Measuring Hierarchy

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Abstract

This paper proposes an axiomatic approach to hierarchy measurement whereby a class of hierarchical orders is characterized in terms of intuitive criteria for hierarchy comparisons.

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JEL classifications: D23, L22.

1. Introduction

Hierarchical structures are pervasive in organizations and exhibit a strong correlation with earnings distributions. Some authors have used theoretical hierarchical models to explain observed worker compensation. Empirical observation suggests that the compensation of a firm's highest paid official is related almost exclusively to firm size, while other variables—and, in particular, profit—have virtually no explanatory power (Roberts, 1956). Simon (1957) proposes a simple hierarchical structure to explain the observed relationship between CEO compensation and firm size. Using a very similar hierarchical mechanism, Lydall (1959) generates a labor income distribution whose upper tail is consistent with the empirical power law.

More recently, Fix (2018, 2019) adapts the hierarchical model of Simon (1957) and Lydall (1959) in light of recent data and finds that relative income within firms scales strongly with the average number of subordinates under an individual's control.

Both Simon (1957) and Lydall (1959) postulate a particular hierarchical structure whereby each supervisor has the same number of immediate subordinates—a number sometimes referred to as the 'span of control.' Under this assumption, these authors can express worker compensation as a function of the span of control and the ratio of an individual's salary to the salaries of his/her immediate subordinates. As pointed out by Simon (1957), the span of control is a measure of the ""steepness" of organizational hierarchies." Empirically, the span of control is not constant across ranks. This is documented in Fix (2018, 2019), where the span of control is replaced by the average number of subordinates.

These measures of hierarchical "steepness" are informal, since no systematic criterion for hierarchy comparisons has been advanced. The distributional consequences of

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hierarchical structure are not well understood, both theoretically—in the absence of a general theory of hierarchy measurement—and empirically—given the paucity of publicly available data. This paper is a first attempt toward an axiomatic theory of hierarchy measurement, which can be viewed as a precondition for a systematic analysis of the link between income distribution and organizational design. But the paper also contributes to a much broader literature, since hierarchies play an essential role in the evolution and transformation of societies at large. In fact, the transition from small-scale to large-scale societies goes hand-in-hand with growing hierarchical complexity and energy capture per capita (see, e.g., Turchin and Gavrilets, 2009; Fix, 2017; Graeber and Wengrow, 2021; Bichler and Nitzan, 2020).

A simple partial order defined over the set of hierarchies is shown to be equivalent to intuitive criteria for hierarchy comparisons. This equivalence serves as a basis for a full characterization of hierarchy measures that are consistent with the said partial order. Methodologically, the analysis is akin to the axiomatic underpinnings of inequality measurement (see, e.g., Cowell, 2016; Chakravarty, 2009, 2015).

We consider hierarchies that can be represented as a series of nodes linked by paths. Each node represents an individual, and the paths between nodes determine the subordination relations. We confine attention to hierarchies whose subordinates have only one immediate supervisor.

A hierarchical order $\succcurlyeq$ is a reflexive and transitive binary relation over the set of all hierarchies. Because $\succcurlyeq$ need not be complete, $\succcurlyeq$ may render no judgement over some comparisons of hierarchies. When two hierarchies h and h' are comparable under $\succcurlyeq$, we write '$h \succcurlyeq h'$' to mean that 'h is at least as hierarchical as h'.'

For hierarchies with the same number of individuals, we adopt two basic criteria for hierarchy comparisons. First, relabeling the individuals in a hierarchy does not alter its hierarchical structure. This property is called 'Anonymity.' The second criterion is based on the notion of 'Subordination Removal.' We say that a hierarchy h' is obtained from another hierarchy h by removal of a subordination relation if the sub-hierarchy $h(i)$ of h that begins at an immediate subordinate i of a supervisor j in h is moved up one level in the hierarchy, so that i is no longer an immediate subordinate of j, but rather either an unsupervised individual (if j has no supervisors) or an immediate subordinate of j's immediate supervisor. The sub-hierarchy $h(i)$ remains otherwise intact, and the structure of h' is otherwise identical to that of h. The 'Subordination Removal' postulate asserts that removing a subordination relation gives a less hierarchical structure.

The following particular hierarchical order, denoted by $\succcurlyeq_H$ and defined over pairs of hierarchies with the same number of individuals, is instrumental in the formulation of our results: $h \succcurlyeq_H h'$ if there exists a bijection between the individuals of h and those of h' with the following property: for each individual i in h linking (via the bijection) to a subordinate j in h', j's immediate supervisor in h' links (via the bijection) to a supervisor of i in h.

The first main result of this paper (Theorem 1) states that, for any two hierarchies h and h' with the same number of individuals, $h \succ_H h'$ if and only if h' can be obtained from a relabeling of h by successive removal of subordination relations.

A hierarchical order is said to be $\succcurlyeq_H$-consistent if it agrees with $\succcurlyeq_H$ for pairs of hierarchies that are comparable under $\succcurlyeq_H$. The second main result of this paper (Theorem 2) characterizes the class of $\succcurlyeq_H$-consistent hierarchical orders in terms of two axioms: for a fixed number of individuals, n, a hierarchical order on the set of all hierarchies of size n satisfies 'Anonymity' and 'Subordination Removal' if and only if it is $\succcurlyeq_H$-consistent.

Similar results can be obtained for extensions of hierarchical orders to pairs of hierarchies of varying size. For hierarchies of varying size, a third basic criterion for hierarchy comparisons is postulated (in addition to the 'Anonymity' and 'Subordination Removal' conditions): replicating a hierarchy—which yields two identical hierarchies, a superstructure that is itself a hierarchy in its own right—does not alter its hierarchical structure. This property is called 'Replication Principle.'

Using the 'Replication Principle,' the hierarchical order $\succcurlyeq_H$ can be extended to pairs of hierarchies of varying size as follows: $h \succcurlyeq_H h'$ if there exist two equally-sized replications, h_r and h'_r, of h and h', respectively, such that $h_r \succcurlyeq_H h'_r$.

The third main result of this paper (Theorem 3) states that a hierarchical order on the set of all hierarchies (of any size) satisfies 'Anonymity,' 'Subordination Removal,' and the 'Replication Principle' if and only if it is $\succcurlyeq_H$-consistent.

Two examples of $\succcurlyeq_H$-consistent hierarchical orders are considered. The first one compares the number of supervisors between pairs of linked individuals across hierarchies and is shown to be a partial completion of $\succcurlyeq_H$. The second one is a completion of $\succcurlyeq_H$: the hierarchical order induced by a hierarchical index that counts the average number of supervisors for each hierarchy.

2. Hierarchies

A ***hierarchy*** is defined as a set of $n \in \mathbb{N}$ individuals satisfying the following:

- There exists a set of level-0 individuals, I_0, such that each $i \in I_0$ has either no subordinates or a set of level-1 subordinates, S_i, satisfying the following:
  - $S_i \cap I_0 = \emptyset$ for each $i \in I_0$.
  - $S_i \cap S_{i'} = \emptyset$ for each $i, i' \in I_0$ with $i \neq i'$.
- Suppose that the set of level-k individuals has been defined, where $k \geq 0$. The set of level-$(k+1)$ individuals is defined as follows. Each level-$(k+1)$ subordinate, j, has either no subordinates or a set of level-$(k+2)$ subordinates, S_j, satisfying the following:
  - For each level-$(k+1)$ subordinate j, the set of level-$(k+2)$ subordinates of j, S_j, contains no level-κ subordinates, where $\kappa \in \{0, \ldots, k+1\}$.
  - For any two distinct level-$(k+1)$ subordinates j and j', the sets of level-$(k+2)$ subordinates S_j and $S_{j'}$ of j and j', respectively, are disjoint.

For each level-k subordinate i in a given hierarchy, where $k > 0$, there is one level-$(k-1)$ supervisor, $p(i)$, one level-$(k-2)$ supervisor, $p^2(i)$, etc. The supervisor $p(i)$ of i is called i's immediate supervisor.

Individuals i and j in a given hierarchy are related if there is a path linking them, i.e., if either $i = j$ or $j = p^l(i)$ for some l. If $i \neq j$, we say that i is a subordinate of j. If $i \neq j = p(i)$, we say that i is an immediate subordinate of j.

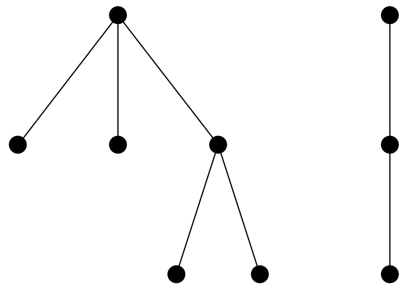

Figure 1: A hierarchy.

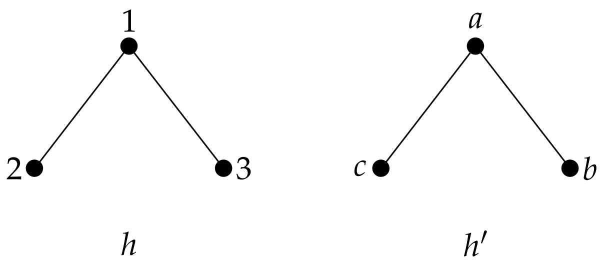

Figure 2: Relabeling.

For each individual i in a hierarchy h, the ***sub-hierarchy*** containing i and all of i's subordinates constitutes a properly defined hierarchy, denoted by $h(i)$. The subordinates of i are the members of the sub-hierarchy $h(i)$ other than i.

Note that, given a hierarchy h and its set I_0 of level-0 individuals, h can be represented as a vector $(h(i))_{i \in I_0}$. In general, a ***sub-hierarchy*** of $h = (h(i))_{i \in I_0}$ is a hierarchy $(h(i))_{i \in I}$, where $I \subseteq I_0$.

Hierarchies can be conveniently represented graphically as a series of nodes linked by paths. Each node represents an individual in the hierarchy. Figure 1 presents a hierarchy with two level-0 individuals, four level-1 individuals, and three level-3 individuals.

3. Hierarchical orders

To begin, we consider orders on hierarchies of a fixed size. Extensions of these orders to hierarchies of varying size are studied in Section 4.

Let $\mathscr{H}_n$ be the set of all n-person hierarchies. A ***hierarchical order*** is a reflexive and transitive binary relation $\succcurlyeq$ on $\mathscr{H}_n$. For $h, h' \in \mathscr{H}_n$, '$h \succcurlyeq h'$' means that 'h is at least as hierarchical as h'.'

The symmetric and asymmetric parts of $\succcurlyeq$ are denoted by $\sim$ and $\succ$, respectively.

We now present two basic properties of hierarchical orders.

A hierarchy h' is said to be a ***relabeling*** of another hierarchy h if h' is obtained from h by relabeling the individuals in h. For example, each of the two hierarchies in Figure 2, h and h', is a relabeling of the other hierarchy.

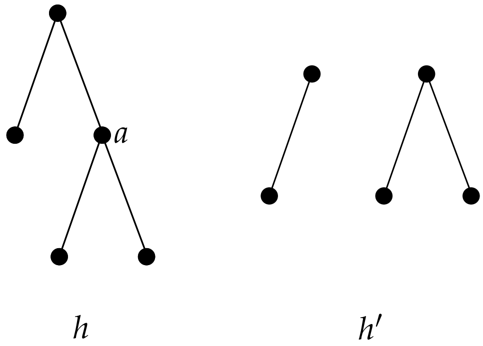

Figure 3: Removing a subordination relation.

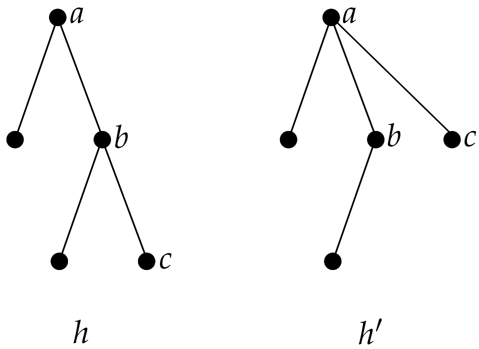

Figure 4: Removing a subordination relation.

Anonymity (A). $h \sim h'$ whenever h' is a relabeling of h.

This axiom asserts that, for a hierarchical order, all relabelings of a given hierarchy belong to the same equivalence class.

We say that h' is obtained from a hierarchy h by ***removing a subordination relation*** if there exists a level-k subordinate i in h, where $k > 0$, satisfying the following:

- If i's immediate supervisor, $p(i)$, is a level-0 individual, then h' is the hierarchy in which i is no longer a level-1 subordinate, but rather a level-0 individual, and the sub-hierarchy that begins at i is $h(i)$; h' is otherwise equal to h.

- If i's immediate supervisor, $p(i)$, is a not level-0 individual, then $p(i) \in S_{p^2(i)}$. In this case, h' is the hierarchy in which i is no longer a level-k subordinate, but rather a level-$(k-1)$ subordinate in $S_{p^2(i)}$, and the sub-hierarchy that begins at i is $h(i)$; h' is otherwise equal to h.

Figure 3 illustrates the previous definition. In Figure 3, h' can be obtained from h by taking the sub-hierarchy $h(a)$ that starts at the node labeled 'a' and moving it up so that the individuals in $h(a)$ are no longer subordinates of the level-0 individual in h.

Figure 4 presents another example where individual b loses one subordinate, individual c, who becomes a direct subordinate of b's supervisor, a.

The following axiom states that removing a subordination relation gives a less hierarchical structure.

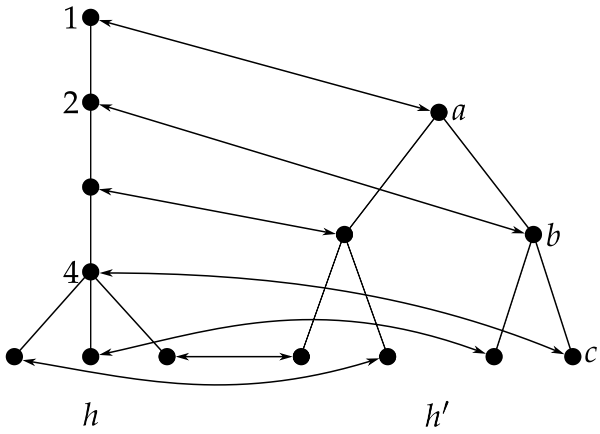

Figure 5: $h \succsim_H h'$.

Subordination Removal (SR). If h' is obtained from h by removing a subordination relation, then $h \succ h'$.

We now define a particular hierarchical order, denoted by $\succsim_H$.

For any two hierarchies h and h' in $\mathscr{H}_n$, $h \succsim_H h'$ if and only if there exists a bijection ϕ from the individuals in h to those in h' satisfying the following: for each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p(\phi(i))$, links (via ϕ^{-1}) to a supervisor j of i in h, i.e., j is in the path from i to i's level-0 supervisor: $\phi^{-1}(p(\phi(i))) = j = p^l(i)$ for some l.

As an example, consider the two hierarchies given in Figure 5. A bijection is represented by means of double-arrowed lines connecting nodes across hierarchies. As can be verified, the linked pairs across hierarchies satisfy the condition from the definition of $\succsim_H$. For example, take individual 2 in h, who links to individual b in h', and whose immediate supervisor, a, links to individual 1 in h, who is a supervisor of 2. As another example, take individual 4, who links to individual c, whose immediate supervisor, b, links to 2, a supervisor of 4. Similar conditions can be verified for the other nodes in h. Thus, $h \succsim_H h'$.

The symmetric and asymmetric parts of $\succsim_H$ are denoted by $\sim_H$ and $\succ_H$, respectively.

The proofs of the results stated below are relegated to Section 5.

Lemma 1. *The hierarchical order $\succsim_H$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies A and SR.*

For $h_1, h_L \in \mathscr{H}_n$, h_1 is obtained from h_L by ***successive removal of subordination relations*** if there are finitely many hierarchies $h_2, \ldots, h_{L-1}$ such that h_l is obtained from h_{l+1} by removing a subordination relation, for each $l \in \{1, \ldots, L-1\}$.

The following result characterizes successive removal of subordination relations in terms of the hierarchical order $\succsim_H$.

Theorem 1. *For $h, h' \in \mathscr{H}_n$, $h \succ_H h'$ if and only if h' can be obtained from some relabeling of h by successive removal of subordination relations.*

A hierarchical order $\succsim$ on $\mathscr{H}_n$ is $\succsim_H$-***consistent*** if the following two conditions are satisfied for every pair h, h' in $\mathscr{H}_n$:

- $h \succ_H h' \Rightarrow h \succ h'$.

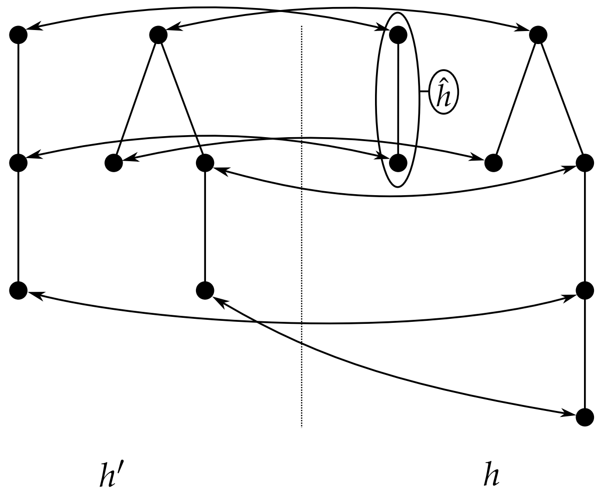

Figure 6: $h \succ_s h'$ and $h \not\succcurlyeq_H h'$.

- $h \sim_H h' \Rightarrow h \sim h'$.

Theorem 2. *A hierarchical order on $\mathscr{H}_n$ satisfies A and SR if and only if it is $\succcurlyeq_H$-consistent.*

As an example of a $\succcurlyeq_H$-consistent hierarchical order on $\mathscr{H}_n$, consider the order $\succcurlyeq_s$ defined as follows: $h \succcurlyeq_s h'$ if and only if there exists a bijection ϕ from the individuals in h to those in h' such that, for each i in h, the number of supervisors of i in h is greater than or equal to the number of supervisors of $\phi(i)$ in h'.

The symmetric and asymmetric parts of $\succcurlyeq_s$ are denoted, as usual, by $\sim_p$ and $\succ_p$, respectively.

To illustrate, consider again the hierarchy from Figure 5. It is easy to see that, for the bijection represented via the double-arrowed lines connecting nodes in the figure, each linked pair of individuals has (weakly) more supervisors in h than in h'. Thus, $h \succcurlyeq_s h'$.

Proposition 1. *The hierarchical order $\succcurlyeq_s$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies A and SR.*

By Proposition 1 and Theorem 2, $\succcurlyeq_s$ is $\succcurlyeq_H$-consistent, and so $\succcurlyeq_H \subseteq \succcurlyeq_s$.

To see that the reverse containment does not hold, so that $\succcurlyeq_H \subsetneqq \succcurlyeq_s$, consider the two seven-person hierarchies in Figure 6. It is easy to see that, for the bijection represented in the figure, each linked pair of individuals has (weakly) more supervisors in h than in h'. Thus, $h \succcurlyeq_s h'$. Moreover, since there is an individual with three supervisors in h and no individual with more than two supervisors in h', we have $h' \not\succcurlyeq_s h$. Hence, $h \succ_s h'$, and yet $h \not\succcurlyeq_H h'$. To see that $h \not\succcurlyeq_H h'$, we use the following result, which is proven in Lemma 6 (Section 5.1): $h \succcurlyeq_H h'$ if and only if for each $i \in I_0$, there exists $I' \subseteq I'_0$ such that $h(i) \succcurlyeq_H (h'(j))_{j \in I'}$, where I_0 (resp., I'_0) represents the set of level-0 individuals in h (resp., h'). Note that, for the sub-hierarchy of h consisting of two individuals, call it $\hat{h}$ (see Figure 6), we cannot have $\hat{h} \succcurlyeq_H (h'(j))_{j \in I'}$ for some $I' \subseteq I'_0$, since all the sub-hierarchies of h' with only one level-0 supervisor have more than two individuals. Hence, $h \not\succcurlyeq_H h'$, and Theorem 1 implies that h' cannot be obtained from some relabeling of h by successive removal of subordination relations.

A *hierarchical index* on $\mathscr{H}_n$ is a map $I : \mathscr{H}_n \to \mathbb{R}$ that assigns a "hierarchical degree" $I(h)$ to every hierarchy $h \in \mathscr{H}_n$. The index I gives rise to a hierarchical order on $\mathscr{H}_n$, $\succcurlyeq_I$, defined as follows:

$$h \succcurlyeq_I h' \Leftrightarrow I(h) \geq I(h').$$

For example, given $h \in \mathscr{H}_n$, let $s_h(i)$ represent the number of supervisors of i in h and define

$$I_s(h) = \frac{1}{n} \sum_{i \in h} s_h(i), \tag{1}$$

which represents the average number of supervisors per individual.

Clearly, $\succcurlyeq_s \subseteq \succcurlyeq_{I_s}$, and it is easy to see that $\succcurlyeq_s \subsetneq \succcurlyeq_{I_s}$. Hence,

$$\succcurlyeq_H \subsetneq \succcurlyeq_s \subsetneq \succcurlyeq_{I_s},$$

and $\succcurlyeq_{I_s}$ is a completion of $\succcurlyeq_H$.

The proof of the following proposition is analogous to that of Proposition 1 (which is proven in Section 5.5).

Proposition 2. *The hierarchical order $\succcurlyeq_{I_s}$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies A and SR.*

By Proposition 2 and Theorem 2, $\succcurlyeq_{I_s}$ is $\succcurlyeq_H$-consistent.

4. Comparing hierarchies of varying size

Recall that $\mathscr{H}_n$ is the set of n-person hierarchies. The superset

$$\mathscr{H} = \bigcup_n \mathscr{H}_n$$

represents the set of hierarchies of any size.

A hierarchical order $\succcurlyeq$ on $\mathscr{H}$ is a reflexive and transitive binary relation on $\mathscr{H}$.

A ***replication*** of a hierarchy $h \in \mathscr{H}$ is a hierarchy in $\mathscr{H}$ of the form $(h, \ldots, h)$. By convention, h is a replication of itself.

For example, the ten-person hierarchy (h, h) in Figure 7 is a replication of the five-person hierarchy h.

Replication Principle (RP). If h' is a replication of $h \in \mathscr{H}$, then $h' \sim h$.

This axiom asserts that, for a hierarchical order on $\mathscr{H}$, any replication of a given hierarchy h is as hierarchical as h.

The hierarchical order $\succcurlyeq_H$ on $\mathscr{H}_n$ introduced in Section 3 can be extended to $\mathscr{H}$ as follows: for $h, h' \in \mathscr{H}$, $h' \succcurlyeq_H h$ if and only if there exists n such that h_r (resp., h'_r) is a replication of h (resp., h') in $\mathscr{H}_n$ and $h'_r \succcurlyeq_H h_r$.

Lemma 1 implies that the extension of $\succcurlyeq_H$ to $\mathscr{H}$ is reflexive and transitive and satisfies A and SR. Moreover, the extension $\succcurlyeq_H$ satisfies RP. Indeed, if $h' = (h, \ldots, h)$ is a replication of h, then $h' \sim_H h$ because $(h, \ldots, h) \sim_H (h, \ldots, h)$.

Lemma 2. *The hierarchical order $\succcurlyeq_H$ defined on $\mathscr{H}$ is reflexive and transitive and satisfies A, SR, and RP.*

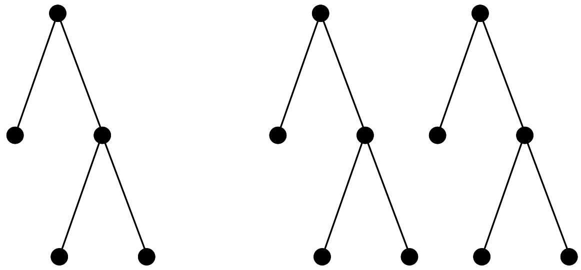

Figure 7: (h,h) is a replication of h.

A hierarchical order $\succcurlyeq$ on $\mathscr{H}$ is ***$\succcurlyeq_H$-consistent*** if the following two conditions are satisfied for every pair h, h' in $\mathscr{H}$:

- $h \succ_H h' \Rightarrow h \succ h'$.
- $h \sim_H h' \Rightarrow h \sim h'$.

The following result extends Theorem 2 to the domain $\mathscr{H}$. Its proof is relegated to Section 5.6.

Theorem 3. *A hierarchical order on $\mathscr{H}$ satisfies A, SR, and RP if and only if it is $\succcurlyeq_H$-consistent.*

Recall the hierarchical order $\succcurlyeq_s$ on $\mathscr{H}_n$ introduced in Section 3: $h \succcurlyeq_s h'$ if and only if there exists a bijection ϕ from the individuals in h to those in h' such that, for each i in h, the number of supervisors of i in h is greater than or equal to the number of supervisors of $\phi(i)$ in h'.

This order can be extended to $\mathscr{H}$ as follows: for $h, h' \in \mathscr{H}$, $h' \succcurlyeq_s h$ if and only if there exists n such that h_r (resp., h'_r) is a replication of h (resp., h') in $\mathscr{H}_n$ and $h'_r \succcurlyeq_s h_r$.

Proposition 1 implies that the extension of $\succcurlyeq_s$ to $\mathscr{H}$ is reflexive and transitive and satisfies A and SR. In addition, the extension $\succcurlyeq_s$ satisfies RP. Indeed, if $h' = (h, \dots, h)$ is a replication of h, then $h' \sim_s h$ because $(h, \dots, h) \sim_s (h, \dots, h)$.

Proposition 3. *The hierarchical order $\succcurlyeq_s$ defined on $\mathscr{H}$ is reflexive and transitive and satisfies A, SR, and RP.*

By Proposition 3 and Theorem 3, $\succcurlyeq_s$ is $\succcurlyeq_H$-consistent. From the example in Figure 6 (in Section 3), we know that $\succcurlyeq_H \subsetneq \succcurlyeq_s$ also for the extensions $\succcurlyeq_H$ and $\succcurlyeq_s$ defined on $\mathscr{H}$.

The hierarchical index I_s defined in (1) can be defined on $\mathscr{H}$ similarly: for $h \in \mathscr{H}$, let

$$I_s(h) = \frac{1}{n} \sum_{i \in h} s_h(i),$$

where $s_h(i)$ denotes the number of supervisors of i in h. For each $h \in \mathscr{H}$, $I_s(h)$ represents the average number of supervisors per individual.

We have $\succcurlyeq_s \subseteq \succcurlyeq_{I_s}$, where $\succcurlyeq_{I_s}$ is the hierarchical order induced by I_s on $\mathscr{H}$:

$$h \succcurlyeq_{I_s} h' \Leftrightarrow I_s(h) \geq I_s(h').$$

To see that $\succsim_s \subseteq \succsim_{I_s}$ on $\mathscr{H}$, suppose that $h \succsim_s h'$ for $h, h' \in \mathscr{H}$. Then there exists n such that h_r (resp., h'_r) is a replication of h (resp., h') in $\mathscr{H}_n$ and $h_r \succsim_s h'_r$. Since $h_r, h'_r \in \mathscr{H}_n$ and $h_r \succsim_s h'_r$, we have $I_s(h_r) \geq I_s(h'_r)$.

Note that, if $h \in \mathscr{H}_m$ and h_r is a k-times replication of h, so that $mk = n$, then

$$I_s(h_r) = \frac{1}{n} \sum_{i \in h_r} s_{h_r}(i) = \frac{1}{mk} k \sum_{i \in h} s_h(i) = I_s(h).$$

Similarly, $I_s(h'_r) = I_s(h')$. Consequently,

$$I_s(h) = I_s(h_r) \geq I_s(h'_r) = I_s(h'),$$

whence $h \succsim_{I_s} h'$.

Thus, $\succsim_s \subseteq \succsim_{I_s}$, and it is easy to see that $\succsim_s \subsetneq \succsim_{I_s}$. Hence,

$$\succsim_H \subsetneq \succsim_s \subsetneq \succsim_{I_s},$$

and $\succsim_{I_s}$ is a completion of $\succsim_H$ on $\mathscr{H}$.

The proof of the following proposition is analogous to that of Proposition 3.

Proposition 4. *The hierarchical order $\succsim_{I_s}$ defined on $\mathscr{H}$ is reflexive and transitive and satisfies A, SR, and RP.*

By Proposition 4 and Theorem 3, $\succsim_{I_s}$, defined on $\mathscr{H}$, is $\succsim_H$-consistent.

5. Proofs

5.1. Preliminary lemmata

Lemma 3. *For $h, h' \in \mathscr{H}_n$, $h \sim_H h'$ implies that there exists a bijection ϕ from the individuals in h to those in h' satisfying the following:*

(a) For each level-k individual i in h, $\phi(i)$ is a level-k individual in h'.

(b) For each individual i in h, the number of immediate subordinates of i in h equals the number of immediate subordinates of $\phi(i)$ in h'.

(c) For each individual i in h, the set $\phi(I_{h(i)})$, where $I_{h(i)}$ denotes the set of all individuals in the sub-hierarchy $h(i)$, is equal to the set of all individuals in the sub-hierarchy $h'(\phi(i))$.

Proof. Since $h \succsim_H h'$, there exists a bijection ϕ from the individuals in h to those in h' satisfying the following:

(I) For each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p(\phi(i))$, links (via ϕ^{-1}) to an a supervisor j of i in h: $\phi^{-1}(p(\phi(i))) = j = p^l(i)$ for some l.

Similarly, since $h' \succsim_H h$, there exists a bijection ϕ' from the individuals in h' to those in h satisfying the following: for each individual i in h' such that $\phi'(i)$ is not a level-0 individual, the immediate supervisor of $\phi'(i)$ in h, $p(\phi'(i))$, links (via ϕ'^{-1}) to a supervisor j of i in h': $\phi'^{-1}(p(\phi'(i))) = j = p^l(i)$ for some l.

Let I_0 (resp., I'_0) be the set of all level-0 individuals in h (resp., h').

First, we show that $\phi(I_0) \subseteq I'_0$. To see this, note that $j \in \phi(I_0) \setminus I'_0$ implies that there exist $i \in I_0$ and a level-k individual j in h', where $k > 0$, such that $\phi(i) = j$. But then $\phi^{-1}(p(\phi(i))) \neq p^l(i)$ for any l, which contradicts (I). Therefore, $\phi(I_0) \setminus I'_0 = \emptyset$, which implies that $\phi(I_0) \subseteq I'_0$.

Similarly, we can show that $\phi'(I'_0) \subseteq I_0$.

Next, let I_l (resp., I'_l) be the set of all level-l individuals in h (resp., h'). Suppose that the containments $\phi(I_l) \subseteq I'_l$ and $\phi'(I'_l) \subseteq I_l$ have been proven for each $l \in \{0, \ldots, k\}$ and some $k \geq 0$. Then the containments $\phi(I_{k+1}) \subseteq I'_{k+1}$ and $\phi'(I'_{k+1}) \subseteq I_{k+1}$ can be proven along the lines of the previous argument.

To see this, note first that, for each l, the two containments $\phi(I_l) \subseteq I'_l$ and $\phi'(I'_l) \subseteq I_l$ imply that h and h' have the same number of level-l individuals. Indeed, if there were more level-l individuals in h', then $\phi(I_l)$ would be a strict subset of I'_l, and, since both I_l and $\phi(I_l)$ have the same cardinality, I_l would be a smaller set than I'_l, contradicting the containment $\phi'(I'_l) \subseteq I_l$. A similar contradiction can be obtained under the assumption that there are more level-l individuals in h.

Now, if $j \in \phi(I_{k+1}) \setminus I'_{k+1}$, since $\phi(I_l) \subseteq I'_l$ and I_l and I'_l have the same cardinality for each $l \in \{0, \ldots, k\}$, we see that $j \in \phi(I_{k+1}) \setminus (\bigcup_{l=0}^{k+1} I'_l)$. Consequently, there exist $i \in I_{k+1}$ and a level-κ' individual j in h', where $\kappa' > k+1$, such that $\phi(i) = j$. But then $\phi^{-1}(p(\phi(i)))$ must be a level-κ individual in h, where $\kappa \geq k+1$. Indeed, if $\phi^{-1}(p(\phi(i)))$ where a level-l individual in h for some $l \in \{0, \ldots, k\}$, then $\phi(I_l) \nsubseteq I'_l$ (since $p(\phi(i)) = p(j)$ is a level-$(k+1)$ individual in h'), contradicting the assumed containment $\phi(I_l) \subseteq I'_l$. Consequently, $\phi^{-1}(p(\phi(i))) \neq p^{\ell}(i)$ for any ℓ, which contradicts (I). Therefore, $\phi(I_{k+1}) \setminus I'_{k+1} = \emptyset$, which implies that $\phi(I_{k+1}) \subseteq I'_{k+1}$.

Next, fix a level-k individual i in h. Since $\phi(I_k) \subseteq I'_k$, it follows that $\phi(i)$ is a level-k individual in h'. This establishes (a).

To see that (b) holds, let i be an individual in h. Suppose that i is a level-k individual. Proceeding by contradiction, suppose that the number of immediate subordinates of i in h is not equal to the number of immediate subordinates of $\phi(i)$ in h'. If he number of immediate subordinates of $\phi(i)$ is greater, then there exists a subordinate j of $\phi(i)$ linking (via ϕ^{-1}) to a level-$(k+1)$ subordinate ι in h whose immediate supervisor, i^*, is not i. But then $\phi(\iota) = j$ and $\phi(i)$ is j's immediate supervisor in h', and yet $\phi(i)$ links (via ϕ^{-1}) to $i \neq i^*$, implying that i is not a supervisor of ι, which contradicts (I).

Hence, the number of immediate subordinates of i in h must be greater than or equal to the number of immediate subordinates of $\phi(i)$ in h'.

If the number of immediate subordinates of i in h is greater than the number of immediate subordinates of $\phi(i)$ in h', there exists an immediate subordinate ι of i such that $\phi(\iota)$'s immediate supervisor in h', $p(\phi(\iota))$, is not $\phi(i)$. But then $\phi^{-1}(p(\phi(\iota)))$ is a level-k individual different from i, implying that $\phi^{-1}(p(\phi(\iota)))$ is not a supervisor of ι, which contradicts (I).

Thus, the number of immediate subordinates of i in h is equal to the number of immediate subordinates of $\phi(i)$ in h'. This establishes (b).

It only remains to prove (c). Fix an individual i in h, and let $I_{h(i)}$ (resp., $I_{h'(\phi(i))}$) be the set of all individuals in the hierarchy $h(i)$ (resp., $h'(\phi(i))$). We must show that $\phi(I_{h(i)}) = I_{h'(\phi(i))}$.

Suppose that i is a level-k individual. Note that it suffices to prove the following: Suppose that j is a level-$(k+l)$ individual in $h(i)$ for $l \geq 0$. Then $\phi(S_j) = S_{\phi(j)}$, where S_j (resp., $S_{\phi(j)}$) represents the set of immediate subordinates of j (resp., $\phi(j)$) in h (resp., h').

Suppose that j is a level-$(k+l)$ individual in $h(i)$ for $l \geq 0$. Suppose that there exists $\iota \in S_j$ such that $\phi(\iota) \notin S_{\phi(j)}$. Then, since ι is a level-$(k+l+1)$ individual in h, so that $\phi(\iota)$ is a level-$(k+l+1)$ individual in h' (by (a)), $\phi(\iota)$'s immediate supervisor in h', $p(\phi(\iota))$, is a level-$(k+l)$ individual in h' who links (via ϕ^{-1}) to a level-$(k+l)$ individual in h, $\phi^{-1}(p(\phi(\iota)))$, that, being different from j, is not a supervisor of ι in h. Since this contradicts (I), we see that $\phi(S_j) \subseteq S_{\phi(j)}$. But then $\phi(S_j) = S_{\phi(j)}$, since S_j and $S_{\phi(j)}$ (and hence $\phi(S_j)$ and $S_{\phi(j)}$) have the same cardinality (by (b)). ■

Lemma 4. *For $h, h' \in \mathscr{H}_n$, $h \sim_H h'$ implies that h is a relabeling of h'.*

Proof. It suffices to show that there exists a bijection ϕ from the individuals in h to those in h' satisfying the following:

$$h(i) = h'(\phi(i)), \qquad \text{for each } i \text{ in } h.$$

Let ϕ be the bijection given by Lemma 3. Let K be the largest level for which there are level-K individuals in h. Then all the level-K individuals in h have zero subordinates. By item (a) of Lemma 3, for any level-K individual i in h, $\phi(i)$ is a level-K individual in h'; moreover, by item (b) of Lemma 3, $\phi(i)$ has zero subordinates, implying that $h(i) = h'(\phi(i))$.

Suppose that the equality $h(i) = h'(\phi(i))$ has been established for each level-k individuals in h, where $k \in \{K, K-1, \ldots, 1\}$. Then $h(i) = h'(\phi(i))$ for each level-$(k-1)$ individual i in h.

To see this, fix a level-$(k-1)$ individual i in h. Let S_i (resp., S'_i) be the set of level-k subordinates of i (resp., $\phi(i)$) in h (resp., h'). If $\phi(S_i) = S'_i$, then, because $h(j) = h'(\phi(j))$ for each $j \in S_i$, it follows that $h(i) = h'(\phi(i))$. Thus, it suffices to show that $\phi(S_i) = S'_i$.

By items (a) and (c) of Lemma 3, we know that $\phi(S_i)$ is a set of level-k individuals in h' contained in the set of all individuals in the sub-hierarchy $h'(\phi(i))$. Since S'_i is the set of all level-k individuals in $h'(\phi(i))$, item (a) of Lemma 3 gives $\phi(S_i) \subseteq S'_i$. But then $\phi(S_i) = S'_i$, since S_i and S'_i (and hence $\phi(S_i)$ and S'_i) have the same cardinality (by item (b) of Lemma 3). ■

Lemma 5. *For $h, h' \in \mathscr{H}_n$, if h' can be obtained from h by removing a subordination relation, then $h \succ_H h'$.*

Proof. We proceed by induction on n. The statement is clearly true if $n = 1$. We now prove the statement for any $n > 1$ under the assumption that it is true for m-person hierarchies, where $m \in \{1, \ldots, n-1\}$.

Because h' can be obtained from h by removing a subordination relation, there exists a level-k subordinate i^* in h, where $k > 0$, satisfying the following:

(i) If i^*'s immediate supervisor, $p(i^*)$, is a level-0 individual, then h' is the hierarchy in which i^* is no longer a level-1 subordinate, but rather a level-0 individual, and the sub-hierarchy that begins at i^* is $h(i^*)$; h' is otherwise equal to h.

(ii) If i^*'s immediate supervisor, $p(i^*)$, is a not level-0 individual, then $p(i^*) \in S_{p^2(i^*)}$. In this case, h' is the hierarchy in which i^* is no longer a level-k subordinate, but rather a level-$(k-1)$ subordinate in $S_{p^2(i^*)}$, and the sub-hierarchy that begins at i^* is $h(i^*)$; h' is otherwise equal to h.

First, we show that $h \succ_H h'$. To see this, let ϕ be the identity map from the individuals in h to those in h'. It suffices to prove the following:

($*$) For each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p(\phi(i))$, links (via ϕ^{-1}) to a supervisor j of i in h: $\phi^{-1}(p(\phi(i))) = j = p^l(i)$ for some l.

Note that if the sub-hierarchy $h(i^*)$ is removed from h and the sub-hierarchy $h'(\phi(i^*)) = h'(i^*)$ is removed from h', the remaining hierarchies are identical. Therefore, for any individual i in h not in $h(i^*)$, ($*$) holds.

Next, fix an individual i in $h(i^*)$. If $i \neq i^*$, then, since the two sub-hierarchies $h(i^*)$ and $h'(i^*)$ are identical, and since ϕ is the identity map, ($*$) holds.

It remains to prove ($*$) for $i = i^*$. Note that if $\phi(i^*) = i^*$ is not a level-0 individual in h', then (ii) must hold. But then the immediate supervisor of $\phi(i^*) = i^*$ in h' is $p^2(i^*)$, which links (via ϕ^{-1}) to $p^2(i^*)$ in h, a supervisor of i^* in h, implying that ($*$) holds.

Since $h \succcurlyeq_H h'$, it remains to show that $h' \not\succcurlyeq_H h$. Proceeding by contradiction, $h' \succcurlyeq_H h$ implies that $h' \sim_H h$. Consequently, h' is a relabeling of h (Lemma 4), contradicting that h' can be obtained from h by removing a subordination relation. ■

Lemma 6. *Suppose that $h, h' \in \mathscr{H}_n$, and let I_0 (resp., I_0') be the set of level-0 individuals in h (resp., h'). The following two conditions are equivalent:*

(i) $h \succcurlyeq_H h'$.

(ii) For each $i \in I_0$, there exists $I' \subseteq I_0'$ such that $h(i) \succcurlyeq_H (h'(j))_{j \in I'}$.

Proof. Suppose that (ii) holds. Then, for each $i \in I_0$, there exist $I_i' \subseteq I_0'$ and a bijection ϕ_i from the individuals in $h(i)$ to those in $(h'(j))_{j \in I_i'}$ satisfying the following: for each individual j in $h(i)$ such that $\phi_i(j)$ is not a level-0 individual, $\phi_i^{-1}(p(\phi_i(j))) = p^l(j)$ for some l.

Define a bijection ϕ from the individuals in h to those in h' as follows: $\phi(j) = \phi_i(j)$ if $j \in h(i)$, $i \in I_0$.

Fix an individual j in h such that $\phi(j)$ is not a level-0 individual. Then $j \in h(i)$ for some $i \in I_0$ and $\phi(j) = \phi_i(j)$, implying that

$$\phi^{-1}(p(\phi(j))) = \phi^{-1}(p(\phi_i(j))) = p^l(j), \quad \text{for some } l.$$

Hence, (i) holds.

Now suppose that (i) holds. Then, there exists a bijection ϕ from the individuals in h to those in h' satisfying the following:

($*$) For each individual i in h such that $\phi(i)$ is not a level-0 individual, $\phi^{-1}(p(\phi(i))) = p^l(i)$ for some l.

For each $i \in I_0$ (resp., $i \in I_0'$), let $I_{h(i)}$ (resp., $I_{h'(i)}$) be the set of all individuals in $h(i)$ (resp., $h'(i)$).

First, we show that

$$\forall i \in I_0', \exists j \in I_0 : \phi^{-1}(I_{h'(i)}) \subseteq I_{h(j)}. \tag{2}$$

Fix $i \in I_0'$. Then $\phi^{-1}(i)$ is an individual in h. Let j be the level-0 supervisor of $\phi^{-1}(i)$ in h. It suffices to show that $\phi^{-1}(I_{h'(i)}) \subseteq I_{h(j)}$. Proceeding by contradiction, suppose that there exists $\iota \in \phi^{-1}(I_{h'(i)})$ in $I_{h(\iota^*)}$ for some $\iota^* \in I_0 \setminus \{j\}$. Note that $\phi(\iota) \neq i$, since $\phi^{-1}(i) \neq \iota$. Since i is the only level-0 individual in $h'(i)$, and since $I_{h'(i)} \ni \phi(\iota) \neq i$, $\phi(\iota)$, an individual in $h'(i)$, is not a level-0 individual. Therefore, by (∗), $\phi^{-1}(p(\phi(\iota))) = p^l(\iota)$ for some l, implying that $\phi^{-1}(p(\phi(\iota))) \in I_{h(\iota^*)}$. If $p(\phi(\iota))$ is a level-0 individual, since $p(\phi(\iota)) \in I_{h'(i)}$, then $p(\phi(\iota)) = i$; in this case, since $\phi^{-1}(i) \in I_{h(j)}$ and $j \neq \iota^*$, $\phi^{-1}(p(\phi(\iota))) = \phi^{-1}(i)$ cannot be a member of $I_{h(\iota^*)}$, a contradiction. If $p(\phi(\iota))$ is not a level-0 individual, then, again applying (∗), we see that $\phi^{-1}(p^2(\phi(\iota))) = p^l(\iota)$ for some l, implying that $\phi^{-1}(p^2(\phi(\iota))) \in I_{h(\iota^*)}$. If $p^2(\phi(\iota))$ is a level-0 individual, since $p^2(\phi(\iota)) \in I_{h'(i)}$, then $p^2(\phi(\iota)) = i$; in this case, since $\phi^{-1}(i) \in I_{h(j)}$ and $j \neq \iota^*$, $\phi^{-1}(p^2(\phi(\iota))) = \phi^{-1}(i)$ cannot be a member of $I_{h(\iota^*)}$, a contradiction. If $p^2(\phi(\iota))$ is not a level-0 individual, again applying (∗), we see that $\phi^{-1}(p^3(\phi(\iota))) = p^l(\iota)$ for some l, implying that $\phi^{-1}(p^3(\phi(\iota))) \in I_{h(\iota^*)}$. This argument can be reiterated until a contradiction is reached in finitely many steps. This proves (2).

Next, fix $i \in I_0$. We have

$$\phi(I_{h(i)}) = \bigcup_{j \in I'} I_{h'(j)}, \quad \text{for some } I' \subseteq I_0'. \tag{3}$$

To see this, note that, by (2), for each $\iota \in I_0'$, there exists $j_\iota \in I_0$ such that $\phi^{-1}(I_{h'(\iota)}) \subseteq I_{h(j_\iota)}$. In addition, because ϕ is a bijection, each j_ι must be unique. Define

$$I' = \{\iota \in I_0' : j_\iota = i\}.$$

It suffices to show that

$$\phi(I_{h(i)}) = \bigcup_{\iota \in I'} I_{h'(\iota)}. \tag{4}$$

Suppose that $j' \in \phi(I_{h(i)})$. Then there exists $j \in I_{h(i)}$ such that $j' = \phi(j)$, implying that $j' \in I_{h'(\iota)}$ for some $\iota \in I_0'$. If $\iota \notin I'$, then there exists $j_\iota \in I_0 \setminus \{i\}$ such that $\phi^{-1}(I_{h'(\iota)}) \subseteq I_{h(j_\iota)}$. Since $j' \in I_{h'(\iota)}$ and $j_\iota \neq i$, this implies that

$$j = \phi^{-1}(j') \in I_{h(j_\iota)} \neq I_{h(i)},$$

a contradiction. Therefore, we must have $j' \in I_{h'(\iota)}$ for some $\iota \in I'$.

Hence, $\phi(I_{h(i)}) \subseteq \bigcup_{\iota \in I'} I_{h'(\iota)}$. Conversely, if $j' \in I_{h'(\iota)}$ for some $\iota \in I'$, then $\phi^{-1}(I_{h'(\iota)}) \subseteq I_{h(i)}$, implying that $\phi^{-1}(j') \in I_{h(i)}$, and so $j' \in \phi(I_{h(i)})$. Consequently, $\phi(I_{h(i)}) \supseteq \bigcup_{\iota \in I'} I_{h'(\iota)}$. This establishes (4).

We conclude that (3) holds. Now let $\phi|_{I_{h(i)}}$ be the restriction of ϕ to $I_{h(i)}$. It is easy to see that, by (3) and (∗), $\phi|_{I_{h(i)}}$ is a bijection from $I_{h(i)}$ to $\bigcup_{j \in I'} I_{h'(j)}$ satisfying the following: for each individual i in h such that $\phi|_{I_{h(i)}}(i)$ is not a level-0 individual, $\phi|_{I_{h(i)}}{}^{-1}(p(\phi|_{I_{h(i)}}(i))) = p^l(i)$ for some l.

Consequently, $h(i) \succsim_H (h'(j))_{j \in I'}$. Since i was arbitrary in I_0, this establishes (ii). ■

Lemma 7. *Suppose that $h, h' \in \mathscr{H}_n$, and let I_0 (resp., I'_0) be the set of level-0 individuals in h (resp., h'). Then (i) implies (ii):*

(i) $h \succ_H h'$.

(ii) *For each $i \in I_0$, there exists $I'_i \subseteq I'_0$ such that $h(i) \succsim_H (h'(j))_{j \in I'_i}$; and there exists $i^* \in I_0$ such that $h(i^*) \succ_H (h'(j))_{j \in I'_{i^*}}$.*

Proof. Suppose that $h \succ_H h'$. Then $h \succsim_H h'$, and so Lemma 6 gives the following: for each $i \in I_0$, there exists $I'_i \subseteq I'_0$ such that $h(i) \succsim_H (h'(j))_{j \in I'_i}$. If there exists $i^* \in I_0$ such that $(h'(j))_{j \in I'_{i^*}} \not\succsim_H h(i^*)$, then $h(i^*) \succ_H (h'(j))_{j \in I'_{i^*}}$, implying that (ii) holds. If, on the other hand, $(h'(j))_{j \in I'_i} \succsim_H h(i)$ for each $i \in I_0$, then $h(i) \sim_H (h'(j))_{j \in I'_i}$ for each $i \in I_0$, implying that $h \sim_H h'$, a contradiction. ∎

5.2. Proof of Lemma 1

Lemma 1 is restated here for the convenience of the reader.

Lemma 1. *The hierarchical order $\succsim_H$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies A and SR.*

Proof. Reflexivity follows immediately from the definition of $\succsim_H$.

Let $I_{\tilde{h}}$ represent the set of individuals in $\tilde{h} \in \mathscr{H}_n$.

To see that $\succsim_H$ is transitive, suppose that $h \succsim_H h' \succsim_H h''$ for $h, h', h'' \in \mathscr{H}_n$. Then, there exist bijections $\phi : I_h \to I_{h'}$ and $\phi' : I_{h'} \to I_{h''}$ satisfying the following:

{1} For each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p(\phi(i))$, links (via ϕ^{-1}) to a supervisor j of i in h, i.e., $\phi^{-1}(p(\phi(i))) = j = p^l(i)$ for some l.

{2} For each individual i in h' such that $\phi'(i)$ is not a level-0 individual, the immediate supervisor of $\phi'(i)$ in h'', $p(\phi'(i))$, links (via ${\phi'}^{-1}$) to a supervisor j of i in h', i.e., ${\phi'}^{-1}(p(\phi'(i))) = j = p^l(i)$ for some l.

Since ϕ and ϕ' are bijections, the composition $\phi^* := \phi' \circ \phi$ is also a bijection (see, e.g., Blyth, 1975, Theorem 5.10, p. 37). Thus, it suffices to show the following:

(∘) For each individual i in h such that $\phi^*(i)$ is not a level-0 individual, the immediate supervisor of $\phi^*(i)$ in h'', $p(\phi^*(i))$, links (via ${\phi^*}^{-1}$) to a supervisor j of i in h, i.e., ${\phi^*}^{-1}(p(\phi^*(i))) = j = p^l(i)$ for some l.

Fix an individual i in h such that $\phi^*(i)$ is not a level-0 individual, and suppose, by contradiction, that

$$ {\phi^*}^{-1}(p(\phi^*(i))) \neq p^l(i), \quad \text{for any } l. \tag{5}$$

Consider the sequence

$$ i, \phi(i), p(\phi(i)), \phi^{-1}(p(\phi(i))), \phi^{-1}(p(\phi(i))), p(\phi(i)), p^2(\phi(i)), \phi^{-1}(p^2(\phi(i))), \\ \phi^{-1}(p^2(\phi(i))), p^2(\phi(i)), p^3(\phi(i)), \phi^{-1}(p^3(\phi(i))), \ldots. $$

This sequence can be subdivided into four-element cycles as follows:

$$\begin{array}{ll} \text{Cycle 1:} & i, \phi(i), p(\phi(i)), \phi^{-1}(p(\phi(i))). \\ \text{Cycle 2:} & \phi^{-1}(p(\phi(i))), p(\phi(i)), p^2(\phi(i)), \phi^{-1}(p^2(\phi(i))) \\ \text{Cycle 3:} & \phi^{-1}(p^2(\phi(i))), p^2(\phi(i)), p^3(\phi(i)), \phi^{-1}(p^3(\phi(i))) \\ \text{Cycle 4:} & \phi^{-1}(p^3(\phi(i))), p^3(\phi(i)), p^4(\phi(i)), \phi^{-1}(p^4(\phi(i))) \\ \vdots & \qquad\qquad \vdots \end{array}$$

The first and last elements of each cycle are individuals in h, while the second and third elements of each cycle are individuals in h'. Moreover, by {1}, the first and last elements of each cycle belong to the path connecting i and i's level-0 supervisor in h, i.e., if j is the first or the fourth element of a cycle, we have $j = p^l(i)$ for some l. In addition, by construction, the second and third elements of every cycle belong to the path connecting $\phi(i)$ and $\phi(i)$'s level-0 supervisor in h', i.e., if j is the second or the third element of a cycle, we have $j = p^l(\phi(i))$ for some l.

Note that each individual in the path connecting $\phi(i)$ and $\phi(i)$'s level-0 supervisor in h' must eventually become the third element of a cycle. Hence, because $\phi'^{-1}(p(\phi^*(i))) = p^l(\phi(i))$ for some l (by {2}), $\phi'^{-1}(p(\phi^*(i)))$ is equal to the third element of some cycle ℓ. But then the fourth element of cycle ℓ, $\phi^{-1}(\phi'^{-1}(p(\phi^*(i))))$, belongs to the path connecting i and i's level-0 supervisor in h (as noted in the previous paragraph). Noting that $\phi^{-1}(\phi'^{-1}(p(\phi^*(i)))) = \phi^{*-1}(p(\phi^*(i))))$, this contradicts our initial assumption in (5).

We conclude that (○) holds, implying that $\succsim_H$ is transitive.

By Lemma 5, $\succsim_H$ satisfies SR.

To see that $\succsim_H$ satisfies A, let h' be a relabeling of h. Then there exists a bijection $\phi : I_h \to I_{h'}$ with the following property: if each individual i in h' is assigned the label '$\phi^{-1}(i)$,' then the resulting hierarchy is identical to h.

It is easy to see that, for the bijection ϕ, the following condition is satisfied: for each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p(\phi(i))$, links (via ϕ^{-1}) to a supervisor j of i in h, i.e., $\phi^{-1}(p(\phi(i))) = j = p^l(i)$ for some l.

Hence, $h \succsim_H h'$. A similar condition can be verified for the bijection $\phi^{-1} : I_{h'} \to I_h$: for each individual i in h' such that $\phi^{-1}(i)$ is not a level-0 individual, the immediate supervisor of $\phi^{-1}(i)$ in h, $p(\phi^{-1}(i))$, links (via ϕ) to a supervisor j of i in h', i.e., $\phi(p(\phi^{-1}(i))) = j = p^l(i)$ for some l.

Consequently, $h' \succsim_H h$ and $h \succsim_H h'$, implying that $h \sim_H h'$. ∎

5.3. Proof of Theorem 1

Theorem 1. *For $h, h' \in \mathscr{H}_n$, $h \succ_H h'$ if and only if h' can be obtained from some relabeling of h by successive removal of subordination relations.*

Proof. [*Necessity.*] First, we prove the 'only if' part of the statement under the assumption that h has only one level-0 individual.

We proceed by induction on n. The statement is clearly true if $n = 1$. We now prove the statement for any $n > 1$ under the assumption that it is true for m-person hierarchies, where $m \in \{1, \dots, n-1\}$.

Suppose that $h \succ_H h'$. We must show that h' can be obtained from some relabeling of h by successive removal of subordination relations.

Since $h \succcurlyeq_H h'$, there exists a bijection ϕ from the individuals in h to those in h' satisfying the following:

(■) For each individual i in h such that $\phi(i)$ is not a level-0 individual, the immediate supervisor of $\phi(i)$ in h', $p(\phi(i))$, links (via ϕ^{-1}) to a supervisor j of i in h, i.e., $\phi^{-1}(p(\phi(i))) = j = p^l(i)$ for some l.

Fix a level-0 individual ι in h. Then $\phi(\iota)$ is a level-0 individual in h' (otherwise $p(\phi(\iota))$ would not link (via ϕ^{-1}) to a supervisor of ι, contradicting (■)).

If $\phi(\iota) \neq \iota$, the individuals in h can be relabeled so that $\phi(\iota) = \iota$. The resulting relabeling will be denoted again by h.

Let $h \setminus \iota$ be the hierarchy resulting from removing individual ι from h:

- In $h \setminus \iota$, every j in the set S_ι of all level-1 subordinates of ι becomes a level-0 individual, and the sub-hierarchy that begins at j is $h(j)$.
- The structure of h remains otherwise intact, i.e., the sub-hierarchy that begins at any level-0 i other than ι is $h(i)$.

The hierarchy $h' \setminus \iota$ is defined similarly.

Let ϕ^* be the restriction of ϕ to the individuals in $h \setminus \iota$. Note that ϕ^* is a bijection between the individuals in $h \setminus \iota$ and those in $h' \setminus \iota$. Moreover, it is easy to see that, because ϕ satisfies (■) and $\phi(\iota) = \iota$, ϕ^* satisfies the following: for each individual i in $h \setminus \iota$ such that $\phi^*(i)$ is not a level-0 individual, ${\phi^*}^{-1}(p(\phi^*(i))) = p^l(i)$ for some l.

Consequently, $h \setminus \iota, h' \setminus \iota \in \mathscr{H}_{n-1}$ and $h \setminus \iota \succcurlyeq_H h' \setminus \iota$.

Suppose first that $h' \setminus \iota \succcurlyeq_H h \setminus \iota$. Then, $h \setminus \iota \sim_H h' \setminus \iota$, and Lemma 4 implies that $h \setminus \iota$ is a relabeling of $h' \setminus \iota$.

Since ι is the only level-0 individual in h, we can write

$$h = h(\iota) \quad \text{and} \quad h' = (h'(\iota), (h'(j))_{j \in I_0' \setminus \{\iota\}}),$$

where I_0' denotes the set of all level-0 individuals in h'. Now, letting S_ι (resp., S_ι') be the set of level-1 subordinates of ι in h (resp., h'), we can write

$$h \setminus \iota = (h(j))_{j \in S_\iota} \quad \text{and} \quad h' \setminus \iota = ((h'(j))_{j \in S_\iota'}, (h'(j))_{j \in I_0' \setminus \{\iota\}}).$$

Since $h \setminus \iota$ is a relabeling of $h' \setminus \iota$, there is no loss of generality in assuming that $h \setminus \iota$ and $h' \setminus \iota$ are identical (since the individuals in $h \setminus \iota$ can always be relabeled in such a way that $h \setminus \iota$ and $h' \setminus \iota$ are identical). Hence,

$$S_\iota = S_\iota' \cup (I_0' \setminus \{\iota\}) \quad \text{and} \quad h(j) = h'(j) \text{ for all } j \in S_\iota. \tag{6}$$

Now let $\{i_1, \dots, i_m\}$ be an enumeration of $I_0' \setminus \{\iota\}$ and define the sequence of hierarchies $h_0, \dots, h_m$ as follows:

- $h_0 = h$.
- h_1 is the hierarchy in which i_1 is no longer a level-1 subordinate in h, but rather a level-0 individual, and the sub-hierarchy that begins at i_1 is $h(i_1)$; h_1 is otherwise equal to h.
- h_2 is the hierarchy in which i_2 is no longer a level-1 subordinate in h_1, but rather a level-0 individual, and the sub-hierarchy that begins at i_2 is $h(i_2)$; h_2 is otherwise equal to h_1.

$$\vdots$$

- h_m is the hierarchy in which i_m is no longer a level-1 subordinate in h_{m-1}, but rather a level-0 individual, and the sub-hierarchy that begins at i_m is $h(i_m)$; h_m is otherwise equal to h_{m-1}.

Observe that, given (6), h_ℓ is obtained from $h_{\ell-1}$ by removing a subordination relation for each $\ell \in \{1, \ldots, m\}$, and $h_m = h'$. Consequently, h' can be obtained from some relabeling of h by successive removal of subordination relations.

Next, suppose that $h' \setminus \iota \not\succsim_H h \setminus \iota$. Since $h \setminus \iota \succsim_H h' \setminus \iota$, we see that $h \setminus \iota \succ_H h' \setminus \iota$. Since $h \setminus \iota, h' \setminus \iota \in \mathscr{H}_{n-1}$ and $h \setminus \iota \succ_H h' \setminus \iota$, the induction hypothesis gives some relabeling of $h \setminus \iota$, denoted again by $h \setminus \iota$, such that

$$h' \setminus \iota \Leftarrow_{RS} h \setminus \iota; \tag{7}$$

here (and in the remainder of the proof), for any two hierarchies $\hat{h}$ and $\bar{h}$, '$\hat{h} \Leftarrow_{RS} \bar{h}$' means that '$\hat{h}$ can be obtained from $\bar{h}$ by successive removal of subordination relations.'

Recall that h and h' can be expressed as

$$h = h(\iota) \quad \text{and} \quad h' = (h'(\iota), (h'(j))_{j \in I_0' \setminus \{\iota\}}),$$

and that $h \setminus \iota$ and $h' \setminus \iota$ are expressible as

$$h \setminus \iota = (h(j))_{j \in S_\iota} \quad \text{and} \quad h' \setminus \iota = ((h'(j))_{j \in S_\iota'}, (h'(j))_{j \in I_0' \setminus \{\iota\}}),$$

where S_ι (resp., S_ι') represents the set of level-1 subordinates of ι in h (resp., h').

Because every removal of a subordination relation in $h \setminus \iota$ changes one and only one of the sub-hierarchies in $(h(j))_{j \in S_\iota}$, (7) implies that there exists a partition

$$(I_j)_{j \in S_\iota}$$

of the set $S_\iota' \cup (I_0' \setminus \{\iota\})$ such that each I_j is a subset of the set of individuals in $h(j)$ and

$$(h'(j'))_{j' \in I_j} \Leftarrow_{RS} h(j), \quad \text{for all } j \in S_\iota. \tag{8}$$

Each partition member I_j with $j \in S_\iota$ can be further partitioned into two sets: the members of I_j that are level-1 subordinates of ι in h', I_j^s, and the members of I_j that are not level-1 subordinates of ι in h', I_j^{ns}:

$$I_j^s = I_j \cap S_\iota' \quad \text{and} \quad I_j^{ns} = I_j \cap (I_0' \setminus \{\iota\}).$$

Using this notation, (8) can be rewritten as

$$((h'(j'))_{j' \in I_j^s}, (h'(j'))_{j' \in I_j^{ns}}) \Leftarrow_{RS} h(j), \quad \text{for all } j \in S_\iota.$$

Consequently,

$$h^* \Leftarrow_{RS} h(\iota) = h, \tag{9}$$

where h^* is a hierarchy defined as follows:

- h^* has only one level-0 individual, ι.
- The level-1 subordinates of ι are the members of

$$S'_\iota \cup (I'_0 \setminus \{\iota\}) = \bigcup_{j \in S_\iota} I_j = \left(\bigcup_{j \in S_\iota} I_j^s \right) \cup \left(\bigcup_{j \in S_\iota} I_j^{ns} \right),$$

  and the sub-hierarchy that begins at any such level-1 subordinate j' is given by $h'(j')$.

Note that, by successive removal of subordination relations in h^*, we can, for any level-1 subordinate j' in $\bigcup_{j \in S_\iota} I_j^{ns}$, move the sub-hierarchy $h'(j')$ to level 0, thus obtaining the hierarchy

$$\hat{h} = (h^{**}, (h'(j'))_{j' \in \bigcup_{j \in S_\iota} I_j^{ns}}),$$

where h^{**} is a hierarchy defined as follows:

- h^{**} has only one level-0 individual, ι.
- The level-1 subordinates of ι are the members of $\bigcup_{j \in S_\iota} I_j^s$, and the sub-hierarchy that begins at any such level-1 subordinate j' is given by $h'(j')$.

We therefore have

$$\hat{h} \Leftarrow_{RS} h^*. \tag{10}$$

Moreover, since all the subordinates of ι in h' are in h^{**} and all the non-subordinates of ι in h' are in

$$(h'(j'))_{j' \in \bigcup_{j \in S_\iota} I_j^{ns}},$$

we see that $\hat{h} = h'$. This, together with (9)-(10), gives $h' \Leftarrow_{RS} h$, as desired.

It remains to prove the 'only if' part of the statement when h has more than one level-0 individual.

Suppose that $h \succ_H h'$. We must show that h' can be obtained from some relabeling of h by successive removal of subordination relations.

Let I_0 (resp., I'_0) be the set of level-0 individuals in h (resp., h'). By Lemma 7, for each $i \in I_0$, there exists $I'_i \subseteq I'_0$ such that $h(i) \succsim_H (h'(j))_{j \in I'_i}$; and there exists $i^* \in I_0$ such that $h(i^*) \succ_H (h'(j))_{j \in I'_{i^*}}$.

Let I^* be the set of all $i \in I_0$ such that $h(i) \succ_H (h'(j))_{j \in I'_i}$. The set I^* is nonempty, since $i^* \in I^*$. Note that, for each $i \in I_0 \setminus I^*$, we have $h(i) \sim_H (h'(j))_{j \in I'_i}$.

From the first part of this proof we obtain the following:

$$(h'(j))_{j \in I'_i} \Leftarrow_{RS} h(i), \quad \text{for all } i \in I^*.$$

Therefore, since $h(i) \sim_H (h'(j))_{j \in I'_i}$ for each $i \in I_0 \setminus I^*$, and since the relation $h(i) \sim_H (h'(j))_{j \in I'_i}$ implies that $(h'(j))_{j \in I'_i}$ is a relabeling of $h(i)$ (Lemma 4), it follows that h' can be obtained from some relabeling of h by successive removal of subordination relations.

[*Sufficiency*.] Suppose that h' can be obtained from some relabeling of h, denoted by $\overline{h}$, by successive removal of subordination relations, i.e.,

$$h' \leftarrow_{RS} h_1 \leftarrow_{RS} \cdots \leftarrow_{RS} h_L \leftarrow_{RS} \overline{h}$$

for finitely many hierarchies $h_1, \ldots, h_L$; here (and in the remainder of the proof), for any two hierarchies $\hat{h}$ and $\underline{h}$, '$\hat{h} \leftarrow_{RS} \underline{h}$' means that '$\hat{h}$ can be obtained from $\underline{h}$ by removing a subordination relation.' We must show that $h \succ_H h'$.

By Lemma 5,

$$\overline{h} \succ_H h_L \succ_H \cdots \succ_H h_1 \succ_H h'.$$

By reflexivity and transitivity of $\succcurlyeq_H$ (Lemma 1), it follows that $\overline{h} \succ_H h'$ (Sen, 2017, Lemma 1*a, p. 56). Moreover, since $\overline{h}$ is a relabeling of h, Lemma 1 gives $h \sim_H \overline{h}$. Consequently,

$$h \sim_H \overline{h} \succ_H h',$$

implying that $h \succ_H h'$ (Sen, 2017, Lemma 1*a, p. 56). ■

5.4. Proof of Theorem 2

Theorem 2. *A hierarchical order on $\mathscr{H}_n$ satisfies* A *and* SR *if and only if it is $\succcurlyeq_H$-consistent.*

Proof. [*Sufficiency*.] Suppose that $\succcurlyeq$ is $\succcurlyeq_H$-consistent. Because $\succcurlyeq_H$ satisfies A and SR (Lemma 1), and since $\succcurlyeq$ is $\succcurlyeq_H$-consistent, it follows that $\succcurlyeq$ also satisfies A and SR.

[*Necessity*.] Suppose that $\succcurlyeq$ is a hierarchical order on $\mathscr{H}_n$ satisfying A and SR. We must show that $\succcurlyeq$ is $\succcurlyeq_H$-consistent, i.e., that the following two conditions are satisfied for every pair h, h' in $\mathscr{H}_n$:

- $h \succ_H h' \Rightarrow h \succ h'$.
- $h \sim_H h' \Rightarrow h \sim h'$.

Suppose first that $h \sim_H h'$. Then h is a relabeling of h' (Lemma 4). Because $\succcurlyeq$ satisfies A, it follows that $h \sim h'$.

Now suppose that $h \succ_H h'$. By Theorem 1, h' can be obtained from some relabeling of h, denoted by h^*, by successive removal of subordination relations. Therefore, there exist hierarchies $h_1, \ldots, h_L$ in $\mathscr{H}_n$ such that

$$h' \leftarrow_{RS} h_1 \leftarrow_{RS} \cdots \leftarrow_{RS} h_L \leftarrow_{RS} h^*,$$

where, for $\hat{h}, \overline{h} \in \mathscr{H}_n$, '$\hat{h} \leftarrow_{RS} \overline{h}$' means that '$\hat{h}$ can be obtained from $\overline{h}$ by removing a subordination relation.'

Consequently, because $\succcurlyeq$ satisfies SR,

$$h' \succ h_1 \succ \cdots \succ h_L \succ h^*,$$

and since h^* is a relabeling of h and $\succcurlyeq$ satisfies A, we see that

$$h' \succ h_1 \succ \cdots \succ h_L \succ h^* \sim h. \tag{11}$$

Because $\succsim$ is reflexive and transitive, (11) implies that $h' \succ h$ (Sen, 2017, Lemma 1*a, p. 56), as desired. ■

5.5. Proof of Proposition 1

Proposition 1. *The hierarchical order $\succsim_s$ defined on $\mathscr{H}_n$ is reflexive and transitive and satisfies A and SR.*

Proof. Reflexivity follows immediately from the definition of $\succsim_s$.

To see that $\succsim_s$ is transitive, suppose that $h \succsim_s h' \succsim_s h''$ for $h, h', h'' \in \mathscr{H}_n$. Then, there exist bijections ϕ and ϕ' satisfying the following:

- For each individual i in h, the number of supervisors of i in h, $\#_h i$, is greater than or equal to the number of supervisors of $\phi(i)$ in h', $\#_{h'}\phi(i)$.
- For each individual i in h', the number of supervisors of i in h', $\#_{h'} i$, is greater than or equal to the number of supervisors of $\phi'(i)$ in h'', $\#_{h''}\phi'(i)$.

Since ϕ and ϕ' are bijections, the composition $\phi^* := \phi' \circ \phi$ is also a bijection (see, e.g., Blyth, 1975, Theorem 5.10, p. 37). Moreover, for each individual i in h, we have $\#_h i \geq \#_{h'}\phi(i) \geq \#_{h''}\phi'(\phi(i))$. Consequently, $\#_h i \geq \#_{h''}[\phi' \circ \phi](i)$, implying that $h \succsim_s h''$.

To see that $\succsim_s$ satisfies A, let $I_{\tilde{h}}$ represent the set of individuals in $\tilde{h} \in \mathscr{H}_n$. Suppose that h' is a relabeling of h. Then there exists a bijection $\phi : I_h \to I_{h'}$ with the following property: if each individual i in h' is assigned the label '$\phi^{-1}(i)$,' then the resulting hierarchy is identical to h.

It is easy to see that, for the bijection ϕ, the following condition is satisfied: for each i in h, the number of supervisors of i in h is equal to the number of supervisors of $\phi(i)$ in h'.

Hence, $h \succsim_s h'$. A similar condition can be verified for the bijection $\phi^{-1} : I_{h'} \to I_h$: for each individual i in h', the number of supervisors of i in h' is equal to the number of supervisors of $\phi^{-1}(i)$ in h.

Consequently, $h' \succsim_s h$ and $h \succsim_s h'$, implying that $h \sim_p h'$.

It remains to show that $\succsim_s$ satisfies SR. Suppose that h' can be obtained from h by removing a subordination relation. Then there exists a level-k subordinate i^* in h, where $k > 0$, satisfying the following:

(i) If i^*'s immediate supervisor, $p(i^*)$, is a level-0 individual, then h' is the hierarchy in which i^* is no longer a level-1 subordinate, but rather a level-0 individual, and the sub-hierarchy that begins at i^* is $h(i^*)$; h' is otherwise equal to h.

(ii) If i^*'s immediate supervisor, $p(i^*)$, is a not level-0 individual, then $p(i^*) \in S_{p^2(i^*)}$. In this case, h' is the hierarchy in which i^* is no longer a level-k subordinate, but rather a level-$(k-1)$ subordinate in $S_{p^2(i^*)}$, and the sub-hierarchy that begins at i^* is $h(i^*)$; h' is otherwise equal to h.

We must show that $h \succ_p h'$.

Note that the only individuals whose set of supervisors is altered as a result of the removal of a subordination relation specified in items (i) and (ii) are those in the sub-hierarchy $h(i^*)$ of h containing i^* and all of i^*'s subordinates. Moreover, after the removal of a subordination relation, the individuals in $h(i^*)$ are left with less supervisors. Consequently, the number of supervisors of any i in h is equal to the number of supervisors of i in h' if i is not in $h(i^*)$, and greater than the number of supervisors of i in h' if i is in $h(i^*)$, implying that $h \succsim_s h'$.

It remains to show that $h' \not\succsim_s h$. Proceeding by contradiction, suppose that $h' \succsim_s h$. Then there exists a bijection φ from the individuals in h' to those in h such that, for each i in h', the number of supervisors of i in h' is greater than or equal to the number of supervisors of $\varphi(i)$ in h.

Let I^* be the (nonempty) set of all individuals in $h(i^*)$ who have the most supervisors in h among all the individuals in $h(i^*)$. Let s^* be the number of supervisors in h for the individuals in I^*. Then $s^* - 1$ is the number of supervisors in h' for the individuals in I^*.

Let $\bar{s}$ be the maximum number of supervisors that an individual in h can have. Note that $\bar{s} \geq s^*$. If $\bar{s} > s^*$, then all the individuals in h with $\bar{s}$ supervisors are not in $h(i^*)$, and so the number of individuals in h with $\bar{s}$ supervisors is equal to the number of individuals in h' with $\bar{s}$ supervisors. Therefore, letting $I_{\bar{s}}$ (resp., $I'_{\bar{s}}$) be the set of all individuals in h (resp., h') with $\bar{s}$ supervisors, we have $\varphi(I'_{\bar{s}}) \supseteq I_{\bar{s}}$ (otherwise there would exist an individual in h' with less supervisors than $\bar{s}$ linking, via φ, to an individual in $I_{\bar{s}}$, contradicting the fact that, for each i in h', the number of supervisors of i in h' is greater than or equal to the number of supervisors of $\varphi(i)$ in h). Since $I_{\bar{s}}$ and $I'_{\bar{s}}$ have the same cardinality, so do $\varphi(I'_{\bar{s}})$ and $I_{\bar{s}}$, and so the containment $\varphi(I'_{\bar{s}}) \supseteq I_{\bar{s}}$ implies that $\varphi(I'_{\bar{s}}) = I_{\bar{s}}$.

Similarly, for any $\ell \in \mathbb{N}$ for which $\bar{s} - \ell > s^*$, we have $\varphi(I'_{\bar{s}-\ell}) = I_{\bar{s}-\ell}$ (where $I_{\bar{s}-\ell}$ (resp., $I'_{\bar{s}-\ell}$) represents the set of all individuals in h (resp., h') with $\bar{s} - \ell$ supervisors).

Note that there exists $\ell^* \in \{0, 1, 2, \dots\}$ such that $\bar{s} - \ell^* = s^*$. Moreover, $I_{\bar{s}-\ell^*} = I_{s^*}$ contains I^* and $I'_{s^*} = I_{s^*} \setminus I^*$.

To see that $I'_{s^*} = I_{s^*} \setminus I^*$, suppose that $i \in I'_{s^*}$. Then i is not in $h(i^*)$. Indeed, if i were in $h(i^*)$, since i has s^* supervisors in h', then i would have $s^* + 1$ supervisors in h, contradicting the fact that those individuals in $h(i^*)$ who have the most supervisors in h have s^* supervisors. Since i is not in $h(i^*)$, we have $i \notin I^*$ (since the members of I^* are also in $h(i^*)$) and, because the removal of a subordination relation specified in items (i) and (ii) does not affect the number of supervisors for those individuals not in $h(i^*)$, $i \in I_{s^*}$, so that $i \in I_{s^*} \setminus I^*$. Hence, $I'_{s^*} \subseteq I_{s^*} \setminus I^*$.

Conversely, suppose that $i \in I_{s^*} \setminus I^*$. Then i is not in $h(i^*)$ (since I^* is the set of all individuals in $h(i^*)$ who have s^* supervisors in h). Therefore, because the removal of a subordination relation specified in items (i) and (ii) does not affect the number of supervisors for those individuals not in $h(i^*)$, we have $i \in I'_{s^*}$. Hence, $I'_{s^*} \supseteq I_{s^*} \setminus I^*$.

Now, since $I'_{s^*} = I_{s^*} \setminus I^*$ and I^* is nonempty, it follows that the number of individuals in h with s^* supervisors exceeds the number of individuals in h' with s^* supervisors. Consequently, because $\varphi(I'_{\bar{s}-\ell}) = I_{\bar{s}-\ell}$ for any $\ell \in \mathbb{N}$ for which $\bar{s} - \ell > s^*$, there exists $j \in I_{s^*}$ such that $\varphi^{-1}(j)$ has less than s^* supervisors in h', a contradiction.

We conclude that $h' \not\succsim_s h$.

Since $h \succsim_s h'$ and $h' \not\succsim_s h$, we see that $h \succ_p h'$. ∎

5.6. Proof of Theorem 3

Theorem 3. *A hierarchical order on $\mathscr{H}$ satisfies A, SR, and RP if and only if it is $\succsim_H$-consistent.*

Proof. [*Sufficiency.*] Suppose that $\succsim$ is $\succsim_H$-consistent. Because $\succsim_H$ satisfies A, SR, and RP (Lemma 2), and since $\succsim$ is $\succsim_H$-consistent, it follows that $\succsim$ also satisfies A, SR, and RP.

[*Necessity*.] Suppose that $\succcurlyeq$ is a hierarchical order on $\mathscr{H}$ satisfying A, SR, and RP. We must show that $\succcurlyeq$ is $\succcurlyeq_H$-consistent, i.e., that the following two conditions are satisfied for every pair h, h' in $\mathscr{H}$:

(a) $h \succ_H h' \Rightarrow h \succ h'$.

(b) $h \sim_H h' \Rightarrow h \sim h'$.

Fix h and h' in $\mathscr{H}$. Suppose that $h \in \mathscr{H}_m$ and $h' \in \mathscr{H}_n$. Let h_r (resp., h'_r) be an n-times (resp., m-times) replication of h (resp., h'). Then h_r and h'_r are hierarchies in $\mathscr{H}_{mn}$.

Suppose first that $h \sim_H h'$. Since $\succcurlyeq_H$ satisfies RP (Lemma 2),

$$h_r \sim_H h \sim_H h' \sim_H h'_r. \tag{12}$$

Because $\succcurlyeq_H$ is reflexive and transitive (Lemma 2), $\sim_H$ is transitive (Sen, 2017, Lemma 1*a, p. 56). Consequently, (12) implies that $h_r \sim_H h'_r$.

Since $h_r, h'_r \in \mathscr{H}_{mn}$ and $h_r \sim_H h'_r$, h_r is a relabeling of h'_r (Lemma 4). Because $\succcurlyeq$ satisfies A and RP, it follows that

$$h \sim h_r \sim h'_r \sim h'.$$

By transitivity of $\sim$, we see that $h \sim h'$. This establishes (a).

Now suppose that $h \succ_H h'$. Since $\succcurlyeq_H$ satisfies RP (Lemma 2),

$$h_r \sim_H h \succ_H h' \sim_H h'_r. \tag{13}$$

Because $\succcurlyeq_H$ is reflexive and transitive (Lemma 2), (13) gives $h_r \succ_H h'_r$ (Sen, 2017, Lemma 1*a, p. 56).

Since $h_r, h'_r \in \mathscr{H}_{mn}$ and $h_r \succ_H h'_r$, h'_r can be obtained from some relabeling of h_r, denoted by h^*_r, by successive removal of subordination relations (Theorem 1). Therefore, there exist hierarchies $h_1, \ldots, h_L$ in $\mathscr{H}_n$ such that

$$h'_r \leftarrow_{RS} h_1 \leftarrow_{RS} \cdots \leftarrow_{RS} h_L \leftarrow_{RS} h^*_r,$$

where, for $\hat{h}, \bar{h} \in \mathscr{H}_{mn}$, '$\hat{h} \leftarrow_{RS} \bar{h}$' means that '$\hat{h}$ can be obtained from $\bar{h}$ by removing a subordination relation.'

Consequently, because $\succcurlyeq$ satisfies SR,

$$h'_r \succ h_1 \succ \cdots \succ h_L \succ h^*_r,$$

and since h^*_r is a relabeling of h_r and $\succcurlyeq$ satisfies A, we see that

$$h'_r \succ h_1 \succ \cdots \succ h_L \succ h^*_r \sim h_r. \tag{14}$$

Because $\succcurlyeq$ is reflexive and transitive, (14) implies that $h'_r \succ h_r$ (Sen, 2017, Lemma 1*a, p. 56). Since $\succcurlyeq$ satisfies RP, and since h_r (resp., h'_r) is a replication of h (resp., h'), it follows that

$$h \sim h_r \succ h'_r \sim h',$$

implying that $h \succ h'$. This establishes (b). ∎

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