(\boldsymbol{p},T,a,y),y)$ is well defined for each k, $\boldsymbol{p}$, $T\in\mathscr{T}$, $a>0$, and $y>0$ and continuous in (a,y) for each $\boldsymbol{p}$ and T; and (b) for each k, $\boldsymbol{p}$, $T\in\mathscr{T}$, and $y>0$,

$$\lim_{a\searrow y/L} MRS^a_k(\boldsymbol{x}(\boldsymbol{p},T,a,y),y)=\infty \quad\text{and}\quad \lim_{a\to\infty} MRS^a_k(\boldsymbol{x}(\boldsymbol{p},T,a,y),y)=0.$$

(iv) u is quasiconcave and exhibits the following form of 'strict quasiconcavity:' given $\boldsymbol{p}$, $T\in\mathscr{T}$, $a>0$, $aL>y>0$, $aL>y'>0$, and solutions $\boldsymbol{x}$ and $\boldsymbol{x}'$ to (9) and

$$\begin{aligned} &\max_{(x_1,\dots,x_K)\in\mathbb{R}^K_+} \quad u(x_1,\dots,x_K,y'/a) \\ &\text{s.t. } p_1x_1+\cdots+p_Kx_K \le y' - T(y'), \end{aligned}$$

respectively, the following condition is satisfied:

$$u(\alpha(\boldsymbol{x},y/a)+(1-\alpha)(\boldsymbol{x}',y'/a)) > \min\{u(\boldsymbol{x},y/a),u(\boldsymbol{x}',y'/a)\}$$

[7] Because u is continuous and the feasible set is compact, a solution exists. Under the condition (iv) below, the solution is unique (see Footnote 8).

for all $\alpha \in (0,1)$ whenever $y \neq y'$ or $\boldsymbol{x} \neq \boldsymbol{x}'$.[8]

(v) For each $\boldsymbol{p}$, $T \in \mathscr{T}$, and $a > 0$, there exist $\boldsymbol{x} \in \mathbb{R}^K_{++}$ and $l > 0$ such that

$$p_1 x_1 + \cdots + p_K x_K \leq al - T(al)$$

and $u(\boldsymbol{x}, l) > u(\boldsymbol{0}, 0)$.

The condition (iii) states that the compensation (in terms of good k) required by an individual for an extra unit of labor income at labor income level y and at a utility maximizing bundle $\boldsymbol{x}(\boldsymbol{p}, T, a, y)$ (a) tends to infinity as aL approaches y from above; and (b) tends to zero as a diverges to ∞.

The last condition, (v), implies that, under an income tax schedule $T \in \mathscr{T}$, an individual whose wage rate is $a > 0$ always consumes a positive amount.

A solution to (8) is denoted by

$$(x^u_1(\boldsymbol{p}, T, \boldsymbol{\tau}, a), \ldots, x^u_K(\boldsymbol{p}, T, \boldsymbol{\tau}, a), l^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)).^{9} \tag{11}$$

The notation used in this section makes the dependence of a solution to (8) on the utility function, u, explicit. Keeping this dependence in mind will be convenient when

[8]This condition implies that the problem (9) has a unique solution. To see this, suppose that there are two distinct solutions, $\boldsymbol{x}$ and $\boldsymbol{x}'$, to (9). Then (iv) implies that

$$u(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{x}', y/a) > u(\boldsymbol{x}, y/a) = u(\boldsymbol{x}', y/a) \tag{10}$$

for $\alpha \in (0,1)$. But since

$$p_1 x_1 + \cdots + p_K x_K \leq y - T(y) \quad \text{and} \quad p_1 x'_1 + \cdots + p_K x'_K \leq y - T(y),$$

it follows that

$$p_1(\alpha x_1 + (1-\alpha)x'_1) + \cdots + p_K(\alpha x_K + (1-\alpha)x'_K) \leq y - T(y).$$

This, together with (10), contradicts that $\boldsymbol{x}$ and $\boldsymbol{x}'$ solve (9).

[9]The problem (8) has at least one solution. To see this, consider first the following problem:

$$\max_{y \in [0, aL)} u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y), y/a). \tag{12}$$

Note that if this problem has a solution, y^*, then

$$(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y^*), y^*/a)$$

solves (8). Thus, it suffices to show that (12) has a solution.

A necessary condition for y^* to solve (12) is

$$(p_k + \tau_k) MRS^a_k(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y^*), y^*) \leq 1 - t, \quad \text{for each } k,$$

where $t < 1$ denotes the maximum marginal tax rate in T (recall that $T \in \mathscr{T}$). Because

$$\lim_{a' \searrow y/L} MRS^a_k(\boldsymbol{x}(\boldsymbol{p}, T, a', y), y) = \infty$$

for each $y < a'L$ (see condition (iii)), there exists $\overline{y}$ close enough to aL such that

$$(p_k + \tau_k) MRS^a_k(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, \overline{y}), \overline{y}) > 1 - t.$$

Hence, because the indifference curves for the utility function

$$(\boldsymbol{x}, y) \in \mathbb{R}^K_+ \times [0, aL) \mapsto u(\boldsymbol{x}, y/a)$$

characterizing the inequality reducing properties of tax schedules in terms of conditions on u.

A corresponding post-tax income function is denoted by

$$\begin{aligned} z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) &= a l^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) - T(a l^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)) - \sum_{k=1}^{K} \tau_k x_k^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) \\ &= \sum_{k=1}^{K} p_k x_k^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a). \end{aligned} \tag{14}$$

In the absence of taxes, the solution in (11) is denoted by

$$(x_1^u(\boldsymbol{p}, 0, 0, a), \ldots, x_K^u(\boldsymbol{p}, 0, 0, a), l^u(\boldsymbol{p}, 0, 0, a)).$$

We now formulate the last two conditions on the utility function u:

(vi) Given $\boldsymbol{p}$ and $(T, \boldsymbol{\tau})$ with $T \in \mathscr{T}$, if $l^u(\boldsymbol{p}, T, \boldsymbol{\tau}, \cdot)$ has a discontinuity point at some $a > 0$, so does $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, \cdot)$.

(vii) Given $b \geq 0$ and $\boldsymbol{p}$, the map $a \mapsto a l^u(\boldsymbol{p}, T, 0, a) + b$ is nondecreasing, where T is defined by $T(y) = -b$ for all $y \geq 0$.

Condition (vi) says that discontinuities in gross income with respect to a translate into similar discontinuities in disposable income.[10]

Condition (vii) states that, for any fixed subsidy $b \geq 0$ and in the absence of commodity taxation, disposable income is nondecreasing with a.[11]

The set of all utility functions satisfying the conditions (i)-(vi) is denoted by $\mathscr{U}$.

We have $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) > 0$ for each $(\boldsymbol{p}, T, \boldsymbol{\tau})$ with $T \in \mathscr{T}$ and every $a > 0$. Indeed, the condition (v) guarantees that $x_k^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) > 0$ for at least one k, implying that $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) > 0$.

A *wage distribution* is a vector $(a_1, \ldots, a_n) \in \mathbb{R}^n_{++}$, where n is the population size and a_i represents individual i's wage rate.

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are convex (by quasiconcavity of u), there exists $\widehat{y} \in [0, \overline{y}]$ such that

$$u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, a, T, \widehat{y}), \widehat{y}/a) > u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, a, T, y), y/a), \quad \text{for all } y \in (\overline{y}, aL).$$

Consequently, any solution to

$$\max_{y \in [0, \overline{y}]} u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y), y/a) \tag{13}$$

is also a solution to (12). But (13) has a solution because the objective function is continuous in y and the feasible set is a closed interval.

[10] By equation (14), condition (vi) holds if there are no inferior goods. More generally, if $l^u(\boldsymbol{p}, T, \boldsymbol{\tau}, \cdot)$ has a discontinuity at a, gross (hence net) labor income is also discontinuous at a, and so the demands for the K commodities will generally exhibit a discontinuity at a, which will generally translate into a discontinuity in $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, \cdot)$.

[11] An increase in a represents an increase in the "price" of leisure, which triggers a substitution effect in demand toward more labor income (and more goods and services) and away from leisure, and an income effect, which increases the demand for leisure (if leisure is a normal good). If the income effect does not outweigh the substitution effect, condition (vii) holds. More generally, even when the income effect dominates, condition (vii) holds if the reduction in the labor supply is "small" relative to the increase in the wage rate.

Given a price vector $\boldsymbol{p}$ and a wage distribution $(a_1, \ldots, a_n)$, a mixed tax system $(T, \boldsymbol{\tau})$ gives rise to an income distribution

$$(z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n)).$$

In the absence of taxation, i.e., when both T and $\boldsymbol{\tau}$ are identically zero, the resulting income distribution is

$$(z^u(\boldsymbol{p}, 0, 0, a_1), \ldots, z^u(\boldsymbol{p}, 0, 0, a_n)).$$

A mixed tax system $(T, \boldsymbol{\tau})$ is *inequality reducing with respect to* $\boldsymbol{p}$ *and* u, or $(\boldsymbol{p}, u)$-*ir*, if

$$(z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n)) \succcurlyeq_L (z^u(\boldsymbol{p}, 0, 0, a_1), \ldots, z^u(\boldsymbol{p}, 0, 0, a_n))$$

for each wage distribution $(a_1, \ldots, a_n)$, every post-tax income function z^u, and every population size n.

When the underlying $\boldsymbol{p}$ and u are clear from the context, we sometimes refer to $(\boldsymbol{p}, u)$-ir mixed tax systems simply as inequality reducing mixed tax systems.

An income tax schedule is *marginal-rate progressive* if it is convex.

If $T \in \mathscr{T}$ is marginal-rate progressive, then the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is uniquely defined.[12]

[12] To see this, note first that if the problem

$$\max_{y \in [0, aL)} u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y), y/a) \tag{15}$$

has a unique solution, then so does the problem (8), implying that the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is uniquely defined. Indeed, because the function $\boldsymbol{x}(\boldsymbol{p}, T, a, y)$ is uniquely defined for each $(\boldsymbol{p}, T, a, y)$ (Footnote 8), if there exist two distinct solutions, $(\boldsymbol{x}', l')$ and $(\boldsymbol{x}'', l'')$, to the problem (8), then $l' \neq l''$ (otherwise, i.e., if $l' = l''$, then $\boldsymbol{x}' = \boldsymbol{x}'' = \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, al)$) and there exist two distinct solutions, al and al', to (15).

Thus, it suffices to show that (15) has a unique solution whenever T is convex. (The problem (15) was shown to have a solution in Footnote 9.) To see that (15) has a unique solution, suppose that there are two distinct solutions, y and y'. Then (iv) implies that

$$\begin{aligned} &u(\alpha(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y), y/a) + (1 - \alpha)(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y'), y'/a)) \\ &\qquad = u(\alpha \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y) + (1 - \alpha)\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y'), (\alpha y + (1 - \alpha) y')/a) \\ &\qquad > u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y), y/a) = u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y'), y'/a) \end{aligned} \tag{16}$$

for $\alpha \in (0, 1)$. Because $y, y' \in [0, aL)$, we have

$$\alpha y + (1 - \alpha) y' \in [0, aL).$$

In addition, because T is a convex function,

$$\begin{aligned} \boldsymbol{p} \cdot (\alpha \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y) &+ (1 - \alpha)\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y')) \\ &= \alpha(\boldsymbol{p} \cdot \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y)) + (1 - \alpha)(\boldsymbol{p} \cdot \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y')) \\ &\leq \alpha(y - T(y)) + (1 - \alpha)(y' - T(y')) \\ &= \alpha y + (1 - \alpha) y' - (\alpha T(y) + (1 - \alpha) T(y')) \\ &\leq \alpha y + (1 - \alpha) y' - T(\alpha y + (1 - \alpha) y'); \end{aligned}$$

here, for $\boldsymbol{x} = (x_1, \ldots, x_K) \in \mathbb{R}^K_+$,

$$\boldsymbol{p} \cdot \boldsymbol{x} = p_1 x_1 + \cdots + p_K x_K.$$

Consequently, (16) contradicts that y and y' solve (15).

Theorem 4. *For $T \in \mathscr{T}$, a mixed tax system $(T, \boldsymbol{\tau})$ is inequality reducing with respect to $\boldsymbol{p}$ and $u \in \mathscr{U}$ only if the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a and T is marginal-rate progressive.*

The proof of Theorem 4 is relegated to Appendix C.

For any income tax schedule $T \in \mathscr{T}$, we have $T(0) = -b$ for some $b \geq 0$. Here, b can be viewed as a subsidy deducted from any tax liability. If b exceeds an individual's tax obligations, the individual receives the excess subsidy as a direct payment.

Let $\mathscr{T}_{m\text{-}prog}$ be the set of all marginal-rate progressive income tax schedules in $\mathscr{T}$. Every income tax schedule T in $\mathscr{T}_{m\text{-}prog}$ is piecewise linear, and so $\mathbb{R}_+$ can be partitioned into finitely many intervals $I_1, \ldots, I_M$ satisfying the following: for each m, there exist $b \in \mathbb{R}$ and $t \in [0, 1)$ such that $T(y) = -b + ty$ for all $y \in I_m$. Note that, because T is convex, $b \geq 0$. Note also that the extension of $-b + ty$ to the entire domain $\mathbb{R}_+$ is itself an income tax schedule in $\mathscr{T}_{m\text{-}prog}$. We call each such linear income tax schedules a *linear transformation* of T. Thus, there are M many linear transformations of T in $\mathscr{T}_{m\text{-}prog}$. More generally, the number of linear transformations of $T \in \mathscr{T}_{m\text{-}prog}$ is equal to the number of tax brackets in T, and the set of all linear transformations of T is contained in $\mathscr{T}_{m\text{-}prog}$.

A subset of tax schedules $\mathscr{S} \subseteq \mathscr{T}_{m\text{-}prog}$ is *closed under linear transformations* if $\mathscr{S}$ contains the linear transformations of its members.

We now characterize the set of inequality reducing mixed tax systems $(T, \boldsymbol{\tau})$, where $T \in \mathscr{S} \subseteq \mathscr{T}_{m\text{-}prog}$ and $\mathscr{S}$ is closed under linear transformations, in terms of conditions on the utility function u.

Consider the tax schedule $T \in \mathscr{T}$ defined by $T(y) = -b$ for all $y \geq 0$ and some $b \geq 0$. For this particular tax schedule, we write

$$\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y) = \boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, b, a, y).$$

Given a price vector $\boldsymbol{p} = (p_1, \ldots, p_K)$, $b \geq 0$, a commodity tax system $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_K)$ with $p_k + \tau_k > 0$ for each k, and $a > 0$, the problem

$$\max_{y \in [0, aL)} u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, b, a, y), y/a)$$

has a unique solution, $y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a)$, which represents the gross income for an individual whose wage rate is a and who receives a subsidy of size b and faces a commodity tax system $\boldsymbol{\tau}$.[13] When consumption is not taxed, i.e., when $\boldsymbol{\tau} = 0$, we write $y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a) = y^u(\boldsymbol{p}, b, 0, a)$.

The corresponding after-tax income is given by

$$z^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a) = y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a) + b - \sum_{k=1}^{K} \tau_k x_k^u(\boldsymbol{p} + \boldsymbol{\tau}, y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a) + b),$$

where $x_k^u(\boldsymbol{p}', y')$ denotes the standard (Marshallian) demand function at price vector $\boldsymbol{p}'$ and income level y'. When $\boldsymbol{\tau} = 0$ and $b = 0$,

$$z^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a) = z^u(\boldsymbol{p}, 0, 0, a) = y^u(\boldsymbol{p}, 0, 0, a)$$

[13]For the proof of uniqueness of the solution, refer to Footnote 12.

represents the solution to

$$\max_{y\in[0,aL)} u(\boldsymbol{x}(\boldsymbol{p},0,a,y),y/a).$$

Let $\mathcal{S}$ be a subset of income tax schedules in $\mathcal{T}_{m\text{-}prog}$, and let $\mathcal{S}'$ be a subset of commodity tax systems. The set of all linear transformations of the elements of $\mathcal{S}$ is denoted by $\mathcal{L}_{\mathcal{S}}$.

Each linear transformation in $\mathcal{L}_{\mathcal{S}}$ is of the form $-b+ty$, where $b\geq 0$ and $t\in[0,1)$. Define

$$B(\mathcal{L}_{\mathcal{S}})=\{b\geq 0: -b+ty\in\mathcal{L}_{\mathcal{S}},\ \text{some } t\}$$

and

$$R(\mathcal{L}_{\mathcal{S}})=\{t\in[0,1): -b+ty\in\mathcal{L}_{\mathcal{S}},\ \text{some } b\}.$$

Theorem 5. *Suppose that $\mathcal{S}\subseteq\mathcal{T}_{m\text{-}prog}$ is closed under linear transformations. Suppose that $\mathcal{S}'$ is a subset of commodity tax systems. Then the mixed tax systems in $\mathcal{S}\times\mathcal{S}'$ are inequality reducing with respect to $\boldsymbol{p}$ and $u\in\mathcal{U}$ if and only if the following two conditions are satisfied:*

(i) the disposable income function $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a for each $T\in\mathcal{L}_{\mathcal{S}}\cup\{0\}$ and $\boldsymbol{\tau}\in\mathcal{S}'$;[14] and

(ii) the quotient

$$\frac{z^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)}{z^u(\boldsymbol{p},0,0,a)}$$

is nonincreasing in a for every $(b,t,\boldsymbol{\tau})\in B(\mathcal{L}_{\mathcal{S}})\times R(\mathcal{L}_{\mathcal{S}})\times\mathcal{S}'$.

The proof of Theorem 5 is presented in Appendix D.

Let us now take a closer look at condition (ii) in Theorem 5. Under differentiability of z^u with respect to a, this condition can be expressed as follows:

$$\frac{\partial z^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)}{\partial a}\Bigg/\frac{\partial z^u(\boldsymbol{p},0,0,a)}{\partial a}\leq\frac{z^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)}{z^u(\boldsymbol{p},0,0,a)},$$
$$\text{for each } a>0 \text{ and } (b,t,\boldsymbol{\tau})\in B(\mathcal{L}_{\mathcal{S}})\times R(\mathcal{L}_{\mathcal{S}})\times\mathcal{S}',$$

which states that the ratio of the marginal effects is less than the ratio of levels. This condition can be equivalently formulated in terms of elasticities:

$$\zeta^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)\leq\zeta^u(\boldsymbol{p},0,0,a),$$
$$\text{for each } a>0 \text{ and } (b,t,\boldsymbol{\tau})\in B(\mathcal{L}_{\mathcal{S}})\times R(\mathcal{L}_{\mathcal{S}})\times\mathcal{S}', \quad (17)$$

where

$$\zeta^u(\boldsymbol{p}',b',\boldsymbol{\tau}',a')=\frac{\partial z^u(\boldsymbol{p}',b',\boldsymbol{\tau}',a')}{\partial a}\cdot\frac{a'}{z^u(\boldsymbol{p}',b',\boldsymbol{\tau}',a')}$$

represents the *wage elasticity of disposable income* at $(\boldsymbol{p}',b',\boldsymbol{\tau}',a')$.

The right-hand side of the inequality in (17) is the elasticity of untaxed income at $\boldsymbol{p}$ and a:

$$\zeta^u(\boldsymbol{p},0,0,a)=\frac{\partial y^u(\boldsymbol{p},0,0,a)}{\partial a}\cdot\frac{a}{y^u(\boldsymbol{p},0,0,a)}.$$

[14]Here 0 denotes the linear tax schedule T defined by $T(y)=0$ for all y.

Since
$$\frac{\partial y^u(\boldsymbol{p},0,0,a)}{\partial a} = l^u(\boldsymbol{p},0,0,a) + a \cdot \frac{\partial l^u(\boldsymbol{p},0,0,a)}{\partial a},$$
we have
$$\zeta^u(\boldsymbol{p},0,0,a) = 1 + \epsilon^u(\boldsymbol{p},0,0,a),$$
where
$$\epsilon^u(\boldsymbol{p}',b',\boldsymbol{\tau}',a') = \frac{\partial l^u(\boldsymbol{p}',b',\boldsymbol{\tau}',a')}{\partial a} \cdot \frac{a'}{l^u(\boldsymbol{p}',b',\boldsymbol{\tau}',a')}$$
represents the *wage elasticity of the labor supply* at $(\boldsymbol{p}',b',\boldsymbol{\tau}',a')$.

The left-hand side of the inequality in (17) is the elasticity of (directly and indirectly) taxed income at $(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)$, where the income tax consists of a fixed subsidy $b \geq 0$ and the commodity tax system is given by $\boldsymbol{\tau}$. This elasticity can be expressed as follows:

$$\zeta^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a) = \zeta^u(\boldsymbol{p}+\boldsymbol{\tau},b,0,(1-t)a) \cdot \frac{1-\sum_{k=1}^K \tau_k \cdot \frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)+b)}{\partial y}}{1-\sum_{k=1}^K \tau_k \cdot \frac{x_k^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)+b)}{y^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)+b}}. \quad (18)$$

Note the difference between the two elasticities on each side of the equality. The first is the wage elasticity of disposable income, while the second is the wage elasticity of income net of income taxes/subsidies and *gross* of consumption taxes. There are two effects of consumption taxes on disposable income. First, consumption taxes distort the relative prices of goods and services, which, together with the subsidy b, influences the optimal supply of labor (and hence labor income) at the individual level. The elasticity of income on the right-hand side of (18) captures this effect. Second, consumption taxes, together with an individual's demand for goods and services, determine the individual's consumption tax liabilities, and therefore his/her disposable income. The ratio on the right-hand side of (18) is related to the second effect.

Note that
$$\sum_{k=1}^K \tau_k \cdot \frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)+b)}{\partial y}$$
represents the fraction of an extra dollar of income (at the income level $y^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)+b$) paid as consumption tax, and so

$$1-\sum_{k=1}^K \tau_k \cdot \frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)+b)}{\partial y}$$

is the fraction of an extra dollar (at the income level $y^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)+b$), net of consumption taxes, spent on goods and services.

Similarly,
$$1-\sum_{k=1}^K \tau_k \cdot \frac{x_k^u(\boldsymbol{p}+\boldsymbol{\tau},y^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)+b)}{y^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)+b}$$
represents the average fraction of every dollar (at the income level $y^u(\boldsymbol{p},b,\boldsymbol{\tau},(1-t)a)+b$), net of consumption taxes, spent on goods and services.

For a luxury good k, we have

$$\frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, (1-t)a) + b)}{\partial y} > \frac{x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, (1-t)a) + b)}{y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, (1-t)a) + b}.$$

Thus, taxing luxury goods ensures that the ratio of the fraction of an extra dollar spent on goods and services to the average fraction of every dollar spent on goods and services,

$$\frac{1 - \sum_{k=1}^{K} \tau_k \cdot \frac{\partial x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, (1-t)a) + b)}{\partial y}}{1 - \sum_{k=1}^{K} \tau_k \cdot \frac{x_k^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, (1-t)a) + b)}{y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, (1-t)a) + b}},$$

is less than one, which helps reduce the wage elasticity of disposable income, $\zeta^u(\boldsymbol{p}, b, \boldsymbol{\tau}, (1-t)a)$ (see (18)), and therefore relaxes the constraints in (17).[15]

While taxing luxury goods relaxes the constraints in (17), the net effect of the tax on the wage elasticity of disposable income depends on its interaction with two other forces shaping the said elasticity: the effects of an income subsidy b, together with a proportional tax rate t, and the price distortions resulting from commodity taxation. These may act as countervailing forces, and the net effect is, in general, ambiguous.

Next, we turn to the equivalence between inequality reducing and bipolarization reducing mixed tax systems in the case of endogenous income.

Recall that, given a price vector $\boldsymbol{p}$ and a wage distribution $(a_1, \ldots, a_n)$, a mixed tax system $(T, \boldsymbol{\tau})$ gives rise to an income distribution

$$(z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n)).$$

In the absence of taxation, the resulting income distribution is

$$(z^u(\boldsymbol{p}, 0, 0, a_1), \ldots, z^u(\boldsymbol{p}, 0, 0, a_n)).$$

A mixed tax system $(T, \boldsymbol{\tau})$ is *bipolarization reducing with respect to* $\boldsymbol{p}$ *and* u, or $(\boldsymbol{p}, u)$*-bpr*, if

$$(z^u(\boldsymbol{p}, 0, 0, a_1), \ldots, z^u(\boldsymbol{p}, 0, 0, a_n)) \succcurlyeq_{FW} (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n))$$

for each wage distribution $(a_1, \ldots, a_n)$, every post-tax income function z^u, and every population size n.

Theorem 6. *For* $T \in \mathscr{T}$*, a mixed tax system* $(T, \boldsymbol{\tau})$ *is inequality reducing with respect to* $\boldsymbol{p}$ *and* $u \in \mathscr{U}$ *if and only if it is bipolarization reducing with respect to* $\boldsymbol{p}$ *and* u*.*

The proof of Theorem 6 is given in Appendix E.

We conclude this section with a brief discussion on the joint effect of direct and indirect taxation on inequality and bipolarization, relative to the effect of direct taxation considered in isolation.

Unlike in the case of exogenous income, here commodity taxation is *not*, in general, superfluous. Indeed, a mixed tax system may be inequality reducing in cases when income taxation lacks any equalizing power. We relegate the details of a (somewhat technical) example illustrating this point to Appendix G and briefly highlight its main features here.

[15] A similar argument can be made if necessities are subsidized.

As illustrated in Appendix G, there exist (quasilinear) preferences for which no income tax schedule (other than a pure subsidy) is inequality (and bipolarization) reducing. Theorem 5 provides necessary and sufficient conditions for a mixed tax system to be inequality reducing, and these conditions can of course be particularized to the case of pure direct taxation. As shown in Appendix G, condition (ii) in Theorem 5 is violated in the special case of pure income taxation (other than a pure subsidy). Equivalently, condition (17) fails when $\boldsymbol{\tau} = 0$. Under pure direct taxation, there is no relative price distortion between goods and services, and the ratio on the right-hand side of (18) vanishes. By contrast, a tax on luxuries, together with a pure income subsidy, relaxes condition (17) (via the ratio on the right-hand side of (18)) and gives an inequality (and bipolarization) reducing mixed tax system (provided that the income subsidy is sufficiently large).

3. Concluding remarks

We have studied mixed tax systems, i.e., tax systems subjecting both income and consumption to taxation, and their ability to reduce income inequality and bipolarization. We have identified necessary and sufficient conditions on the structure of mixed tax systems ensuring a reduction in income inequality and bipolarization, both in the cases of exogenous and endogenous income.

While commodity taxation has been shown to be "redundant" in the case of exogenous income, in the sense that any post-tax income distribution resulting from the implementation of a mixed tax system is attainable by means of an income tax, there are cases when relying solely on income taxes does not always effect more equal (or less bipolarized) endogenous income distributions, whereas mixed tax structures have equalizing and depolarizing power.

We conclude with two comments. First, it perhaps bears reiterating the problems associated with studying *welfare*—as opposed to income—inequality (resp., bipolarization) reducing tax systems, which stem from the fact that Lorenz (resp., Foster-Wolfson) dominance is not generally invariant to order preserving utility transformations. Second, while the present analysis treats exogenous and endogenous income separately, a natural extension—notwithstanding the challenges it would bring to the fore—would allow for heterogeneous sources of income at the individual level (e.g., "capital" income vs. labor income).

A. Proof of Theorem 1

The proof of Theorem 1 is based on the following result, which is well-known in the literature.

Lemma 1. *Suppose that $\boldsymbol{y}' = (y'_1, \ldots, y'_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ are two income distributions with*

$$y_1 \leq \cdots \leq y_n \quad \text{and} \quad y'_1 \leq \cdots \leq y'_n.$$

If y_ι is the first positive income level in $\boldsymbol{y}$, then

$$\frac{y'_\iota}{y_\iota} \geq \cdots \geq \frac{y'_n}{y_n} \Rightarrow \boldsymbol{y}' \succcurlyeq_L \boldsymbol{y}.$$

Proof. The case when $\iota = 1$ is proven in Marshall et al. (1967, Theorem 2.4).

Suppose that $\iota > 1$. We must show that

$$\frac{\sum_{i=1}^{l} y'_i}{\sum_{i=1}^{n} y'_i} \geq \frac{\sum_{i=1}^{l} y_i}{\sum_{i=1}^{n} y_i}, \quad \text{for all } l \in \{1, \ldots, n\}.$$

For fixed l, the inequality is equivalent to

$$\left(\sum_{i=l+1}^{n} y_i\right)\left(\sum_{i=1}^{l} y'_i\right) - \left(\sum_{i=1}^{l} y_i\right)\left(\sum_{i=l+1}^{n} y'_i\right) \geq 0. \tag{19}$$

This is clearly true if $l \in \{1, \ldots, \iota - 1\}$, since, in this case, $\sum_{i=1}^{l} y_i = 0$.

For $l \geq \iota$, (19) can be expressed as

$$\left(\sum_{i=l+1}^{n} y_i\right)\left(\sum_{i=1}^{\iota-1} y'_i + \sum_{i=\iota}^{l} y'_i\right) - \left(\sum_{i=1}^{l} y_i\right)\left(\sum_{i=l+1}^{n} y'_i\right) \geq 0. \tag{20}$$

For the sub-distributions

$$(y'_\iota, \ldots, y'_n) \quad \text{and} \quad (y_\iota, \ldots, y_n),$$

we know that $(y'_\iota, \ldots, y'_n) \succcurlyeq_L (y_\iota, \ldots, y_n)$, and so

$$\frac{\sum_{i=\iota}^{l} y'_i}{\sum_{i=\iota}^{n} y'_i} \geq \frac{\sum_{i=\iota}^{l} y_i}{\sum_{i=\iota}^{n} y_i},$$

whence

$$\left(\sum_{i=l+1}^{n} y_i\right)\left(\sum_{i=\iota}^{l} y'_i\right) - \left(\sum_{i=\iota}^{l} y_i\right)\left(\sum_{i=l+1}^{n} y'_i\right) \geq 0,$$

implying that (20) holds, as we sought. ■

Using Lemma 1, Theorem 1 can be proven as follows.

Theorem 1. *Given a price vector $\boldsymbol{p}$, a mixed tax system $(T, \boldsymbol{\tau})$ is inequality reducing if and only if the disposable income function $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in the pre-tax income level y and $(T, \boldsymbol{\tau})$ is jointly average-rate progressive.*

Proof. Suppose that the mixed tax system $(T, \boldsymbol{\tau})$ is inequality reducing. We must show that $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in y and

$$\frac{1}{y}\left(T(y) + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}, T, \boldsymbol{\tau}, y)\right) \leq \frac{1}{y'}\left(T(y') + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}, T, \boldsymbol{\tau}, y')\right) \text{ for } y' > y > 0. \tag{21}$$

To begin, we assume that

$$z(\boldsymbol{p}, T, \boldsymbol{\tau}, y') < z(\boldsymbol{p}, T, \boldsymbol{\tau}, y), \quad \text{for some } y' > y \geq 0,$$

and derive a contradiction. Note that $y > 0$ (otherwise $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) = 0$) and, for large enough n,

$$\frac{y}{(n-1)y + y'} > \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')}{(n-1)z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) + z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')}.$$

Consequently, for the income distributions

$$\boldsymbol{y}^* = (y_1^*, \ldots, y_n^*) = (y, \ldots, y, y')$$

and

$$\boldsymbol{z}^* = (z_1^*, \ldots, z_n^*) = (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y), \ldots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y), z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')),$$

we have $\boldsymbol{z}^* \not\succcurlyeq_L \boldsymbol{y}^*$, contradicting that $(T, \boldsymbol{\tau})$ is inequality reducing.

It remains to prove (21). Fix $y' > y > 0$. Define the distribution $\boldsymbol{y}'' = (y_1'', \ldots, y_n'')$ by $y_1'' = y$ and $y_i'' = y'$ for $i \neq 1$. Since $(T, \boldsymbol{\tau})$ is inequality reducing, we have

$$(z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1''), \ldots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n'')) \succcurlyeq_L (y_1'', \ldots, y_n''), \tag{22}$$

implying that

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1'')}{\sum_{i=1}^n z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i'')} \geq \frac{y_1''}{\sum_{i=1}^n y_i''},$$

i.e.,

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) + (n-1)z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')} \geq \frac{y}{y + (n-1)y'}. \tag{23}$$

Equation (22) also implies

$$\frac{\sum_{i=1}^{n-1} z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i'')}{\sum_{i=1}^n z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i'')} \geq \frac{\sum_{i=1}^{n-1} y_i''}{\sum_{i=1}^n y_i''},$$

i.e.,

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) + (n-2)z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) + (n-1)z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')} \geq \frac{y + (n-2)y'}{y + (n-1)y'},$$

whence

$$1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) + (n-2)z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) + (n-1)z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')} \leq 1 - \frac{y + (n-2)y'}{y + (n-1)y'},$$

or

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) + (n-1)z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')} \leq \frac{y'}{y + (n-1)y'}.$$

Combining this inequality with (25) yields

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)}{y} \geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) + (n-1)z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')}{y + (n-1)y'} \geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')}{y'},$$

or

$$\frac{1}{y}\left(T(y)+\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y)\right)\leq\frac{1}{y'}\left(T(y')+\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y')\right),$$

as we sought.

Conversely, suppose that the mixed tax system $(T,\boldsymbol{\tau})$ is jointly average-rate progressive and $z(\boldsymbol{p},T,\boldsymbol{\tau},y)$ is nondecreasing in y. Fix any income distribution $\boldsymbol{y}=(y_1,\ldots,y_n)$. Without loss of generality, suppose that

$$y_1\leq\cdots\leq y_n.$$

Let y_ι be the first positive coordinate in $\boldsymbol{y}$. Then

$$\frac{1}{y_\iota}\left(T(y_\iota)+\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y_\iota)\right)\leq\cdots\leq\frac{1}{y_n}\left(T(y_n)+\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y_n)\right).$$

Equivalently,

$$\frac{1}{y_\iota}\left(y_\iota-T(y_\iota)-\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y_\iota)\right)\geq\cdots\geq\frac{1}{y_n}\left(y_n-T(y_n)-\sum_{k=1}^{K}\tau_k x_k(\boldsymbol{p},T,\boldsymbol{\tau},y_n)\right),$$

or

$$\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_\iota)}{y_\iota}\geq\cdots\geq\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_n)}{y_n}.$$

By Lemma 1, it follows that

$$(z(\boldsymbol{p},T,\boldsymbol{\tau},y_1),\ldots,z(\boldsymbol{p},T,\boldsymbol{\tau},y_n))\succcurlyeq_L(y_1,\ldots,y_n).$$

Since $\boldsymbol{y}$ was arbitrary, $(T,\boldsymbol{\tau})$ is inequality reducing. ∎

B. Proof of Theorem 2

Theorem 2. *Given a price vector $\boldsymbol{p}$, a mixed tax system $(T,\boldsymbol{\tau})$ is bipolarization reducing if and only if the disposable income function $z(\boldsymbol{p},T,\boldsymbol{\tau},y)$ is nondecreasing in the pre-tax income level y and $(T,\boldsymbol{\tau})$ is jointly average-rate progressive.*

Proof. Choose a bipolarization reducing mixed tax system $(T,\boldsymbol{\tau})$. First, we show that $z(\boldsymbol{p},T,\boldsymbol{\tau},y)$ is nondecreasing in y. Proceeding by contradiction, suppose that

$$z(\boldsymbol{p},T,\boldsymbol{\tau},y')<z(\boldsymbol{p},T,\boldsymbol{\tau},y),\quad\text{for some } y'>y\geq 0.$$

Note that $y>0$ (otherwise $z(\boldsymbol{p},T,\boldsymbol{\tau},y)=0$). Consider the income distributions

$$\boldsymbol{y}^*=(y_1^*,\ldots,y_n^*)=(y,\ldots,y,y')$$

and

$$\boldsymbol{z}^*=(z_1^*,\ldots,z_n^*)=(z(\boldsymbol{p},T,\boldsymbol{\tau},y),\ldots,z(\boldsymbol{p},T,\boldsymbol{\tau},y),z(\boldsymbol{p},T,\boldsymbol{\tau},y')).$$

For $n > 2$, we have $m(\boldsymbol{y}^*) = y$ and $m(\boldsymbol{z}^*) = z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$. Therefore,

$$\frac{1}{m(\boldsymbol{y}^*)} \sum_{1 \le i < \frac{n+1}{2}} (m(\boldsymbol{y}^*) - y^*_{[i]}) = 0 < \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y) - z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')}{m(\boldsymbol{z}^*)}$$
$$= \frac{1}{m(\boldsymbol{z}^*)} \sum_{1 \le i < \frac{n+1}{2}} (m(\boldsymbol{z}^*) - z^*_{[i]}),$$

and so $\frac{m(z^*)}{m(y^*)}\boldsymbol{y}^* \not\succcurlyeq_{FW} \boldsymbol{z}^*$, whence $\boldsymbol{y}^* \not\succcurlyeq_{FW} \boldsymbol{z}^*$, contradicting that $(T, \boldsymbol{\tau})$ is bipolarization reducing.

It remains to show that $(T, \boldsymbol{\tau})$ is jointly average-rate progressive. Suppose that (T, τ) is not jointly average-rate progressive, then there exist $0 < y < y'$ such that

$$\frac{1}{y}\left(T(y) + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}, T, \boldsymbol{\tau}, y)\right) > \frac{1}{y'}\left(T(y') + \sum_{k=1}^{K} \tau_k x_k(\boldsymbol{p}, T, \boldsymbol{\tau}, y')\right),$$

or

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)}{y} < \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')}{y'}. \tag{24}$$

Suppose first that n is odd. Take an income distribution $\boldsymbol{y} = (y_1, \ldots, y_n)$ with

$$y_1 \le \cdots \le y_n,$$

$y_{m-1} = y$, and $y_m = y'$, where $m = (n+1)/2$, so that y_m is the median income in $\boldsymbol{y}$. Then

$$\begin{aligned}
\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m) - z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{m-1})}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} &= 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{m-1})}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} \\
&= 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y')} \\
&> 1 - \frac{y}{y'} \\
&= \frac{y_m - y_{m-1}}{y_m},
\end{aligned}$$

where the inequality uses (24). Consequently,

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)}{y_m}(y_1, \ldots, y_n) \not\succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \ldots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)),$$

whence

$$(y_1, \ldots, y_n) \not\succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \ldots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)),$$

a contradiction.

If n is even, set $y_{n/2} = y$ and $y_{(n/2)+1} = y'$ and note that

$$m = m(z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \ldots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)) = \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2}) + z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{(n/2)+1})}{2}$$

and

$$m' = m(\boldsymbol{y}) = \frac{y_{n/2} + y_{(n/2)+1}}{2}.$$

Hence,

$$\begin{aligned}
\frac{m - z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2})}{m} &= 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_{n/2})}{m} \\
&= 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)}{\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y)+z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{2}} \\
&= 1 - 2\frac{1}{1 + \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y')}{z(\boldsymbol{p},T,\boldsymbol{\tau},y)}} \\
&> 1 - 2\frac{1}{1 + \frac{y'}{y}} \\
&= \frac{m' - y}{m'},
\end{aligned}$$

where the inequality uses (24). Consequently,

$$(y_1, \dots, y_n) \not\succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)),$$

a contradiction.

Conversely, suppose that the mixed tax system $(T, \boldsymbol{\tau})$ is jointly average-rate progressive and $z(\boldsymbol{p}, T, \boldsymbol{\tau}, y)$ is nondecreasing in y. Choose an income distribution $\boldsymbol{y} = (y_1, \dots, y_n)$ with

$$y_1 \leq \cdots \leq y_n.$$

Suppose first that n is odd. Let y_m represent the median income in $\boldsymbol{y}$. Because $(T, \boldsymbol{\tau})$ is jointly average-rate progressive, we have

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{y_i} \geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)}{y_m} \geq \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_j)}{y_j}, \quad \text{for } i < m \text{ and } j > m.$$

Therefore,

$$\begin{aligned}
\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m) - z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} &= 1 - \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} \leq 1 - \frac{y_i}{y_m} = \frac{y_m - y_i}{y_m}, \quad \text{for } i < m, \\
\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i) - z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} &= \frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_i)}{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)} - 1 \leq \frac{y_i}{y_m} - 1 = \frac{y_i - y_m}{y_m}, \quad \text{for } i > m.
\end{aligned}$$

Consequently,

$$\frac{z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_m)}{y_m}(y_1, \dots, y_n) \succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)),$$

whence

$$(y_1, \dots, y_n) \succcurlyeq_{FW} (z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_1), \dots, z(\boldsymbol{p}, T, \boldsymbol{\tau}, y_n)).$$

Since $\boldsymbol{y}$ was arbitrary, $(T, \boldsymbol{\tau})$ is bipolarization reducing.

Now suppose that n is even. Then the median incomes for $\boldsymbol{y}$ and

$$(z(\boldsymbol{p},T,\boldsymbol{\tau},y_1),\ldots,z(\boldsymbol{p},T,\boldsymbol{\tau},y_n))$$

are given by

$$m = \frac{y_{n/2} + y_{(n/2)+1}}{2} \quad \text{and} \quad m' = \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{n/2}) + z(\boldsymbol{p},T,\boldsymbol{\tau},y_{(n/2)+1})}{2},$$

respectively, and so

$$\begin{aligned}\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_i)}{y_i} \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{n/2})}{y_{n/2}} \geq \frac{m'}{m} \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{(n/2)+1})}{y_{(n/2)+1}} \\ \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_j)}{y_j}, \quad \text{for } i \leq n/2 \text{ and } j \geq (n/2)+1,\end{aligned}$$

where the second and third inequalities hold because

$$\begin{aligned}\frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{n/2})}{y_{n/2}} \geq \frac{m'}{m} &\Leftrightarrow \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{n/2})}{y_{n/2}} \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{n/2}) + z(\boldsymbol{p},T,\boldsymbol{\tau},y_{(n/2)+1})}{y_{n/2} + y_{(n/2)+1}} \\ &\Leftrightarrow \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{n/2})}{y_{n/2}} \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{(n/2)+1})}{y_{(n/2)+1}}\end{aligned}$$

and

$$\begin{aligned}\frac{m'}{m} \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{(n/2)+1})}{y_{(n/2)+1}} &\Leftrightarrow \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{n/2}) + z(\boldsymbol{p},T,\boldsymbol{\tau},y_{(n/2)+1})}{y_{n/2} + y_{(n/2)+1}} \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{(n/2)+1})}{y_{(n/2)+1}} \\ &\Leftrightarrow \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{n/2})}{y_{n/2}} \geq \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_{(n/2)+1})}{y_{(n/2)+1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{m' - z(\boldsymbol{p},T,\boldsymbol{\tau},y_i)}{m'} &= 1 - \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_i)}{m'} \leq 1 - \frac{y_i}{m} = \frac{m - y_i}{m}, \quad \text{for } i \leq n/2, \\ \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_i) - m'}{m'} &= \frac{z(\boldsymbol{p},T,\boldsymbol{\tau},y_i)}{m'} - 1 \leq \frac{y_i}{m} - 1 = \frac{y_i - m}{m}, \quad \text{for } i \geq (n/2)+1.\end{aligned}$$

Consequently,

$$(y_1,\ldots,y_n) \succcurlyeq_{FW} (z(\boldsymbol{p},T,\boldsymbol{\tau},y_1),\ldots,z(\boldsymbol{p},T,\boldsymbol{\tau},y_n)).$$

Since $\boldsymbol{y}$ was arbitrary, $(T,\boldsymbol{\tau})$ is bipolarization reducing. ■

C. Proof of Theorem 4

The following lemma will be used in the proof of Theorem 4.

Lemma 2. *Given $u \in \mathscr{U}$, $\boldsymbol{p}$, and a mixed tax system $(T, \boldsymbol{\tau})$, suppose that $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a. Then $(T, \boldsymbol{\tau})$ is inequality reducing if and only if*

$$\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)}{z^u(\boldsymbol{p}, 0, 0, a)} \geq \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, 0, 0, a')}, \quad \textit{whenever } a' > a > 0.$$

Proof. The sufficiency part follows from condition (vii) and Lemma 1.

Now suppose that

$$\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)}{z^u(\boldsymbol{p}, 0, 0, a)} < \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, 0, 0, a')}, \quad \text{for some } a' > a > 0.$$

Then, for the wage distribution

$$(a_1^*, \ldots, a_n^*) = (a, a' \ldots, a'),$$

we have

$$\frac{z^u(\boldsymbol{p}, 0, 0, a_1^*)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*)} > \frac{z^u(\boldsymbol{p}, 0, 0, a_2^*)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_2^*)} = \cdots = \frac{z^u(\boldsymbol{p}, 0, 0, a_n^*)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n^*)}. \tag{25}$$

Therefore, Lemma 1 gives

$$(z^u(\boldsymbol{p}, 0, 0, a_1^*), \ldots, z^u(\boldsymbol{p}, 0, 0, a_n^*)) \succcurlyeq_L (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n^*)). \tag{26}$$

If

$$(z^u(\boldsymbol{p}, 0, 0, a_1^*), \ldots, z^u(\boldsymbol{p}, 0, 0, a_n^*)) \succ_L (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n^*)), \tag{27}$$

then $(T, \boldsymbol{\tau})$ is not inequality reducing and the proof is complete.

To see that (27) holds, note that, if

$$\frac{z^u(\boldsymbol{p}, 0, 0, a_1^*)}{\sum_i z^u(\boldsymbol{p}, 0, 0, a_i^*)} > \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*)}{\sum_i z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i^*)},$$

then, by (26), we see that (27) holds. If, on the other hand,

$$\frac{z^u(\boldsymbol{p}, 0, 0, a_1^*)}{\sum_i z^u(\boldsymbol{p}, 0, 0, a_i^*)} = \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*)}{\sum_i z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i^*)},$$

then the inequality in (25) implies that

$$\frac{z^u(\boldsymbol{p}, 0, 0, a_2^*)}{\sum_i z^u(\boldsymbol{p}, 0, 0, a_i^*)} < \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_2^*)}{\sum_i z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i^*)},$$

whence

$$\frac{z^u(\boldsymbol{p}, 0, 0, a_1^*) + z^u(\boldsymbol{p}, 0, 0, a_2^*)}{\sum_i z^u(\boldsymbol{p}, 0, 0, a_i^*)} < \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1^*) + z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_2^*)}{\sum_i z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i^*)},$$

which contradicts (26). ∎

Theorem 4. *For $T \in \mathscr{T}$, a mixed tax system $(T, \boldsymbol{\tau})$ is inequality reducing with respect to $\boldsymbol{p}$ and $u \in \mathscr{U}$ only if the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a and T is marginal-rate progressive.*

Proof. Choose $\boldsymbol{p}$, a mixed tax system $(T, \boldsymbol{\tau})$ with $T \in \mathscr{T}$, and $u \in \mathscr{U}$. Suppose that

$$z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a') < z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), \quad \text{for } a' > a > 0.$$

By condition (vii), the map $a \mapsto z^u(\boldsymbol{p}, 0, 0, a) = a l^u(\boldsymbol{p}, 0, 0, a)$ is nondecreasing.

For the income distributions

$$\boldsymbol{z} = (z_1, \ldots, z_n) = (z^u(\boldsymbol{p}, 0, 0, a), \ldots, z^u(\boldsymbol{p}, 0, 0, a), z^u(\boldsymbol{p}, 0, 0, a'))$$

and

$$\boldsymbol{z}' = (z_1', \ldots, z_n') = (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')),$$

we have, for large enough n,

$$\frac{z^u(\boldsymbol{p}, 0, 0, a)}{(n-1) z^u(\boldsymbol{p}, 0, 0, a) + z^u(\boldsymbol{p}, 0, 0, a')} > \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a') + (n-1) z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)},$$

implying that $\boldsymbol{z}' \not\geq_L \boldsymbol{z}$, and so $(T, \boldsymbol{\tau})$ is not inequality reducing.

We now assume that $(T, \boldsymbol{\tau})$ is inequality reducing and T is not marginal-rate progressive, and derive a contradiction. By the previous argument, we know that $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a.

Since $T \in \mathscr{T}$, we can partition $\mathbb{R}_+$ into finitely many intervals $I_1, \ldots, I_J$ such that T is linear on I_j for each j. Because T is not convex, there exist two contiguous intervals,

$$I_j = [\underline{y}, y^*] \quad \text{and} \quad I_{j'} = [y^*, \overline{y}],$$

such that T is concave on $I_j \cup I_{j'}$. Therefore, the restriction of the map

$$y \in \mathbb{R}_+ \mapsto f(y)$$

from pre-tax income y to post-tax income $f(y) = y - T(y)$ to the set $I_j \cup I_{j'}$ can be expressed as follows:

$$f(y) = \begin{cases} \alpha + \beta y & \text{if } y \in I_j, \\ \alpha' + \beta' y & \text{if } y \in I_{j'}, \end{cases}$$

where $\alpha, \alpha' \in \mathbb{R}$, $\alpha > \alpha'$, and $\beta' > \beta > 0$. Note that $f(y) > 0$ if $y > 0$ (since marginal tax rates are less than unity).

Recall that the marginal rate of substitution of x_k for y for an 'a-individual' is given by

$$MRS_k^a(\boldsymbol{x}, y) = -\frac{(1/a)(\partial u(\boldsymbol{x}, y/a)/\partial l)}{\partial u(\boldsymbol{x}, y/a)/\partial x_k}.$$

It represents the amount of extra good k an individual should receive as compensation for an extra unit of gross labor income. Note that $y < aL$.

Recall that

$$(x_1^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), \ldots, x_K^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), l^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a))$$

represents a solution to

$$\max_{(x_1,\ldots,x_K,l)\in\mathbb{R}^K_+\times[0,L)} u(x_1,\ldots,x_K,l)$$
$$\text{s.t. } (p_1+\tau_1)x_1+\cdots+(p_K+\tau_K)x_K \le al - T(al).$$

An individual whose wage is $a > \underline{y}/L$ and whose labor supply is $l = \underline{y}/a$ earns gross (resp., net) labor income $\underline{y}$ (resp., $x(\underline{y})$).

Let $\boldsymbol{x}(a,y)$ solve

$$\max_{(x_1,\ldots,x_K)\in\mathbb{R}^K_+} u(x_1,\ldots,x_K,y/a)$$
$$\text{s.t. } (p_1+\tau_1)x_1+\cdots+(p_K+\tau_K)x_K \le f(y).$$

By the condition (iii),

$$\lim_{a\searrow\underline{y}/L} MRS^a_k(\boldsymbol{x}(a,\underline{y}),\underline{y}) = \infty.$$

Therefore (since the indifference curves for the utility function

$$(\boldsymbol{x},y)\in\mathbb{R}^K_+\times[0,aL)\mapsto u(\boldsymbol{x},y/a)$$

are convex by quasiconcavity of u), there exists $\underline{a} > 0$ such that $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \le \underline{y} < y^*$ for all $a \le \underline{a}$.

An individual whose wage is $a > \overline{y}/L$ and whose labor supply is $l = \overline{y}/a$ earns gross (resp., net) labor income $\overline{y}$ (resp., $x(\overline{y})$).

Because

$$\lim_{a\to\infty} MRS^a_k(\boldsymbol{x}(a,\overline{y}),\overline{y}) = 0,$$

there exists $\overline{a} > 0$ such that $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \ge \overline{y} > y^*$ for all $a \ge \overline{a}$.

Let

$$a^* = \inf\{a > 0 : al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) > y^*\}.$$

Then $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) > y^*$ for all $a > a^*$ and $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \le y^*$ for all $a < a^*$.

We claim that

$$\sup\{al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) : a < a^*\} < y^*. \tag{28}$$

To see this, note that $\sup\{al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) : a < a^*\} = y^*$ implies that there are sequences (a_n) and (y_n) such that $a_n \nearrow a^*$, each y_n is a solution to

$$\max_{y\in[0,a_nL)} u(\boldsymbol{x}(a_n,y),y/a_n), \tag{29}$$

and $y_n \nearrow y^*$. A necessary condition for y_n to solve (29) is

$$MRS^{a_n}_k(\boldsymbol{x}(a_n,y_n),y_n) = \frac{\beta}{p_k+\tau_k} > 0. \tag{30}$$

Because $a_n \to a^*$, $y_n \to y^*$, and $MRS^a_k(\boldsymbol{x}(a,y),y)$ is continuous in (a,y), we have

$$MRS^{a_n}_k(\boldsymbol{x}(a_n,y_n),y_n) \to MRS^{a^*}_k(\boldsymbol{x}(a^*,y^*),y^*).$$

Consequently, (30) gives

$$MRS_k^{a^*}(\boldsymbol{x}(a^*,y^*),y^*) = \frac{\beta}{p_k+\tau_k}.$$

Hence,

$$(p_k+\tau_k)MRS_k^{a^*}(\boldsymbol{x}(a^*,y^*),y^*) = \beta < \beta' = \frac{dx(y^*)}{dy},$$

and so there exists $\hat{y} \in I_{k'}$ such that $\hat{y} < a^*L$ and

$$u(\boldsymbol{x}(a^*,y^*),y^*/a^*) < u(\boldsymbol{x}(a^*,\hat{y}),\hat{y}/a^*).$$

Consequently, since the map

$$(a,y) \in \mathbb{R}^2_{++} \mapsto u(\boldsymbol{x}(a,y),y/a)$$

is continuous and $a_n \to a^*$ and $y_n \to y^*$,

$$u(\boldsymbol{x}(a_n,y_n),y_n/a_n) \to u(\boldsymbol{x}(a^*,y^*),y^*/a^*) < u(\boldsymbol{x}(a^*,\hat{y}),\hat{y}/a^*).$$

Since $u(\boldsymbol{x}(a^*,y^*),y^*/a^*) < u(\boldsymbol{x}(a^*,\hat{y}),\hat{y}/a^*)$, we may choose $\varepsilon > 0$ such that

$$u(\boldsymbol{x}(a^*,y^*),y^*/a^*) + 2\varepsilon < u(\boldsymbol{x}(a^*,\hat{y}),\hat{y}/a^*).$$

Since $u(\boldsymbol{x}(a_n,y_n),y_n/a_n) \to u(\boldsymbol{x}(a^*,y^*),y^*/a^*)$, there exists N such that

$$u(\boldsymbol{x}(a_n,y_n),y_n/a_n) < u(\boldsymbol{x}(a^*,y^*),y^*/a^*) + \varepsilon, \quad \text{for all } n \geq N.$$

Moreover, since $a_n \to a^*$, there exists M such that

$$u(\boldsymbol{x}(a^*,\hat{y}),\hat{y}/a^*) - \varepsilon < u(\boldsymbol{x}(a_n,\hat{y}),\hat{y}/a_n), \quad \text{for all } n \geq M.$$

Consequently, for $n \geq \max\{M,N\}$,

$$u(\boldsymbol{x}(a_n,y_n),y_n/a_n) < u(\boldsymbol{x}(a_n,\hat{y}),\hat{y}/a_n),$$

contradicting that y_n solves (29).

Hence, (28) holds. Therefore, because $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) > y^*$ for all $a > a^*$ and $al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \leq y^*$ for all $a < a^*$, we see that there exist $0 < y < y'$ such that

$$\begin{aligned} al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) &\leq y \quad \text{for } a < a^*, \\ al^u(\boldsymbol{p},T,\boldsymbol{\tau},a) &\geq y' \quad \text{for } a > a^*. \end{aligned}$$

Thus, $l^u(\boldsymbol{p},T,\boldsymbol{\tau},\cdot)$ has a discontinuity at a^*, and so, by the condition (vi), $z^u(\boldsymbol{p},T,\boldsymbol{\tau},\cdot)$ has a discontinuity at a^*. Hence, because $z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)$ is nondecreasing in a, there exist $z < z'$ such that

$$z^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \leq z \quad \text{for } a < a^*, \tag{31}$$

$$z^u(\boldsymbol{p},T,\boldsymbol{\tau},a) \geq z' \quad \text{for } a > a^*. \tag{32}$$

Next, we show that the map $z^u(\boldsymbol{p},0,0,\cdot)$ is continuous. Note that $z^u(\boldsymbol{p},0,0,a)$ solves

$$\max_{y\in[0,aL)} u(\boldsymbol{x}(\boldsymbol{p},0,a,y),y/a).$$

By the argument in Footnote 9, for each $a>0$ there exists $\overline{y}(a)\in(0,aL)$ such that $z^u(\boldsymbol{p},0,0,a)$ solves

$$\max_{y\in[0,\overline{y}(a)]} u(\boldsymbol{x}(\boldsymbol{p},0,a,y),y/a).$$

By the Maximum Theorem, the correspondence

$$a>0\rightrightarrows \operatorname*{arg\,max}_{y\in[0,\overline{y}(a)]} u(\boldsymbol{x}(\boldsymbol{p},0,a,y),y/a)$$

is upper hemicontinuous. We will show that this correspondence is, in fact, single-valued, implying that it is a continuous function, i.e., the map $z^u(\boldsymbol{p},0,0,\cdot)$ is continuous.

Suppose that there exist $a>0$ and distinct $y'>0$ and $y''>0$ in

$$\operatorname*{arg\,max}_{y\in[0,\overline{y}(a)]} u(\boldsymbol{x}(\boldsymbol{p},0,a,y),y/a).$$[16]

Fix $\alpha\in(0,1)$. By the condition (iv),

$$\begin{aligned}
&u(\alpha(\boldsymbol{x}(\boldsymbol{p},0,a,y'),y'/a)+(1-\alpha)(\boldsymbol{x}(\boldsymbol{p},0,a,y''),y''/a))\\
&\qquad=u(\alpha\boldsymbol{x}(\boldsymbol{p},0,a,y')+(1-\alpha)\boldsymbol{x}(\boldsymbol{p},0,a,y''),(\alpha y'+(1-\alpha)y'')/a)\\
&\qquad\qquad>\min\{u(\boldsymbol{x}(\boldsymbol{p},0,a,y'),y'/a),u(\boldsymbol{x}(\boldsymbol{p},0,a,y''),y''/a)\}. \qquad (33)
\end{aligned}$$

Because $y',y''\in[0,\overline{y}(a)]$, we have

$$\alpha y'+(1-\alpha)y''\in[0,\overline{y}(a)].$$

In addition,

$$\begin{aligned}
\boldsymbol{p}\cdot(\alpha\boldsymbol{x}(\boldsymbol{p},0,a,y')+(1-\alpha)\boldsymbol{x}(\boldsymbol{p},0,a,y''))&=\alpha(\boldsymbol{p}\cdot\boldsymbol{x}(\boldsymbol{p},0,a,y'))+(1-\alpha)(\boldsymbol{p}\cdot\boldsymbol{x}(\boldsymbol{p},0,a,y''))\\
&\le\alpha y'+(1-\alpha)y''.
\end{aligned}$$[17]

Therefore, (33) contradicts that y' and y'' are members of $\operatorname{arg\,max}_{y\in[0,\overline{y}(a)]} u(\boldsymbol{x}(\boldsymbol{p},a,y),y/a)$.

Given (31)-(32) and the continuity of the map $z^u(\boldsymbol{p},0,0,\cdot)$, we see that, for $0<a<a^*<a'$ with a and a' close enough to a^*, we have

$$\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a)}{z^u(\boldsymbol{p},0,0,a)}<\frac{z^u(\boldsymbol{p},T,\boldsymbol{\tau},a')}{z^u(\boldsymbol{p},0,0,a')}.$$

By Lemma 2, $(T,\boldsymbol{\tau})$ is not inequality reducing, a contradiction. ∎

[16]Recall that the condition (v) guarantees that $y'>0$ and $y''>0$.

[17]Here, for $\boldsymbol{x}=(x_1,\dots,x_K)\in\mathbb{R}^K_+$,

$$\boldsymbol{p}\cdot\boldsymbol{x}=p_1x_1+\cdots+p_Kx_K.$$

D. Proof of Theorem 5

The proof of Theorem 5 is based on the following lemma.

Lemma 3. *Suppose that $\mathcal{S} \subseteq \mathcal{T}_{m\text{-}prog}$ is closed under linear transformations. Suppose that $\mathcal{S}'$ is a subset of commodity tax systems. Then the mixed tax systems in $\mathcal{S} \times \mathcal{S}'$ are inequality reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$ if and only if the following two conditions are satisfied:*

(i) the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a for each $T \in \mathcal{L}_{\mathcal{S}} \cup \{0\}$ and $\boldsymbol{\tau}$; and

(ii) the members of $\mathcal{L}_{\mathcal{S}} \times \mathcal{S}'$ are inequality reducing.

Proof. Suppose that the mixed tax systems in $\mathcal{S} \times \mathcal{S}'$ are inequality reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$. Then the members of $\mathcal{L}_{\mathcal{S}} \times \mathcal{S}'$ are inequality reducing (since $\mathcal{L}_{\mathcal{S}} \subseteq \mathcal{S}$) and, by Theorem 4, the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a for each $T \in \mathcal{L}_{\mathcal{S}}$ and $\boldsymbol{\tau} \in \mathcal{S}'$.

Now assume (i)-(ii). Fix $(T, \boldsymbol{\tau}) \in \mathcal{S} \times \mathcal{S}'$. We must show that $(T, \boldsymbol{\tau})$ is inequality reducing with respect to $\boldsymbol{p}$ and u. By Lemma 2, it suffices to show that $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a and

$$\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)}{z^u(\boldsymbol{p}, 0, 0, a)} \geq \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, 0, 0, a')}, \quad \text{whenever } a' > a > 0.$$

Because T is piecewise linear in $\mathcal{T}_{m\text{-}prog}$, there exist

$$0 < e_1 < \cdots < e_M < \infty$$

and intervals

$$I_1 = [0, e_1], \ldots, I_M = [e_{M-1}, e_M], I_{M+1} = [e_M, \infty)$$

satisfying the following: for each m, there exist $b_m \geq 0$ and $t_m \in [0, 1)$ such that $T(y) = -b_m + t_m y$ for all $y \in I_m$. Moreover,

$$b_1 < \cdots < b_{M+1} \quad \text{and} \quad t_1 < \cdots < t_{M+1}.$$

For $m \in \{1, \ldots, M+1\}$, let $T_m(y) = -b_m + t_m y$. Suppose that $y_m(a)$ and $y(a)$ denote the solutions to

$$\max_{y \in [0, aL)} u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T_m, a, y), y/a)$$

and

$$\max_{y \in [0, aL)} u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T, a, y), y/a),$$

respectively.[18] By the argument in Footnote 9, for each m and $a > 0$ there exists $\bar{y}_m(a) \in (0, aL)$ such that $y_m(a)$ solves

$$\max_{y \in [0, \bar{y}_m(a)]} u(\boldsymbol{x}(\boldsymbol{p} + \boldsymbol{\tau}, T_m, a, y), y/a).$$

By the Maximum Theorem, the function $y_m(a)$ is continuous.

[18] These problems have a unique solution. See Footnote 12.

There exists $\bar{a}_1 > 0$ such that $y_1(\bar{a}_1) = e_1$. To see this, note that there exists a small enough $\alpha > 0$ such that $y_1(\alpha) < e_1$. Moreover, because

$$\lim_{a\to\infty} MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_1, a, e_1), e_1) = 0$$

for each k, there exists a large enough $\beta > 0$ such that $y_1(\beta) > e_1$. Consequently, the Intermediate Value Theorem gives $\bar{a}_1 > 0$ such that $y_1(\bar{a}_1) = e_1$.

Similarly, we can show the following:

$$\begin{aligned}
&\exists \bar{a}_1 > 0 : y_1(\bar{a}_1) = e_1, \\
&\exists \underline{a}_2 > 0, \bar{a}_2 > 0 : y_2(\underline{a}_2) = e_1 \text{ and } y_2(\bar{a}_2) = e_2, \\
&\;\vdots \\
&\exists \underline{a}_M > 0, \bar{a}_M > 0 : y_M(\underline{a}_M) = e_{M-1} \text{ and } y_M(\bar{a}_M) = e_M, \\
&\exists \underline{a}_{M+1} > 0 : y_{M+1}(\underline{a}_{M+1}) = e_M.
\end{aligned} \tag{34}$$

Moreover, there is no loss of generality in assuming that

$$\bar{a}_1 \le \underline{a}_2 \le \bar{a}_2 \le \cdots \le \underline{a}_M \le \bar{a}_M \le \underline{a}_{M+1}. \tag{35}$$

We prove this in two steps.

First, we show that

$$\underline{a}_m \le \bar{a}_m, \quad m \in \{2, \ldots, M\}. \tag{36}$$

Note that, for $m \in \{1, \ldots, M+1\}$, the map

$$a \mapsto l^u(\boldsymbol{p}+\boldsymbol{\tau}, T_m, 0, a)$$

is nondecreasing. Indeed, for $a' > a > 0$, we have

$$\begin{aligned}
al^u(\boldsymbol{p}+\boldsymbol{\tau}, T_m, 0, a) = al^u(\boldsymbol{p}+\boldsymbol{\tau}, -b_m, 0, (1-t_m)a)& \\
\le al^u(\boldsymbol{p}+\boldsymbol{\tau}, -b_m, 0, (1-t_m)a') = al^u(\boldsymbol{p}+\boldsymbol{\tau}, T_m, 0, a'),&
\end{aligned}$$

where the inequality follows from condition (vii). Hence, the map $a \mapsto y_m(a)$ is nondecreasing. Consequently, for $m \in \{2, \ldots, M\}$, since

$$y_m(\underline{a}_m) = e_{m-1} < e_m = y_m(\bar{a}_m),$$

we see that $\underline{a}_m \le \bar{a}_m$. This establishes (36).

It remains to show that

$$\bar{a}_m \le \underline{a}_{m+1}, \quad m \in \{1, \ldots, M\}.$$

We only show that $\bar{a}_1 \le \underline{a}_2$, since the other inequalities can be handled similarly. Proceeding by contradiction, suppose that $\bar{a}_1 > \underline{a}_2$. Since $y_2(\underline{a}_2) = e_1$, for every k we have

$$(p_k + \tau_k) MRS_k^{\underline{a}_2}(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_2, \underline{a}_2, e_1), e_1) = 1 - t_2.$$

Now since $\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_2, \underline{a}_2, e_1)$ solves

$$\max_{(x_1,\ldots,x_K)\in\mathbb{R}^K_+} u(x_1,\ldots,x_K,e_1/\underline{a}_2)$$
$$\text{s.t. } p_1x_1+\cdots+p_Kx_K \leq e_1 - T_2(e_1),$$

and since $T_1(e_1) = T_2(e_1)$, we have

$$\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_2, \underline{a}_2, e_1) = \boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_1, \underline{a}_2, e_1),$$

and so

$$(p_k+\tau_k)MRS_k^{\underline{a}_2}(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_1, \underline{a}_2, e_1), e_1) = 1-t_2 < 1-t_1,$$

implying that $y_1(\underline{a}_2) > e_1$, a contradiction, since $y_1(a)$ is nondecreasing and $y_1(\bar{a}_1) = e_1$.

We have seen that (34) and (35) hold. Note that

$$y(a) = \begin{cases} y_1(a) & \text{if } a \in (0, \bar{a}_1], \\ y_2(a) & \text{if } a \in [\underline{a}_2, \bar{a}_2], \\ \vdots & \vdots \\ y_M(a) & \text{if } a \in [\underline{a}_M, \bar{a}_M], \\ y_{M+1}(a) & \text{if } a \geq \underline{a}_{M+1}. \end{cases} \tag{37}$$

Moreover, for $a \in [\bar{a}_1, \underline{a}_2]$ we have $y(a) = e_1$. To see this, fix $a \in [\bar{a}_1, \underline{a}_2]$. For each k we have

$$(p_k+\tau_k)MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1) \in [1-t_2, 1-t_1], \tag{38}$$

implying that $y(a) = e_1$. Prior to verifying the inclusion in (38), we argue that $y(a) = e_1$ is a consequence of (38).

Under (38), an increase in labor income, dy, starting at $(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1)$, requires at least

$$MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1)dy \geq \frac{1-t_2}{p_k+\tau_k}dy$$

extra units of good k to keep utility constant, since the indifference curves for the utility function

$$(\boldsymbol{x}, y) \mapsto u(\boldsymbol{x}, y/a)$$

are convex (by quasiconcavity of u). But increasing labor income by dy only brings about an extra (net) labor income of at most $(1-t_2)dy$ (recall that

$$1-t_1 > \cdots > 1-t_{M+1}),$$

which can afford at most $\frac{1-t_2}{p_k+\tau_k}dy$ extra units of good k. Thus, an increase in labor income is not welfare improving. A similar argument can be made for a reduction (rather than an increase) in labor income, taking the bundle $(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1)$ as the initial point.

To see that (38) holds, we assume that

$$(p_k+\tau_k)MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1) < 1-t_2 \tag{39}$$

and derive a contradiction (the case when

$$(p_k + \tau_k) MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1), e_1) > 1 - t_1$$

can be handled similarly). Since

$$\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T, a, e_1) = \boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_2, a, e_1),$$

(39) implies that

$$(p_k + \tau_k) MRS_k^a(\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{\tau}, T_2, a, e_1), e_1) < 1 - t_2,$$

whence $y_2(a) > e_1$, a contradiction, since $y_2(a)$ is nondecreasing, $a \le \underline{a}_2$, and $y_2(\underline{a}_2) = e_1$.

We have seen that $y(a) = e_1$ whenever $a \in [\bar{a}_1, \underline{a}_2]$. Similarly, we can show that

$$y(a) = \begin{cases} e_1 & \text{if } a \in [\bar{a}_1, \underline{a}_2], \\ \vdots & \vdots \\ e_M & \text{if } a \in [\bar{a}_M, \underline{a}_{M+1}]. \end{cases} \tag{40}$$

We are now ready to show that to show that $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a and

$$\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)}{z^u(\boldsymbol{p}, 0, 0, a)} \ge \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, 0, 0, a')}, \quad \text{whenever } a' > a > 0.$$

Choose $a' > a > 0$. Suppose that $a \in (0, \bar{a}_1)$ and $a' > \underline{a}_{M+1}$ (the other cases can be handled similarly). Then

$$\begin{aligned} z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a') &= z^u(\boldsymbol{p}, T_{M+1}, \boldsymbol{\tau}, a') && \text{(by (37))} \\ &\ge z^u(\boldsymbol{p}, T_{M+1}, \boldsymbol{\tau}, \underline{a}_{M+1}) && \text{(by (i))} \\ &= z^u(\boldsymbol{p}, T_M, \boldsymbol{\tau}, \bar{a}_M) && \text{(by (37) and (40))} \\ &\ge z^u(\boldsymbol{p}, T_M, \boldsymbol{\tau}, \underline{a}_M) && \text{(by (i))} \\ &\vdots \\ &\ge z^u(\boldsymbol{p}, T_2, \boldsymbol{\tau}, \underline{a}_2) \\ &= z^u(\boldsymbol{p}, T_1, \boldsymbol{\tau}, \bar{a}_1) && \text{(by (37) and (40))} \\ &\ge z^u(\boldsymbol{p}, T_1, \boldsymbol{\tau}, a) && \text{(by (i))}. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, 0, 0, a')} &= \frac{z^u(\boldsymbol{p}, T_{M+1}, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, 0, 0, a')} && \text{(by (37))} \\ &\le \frac{z^u(\boldsymbol{p}, T_{M+1}, \boldsymbol{\tau}, \underline{a}_{M+1})}{z^u(\boldsymbol{p}, 0, 0, \underline{a}_{M+1})} && \text{(by (ii) and Lemma 2)} \\ &= \frac{z^u(\boldsymbol{p}, T_M, \boldsymbol{\tau}, \bar{a}_M)}{z^u(\boldsymbol{p}, 0, 0, \underline{a}_{M+1})} && \text{(by (37) and (40))} \\ &\le \frac{z^u(\boldsymbol{p}, T_M, \boldsymbol{\tau}, \bar{a}_M)}{z^u(\boldsymbol{p}, 0, 0, \bar{a}_M)} && \text{(by (i))} \\ &\vdots \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{z^u(\boldsymbol{p}, T_2, \boldsymbol{\tau}, \underline{a}_2)}{z^u(\boldsymbol{p}, 0, 0, \underline{a}_2)} && \\
&= \frac{z^u(\boldsymbol{p}, T_1, \boldsymbol{\tau}, \bar{a}_1)}{z^u(\boldsymbol{p}, 0, 0, \underline{a}_2)} && \text{(by (37) and (40))} \\
&\leq \frac{z^u(\boldsymbol{p}, T_1, \boldsymbol{\tau}, \bar{a}_1)}{z^u(\boldsymbol{p}, 0, 0, \bar{a}_1))} && \text{(by (i))} \\
&\leq \frac{z^u(\boldsymbol{p}, T_1, \boldsymbol{\tau}, a)}{z^u(\boldsymbol{p}, 0, 0, a))} && \text{(by (ii) and Lemma 2).}
\end{aligned}
$$

This completes the proof. ■

Theorem 5. *Suppose that $\mathcal{S} \subseteq \mathcal{T}_{m\text{-}prog}$ is closed under linear transformations. Suppose that $\mathcal{S}'$ is a subset of commodity tax systems. Then the mixed tax systems in $\mathcal{S} \times \mathcal{S}'$ are inequality reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$ if and only if the following two conditions are satisfied:*

(i) the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a for each $T \in \mathcal{L}_{\mathcal{S}} \cup \{0\}$ and $\boldsymbol{\tau} \in \mathcal{S}'$;[19] *and*

(ii) the quotient

$$\frac{z^u(\boldsymbol{p}, b, \boldsymbol{\tau}, (1-t)a)}{z^u(\boldsymbol{p}, 0, 0, a)}$$

is nonincreasing in a for every $(b, t, \boldsymbol{\tau}) \in B(\mathcal{L}_{\mathcal{S}}) \times R(\mathcal{L}_{\mathcal{S}}) \times \mathcal{S}'$.

Proof. By Lemma 3, the mixed tax systems in $\mathcal{S} \times \mathcal{S}'$ are inequality reducing with respect to $\boldsymbol{p}$ and $u \in \mathcal{U}$ if and only if the following two conditions are satisfied:

- the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a for each $T \in \mathcal{L}_{\mathcal{S}} \cup \{0\}$ and $\boldsymbol{\tau} \in \mathcal{S}'$; and
- the members of $\mathcal{L}_{\mathcal{S}} \times \mathcal{S}'$ are inequality reducing.

Given the first bullet point and using Lemma 2, the second bullet point is expressible as follows:

$$\frac{z^u(\boldsymbol{p}, -b + ty, \boldsymbol{\tau}, a)}{z^u(\boldsymbol{p}, 0, 0, a)}$$

is nonincreasing in a every $(b, t, \boldsymbol{\tau}) \in B(\mathcal{L}_{\mathcal{S}}) \times R(\mathcal{L}_{\mathcal{S}}) \times \mathcal{S}'$. Since

$$z^u(\boldsymbol{p}, -b + ty, \boldsymbol{\tau}, a) = (1-t)al^u(\boldsymbol{p}, -b + ty, \boldsymbol{\tau}, a) + b - \sum_{k=1}^{K} \tau_k x_k^u(\boldsymbol{p}, -b + ty, \boldsymbol{\tau}, a)$$

and

$$(x_1^u(\boldsymbol{p}, -b + ty, \boldsymbol{\tau}, a), \ldots, x_K^u(\boldsymbol{p}, -b + ty, \boldsymbol{\tau}, a), l^u(\boldsymbol{p}, -b + ty, \boldsymbol{\tau}, a))$$

solves

$$
\begin{aligned}
&\max_{(x_1, \ldots, x_K, l) \in \mathbb{R}_+^K \times [0, L)} \quad u(x_1, \ldots, x_K, l) \\
&\text{s.t. } (p_1 + \tau_1)x_1 + \cdots + (p_K + \tau_K)x_K \leq b + (1-t)al,
\end{aligned}
$$

[19] Here 0 denotes the linear tax schedule T defined by $T(y) = 0$ for all y.

it follows that $z^u(\boldsymbol{p}, -b+ty, \boldsymbol{\tau}, a)$ can also be expressed as $z^u(\boldsymbol{p}, b, \boldsymbol{\tau}, (1-t)a)$. Therefore, the second bullet point can be written as follows:

$$\frac{z^u(\boldsymbol{p}, b, \boldsymbol{\tau}, (1-t)a)}{z^u(\boldsymbol{p}, 0, 0, a)}$$

is nonincreasing in a every $(b, t, \boldsymbol{\tau}) \in B(\mathscr{L}_{\mathcal{S}}) \times R(\mathscr{L}_{\mathcal{S}}) \times \mathcal{S}'$. ∎

E. Proof of Theorem 6

First, we state and prove the following intermediate result.

Lemma 4. *For $T \in \mathscr{T}$, suppose that $(T, \boldsymbol{\tau})$ is bipolarization reducing with respect to $\boldsymbol{p}$ and $u \in \mathscr{U}$. Then, the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a.*

Proof. Suppose that $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is not nondecreasing in a. Then, there exist $a' > a > 0$ such that

$$z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a') < z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a).$$

Consider the income distributions

$$\boldsymbol{z}^* = (z_1^*, \ldots, z_n^*) = (z^u(\boldsymbol{p}, 0, 0, a), \ldots, z^u(\boldsymbol{p}, 0, 0, a), z^u(\boldsymbol{p}, 0, 0, a'))$$

and

$$\boldsymbol{z}^{**} = (z_1^{**}, \ldots, z_n^{**}) = (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a), z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')).$$

For $n > 2$, we have $m(\boldsymbol{z}^*) = z^u(\boldsymbol{p}, 0, 0, a)$ and $m(\boldsymbol{z}^{**}) = z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$. Therefore, since

$$z^u(\boldsymbol{p}, 0, 0, a) \leq \cdots \leq z^u(\boldsymbol{p}, 0, 0, a) \leq z^u(\boldsymbol{p}, 0, 0, a'),$$

where the last inequality follows from the condition (vii), we have

$$\begin{aligned}\frac{1}{m(\boldsymbol{z}^*)} \sum_{1 \leq i < \frac{n+1}{2}} (m(\boldsymbol{z}^*) - z_{[i]}^*) = 0 < \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a) - z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{m(\boldsymbol{z}^{**})} \\ = \frac{1}{m(\boldsymbol{z}^{**})} \sum_{1 \leq i < \frac{n+1}{2}} (m(\boldsymbol{z}^{**}) - z_{[i]}^{**}),\end{aligned}$$

and so $\frac{m(\boldsymbol{z}^{**})}{m(\boldsymbol{z}^*)} \boldsymbol{z}^* \not\succcurlyeq_{FW} \boldsymbol{z}^{**}$, whence $\boldsymbol{z}^* \not\succcurlyeq_{FW} \boldsymbol{z}^{**}$. ∎

Next, we prove Theorem 6.

Theorem 6. *For $T \in \mathscr{T}$, a mixed tax system $(T, \boldsymbol{\tau})$ is inequality reducing with respect to $\boldsymbol{p}$ and $u \in \mathscr{U}$ if and only if it is bipolarization reducing with respect to $\boldsymbol{p}$ and u.*

Proof. For $T \in \mathscr{T}$, suppose that a mixed tax system $(T, \boldsymbol{\tau})$ is inequality reducing with respect to $\boldsymbol{p}$ and $u \in \mathscr{U}$. Fix a wage distribution $(a_1, \ldots, a_n)$, an income function z^u, and a population size n. We must show that

$$(z^u(\boldsymbol{p}, 0, 0, a_1), \ldots, z^u(\boldsymbol{p}, 0, 0, a_n)) \succcurlyeq_{FW} (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n)). \tag{41}$$

By Theorem 4, the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a. By the condition (vii), the disposable income function $z^u(\boldsymbol{p}, 0, 0, a)$ is nondecreasing in a.

We prove (41) when n is odd (the case when n is even can be handled similarly). Choose a wage distribution $(a_1, \dots, a_n)$ with

$$a_1 \leq \cdots \leq a_n.$$

Let $m = (n+1)/2$, so that a_m is the median wage. Because $(T, \boldsymbol{\tau})$ is inequality reducing with respect to $\boldsymbol{p}$ and $u \in \mathscr{U}$, and since the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ (resp., $z^u(\boldsymbol{p}, 0, 0, a)$) is nondecreasing in a, Lemma 2 implies that

$$\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i)}{z^u(\boldsymbol{p}, 0, 0, a_i)} \geq \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m)}{z^u(\boldsymbol{p}, 0, 0, a_m)} \geq \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_j)}{z^u(\boldsymbol{p}, 0, 0, a_j)}, \quad \text{for } i < m \text{ and } j > m.$$

Hence,

$$\begin{aligned}
\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m) - z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m)} &= 1 - \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m)} \\
&\leq 1 - \frac{z^u(\boldsymbol{p}, 0, 0, a_i)}{z^u(\boldsymbol{p}, 0, 0, a_m)} = \frac{z^u(\boldsymbol{p}, 0, 0, a_m) - z^u(\boldsymbol{p}, 0, 0, a_i)}{z^u(\boldsymbol{p}, 0, 0, a_m)}, \quad \text{for } i < m, \\
\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i) - z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m)} &= \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_i)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m)} - 1 \\
&\leq \frac{z^u(\boldsymbol{p}, 0, 0, a_i)}{z^u(\boldsymbol{p}, 0, 0, a_m)} - 1 = \frac{z^u(\boldsymbol{p}, 0, 0, a_i) - z^u(\boldsymbol{p}, 0, 0, a_m)}{z^u(\boldsymbol{p}, 0, 0, a_m)}, \quad \text{for } i > m.
\end{aligned}$$

Consequently,

$$\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m)}{z^u(\boldsymbol{p}, 0, 0, a_m)}(z^u(\boldsymbol{p}, 0, 0, a_1), \dots, z^u(\boldsymbol{p}, 0, 0, a_n)) \succcurlyeq_{FW} (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \dots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n)),$$

implying (41).

Conversely, suppose that $(T, \boldsymbol{\tau})$ is bipolarization reducing with respect to $\boldsymbol{p}$ and u. Fix a wage distribution $(a_1, \dots, a_n)$, an income function z^u, and a population size n. We must show that

$$(z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \dots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n)) \succcurlyeq_L (z^u(\boldsymbol{p}, 0, 0, a_1), \dots, z^u(\boldsymbol{p}, 0, 0, a_n)). \tag{42}$$

By Lemma 4, the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a. By the condition (vii), the disposable income function $z^u(\boldsymbol{p}, 0, 0, a)$ is nondecreasing in a.

We only prove (42) when n is odd (the case when n is even can be proven similarly). Proceeding by contradiction, suppose that (42) is false. Then Lemma 2 implies that there exist $a' > a > 0$ such that

$$\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)}{z^u(\boldsymbol{p}, 0, 0, a)} < \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')}{z^u(\boldsymbol{p}, 0, 0, a')}. \tag{43}$$

Pick a wage distribution $(a_1, \dots, a_n)$ with

$$a_1 \leq \cdots \leq a_n$$

such that the median wage, a_m, where $m = (n+1)/2$, is equal to a', and $a_{m-1} = a$. Then

$$\begin{aligned}\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m) - z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_{m-1})}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m)} &= 1 - \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_{m-1})}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m)} \\ &= 1 - \frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)}{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a')} \\ &> 1 - \frac{z^u(\boldsymbol{p}, 0, 0, a)}{z^u(\boldsymbol{p}, 0, 0, a')} \\ &= \frac{z^u(\boldsymbol{p}, 0, 0, a_m) - z^u(\boldsymbol{p}, 0, 0, a_{m-1})}{z^u(\boldsymbol{p}, 0, 0, a_m)},\end{aligned}$$

where the inequality follows from (43). Consequently,

$$\frac{z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_m)}{z^u(\boldsymbol{p}, 0, 0, a_m)}(z^u(\boldsymbol{p}, 0, 0, a_1), \ldots, z^u(\boldsymbol{p}, 0, 0, a_n)) \not\succcurlyeq_{FW} (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n)),$$

whence

$$(z^u(\boldsymbol{p}, 0, 0, a_1), \ldots, z^u(\boldsymbol{p}, 0, 0, a_n)) \not\succcurlyeq_{FW} (z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_1), \ldots, z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a_n)),$$

a contradiction. ∎

F. Joint and separate average-rate progressivity

In this section, we show that separate average-rate progressivity implies joint average-rate progressivity, but the converse is not true.

For an example of a jointly average-rate progressive tax system that is not separately average-rate progressive, consider the utility function

$$u(x_1, x_2) = 2\sqrt{x_1} + x_2,$$

whose associated (Marshallian) demand functions are given by

$$x_1(p_1, p_2, y) = \begin{cases} (p_2/p_1)^2 & \text{if } y \geq p_2^2/p_1, \\ y/p_1 & \text{if } y < p_2^2/p_1, \end{cases} \quad \text{and} \quad x_2(p_1, p_2, y) = \begin{cases} \frac{y}{p_2} - \frac{p_2}{p_1} & \text{if } y \geq p_2^2/p_1, \\ 0 & \text{if } y < p_2^2/p_1. \end{cases}$$

Let $(T, \boldsymbol{\tau})$ be a mixed tax system such that $\boldsymbol{\tau} = (0, \tau_2)$ and

$$T(y) = \begin{cases} \beta' y & \text{if } (1-\beta')y < (p_2+\tau_2)^2/p_1, \\ \frac{(p_2+\tau_2)^2}{(1-\beta')p_1}(\beta' - \beta) + \beta y & \text{if } (1-\beta')y \geq (p_2+\tau_2)^2/p_1, \end{cases}$$

where $1 > \beta' > \beta > 0$.

Because T is concave, $(T, \boldsymbol{\tau})$ fails to be separately average-rate progressive, and yet $(T, \boldsymbol{\tau})$ is jointly average-rate progressive if $\tau_2(1-\beta') \geq p_2(\beta'-\beta)$. To see this, note that

$$\frac{1}{y}(T(y) + \tau_2 x_2(\boldsymbol{p}, T, \boldsymbol{\tau}, y))$$
$$= \begin{cases} \beta' & \text{if } (1-\beta')y < (p_2+\tau_2)^2/p_1, \\ \beta + \frac{\tau_2(1-\beta)}{p_2+\tau_2} + \frac{p_2+\tau_2}{(1-\beta')p_1 y}(p_2(\beta'-\beta) - \tau_2(1-\beta')) & \text{if } (1-\beta')y \geq (p_2+\tau_2)^2/p_1. \end{cases}$$

Hence, $\frac{1}{y}(T(y) + \tau_2 x_2(\boldsymbol{p}, T, \boldsymbol{\tau}, y))$ is nondecreasing in y if $\tau_2(1-\beta') \geq p_2(\beta'-\beta)$.

We now show that separate average-rate progressivity implies joint average-rate progressivity.

Suppose that $(T, \boldsymbol{\tau})$ is separately average-rate progressive. Recall that, under differentiability of T and the demand functions $x_k(\boldsymbol{p}', y')$ ($k \in \{1, \ldots, K\}$), joint average-rate progressivity is expressible as

$$T'(y) + (1 - T'(y))\left(\sum_{k=1}^{K} \tau_k \partial_2 x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))\right)$$
$$\geq \frac{T(y)}{y} + \sum_{k=1}^{K} \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))}{y}, \quad y > 0. \tag{44}$$

Recall from (6) that $\boldsymbol{\tau}$ taxes luxuries and/or subsidizes necessities (since $\boldsymbol{\tau}$ is average-rate progressive). Since luxury goods are normal and inferior goods are necessities, it follows that the bracketed summation on the left-hand side of expression (44) is nonnegative. Consequently, it suffices to show that

$$T'(y) \geq \frac{T(y)}{y} + \sum_{k=1}^{K} \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))}{y}, \quad y > 0,$$

i.e.,

$$T'(y) - \frac{T(y)}{y} \geq \sum_{k=1}^{K} \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))}{y}, \quad y > 0. \tag{45}$$

Note that the left-hand side of (45) is greater than or equal to one, since T is average-rate progressive. Therefore,

$$1 \geq \sum_{k=1}^{K} \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))}{y}, \quad y > 0,$$

is a sufficient condition for (45) to hold. But the last inequality is true, since

$$\sum_{k=1}^{K} \tau_k \cdot \frac{x_k(\boldsymbol{p}+\boldsymbol{\tau}, y - T(y))}{y}$$

is the fraction of every dollar paid as consumption tax at $y > 0$.

G. On mixed vs. pure direct taxation

As stated at the end of Section 2.2, commodity taxation is *not*, in general, superfluous when income is endogenous, in contrast with the case of exogenous income. Indeed, a mixed tax system may be inequality reducing in cases when income taxation lacks any equalizing power.

To illustrate this point, suppose that there are two goods (i.e., $K = 2$) and consider the following utility function:

$$u(x_1, x_2, l) = -\frac{1}{x_1} + x_2 - \frac{1}{1-l}, \tag{46}$$

where $l \in [0, 1)$.

For this utility function, we have

$$x_1^u(\boldsymbol{p}, y) = \begin{cases} \frac{y}{p_1} & \text{if } y < \sqrt{p_1 p_2}, \\ \sqrt{\frac{p_2}{p_1}} & \text{if } y \geq \sqrt{p_1 p_2}, \end{cases}$$

$$x_2^u(\boldsymbol{p}, y) = \begin{cases} 0 & \text{if } y < \sqrt{p_1 p_2}, \\ \frac{y - \sqrt{p_1 p_2}}{p_2} & \text{if } y \geq \sqrt{p_1 p_2}, \end{cases} \tag{47}$$

$$\boldsymbol{x}(\boldsymbol{p}, T, a, y) = \left(\min\left\{ \sqrt{\frac{p_2}{p_1}}, \frac{y - T(y)}{p_1} \right\}, \max\left\{ 0, \frac{y - T(y) - \sqrt{p_1 p_2}}{p_2} \right\} \right),$$

$$y^u(\boldsymbol{p}, b, 0, a) + b = \begin{cases} b & \text{if } b \geq \sqrt{p_1 p_2},\ a < p_2, \\ a\left(1 - \sqrt{\frac{p_2}{a}}\right) + b & \text{if } b \geq \sqrt{p_1 p_2},\ a \geq p_2, \\ b & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b < \sqrt{p_1 p_2}, \\ & b > \sqrt{a p_1}, \\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b < \sqrt{p_1 p_2}, \\ & b \leq \sqrt{a p_1}, \\ \sqrt{p_1 p_2} & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b \geq \sqrt{p_1 p_2}, \\ & b \leq \sqrt{a p_1},\ a < p_2, \\ \sqrt{p_1 p_2} & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b \geq \sqrt{p_1 p_2}, \\ & b \leq \sqrt{a p_1},\ a \geq p_2, \\ & a\left(1 - \sqrt{\frac{p_2}{a}}\right) + b < \sqrt{p_1 p_2}, \\ a\left(1 - \sqrt{\frac{p_2}{a}}\right) + b & \text{if } b < \sqrt{p_1 p_2},\ \frac{a(\sqrt{a p_1} - b)}{a + \sqrt{a p_1}} + b \geq \sqrt{p_1 p_2}, \\ & b \leq \sqrt{a p_1},\ a \geq p_2, \\ & a\left(1 - \sqrt{\frac{p_2}{a}}\right) + b \geq \sqrt{p_1 p_2}, \end{cases} \tag{48}$$

and

$$\zeta^u(\boldsymbol{p},b,0,a)=\begin{cases}0 & \text{if } b\geq\sqrt{p_1p_2},\ a<p_2,\\ \frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}+b} & \text{if } b\geq\sqrt{p_1p_2},\ a\geq p_2,\\ 0 & \text{if } b<\sqrt{p_1p_2},\ \frac{a(\sqrt{ap_1}-b)}{a+\sqrt{ap_1}}+b<\sqrt{p_1p_2},\\ & b>\sqrt{ap_1},\\ \frac{a(\sqrt{ap_1}-b+\frac{a+b}{2})}{(a+\sqrt{ap_1})(a+b)} & \text{if } b<\sqrt{p_1p_2},\ \frac{a(\sqrt{ap_1}-b)}{a+\sqrt{ap_1}}+b<\sqrt{p_1p_2},\\ & b\leq\sqrt{ap_1},\\ 0 & \text{if } b<\sqrt{p_1p_2},\ \frac{a(\sqrt{ap_1}-b)}{a+\sqrt{ap_1}}+b\geq\sqrt{p_1p_2},\\ & b\leq\sqrt{ap_1},\ a<p_2,\\ 0 & \text{if } b<\sqrt{p_1p_2},\ \frac{a(\sqrt{ap_1}-b)}{a+\sqrt{ap_1}}+b\geq\sqrt{p_1p_2},\\ & b\leq\sqrt{ap_1},\ a\geq p_2,\\ & a\left(1-\sqrt{\frac{p_2}{a}}\right)+b<\sqrt{p_1p_2},\\ \frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}+b} & \text{if } b<\sqrt{p_1p_2},\ \frac{a(\sqrt{ap_1}-b)}{a+\sqrt{ap_1}}+b\geq\sqrt{p_1p_2},\\ & b\leq\sqrt{ap_1},\ a\geq p_2,\\ & a\left(1-\sqrt{\frac{p_2}{a}}\right)+b\geq\sqrt{p_1p_2}.\end{cases}\tag{49}$$

It is straightforward to verify that u satisfies conditions (i)-(vii).[20]
Since

$$\frac{\partial x_1^u(\boldsymbol{p},y)}{\partial y}=\begin{cases}\frac{1}{p_1} & \text{if } y<\sqrt{p_1p_2},\\ 0 & \text{if } y\geq\sqrt{p_1p_2},\end{cases}\quad\text{and}\quad\frac{x_1^u(\boldsymbol{p},y)}{y}=\begin{cases}\frac{1}{p_1} & \text{if } y<\sqrt{p_1p_2},\\ \sqrt{\frac{1}{y}\frac{p_2}{p_1}} & \text{if } y\geq\sqrt{p_1p_2},\end{cases}$$

and

$$\frac{\partial x_2^u(\boldsymbol{p},y)}{\partial y}=\begin{cases}0 & \text{if } y<\sqrt{p_1p_2},\\ \frac{1}{p_2} & \text{if } y\geq\sqrt{p_1p_2},\end{cases}\quad\text{and}\quad\frac{x_2^u(\boldsymbol{p},y)}{y}=\begin{cases}0 & \text{if } y<\sqrt{p_1p_2},\\ \frac{1}{p_2}-\frac{1}{y}\sqrt{\frac{p_1}{p_2}} & \text{if } y\geq\sqrt{p_1p_2},\end{cases}$$

good 1 is a necessity and good 2 is a luxury.

In the special case when $\mathcal{S}'=\{\boldsymbol{\tau}=0\}$ (no commodity taxes), Theorem 5 (together with (17)) implies that the following is a necessary condition for a subset $\mathcal{S}\subseteq\mathcal{T}_{m\text{-}prog}$ to be inequality reducing with respect to $\boldsymbol{p}$ and u:

$$\zeta^u(\boldsymbol{p},b,0,(1-t)a)\leq\zeta^u(\boldsymbol{p},0,0,a),\quad\text{for each } a>0 \text{ and } (b,t)\in B(\mathcal{L}_{\mathcal{S}})\times R(\mathcal{L}_{\mathcal{S}}).$$

We will show that, for any $(b,t)\in B(\mathcal{L}_{\mathcal{S}})\times R(\mathcal{L}_{\mathcal{S}})$ with $t>0$, there exists $a>0$ such that

$$\zeta^u(\boldsymbol{p},b,0,(1-t)a)>\zeta^u(\boldsymbol{p},0,0,a),\tag{50}$$

[20]Condition (vi) holds because no good is inferior (see Footnote 10). Condition (vii) holds because $y^u(\boldsymbol{p},b,0,a)+b$ is nondecreasing in a.

implying that no income tax schedule other than a pure subsidy, $T(y) = -b$, is inequality reducing.

Choose $(b, t) \in B(\mathscr{L}_S) \times R(\mathscr{L}_S)$. Using (49), we see that, for any sufficiently large a, (50) can be written as

$$\frac{(1-t)a - \sqrt{(1-t)ap_2} + \frac{1}{2}\sqrt{\frac{(1-t)a}{p_2}}}{(1-t)a - \sqrt{(1-t)ap_2} + b} > \frac{a - \sqrt{ap_2} + \frac{1}{2}\sqrt{\frac{a}{p_2}}}{a - \sqrt{ap_2}}.$$

Arranging terms yields

$$\frac{a}{2}\sqrt{\frac{a}{p_2}}(\sqrt{1-t} - (1-t)) > b(a - \sqrt{ap_2}) + \frac{b}{2}\sqrt{\frac{a}{p_2}}. \tag{51}$$

Since $\sqrt{1-t} - (1-t) > 0$, we see that there is a large enough a such that (51) holds.[21]

Thus, for the utility function in (46), there is no inequality reducing income tax schedule other than a pure subsidy. We now show that there are *mixed* tax systems that are inequality reducing.

Consider a mixed tax system $(T, \boldsymbol{\tau})$ such that $T(y) = -b$, for $b \geq 0$, and $\boldsymbol{\tau} = (\tau_1, \tau_2) = (0, \tau_2)$, where $\tau_2 > 0$, i.e., the commodity tax system taxes the luxury good.

Using (17), (18), and Theorem 5, we see that $(T, \boldsymbol{\tau})$ is inequality reducing with respect to $\boldsymbol{p}$ and u if and only if the following two conditions are satisfied:

- the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a for $T = 0$ and $T = -b$; and
- the inequality

$$\zeta^u(\boldsymbol{p}+\boldsymbol{\tau}, b, 0, a) \cdot \frac{1 - \tau_2 \cdot \frac{\partial x_2^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}+\boldsymbol{\tau}, b, 0, a)+b)}{\partial y}}{1 - \tau_2 \cdot \frac{x_2^u(\boldsymbol{p}+\boldsymbol{\tau}, y^u(\boldsymbol{p}+\boldsymbol{\tau}, b, 0, a)+b)}{y^u(\boldsymbol{p}+\boldsymbol{\tau}, b, 0, a)+b}} \leq \zeta^u(\boldsymbol{p}, 0, 0, a) \tag{53}$$

  holds for each $a > 0$.

We will show that, given $\boldsymbol{p}$ and $\tau_2 > 0$, there exists $\underline{b} \geq 0$ such that these two conditions are satisfied if $b \geq \underline{b}$, implying that any mixed tax system $(T, \boldsymbol{\tau})$ such that $T(y) = -b$, for $b \geq \underline{b}$, and $\boldsymbol{\tau} = (\tau_1, \tau_2) = (0, \tau_2)$, is inequality reducing with respect to $\boldsymbol{p}$ and u.

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[21]The inequality

$$\frac{(\alpha a^{3/2})/(2\sqrt{p_2})}{ba + \frac{ba^{1/2}}{2\sqrt{p_2}}} > 1, \tag{52}$$

where $\alpha = \sqrt{1-t} - (1-t)$, is sufficient for (51) to hold. Since

$$\lim_{a\to\infty} \frac{(\alpha a^{3/2})/(2\sqrt{p_2})}{ba + \frac{ba^{1/2}}{2\sqrt{p_2}}} = \lim_{a\to\infty} \frac{(\frac{3\alpha}{2}a^{1/2})/(2\sqrt{p_2})}{b + \frac{(b/2)a^{-1/2}}{2\sqrt{p_2}}} = \infty$$

(by l'Hôpital's rule), it follows that (52) holds for large enough a.

To see that the disposable income function $z^u(\boldsymbol{p}, T, \boldsymbol{\tau}, a)$ is nondecreasing in a for $T = 0$ and $T = -b$, note first that

$$\begin{aligned} z^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a) &= y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a) + b - \tau_2 x_2^u(\boldsymbol{p} + \boldsymbol{\tau}, y^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a) + b) \\ &= y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b - \tau_2 x_2^u(\boldsymbol{p} + \boldsymbol{\tau}, y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, a) + b). \end{aligned}$$

It suffices to show that $z^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a)$ is nondecreasing in a for any $b \geq 0$. Note that, by (47),

$$\begin{aligned} &z^u(\boldsymbol{p}, b, \boldsymbol{\tau}, a) \\ &= \begin{cases} y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b & \text{if } y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b < \sqrt{p_1(p_2 + \tau_2)}, \\ (y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b)(1 - \frac{\tau_2}{p_2 + \tau_2}) & \\ \qquad\qquad\qquad + \tau_2 \sqrt{\frac{p_1}{p_2 + \tau_2}} & \text{if } y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b \geq \sqrt{p_1(p_2 + \tau_2)}. \end{cases} \end{aligned}$$

Consequently, it suffices to show that $y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b$ is nondecreasing in a for any $b \geq 0$. This can be verified using (48).[22]

The equation (53) can be expressed as

$$\begin{cases} \zeta^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) \leq \zeta^u(\boldsymbol{p}, 0, 0, a) & \text{if } y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b \leq \sqrt{p_1(p_2 + \tau_2)}, \\ \zeta^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) \cdot \dfrac{1 - \frac{\tau_2}{p_2 + \tau_2}}{1 - \frac{\tau_2}{p_2 + \tau_2}\left(1 - \frac{\sqrt{p_1(p_2 + \tau_2)}}{y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b}\right)} & \\ \qquad\qquad\qquad \leq \zeta^u(\boldsymbol{p}, 0, 0, a) & \text{if } y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b > \sqrt{p_1(p_2 + \tau_2)}. \end{cases}$$

Since

$$\frac{1 - \frac{\tau_2}{p_2 + \tau_2}}{1 - \frac{\tau_2}{p_2 + \tau_2}\left(1 - \frac{\sqrt{p_1(p_2 + \tau_2)}}{y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b}\right)} \leq 1$$

whenever $y^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) + b > \sqrt{p_1(p_2 + \tau_2)}$, it suffices to show that there exists $\underline{b} \geq 0$ such that

$$\zeta^u(\boldsymbol{p} + \boldsymbol{\tau}, b, 0, a) \leq \zeta^u(\boldsymbol{p}, 0, 0, a), \quad \text{for } a > 0 \text{ and } b \geq \underline{b}. \tag{54}$$

Let

$$\underline{b} = \max\left\{\frac{p_2}{1 + 2p_2}, a^* + \frac{\sqrt{a^*}}{\sqrt{p_2 + \tau_2}} + \sqrt{\frac{p_1}{p_2 + \tau_2}}, \sqrt{p_1(p_2 + \tau_2)}\right\}, \tag{55}$$

[22]Note that both $a(1 - \sqrt{p_2/a}) + b$ and $\frac{a(\sqrt{ap_1} - b)}{a + \sqrt{ap_1}} + b$ are nondecreasing in a. To see that $\frac{a(\sqrt{ap_1} - b)}{a + \sqrt{ap_1}} + b$ is nondecreasing in a, note that

$$\partial\left(\frac{a(\sqrt{ap_1} - b)}{a + \sqrt{ap_1}}\right) \Big/ \partial a = \frac{(\sqrt{ap_1} - b + \frac{1}{2}\sqrt{ap_1})(a + \sqrt{ap_1}) - a(\sqrt{ap_1} - b)\left(1 + \frac{p_1}{2\sqrt{ap_1}}\right)}{(a + \sqrt{ap_1})^2}.$$

This expression is nonnegative if and only if

$$\frac{1}{2}\sqrt{ap_1}(\sqrt{ap_1} - b + a + \sqrt{ap_1}) \geq 0.$$

The last inequality holds if $\sqrt{ap_1} - b \geq 0$.

where $a^* > 0$ is implicitly defined by the following equation:

$$a^* = \sqrt{a^* p_2} + \sqrt{p_1 p_2}. \tag{56}$$

Note that

$$\underline{b} \geq \frac{p_2}{1+2p_2} \geq \frac{2\sqrt{ap_2} - a}{1+2p_2}, \quad \text{for all } a > 0. \tag{57}$$

This inequality will be used later.

Using (49), $\zeta^u(\boldsymbol{p}, 0, 0, a)$ can be expressed as follows:

$$\zeta^u(\boldsymbol{p}, 0, 0, a) = \begin{cases} \frac{\sqrt{ap_1}+\frac{a}{2}}{a+\sqrt{ap_1}} & \text{if } a < \sqrt{ap_2} + \sqrt{p_1 p_2}, \\ 0 & \text{if } a \geq \sqrt{ap_2} + \sqrt{p_1 p_2},\, a < p_2, \\ \frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}} & \text{if } a \geq \sqrt{ap_2} + \sqrt{p_1 p_2},\, a \geq p_2. \end{cases} \tag{58}$$

Using (49), (58), and the inequalities $b \geq \underline{b} \geq \sqrt{p_1(p_2+\tau_2)}$ (from (55)), $\zeta^u(\boldsymbol{p}+\boldsymbol{\tau}, b, 0, a) - \zeta^u(\boldsymbol{p}, 0, 0, a)$ can be expressed as follows:

$$\zeta^u(\boldsymbol{p}+\boldsymbol{\tau}, b, 0, a) - \zeta^u(\boldsymbol{p}, 0, 0, a) = \begin{cases} -\frac{\sqrt{ap_1}+\frac{a}{2}}{a+\sqrt{ap_1}} & \text{if } a < p_2+\tau_2, \\ & a < \sqrt{ap_2} + \sqrt{p_1 p_2}, \\ \frac{a-\sqrt{a(p_2+\tau_2)}+\frac{1}{2}\sqrt{\frac{a}{p_2+\tau_2}}}{a-\sqrt{a(p_2+\tau_2)}+b} - \frac{\sqrt{ap_1}+\frac{a}{2}}{a+\sqrt{ap_1}} & \text{if } a \geq p_2+\tau_2, \\ & a < \sqrt{ap_2} + \sqrt{p_1 p_2}, \\ 0 & \text{if } a < p_2, \\ & a \geq \sqrt{ap_2} + \sqrt{p_1 p_2}, \\ 0 - \frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}} & \text{if } a < p_2+\tau_2, a \geq p_2, \\ & a \geq \sqrt{ap_2} + \sqrt{p_1 p_2}, \\ \frac{a-\sqrt{a(p_2+\tau_2)}+\frac{1}{2}\sqrt{\frac{a}{p_2+\tau_2}}}{a-\sqrt{a(p_2+\tau_2)}+b} & \\ \qquad - \frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}} & \text{if } a \geq p_2+\tau_2, \\ & a \geq \sqrt{ap_2} + \sqrt{p_1 p_2}, \end{cases}$$

To establish (54), consider the following cases for the expression

$$\zeta^u(\boldsymbol{p}+\boldsymbol{\tau}, b, 0, a) - \zeta^u(\boldsymbol{p}, 0, 0, a)$$

given above.

In the second case, we have

$$\frac{a - \sqrt{a(p_2+\tau_2)} + \frac{1}{2}\sqrt{\frac{a}{p_2+\tau_2}}}{a - \sqrt{a(p_2+\tau_2)} + b} \leq \frac{\sqrt{ap_1} + \frac{a}{2}}{a + \sqrt{ap_1}} \tag{59}$$

if and only if

$$a+\sqrt{a}\left(\frac{1}{\sqrt{p_2+\tau_2}}-\sqrt{p_2+\tau_2}\right)+\sqrt{\frac{p_1}{p_2+\tau_2}}\leq\frac{2b\sqrt{p_1}}{\sqrt{a}}+b.$$

A sufficient condition for this inequality to hold is

$$a+\frac{\sqrt{a}}{\sqrt{p_2+\tau_2}}+\sqrt{\frac{p_1}{p_2+\tau_2}}\leq b,$$

which holds for every $0<a<\sqrt{ap_2}+\sqrt{p_1p_2}$ if

$$a^*+\frac{\sqrt{a^*}}{\sqrt{p_2+\tau_2}}+\sqrt{\frac{p_1}{p_2+\tau_2}}\leq b,$$

where, recall, a^* is defined by the equation (56). Since this inequality is true (see (55)), it follows that (59) holds, and so, (54) holds.

In the fifth case, we have

$$\frac{a-\sqrt{a(p_2+\tau_2)}+\frac{1}{2}\sqrt{\frac{a}{p_2+\tau_2}}}{a-\sqrt{a(p_2+\tau_2)}+b}\leq\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}} \tag{60}$$

for $b\geq\underline{b}$. To see this, note first that

$$\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}+b}\leq\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}},$$

since $b\geq 0$. Moreover, because

$$\frac{\partial}{\partial p_2}\left(\frac{a-\sqrt{ap_2}+\frac{1}{2}\sqrt{\frac{a}{p_2}}}{a-\sqrt{ap_2}+b}\right)\leq 0$$

is equivalent to

$$b\geq\frac{2\sqrt{ap_2}-a}{1+2p_2},$$

and since this inequality holds for $b\geq\underline{b}$ by virtue of (57), it follows that (60) holds for $b\geq\underline{b}$. Thus, in the fifth case, (54) holds.

We conclude that, given $\boldsymbol{p}$ and $\tau_2>0$, there exists $\underline{b}\geq 0$ such that any mixed tax system $(T,\boldsymbol{\tau})$ such that $T(y)=-b$, for $b\geq\underline{b}$, and $\boldsymbol{\tau}=(\tau_1,\tau_2)=(0,\tau_2)$, is inequality reducing with respect to $\boldsymbol{p}$ and u. By contrast, no income tax schedule (other than a pure subsidy) is inequality reducing.

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