Poverty-Reducing Income Taxation*

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Abstract

This paper develops a theoretical framework for evaluating poverty-reducing tax policies in endowment economies. Using a comprehensive class of poverty measures, we characterize the complete set of tax schedules that reduce both absolute and relative poverty, regardless of the underlying distribution of income or wealth. Our main theorems establish necessary and sufficient conditions for universal poverty reduction through taxation. For relative poverty reduction, we show that tax schedules must preserve ranks and exhibit average-rate progressivity among the poor, while maintaining the non-poor status of all individuals. Absolute poverty reduction requires these conditions plus the additional requirement that tax liabilities be non-decreasing among the poor. These results provide insights into effective tax policy design for poverty alleviation.

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1. Introduction

The conditions under which nonlinear taxation reduces inequality are well established in the literature. A seminal series of papers by Jakobsson (1976), Fellman (1976), and Kakwani (1977) provided a complete characterization of tax schedules that universally reduce inequality, as measured by the relative Lorenz pre-order, in economies with exogenously given incomes. These works established that a tax schedule preserves rank and reduces inequality if and only if it exhibits average-rate progressivity—that is, the average tax rate increases weakly with pre-tax income or wealth. Later, Eichhorn et al. (1984) extended this result by demonstrating that preservation of income ranks is necessary for inequality-reducing tax schedules.

These two conditions have intuitive interpretations. Average-rate progressivity requires that as income rises, the proportion of income paid in taxes does not decrease—ensuring that richer individuals contribute at least as much proportionally as poorer ones, thereby compressing relative income gaps. Income rank preservation ensures that taxation does not reorder individuals' positions in the income hierarchy; if person A earns less than person

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B before taxes, then A must still earn less than B after taxes, preventing "leapfrogging" where initially poorer individuals end up richer after taxation.

Moyes (1988) provides a comprehensive characterization of tax schedules that unambiguously reduce inequality under the absolute Lorenz criterion. This characterization establishes that a tax schedule is inequality-reducing if and only if it exhibits two properties: monotonically non-decreasing tax liabilities and preservation of income ranks.[1]

The first condition requires that tax payments do not fall as income rises—higher earners pay at least as much in absolute dollars as lower earners, preventing the tax system from imposing heavier burdens on the poor. The second condition, rank preservation, ensures that the relative ordering of individuals by income remains unchanged after taxation. These conditions guarantee that absolute income gaps do not widen after taxes.

While the properties of inequality-reducing income tax systems are well-understood, the literature has not thoroughly explored how income taxation specifically affects poverty reduction. This paper addresses this gap by examining the relationship between taxation and universal poverty alleviation. Specifically, we seek to characterize tax schedules that effectively reduce poverty regardless of the pre-tax distribution of income or wealth.

Our analysis builds on a general class of relative and absolute poverty metrics encompassing most comparative measurement methods. These criteria are less restrictive than the standard approach based on Shorrocks' generalized Lorenz dominance concept (Shorrocks, 1983). As a result, any tax schedule deemed poverty-reducing in our framework will also reduce poverty according to any refinement of our measures. The broad applicability of our approach is further elaborated in Section 2.5.

We represent our poverty measures as pre-orders defined on income distributions. These pre-orders impose minimal restrictions because they avoid forcing trade-offs between income growth among the poor and distributional equity. Our approach adapts two measures that Shorrocks (1983) originally developed for income distribution analysis. Unlike generalized Lorenz dominance, which treats all income increases as welfare-enhancing, these measures consider income increases welfare-improving only if they do not worsen inequality in relative or absolute terms.

We adapt these measures to poverty analysis by applying them to income distributions "censored" at the poverty line—essentially focusing only on incomes of those below the poverty threshold. We consider two distinct poverty criteria.

Relative poverty measures focus on proportional disparities among the poor. For example, they would reject a policy where some poor individuals' incomes double while others gain only 10%, even though everyone becomes better off, because this widens relative inequality among the poor.

Absolute poverty measures emphasize fixed-dollar gaps between the poor. They would reject policies that widen absolute income differences among the poor, even if all poor individuals experience income gains. Both measures ensure that poverty reduction efforts focus exclusively on the poor while requiring that income growth not come at the expense of distributional equity.

[1] The literature on taxation and inequality branches out in several directions (see, e.g., Hemming and Keen, 1983; Liu, 1985; Formby et al., 1986; Thon, 1987; Latham, 1988; Thistle, 1988; Moyes, 1989, 1994; Le Breton et al., 1996; Ebert and Moyes, 2000; Ju and Moreno-Ternero, 2008; Carbonell-Nicolau and Llavador, 2018, 2021a,b, 2025; Carbonell-Nicolau, 2019, 2024, 2025a).

In Appendix A, we fully characterize the relative and absolute poverty measures used in the analysis through fundamental, well-accepted poverty axioms, demonstrating that these axioms are compatible with any weak extension of our core pre-orders.

We then define our central concept: *poverty-reducing* tax schedules. A tax schedule is poverty-reducing if, for any pre-tax income distribution, the resulting post-tax distribution exhibits no more poverty than the original distribution, according to our core poverty pre-orders. This definition is universal in scope, requiring poverty alleviation *regardless* of the initial distributional conditions. Such robustness enhances both the practical relevance and policymaking flexibility of our framework.[2]

We define tax schedules broadly, requiring only continuity and tax liabilities not exceeding pre-tax income. We deliberately avoid imposing additional properties like income rank preservation or monotonicity, as variants of these properties naturally arise in our characterization of poverty-reducing tax schedules.

Our main results (Theorem 1 and Theorem 2) provide complete characterizations of poverty-reducing tax schedules. For relative poverty reduction, three fundamental properties emerge:

1. Income rank preservation among poor individuals whose post-tax income remains below the poverty line.

2. Average-rate progressivity over this same restricted income domain.

3. Preservation of non-poor status, ensuring the tax code never pushes non-poor individuals into poverty.

For absolute poverty reduction, rank preservation and non-poor status preservation remain essential, but average-rate progressivity is replaced by monotonicity of tax liabilities among poor individuals.

These properties can be understood as safeguards. Rank preservation among the poor ensures that taxation does not reshuffle their relative positions. Restricted average-rate progressivity guarantees that poorer individuals face proportionally lighter tax burdens. Non-poor status preservation prevents "poverty traps" by ensuring that taxation does not push borderline non-poor individuals below the poverty line. Finally, for absolute poverty reduction, the additional requirement of monotonicity stipulates that absolute taxes among the poor must not decrease with income, thereby preventing the widening of poverty gaps in absolute dollar terms.

Through a series of examples, we demonstrate that these properties are independent of each other—the failure of any single condition renders the main results invalid.

While the characterizations differ for relative and absolute poverty, we show that the properties required for absolute poverty reduction have stronger implications. Specifically, restricted monotonicity and preservation of non-poor status imply both weak subsidization of the poor and restricted average-rate progressivity. Additionally, subsidization of the poor also emerges due to the weaker conditions for relative poverty reduction.

The relationship between income taxation and poverty has been examined by other authors, including Bourguignon and Fields (1990) and Kanbur et al. (1994). Bourguignon and Fields (1990) focused on identifying budget-balanced tax policies that minimize poverty,

[2]Section 3.3 provides a detailed discussion of the implications and scope of this "strong" criterion (see Remark 4).

using specific poverty indices to create complete rankings of income distributions. Their research demonstrated that poverty-minimizing tax schedule characteristics fundamentally depend on the selected poverty index.

In contrast to Bourguignon and Fields' approach, our analysis provides a characterization that applies to all poverty measures encompassed by our core poverty pre-orders. Our framework differs in three fundamental aspects: first, we consider both budget-balanced and non-balanced tax schedules aimed at alleviating poverty, rather than focusing solely on minimization; second, we require poverty reduction to be universal, independent of the initial distribution, whereas Bourguignon and Fields maintained a fixed initial distribution throughout their analysis; third, they assume perfect targeting of poverty alleviation measures, allowing for personalized taxes and transfers among the poor, whereas our tax schedules are solely based on pre-tax incomes.

In their study, Kanbur et al. (1994) examined poverty reduction within a Mirrleesian framework that incorporates endogenous income, partially characterizing tax schedules designed to alleviate poverty.[3] This work utilized the additively separable poverty indices introduced by Atkinson (1987) as a measurement tool. While our current framework assumes exogenous incomes, we provide a comprehensive characterization of tax schedules that effectively reduce poverty across a broader spectrum of poverty measures.[4]

Another related paper by Chambers and Moreno-Ternero (2017) offers a different perspective by axiomatizing poverty-sensitive taxation methods through the lens of Young's equal sacrifice principle (Young, 1987, 1988, 1990). Their approach refines Young's original framework by relaxing conditions that previously precluded tax policies from exempting low-income individuals.

This paper is organized as follows. Section 2 establishes the foundational framework for poverty measurement and formulates the measures used in our analysis. The theoretical underpinnings of these measures, including their complete axiomatization, are developed in Appendix A. Section 3 presents our central contribution—a comprehensive characterization of poverty-reducing tax schedules—accompanied by illustrative examples and an in-depth discussion of their key properties. Finally, we conclude in Section 4 by outlining directions for future research.

2. Measuring poverty

In this section, we formulate the poverty criteria adopted in this paper. We begin by introducing essential terminology and definitions.

2.1. Basic definitions and notation

An *income distribution* is represented by an n-dimensional vector

$$\boldsymbol{x} = (x_1, \ldots, x_n),$$

[3]Kanbur et al. (1994) extended the analysis presented in Kanbur and Keen (1989) to non-linear income taxation. Pirttilä and Tuomala (2004) further enriched the model by introducing both commodity taxation and public good provision, contrasting their findings to the optimal taxation results for mixed tax structures established by Atkinson and Stiglitz (1976).

[4]Future research directions, incorporating behavioral responses to income taxation, are discussed in Section 4.

where each coordinate is a nonnegative real number and $\sum_i x_i > 0$.[5] Here, $n \in \mathbb{N}$ denotes the population size, and x_i represents the income of individual i.

The mean of an income distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ is denoted by $\mu_{\boldsymbol{x}}$:

$$\mu_{\boldsymbol{x}} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Without loss of generality, we can assume that the coordinates in any given income distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ are arranged in nondecreasing order:

$$x_1 \leq \cdots \leq x_n.$$

This assumption is justified because the poverty measures considered in this analysis are invariant to any reshuffling of incomes across individuals. In other words, the identities of the individuals do not play any role in the poverty assessment, as long as the overall income distribution remains unchanged.

Given an income distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ and a poverty line $z > 0$, we define the *poor* as the set of individuals i whose income x_i is less than or equal to z.

2.2. Censored income distributions

To evaluate poverty in a way that isolates the circumstances of the poor, we transform an income distribution by censoring it at the poverty line. Specifically, we replace every income above the threshold with the poverty line itself, so that comparisons disregard variations among the non-poor and focus solely on the distribution of resources below the threshold. The resulting censored distribution ensures that poverty measures capture only changes affecting those deemed poor.

The distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ *censored at* z, denoted as

$$\boldsymbol{x}^z = (x_1^z, \ldots, x_n^z),$$

is defined by

$$x_i^z = \begin{cases} x_i & \text{if } x_i \leq z, \\ z & \text{if } x_i > z, \end{cases}$$

for each $i \in \{1, \ldots, n\}$.

[5]The analysis remains valid if $\boldsymbol{x}$ is interpreted as a wealth distribution instead.

2.3. Relative and absolute poverty pre-orders

The *relative Lorenz pre-order*, denoted by $\succcurlyeq_{RL}$, is a binary relation on the set of income distributions. It is formally expressed as:

$$\boldsymbol{x} \succcurlyeq_{RL} \boldsymbol{y} \Leftrightarrow \begin{cases} \frac{x_1}{\sum_i x_i} \geq \frac{y_1}{\sum_i y_i}, \\ \frac{x_1+x_2}{\sum_i x_i} \geq \frac{y_1+y_2}{\sum_i y_i}, \\ \vdots \\ \frac{\sum_i x_i}{\sum_i x_i} \geq \frac{\sum_i y_i}{\sum_i y_i}. \end{cases}$$

The *absolute Lorenz pre-order* (Moyes, 1987), denoted by $\succcurlyeq_{AL}$, is a binary relation on the set of income distributions defined as follows:

$$\boldsymbol{x} \succcurlyeq_{AL} \boldsymbol{y} \Leftrightarrow \begin{cases} x_1 - \mu_{\boldsymbol{x}} \geq y_1 - \mu_{\boldsymbol{y}}, \\ x_1 - \mu_{\boldsymbol{x}} + x_2 - \mu_{\boldsymbol{x}} \geq y_1 - \mu_{\boldsymbol{y}} + y_2 - \mu_{\boldsymbol{y}}, \\ \vdots \\ \sum_{i=1}^{n}(x_i - \mu_{\boldsymbol{x}}) \geq \sum_{i=1}^{n}(y_i - \mu_{\boldsymbol{y}}). \end{cases}$$

Shorrocks (1983) introduced the following pre-orders on the set of income distributions:

$$\boldsymbol{x} \succcurlyeq_{RS} \boldsymbol{y} \Leftrightarrow \left[\boldsymbol{x} \succcurlyeq_{RL} \boldsymbol{y} \text{ and } \sum_i x_i \geq \sum_i y_i\right] \text{ and } \boldsymbol{x} \succcurlyeq_{AS} \boldsymbol{y} \Leftrightarrow \left[\boldsymbol{x} \succcurlyeq_{AL} \boldsymbol{y} \text{ and } \sum_i x_i \geq \sum_i y_i\right].$$

In words, a distribution $\boldsymbol{x}$ $\succcurlyeq_{RS}$-dominates $\boldsymbol{y}$ if $\boldsymbol{x}$ is at least as equal as $\boldsymbol{y}$, according to the relative Lorenz pre-order, and has higher total income than $\boldsymbol{y}$. This concept captures the idea that proportional increases in individual incomes, which maintain the relative degree of inequality, generate a welfare-superior income distribution.

The relation $\succcurlyeq_{RS}$ diverges from the concept of generalized Lorenz dominance from Shorrocks (1983). The distinction lies in how the two approaches evaluate "welfare" improvements. While generalized Lorenz dominance allows for trade-offs between rising incomes and increased inequality, $\succcurlyeq_{RS}$ is uncompromising: it requires that growth also preserve or enhance distributional equity. By requiring that economic growth maintain or enhance distributional equity, $\succcurlyeq_{RS}$ offers a more stringent standard for assessing improvements in a society's income distribution.

Similarly, an income distribution $\boldsymbol{x}$ $\succcurlyeq_{AS}$-dominates another distribution $\boldsymbol{y}$ if it satisfies analogous conditions: $\boldsymbol{x}$ is at least as equal as $\boldsymbol{y}$ according to the absolute Lorenz criterion, and $\boldsymbol{x}$ has a higher total income than $\boldsymbol{y}$. In this context, "welfare" improvements are associated with progressive transfers between individuals or translational (additive) increases in individual incomes that preserve the absolute degree of inequality.

For a fixed poverty line $z > 0$, we establish a ranking system for income distributions using the following poverty pre-orders, $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$:

$$\boldsymbol{x} \succcurlyeq_R^z \boldsymbol{y} \Leftrightarrow \boldsymbol{y}^z \succcurlyeq_{RS} \boldsymbol{x}^z \quad \text{and} \quad \boldsymbol{x} \succcurlyeq_A^z \boldsymbol{y} \Leftrightarrow \boldsymbol{y}^z \succcurlyeq_{AS} \boldsymbol{x}^z,$$

where $\boldsymbol{x}^z$ and $\boldsymbol{y}^z$ represent the censored distributions of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, at the poverty line z.

The relation "$\boldsymbol{x} \succcurlyeq_R^z \boldsymbol{y}$" should be interpreted as follows: "The income distribution $\boldsymbol{x}$ exhibits no less poverty than the income distribution $\boldsymbol{y}$, relative to the poverty line z and according to the poverty measure $\succcurlyeq_R^z$." An analogous interpretation applies to the poverty measure $\succcurlyeq_A^z$.

2.4. Comparison: $\succcurlyeq_R^z$, $\succcurlyeq_A^z$, and generalized Lorenz dominance

To illustrate the practical differences between the interpretations of $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$, consider a simple example.

Suppose a society with two individuals, both below the poverty line $z = 10$: one earning \$2 and the other \$4 (denote this distribution by $\boldsymbol{x} = (2, 4)$). Now imagine their incomes double proportionally to \$4 and \$8, respectively, while both remain below the poverty line (denote this distribution by $\boldsymbol{y} = (4, 8)$).

Under $\succcurlyeq_R^z$ (relative poverty), $\boldsymbol{y}$ clearly exhibits less poverty than $\boldsymbol{x}$: the relative gap is unchanged (the higher earner still has twice as much), while mean income among the poor rises, yielding an unambiguously preferable distribution.

Under $\succcurlyeq_A^z$ (absolute poverty), however, $\boldsymbol{x}$ and $\boldsymbol{y}$ are incomparable: total income increases but the absolute gap widens (from \$2 to \$4). Because the rise in aggregate income occurs alongside a larger absolute gap, $\succcurlyeq_A^z$ cannot rank the two distributions. In such a trade-off, the absolute pre-order does not resolve the conflict—it treats the two censored distributions as incomparable.

This example highlights that $\succcurlyeq_R^z$ favors proportional growth that preserves relative equality, whereas $\succcurlyeq_A^z$ requires non-increasing absolute differences for an unambiguous improvement. These distinctions provide helpful context for the tax-policy characterizations that follow.

For completeness, we note how the generalized Lorenz criterion ranks this pair. Because $\boldsymbol{y} = 2\boldsymbol{x}$ the (relative) Lorenz curve of $\boldsymbol{y}$ coincides pointwise with that of $\boldsymbol{x}$, while the mean doubles: $\mu_{\boldsymbol{x}} = 3$ and $\mu_{\boldsymbol{y}} = 6$. The generalized Lorenz curve is defined as the mean times the Lorenz curve, so for every cumulative population share p

$$GL_{\boldsymbol{y}}(p) = \mu_{\boldsymbol{y}} L_{\boldsymbol{y}}(p) = 6L_{\boldsymbol{x}}(p) = (2 \cdot 3)L_{\boldsymbol{x}}(p) = 2GL_{\boldsymbol{x}}(p).$$

Hence $GL_{\boldsymbol{y}}(p) \geq GL_{\boldsymbol{x}}(p)$ for all p (strictly so for some p), and $\boldsymbol{y}$ generalized-Lorenz-dominates $\boldsymbol{x}$. Put differently, generalized Lorenz dominance accepts proportional, mean-increasing improvements as unambiguously welfare-improving, so it ranks $\boldsymbol{y}$ above $\boldsymbol{x}$ in this example.

As a second, straightforward contrast, consider three-person distributions $\boldsymbol{a} = (2, 9, 9)$ and $\boldsymbol{b} = (5, 5, 10)$ (all coordinates below the poverty line so that censoring is irrelevant). The two Lorenz curves cross, so the relative criterion $\succcurlyeq_R^z$ leaves $\boldsymbol{a}$ and $\boldsymbol{b}$ incomparable. In contrast, the absolute pre-order $\succcurlyeq_A^z$ compares absolute (additive) deviations from the mean; because $\boldsymbol{b}$ has both a larger total income and a smaller absolute dispersion around its mean in the lower tail, $\boldsymbol{b}$ is preferred under the absolute ordering (so $\boldsymbol{b}$ shows strictly less poverty than $\boldsymbol{a}$ according to $\succcurlyeq_A^z$). The generalized Lorenz criterion—which weights the Lorenz curve by the mean—also ranks $\boldsymbol{b}$ above $\boldsymbol{a}$ in this case: the higher mean of $\boldsymbol{b}$ compensates the crossing of the Lorenz curves and yields $GL_{\boldsymbol{b}}(p) \geq GL_{\boldsymbol{a}}(p)$ for all p. That generalized

Lorenz dominance agrees with the absolute ordering here is not accidental: both $\succcurlyeq_{RS}$- and $\succcurlyeq_{AS}$-dominance imply generalized Lorenz dominance whenever they apply.[6]

2.5. Axiomatic basis and scope

In Appendix A, we provide a characterization of all poverty measures compatible with the pre-orders $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$ by means of foundational axioms of poverty measurement. This axiomatization provides a unified theoretical framework encompassing a wide range of established poverty measures in the literature. Our approach achieves broad applicability because our poverty orders are coarser than generalized Lorenz dominance, which is equivalent to second-order stochastic dominance and is regarded as a fundamental criterion in poverty measurement (see Zheng (2000) for a comprehensive survey).

The relationship between our measures and generalized Lorenz dominance deserves emphasis. Applied to censored distributions, generalized Lorenz dominance generates a finer ordering that coincides with our poverty measures whenever the latter judge two income distributions comparable.

Formally, denoting the generalized Lorenz pre-order by $\succcurlyeq_{GL}$, we have

$$\boldsymbol{y}^z \succcurlyeq_{RS} \boldsymbol{x}^z \Rightarrow \boldsymbol{y}^z \succcurlyeq_{GL} \boldsymbol{x}^z \quad \text{and} \quad \boldsymbol{y}^z \succcurlyeq_{AS} \boldsymbol{x}^z \Rightarrow \boldsymbol{y}^z \succcurlyeq_{GL} \boldsymbol{x}^z$$

for any two income distributions $\boldsymbol{x}$ and $\boldsymbol{y}$ and any poverty line $z > 0$.[7]

This hierarchical relationship between our poverty measures and generalized Lorenz dominance strengthens our poverty-reducing taxation results. Specifically, tax schedules that are poverty-reducing under our definition remain poverty-reducing under any refinement of our poverty measures. This robustness lends greater generality to our analysis.

To contextualize our approach, we examine several poverty measures from the existing literature and analyze when they align with either the generalized Lorenz ordering or our proposed measures. Though not comprehensive, this survey demonstrates the breadth of our poverty measurement framework and shows how using measures that are less restrictive than generalized Lorenz dominance can provide greater analytical flexibility.

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[6]Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ be nonnegative incomes sorted nondecreasingly. Recall $GL_{\boldsymbol{x}}(p) = \mu_{\boldsymbol{x}} L_{\boldsymbol{x}}(p)$ and, for $p = k/n$, $GL_{\boldsymbol{x}}(k/n) = \frac{1}{n}\sum_{i=1}^k x_i$.

If $\boldsymbol{x} \succcurlyeq_{RS} \boldsymbol{y}$ then $\boldsymbol{x} \succcurlyeq_{RL} \boldsymbol{y}$ and $\sum_i x_i \geq \sum_i y_i$, hence $\mu_{\boldsymbol{x}} \geq \mu_{\boldsymbol{y}}$ and $L_{\boldsymbol{x}}(p) \geq L_{\boldsymbol{y}}(p)$ for all p. Therefore, for every p

$$GL_{\boldsymbol{x}}(p) = \mu_{\boldsymbol{x}} L_{\boldsymbol{x}}(p) \geq \mu_{\boldsymbol{x}} L_{\boldsymbol{y}}(p) \geq \mu_{\boldsymbol{y}} L_{\boldsymbol{y}}(p) = GL_{\boldsymbol{y}}(p),$$

so $\boldsymbol{x}$ generalized Lorenz dominates $\boldsymbol{y}$.

If $\boldsymbol{x} \succcurlyeq_{AS} \boldsymbol{y}$ then $\boldsymbol{x} \succcurlyeq_{AL} \boldsymbol{y}$ and $\sum_i x_i \geq \sum_i y_i$. The absolute Lorenz inequalities state

$$\sum_{i=1}^k (x_i - \mu_{\boldsymbol{x}}) \geq \sum_{i=1}^k (y_i - \mu_{\boldsymbol{y}}), \quad k \in \{1, \ldots, n\}.$$

Rearranging gives $\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i + k(\mu_{\boldsymbol{x}} - \mu_{\boldsymbol{y}}) \geq \sum_{i=1}^k y_i$ since $\mu_{\boldsymbol{x}} \geq \mu_{\boldsymbol{y}}$. Dividing by n yields $GL_{\boldsymbol{x}}(k/n) \geq GL_{\boldsymbol{y}}(k/n)$ for every k.

Finally, because each generalized Lorenz curve is a straight (affine) segment on every interval $[k/n, (k+1)/n]$, any interior value $GL_{\boldsymbol{x}}(p)$ and $GL_{\boldsymbol{y}}(p)$ is the same convex combination of the two neighboring knot values for both $\boldsymbol{x}$ and $\boldsymbol{y}$ (the weight $t = np - k \in [0, 1]$ depends only on p and n, not on the data). Hence, the pointwise inequalities at the knots extend by the identical convex combination to every $p \in [k/n, (k+1)/n]$, and therefore to all $p \in [0, 1]$. Thus both $\succcurlyeq_{RS}$- and $\succcurlyeq_{AS}$-dominance imply generalized Lorenz dominance.

[7]See Footnote 6.

2.5.1. The Foster–Greer–Thorbecke (FGT_α) Poverty Measures

The Foster–Greer–Thorbecke (FGT) class is a widely used parametric family of poverty indices introduced in Foster et al. (1984).

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ denote nonnegative individual incomes for a population of size n. Fix a poverty line $z > 0$. Define the *shortfall* (gap) of individual i by

$$g_i := \max\{0, z - x_i\} = (z - x_i)\mathbf{1}_{\{x_i < z\}},$$

where

$$\mathbf{1}_{\{x_i < z\}} := \begin{cases} 1 & \text{if } x_i < z, \\ 0 & \text{if } x_i \geq z. \end{cases}$$

We will also use the censored income notation

$$x_i^z := \min\{x_i, z\},$$

so that $g_i = z - x_i^z$.

For $\alpha \geq 0$ the FGT_α index is

$$\mathrm{FGT}_\alpha(x; z) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{z - x_i}{z} \right)^\alpha \mathbf{1}_{\{x_i < z\}}.$$

Equivalently, using censored incomes,

$$\mathrm{FGT}_\alpha(x; z) = \frac{1}{n z^\alpha} \sum_{i=1}^{n} (z - x_i^z)^\alpha.$$

This form highlights that the index depends only on the vector of censored shortfalls $(z - x_i^z)_{i=1}^n$.

The parameter α controls sensitivity to the *depth* (intensity) and *severity* of poverty:

- $\alpha = 0$ counts the poor (headcount): each poor person contributes 1.
- $\alpha = 1$ weights shortfalls linearly: contributions are proportional to the gap size.
- $\alpha > 1$ gives extra weight to larger shortfalls: the index is increasingly sensitive to the poorest among the poor.

More generally, raising gaps to the power α emphasizes larger gaps when $\alpha > 1$ (convex emphasis) and deemphasizes them when $0 \leq \alpha < 1$.

The FGT family has the following properties:

Additive decomposability. It is additively decomposable by population subgroups: the aggregate index can be expressed as a population-weighted average of subgroup indices, facilitating subgroup and between/within decomposition (see Foster et al. (1984)).

Monotonicity. If a poor person's income falls (holding others fixed), all FGT indices (for $\alpha \geq 0$) worsen (increase).

Sensitivity to transfers among the poor. For $\alpha > 0$, the index becomes responsive to transfers among the poor because such transfers alter the measured poverty level. For $\alpha > 1$, this sensitivity takes an inequality-averse form: regressive transfers that widen disparities among the poor raise the poverty index, while progressive transfers reduce it.

Connection to the generalized Lorenz pre-order. Expressed in censored form, FGT_α is (up to a monotone transform) an additive aggregation of the function

$$u(t) = -(z-t)^\alpha, \qquad t \in [0, z], \tag{1}$$

evaluated at the censored incomes $t = x_i^z$. For $\alpha \geq 1$, the function u is nondecreasing and concave on $[0, z]$, hence generalized Lorenz dominance on the censored-income vectors implies the same FGT_α ranking.[8]

When $0 \leq \alpha < 1$, the power function underlying FGT_α is nondecreasing but convex rather than concave, implying that FGT_α cannot be written as an additive welfare functional belonging to the concave class covered by Theorem 2 of Shorrocks (1983). Consequently, generalized Lorenz dominance of censored distributions does not in general guarantee consistent rankings by FGT_α when $\alpha < 1$.[9]

Since the case $\alpha < 1$ falls outside the concave-additive class characterized by censored generalized Lorenz dominance, a natural question is whether an FGT_α index with $0 \leq \alpha < 1$ might nevertheless serve as a completion (total extension) of either $\succcurlyeq_R^z$ or $\succcurlyeq_A^z$, which are coarser pre-orders. The answer is negative.

To see this, fix $0 < \alpha < 1$ and consider a small regressive transfer among two poor individuals. Working with two poor persons (the same local argument extends by perturbation in larger populations), start from the egalitarian censored pair with equal gaps $g = z - a > 0$ and apply a transfer $d > 0$ from the poorer to the richer poor, producing gaps $g + d$ and $g - d$. Up to the common positive normalizing factor, the FGT_α aggregate

[8]By Theorem 2 in Shorrocks (1983) we have the following equivalence for any two income vectors $\boldsymbol{x}, \boldsymbol{y}$:

$$GL_{\boldsymbol{y}}(p) \geq GL_{\boldsymbol{x}}(p) \text{ for every } p \in [0,1] \Leftrightarrow W(\boldsymbol{y}) \geq W(\boldsymbol{x}) \text{ for every } W \in \mathcal{W}_1,$$

where $\mathcal{W}_1$ denotes the class of nondecreasing S-concave (Schur-concave) welfare functionals considered by Shorrocks.

Since any additive utilitarian welfare functional of the form

$$W_u(\boldsymbol{x}') = \sum_{i=1}^{n} u(x_i')$$

with $u : [0, \infty) \to \mathbb{R}$ nondecreasing and concave belongs to $\mathcal{W}_1$, Theorem 2 implies the following corollary used in the text:

$$GL_{\boldsymbol{y}}(p) \geq GL_{\boldsymbol{x}}(p) \text{ for all } p \Leftrightarrow \sum_{i=1}^{n} u(y_i) \geq \sum_{i=1}^{n} u(x_i) \text{ for every function } u \text{ that is nondecreasing and concave.}$$

[9]In the boundary case $\alpha = 0$, the function $u(t)$ defined in (1) is constant, hence both concave and nondecreasing in the weak sense. However, the resulting welfare functional is constant across distributions and therefore does not reproduce FGT_0. In particular, the headcount ratio cannot be expressed as a member of $\mathcal{W}_1$, so FGT_0 is not implied by censored generalized Lorenz dominance.

of the gaps is proportional to

$$F(d) \;=\; (g+d)^{\alpha} + (g-d)^{\alpha}, \qquad d \in (-g, g).$$

Differentiating gives

$$F'(d) = \alpha\big[(g+d)^{\alpha-1} - (g-d)^{\alpha-1}\big], \qquad F''(d) = \alpha(\alpha-1)\big[(g+d)^{\alpha-2} + (g-d)^{\alpha-2}\big].$$

Since $F'(0) = 0$ and

$$F''(0) = 2\alpha(\alpha-1)g^{\alpha-2} < 0,$$

$d = 0$ is a strict local maximum of F. Hence, any sufficiently small regressive transfer (a primitive worsening move for $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$; see Lemma 3 in Appendix A) strictly *reduces* the FGT$_\alpha$ sum, so FGT$_\alpha$ can report strictly *less* poverty after a move that those pre-orders declare strictly worse.

For the boundary case $\alpha = 0$, the headcount ratio is insensitive to pure redistributions among the poor and therefore likewise fails to reflect these primitive worsening moves. Collecting these observations, no FGT$_\alpha$ with $\alpha \in [0, 1)$ can be a completion (total extension) of $\succcurlyeq_R^z$ or $\succcurlyeq_A^z$.

In intuitive terms, the case $\alpha < 1$ in the FGT family means that smaller gaps are given relatively more weight than larger ones. Intuitively, it "softens" the impact of extreme poverty and focuses more on the number of poor and moderate shortfalls, rather than the severity of the deepest poverty. This characteristic leads to weaker sensitivity to transfers among the poor, causing the measure sometimes to fail to recognize worsening redistributions within the poor group. Consequently, such indices are generally considered less appealing for poverty measurement due to their failure to satisfy important axiomatic properties such as Minimal Transfer (see Appendix A).

2.5.2. Atkinson-style poverty indices

Atkinson-style poverty indices are based on Atkinson's inequality-sensitive welfare framework (Atkinson, 1970), which he later adapted for poverty analysis in Atkinson (1987). The construction uses the idea of an *equally distributed equivalent* (EDE) income: the level of equally shared income that would generate the same social welfare as the observed (possibly unequal) distribution when evaluated with a given inequality-aversion parameter ε. Poverty is then assessed by how far this EDE of the censored incomes falls short of the poverty line z.

Working with censored incomes

$$x_i^z := \min\{x_i, z\}, \qquad i = 1, \ldots, n,$$

define the Atkinson utility function

$$u_\varepsilon(t) = \begin{cases} \dfrac{t^{1-\varepsilon}}{1-\varepsilon}, & \varepsilon \neq 1, \\ \ln t, & \varepsilon = 1, \end{cases}$$

for $t > 0$ and $\varepsilon \geq 0$. The associated welfare of the censored income vector $\boldsymbol{x}^z = (x_1^z, \ldots, x_n^z)$ is

$$W_\varepsilon(\boldsymbol{x}^z) = \frac{1}{n}\sum_{i=1}^{n} u_\varepsilon(x_i^z).$$

Formally, the equally distributed equivalent (EDE) income $\mathrm{EDE}_u(\boldsymbol{x}^z)$ is defined implicitly by

$$u\big(\mathrm{EDE}_u(\boldsymbol{x}^z)\big) = \frac{1}{n}\sum_{i=1}^{n} u(x_i^z),$$

that is, it is the level of equal income which, if received by everyone, yields the same average utility as the actual (possibly unequal) censored distribution.

The EDE corresponding to the welfare level $W_\varepsilon(\boldsymbol{x}^z)$ is

$$\mathrm{EDE}_\varepsilon(\boldsymbol{x}^z) = \begin{cases} \left(\frac{1}{n}\sum_{i=1}^{n}(x_i^z)^{1-\varepsilon}\right)^{1/(1-\varepsilon)}, & \varepsilon \neq 1, \\ \exp\left(\frac{1}{n}\sum_{i=1}^{n}\ln x_i^z\right), & \varepsilon = 1. \end{cases}$$

The *Atkinson-style poverty index* is then defined by normalizing the gap between the poverty line and the EDE:

$$\mathrm{A}_\varepsilon(\boldsymbol{x}; z) = 1 - \frac{\mathrm{EDE}_\varepsilon(\boldsymbol{x}^z)}{z}.$$ [10]

This index measures the fraction of the poverty line "lost" due to shortfalls in censored income, paralleling Atkinson's inequality index. Thus $\mathrm{A}_\varepsilon(\boldsymbol{x}; z) = 0$ when every censored income equals z (no poverty), in which case the EDE equals the poverty line, and the index approaches 1 as the EDE becomes small relative to z.

To understand the relationship between this family of indices and generalized Lorenz dominance (applied to censored incomes), it is essential to analyze the monotonicity and concavity properties of the underlying utility u_ε on the interval $[0, z]$. For the utility specification above, one has

$$u_\varepsilon''(t) = \begin{cases} -\varepsilon\, t^{-1-\varepsilon}, & \varepsilon \neq 1, \\ -\frac{1}{t^2}, & \varepsilon = 1, \end{cases}$$

for all $t > 0$.

Consequently, u_ε is increasing and concave on $[0, z]$. By the standard characterization generalized Lorenz dominance (Shorrocks, 1983, Theorem 2), $\boldsymbol{y}^z \succcurlyeq_{GL} \boldsymbol{x}^z$ implies

$$W_\varepsilon(\boldsymbol{y}^z) \geq W_\varepsilon(\boldsymbol{x}^z),$$

and hence any poverty index that decreases as W_ε increases (such as A_ε above) will rank $\boldsymbol{y}$ as weakly less poor than $\boldsymbol{x}$. In short, the Atkinson-type poverty indices are consistent with rankings induced by censored generalized Lorenz dominance.

[10] Note that the index is only defined for censored vectors where every censored income is positive.

2.5.3. The Sen (1976) poverty index

The *Sen index* (Sen, 1976) is a classic composite poverty measure that blends three intuitive ingredients: the incidence of poverty (headcount), the average depth of poverty among the poor (average shortfall), and inequality among the poor. As such, it is often viewed as an attempt to combine extensive and intensive poverty features into a single scalar. Here "extensive" refers to the sheer incidence of poverty (whether individuals fall below the line at all), while "intensive" refers to the depth and severity of the shortfalls among those who are poor. In addition, the Sen index incorporates an explicit inequality component that adjusts for how unevenly poverty is distributed among the poor themselves, beyond mere incidence and average depth.

Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be individual incomes, and $z > 0$ the poverty line. Denote by

$$q = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{x_i < z\}}$$

the *headcount ratio* (proportion of poor individuals).[11] Let μ_p be the mean income among the poor; define the *average normalized shortfall among the poor*

$$A = \frac{z - \mu_p}{z},$$

so that the usual population-level poverty gap (FGT_1) equals qA.[12] Finally, let G_p denote the Gini coefficient computed over the (relevant) incomes of the poor; $G_p \in [0, 1]$ captures inequality among the poor.

The Sen index is defined as follows:

$$S = qA + q(1 - A)G_p,$$

which captures the mean poverty gap qA and includes an adjustment for inequality among the poor. The adjustment term vanishes when $G_p = 0$.

The first factor q scales the measure by the proportion of poor individuals (incidence). The term A measures the average depth of poverty among the poor (intensity). The multiplier involving G_p increases the index when there is greater inequality among the poor: for fixed incidence q and average shortfall A, a distribution that is more unequal among the poor produces a larger Sen index. Thus, the Sen index rewards policies that both reduce the number of poor and reduce deep or very unequal poverty among the poor.

Regarding its relationship to censored generalized Lorenz dominance, the Sen index differs from measures that can be represented as $H\big(\sum_i u(x_i^z)\big)$ for some single concave and nondecreasing function u, where H is a monotone transformation. Specifically, the multiplicative combination of headcount/gap components with the inequality term G_p implies that the Sen index *cannot* be expressed in this additive utility form. Consequently, generalized Lorenz dominance on censored incomes *does not* generally imply the same ranking as the Sen index.

[11] The headcount ratio corresponds exactly to FGT_0, the $\alpha = 0$ member of the Foster–Greer–Thorbecke family introduced earlier.

[12] The poverty gap index qA is precisely FGT_1, the $\alpha = 1$ case of the Foster–Greer–Thorbecke class.

In addition to failing to be representable in the concave-additive form associated with censored generalized Lorenz dominance, the Sen index cannot serve as a completion (total extension) of either $\succcurlyeq^z_R$ or $\succcurlyeq^z_A$ because it violates the Continuity axiom (see Appendix A). The reason is simple and structural: the Sen index contains the headcount ratio

$$q = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{x_i < z\}},$$

which is a discontinuous functional of the income vector (it depends on the indicators $\mathbf{1}_{\{x_i < z\}}$). Thus, small, continuous perturbations of incomes that push an individual across the poverty line produce a discrete jump in q (and hence in S), whereas any ordering that satisfies Continuity must be stable under such vanishing perturbations.

Second, and perhaps more normatively concerning, the Sen index also fails to satisfy standard weak monotonicity axioms. These axioms require that any uniform proportional or absolute reduction in the incomes of the poor should increase measured poverty. To see why the Sen index can violate this principle, recall its functional form:

$$S = qA + q(1 - A)G_p.$$

The violation occurs due to the interaction between the average shortfall term A and the inequality component G_p. When the poorest individuals have zero income, the Gini coefficient among the poor reaches its maximum value of $G_p = 1$. In this boundary case, the Sen index simplifies to:

$$S = qA + q(1 - A) \cdot 1 = qA + q - qA = q$$

Thus, when inequality among the poor is already at its maximum ($G_p = 1$), the Sen index reduces to just the headcount ratio, q. Now consider a uniform proportional reduction in the incomes of all poor individuals with positive income. This reduction will:

- Increase the average shortfall A (since gaps become larger).
- Leave G_p unchanged at its maximum value of 1 (since relative inequality among the poor cannot increase further).
- Keep the headcount ratio q constant (assuming no one crosses the poverty line).

Since $S = q$ when $G_p = 1$ and q remains unchanged, the Sen index registers no increase in poverty despite the unambiguous worsening of the poor's situation. This fact demonstrates how the bounded nature of the Gini coefficient can cause the Sen index to violate weak monotonicity.

2.5.4. The Sen–Shorrocks–Thon (SST) index

The Sen-Shorrocks-Thon (SST) index (Thon, 1979; Shorrocks, 1995) is a widely used refinement of Sen's (1976) composite poverty measure. The SST index was developed to address two fundamental shortcomings of the original Sen index: its violation of the Continuity axiom and its failure to satisfy basic monotonicity requirements. The SST index preserves the three essential components of poverty measurement—incidence, depth, and inequality—while resolving both the discontinuity and monotonicity issues inherent in the original Sen index.

The SST is commonly written in the multiplicative form

$$\mathrm{SST}(x;z) = qA(1+G_g),$$

where

$$q = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{x_i<z\}}, \quad g_i = \max\{0, z-x_i\}, \quad \text{and } A = \frac{1}{qn}\sum_{i=1}^{n} g_i,$$

and where G_g denotes the Gini coefficient computed over the vector of gaps $(g_1,\ldots,g_n)$ (including zeros for the non-poor). Thus SST aggregates incidence (q), average gap (A) and the inequality of gaps (G_g) in a decomposable way (Shorrocks, 1995; Thon, 1979).

Intuitively, the SST index can be understood as the *average normalized poverty gap in the population*—that is, the FGT_1 measure qA—further adjusted by a factor $(1+G_g)$ that captures the degree of inequality in the distribution of poverty gaps: when gaps are small or concentrated near zero the adjustment is mild, while larger or more unequal shortfalls raise the index.

Censored generalized Lorenz dominance characterizes rankings that can be represented by additive utilitarian functionals applied to incomes censored at the poverty line. The SST index is not, in general, a member of this concave, additive class: its multiplicative incidence–gap–inequality structure cannot be expressed in the form

$$H\left(\sum_{i=1}^{n} u(x_i^z)\right)$$

for a single concave nondecreasing u and monotone H. Consequently, it does *not* provide a completion of the censored generalized Lorenz pre-order.

A manifestation of this inconsistency between the SST index and generalized Lorenz dominance over censored income vectors appears in their divergent treatment of certain income transfers. Unlike censored generalized Lorenz dominance, the SST index violates the Strong Transfer axiom (Donaldson and Weymark, 1986), which requires that poverty increase following any regressive transfer from a poor person, regardless of whether the recipient is poor or non-poor. The intuition behind this violation is straightforward: a regressive transfer from the richest among the poor may decrease inequality within the poor population, thereby reducing the Gini coefficient G_g that measures inequality in poverty gaps. While censored generalized Lorenz dominance would unambiguously treat such transfers as welfare-reducing (since they worsen the overall distribution), the SST index faces a trade-off between the transfer's direct effect on deepening poverty gaps and its indirect effect of reducing gap inequality. The index's structure may cause the inequality-reducing effect to more than offset the gap-deepening effect, potentially leading the SST index to register a decrease in poverty despite the regressive transfer—a clear violation of the Strong Transfer axiom.

Although the SST index fails to align with generalized Lorenz dominance over censored income vectors, it is nevertheless a complete representation of the poverty pre-orders $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$. Indeed, the SST index satisfies every axiom laid out in Appendix A to characterize these orderings. Thus, by grounding poverty comparisons in a minimal set of axioms, we obtain a class of orderings that is *strictly larger* than the one generated by Lorenz dominance. The SST index is a prominent member of this enlarged class that lies outside the utilitarian welfare framework. The existence of such measures underscores the scope and practical implications of our analysis.

2.5.5. Censored first-order stochastic dominance

Censored first-order stochastic dominance ("censored FOSD") is a coarser ordering than censored generalized Lorenz dominance. A natural question is whether censored FOSD has a nested relationship with the poverty pre-orders introduced above, $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$. The answer is *no*: neither direction of implication holds in general. We demonstrate this by two complementary observations.

(i) **Censored FOSD does not imply $\succcurlyeq_R^z$ or $\succcurlyeq_A^z$.** Members of the Foster–Greer–Thorbecke (FGT) family FGT_α with $\alpha > 0$ refine censored FOSD (see, e.g., Zheng, 2000, Proposition 3.1 and following remarks). However, as discussed in Section 2.5.1, the subfamily with $0 < \alpha < 1$ can yield rankings that conflict with the pre-orders $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$. This discrepancy shows that censored FOSD need not imply dominance according to $\succcurlyeq_R^z$ or $\succcurlyeq_A^z$.

To see this concretely, consider the pair of distributions

$$\boldsymbol{x} = (1, 1) \quad \text{and} \quad \boldsymbol{y} = (1, 2),$$

with poverty line $z = 3$. Under censored FOSD, $\boldsymbol{y}$ is clearly ranked above $\boldsymbol{x}$. Accordingly, $\boldsymbol{y}$ is deemed "less poor" by every FGT_α with $\alpha > 0$. However, under the poverty pre-orders $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$, the two distributions are incomparable: $\boldsymbol{y}$ has a higher mean among the poor but also greater inequality.

Hence, censored FOSD does not imply dominance according to $\succcurlyeq_R^z$ or $\succcurlyeq_A^z$.

(ii) **$\succcurlyeq_R^z$ and $\succcurlyeq_A^z$ do not imply censored FOSD.** The converse discrepancy arises with regressive transfers among the poor. Suppose a poorer individual transfers income to a less poor individual. Such a move increases inequality within the poor population and is therefore judged unambiguously worse by both the pre-orders $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$, so that the more equal distribution is strictly preferred.

To illustrate, let $z = 10$ and consider

$$\boldsymbol{y} = (6, 6), \qquad \boldsymbol{x} = (5, 7).$$

Distribution $\boldsymbol{x}$ is obtained from $\boldsymbol{y}$ by a regressive transfer from the poorer poor to the richer poor. Both pre-orders rank $\boldsymbol{y}$ as "less poor" than $\boldsymbol{x}$:

$$\boldsymbol{x} \succ_R^z \boldsymbol{y} \quad \text{and} \quad \boldsymbol{x} \succ_A^z \boldsymbol{y}.$$

Censored FOSD, however, leaves them non-comparable. After censoring at $z = 10$ and sorting in ascending order, both distributions remain unchanged: $\boldsymbol{y}$ becomes $(6, 6)$ and $\boldsymbol{x}$ becomes $(5, 7)$. For censored FOSD, we compare corresponding order statistics: the smallest values are $5 < 6$, but the largest values are $7 > 6$. Since the comparisons go in opposite directions, the distributions are incomparable under censored FOSD.

Furthermore, recall that the SST family provides a complete representation of $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$ (see Section 2.5.4). Because the SST index incorporates an explicit inequality-of-gaps component, its value can move in the opposite direction to a censored-FOSD comparison. Thus, while our characterization of poverty-reducing tax schedules applies directly to the SST family, analogous results based solely on censored FOSD would not encompass this class of measures.

Taken together, the two points above show that censored FOSD and the pre-orders $\succcurlyeq_R^z, \succcurlyeq_A^z$ are incomparable in general: each captures different normative sensitivities (to transfers among the poor, to headcount effects, and to gap inequality), and neither can be substituted for the other without additional normative commitment.

3. Theoretical characterization of poverty-reducing tax schedules

Having established the poverty measures $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$, we now focus on their implications for nonlinear income taxation. This section explores the relationship between our poverty criteria and the structure of tax policies in endowment economies characterized by an initial income or wealth distribution.

Our primary objective is to characterize tax schedules that universally reduce poverty according to $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$, i.e., those that reduce poverty irrespective of the initial distribution. This robustness requirement ensures applicability across various economic contexts.

3.1. Tax schedules

We begin by introducing the definition of a nonlinear tax schedule.

A *tax schedule* is a continuous function $t : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $t(x) \leq x$ for all $x \geq 0$.

For any income level x, the value $t(x)$ represents the corresponding tax liability. The schedule allows for negative tax liabilities, interpreted as subsidies. The constraint $t(x) \leq x$ ensures that tax liabilities never exceed pre-tax income—a natural economic requirement.

Our definition of a tax schedule is intentionally broad, avoiding two commonly imposed assumptions: monotonicity of tax liabilities and preservation of income ranks. Restricted versions of these properties, first formalized by Fei (1981), emerge naturally in our characterization of poverty-reducing tax schedules.

3.2. Core properties of poverty-reducing tax schedules

Traditionally, monotonicity stipulates that the tax burden should not decrease as income increases. In other words, the tax function $t(x)$ must be non-decreasing over the entire income domain. However, our characterization of poverty-reducing tax schedules employs a more refined concept: the "constrained non-decreasing" tax schedule.

Formally, given a tax schedule t, define

$$X_t = \{x \in [0, z] : x - t(x) \leq z\}.$$

The set X_t contains all pre-tax income levels up to and including the poverty line for which the corresponding post-tax income does not exceed the poverty line threshold. A tax schedule is defined as *non-decreasing on* X_t if the map

$$x \in X_t \mapsto t(x)$$

is non-decreasing. This concept satisfies the principle of vertical equity specifically in the restricted domain X_t, preventing scenarios where earning more within X_t results in a lower tax burden.

Remark 1. The requirement that t be non-decreasing on X_t has a simple differential interpretation: when t is differentiable on X_t the condition is equivalent to

$$t'(x) \geq 0, \quad \text{for all } x \in X_t,$$

so marginal tax rates are non-negative on the restricted domain. In economic terms, this rules out "perverse" negative marginal taxation on X_t—i.e., situations where an extra dollar of pre-tax income would increase an individual's net receipts by more than one dollar.

Similarly, we consider a constrained version of rank preservation. Typically, rank preservation requires that the ordering of after-tax incomes mirrors that of pre-tax incomes. Mathematically, this is equivalent to requiring that the function representing post-tax income, $x \mapsto x - t(x)$, be strictly increasing over the entire income domain. In our framework, a tax schedule is termed *rank-preserving on* X_t if the map

$$x \in X_t \mapsto x - t(x)$$

is non-decreasing.

Remark 2. The rank-preservation condition for a tax schedule admits a simple interpretation in terms of marginal tax rates. If t is differentiable on X_t, then

$$x \mapsto x - t(x) \text{ is non-decreasing on } X_t \Leftrightarrow 1 - t'(x) \geq 0 \text{ for all } x \in X_t,$$

i.e.

$$t'(x) \leq 1, \quad \text{for all } x \in X_t.$$

Economically, this equivalent condition states that the marginal tax on an extra dollar of pre-tax income never exceeds one dollar—earning an additional dollar cannot lower an individual's after-tax income, which is precisely what preserves ranks among the poor.

This property ensures that the relative economic positions of poor individuals are maintained post-taxation. This principle operates in two ways. First, the tax system preserves the relative economic standing of those who remain impoverished after taxation. Second, when taxation lifts someone above the poverty threshold z, all poor individuals with higher pre-tax incomes (up to z) must also escape poverty or reach exactly the poverty line after taxation.

We can prove this mathematically. Consider an individual with pre-tax income $x \leq z$ whose post-tax income rises above the poverty line, i.e., $x - t(x) \geq z$. We claim that for any income y where $x < y \leq z$, the post-tax income must satisfy $y - t(y) \geq z$. To prove this by contradiction, suppose there existed some y where this was not true. Then consider the continuous function mapping incomes to their post-tax values:

$$x' \in [x, y] \mapsto x' - t(x')$$

This continuous function starts at a value greater than or equal to z (at x) and ends at a value less than z (at y). By the Intermediate Value Theorem, there must exist some income level x^* between x and y where

$$x^* - t(x^*) = z > y - t(y)$$

However, this would violate the rank-preservation property, as someone with a higher pre-tax income (y) would end up with a lower post-tax income than someone with a lower pre-tax income (x^*). Therefore, our original claim must be valid.

This targeted approach to rank preservation specifically addresses how taxation affects the poor population, preventing rank re-orderings at the lower end of the income scale.

An additional critical property of a tax schedule is ensuring that no individual is taxed into poverty. Formally, a tax schedule t *preserves the non-poor status of individuals* if it satisfies the following condition:

$$x > z \Rightarrow x - t(x) \geq z.$$

In other words, any non-poor individual remains at or above the poverty line after taxes are applied. This condition ensures the tax system does not push individuals below the poverty line. The property represents a fundamental principle in designing tax policies that aim to reduce poverty without creating new instances of economic vulnerability.

Remark 3. Non-poor status preservation follows from two conditions: (i) weak poor subsidization, meaning $t(x) \leq 0$ for all $x \leq z$; and (ii) rank preservation among non-poor pre-tax incomes, that is, the map $x \mapsto x - t(x)$ is non-decreasing for all $x \geq z$.

To see this, suppose the tax schedule t satisfies $t(x) \leq 0$ for all $x \leq z$. In particular, at the poverty line z, we then have

$$z - t(z) \geq z.$$

Next, assume the schedule is rank-preserving among non-poor incomes, so that the post-tax income function

$$x \mapsto x - t(x)$$

is non-decreasing for all $x \geq z$. It follows that for any non-poor income $x \geq z$,

$$x - t(x) \geq z - t(z) \geq z.$$

Consequently, every individual with pre-tax income $x > z$ ends up with post-tax income above (or equal to) z, so no non-poor individual is pushed into poverty. Thus, weak subsidization together with rank preservation among the non-poor implies preservation of the non-poor status.

Later in this section, we provide a graphical illustration of how violations of this condition—where the slope of the tax function exceeds unity—can result in individuals being taxed into poverty. See Figure 3.

Our final definition pertains to the concept of average-rate progressivity in tax schedules.

A tax schedule t is deemed *average-rate progressive on* X_t if the map

$$x \in X_t \mapsto t(x)/x$$

is non-decreasing. In other words, a tax schedule t is considered average-rate progressive over X_t if the average tax rate is non-decreasing for income levels within the restricted domain X_t.

3.3. Proverty-reducing tax schedules

With this framework in place, we can now analyze how different tax structures impact poverty levels as measured by $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$ and identify the key features of tax schedules that consistently lead to poverty reduction.

Central to this analysis is the concept of a poverty-reducing tax schedule. We define a tax schedule t as *poverty-reducing in the relative sense* if, for all pre-tax distributions of income $\boldsymbol{x} = (x_1, \ldots, x_n)$ and all population sizes n, we have

$$\boldsymbol{x} \succcurlyeq_R^z (x_1 - t(x_1), \ldots, x_n - t(x_n)).$$

In other words, the tax schedule t is considered poverty-reducing if the post-tax distribution

$$(x_1 - t(x_1), \ldots, x_n - t(x_n))$$

exhibits no more poverty than the pre-tax distribution $\boldsymbol{x}$ for every distribution $\boldsymbol{x}$.

This concept is "universal" in nature, requiring (at least weak) poverty reduction regardless of the initial distribution.

The notion of absolute poverty reduction is defined analogously: a tax schedule t is *poverty-reducing in the absolute sense* if, for all pre-tax distributions of income $\boldsymbol{x} = (x_1, \ldots, x_n)$ and all population sizes n, we have

$$\boldsymbol{x} \succcurlyeq_A^z (x_1 - t(x_1), \ldots, x_n - t(x_n)).$$

Remark 4. The universality built into the definition of a poverty-reducing tax schedule—requiring that the post-tax distribution always exhibit less poverty than the pre-tax distribution, *regardless* of the initial distribution—merits further discussion.

First, the robustness of such tax systems has clear practical value. Policymakers need not know the detailed features of the pre-tax income distribution, nor must they constantly adjust to circumstantial changes in income distributions, such as economic shocks or recessions, which tend to arise more rapidly than tax policies can be revised. Even structural changes, unfolding over longer horizons, do not require immediate policy adaptation in a universal framework. This "worry-free" robustness feature greatly enhances both the flexibility and the practical applicability of poverty-reducing tax design.

One might worry, however, that universality is too restrictive—that it narrows the set of feasible, robust tax policies to such an extent that little remains for effective policymaking. This concern proves unwarranted. Our main characterizations of poverty-reducing tax schedules (Theorem 1 and Theorem 2) show that the robust set is in fact "large." For instance, Theorem 1 reveals that relative poverty-reducing schedules are characterized by just a few essential properties: rank preservation and average-rate progressivity among the poor, and protection of non-poor individuals from being taxed into poverty. These broad principles still leave ample scope for tax design within the feasible domain. Once a tax system is shown to be robust in this sense, it may be further fine-tuned to pursue complementary goals, such as revenue neutrality.

Which brings us to a second point. Budget balance considerations are inherently at odds with our distribution-free framework. By construction, revenue neutrality is distribution-specific: a tax schedule that balances the budget for one initial distribution will almost surely fail to do so for another. Imposing a budget constraint, therefore, alters the problem fundamentally, forcing attention to a single fixed pre-tax distribution (as in Bourguignon and Fields (1990)). Nevertheless, our approach does not preclude such considerations: budget constraints can be incorporated as an additional refinement to the robust set, narrowing the range of feasible schedules to satisfy revenue neutrality.[13]

We are now prepared to present our main results, which offer a complete characterization of poverty-reducing tax schedules evaluated through the dual lenses of relative poverty reduction ($\succcurlyeq_R^z$) and absolute poverty reduction ($\succcurlyeq_A^z$).

3.4. Relative poverty reduction (Theorem 1)

Our first main theorem establishes that three fundamental properties are both necessary and sufficient for a tax schedule to be poverty-reducing according to the relative poverty measure $\succcurlyeq_R^z$, regardless of the initial income distribution. Specifically, these conditions ensure that the tax system preserves economic ranks among the poor, maintains average-rate progressivity over the relevant income domain, and prevents non-poor individuals from falling below the poverty line due to taxation. The proof of this result is relegated to Appendix B.1.

Theorem 1. *A tax schedule t is poverty-reducing in the relative sense if and only if it is rank-preserving on X_t, average-rate progressive on X_t, and preserves the non-poor status of individuals.*

Remark 5. The three properties of tax schedules established in Theorem 1 are independent of each other, as demonstrated below through a series of examples. When preservation of the non-poor status and average-rate progressivity on X_t are combined, poverty-reducing tax schedules exhibit an additional property: they must (weakly) subsidize the poor, i.e., $t(x) \leq 0$ for all incomes $x \leq z$.

To prove this implication, suppose by contradiction that $t(x) > 0$ for some $x \leq z$. Note that $x > 0$ must hold, since by definition, tax schedules satisfy $t(x') \leq x'$ for all $x' \geq 0$. We proceed in two steps.

First, we establish that $t(z) > 0$. If instead $t(z) \leq 0$, the Intermediate Value Theorem would guarantee the existence of some $x^* \in [x, z]$ where $t(x^*) = 0$. Since

$$x - t(x) < x^* - t(x^*) = x^* \leq z \quad \text{and} \quad \frac{t(x)}{x} > 0 = \frac{t(x^*)}{x^*},$$

this would violate average-rate progressivity on X_t.

Second, given $t(z) > 0$, the continuity of t ensures the existence of income levels $y > z$ arbitrarily close to z where $t(y) > 0$ and $y - t(y) < z$. This implies that some individuals

[13] If the initial distribution is fixed and revenue neutrality is imposed, then pre- and post-tax aggregate incomes are equal. In this case, the relative and absolute poverty pre-orders $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$ coincide.

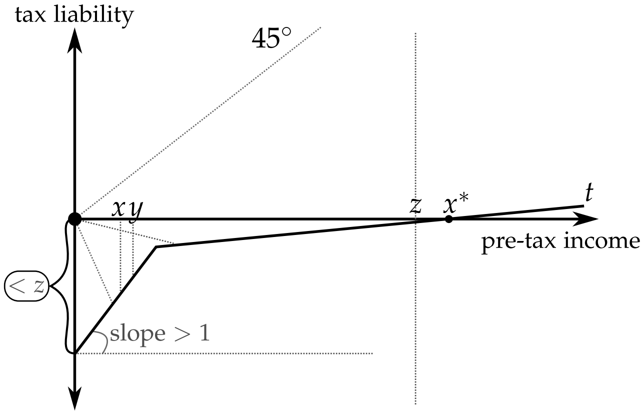

Figure 1: A failure of rank preservation on X_t.

above the poverty line would be taxed into poverty, contradicting the preservation of non-poor status.

3.4.1. Examples and counterexamples

To illustrate the independence of the three conditions in Theorem 1, we present a series of examples. In each case, exactly one of the conditions fails, demonstrating how this singular failure renders the tax schedule ineffective in reducing relative poverty. These examples are complemented by visual representations and intuitive explanations, offering an accessible interpretation of Theorem 1.

Consider the tax schedule t illustrated in Figure 1. This schedule satisfies all the conditions of Theorem 1 but the rank preservation property. This failure occurs because the slope of the initial tax bracket exceeds unity: for the points x and y shown in the figure, we have $y - t(y) < x - t(x)$, $x - t(x) < z$, and $y - t(y) < z$, which violates the rank preservation property.

The schedule exhibits average-rate progressivity across its entire domain, as the slopes of rays extending from the origin to points along the graph increase with pre-tax income. The preservation of non-poor status is guaranteed since the mapping $x' \mapsto x' - t(x')$ increases over the interval $[x^*, +\infty)$.

Let us now illustrate how the violation of the rank preservation implies the existence of an income distribution whose corresponding post-tax distribution increases poverty.

Consider the n-person distribution $(x, \ldots, x, y)$, whose corresponding post-tax distribution is given by

$$(y - t(y), x - t(x), \ldots, x - t(x)).$$

Note that, because $y - t(y) < x - t(x)$, for large enough n we have

$$\frac{x}{(n-1)x + y} > \frac{y - t(y)}{y - t(y) + (n-1)(x - t(x))}.$$

Consequently, for large enough n,

$$(y - t(y), x - t(x), \ldots, x - t(x)) \not\succcurlyeq_{RL} (x, \ldots, x, y),$$

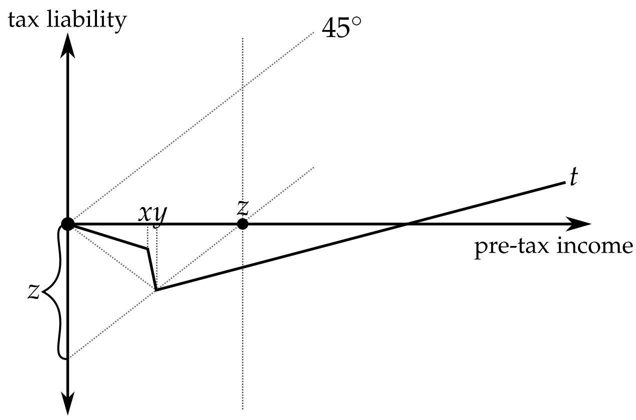

Figure 2: A failure of average-rate progressivity on X_t.

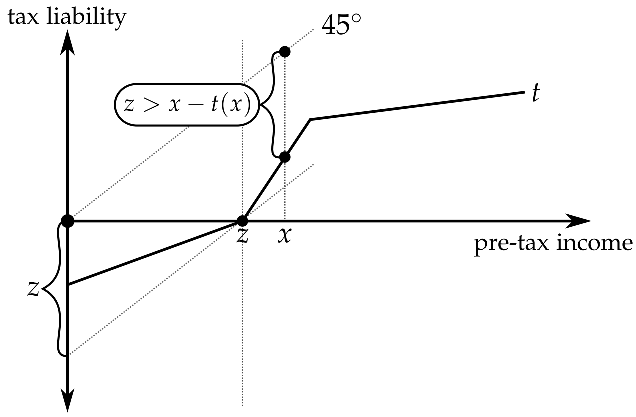

Figure 3: A failure of non-poor status preservation.

implying that t fails to mitigate poverty according to $\succcurlyeq_R^z$.

The tax schedule t depicted in Figure 2 illustrates another instance where t satisfies all the conditions of Theorem 1 except for average-rate progressivity on the set X_t.

To see that t violates average-rate progressivity on X_t, note that, for the points x and y shown in the figure, we have $t(x)/x > t(y)/y$.

Note that because the map $x' \mapsto x' - t(x')$ is increasing, t is rank-preserving on X_t and maintains the non-poor status of individuals.

To see that t is not poverty-reducing in the relative sense, consider the two-person income distribution (x, y), whose post-tax distribution is given by

$$(x - t(x), y - t(y)).$$

Note that

$$\frac{x}{x+y} > \frac{x - t(x)}{x - t(x) + y - t(y)} \Leftrightarrow \frac{t(y)}{y} < \frac{t(x)}{x}.$$

Since the last inequality is true, it follows that

$$(x - t(x), y - t(y)) \not\succcurlyeq_{RL} (x, y),$$

implying that t is not poverty-reducing in the relative sense.

Figure 3 illustrates a tax schedule that meets the conditions of Theorem 1 with one exception: it fails to preserve the non-poor status of individuals. This failure occurs at specific income levels, such as x, where the resulting after-tax income falls below the poverty line.

The reason why t is not poverty-reducing is simple: for the two-person income distribution $\boldsymbol{x} = (x, x)$, we have

$$(x - t(x), x - t(x)) \not\succcurlyeq_R^z \boldsymbol{x}$$

because $x - t(x) < z$.

3.5. Absolute poverty reduction (Theorem 2)

Our next principal finding characterizes poverty-reducing tax schedules based on the absolute poverty measure $\succcurlyeq_A^z$. This measure balances additive increases in overall income with the income distribution among society's members, without imposing trade-offs between absolute equity and income growth. Under this normative criterion, uniform additive increases in everyone's income—which maintain the absolute degree of equality—are deemed preferable.

Interestingly, the tax schedules that reduce poverty in the absolute sense form a strict subset of those that reduce poverty in the relative sense, since, unlike the latter, the former must exhibit monotonically increasing tax liabilities on X_t (see Remark 6 below).

Theorem 2. *A tax schedule t is poverty-reducing in the absolute sense if and only if it is non-decreasing on X_t, rank-preserving on X_t, and preserves the non-poor status of individuals.*

The proof of Theorem 2 can be found in Appendix B.2.

Remark 6. The tax schedule properties established in Theorem 2 have important additional implications.

First, we show that if a tax schedule t is non-decreasing on X_t and preserves the non-poor status of individuals, then it must (weakly) subsidize the poor—that is, $t(x) \leq 0$ for all $x \in [0, z]$. We prove this by contradiction.

Suppose there exists some $x \in [0, z]$ such that $t(x) > 0$. This implies $t(z) > 0$. For if instead $t(z) \leq 0$, the Intermediate Value Theorem would guarantee the existence of some $x^* \in [x, z]$ where $t(x^*) = 0$. This would mean $t(x) > t(x^*)$ while both $x - t(x) \leq z$ and $x^* - t(x^*) = x^* \leq z$, contradicting the monotonicity of t on X_t.

However, if $t(z) > 0$, then by continuity of t, there exist non-poor income levels $y > z$ arbitrarily close to z where $y - t(y) < z$, contradicting the assumption that t preserves the non-poor status of individuals.

Second, we show that if a tax schedule t (weakly) subsidizes the poor and is non-decreasing on X_t, then it must be average-rate progressive on X_t. Proceeding by contradiction, assume that t is not average-rate progressive on X_t. Then there exist $x, y \in X_t$ with $x < y$ and

$$\frac{t(x)}{x} > \frac{t(y)}{y}.$$

Given that $t(x') \leq 0$ for all $x' \in [0, z]$, and since $y \leq z$, this inequality implies

$$t(y) \leq t(y)(x/y) < t(x).$$

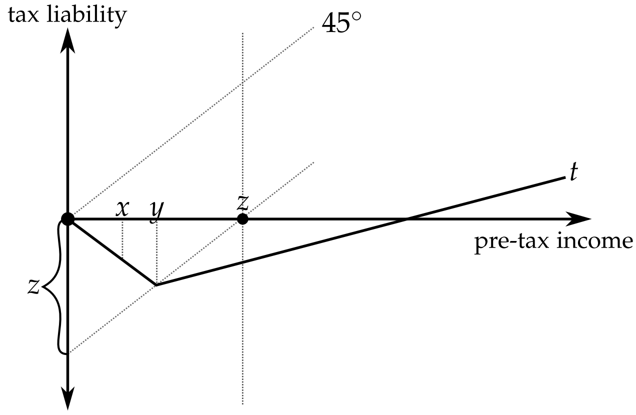

Figure 4: A failure monotonicity on X_t.

This would mean that the tax liability strictly decreases as income increases from x to y, or, in other words, that the subsidy increases as the income increases from x to y, which contradicts the assumed monotonicity of t on X_t.

From these observations, together with Theorem 1 and Theorem 2, we deduce that if a tax schedule t is poverty-reducing in the absolute sense, then it is also poverty-reducing in the relative sense and satisfies the following conditions: it is non-decreasing on X_t, rank-preserving on X_t, average-rate progressive on X_t, it preserves the non-poor status of individuals, and it (weakly) subsidizes the poor.[14]

3.5.1. Examples and counterexamples

We now demonstrate that the conditions outlined in Theorem 2 are independent.

First, we examine a tax schedule t that violates the non-decreasing property on X_t. Consider the tax function illustrated in Figure 4. This function decreases in its first bracket, thus violating the non-decreasing property. However, the function $x' \mapsto x' - t(x')$ remains increasing over the entire income domain, ensuring that t is both rank-preserving and preserves the non-poor status of individuals.

To prove that t is not poverty-reducing in the absolute sense, let us analyze the income distribution (x, y), where x and y are marked in Figure 4. After applying the tax function, we obtain the post-tax distribution

$$(x - t(x), y - t(y)).$$

We can show that this tax schedule increases poverty through the following argument. First, observe that

$$x - \frac{1}{2}(x + y) > x - t(x) - \frac{1}{2}[x - t(x) + y - t(y)] \Leftrightarrow t(x) > t(y).$$

[14]The nesting of the absolute poverty reduction criterion within its relative counterpart, as established here, contrasts with analogous criteria in the inequality context. Specifically, when comparing the properties of inequality-reducing tax schedules in endowment economies—detailed in Eichhorn et al. (1984) and Moyes (1988)—one can easily find a tax schedule that reduces inequality in the absolute sense but increases it in the relative sense. For example, a strictly convex (marginal-rate regressive), rank-preserving tax function with monotonically increasing tax liabilities illustrates this disparity.

This inequality implies

$$(x - t(x), y - t(y)) \not\succcurlyeq_{AL} (x, y).$$

Moreover, since

$$x - t(x) \leq y - t(y) \leq z \quad \text{and} \quad x \leq y \leq z,$$

we can conclude that

$$(x, y) \not\succcurlyeq_A^z (x - t(x), y - t(y)).$$

Therefore, t fails to be poverty-reducing in the absolute sense, despite preserving ranks and the non-poor status of individuals.

It is worth noting that the tax schedule depicted in Figure 4 exhibits average-rate progressivity, thereby satisfying all conditions of Theorem 1 while failing to meet those of Theorem 2. Consequently, in light of Theorem 1 and Theorem 2, the tax schedule t reduces poverty in the relative sense but fails to do so in the absolute sense.

The example presented in Figure 1 illustrates a tax schedule that meets all the criteria outlined in Theorem 2, except for the rank preservation property on X_t. It can be demonstrated that this tax schedule fails to achieve poverty reduction in an absolute sense, following a similar argument to that used for the case of relative poverty reduction.

The tax schedule t from Figure 3 illustrates how failing to preserve non-poor status undermines absolute poverty reduction. While t is monotonically increasing and preserves ranks on the income domain X_t, it fails to preserve the non-poor status of individuals, as previously demonstrated. We can show that this tax schedule fails to reduce absolute poverty through an argument parallel to that used in the relative case.

3.6. Illustration: constant progression tax schedules

Before linking our results to features of real-world tax systems, we illustrate them with an example: tax schedules that display constant proportional progressivity.

For parameters $b > 0$ and $a \in \mathbb{R}$, define the tax schedule

$$t(x) = x - bx^a, \quad \text{for all } x > 0.$$

This functional form was introduced by Jakobsson (1976), who defined it in terms of *residual progression*. In his terminology, residual progression measures the elasticity of disposable income with respect to gross income. In this family, disposable income is given by $x - t(x) = bx^a$, which has a constant elasticity across all income levels:

$$\frac{d \ln(x - t(x))}{d \ln x} = a.$$

Thus, a 1% increase in gross income always yields an $a\%$ increase in disposable income, regardless of the level of x. This constant elasticity is what Jakobsson refers to as a constant rate of progression. If $a < 1$ then average tax rates rise with income (progressive taxation); if $a = 1$ the tax is proportional; and if $a > 1$ average tax rates fall with income (regressive taxation). The parameter b scales disposable income and determines the overall level of taxation, while a captures the strength and direction of progressivity in a single, uniform measure.

Theorem 1 requires three conditions:

1. Rank preservation on X_t: The function $x \mapsto x - t(x) = bx^a$ must be non-decreasing on X_t.

   Compute the derivative:
$$\frac{d}{dx}(bx^a) = abx^{a-1}.$$
   For $x > 0$, this is non-negative iff $a \geq 0$. Thus, rank preservation holds if $a \geq 0$.

2. Average-rate progressivity on X_t: The function $x \mapsto \frac{t(x)}{x} = 1 - bx^{a-1}$ must be non-decreasing on X_t.

   Compute the derivative:
$$\frac{d}{dx}\left(1 - bx^{a-1}\right) = -b(a-1)x^{a-2}.$$
   For $x > 0$, this is non-negative iff $-(a-1) \geq 0$, i.e., $a \leq 1$. Thus, average-rate progressivity holds if $a \leq 1$.

3. Preservation of non-poor status: For all $x > z$, $x - t(x) = bx^a \geq z$.

   Consider two cases:

   - If $a = 0$, then $b \geq z$.
   - If $0 < a \leq 1$, the function $f(x) = bx^a$ is non-decreasing (since $a > 0$), so its minimum on (z, ∞) occurs at $x = z$. Thus, $bz^a \geq z$, i.e., $b \geq z^{1-a}$.

Combining all conditions:

$$a \geq 0 \quad \text{(from rank preservation)}.$$
$$a \leq 1 \quad \text{(from average-rate progressivity)}.$$
$$b \geq z \text{ if } a = 0, b \geq z^{1-a} \text{ if } 0 < a \leq 1 \quad \text{(from non-poor status preservation)}.$$

Recall that Theorem 2 requires three conditions:

- t is rank-preserving on X_t. We know that this condition holds if $a \geq 0$.
- t preserves non-poor status.

  We require that for all $x > z$, $bx^a \geq z$.

  - If $a = 0$, then $bx^a = b$, so we need $b \geq z$.
  - If $a > 0$, then bx^a is increasing in x, so the minimum occurs at $x = z$. Thus, we require $bz^a \geq z$, i.e., $b \geq z^{1-a}$.

- t is non-decreasing on X_t. We require that $t(x) = x - bx^a$ be non-decreasing on X_t. Compute the derivative:
$$t'(x) = 1 - abx^{a-1}.$$
  We need $t'(x) \geq 0$ for all $x \in X_t$. Note that $X_t = \{x \in [0, z] : bx^a \leq z\}$. Consider cases:

  - Case $a = 0$: Then $t(x) = x - b$, so $t'(x) = 1 \geq 0$.

– Case $a = 1$: Then $t(x) = x - bx = (1-b)x$, so $t'(x) = 1 - b$. Thus, we need $1 - b \geq 0$, i.e., $b \leq 1$. From preservation of non-poor status, we require $b \geq z^0 = 1$. Thus, $b = 1$.
– Case $a > 1$: As $x \to 0^+$, $abx^{a-1} \to \infty$, so $t'(x) \to -\infty$. Hence, the required condition fails for some $x \in X_t$.
– Case $0 < a < 1$: As $x \to 0^+$, $x^{a-1} \to \infty$, so $t'(x) \to -\infty$. The required condition fails again.

Thus, the monotonicity condition is quite restrictive for this family, since it only admits the cases $a = 0$ and $a = 1$. The first case corresponds to linear taxation with a negative intercept and a 100% marginal tax rate. In the second case, preservation of non-poor status further requires $b = 1$. When $a = 1$ and $b = 1$, t is identically zero.

The only parameter values satisfying all conditions are:

- $a = 0$ and $b \geq z$, implying $t(x) = x - b$, and
- $a = 1$ and $b = 1$, i.e., t is identically zero.

3.7. Implications for tax policy design

The properties of poverty-reducing tax schedules derived in our results are often reflected in real-world tax systems, though practical violations may occur. Tiered piecewise-linear *statutory* tax schedules, prevalent in developed countries, generally satisfy three key conditions:

- Rank preservation, as marginal tax rates typically remain below unity.
- Average-rate progressivity, with graduated increasing marginal tax rates.
- Monotonicity, with non-negative marginal tax rates.

The preservation of non-poor status mirrors standard statutory exemption rates followed by rank-preserving graduated tax rates. In practice, however, statutory exemptions are not always calibrated to coincide with poverty thresholds.

These properties may fail when we consider *effective* tax rates—which account for deductions, credits, and other provisions—rather than statutory rates. Refundable tax credits, such as the U.S. Earned Income Tax Credit (EITC) and Child Tax Credit (CTC), are designed to alleviate poverty but can create complications. These programs typically feature three income regions: a phase-in range where the credit increases with income, a plateau where the credit remains constant, and a phase-out range where the credit decreases.

During the phase-in period, the effective marginal tax rate becomes negative because each additional dollar of earned income increases the tax credit received. This negative rate can violate the monotonicity condition when it is sufficiently large in magnitude, meaning that higher-income individuals within this range can have lower net tax burdens than those earning less—contradicting the principle that after-tax income should increase monotonically with pre-tax income.[15]

[15]Another example comes from the effective tax rates estimated by Saez and Zucman (2020) for the U.S. economy, which exhibit regressivity at the top of the income distribution. This feature does not necessarily violate our conditions for poverty reduction, since these conditions apply only across the poverty income range. However, this regressivity does imply a failure of average-rate progressivity for high incomes.

Thus, while statutory tax systems often align with the properties we identify for poverty reduction, effective tax systems may deviate in ways that undermine these objectives.

While violating the monotonicity condition required for tax-induced absolute poverty reduction, negative effective marginal tax rates can still reduce poverty when labor supply is endogenous. In such cases, individuals may increase their work effort to achieve higher net incomes, thereby alleviating poverty. This dynamic, absent under exogenous labor supply, is discussed further in the concluding section.

3.8. Remarks on the exogeneity of the poverty line

A potential limitation of our analysis is that the poverty line z is fixed exogenously. This raises questions about the relevance of our results when the poverty line is determined as a function of the income distribution. A common approach in policy contexts is to define the poverty line as a fraction $\alpha > 0$ of median income, $m_{\boldsymbol{x}} > 0$, where $m_{\boldsymbol{x}}$ is the median of the distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$. In this case, the poverty line is endogenous and given by $z_{\boldsymbol{x}} = \alpha m_{\boldsymbol{x}}$.

To provide some guidance on this more complex endogenous setting, we rely on the scale-invariance property: proportional scaling of an income distribution should leave poverty rankings unchanged. Under this assumption, one can normalize all pre-tax income distributions by their median incomes, transforming each distribution $\boldsymbol{x}$ into a scaled distribution $\gamma_{\boldsymbol{x}}\boldsymbol{x}$, with scale factor $\gamma_{\boldsymbol{x}} = 1/m_{\boldsymbol{x}}$. The transformed distributions share a common poverty line $z^* = \alpha$.

By normalizing distributions by their median, the poverty line becomes fixed at $z^* = \alpha$, allowing application of Theorem 1 to these standardized distributions.

Suppose a linear tax schedule $t(x) = -b + \tau x$, with parameters $b \geq 0$ and $\tau \in (0, 1)$, is poverty-reducing for this normalized set of distributions. Each scaled distribution satisfies

$$(\gamma_{\boldsymbol{x}}\boldsymbol{x})^{z^*} \succcurlyeq_{RS} (\gamma_{\boldsymbol{x}} x_1 + b - \tau\gamma_{\boldsymbol{x}} x_1, \ldots, \gamma_{\boldsymbol{x}} x_n + b - \tau\gamma_{\boldsymbol{x}} x_n)^{z^*}.$$

By scale invariance, and since

$$\frac{1}{\gamma_{\boldsymbol{x}}}(\gamma_{\boldsymbol{x}}\boldsymbol{x})^{z^*} = \boldsymbol{x}^{z^*/\gamma_{\boldsymbol{x}}} = \boldsymbol{x}^{z_{\boldsymbol{x}}}$$

and

$$\frac{1}{\gamma_{\boldsymbol{x}}}(\gamma_{\boldsymbol{x}} x_1 + b - \tau\gamma_{\boldsymbol{x}} x_1, \ldots, \gamma_{\boldsymbol{x}} x_n + b - \tau\gamma_{\boldsymbol{x}} x_n)^{z^*} = \left(x_1(1-\tau) + \frac{b}{\gamma_{\boldsymbol{x}}}, \ldots, x_n(1-\tau) + \frac{b}{\gamma_{\boldsymbol{x}}}\right)^{z_{\boldsymbol{x}}},$$

it follows that

$$\boldsymbol{x} \succcurlyeq_R^{z_{\boldsymbol{x}}} \left(x_1(1-\tau) + \frac{b}{\gamma_{\boldsymbol{x}}}, \ldots, x_n(1-\tau) + \frac{b}{\gamma_{\boldsymbol{x}}}\right).$$

Consequently, the tax schedule

$$t_{\boldsymbol{x}}(y) = -\frac{b}{\gamma_{\boldsymbol{x}}} + \tau y = -m_{\boldsymbol{x}} b + \tau y$$

is poverty-reducing for each original distribution $\boldsymbol{x}$.

Thus, while a single universal tax schedule may not exist under endogenous poverty lines, our analysis suggests a family of linear tax schedules with a common marginal tax rate, τ, but varying exemption rates linked to the median income, $-m_x b$. By tying the exemption rate to the median income, the tax system adjusts to changes in the poverty threshold, ensuring that the base income level exempted from taxation moves consistently with the prevailing poverty line.

More generally, the above characterization extends to any poverty line based on a positional statistic derived from the income distribution, such as quantiles or other order statistics.

A similar argument applies under translation invariance, where poverty-reducing tax schedules would be characterized from Theorem 2 instead of Theorem 1.

Although the present discussion does not fully characterize endogenous poverty lines, it offers a useful conceptual bridge from the exogenous to the endogenous poverty line case.

4. Concluding remarks

This paper provides the first complete characterization of nonlinear tax schedules that universally reduce poverty across a broad class of relative and absolute poverty measures. In doing so, it extends the scope of the classical results on inequality-reducing taxation to the context of poverty alleviation.

We establish necessary and sufficient conditions for nonlinear income tax schedules to guarantee poverty reduction, regardless of the initial income distribution. Our results are valid for a broad class of absolute and relative poverty measures, extending beyond the traditional application of generalized Lorenz dominance to poverty analysis.

For relative poverty reduction, three key properties emerge as necessary and sufficient: income rank preservation, average-rate progressivity among the poor, and the principle that taxation should not push individuals into poverty. In the absolute poverty case, while rank preservation and protection of non-poor status remain crucial, average-rate progressivity is superseded by the requirement for monotonic tax liabilities among the poor.

These properties are demonstrably independent of each other and imply that the poor must receive subsidies in both relative and absolute frameworks. Notably, the absolute poverty case imposes stricter conditions, as the combination of monotonic tax liabilities and preservation of non-poor status necessarily implies restricted average-rate progressivity.

This finding stands in contrast to the conventional analysis of inequality-reducing income taxation, where the properties of tax schedules derived by Eichhorn et al. (1984) for the relative Lorenz ordering and by Moyes (1988) for the absolute Lorenz criterion exhibit no such nested relationship.

Our main contributions concern tax schedules under exogenous incomes. When incomes respond to taxation, however, additional considerations arise. In the presence of disincentive effects of taxation, a schedule qualifies as poverty-reducing if it generates an income distribution with less poverty than the corresponding tax-free distribution. Kanbur et al. (1994) provides a partial analysis of this case, while Carbonell-Nicolau (2025b) offers a complete characterization using censored first-order stochastic dominance as the poverty criterion. However, this criterion is non-nested with the poverty pre-orders $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$

employed in our paper (see Section 2.5.5), leaving the comprehensive characterization for our measures as an open question.

Two important considerations emerge when extending to endogenous labor responses. First, negative marginal tax rates—which violate the monotonicity condition from Theorem 2—feature prominently in both optimal taxation literature and poverty-focused tax design (see Carbonell-Nicolau, 2025b, and references therein). These rates incentivize work and increase income, thereby alleviating *income* poverty.[16] This suggests that the monotonicity property derived in Theorem 2 may not extend to the endogenous framework.

Second, rank preservation—an essential property of poverty-reducing tax schedules in our framework—can be viewed as a weaker analogue of the monotonicity or implementability constraints central to the Mirrleesian model of optimal income taxation with endogenous income (Mirrlees, 1971). Under the standard single-crossing ("agent monotonicity") condition introduced by Mirrlees (1971), intervals where rank preservation fails are endogenously avoided, as no agent type would optimally choose an income level where their ranking is reversed. Consequently, rank preservation emerges as a necessary condition for implementability in settings with behavioral responses, and equilibria naturally preclude allocations that violate it.

In closing, we briefly comment on the practical challenges of implementing poverty-reducing tax systems. In reality, tax targeting is inevitably noisy: incomes are imperfectly observed, administrative costs are nontrivial, enforcement is costly, and interactions with other tax and transfer programs can create unintended effects. The optimal taxation literature has developed rich frameworks to model such frictions, explicitly incorporating avoidance, evasion, compliance costs, and enforcement constraints (see, e.g., Slemrod and Yitzhaki, 2002, and references therein). These frameworks can serve as a modeling foundation for poverty-reducing taxation—which differs fundamentally from the standard welfare-maximization approach—but adapting our characterizations to this enriched setting requires new analysis and remains an important open problem.

A. Axiomatic characterization of poverty measures

This section fully characterizes the poverty orderings used in our analysis. We focus on the partial orders $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$ introduced in Section 2.

We begin our analysis with a poverty pre-order denoted by $\succcurlyeq^z$, which is associated with a specific poverty line $z > 0$. The symbol $\succcurlyeq^z$ represents a reflexive and transitive binary relation defined over the set of income distributions for a population of size n. Formally, for any two income distributions $\boldsymbol{x}$ and $\boldsymbol{y}$, the relation $\boldsymbol{x} \succcurlyeq^z \boldsymbol{y}$ indicates that distribution $\boldsymbol{y}$ exhibits no more poverty than distribution $\boldsymbol{x}$, given the poverty line z.

The symmetric and asymmetric parts of $\succcurlyeq^z$ are denoted by $\sim^z$ and $\succ^z$ respectively, where $\boldsymbol{x} \sim^z \boldsymbol{y}$ means that $\boldsymbol{x}$ and $\boldsymbol{y}$ are equivalent in terms of poverty, and $\boldsymbol{x} \succ^z \boldsymbol{y}$ means that $\boldsymbol{x}$ exhibits strictly more poverty than $\boldsymbol{y}$.

Our characterization of poverty measures will be based on a minimal set of axioms.

We say that an income distribution $\boldsymbol{y} = (y_1, \ldots, y_n)$ is obtained from another distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ by an *increment* if there exists i such that $x_j = y_j$ for all $j \neq i$ and $y_i > x_i$.

Recall that a person is considered non-poor if their income exceeds the poverty line z.

[16] A welfare analysis of income tax schemes based on axioms of responsibility and poverty reduction can be found in Henry de Frahan and Maniquet (2021).

Focus (F) (Sen, 1981)**.** $\boldsymbol{x} \sim^z \boldsymbol{y}$ whenever $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by an increment to a non-poor person.

The Focus axiom asserts that poverty measures should be insensitive to changes in income above the poverty line. Specifically, it states that the measured poverty level remains unchanged due to income increases among non-poor individuals.

The next axiom introduces the concept of a *permutation* of an income distribution, which is simply a reordering of individual incomes within the distribution.

Symmetry (S). $\boldsymbol{x} \sim^z \boldsymbol{y}$ whenever $\boldsymbol{y}$ is a permutation of $\boldsymbol{x}$.

This axiom asserts that the poverty measure should be invariant to the ordering of individuals. In other words, it should treat all individuals anonymously, focusing solely on the income levels rather than the identities of the individuals holding them.

The following technical axiom is standard in the poverty measurement literature:

Continuity (C). The poverty ordering $\succcurlyeq^z$ is continuous. Formally, $\boldsymbol{y} \succcurlyeq^z \boldsymbol{x}$ whenever there exist sequences $(\boldsymbol{y}^k)$ and $(\boldsymbol{x}^k)$ such that $\boldsymbol{y}^k \succcurlyeq^z \boldsymbol{x}^k$ for all k, $\boldsymbol{x}^k \to \boldsymbol{x}$, and $\boldsymbol{y}^k \to \boldsymbol{y}$.

Intuitively, this axiom ensures that small changes in income distributions do not lead to abrupt reversals in poverty comparisons. It implies that if one distribution is judged to have no more poverty than another, this relationship should be preserved under small perturbations of both distributions.

The following axiom, introduced by Donaldson and Weymark (1986), is based on the concept of a *regressive transfer*, which refers to an income transfer from a poorer individual to a richer individual.

Minimal Transfer (MT). $\boldsymbol{y} \succ^z \boldsymbol{x}$ whenever $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ through a regressive transfer between two poor individuals who remain poor after the transfer.

This axiom asserts that poverty should increase when income is transferred from a poorer individual to a richer one, provided the recipient's new income does not exceed the poverty line.

The last two axioms represent relaxations of the weak monotonicity axiom proposed by Sen (1976). We define a distribution $\boldsymbol{y} = (y_1, \ldots, y_n)$ as being derived from a distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ through a *proportional decrement* in the income of the poor if there exists j such that $0 < x_j \leq z$ and $y_i = \alpha x_i$ for all individuals i with $x_i \leq z$, where $\alpha \in (0, 1)$. In other words, there is a poor person in $\boldsymbol{x}$ whose income is positive, and the incomes of the poor individuals in $\boldsymbol{x}$ are scaled down by a proportional factor α.

Relative Weak Monotonicity (RWM). $\boldsymbol{y} \succ^z \boldsymbol{x}$ whenever $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a proportional decrement in the income of the poor.

A corresponding axiom can be formulated in terms of additive decrements in the income of the poor. A distribution $\boldsymbol{y} = (y_1, \ldots, y_n)$ is derived from a distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ through a *translational decrement* in the income of the poor if there exists j such that $0 < x_j \leq z$ and $y_i = x_i - \alpha$ for all individuals i with $x_i \leq z$, where $\alpha > 0$. In this case, the incomes of the poor individuals in $\boldsymbol{x}$ are reduced by an additive factor of α.

Absolute Weak Monotonicity (AWM). $\boldsymbol{y} \succ^z \boldsymbol{x}$ whenever $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a translational decrement in the income of the poor.

We first show that the poverty pre-order $\succcurlyeq_R^z$ (respectively, $\succcurlyeq_A^z$) is reflexive and transitive and satisfies F, S, C, MT, and RWM (respectively, F, S, C, MT, and AWM).

Lemma 1. *The poverty pre-order $\succcurlyeq_R^z$ is reflexive and transitive and satisfies F, S, C, MT, and RWM.*

Proof. $\succcurlyeq_R^z$ inherits reflexivity and transitivity from $\succcurlyeq_{RS}$.

[F] Suppose that $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by an increment to a non-poor person. Then $\boldsymbol{y}^z = \boldsymbol{x}^z$, and so $\boldsymbol{y}^z \sim_{RS} \boldsymbol{x}^z$, i.e., $\boldsymbol{y} \sim_R^z \boldsymbol{x}$.

[S] Let $\boldsymbol{y}$ be a permutation of $\boldsymbol{x}$. Then $\boldsymbol{y}^z$ is a permutation of $\boldsymbol{x}^z$ and both $\boldsymbol{y}^z$ and $\boldsymbol{x}^z$ have the same total income. Since $\succcurlyeq_{RL}$ satisfies S, we have $\boldsymbol{x}^z \sim_{RL} \boldsymbol{y}^z$. Hence, $\boldsymbol{y}^z \sim_{RS} \boldsymbol{x}^z$, i.e., $\boldsymbol{y} \sim_R^z \boldsymbol{x}$.

[C] Suppose that $\boldsymbol{x}^k \to \boldsymbol{x}$, $\boldsymbol{y}^k \to \boldsymbol{y}$, and $\boldsymbol{y}^k \succcurlyeq_R^z \boldsymbol{x}^k$ for each k. By contradiction, suppose that $\boldsymbol{y} \not\succcurlyeq_R^z \boldsymbol{x}$. Then $\boldsymbol{x}^z \not\succcurlyeq_{RS} \boldsymbol{y}^z$, implying that either

$$\frac{\sum_{i=1}^{l} x_i^z}{\sum_{i=1}^{n} x_i^z} < \frac{\sum_{i=1}^{l} y_i^z}{\sum_{i=1}^{n} y_i^z}, \quad \text{some } l, \tag{2}$$

or

$$\sum_{i=1}^{n} x_i^z < \sum_{i=1}^{n} y_i^z. \tag{3}$$

Define the sequences

$$(\hat{\boldsymbol{x}}^k) = (\hat{x}_1^k, \ldots, \hat{x}_n^k) \quad \text{and} \quad (\hat{\boldsymbol{y}}^k) = (\hat{y}_1^k, \ldots, \hat{y}_n^k)$$

as follows: for each i,

$$\hat{x}_i^k = \begin{cases} x_i^k & \text{if } x_i^k \le z, \\ z & \text{if } x_i^k > z, \end{cases} \quad \text{and} \quad \hat{y}_i^k = \begin{cases} y_i^k & \text{if } y_i^k \le z, \\ z & \text{if } y_i^k > z. \end{cases}$$

Note that

$$\hat{\boldsymbol{x}}^k \to \boldsymbol{x}^z \quad \text{and} \quad \hat{\boldsymbol{y}}^k \to \boldsymbol{y}^z.$$

Therefore, if (2) holds, then there exists a large enough k such that

$$\frac{\sum_{i=1}^{l} \hat{x}_i^k}{\sum_{i=1}^{n} \hat{x}_i^k} < \frac{\sum_{i=1}^{l} \hat{y}_i^k}{\sum_{i=1}^{n} \hat{y}_i^k},$$

implying that $\hat{\boldsymbol{x}}^k \not\succcurlyeq_{RS} \hat{\boldsymbol{y}}^k$, and hence $\boldsymbol{y}^k \not\succcurlyeq_R^z \boldsymbol{x}^k$, a contradiction.

Similarly, if (3) holds, there is a large enough k such that

$$\sum_{i=1}^{n} \hat{x}_i^k < \sum_{i=1}^{n} \hat{y}_i^k,$$

implying that $\hat{\boldsymbol{x}}^k \not\succcurlyeq_{RS} \hat{\boldsymbol{y}}^k$, and hence $\boldsymbol{y}^k \not\succcurlyeq_R^z \boldsymbol{x}^k$, a contradiction.

[MT] Let $\boldsymbol{y}$ be obtained from $\boldsymbol{x}$ through a regressive transfer between two poor individuals who both remain poor after the transfer. Then $\boldsymbol{y}^z$ can be obtained from $\boldsymbol{x}^z$ through a regressive transfer and both $\boldsymbol{y}^z$ and $\boldsymbol{x}^z$ have the same total income. Now Lemma

3 in Foster (1985) implies that $\boldsymbol{x}^z \succ_{RL} \boldsymbol{y}^z$. Since both $\boldsymbol{y}^z$ and $\boldsymbol{x}^z$ have the same total income, it follows that $\boldsymbol{x}^z \succ_{RS} \boldsymbol{y}^z$, i.e., $\boldsymbol{y} \succ_R^z \boldsymbol{x}$.

[RWM] Suppose that $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a proportional decrement in the income of the poor. Suppose that the incomes below or at the poverty line in $\boldsymbol{x}$ are

$$x_1 \leq \cdots \leq x_j \leq z$$

and $x_{j'} > z$ for all $j' > j$. Then

$$\boldsymbol{x}^z = (x_1, \ldots, x_j, z, \ldots, z) \quad \text{and} \quad \boldsymbol{y}^z = (\alpha x_1, \ldots, \alpha x_j, z, \ldots, z),$$

implying that

$$\sum_i x_i^z > \sum_i y_i^z \tag{4}$$

(since there exists i such that $z \geq x_i > \alpha x_i$).

We claim that $\boldsymbol{x}^z \succcurlyeq_{RL} \boldsymbol{y}^z$, i.e.,

$$\frac{\sum_{i=1}^k x_i^z}{\sum_{i=1}^n x_i^z} \geq \frac{\sum_{i=1}^k y_i^z}{\sum_{i=1}^n y_i^z}, \quad \text{for all } k \in \{1, \ldots, n\}.$$

Indeed, for $k \leq j$ we have

$$\frac{\sum_{i=1}^k x_i^z}{\sum_{i=1}^n x_i^z} = \frac{\sum_{i=1}^k x_i}{\sum_{i=1}^j x_i + (n-j)z} \geq \frac{\sum_{i=1}^k (\alpha x_i)}{\sum_{i=1}^j (\alpha x_i) + (n-j)(\alpha z)} \geq \frac{\sum_{i=1}^k (\alpha x_i)}{\sum_{i=1}^j (\alpha x_i) + (n-j)z} = \frac{\sum_{i=1}^k y_i^z}{\sum_{i=1}^n y_i^z}.$$

For $k > j$,

$$\begin{aligned}\frac{\sum_{i=1}^k x_i^z}{\sum_{i=1}^n x_i^z} &= \frac{\sum_{i=1}^j x_i + (k-j)z}{\sum_{i=1}^j x_i + (n-j)z} \\ &\geq \frac{\sum_{i=1}^j (\alpha x_i) + (k-j)(\alpha z)}{\sum_{i=1}^j (\alpha x_i) + (n-j)(\alpha z)} \geq \frac{\sum_{i=1}^j (\alpha x_i) + (k-j)z}{\sum_{i=1}^j (\alpha x_i) + (n-j)z} = \frac{\sum_{i=1}^k y_i^z}{\sum_{i=1}^n y_i^z}.\text{[17]}\end{aligned}$$

Since $\boldsymbol{x}^z \succcurlyeq_{RL} \boldsymbol{y}^z$ and (4) holds, we see that $\boldsymbol{x}^z \succ_{RS} \boldsymbol{y}^z$, i.e., $\boldsymbol{y} \succ_R^z \boldsymbol{x}$. ■

[17] The second inequality follows from the fact that the map

$$a > 0 \mapsto f(a) = \frac{\sum_{i=1}^j (\alpha x_i) + (k-j)a}{\sum_{i=1}^j (\alpha x_i) + (n-j)a}$$

is non-increasing, since

$$f'(a) = \frac{(k-j)(\sum_{i=1}^j (\alpha x_i) + (n-j)a) - (n-j)(\sum_{i=1}^j (\alpha x_i) + (k-j)a)}{(\sum_{i=1}^j (\alpha x_i) + (n-j)a)^2}$$

and

$$f'(a) \leq 0 \Leftrightarrow (k-j)\left(\sum_{i=1}^j (\alpha x_i)\right) \leq (n-j)\left(\sum_{i=1}^j (\alpha x_i)\right).$$

Lemma 2. *The poverty pre-order $\succcurlyeq_A^z$ is reflexive and transitive and satisfies F, S, C, MT, and AWM.*

Proof. $\succcurlyeq_A^z$ inherits reflexivity and transitivity from $\succcurlyeq_{AS}$.

[F] Suppose that $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by an increment to a non-poor person. Then $\boldsymbol{y}^z = \boldsymbol{x}^z$, and so $\boldsymbol{y}^z \sim_{AS} \boldsymbol{x}^z$, i.e., $\boldsymbol{y} \sim_A^z \boldsymbol{x}$.

[S] Let $\boldsymbol{y}$ be a permutation of $\boldsymbol{x}$. Then $\boldsymbol{y}^z$ is a permutation of $\boldsymbol{x}^z$ and both $\boldsymbol{y}^z$ and $\boldsymbol{x}^z$ have the same total income. Since $\succcurlyeq_{AL}$ satisfies S, we have $\boldsymbol{x}^z \sim_{AL} \boldsymbol{y}^z$. Hence, $\boldsymbol{y}^z \sim_{AS} \boldsymbol{x}^z$, i.e., $\boldsymbol{y} \sim_A^z \boldsymbol{x}$.

[C] Suppose that $\boldsymbol{x}^k \to \boldsymbol{x}$, $\boldsymbol{y}^k \to \boldsymbol{y}$, and $\boldsymbol{y}^k \succcurlyeq_A^z \boldsymbol{x}^k$ for each k. By contradiction, suppose that $\boldsymbol{y} \not\succcurlyeq_A^z \boldsymbol{x}$. Then $\boldsymbol{x}^z \not\succcurlyeq_{AS} \boldsymbol{y}^z$, implying that either

$$\sum_{i=1}^{l}(x_i^z - \mu_{\boldsymbol{x}^z}) < \sum_{i=1}^{l}(y_i^z - \mu_{\boldsymbol{y}^z}), \quad \text{some } l, \tag{5}$$

or

$$\sum_{i=1}^{n} x_i^z < \sum_{i=1}^{n} y_i^z. \tag{6}$$

Define the sequences

$$(\hat{\boldsymbol{x}}^k) = (\hat{x}_1^k, \ldots, \hat{x}_n^k) \quad \text{and} \quad (\hat{\boldsymbol{y}}^k) = (\hat{y}_1^k, \ldots, \hat{y}_n^k)$$

as follows: for each i,

$$\hat{x}_i^k = \begin{cases} x_i^k & \text{if } x_i^k \le z, \\ z & \text{if } x_i^k > z, \end{cases} \quad \text{and} \quad \hat{y}_i^k = \begin{cases} y_i^k & \text{if } y_i^k \le z, \\ z & \text{if } y_i^k > z. \end{cases}$$

Note that

$$\hat{\boldsymbol{x}}^k \to \boldsymbol{x}^z \quad \text{and} \quad \hat{\boldsymbol{y}}^k \to \boldsymbol{y}^z.$$

Therefore, if (5) holds, then there exists a large enough k such that

$$\sum_{i=1}^{l}(\hat{x}_i^k - \mu_{\hat{\boldsymbol{x}}^k}) < \sum_{i=1}^{l}(\hat{y}_i^k - \mu_{\hat{\boldsymbol{y}}^k}),$$

implying that $\hat{\boldsymbol{x}}^k \not\succcurlyeq_{AS} \hat{\boldsymbol{y}}^k$, and hence $\boldsymbol{y}^k \not\succcurlyeq_A^z \boldsymbol{x}^k$, a contradiction.

Similarly, if (6) holds, there is a large enough k such that

$$\sum_{i=1}^{n} \hat{x}_i^k < \sum_{i=1}^{n} \hat{y}_i^k,$$

implying that $\hat{\boldsymbol{x}}^k \not\succcurlyeq_{RS} \hat{\boldsymbol{y}}^k$, and hence $\boldsymbol{y}^k \not\succcurlyeq_R^z \boldsymbol{x}^k$, a contradiction.

[MT] Let $\boldsymbol{y}$ be obtained from $\boldsymbol{x}$ through a regressive transfer between two poor individuals who both remain poor after the transfer. Then $\boldsymbol{y}^z$ can be obtained from $\boldsymbol{x}^z$ through a regressive transfer and both $\boldsymbol{y}^z$ and $\boldsymbol{x}^z$ have the same total income. Now Lemma 3 in Foster (1985) implies that $\boldsymbol{x}^z \succ_{RL} \boldsymbol{y}^z$. Hence, because the absolute Lorenz pre-order is equivalent to the relative Lorenz pre-order for pairs of distributions with the same total income, we have $\boldsymbol{x}^z \succ_{AL} \boldsymbol{y}^z$. Since both $\boldsymbol{y}^z$ and $\boldsymbol{x}^z$ have the same total income, it follows that $\boldsymbol{x}^z \succ_{AS} \boldsymbol{y}^z$, i.e., $\boldsymbol{y} \succ_A^z \boldsymbol{x}$.

[AWM] Suppose that $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a translational decrement in the income of the poor. Suppose that the incomes below or at the poverty line in $\boldsymbol{x}$ are

$$x_1 \leq \cdots \leq x_j \leq z$$

and $x_{j'} > z$ for all $j' > j$. Then

$$\boldsymbol{x}^z = (x_1, \ldots, x_j, z, \ldots, z) \quad \text{and} \quad \boldsymbol{y}^z = (x_1 - \alpha, \ldots, x_j - \alpha, z, \ldots, z),$$

implying that

$$\sum_i x_i^z > \sum_i y_i^z \tag{7}$$

(since there exists i such that $z \geq x_i > x_i - \alpha$).

We claim that $\boldsymbol{x}^z \succcurlyeq_{AL} \boldsymbol{y}^z$, i.e.,

$$\sum_{i=1}^k (x_i^z - \mu_{\boldsymbol{x}^z}) \geq \sum_{i=1}^k (y_i^z - \mu_{\boldsymbol{y}^z}), \quad \text{for all } k \in \{1, \ldots, n\},$$

with the inequality strict for some k. Indeed, for $k \leq j$ we have

$$\begin{aligned}
\sum_{i=1}^k (x_i^z - \mu_{\boldsymbol{x}^z}) &= \sum_{i=1}^k \left(x_i - \frac{1}{n} \left(\sum_{i'=1}^j x_{i'} + (n-j)z \right) \right) \\
&= \sum_{i=1}^k \left((x_i - \alpha) - \frac{1}{n} \left(\sum_{i'=1}^j (x_{i'} - \alpha) + (n-j)(z-\alpha) \right) \right) \\
&\geq \sum_{i=1}^k \left((x_i - \alpha) - \frac{1}{n} \left(\sum_{i'=1}^j (x_{i'} - \alpha) + (n-j)z \right) \right) = \sum_{i=1}^k (y_i^z - \mu_{\boldsymbol{y}^z}).
\end{aligned}$$

For $k > j$,

$$\begin{aligned}
\sum_{i=1}^k (x_i^z - \mu_{\boldsymbol{x}^z}) &= \sum_{i=1}^j \left(x_i - \frac{1}{n} \left(\sum_{i'=1}^j x_{i'} + (n-j)z \right) \right) + (k-j) \left(z - \frac{1}{n} \left(\sum_{i'=1}^j x_{i'} + (n-j)z \right) \right) \\
&= \sum_{i=1}^j \left((x_i - \alpha) - \frac{1}{n} \left(\sum_{i'=1}^j (x_{i'} - \alpha) + (n-j)(z-\alpha) \right) \right) \\
&\qquad + (k-j) \left((z-\alpha) - \frac{1}{n} \left(\sum_{i'=1}^j (x_{i'} - \alpha) + (n-j)(z-\alpha) \right) \right) \\
&\geq \sum_{i=1}^j \left((x_i - \alpha) - \frac{1}{n} \left(\sum_{i'=1}^j (x_{i'} - \alpha) + (n-j)z \right) \right) \\
&\qquad + (k-j) \left(z - \frac{1}{n} \left(\sum_{i'=1}^j (x_{i'} - \alpha) + (n-j)z \right) \right)
\end{aligned}$$

$$= \sum_{i=1}^{k} (y_i^z - \mu_{y^z}).^{18}$$

Since $x^z \succcurlyeq_{AL} y^z$ and (7) holds, we see that $x^z \succ_{AS} y^z$, i.e., $y \succ_A^z x$. ■

We now state and prove two lemmas that play an essential role in the axiomatization of the poverty measures $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$.

We say that an income distribution y is obtained from another distribution x through a *proportional decrement* if $y_i = \alpha x_i$ for all i and some $\alpha \in (0, 1)$.

Lemma 3. *$y \succ_R^z x$ if and only if y^z can be obtained from x^z through a finite sequence of successive regressive transfers and/or proportional decrements.*

Proof. Suppose that y^z can be obtained from x^z through a finite sequence of successive regressive transfers and/or proportional decrements. Then, since $\succcurlyeq_R^z$ satisfies MT and RWM (Lemma 1), we have

$$y^z \succ_R^z a^1 \succ_R^z \cdots \succ_R^z a^k \succ_R^z x^z \tag{8}$$

for some finite sequence of income distributions $a^1, \ldots, a^k$. Because $\succcurlyeq_R^z$ is reflexive and transitive (Lemma 1), the asymmetric part $\succ_R^z$ is also transitive (see, e.g., Sen, 2017, Lemma 1*a, p. 56). Therefore, (8) gives $y^z \succ_R^z x^z$, which implies $y \succ_R^z x$.

Conversely, suppose that $y \succ_R^z x$. Then $x^z \succ_{RS} y^z$, which implies

$$\sum_i x_i^z \geq \sum_i y_i^z. \tag{9}$$

Define

$$\tilde{x} = \left(\frac{\sum_i y_i^z}{\sum_i x_i^z} \right) x^z. \tag{10}$$

Note that $\tilde{x}$ and y^z have the same total income.

Since $x^z \succ_{RS} y^z$, we have $x^z \succcurlyeq_{RL} y^z$. Therefore, since the relative Lorenz pre-order $\succcurlyeq_{RL}$ is scale invariant,

$$\tilde{x} \sim_{RL} x^z \succcurlyeq_{RL} y^z,$$

implying $\tilde{x} \succcurlyeq_{RL} y^z$ (by reflexivity and transitivity of $\succcurlyeq_{RL}$ and Lemma 1*a in Sen (2017)).

There are two cases to consider: $\tilde{x} \sim_{RL} y^z$ and $\tilde{x} \succ_{RL} y^z$. The first case implies $\tilde{x} = y^z$, since both $\tilde{x}$ and y^z have the same total income. But then (9) holds with strict inequality and (10) implies that $y^z = \tilde{x}$ can be obtained from x^z through a proportional decrement.

If $\tilde{x} \succ_{RL} y^z$, then (since both $\tilde{x}$ and y^z have the same total income) Lemma 3 in Foster (1985) implies that y^z can be obtained from $\tilde{x}$ by a finite sequence of regressive transfers.

[18]The inequality holds because the map

$$a > 0 \mapsto g(a) = \sum_{i=1}^{j} \left((x_i - \alpha) - \frac{1}{n} \left(\sum_{i'=1}^{j} (x_{i'} - \alpha) + (n-j)a \right) \right) + (k-j) \left(a - \frac{1}{n} \left(\sum_{i'=1}^{j} (x_{i'} - \alpha) + (n-j)a \right) \right)$$

is non-increasing. Indeed,

$$g'(a) = k - j - \frac{k(n-j)}{n} \leq 0 \Leftrightarrow n \geq k.$$

Since $\tilde{\boldsymbol{x}}$ can be obtained from $\boldsymbol{x}^z$ through a proportional decrement (whenever $\tilde{\boldsymbol{x}} \neq \boldsymbol{x}^z$), it follows that $\boldsymbol{y}^z$ can be obtained from $\boldsymbol{x}^z$ through a finite sequence of successive regressive transfers and/or proportional decrements. ■

An income distribution $\boldsymbol{y}$ is obtained from another distribution $\boldsymbol{x}$ through a *translational decrement* if $y_i = x_i - \alpha$ for all i and some $\alpha > 0$.

Lemma 4. *$\boldsymbol{y} \succ_A^z \boldsymbol{x}$ if and only if $\boldsymbol{y}^z$ can be obtained from $\boldsymbol{x}^z$ through a finite sequence of successive regressive transfers and/or translational decrements.*

Proof. Suppose that $\boldsymbol{y}^z$ can be obtained from $\boldsymbol{x}^z$ through a finite sequence of successive regressive transfers and/or translational decrements. Then, since $\succcurlyeq_A^z$ satisfies MT and AWM (Lemma 2), we have

$$\boldsymbol{y}^z \succ_A^z \boldsymbol{a}^1 \succ_A^z \cdots \succ_A^z \boldsymbol{a}^k \succ_A^z \boldsymbol{x}^z \tag{11}$$

for some finite sequence of income distributions $\boldsymbol{a}^1, \ldots, \boldsymbol{a}^k$. Because $\succcurlyeq_A^z$ is reflexive and transitive (Lemma 2), the asymmetric part $\succ_A^z$ is also transitive (see, e.g., Sen, 2017, Lemma 1*a, p. 56). Therefore, (11) gives $\boldsymbol{y}^z \succ_A^z \boldsymbol{x}^z$, which implies $\boldsymbol{y} \succ_A^z \boldsymbol{x}$.

Conversely, suppose that $\boldsymbol{y} \succ_A^z \boldsymbol{x}$. Then $\boldsymbol{x}^z \succ_{AS} \boldsymbol{y}^z$, which implies

$$\sum_i x_i^z \geq \sum_i y_i^z. \tag{12}$$

Define

$$\tilde{\boldsymbol{y}} = (y_1^z + \alpha, \ldots, y_n^z + \alpha),$$

where

$$\alpha = \frac{1}{n}\left(\sum_i x_i^z - \sum_i y_i^z\right).$$

Note that $\tilde{\boldsymbol{y}}$ and $\boldsymbol{x}^z$ have the same total income and that $\boldsymbol{y}^z$ can be obtained from $\tilde{\boldsymbol{y}}$ through a translational decrement if (12) holds with strict inequality.

Since $\boldsymbol{x}^z \succ_{AS} \boldsymbol{y}^z$, we have $\boldsymbol{x}^z \succcurlyeq_{AL} \boldsymbol{y}^z$. Therefore, since the absolute Lorenz pre-order $\succcurlyeq_{AL}$ is translation invariant,

$$\boldsymbol{x}^z \succcurlyeq_{AL} \boldsymbol{y}^z \sim_{AL} \tilde{\boldsymbol{y}},$$

implying $\boldsymbol{x}^z \succcurlyeq_{AL} \tilde{\boldsymbol{y}}$ (by reflexivity and transitivity of $\succcurlyeq_{AL}$ and Lemma 1*a in Sen (2017)).

There are two cases to consider: $\boldsymbol{x}^z \sim_{AL} \tilde{\boldsymbol{y}}$ and $\boldsymbol{x}^z \succ_{AL} \tilde{\boldsymbol{y}}$. The first case implies $\boldsymbol{x}^z = \tilde{\boldsymbol{y}}$, since both $\tilde{\boldsymbol{y}}$ and $\boldsymbol{x}^z$ have the same total income. But then (12) holds with strict inequality and $\boldsymbol{y}^z$ can be obtained from $\boldsymbol{x}^z = \tilde{\boldsymbol{y}}$ through a translational decrement.

If $\boldsymbol{x}^z \succ_{AL} \tilde{\boldsymbol{y}}$, then (since both $\tilde{\boldsymbol{y}}$ and $\boldsymbol{x}^z$ have the same total income) $\boldsymbol{x}^z \succ_{RL} \tilde{\boldsymbol{y}}$ and so Lemma 3 in Foster (1985) implies that $\tilde{\boldsymbol{y}}$ can be obtained from $\boldsymbol{x}^z$ by a finite sequence of regressive transfers. Since $\boldsymbol{y}^z$ can be obtained from $\tilde{\boldsymbol{y}}$ through a translational decrement (whenever $\boldsymbol{y}^z \neq \tilde{\boldsymbol{y}}$), it follows that $\boldsymbol{y}^z$ can be obtained from $\boldsymbol{x}^z$ through a finite sequence of successive regressive transfers and/or translational decrements. ■

We aim to characterize poverty measures consistent with $\succcurlyeq_R^z$ and $\succcurlyeq_A^z$ in the following precise sense.

Definition 1. A poverty pre-order $\succcurlyeq'$ is said to be *$\succcurlyeq$-consistent*, where $\succcurlyeq$ is another poverty pre-order, if the following two conditions are satisfied for any two income distributions $\boldsymbol{x}$ and $\boldsymbol{y}$:

- $\boldsymbol{x} \succ \boldsymbol{y} \Rightarrow \boldsymbol{x} \succ' \boldsymbol{y}$.
- $\boldsymbol{x} \sim \boldsymbol{y} \Rightarrow \boldsymbol{x} \sim' \boldsymbol{y}$.

The following theorems present the principal characterizations of $\succcurlyeq^z_R$-consistent and $\succcurlyeq^z_A$-consistent poverty pre-orders, expressed in terms of the aforementioned axioms.

Theorem 3. *Given a poverty line $z > 0$, a reflexive and transitive poverty pre-order $\succcurlyeq^z$ satisfies F, S, C, MT, and RWM if and only if it is $\succcurlyeq^z_R$-consistent.*

Proof. [*Sufficiency*.] Suppose that $\succcurlyeq^z$ is $\succcurlyeq^z_R$-consistent. By Lemma 1, $\succcurlyeq^z_R$ satisfies F, S, C, MT, and RWM. By $\succcurlyeq^z_R$-consistency, so does $\succcurlyeq^z$.

[*Necessity*.] Suppose that $\succcurlyeq^z$ satisfies F, S, C, MT, and RWM.

First, suppose that $\boldsymbol{x} \sim^z_R \boldsymbol{y}$. We must show that $\boldsymbol{x} \sim^z \boldsymbol{y}$. Because $\boldsymbol{x} \sim^z_R \boldsymbol{y}$, we have $\boldsymbol{x}^z \sim_{RS} \boldsymbol{y}^z$, implying $\boldsymbol{x}^z \sim_{RL} \boldsymbol{y}^z$ and

$$\sum_i x^z_i = \sum_i y^z_i.$$

Consequently, $\boldsymbol{x}^z = \boldsymbol{y}^z$, whence $\boldsymbol{x}^z \sim^z \boldsymbol{y}^z$ (by S and reflexivity of $\succcurlyeq^z$).[19]

Note that there are perturbations $\tilde{\boldsymbol{x}}$ of $\boldsymbol{x}^z$ (respectively, $\tilde{\boldsymbol{y}}$ of $\boldsymbol{y}^z$) arbitrarily close to $\boldsymbol{x}^z$ (respectively, $\boldsymbol{y}^z$) with the following properties: (i) the set of non-poor persons in $\tilde{\boldsymbol{x}}$ (respectively, $\tilde{\boldsymbol{y}}$) is the same as the set of non-poor persons in $\boldsymbol{x}$ (respectively, $\boldsymbol{y}$); and (ii) for any non-poor person i in $\tilde{\boldsymbol{x}}$ (respectively, $\tilde{\boldsymbol{y}}$), i's income in $\tilde{\boldsymbol{x}}$ (respectively, $\tilde{\boldsymbol{y}}$) is less than i's income in $\boldsymbol{x}$ (respectively, $\boldsymbol{y}$).

Note that, for any such perturbations $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$, we have, by the F axiom,

$$\tilde{\boldsymbol{x}} \sim^z \boldsymbol{x} \quad \text{and} \quad \tilde{\boldsymbol{y}} \sim^z \boldsymbol{y}. \tag{13}$$

Moreover, because $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$ can be taken arbitrarily close to $\boldsymbol{x}^z$ and $\boldsymbol{y}^z$, respectively, and since (13) holds, the C axiom gives

$$\boldsymbol{x}^z \sim^z \boldsymbol{x} \quad \text{and} \quad \boldsymbol{y}^z \sim^z \boldsymbol{y}.$$

Consequently,

$$\boldsymbol{x} \sim^z \boldsymbol{x}^z \sim^z \boldsymbol{y}^z \sim^z \boldsymbol{y}. \tag{14}$$

Since $\succcurlyeq^z$ is reflexive and transitive, so is $\sim^z$ (Sen, 2017, Lemma 1*a). Therefore, (14) implies $\boldsymbol{x} \sim^z \boldsymbol{y}$, as we sought.

It remains to show that $\boldsymbol{x} \succ^z_R \boldsymbol{y}$ implies $\boldsymbol{x} \succ^z \boldsymbol{y}$. If $\boldsymbol{x} \succ^z_R \boldsymbol{y}$, Lemma 3 implies that $\boldsymbol{x}^z$ can be obtained from $\boldsymbol{y}^z$ through a finite sequence of successive regressive transfers and/or proportional decrements. Since all the indiviuals in the censored distributions $\boldsymbol{x}^z$ and $\boldsymbol{y}^z$ are poor, and since $\succcurlyeq^z$ satisfies MT and RWM, it follows that there is a finite sequence of income distributions $\boldsymbol{a}^1, \ldots, \boldsymbol{a}^k$ such that

$$\boldsymbol{x}^z \succ^z \boldsymbol{a}^1 \succ^z \cdots \succ^z \boldsymbol{a}^k \succ^z \boldsymbol{y}^z.$$

[19]Since the equality $\boldsymbol{x}^z = \boldsymbol{y}^z$ refers to these vectors with their coordinates arranged in non-decreasing order, we must invoke S to conclude that $\boldsymbol{x}^z \sim^z \boldsymbol{y}^z$, as the original vectors may have had different orderings.

This implies $\boldsymbol{x}^z \succ^z \boldsymbol{y}^z$ (since $\succ^z$ is transitive by reflexivity and transitivity of $\succcurlyeq^z$ (Sen, 2017, Lemma 1*a)). Using the continuity argument from the previous paragraph, we see that

$$\boldsymbol{x} \sim^z \boldsymbol{x}^z \succ^z \boldsymbol{y}^z \sim^z \boldsymbol{y},$$

which implies $\boldsymbol{x} \succ^z \boldsymbol{y}$ by transitivity of $\succ^z$. This completes the proof. ■

The axiomatization of $\succcurlyeq^z_A$ can be established through a process that closely parallels the proof of Theorem 3. Given the similarity in approach, we omit the detailed proof.

Theorem 4. *Given a poverty line $z > 0$, a reflexive and transitive poverty pre-order $\succcurlyeq^z$ satisfies F, S, C, MT, and AWM if and only if it is $\succcurlyeq^z_A$-consistent.*

B. Proofs of main theorems

B.1. Proof of Theorem 1

The proof of Theorem 1 relies on the following two results. The first result is well-known:

Lemma 5. *Suppose that $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ are two income distributions with*

$$0 \leq x_1 \leq \cdots \leq x_n \quad \textit{and} \quad 0 \leq y_1 \leq \cdots \leq y_n.$$

If x_ι is the first positive income level in $\boldsymbol{x}$, then

$$\frac{y_\iota}{x_\iota} \geq \cdots \geq \frac{y_n}{x_n} \Rightarrow \boldsymbol{y} \succcurlyeq_{RL} \boldsymbol{x}.$$

The proof of Lemma 5 can be found in Carbonell-Nicolau (2024, Lemma 1).

Lemma 6. *Suppose that t is rank-preserving on X_t. For any income distribution*

$$0 \leq x_1 \leq \cdots \leq x_n$$

such that $x_i - t(x_i) < z$ for some i, there exists $j \in \{1, \ldots, n\}$ such that

$$x_1 - t(x_1) \leq \cdots \leq x_j - t(x_j) < z$$

and $x_{j'} - t(x_{j'}) \geq z$ for any $j' > j$.

Proof. Fix any income distribution

$$0 \leq x_1 \leq \cdots \leq x_n$$

with $x_i - t(x_i) < z$ for some i. If

$$x_\iota - t(x_\iota) < z, \quad \text{for all } \iota,$$

then, since t is rank-preserving on X_t, we have

$$x_1 - t(x_1) \leq \cdots \leq x_n - t(x_n) < z.$$

If $x_\iota - t(x_\iota) \geq z$ for some ι, let ι' be the lowest ι for which $x_\iota - t(x_\iota) \geq z$. Then

$$x_{j'} - t(x_{j'}) \geq z, \quad \text{for all } j' \geq \iota'. \tag{15}$$

Indeed, if $x_{j'} - t(x_{j'}) < z$ for some $j' > \iota'$ we have $x_{\iota'} \leq x_{j'}$ and the Intermediate Value Theorem gives $x^* \in [x_{\iota'}, x_{j'}]$ such that the continuous map

$$x \in [x_{\iota'}, x_{j'}] \mapsto x - t(x)$$

satisfies $x^* - t(x^*) = z$. Hence,

$$x^* - t(x^*) = z > x_{j'} - t(x_{j'}),$$

which violates the rank-preserving property.

Since ι' is the lowest ι for which $x_\iota - t(x_\iota) \geq z$ and $x_i - t(x_i) < z$ for some i, (15) implies that $\iota' \in \{2, \ldots, n\}$. Thus, we have

$$\begin{aligned} x_{j'} - t(x_{j'}) &\geq z, \quad \text{for all } j' \geq \iota' \geq 2, \\ x_{j'} - t(x_{j'}) &< z, \quad \text{for all } j' < \iota'. \end{aligned}$$

Since

$$x_1 - t(x_1) \leq \cdots \leq x_{\iota'-1} - t(x_{\iota'-1}) < z$$

by the rank-preserving property, we see that the lemma holds for $j = \iota' - 1$. ■

Theorem 1. *A tax schedule t is poverty-reducing in the relative sense if and only if it is rank-preserving on X_t, average-rate progressive on X_t, and preserves the non-poor status of individuals.*

Proof. Suppose that t is poverty-reducing in the relative sense. We first demonstrate that t preserves the non-poor status of individuals.

Proceeding by contradiction, suppose that there exists $x > z$ such that $x - t(x) < z$. Consider the income distribution $\boldsymbol{x} = (x, \ldots, x)$. Then

$$(x - t(x), \ldots, x - t(x)) \not\succcurlyeq_{RS} \boldsymbol{x}^z,$$

since $z > x - t(x)$. Hence,

$$\boldsymbol{x} \not\succcurlyeq_R^z (x - t(x), \ldots, x - t(x)),$$

so t is not poverty-reducing, a contradiction.

We now show that t is average-rate progressive on X_t. To this end, we choose arbitrary $x, y \in X_t$ with $x < y$ and show that $t(x)/x \leq t(y)/y$.

Define the distribution

$$\boldsymbol{x}' = (x_1', \ldots, x_n') = (x, y, \ldots, y).$$

Since t is poverty-reducing, we have

$$(x_1', \ldots, x_n') \succcurlyeq_R^z (x_1' - t(x_1'), \ldots, x_n' - t(x_n')),$$

i.e.,

$$(x_1' - t(x_1'), \ldots, x_n' - t(x_n')) \succcurlyeq_{RS} (x_1', \ldots, x_n') \tag{16}$$

(note that $x_i' \leq z$ and $x_i' - t(x_i') \leq z$ for each i, since $x, y \in X_t$). Consequently,

$$\frac{x_1' - t(x_1')}{\sum_i (x_i' - t(x_i'))} = \frac{x - t(x)}{x - t(x) + (n-1)(y - t(y))} \geq \frac{x_1'}{\sum_i x_i'} = \frac{x}{x + (n-1)y},$$

whence

$$\frac{x - t(x)}{x} \geq \frac{x - t(x) + (n-1)(y - t(y))}{x + (n-1)y}. \tag{17}$$

Moreover, (16) also implies

$$\frac{\sum_{i=1}^{n-1}(x_i' - t(x_i'))}{\sum_{i=1}^{n}(x_i' - t(x_i'))} = \frac{x - t(x) + (n-2)(y - t(y))}{x - t(x) + (n-1)(y - t(y))} \geq \frac{\sum_{i=1}^{n-1} x_i'}{\sum_{i=1}^{n} x_i'} = \frac{x + (n-2)y}{x + (n-1)y},$$

whence

$$1 - \frac{x - t(x) + (n-2)(y - t(y))}{x - t(x) + (n-1)(y - t(y))} \leq 1 - \frac{x + (n-2)y}{x + (n-1)y}.$$

The last inequality can be expressed as

$$\frac{y - t(y)}{y} \leq \frac{x - t(x) + (n-1)(y - t(y))}{x + (n-1)y}.$$

Combining this expression with (17) yields

$$\frac{y - t(y)}{y} \leq \frac{x - t(x)}{x},$$

implying $t(y)/y \geq t(x)/x$, as we sought.

Next, we show that t is rank-preserving on X_t. Proceeding by contradiction, suppose that there exist $0 \leq x < y \leq z$ such that

$$z \geq x - t(x) > y - t(y).$$

Define the distribution

$$\boldsymbol{y}' = (y_1', \ldots, y_n') = (x, \ldots, x, y),$$

whose post-tax income distribution is given by

$$(y - t(y), x - t(x), \ldots, x - t(x)).$$

Because $x < y$ and

$$x - t(x) > y - t(y),$$

we have, for large enough n,

$$\frac{x}{(n-1)x + y} > \frac{y - t(y)}{y - t(y) + (n-1)(x - t(x))}.$$

Consequently,

$$\frac{y_1'}{\sum_i y_i'} = \frac{x}{(n-1)x + y} > \frac{y - t(y)}{y - t(y) + (n-1)(x - t(x))},$$

implying that

$$(y - t(y), x - t(x), \dots, x - t(x)) \not\succcurlyeq_{RL} \boldsymbol{y}'. \tag{18}$$

Because $x - t(x) \le z$ and $y'_i \le z$ for each i, (18) implies that

$$\boldsymbol{y}' \not\succcurlyeq^z_R (y - t(y), x - t(x), \dots, x - t(x)),$$

contradicting that t is poverty-reducing in the relative sense.

Conversely, assume t is rank-preserving on X_t, average-rate progressive on X_t, and preserves the non-poor status of individuals. We must prove that t is poverty-reducing in the relative sense.

Fix any pre-tax income distribution $\boldsymbol{x} = (x_1, \dots, x_n)$ with

$$0 \le x_1 \le \cdots \le x_n.$$

Let

$$\boldsymbol{y} = (y_1, \dots, y_n)$$

be the corresponding post-tax income distribution.

If $y_i \ge z$ for all i, it is easy to see that

$$\boldsymbol{y}^z = (z, \dots, z) \succcurlyeq_{RS} \boldsymbol{x}^z,$$

i.e., $\boldsymbol{x} \succcurlyeq^z_R \boldsymbol{y}$.

Suppose that $y_{i^*} < z$ for some i^*. Then, by Lemma 6, $\boldsymbol{y}$ satisfies the following conditions for some $j \in \{1, \dots, n\}$:

$$y_1 \le \cdots \le y_j < z \quad \text{and} \quad y_{j'} \ge z \text{ for } j' > j. \tag{19}$$

We must show that $\boldsymbol{x} \succcurlyeq^z_R \boldsymbol{y}$, i.e., $\boldsymbol{y}^z \succcurlyeq_{RS} \boldsymbol{x}^z$, which is expressible as

$$(y_1, \dots, y_j, z, \dots, z) \succcurlyeq_{RS} (x^z_1, \dots, x^z_n)$$

by (19).

By Lemma 5, it suffices to show that

$$\frac{y_\iota}{x^z_\iota} \ge \cdots \ge \frac{y_j}{x^z_j} \ge \frac{z}{x^z_{j+1}} \ge \cdots \ge \frac{z}{x^z_n}, \tag{20}$$

where x^z_ι represents the first positive income level in $\boldsymbol{x}^z$, and

$$\sum_i y^z_i \ge \sum_i x^z_i. \tag{21}$$

To see that (20) holds, note first that

$$x^z_i = x_i, \quad \text{for each } i \in \{\iota, \dots, j\}. \tag{22}$$

Indeed, $x_i^z \neq x_i$ for some $i \in \{\iota, \ldots, j\}$ implies $x_i^z = z < x_i$, and so $x_i - t(x_i) \geq z$, since t preserves the nonpoor status of individuals. But this is a contradiction, since

$$z > y_i = x_i - t(x_i).$$

Hence, (20) can be written as

$$\frac{y_\iota}{x_\iota} \geq \cdots \geq \frac{y_j}{x_j} \geq \frac{z}{x_{j+1}^z} \geq \cdots \geq \frac{z}{x_n^z}. \tag{23}$$

In addition, (22) implies that

$$x_\iota \leq \cdots \leq x_j \leq z. \tag{24}$$

Therefore, because t is average-rate progressive on X_t, we have

$$\frac{t(x_\iota)}{x_\iota} \leq \cdots \leq \frac{t(x_j)}{x_j},$$

implying that

$$\frac{x_\iota - t(x_\iota)}{x_\iota} \geq \cdots \geq \frac{x_j - t(x_j)}{x_j},$$

i.e.,

$$\frac{y_\iota}{x_\iota} \geq \cdots \geq \frac{y_j}{x_j}. \tag{25}$$

Next, we show that

$$\frac{y_j}{x_j} \geq \frac{z}{x_{j+1}^z} \geq \cdots \geq \frac{z}{x_n^z}. \tag{26}$$

The inequalities

$$\frac{z}{x_{j+1}^z} \geq \cdots \geq \frac{z}{x_n^z}$$

follow immediately from the inequalities

$$x_{j+1}^z \leq \cdots \leq x_n^z.$$

The inequality $y_j/x_j \geq z/x_{j+1}^z$ is clearly true if $x_{j+1}^z = z$, since

$$\frac{y_j}{x_j} = \frac{x_j - t(x_j)}{x_j} \geq 1,$$

where the last inequality follows from (24) and the fact that $t(x) \leq 0$ for all $x \in [0, z]$ (see Remark 5). If, on the other hand, $x_{j+1}^z \neq z$, then $x_{j+1}^z = x_{j+1}$. Therefore, since

$$x_j - t(x_j) < z \leq x_{j+1} - t(x_{j+1}),$$

and since the map $x \mapsto x - t(x)$ is continuous on $[x_j, x_{j+1}]$, the Intermediate Value Theorem gives $x^* \in [x_j, x_{j+1}]$ such that $x^* - t(x^*) = z$. Consequently,

$$\frac{y_j}{x_j} = \frac{x_j - t(x_j)}{x_j} \geq \frac{x^* - t(x^*)}{x^*} = \frac{z}{x^*} \geq \frac{z}{x_{j+1}} = \frac{z}{x^z_{j+1}},$$

where the first inequality follows from the fact that t is average-rate progressive on X_t.

We have established (25) and (26), and hence (23) and (20). It remains to show that (21) holds.

By (19), we have

$$\sum_{i=1}^{n} y_i^z = \sum_{i=1}^{j} y_i + (n-j)z = \sum_{i=1}^{j} (x_i - t(x_i)) + (n-j)z. \tag{27}$$

Moreover, (22) implies that

$$\sum_{i=1}^{n} x_i^z \leq \sum_{i=1}^{j} x_i + (n-j)z. \tag{28}$$

Because $t(x) \leq 0$ for $x \leq z$ (Remark 5) and (24) holds, (27) and (28) imply

$$\sum_{i=1}^{n} y_i^z \geq \sum_{i=1}^{n} x_i^z,$$

as desired. ∎

B.2. Proof of Theorem 2

Theorem 2. *A tax schedule t is poverty-reducing in the absolute sense if and only if it is non-decreasing on X_t, rank-preserving on X_t, and preserves the non-poor status of individuals.*

Proof. Suppose that t is poverty-reducing in the absolute sense. We first demonstrate that t preserves the non-poor status of individuals. Proceeding by contradiction, suppose that there exists $x > z$ such that $x - t(x) < z$. Consider the income distribution $\boldsymbol{x} = (x, \ldots, x)$. Then

$$(x - t(x), \ldots, x - t(x)) \not\succcurlyeq_{AS} \boldsymbol{x}^z,$$

since $z > x - t(x)$. Hence,

$$\boldsymbol{x} \not\succcurlyeq^z_A (x - t(x), \ldots, x - t(x)),$$

so t is not poverty-reducing, a contradiction.

Next, we show that t is rank-preserving on X_t. Proceeding by contradiction, suppose that there exist $0 \leq x < y \leq z$ such that

$$z \geq x - t(x) > y - t(y).$$

Define the distribution

$$\boldsymbol{y}' = (y'_1, \ldots, y'_n) = (x, \ldots, x, y),$$

whose post-tax income distribution is given by

$$(y - t(y), x - t(x), \dots, x - t(x)).$$

Because $x < y$ and

$$x - t(x) > y - t(y),$$

we have, for large enough n,

$$x - \frac{1}{n}[(n-1)x + y] > y - t(y) - \frac{1}{n}[y - t(y) + (n-1)(x - t(x))].$$

Consequently,

$$y'_1 - \mu_{\mathbf{y}'} = x - \frac{1}{n}[(n-1)x + y] > y - t(y) - \frac{1}{n}[y - t(y) + (n-1)(x - t(x))],$$

implying that

$$(y - t(y), x - t(x), \dots, x - t(x)) \not\succcurlyeq_{AL} \mathbf{y}'. \tag{29}$$

Because $x - t(x) \leq z$ and $y'_i \leq z$ for each i, (29) implies that

$$\mathbf{y}' \not\succcurlyeq^z_A (y - t(y), x - t(x), \dots, x - t(x)),$$

contradicting that t is poverty-reducing in the absolute sense.

We now demonstrate that t is non-decreasing on X_t. Suppose, by contradiction, that $x, y \in X_t$, $x < y$, and $t(x) > t(y)$. Define the distribution

$$\mathbf{x}' = (x'_1, \dots, x'_n) = (x, \dots, x, y),$$

whose post-tax income distribution is given by

$$(x - t(x), \dots, x - t(x), y - t(y))$$

(note that $x < y$ and $t(x) > t(y)$ implies $x - t(x) < y - t(y)$).

Note that

$$x - \frac{1}{n}[(n-1)x + y] > x - t(x) - \frac{1}{n}[(n-1)(x - t(x)) + y - t(y)],$$

since this expression reduces to $t(y) < t(x)$. Consequently,

$$x'_1 - \mu_{\mathbf{x}'} = x - \frac{1}{n}[(n-1)x + y] > x - t(x) - \frac{1}{n}[(n-1)(x - t(x)) + y - t(y)],$$

implying that

$$(x - t(x), \dots, x - t(x), y - t(y)) \not\succcurlyeq_{AL} \mathbf{x}'.$$

Since $x - t(x) \leq z$, $y - t(y) \leq z$, and $x'_i \leq z$ for each i, it follows that

$$\mathbf{x}' \not\succcurlyeq^z_A (x - t(x), \dots, x - t(x), y - t(y)),$$

contradicting that t is poverty-reducing in the absolute sense.

Conversely, assume t is non-decreasing on X_t, rank-preserving on X_t, and preserves the non-poor status of individuals. We must prove that t is poverty-reducing in the absolute sense.

First, recall from Remark 6 that t (weakly) subsidizes the poor, i.e., $t(x) \leq 0$ for all $x \leq z$.

Fix any income distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ with

$$0 \leq x_1 \leq \cdots \leq x_n.$$

The corresponding post-tax income distribution is given by

$$\boldsymbol{y} = (y_1, \ldots, y_n) = (x_1 - t(x_1), \ldots, x_n - t(x_n)).$$

We must show that $\boldsymbol{x} \succcurlyeq^z_A \boldsymbol{y}$, i.e., $\boldsymbol{y}^z \succcurlyeq_{AS} \boldsymbol{x}^z$.

If $y_i \geq z$ for all i, we have

$$\boldsymbol{y}^z = (z, \ldots, z) \succcurlyeq_{AS} \boldsymbol{x}^z,$$

since, for each $k \in \{1, \ldots, n\}$,

$$\sum_{i=1}^{k} (y^z_i - \mu_{\boldsymbol{y}^z}) = 0 \geq \sum_{i=1}^{k} (x^z_i - \mu_{\boldsymbol{x}^z}),$$

and $\sum_i y^z_i = nz \geq \sum_i x^z_i$.

Now suppose that $y_{i^*} < z$ for some i^*. Then, by Lemma 6, $\boldsymbol{y}$ satisfies the following conditions for some $j \in \{1, \ldots, n\}$:

$$y_1 \leq \cdots \leq y_j < z \quad \text{and} \quad y_{j'} \geq z \text{ for } j' > j. \tag{30}$$

We must show that $\boldsymbol{x} \succcurlyeq^z_A \boldsymbol{y}$, i.e., $\boldsymbol{y}^z \succcurlyeq_{AS} \boldsymbol{x}^z$, which is expressible as

$$(y_1, \ldots, y_j, z, \ldots, z) \succcurlyeq_{AS} (x^z_1, \ldots, x^z_n) \tag{31}$$

by (30).

Note that

$$x_1 \leq \cdots \leq x_j \leq z. \tag{32}$$

Indeed, $x_{j'} > z$ for some $j' \in \{1, \ldots, j\}$ implies that $y_{j'} \geq z$ (since t preserves the non-poor status of individuals), which contradicts (30). Hence, (31) is expressible as

$$(y_1, \ldots, y_j, z, \ldots, z) \succcurlyeq_{AS} (x_1, \ldots, x_j, x^z_{j+1}, \ldots, x^z_n).$$

Because t is non-decreasing on X_t, $x_1 \leq \cdots \leq x_j$, and (30) holds, we have

$$t(x_1) \leq \cdots \leq t(x_j),$$

which is expressible as

$$x_1 - y_1 \leq \cdots \leq x_j - y_j,$$

or

$$(x_1 - \mu_{\boldsymbol{x}^z}) - (y_1 - \mu_{\boldsymbol{y}^z}) \leq \cdots \leq (x_j - \mu_{\boldsymbol{x}^z}) - (y_j - \mu_{\boldsymbol{y}^z}). \tag{33}$$

Next, we show that

$$x_j - y_j \leq x^z_{j+1} - z \leq \cdots \leq x^z_n - z. \tag{34}$$

The inequalities

$$x^z_{j+1} - z \leq \cdots \leq x^z_n - z$$

follow immediately from the inequalities

$$x^z_{j+1} \leq \cdots \leq x^z_n.$$

The inequality

$$x_j - y_j \leq x^z_{j+1} - z$$

is clearly true if $x^z_{j+1} = z$, since

$$x_j - y_j = t(x_j) \leq 0$$

(recall that $t(x) \leq 0$ for $x \leq z$). If $x^z_{j+1} \neq z$, then $x^z_{j+1} = x_{j+1}$. Therefore, since

$$x_j - t(x_j) < z \leq x_{j+1} - t(x_{j+1}),$$

and since the map $x \mapsto x - t(x)$ is continuous on $[x_j, x_{j+1}]$, the Intermediate Value Theorem gives $x^* \in [x_j, x_{j+1}]$ such that $x^* - t(x^*) = z$. Consequently,

$$x_j - y_j = t(x_j) \leq t(x^*) = x^* - z \leq x_{j+1} - z = x^z_{j+1} - z,$$

where the first inequality follows from the fact that t is non-decreasing on X_t.

We have established (34), which is expressible as

$$(x_j - \mu_{x^z}) - (y_j - \mu_{y^z}) \leq (x^z_{j+1} - \mu_{x^z}) - (z - \mu_{y^z}) \leq \cdots \leq (x^z_n - \mu_{x^z}) - (z - \mu_{y^z}).$$

Combining these inequalities with (33) yields

$$\begin{aligned}(x_1 - \mu_{x^z}) - (y_1 - \mu_{y^z}) \leq \cdots \leq (x_j - \mu_{x^z}) - (y_j - \mu_{y^z})& \\ \leq (x^z_{j+1} - \mu_{x^z}) - (z - \mu_{y^z}) \leq \cdots \leq (x^z_n - \mu_{x^z}) - (z - \mu_{y^z}),&\end{aligned}$$

which is also expressible as

$$(x^z_1 - \mu_{x^z}) - (y^z_1 - \mu_{y^z}) \leq \cdots \leq (x^z_n - \mu_{x^z}) - (y^z_n - \mu_{y^z}). \tag{35}$$

Note that

$$\sum_{i=1}^{n} [(x^z_i - \mu_{x^z}) - (y^z_i - \mu_{y^z})] = 0. \tag{36}$$

Hence, using (35) we see that there exists κ such that

$$\begin{aligned}(x^z_i - \mu_{x^z}) - (y^z_i - \mu_{y^z}) \geq 0, \quad i \leq \kappa,\\ (x^z_i - \mu_{x^z}) - (y^z_i - \mu_{y^z}) \geq 0, \quad i \geq \kappa.\end{aligned} \tag{37}$$

Consequently,

$$\sum_{i=1}^{k}(y_i^z - \mu_{y^z}) \geq \sum_{i=1}^{k}(x_i^z - \mu_{x^z}), \quad k \in \{1, \ldots, \kappa\}. \tag{38}$$

Moreover, for $k \in \{\kappa + 1, \ldots, n-1\}$,

$$\begin{aligned}\sum_{i=1}^{k}[(x_i^z - \mu_{x^z}) - (y_i^z - \mu_{y^z})] &= \sum_{i=1}^{n}[(x_i^z - \mu_{x^z}) - (y_i^z - \mu_{y^z})] - \sum_{i=k+1}^{n}[(x_i^z - \mu_{x^z}) - (y_i^z - \mu_{y^z})] \\ &= 0 - \sum_{i=k+1}^{n}[(x_i^z - \mu_{x^z}) - (y_i^z - \mu_{y^z})] \\ &\leq 0,\end{aligned}$$

where the second equality follows from (36) and the inequality relies on (37).

Therefore, for $k \in \{\kappa + 1, \ldots, n-1\}$,

$$\sum_{i=1}^{k}(x_i^z - \mu_{x^z}) \leq \sum_{i=1}^{k}(y_i^z - \mu_{y^z}).$$

This, together with (38), implies $\boldsymbol{y}^z \succcurlyeq_{AL} \boldsymbol{x}^z$.

It remains to show that

$$\sum_{i} y_i^z \geq \sum_{i} x_i^z. \tag{39}$$

Note first that (30) gives

$$\sum_{i=1}^{n} y_i^z = \sum_{i=1}^{j} y_i + (n-j)z = \sum_{i=1}^{j}(x_i - t(x_i)) + (n-j)z. \tag{40}$$

Moreover, (32) implies that

$$\sum_{i=1}^{n} x_i^z \leq \sum_{i=1}^{j} x_i + (n-j)z. \tag{41}$$

Because $t(x) \leq 0$ for $x \leq z$ and (32) holds, (40) and (41) yield (39), as desired. ■

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