Social Preorders and Tax Progressivity*

Oriol Carbonell-Nicolau† Humberto Llavador‡

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Abstract

Income inequality, bipolarization, and polarization more generally are critical issues that have drawn the attention of economists, policymakers, and social scientists. While related, these phenomena present important conceptual differences. This paper studies the role of nonlinear income taxation as a mechanism for income inequality reduction and depolarization. We introduce a novel and intuitive variance-sensitive axiom defined on perfectly bimodal income distributions, an axiom that serves as the basis for the definition of a social preorder, which is used as the main normative criterion for the evaluation of income distributions and encompasses various inequality and (bi)polarization measures. In an endogenous income framework, we fully characterize the conditions under which income tax schedules effectively reduce income inequality and (bi)polarization, as measured by a wide range of metrics. We show that such tax schedules are necessarily progressive and characterize subsets of tax policies that simultaneously achieve a universal reduction in inequality and (bi)polarization. These results underscore the critical role of progressive taxation in mitigating economic disparities and fostering a more balanced economic landscape.

Keywords: Income Inequality, Income Polarization, Progressive Taxation, Social Preorder, Lorenz Criterion, Endogenous Income.

JEL classifications: D31, D63, E62, H23, H24, D71.

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†Department of Economics, Rutgers University, 75 Hamilton Street, New Brunswick, NJ 08901, USA. E-mail: carbonell-nicolau@rutgers.edu.

‡Universitat Pompeu Fabra and Barcelona GSE, R. Trias Fargas 25–27, 08005 Barcelona, Spain. E-mail: humberto.llavador@upf.edu. Humberto Llavador acknowledges financial support from the Spanish Research Agency (AEI) through the Severo Ochoa Programme for Centres of Excellence in R&D (Barcelona School of Economics CEX2019-000915-S) funded by MCIN/AEI/10.13039/501100011033 and the grant PID2022-138443NB-I00.

1 Introduction

The study of income inequality and polarization has garnered significant attention from economists, policymakers, and social scientists alike. While both phenomena relate to the distribution of income and are perceived as being normatively regrettable, there are conceptual differences between them. Inequality refers to the overall uneven distribution of resources among individuals or groups and is typically measured by Lorenz-consistent metrics (Lambert, 2002; Chakravarty, 2009, 2015; Cowell, 2011; Myles, 2012). Bipolarization, on the other hand, is concerned with the emergence of two distinct groups with a diminishing middle class, reflecting a growing divide between the rich and poor (Chakravarty, 2015). Finally, polarization encompasses a broader phenomenon where society fragments into multiple distinct clusters or extremes, not limited to just two groups, leading to multipolar social and political division (Esteban and Ray, 1994; Duclos et al., 2004). Thus, while inequality focuses on general disparities, bipolarization points to the erosion of the middle class, and polarization measures the clustering around various extremes. Each phenomenon poses unique challenges to societal cohesion and stability.

This paper investigates the relationship between equality and depolarization and the progressivity of tax schedules. By examining the theoretical foundations of inequality and polarization measurement, we seek to characterize the conditions under which tax schedules can effectively reduce income inequality and polarization.

We work with broadly defined social preorders subsuming a wide variety of inequality or polarization measures. A critical contribution of our analysis is Axiom 1, which compares perfectly bimodal income distributions—distributions that are split into two distinct income groups. Axiom 1 only requires a social preorder to identify increases in the spread between two income groups. This axiom is satisfied by any social preorder based on the Lorenz criterion (Proposition 1). Combined with standard notions of invariance, Axiom 1 implies that, for any social preorder, order-reducing income taxation must exhibit increasing marginal tax rates on income (Theorem 1).

Next, we introduce the concepts of inequality, bipolarization, and polarization. We define inequality (resp., bipolarization) as a social preorder satisfying the Transfer Principle (resp., the Increased Spread and Increased Bipolarity axioms). Our measure of polarization is based on the index of Esteban and Ray (1994).

Inequality, bipolarization, and polarization preorders satisfy Axiom 1 (Proposition 2-Proposition 4).

Building on these results, we provide a normative rationale for progressivity based on principles of equality and depolarization. We frame our analysis within the classical Mirrlees (1971) model, which provides a robust foundation for the study of nonlinear income taxation with endogenous labor supply. We restrict our focus to continuous, piecewise linear tax schedules that maintain the ranking of pre-tax incomes, allowing for both subsidies (negative taxes) and an unrestricted range of marginal tax rates.

We show that, even though inequality, bipolarization, and polarization are fundamentally different concepts, a unified foundation for tax progressivity, consistent across various metrics, can be obtained. First, a reduction in inequality, bipolarization, or polarization can only be achieved by means of marginal-rate progressive taxation. Secondly, taxes are progressive if and only if they are inequality-reducing if and only if they are bipolarization-reducing, while the same is true for polarization when we restrict attention to linear tax schedules or when we require tax schedules to be polarization-reducing for any population size.

Our results highlight the critical role of tax progressivity in the design of tax policies that effectively and simultaneously reduce income inequality, bipolarization, and polarization, contributing to a broader discourse on equitable economic policy.

The relationship between income inequality and taxation has been extensively examined in the economic literature. The seminal contributions of Jakobsson (1976), Fellman (1976), and Kakwani (1977) established the foundational principle that average-rate progressive income taxes are necessary and sufficient for a post-tax income distribution to Lorenz dominate, in the relative sense, any pre-tax income distribution.[1] In recent years, this result has been extended to accommodate the endogenous nature of income. Carbonell-Nicolau and Llavador (2018, 2021a) demonstrated that marginal-rate progressivity—i.e., increasing marginal tax rates on income—fully characterizes, under general conditions on preferences, the inequality-reducing principle based on the relative Lorenz dominance criterion. That is, there exists a general class of utility functions for which income taxes are inequality-reducing if and only if they are marginal-rate progressive.

Bipolarization, at least since the 1990s, has been viewed as being intimately related to the size of the middle class (Wolfson, 1994; Deutsch et al., 2013).[2] According to Wolfson (1994), a more bipolarized income distribution is one that is more spread out from the middle so that there are fewer individuals with middle-level incomes. Usually, this spreading out goes hand in hand with a tendency towards bimodality. This is because a smaller middle class is associated with greater separateness of the bottom and top halves of the income distribution and with greater distances between groups. Our analysis identifies the conditions under which tax policies can effectively reduce bipolarization, emphasizing the importance of progressive taxation in creating a more balanced economic landscape. These results extend the analysis in Carbonell-Nicolau and Llavador (2021b) by broadening the scope of the equivalence between inequality and bipolarization-reducing taxation to absolute and relative bipolarization measures satisfying the basic Increased Spread and Increased Bipolarity axioms.

Conceptually, polarization does not necessarily capture the size of the middle class and is distinct from the notion of bipolarization.[3] The literature on polarization is closely linked to the literature on conflict (Reynal-Querol, 2002; Montalvo and Reynal-Querol, 2005; Esteban et al., 2012), and the measures of polarization set forth in the literature (such as those in Esteban and Ray (1994) and Duclos, Esteban and Ray (2004)) rely on the concepts of identification and alienation. Members of each cluster feel a stronger sense of identification with their group the larger the share of the population it represents, while the perceived alienation from other groups increases with the distance between groups. Polarization increases with both identification and alienation.

Our results are valid for a wide array of measures from the literature on inequality and (bi)polarization. We find that, despite their fundamental differences, these measures share a good deal of common ground when it comes to the distributional effects of nonlinear income taxation.

[1] Most of the extensions of the seminal Jakobsson-Fellman-Kakwani result are framed in terms of exogenous income (see, e.g., Hemming and Keen (1983); Eichhorn et al. (1984); Liu (1985); Formby et al. (1986); Thon (1987); Latham (1988); Thistle (1988); Moyes (1988, 1994); Le Breton et al. (1996); Ebert and Moyes (2000); Ju and Moreno-Ternero (2008)). The first papers introducing the disincentive effects of taxation tended to emphasize negative results (Allingham, 1979; Onrubia et al., 2005; Ebert and Moyes, 2003, 2007).

[2] In fact, the literature on bipolarization developed hand in hand with the continuous decline in the size of the middle class in the US and the UK since the 1980s (Wolfson, 1994; Jenkins, 1995).

[3] For a detailed comparison of polarization measures, see Esteban and Ray (2012), Duclos and Taptué (2015) and Chakravarty (2009, 2015)

The paper is structured as follows. Section 2 introduces the main framework of analysis: the concept of a social preorder, Axiom 1, and two notions of invariance; and it provides the definition of a tax schedule and the notions of progressivity and order-reducing taxation. Section 3 states some key results, whose implications for income inequality, bipolarization, and polarization preorders are presented in Section 4, Section 5, and Section 6, respectively. Finally, Section 7 concisely summarizes the main results and outlines potential avenues for future research.

2 The Model

2.1 Social preorders

This section introduces the concept of a social preorder for the comparison of income distributions. A social preorder is a general concept representing an incomplete binary relation defined over the set of income distributions. It ranks certain income distributions according to the degree of divergence between two income groups. We will show that a wide range of inequality and polarization measures are particular instances of social preorders.

In this section, we introduce some terminology and several concepts, including the main axiom that allows us to formally define a social preorder.

An ***income distribution*** is a vector $\boldsymbol{z} = (z_1, ..., z_n)$ in $\mathbb{R}^n_{++}$ with its coordinates arranged in increasing order, *i.e.*, $z_1 \leq \cdots \leq z_n$; here, for each i, z_i represents the income of individual i, and n is a fixed but otherwise arbitrary, even natural number representing the size of the population.[4] The set of all income distributions is denoted by $\mathcal{Z}_n$.

Let $\succcurlyeq \subseteq \mathcal{Z}_n \times \mathcal{Z}_n$ represent a binary relation on $\mathcal{Z}_n$. We are interested in binary relations that measure income dispersion or spread in a very weak sense. We propose the following axiom based on the comparison of perfectly bimodal income distributions.

An income distribution $\boldsymbol{z} = (z_1, \ldots, z_n)$ in $\mathcal{Z}_n$ is said to be ***perfectly bimodal*** if

$$\underline{z} := z_1 = \cdots = z_{\frac{n}{2}} < z_{\frac{n}{2}+1} = \cdots = z_n =: \overline{z}.$$

We sometimes denote $\boldsymbol{z}$ by $(\underline{z}, \overline{z})$, which is a slight abuse of notation.

Axiom 1 (*majorization by bigger-spread perfectly bimodal distributions*)**.** Let $\succcurlyeq$ be a binary relation on $\mathcal{Z}_n$. Suppose that $(\underline{x}, \overline{x})$ and $(\underline{y}, \overline{y})$ are perfectly bimodal income distributions with the same total income. Then

$$(\underline{y}, \overline{y}) \succcurlyeq (\underline{x}, \overline{x}) \Rightarrow \overline{y} - \underline{y} \geq \overline{x} - \underline{x}.$$

We now introduce the notion of a social preorder.[5]

Definition 1. A ***social preorder*** is a reflexive and transitive binary relation on $\mathcal{Z}_n$ satisfying Axiom 1.

The asymmetric and symmetric parts of $\succcurlyeq$ are denoted by $\succ$ and $\sim$, respectively.

Social preorders are not necessarily complete. They measure income divergence or spread between two income groups: "$\boldsymbol{z} \succcurlyeq \boldsymbol{z}'$" means that "the perfectly bimodal income distribution $\boldsymbol{z}$ is more spread out than $\boldsymbol{z}'$." Observe that the ranking induced by $\succcurlyeq$ involves no welfare comparison between income distributions in the standard neoclassical sense.

The preorder induced by the Lorenz dominance relation is a particular example of a social preorder. Indeed, Axiom 1 is weaker than the Lorenz criterion, which is formally defined next.

[4] We omit the analysis for the case when n is odd, which is similar but requires its own separate terminology.

[5] To aid reader comprehension, Appendix A contains a full list of terminologies for the various preorders introduced throughout the paper.

Definition 2. Given two income distributions $\boldsymbol{z} = (z_1, \ldots, z_n)$ and $\boldsymbol{z}' = (z'_1, \ldots, z'_n)$, $\boldsymbol{z}' \succcurlyeq_{RL} \boldsymbol{z}$ if and only if $\boldsymbol{z}$ Lorenz dominates $\boldsymbol{z}'$ in the relative sense, that is, if

$$\frac{\sum_{i=1}^{k} z_i}{\sum_{i=1}^{n} z_i} \geq \frac{\sum_{i=1}^{k} z'_i}{\sum_{i=1}^{n} z'_i}, \quad \text{for all } k \in \{1, \ldots, n\}.$$

The notion of absolute Lorenz dominance can be found in Moyes (1988).

Definition 3. Given two income distributions $\boldsymbol{z} = (z_1, \ldots, z_n)$ and $\boldsymbol{z}' = (z'_1, \ldots, z'_n)$ with $z_n, z'_n > 0$, $\boldsymbol{z}' \succcurlyeq_{AL} \boldsymbol{z}$ if and only if $\boldsymbol{z}$ Lorenz dominates $\boldsymbol{z}'$ in the absolute sense, that is, if

$$\sum_{i=1}^{k} z_i - \sum_{i=1}^{n} z_i \geq \sum_{i=1}^{k} z'_i - \sum_{i=1}^{n} z'_i, \quad \text{for all } k \in \{1, \ldots, n\}.$$

Proposition 1. *The preorder $\succcurlyeq_{RL}$ (resp., $\succcurlyeq_{AL}$) on $\mathcal{Z}_n$ satisfies Axiom 1.*

Proof. Suppose that $(\underline{x}, \overline{x})$ and $(\underline{y}, \overline{y})$ are perfectly bimodal income distributions with the same total income $(\underline{y} + \overline{y} = \underline{x} + \overline{x})$. Suppose further that $(\underline{y}, \overline{y}) \succcurlyeq_{RL} (\underline{x}, \overline{x})$ (resp., $(\underline{y}, \overline{y}) \succcurlyeq_{AL} (\underline{x}, \overline{x})$). Then $\underline{x} \geq \underline{y}$ and $\overline{x} \leq \overline{y}$. Therefore, $\overline{y} - \underline{y} \geq \overline{x} - \underline{x}$, and hence Axiom 1 is satisfied. ■

We now consider invariance properties of a social preorder. Two well-accepted invariance concepts are concerned with invariance with respect to proportional changes (resp., translations) in income levels.

An income distribution $\boldsymbol{z}'$ is obtained from $\boldsymbol{z} \in \mathcal{Z}_n$ by a ***proportional change*** in incomes if

$$(z'_1, \ldots, z'_n) = \alpha(z_1, \ldots, z_n), \quad \text{for some scalar } \alpha > 0.$$

An income distribution $\boldsymbol{z}' \in \mathcal{Z}_n$ is obtained from $\boldsymbol{z} \in \mathcal{Z}_n$ by a ***translation*** in incomes if

$$\boldsymbol{z}' = (z_1 + \alpha, \ldots, z_n + \alpha), \quad \text{for some scalar } \alpha \in \mathbb{R}.$$

The following notions of invariance are standard in the literature.

Scale Invariance (SI). Suppose that $\succcurlyeq$ is a social preorder. If $\boldsymbol{z}'$ is obtained from $\boldsymbol{z} \in \mathcal{Z}_n$ by a proportional change in incomes, then $\boldsymbol{z}' \sim \boldsymbol{z}$.

Translation Invariance (TI). Suppose that $\succcurlyeq$ is a social preorder. If $\boldsymbol{z}' \in \mathcal{Z}_n$ is obtained from $\boldsymbol{z} \in \mathcal{Z}_n$ by a translation in incomes, then $\boldsymbol{z}' \sim \boldsymbol{z}$.

2.2 Tax schedules and progressivity

A ***tax schedule*** is a continuous and nondecreasing map $T : \mathbb{R}_+ \to \mathbb{R}$ that assigns to each income level $z \in \mathbb{R}_+$ a tax liability, $T(z)$, and satisfies the following conditions:

- $T(z) \leq z$ for each $z \in \mathbb{R}_+$; and
- the map $z \mapsto z - T(z)$ is nondecreasing (i.e., T is order-preserving).

A negative tax liability represents a subsidy.

We restrict attention to the class of piecewise linear tax schedules.

Definition 4. A tax schedule T is a $(K+1)$***-bracket piecewise linear tax schedule*** if

$$T(y) := \begin{cases} -\alpha_0 + t_0 y & \text{if } 0 = \overline{y}_0 \leq y \leq \overline{y}_1, \\ -\alpha_0 + t_0 \overline{y}_1 + t_1(y - \overline{y}_1) & \text{if } \overline{y}_1 < y \leq \overline{y}_2, \\ \quad \vdots & \quad \vdots \\ -\alpha_0 + t_0 \overline{y}_1 + t_1(\overline{y}_2 - \overline{y}_1) + \cdots + t_{K-1}(\overline{y}_K - \overline{y}_{K-1}) + t_K(y - \overline{y}_K) & \text{if } \overline{y}_K < y, \end{cases}$$

where $\alpha_0 \geq 0$, $K \in \mathbb{Z}_+$, $t_k \in [0,1)$ for each $k \in \{0, \ldots, K\}$, $t_k \neq t_{k+1}$ whenever $k \in \{0, \ldots, K-1\}$ and $K \geq 1$, and $0 = \overline{y}_0 < \cdots < \overline{y}_K$.

A generic $(K+1)$-bracket piecewise linear tax schedule is completely determined by a tuple

$$(\alpha_0, \boldsymbol{t}, \overline{\boldsymbol{y}}) = (\alpha_0, (t_0, \dots, t_K), (\overline{y}_0, \dots, \overline{y}_K)).$$

For $K \in \mathbb{Z}_+$, the set of $(K+1)$-bracket piecewise linear tax schedules is denoted by $\mathscr{T}_K$, and the set of all piecewise linear tax schedules in $\mathscr{T}$ is defined as

$$\mathscr{T} := \bigcup_{K=1}^{\infty} \mathscr{T}_K.$$

A tax schedule $T \in \mathscr{T}$ is ***linear*** if $T(y) = -\alpha_0 + t_0 y$ for all $y \in \mathbb{R}_+$ and some $\alpha_0 \geq 0$ and $t_0 \in [0,1)$. Denote the set of all linear tax schedules in $\mathscr{T}$ by $\mathscr{T}_{lin}$.

The following notion of progressivity plays a central role in the results.

Definition 5. A tax schedule $T \in \mathscr{T}$ is ***marginal-rate progressive*** if it is a convex function.

In words, a tax schedule is marginal-rate progressive if it exhibits nondecreasing marginal tax rates on income. It is easy to see that a tax schedule $T \in \mathscr{T}$ is marginal-rate progressive only if it is average-rate progressive, but the converse assertion is not generally true.[6]

The set of all marginal-rate progressive tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{m\text{-}prog}$.

It will be useful to define the following subclasses of $\mathscr{T}$. Given $b \geq 0$ and $R \subseteq [0,1)$, let

$$\mathscr{T}(b,R) := \{(\alpha_0, \boldsymbol{t}, \overline{\boldsymbol{y}}) \in \mathscr{T} : \alpha_0 \geq b \text{ and } t_k \in R \text{ for all } k\}.$$

In words, $\mathscr{T}(b,R)$ is the set of all piecewise linear tax schedules that endow all agents with a subsidy of at least b and whose marginal tax rates t_k $(k = 0,1,\dots)$ are contained in the subset R. Note that for $b = 0$ and $R = [0,1)$, $\mathscr{T}(b,R)$ is simply the set of all piecewise linear taxes, $\mathscr{T}$.

The intersection $\mathscr{T}(b,R) \cap \mathscr{T}_{m\text{-}prog}$ will be denoted by $\mathscr{T}_{m\text{-}prog}(b,R)$. Likewise, the intersection $\mathscr{T}(b,R) \cap \mathscr{T}_{lin}$ will be denoted by $\mathscr{T}_{lin}(b,R)$.

2.3 Endogenous income distributions

To allow for potential disincentive effects of taxation on work effort, we adopt the standard Mirrlees model (Mirrlees, 1971). The description of the setup follows Carbonell-Nicolau and Llavador (2021a) closely.

The utility function $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$, defined over consumption-labor pairs $(c,l) \in \mathbb{R}_+ \times [0,1]$, is assumed continuous with $u(\cdot,l)$ strictly increasing in c for each $l \in [0,1)$ and $u(c,\cdot)$ strictly decreasing in l for each $c > 0$. In addition, we assume that u is strictly quasiconcave on $\mathbb{R}_{++} \times [0,1)$ and twice continuously differentiable on $\mathbb{R}_{++} \times (0,1)$, and that there exists $l > 0$ such that $u(c,l) > u(\mathbf{0})$ whenever $c > 0$.[7]

For $(c,l) \in \mathbb{R}_{++} \times (0,1)$, let

$$MRS(c,l) := -\frac{u_l(c,l)}{u_c(c,l)}$$

denote the marginal rate of substitution of labor for consumption, where

$$u_c(c,l) := \frac{\partial u(c,l)}{\partial c} \quad \text{and} \quad u_l(c,l) := \frac{\partial u(c,l)}{\partial l}.$$

[6] A tax schedule is *average-rate progressive* if it exhibits nondecreasing average tax rates on income (i.e., if the map $z \mapsto T(z)/z$ is nondecreasing).

[7] The last assumption is only needed for consistency in the definition of a bipolarization preorder, and it merely implies that individuals consume a positive amount.

The following assumptions will be maintained for each $c > 0$:

$$\lim_{l \to 1^-} MRS(c,l) = +\infty \quad \text{and} \quad \lim_{l \to 0^+} MRS(c,l) < +\infty. \tag{1}$$

The first condition states that the compensation required by an individual for an extra unit of working time tends to infinity as the agent's leisure time approaches zero. The second condition is a mild finiteness condition on the marginal rate of substitution of labor for consumption.

An ***ability distribution*** is a vector $\boldsymbol{a} = (a_1, \ldots, a_n)$ in $\mathbb{R}^n_{++}$ such that the coordinates in $\boldsymbol{a}$ are arranged in increasing order, *i.e.*, $a_1 \leq \cdots \leq a_n$; here, for each i, a_i represents the ability level of agent i. Let $\mathscr{A}_n$ represent the set of all ability distributions with population size n.

An agent of ability $a > 0$ who chooses $l \in [0,1]$ units of labor and faces a tax schedule $T \in \mathscr{T}$ consumes $c = al - T(al)$ units of the good and derives a utility of $u(c,l)$. Thus, the agent's problem is

$$\max_{l \in [0,1]} u(al - T(al), l). \tag{2}$$

A solution function is a map $l^u : \mathbb{R}_{++} \times \mathscr{T} \to [0,1]$ such that $l^u(a,T)$ is a solution to (2) for each $(a,T) \in \mathbb{R}_{++} \times \mathscr{T}$. A ***solution function*** l^u induces ***pre-tax*** and ***post-tax income functions*** $y^u : \mathbb{R}_{++} \times \mathscr{T} \to \mathbb{R}_+$ and $x^u : \mathbb{R}_{++} \times \mathscr{T} \to \mathbb{R}_+$, respectively, defined by

$$y^u(a,T) := al^u(a,T) \quad \text{and} \quad x^u(a,T) := al^u(a,T) - T(al^u(a,T)).$$

We write $y^u(a,0)$ and $x^u(a,0)$ to denote, respectively, the pre-tax and post-tax incomes of an a-type in the absence of taxation, i.e., when the tax schedule is identically zero ($T \equiv 0$).

Given $a > 0$, let $U^a : \mathbb{R}_+ \times [0,a] \to \mathbb{R}$ be defined by $U^a(c,y) := u(c, y/a)$. For $(c,y,a) \in \mathbb{R}^3_{++}$ with $y < a$, define

$$\eta^a(c,y) := -\frac{\partial U^a(c,y)}{\partial y} \bigg/ \frac{\partial U^a(c,y)}{\partial c}.$$

The following is the standard *agent monotonicity* condition introduced by Mirrlees (1971) (see also Seade (1982) and Myles (1995, p. 136)).

Definition 6. A utility function satisfies ***agent monotonicity*** if $\eta^a(c,y) > \eta^{a'}(c,y)$ for each $(c,y) \in \mathbb{R}^2_+$ and $0 < a < a'$ with $y < a$.[8]

The set of utility functions satisfying the conditions in (1) and agent monotonicity is denoted by $\mathscr{U}$. This domain contains the standard utility functions used in the literature (see Carbonell-Nicolau and Llavador, 2018).

For each $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_+$, consider the problem

$$\max_{l \in [0,1]} u(al + b, l). \tag{3}$$

This is the problem faced by an a-agent who receives a subsidy b. Since u is strictly quasiconcave on $\mathbb{R}_{++} \times [0,1)$, for each $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_+$, there is a unique solution $l^u(a,b)$ to (3). For given $b \geq 0$, the derivative of the map $a \mapsto l^u(a,b)$ exists for all but at most one $a > 0$ (see Carbonell-Nicolau and Llavador (2018)).

[8] Mirrlees (1971) uses the weak version of this monotonicity condition (i.e., $\eta^a(c,y) \geq \eta^{a'}(c,y)$ for each $(c,y) \in \mathbb{R}^2_+$ and $0 < a < a'$ with $y < a$). All our results, except for Theorem 7, remain intact under the weaker condition.

3 Order-reducing taxation

The focus of this paper is on the characterization of tax structures inducing order-reducing post-tax income distributions, relative to the taxless distribution, with respect to a social preorder.

Definition 7. Given $u \in \mathscr{U}$ and a social preorder $\succcurlyeq$, a tax schedule $T \in \mathscr{T}$ is $(\succcurlyeq, u)$***-reducing with respect to*** $(\succcurlyeq, u)$, denoted by $(\succcurlyeq, u)$***-r***, if

$$(y^u(a_1,0),\dots,y^u(a_n,0)) \succcurlyeq (x^u(a_1,T),\dots,x^u(a_n,T))$$

for every ability distribution $(a_1,\dots,a_n) \in \mathscr{A}_n$ and every pre-tax and post-tax income functions y^u and x^u.

For $u \in \mathscr{U}$, the set of all $(\succcurlyeq, u)$-r tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{(\succcurlyeq,u)\text{-}r}$.

Our first result states that, for any invariant social preorder, order-reducing tax schedules must necessarily be marginal-rate progressive.

Theorem 1. *For $u \in \mathscr{U}$ and a social preorder $\succcurlyeq$ satisfying either SI or TI,*

$$\mathscr{T}_{(\succcurlyeq,u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}.$$

We use the following lemmas, whose proofs are relegated to Appendix B, to prove the theorem. The first two lemmas show that if a social preorder satisfies SI (resp. TI), then $(\succcurlyeq, u)$-r tax schedules are inequality-reducing according to the relative (resp. absolute) Lorenz criterion. The third lemma provides a way to check whether tax schedules are inequality-reducing with respect to the absolute Lorenz criterion.

Lemma 1. *For $u \in \mathscr{U}$ and a social preorder $\succcurlyeq$ satisfying SI, $\mathscr{T}_{(\succcurlyeq,u)\text{-}r} \subseteq \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r}$.*

Lemma 2. *For $u \in \mathscr{U}$ and a social preorder $\succcurlyeq$ satisfying TI, $\mathscr{T}_{(\succcurlyeq,u)\text{-}r} \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$.*

Lemma 3. *For $u \in \mathscr{U}$, a tax schedule $T \in \mathscr{T}$ is $(\succcurlyeq_{AL}, u)$-r if and only if, for any pre-tax and post-tax income functions x^u and y^u,*

$$y^u(a',0) - y^u(a,0) \geq x^u(a',T) - x^u(a,T), \quad \textit{whenever } a' > a > 0. \tag{4}$$

Proof of Theorem 1. First, from Theorem 3 in Carbonell-Nicolau and Llavador (2021b), $\mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}$. It follows from Lemma 1 that if $\succcurlyeq$ satisfies SI then $\mathscr{T}_{(\succcurlyeq,\,u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}$. Next, let $\succcurlyeq$ satisfy TI. Because of Lemma 2, it is sufficient to show that if $T \notin \mathscr{T}_{m-prog}$ then $T \notin \mathscr{T}_{(\succcurlyeq_{AL},u)-\mathrm{r}}$. Let y^u and x^u be any two pre-tax and post-tax income functions. The map $a \mapsto y^u(a,0)$ is continuous on $\mathbb{R}_{++}$, while the map $a \mapsto x^u(a,T)$ is non-decreasing (Lemma 1 in Carbonell-Nicolau and Llavador, 2018) and has at least one discontinuity point (Lemma 4 in Carbonell-Nicolau and Llavador, 2018). Let $a \in \mathbb{R}_{++}$ be one such discontinuity point. Then

$$\lim_{a\uparrow a} x^u(a,T) < \lim_{a\downarrow a} x^u(a,T) \quad \text{and} \quad \lim_{a\uparrow a} y^u(a,0) = \lim_{a\downarrow a} y^u(a,0),$$

implying that

$$\lim_{a\downarrow a} x^u(a,T) - \lim_{a\uparrow a} x^u(a,T) > \lim_{a\downarrow a} y^u(a,0) = \lim_{a\uparrow a} y^u(a,0) = 0.$$

Hence, we can find $a' > a > 0$ violating condition (4) in Lemma 3, and so $T \notin \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$, as we wanted to prove. ∎

We have seen that marginal-rate progressivity is a necessary condition for the reduction of income dispersion, according to a weak and appealing notion of variance. The following sections deepen the analysis and study of inequality and polarization-based social preorders, providing necessary and sufficient conditions.

4 Inequality and progressivity

In this section, we define inequality preorders as binary relations on $\mathcal{Z}_n$ satisfying the transfer principle and characterize inequality-reducing tax schedules for scale- and translation-invariant inequality preorders.

A distribution $\boldsymbol{z}'$ is obtained from $\boldsymbol{z} \in \mathcal{Z}_n$ by a ***progressive transfer*** if $\boldsymbol{z}'$ results from a transfer of income from one individual to anyone poorer up to the point where both individuals have the same income.

A fundamental tenet in the literature on inequality is that its measurement should adhere to the transfer principle.[9]

Transfer Principle (TP) (Pigou, 1912; Dalton, 1920)**.** Suppose that $\succcurlyeq$ is a social preorder. If $\boldsymbol{z}'$ is obtained from $\boldsymbol{z} \in \mathcal{Z}_n$ by a progressive transfer, then $\boldsymbol{z} \succ \boldsymbol{z}'$.

We define an inequality preorder as a binary relation on income distributions satisfying the Transfer Principle.

Definition 8. An ***inequality preorder*** $\succcurlyeq_I$ is a binary relation on $\mathcal{Z}_n$ satisfying TP, with "$\boldsymbol{z}' \succcurlyeq_I \boldsymbol{z}$" understood as "$\boldsymbol{z}'$ is (weakly) more unequal than $\boldsymbol{z}$."

Because TP is stronger than Axiom 1, we have the following result.

Proposition 2. *Any inequality preorder satisfies Axiom 1.*

Proof. Suppose that $(\underline{x}, \overline{x})$ and $(\underline{y}, \overline{y})$ are perfectly bimodal income distributions with the same total income. Suppose further that $(\underline{y}, \overline{y}) \succcurlyeq_I (\underline{x}, \overline{x})$. We must show that

$$\overline{y} - \underline{y} \geq \overline{x} - \underline{x}. \tag{5}$$

This inequality is clearly true if $(\underline{x}, \overline{x}) = (\underline{y}, \overline{y})$, so suppose that $(\underline{x}, \overline{x}) \neq (\underline{y}, \overline{y})$. Then, since $(\underline{x}, \overline{x})$ and $(\underline{y}, \overline{y})$ have the same total income, it follows that

$$\text{either} \quad [\underline{x} < \underline{y} \;\&\; \overline{x} > \overline{y}] \quad \text{or} \quad [\underline{x} > \underline{y} \;\&\; \overline{x} < \overline{y}].$$

In the first case, $(\underline{y}, \overline{y})$ can be obtained from $(\underline{x}, \overline{x})$ by a progressive transfer, implying (by TP) that $(\underline{x}, \overline{x}) \succ_I (\underline{y}, \overline{y})$, a contradiction.[10] Hence,

$$\underline{x} > \underline{y} \quad \text{and} \quad \overline{x} < \overline{y},$$

which yields (5). ∎

Next, we study the inequality-reducing properties of marginal-rate progressive tax schedules with respect to scale-invariant and translation-invariant inequality preorders. Our characterization of tax progressivity exploits the fact that inequality preorders are Lorenz consistent in the following sense.

Definition 9. Given a binary relation $\succcurlyeq$ on $\mathcal{Z}_n$, an inequality preorder $\succcurlyeq_I$ is $\succcurlyeq$***-consistent*** if the following two conditions are satisfied:

1. $\boldsymbol{x} \sim \boldsymbol{y} \Rightarrow \boldsymbol{x} \sim_I \boldsymbol{y}$.
2. $\boldsymbol{x} \succ \boldsymbol{y} \Rightarrow \boldsymbol{x} \succ_I \boldsymbol{y}$.

We write "$\succcurlyeq_{RL}$*-consistent*" for relative Lorenz consistency and "$\succcurlyeq_{AL}$*-consistent*" for absolute Lorenz consistency.

[9] The anonymity principle, according to which two income distributions are in the same equivalence class whenever one is a permutation of the other, is also widely adopted. Our treatment implicitly assumes anonymity since an income distribution has been defined as a vector with its coordinates arranged in increasing order.

[10] Observe that $(\underline{x}, \overline{x}) \succ_I (\underline{y}, \overline{y})$ implies that $(\underline{x}, \overline{x}) \succcurlyeq_I (\underline{y}, \overline{y})$, which, combined with the relation $(\underline{y}, \overline{y}) \succcurlyeq_I (\underline{x}, \overline{x})$, gives $(\underline{x}, \overline{x}) \sim_I (\underline{y}, \overline{y})$. This contradicts the fact that $(\underline{x}, \overline{x}) \succ_I (\underline{y}, \overline{y})$.

4.1 Scale-invariant inequality preorders

A scale-invariant inequality preorder $\succcurlyeq_{SI}$ is an inequality preorder that satisfies SI. Scale invariance is equivalent to requiring consistency with the preorder induced by the relative Lorenz criterion.

Lemma 4. *An inequality preorder $\succcurlyeq_I$ satisfies SI if and only if it is $\succcurlyeq_{RL}$-consistent.*

Proof. See Appendix C. ■

Because inequality preorders are social preorders (Proposition 2), it follows from Theorem 1 that scale-invariant, inequality-reducing tax schedules are necessarily marginal-rate progressive. A full characterization of tax progressivity for scale-invariant inequality preorders requires a condition on the income elasticity with respect to ability.

For $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_+$, let

$$\zeta^u(a,b) := \frac{\partial(al^u(a,b)+b)}{\partial a} \cdot \frac{a}{al^u(a,b)+b}$$

be the elasticity of income with respect to ability at (a,b).

For $b \geq 0$ and $R \subseteq [0,1)$, let $\mathscr{U}_{SI}(b,R)$ be the set of all $u \in \mathscr{U}$ such that

$$\zeta^u((1-r)a,b') \leq \zeta^u(a,0), \quad \text{for all } (a,b',r) \in \mathbb{R}_{++} \times [b,\infty) \times R.^{11} \tag{6}$$

Theorem 2. *For $u \in \mathscr{U}$, $b \geq 0$, $R \subseteq [0,1)$, and a scale-invariant inequality preorder $\succcurlyeq_{SI}$,*

$$\left[\mathscr{T}_{(\succcurlyeq_{SI},u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}\right] \quad \textit{and} \quad \left[\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{SI},u)\text{-}r} \Leftrightarrow u \in \mathscr{U}_{SI}(b,R)\right].$$

Proof. Proposition 2 and Theorem 1 give the first containment. The bracketed equivalence follows from Theorem 3 in Carbonell-Nicolau and Llavador (2021b) and the fact that $\mathscr{T}_{(\succcurlyeq_{SI},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r}$. To see that $\mathscr{T}_{(\succcurlyeq_{SI},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r}$, note that Proposition 2 and Lemma 1 give $\mathscr{T}_{(\succcurlyeq_{SI},u)\text{-}r} \subseteq \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r}$. The containment $\mathscr{T}_{(\succcurlyeq_{SI},u)\text{-}r} \supseteq \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r}$ follows from Lemma 4. ■

The bracketed equivalence states that the members of the set $\mathscr{T}_{m\text{-}prog}(b,R)$ of all marginal-rate progressive tax schedules in $\mathscr{T}_{m\text{-}prog}$ whose intercept α_0 is greater than or equal to b and whose marginal tax rates lie in R are all inequality-reducing if and only if $u \in \mathscr{U}_{SI}(b,R)$, *i.e.*, if and only if the elasticity of income with respect to ability satisfies condition (6).

The result in Theorem 2 also holds for the preorder induced by the relative Lorenz criterion.

Corollary 1 (to Theorem 2)**.** *For $u \in \mathscr{U}$, $b \geq 0$, and $R \subseteq [0,1)$,*

$$\left[\mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}\right] \quad \textit{and} \quad \left[\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r} \Leftrightarrow u \in \mathscr{U}_{SI}(b,R)\right].$$

Proof. The assertion follows immediately from Theorem 2 and the fact that $\succcurlyeq_{RL}$ satisfies SI and TP. ■

In the special case when the lower bound on the intercept α_0 of a tax schedule is zero and the marginal tax rates can take values anywhere in the interval $[0,1)$, Theorem 2 immediately gives the following result, which was first proven for the relative Lorenz criterion in Carbonell-Nicolau and Llavador (2018, Corollary 3).

Corollary 2 (to Theorem 2)**.** *For $u \in \mathscr{U}$ and a scale-invariant inequality preorder $\succcurlyeq_{SI}$,*

$$\mathscr{T}_{(\succcurlyeq_{SI},u)\text{-}r} = \mathscr{T}_{m\text{-}prog} \Leftrightarrow u \in \mathscr{U}_{SI}(0,[0,1)).$$

The reader is referred to Carbonell-Nicolau and Llavador (2021a) for applications of Theorem 2 to the families of the CES and quasilinear utility functions.

[11] Refer to Carbonell-Nicolau and Llavador (2021a) for an interpretation of condition (6) based on a decomposition of the inequality in (6) into two conditions on the wage elasticity of income, each capturing different aspects of the transition between before-tax and after-tax income distributions.

4.2 Translation-invariant inequality preorders

Similar results can be proven for translation-invariant inequality preorders, $\succcurlyeq_{TI}$, defined as inequality preorders satisfying TI. Under translation invariance, the full characterization of tax progressivity requires a condition on the wage elasticity of the labor supply rather than the wage elasticity of income.

First, it follows from Theorem 1 that translation-invariant, inequality-reducing tax schedules are necessarily marginal-rate progressive. Second, an inequality preorder is translation-invariant if and only if it is absolute Lorenz consistent.

Lemma 5. *An inequality preorder satisfies* TI *if and only if it is* $\succcurlyeq_{AL}$*-consistent.*

Proof. See Appendix C. ■

The following theorem provides the characterization of marginal-rate progressivity for translation-invariant inequality preorders. The condition on the utility function requires the definition of the wage elasticity of the labor supply. For $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$, let

$$\xi^u(a,b) := \frac{\partial l^u(a,b)}{\partial a} \cdot \frac{a}{l^u(a,b)}$$

be the *wage elasticity of the labor supply* at (a,b). For $b \geq 0$ and $R \subseteq [0,1)$, let $\mathscr{U}_{TI}(b,R)$ be the set of all $u \in \mathscr{U}$ such that

$$\begin{aligned} & y^u\left((1-r)a,b'\right)\left[1+\xi^u\left((1-r)a,b'\right)\right] \\ & \qquad \leq y^u(a,0)\left[1+\xi^u(a,0)\right], \quad \text{for all } (a,b',r) \in \mathbb{R}_{++} \times [b,\infty) \times R, \end{aligned} \tag{7}$$

Now, we can state the theorem.

Theorem 3. *For $u \in \mathscr{U}$, $b \geq 0$, $R \subseteq [0,1)$, and a translation-invariant inequality preorder $\succcurlyeq_{TI}$,*

$$\left[\mathscr{T}_{(\succcurlyeq_{TI},u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}\right] \quad \textit{and} \quad \left[\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{TI},u)\text{-}r} \Leftrightarrow u \in \mathscr{U}_{TI}(b,R)\right].$$

The bracketed equivalence states that the members of the set $\mathscr{T}_{m\text{-}prog}(b,R)$ of all marginal-rate progressive tax schedules in $\mathscr{T}_{m\text{-}prog}$ whose intercept a_0 is greater than or equal to b and whose marginal tax rates lie in R are all inequality-reducing if and only if $u \in \mathscr{U}_{TI}(b,R)$, *i.e.*, if and only if the wage elasticity of labor satisfies condition (7).

We use the following two lemmas, whose proofs are relegated to Appendix D, to establish Theorem 3. The two lemmas combined imply that it suffices to obtain conditions on preferences ensuring that the linear progressive tax schedules are inequality-reducing.

Lemma 6. *For $u \in \mathscr{U}$, $b \geq 0$, and $R \subseteq [0,1)$, $\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r} \Leftrightarrow \mathscr{T}_{lin}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$.*

Lemma 7. *For $u \in \mathscr{U}$, $b \geq 0$, and $R \subseteq [0,1)$, $\mathscr{T}_{lin}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r} \Leftrightarrow u \in \mathscr{U}_{TI}(b,R)$.*

Proof of Theorem 3. Proposition 2 and Theorem 1 give the first containment. The bracketed equivalence follows from Lemma 6, Lemma 7, and the fact that $\mathscr{T}_{(\succcurlyeq_{TI},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$. To see that $\mathscr{T}_{(\succcurlyeq_{TI},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$, note that Proposition 2 and Lemma 2 give $\mathscr{T}_{(\succcurlyeq_{TI},u)\text{-}r} \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$. The containment $\mathscr{T}_{(\succcurlyeq_{TI},u)\text{-}r} \supseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$ follows from Lemma 5. ■

The result in Theorem 3 also holds for the preorder induced by the absolute Lorenz criterion.

Corollary 3 (to Theorem 3)**.** *For $u \in \mathscr{U}$, $b \geq 0$, and $R \subseteq [0,1)$,*

$$\left[\mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}\right] \quad \textit{and} \quad \left[\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r} \Leftrightarrow u \in \mathscr{U}_{TI}(b,R)\right].$$

Proof. The assertion follows immediately Theorem 3 and the fact that $\succcurlyeq_{AL}$ satisfies TI and TP. ■

In the special case when the lower bound on the intercept α_0 of a tax schedule is zero and the marginal tax rates can take values anywhere in the interval $[0,1)$, Theorem 3 immediately gives the following equivalence.

Corollary 4 (to Theorem 3)**.** *For $u \in \mathscr{U}$ and a translation-invariant inequality preorder $\succcurlyeq_{TI}$,*

$$\mathscr{T}_{(\succcurlyeq_{TI},u)\text{-}r} = \mathscr{T}_{m\text{-}prog} \Leftrightarrow u \in \mathscr{U}_{TI}(0,[0,1)).$$

5 Bipolarization and progressivity

In this section, we define bipolarization preorders as binary relations on $\mathcal{Z}_n$ satisfying the increased spread and increased bipolarity axioms (Chakravarty, 2015; Foster and Wolfson, 2010). We then define relative and absolute bipolarity, in a way analogous to the formulation of the relative and absolute Lorenz dominance relation. Finally, we characterize bipolarization-reducing tax schedules for scale-invariant and translation-invariant bipolarization preorders.

Prior to defining a bipolarization preorder, we need some preliminary notation.

For any $\boldsymbol{z} \in \mathcal{Z}_n$, let

$$z_m := \left(z_{\frac{n}{2}} + z_{\frac{n}{2}+1}\right) \Big/ 2, \quad \boldsymbol{z}_- = (z_1, \ldots, z_{m-1}), \quad \text{and} \quad \boldsymbol{z}_+ = (z_{m+1}, \ldots, z_n).$$

Let "$\boldsymbol{z} < \boldsymbol{z}'$" mean that "$z_i \leq z'_i$ for all i with at least one strict inequality;"[12] and let "$\boldsymbol{z}' T \boldsymbol{z}$" represent the condition that $\boldsymbol{z}'$ is obtained from $\boldsymbol{z}$ by a progressive transfer.

We now define the two standard axioms for bipolarization measures.

Increased Spread (IS) (Chakravarty (2015); Foster and Wolfson (2010))**.** Let $\succcurlyeq$ be a binary relation on $\mathcal{Z}_n$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ have the same median ($x_m = y_m$) and
[$\boldsymbol{x}_- = \boldsymbol{y}_-$ and $\boldsymbol{y}_+ < \boldsymbol{x}_+$] or
[$\boldsymbol{x}_- < \boldsymbol{y}_-$ and $\boldsymbol{y}_+ = \boldsymbol{x}_+$] or
[$\boldsymbol{x}_- < \boldsymbol{y}_-$ and $\boldsymbol{y}_+ < \boldsymbol{x}_+$], then $\boldsymbol{x} \succ \boldsymbol{y}$.

Increased Bipolarity (IB) (Chakravarty (2015); Foster and Wolfson (2010))**.** Let $\succcurlyeq$ be a binary relation on $\mathcal{Z}_n$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ have the same median ($x_m = y_m$) and
[$\boldsymbol{y}_- = \boldsymbol{x}_-$ and $\boldsymbol{y}_+ T \boldsymbol{x}_+$] or
[$\boldsymbol{y}_- T \boldsymbol{x}_-$ and $\boldsymbol{y}_+ = \boldsymbol{x}_+$] or
[$\boldsymbol{y}_- T \boldsymbol{x}_-$ and $\boldsymbol{y}_+ T \boldsymbol{x}_+$], then $\boldsymbol{x} \succ \boldsymbol{y}$.

Definition 10. A ***bipolarization preorder*** $\succcurlyeq_B$ is a binary relation on $\mathcal{Z}_n$ satisfying IS and IB, with "$\boldsymbol{z}' \succcurlyeq_B \boldsymbol{z}$" understood as "$\boldsymbol{z}'$ is (weakly) more bipolarized than $\boldsymbol{z}$."

The Increased Spread axiom (IS) is stronger than Axiom 1, and so bipolarization preorders are a special case of social preorders.

Proposition 3. *Let $\succcurlyeq$ be a binary relation on $\mathcal{Z}_n$ satisfying IS. Then $\succcurlyeq$ satisfies Axiom 1.*

[12] We write "$\boldsymbol{x} \gg \boldsymbol{y}$" to represent the case when $x_i > y_i$ for all i; "$\boldsymbol{x} > \boldsymbol{y}$" if $x_i \geq y_i$ for all i and $x_j > y_j$ for at least one j; and "$\boldsymbol{x} \geq \boldsymbol{y}$" if $x_i \geq y_i$ for all i, i.e., either $\boldsymbol{x} > \boldsymbol{y}$ or $\boldsymbol{x} = \boldsymbol{y}$.

Proof. Suppose that $(\underline{x},\overline{x})$ and $(\underline{y},\overline{y})$ are perfectly bimodal income distributions with the same total income. Suppose further that $(\underline{y},\overline{y}) \succcurlyeq (\underline{x},\overline{x})$. We must show that

$$\overline{y} - \underline{y} \geq \overline{x} - \underline{x}. \tag{8}$$

This inequality is clearly true if $(\underline{x},\overline{x}) = (\underline{y},\overline{y})$, so suppose that $(\underline{x},\overline{x}) \neq (\underline{y},\overline{y})$. Then, since $(\underline{x},\overline{x})$ and $(\underline{y},\overline{y})$ have the same total income, it follows that

$$\text{either} \quad [\underline{x} < \underline{y} \;\&\; \overline{x} > \overline{y}] \quad \text{or} \quad [\underline{x} > \underline{y} \;\&\; \overline{x} < \overline{y}].$$

In the first case, IS gives $(\underline{x},\overline{x}) \succ (\underline{y},\overline{y})$, a contradiction.[13] Hence,

$$\underline{x} > \underline{y} \quad \text{and} \quad \overline{x} < \overline{y},$$

which yields (8). ■

Next, we study the bipolarization-reducing properties of marginal-rate progressive tax schedules for scale-invariant and translation-invariant measures of bipolarization.

5.1 The relative bipolarization preorder

Given two income distributions $\boldsymbol{z}$ and $\boldsymbol{z}'$ in $\mathscr{Z}_n$, $\boldsymbol{z}'$ is relatively more bipolarized than $\boldsymbol{z}$ if the relative bipolarization curve for $\boldsymbol{z}'$ does not lie below the relative bipolarization curve for $\boldsymbol{z}$. The relative bipolarization curve is a normalized measure of a distribution's aggregate deviation from the median income.

Formally, the relative bipolarization curve RB is constructed as follows (Carbonell-Nicolau and Llavador, 2021b; Chakravarty, 2015). Given $\alpha \in [0,1]$ and any income distribution $\boldsymbol{z} \in \mathscr{Z}_n$, define

$$RB(\boldsymbol{z},\alpha) := \begin{cases} \frac{1}{nz_m}\sum_{j\leq i<m}(z_m - z_i) & \text{if } \alpha = \frac{j}{n} \text{ for some } j \in \{1,\dots,m-1\}, \\ \frac{1}{nz_m}\sum_{m\leq i\leq j}(z_i - z_m) & \text{if } \alpha = \frac{j}{n} \text{ for some } j \in \{m,\dots,n\}, \\ 1 & \text{if } \alpha = 0, \end{cases} \tag{9}$$

and

$$RB(\boldsymbol{z},\alpha) := \lambda RB\left(\boldsymbol{z},\frac{j}{n}\right) + (1-\lambda)RB\left(\boldsymbol{z},\frac{j+1}{n}\right), \quad \text{if } \alpha = \lambda\left(\tfrac{j}{n}\right) + (1-\lambda)\left(\tfrac{j+1}{n}\right), \tag{10}$$

where $\lambda \in (0,1)$ and $j \in \{0,1,\dots,n-1\}$.

Definition 11. The ***relative bipolarization preorder*** $\succcurlyeq_{RB}$ is a binary relation on $\mathscr{Z}_n$ such that, given two income distributions $\boldsymbol{z}$ and $\boldsymbol{z}'$ in $\mathscr{Z}_n$, $\boldsymbol{z}' \succcurlyeq_{RB} \boldsymbol{z}$—with the interpretation that "*$\boldsymbol{z}'$ is relatively more bipolarized than $\boldsymbol{z}$*"—if and only if

$$RB(\boldsymbol{z}',\alpha) \geq RB(\boldsymbol{z},\alpha), \quad \text{for all } \alpha \in [0,1].$$

Observe that $\succcurlyeq_{RB}$ is indeed a bipolarization preorder.

Lemma 8. *The relative bipolarization preorder $\succcurlyeq_{RB}$ satisfies* IS *and* IB.

Proof. It follows from Theorem 2.3 in Chakravarty (2015). ■

[13] $(\underline{x},\overline{x}) \succ (\underline{y},\overline{y})$ implies that $(\underline{x},\overline{x}) \succcurlyeq (\underline{y},\overline{y})$, which, combined with the relation $(\underline{y},\overline{y}) \succcurlyeq (\underline{x},\overline{x})$, gives $(\underline{x},\overline{x}) \sim (\underline{y},\overline{y})$. This contradicts the fact that $(\underline{x},\overline{x}) \succ (\underline{y},\overline{y})$.

5.2 Scale-invariant bipolarization preorders

A scale-invariant bipolarization preorder $\succcurlyeq_{SB}$ is a bipolarization preorder that satisfies SI. A bipolarization preorder is scale-invariant if and only if it is consistent with the relative bipolarization preorder.

Definition 12. A bipolarization preorder $\succcurlyeq_B$ is $\succcurlyeq_{RB}$***-consistent*** if the following two conditions are satisfied:

1. $\boldsymbol{x} \sim_{RB} \boldsymbol{y} \Rightarrow \boldsymbol{x} \sim_B \boldsymbol{y}$.
2. $\boldsymbol{x} \succ_{RB} \boldsymbol{y} \Rightarrow \boldsymbol{x} \succ_B \boldsymbol{y}$.

Lemma 9. *A bipolarization preorder $\succcurlyeq_B$ satisfies SI if and only if it is $\succcurlyeq_{RB}$-consistent.*

Proof. Proved in Chakravarty (2009), Theorem 4.3, page 119. ■

Scale-invariant bipolarization and inequality-reducing tax schedules can be characterized using the same subclass of preferences.

Theorem 4. *For $u \in \mathscr{U}$, $b \geq 0$, $R \subseteq [0,1)$, and a scale-invariant bipolarization preorder $\succcurlyeq_{SB}$,*

$$\left[\mathscr{T}_{(\succcurlyeq_{SB},u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}\right] \quad \text{and} \quad \left[\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{SB},u)\text{-}r} \Leftrightarrow u \in \mathscr{U}_{SI}(b,R)\right].$$

Proof. Proposition 3 and Theorem 1 give the first containment. The bracketed equivalence follows from Theorem 3 in Carbonell-Nicolau and Llavador (2021b) and the fact that

$$\mathscr{T}_{(\succcurlyeq_{SB},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{RB},u)\text{-}r}. \tag{11}$$

The last equality follows from Theorem 4 in Carbonell-Nicolau and Llavador (2021b). To see that $\mathscr{T}_{(\succcurlyeq_{SB},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{RB},u)\text{-}r}$, note that Proposition 3 and Theorem 1 give $\mathscr{T}_{(\succcurlyeq_{SB},u)\text{-}r} \subseteq \mathscr{T}_{(\succcurlyeq_{RB},u)\text{-}r}$. The containment $\mathscr{T}_{(\succcurlyeq_{SB},u)\text{-}r} \supseteq \mathscr{T}_{(\succcurlyeq_{RB},u)\text{-}r}$ follows from Lemma 9. ■

The following corollary to Theorem 4 follows directly from expression (11) in its proof.

Corollary 5 (to Theorem 4)**.** *For $u \in \mathscr{U}$, $b \geq 0$, and $R \subseteq [0,1)$,*

$$\left[\mathscr{T}_{(\succcurlyeq_{RB},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}\right] \text{ and } \left[\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{RB},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r} \Leftrightarrow u \in \mathscr{U}_{SI}(b,R)\right].$$

In the special case when the lower bound on the intercept α_0 of a tax schedule is zero and the marginal tax rates can take values anywhere in the interval $[0,1)$, Theorem 4 immediately gives the following identity between marginal progressivity and relative bipolarization reduction.

Corollary 6 (to Theorem 4)**.** *For $u \in \mathscr{U}$ and a translation-invariant bipolarization preorder $\succcurlyeq_{SB}$,*

$$\mathscr{T}_{(\succcurlyeq_{SB},u)\text{-}r} = \mathscr{T}_{m\text{-}prog} \Leftrightarrow u \in \mathscr{U}_{SI}(0,[0,1)). \tag{12}$$

5.3 The absolute bipolarization preorder

Similarly to the relative bipolarization preorder, we can define the *absolute bipolarization preorder* from the *absolute bipolarization curve* obtained by scaling up the relative bipolarization curve by the median income level: $AB(\boldsymbol{z},\alpha) = z_m \times RB(\boldsymbol{z},\alpha)$ (Chakravarty, 2015, p. 48).

Formally, since the population size n is even, the absolute bipolarization curve is defined by

$$AB(\boldsymbol{z},\alpha) := \begin{cases} \frac{1}{n}\sum_{j\le i<m}(z_m - z_i) & \text{if } \alpha = \frac{j}{n} \text{ for some } j \in \{1,\dots,m-1\}, \\ \frac{1}{n}\sum_{m\le i\le j}(z_i - z_m) & \text{if } \alpha = \frac{j}{n} \text{ for some } j \in \{m,\dots,n\}, \\ z_m & \text{if } \alpha = 0, \end{cases} \tag{13}$$

and

$$AB(\boldsymbol{z},\alpha) := \lambda\, AB\left(\boldsymbol{z},\frac{j}{n}\right) + (1-\lambda)\, AB\left(\boldsymbol{z},\frac{j+1}{n}\right) \tag{14}$$

if $\alpha = \lambda\left(\frac{j}{n}\right) + (1-\lambda)\left(\frac{j+1}{n}\right)$, where $\lambda \in (0,1)$ and $j \in \{0,1,\dots,n-1\}$.

Definition 13. The ***absolute bipolarization preorder*** $\succcurlyeq_{AB}$ is a social preorder such that, given two income distributions $\boldsymbol{z}$ and $\boldsymbol{z}'$ in $\mathcal{Z}_n$, $\boldsymbol{z}' \succcurlyeq_{AB} \boldsymbol{z}$—with the interpretation that "$\boldsymbol{z}'$ *is absolutely more bipolarized than* $\boldsymbol{z}$"—if and only if

$$AB(\boldsymbol{z}',\alpha) \ge AB(\boldsymbol{z},\alpha), \quad \text{for all } \alpha \in [0,1].$$

The preorder $\succcurlyeq_{AB}$ is indeed a bipolarization preorder.

Lemma 10. *The absolute bipolarization preorder $\succcurlyeq_{AB}$ satisfies IS and IB.*

Proof. It follows from Theorem 1 in Chakravarty et al. (2007). ∎

5.4 Translation-invariant bipolarization preorders

A translation-invariant bipolarization preorder $\succcurlyeq_{TB}$ is a bipolarization preorder that satisfies TI. A bipolarization preorder is translation-invariant if and only if it is consistent with the absolute bipolarization preorder.

Definition 14. A bipolarization preorder $\succcurlyeq_B$ is $\succcurlyeq_{AB}$***-consistent*** if the following two conditions are satisfied:

1. $\boldsymbol{x} \sim_{AB} \boldsymbol{y} \Rightarrow \boldsymbol{x} \sim_B \boldsymbol{y}$.
2. $\boldsymbol{x} \succ_{AB} \boldsymbol{y} \Rightarrow \boldsymbol{x} \succ_B \boldsymbol{y}$.

Lemma 11. *A bipolarization preorder $\succcurlyeq_B$ satisfies TI if and only if it is $\succcurlyeq_{AB}$-consistent.*

Proof. See Appendix E. ∎

Moreover, for a given $u \in \mathscr{U}$, a tax schedule is bipolarization-reducing in the absolute sense if and only if it is inequality-reducing with respect to the absolute Lorenz criterion.

Lemma 12. *For $u \in \mathscr{U}$, $\mathscr{T}_{(\succcurlyeq_{AB},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$.*

Proof. See Appendix E. ∎

The previous lemmas are instrumental in the proof of the following main result.

Theorem 5. *For $u \in \mathscr{U}$, $b \geq 0$, $R \subseteq [0,1)$, and a translation-invariant bipolarization preorder $\succcurlyeq_{TB}$,*

$$\left[\mathscr{T}_{(\succcurlyeq_{TB},u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}\right] \quad and \quad \left[\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{TB},u)\text{-}r} \Leftrightarrow u \in \mathscr{U}_{TI}(b,R)\right].$$

Proof. Proposition 3 and Theorem 1 give the first containment. The bracketed equivalence follows from Corollary 3 and the fact that

$$\mathscr{T}_{(\succcurlyeq_{TB},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{AB},u)\text{-}r}. \tag{15}$$

The last equality follows from Lemma 12. To see that $\mathscr{T}_{(\succcurlyeq_{TB},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{AB},u)\text{-}r}$, note that Proposition 3 and Lemma 2 give

$$\mathscr{T}_{(\succcurlyeq_{TB},u)\text{-}r} \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{AB},u)\text{-}r}.$$

The containment $\mathscr{T}_{(\succcurlyeq_{TB},u)\text{-}r} \supseteq \mathscr{T}_{(\succcurlyeq_{AB},u)\text{-}r}$ follows from Lemma 11. ■

The next corollaries follow from Theorem 5.

Corollary 7 (to Theorem 5)**.** *For $u \in \mathscr{U}$, $b \geq 0$, and $R \subseteq [0,1)$,*

$$\left[\mathscr{T}_{(\succcurlyeq_{AB},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}\right] \text{ and } \left[\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{AB},u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r} \Leftrightarrow u \in \mathscr{U}_{TI}(b,R)\right].$$

Proof. The assertion follows immediately from expression (15) in the proof of Theorem 5. ■

Corollary 8 (to Theorem 5)**.** *For $u \in \mathscr{U}$ and a translation-invariant bipolarization preorder $\succcurlyeq_{TB}$,*

$$\mathscr{T}_{(\succcurlyeq_{TB},u)\text{-}r} = \mathscr{T}_{m\text{-}prog} \Leftrightarrow u \in \mathscr{U}_{TI}(0,[0,1)). \tag{16}$$

Proof. It follows immediately from Theorem 5. ■

6 Polarization and progressivity

The literature on polarization has centered on the polarization index in Esteban and Ray (1994). The *relative* version of the index takes the following form:

$$P_R(\boldsymbol{z}) = K \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{\boldsymbol{z}}(z_i)^{\alpha} |\ln z_i - \ln z_j|,$$

where $\pi_{\boldsymbol{z}}(z)$ denotes the number of individuals whose income is z.[14]

[14]Our formulation is equivalent to the expression in Esteban and Ray (1994), which is

$$P_R(\boldsymbol{x},\boldsymbol{\pi}) = K \sum_{i=1}^{k} \sum_{j=1}^{k} \pi_i^{1+\alpha} \pi_j |\ln x_i - \ln x_j|;$$

here, $\boldsymbol{x} = (x_1,\ldots,x_k)$ represents the vector of distinct income levels in the distribution at hand, and $\boldsymbol{\pi} = (\pi_1,\ldots,\pi_k)$ is a vector of densities, where each π_j ($j \in \{1,\ldots,k\}$) represents the number of individuals with income level x_j; the parameter K is positive and $\alpha \in (0,\alpha^*]$, where $\alpha^* \approx 1.6$.

To see that the equivalence holds, consider an income distribution $\boldsymbol{z} = (z_1,\ldots,z_n) \in \mathscr{Z}_n$. Its corresponding *density* is denoted by $(x_1,\ldots,x_{k_{\boldsymbol{z}}};\pi_1,\ldots,\pi_{k_{\boldsymbol{z}}})$, i.e.,

$$\boldsymbol{z} = (\underbrace{x_1,\ldots,x_1}_{\pi_1},\underbrace{x_2,\ldots,x_2}_{\pi_2},\ldots,\underbrace{x_{k_{\boldsymbol{z}}},\ldots,x_{k_{\boldsymbol{z}}}}_{\pi_{k_{\boldsymbol{z}}}}),$$

where $k_{\boldsymbol{z}}$ represents the number of distinct income levels in the vector $\boldsymbol{z}$.

The *absolute* counterpart of P_R, P_A, is given by

$$P_A(\boldsymbol{z}) = K\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{\boldsymbol{z}}(z_i)^{\alpha}|z_i - z_j|.$$

The associated relative and absolute polarization orders, respectively denoted by $\succcurlyeq_{RP}$ and $\succcurlyeq_{AP}$, are defined on $\mathscr{Z}_n$ by

$$\boldsymbol{x} \succcurlyeq_{RP} \boldsymbol{y} \Leftrightarrow P_R(\boldsymbol{x}) \geq P_R(\boldsymbol{y})$$

and

$$\boldsymbol{x} \succcurlyeq_{AP} \boldsymbol{y} \Leftrightarrow P_A(\boldsymbol{x}) \geq P_A(\boldsymbol{y}).$$

Consequently, unlike the inequality and bipolarization preorders, the polarization order is complete. A first results shows that $\succcurlyeq_{RP}$ satisfies SI and Axiom 1.

Proposition 4. *The polarization order $\succcurlyeq_{RP}$ is scale-invariant and satisfies Axiom 1.*

Proof. First, we show that $\succcurlyeq_{RP}$ is scale-invariant. Let $\boldsymbol{z}, \boldsymbol{z}' \in \mathscr{Z}_n$ such that $\boldsymbol{z}' = \beta\boldsymbol{z}$ for some $\beta > 0$. Then,

$$P_R(\boldsymbol{z}') = K\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{\boldsymbol{z}'}(z_i')^{\alpha}|\ln z_i' - \ln z_j'| = K\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{\boldsymbol{z}}(z_i)^{\alpha}|\ln z_i - \ln z_j| = P_R(\boldsymbol{z}).$$

To see that Axiom 1 is satisfied, suppose that $(\underline{x}, \overline{x})$ and $(\underline{y}, \overline{y})$ are perfectly bimodal income distributions with the same total income. We have

$$\begin{aligned}(\underline{y}, \overline{y}) \succcurlyeq_{RP} (\underline{x}, \overline{x}) \Leftrightarrow P_R(\underline{y}, \overline{y}) \geq P_R(\underline{x}, \overline{x}) \Leftrightarrow K\left(\frac{n}{2}\right)^{2+\alpha} 2(\ln\overline{y} - \ln\underline{y}) \geq K\left(\frac{n}{2}\right)^{2+\alpha} 2(\ln\overline{x} - \ln\underline{x}) \\ \Leftrightarrow \ln\overline{y} - \ln\underline{y} \geq \ln\overline{x} - \ln\underline{x} \Leftrightarrow \frac{\overline{y}}{\underline{y}} \geq \frac{\overline{x}}{\underline{x}}.\end{aligned}$$

Because both distributions have the same total income (i.e., $\overline{y} + \underline{y} = \overline{x} + \underline{x}$), we have

$$\overline{y} \geq \overline{x} \geq \underline{x} \geq \underline{y},$$

whence $\overline{y} - \underline{y} \geq \overline{x} - \underline{x}$. Hence, Axiom 1 holds. ■

Letting $\Pi_j = \pi_1 + \cdots + \pi_j$ (note that $\Pi_1 = \pi_1$), we can write

$$\begin{aligned}P_R(\boldsymbol{z}) =& K\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{\boldsymbol{z}}(z_i)^{\alpha}|\ln z_i - \ln z_j| \\ =& K\left(\sum_{i=1}^{\Pi_1}\pi_1^{\alpha}\sum_{j=1}^{n}|\ln x_1 - \ln z_j| + \sum_{i=\Pi_1+1}^{\Pi_2}\pi_2^{\alpha}\sum_{j=1}^{n}|\ln x_2 - \ln z_j| + \cdots + \sum_{i=\Pi_{k_{\boldsymbol{z}}-1}+1}^{\Pi_{k_{\boldsymbol{z}}}}\pi_{k_{\boldsymbol{z}}}^{\alpha}\sum_{j=1}^{n}|\ln x_{k_{\boldsymbol{z}}} - \ln z_j|\right) \\ =& K\left(\pi_1^{\alpha+1}\sum_{j=1}^{n}|\ln x_1 - \ln z_j| + \pi_2^{\alpha+1}\sum_{j=1}^{n}|\ln x_2 - \ln z_j| + \cdots + \pi_{k_{\boldsymbol{z}}}^{\alpha+1}\sum_{j=1}^{n}|\ln x_{k_{\boldsymbol{z}}} - \ln z_j|\right) \\ =& K\left(\pi_1^{\alpha+1}\sum_{j=1}^{k_{\boldsymbol{z}}}\pi_j|\ln x_1 - \ln x_j| + \pi_2^{\alpha+1}\sum_{j=1}^{k_{\boldsymbol{z}}}\pi_j|\ln x_2 - \ln x_j| + \cdots + \pi_{k_{\boldsymbol{z}}}^{\alpha+1}\sum_{j=1}^{k_{\boldsymbol{z}}}\pi_j|\ln x_{k_{\boldsymbol{z}}} - \ln x_j|\right) \\ =& K\sum_{i=1}^{k_{\boldsymbol{z}}}\sum_{j=1}^{k_{\boldsymbol{z}}}\pi_i^{\alpha+1}\pi_j|\ln x_i - \ln x_j| = P_R(\boldsymbol{x}, \boldsymbol{\pi}),\end{aligned}$$

where we have used the fact that $\sum_{j=1}^{n}|\ln x_i - \ln z_j| = \sum_{j=1}^{k_{\boldsymbol{z}}}\pi_j|\ln x_i - \ln x_j|$ for each i.

The following theorems characterize relative polarization-reducing tax schedules. First, we show that only marginal progressive tax schedules can be polarization-reducing and that all polarization-reducing tax schedules are inequality-reducing with respect to the relative Lorenz criterion.

Theorem 6. *For $u \in \mathscr{U}$,*

$$\mathscr{T}_{(\succcurlyeq_{RP},u)\text{-}r} \subseteq \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r} \subseteq \mathscr{T}_{m\text{-}prog}. \tag{17}$$

Proof. Because $\succcurlyeq_{RP}$ is scale-invariant and satisfies Axiom 1 (Proposition 4), Theorem 1 and Lemma 1 imply (17). ■

In general, $\mathscr{T}_{(\succcurlyeq_{RP},u)\text{-}r} \not\supseteq \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r}$, as the next example illustrates. Thus, $\succcurlyeq_{RP}$-order reduction is stronger than $\succcurlyeq_{RL}$-order reduction.

Example 1. Consider the Cobb-Douglas utility function $u(c,l) = c(1-l)$ and the progressive two-bracket tax schedule

$$T(y) = \begin{cases} -0.5 & \text{if } 0 \leq y \leq 0.1, \\ -0.55 + 0.5y & \text{if } y > 0.1. \end{cases}$$

We know that, for the Cobb-Douglas family of utility functions, $\mathscr{T}_{(\succeq_{RL},u)\text{-}r} = \mathscr{T}_{m\text{-}prog}$ (Remark 3 in Carbonell-Nicolau and Llavador, 2021a). Hence $T \in \mathscr{T}_{(\succeq_{RL},u)\text{-}r}$. Choose $\boldsymbol{a} = (0.4, 0.5, 6)$, and compute the income distributions $\boldsymbol{y}(\boldsymbol{a},0) = (0.2, 0.25, 3)$ and $\boldsymbol{x}(\boldsymbol{a},T) = (0.5, 0.5, 3.25)$. The corresponding relative polarization index values (for $\alpha = 1.6$) are $P_R(\boldsymbol{y}(\boldsymbol{a},0)) = 10.8K < 15.1K = P_R(\boldsymbol{x}(\boldsymbol{a},T))$. Consequently, $T \notin \mathscr{T}_{(\succcurlyeq_{RP},u)\text{-}r}$.

On the other hand, when restricting the analysis to linear tax schedules, relative Lorenz inequality-reducing tax schedules are also relative polarization-reducing.

Theorem 7. *For $u \in \mathscr{U}$, $\mathscr{T}_{(\succcurlyeq_{RP},u)\text{-}r} \supseteq \mathscr{T}_{lin} \cap \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r}$.*

Proof. Suppose that $T \in \mathscr{T}_{lin} \cap \mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r}$. Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathscr{A}_n$. Because T is $(\succcurlyeq_{RL}, u)$-r, Lemma 2 in Carbonell-Nicolau and Llavador (2021b) implies that

$$\frac{x^u(a_1,T)}{y^u(a_1,0)} \geq \cdots \geq \frac{x^u(a_n,T)}{y^u(a_n,0)}.$$

Therefore, given i,j with $j \geq i$, we have

$$\frac{x^u(a_i,T)}{y^u(a_i,0)} \geq \frac{x^u(a_j,T)}{y^u(a_j,0)},$$

implying that

$$\ln y^u(a_j,0) - \ln y^u(a_i,0) \geq \ln x^u(a_j,T) - \ln x^u(a_i,T).$$

In addition, because $u \in \mathscr{U}$, $a_j > a_i$ implies

$$y^u(a_i,0) < y^u(a_j,0) \quad \text{and} \quad x^u(a_i,T) < x^u(a_j,T),$$

where the last inequality uses the fact that T is linear. Consequently, letting

$$\boldsymbol{y} = (y^u(a_1,0), \ldots, y^u(a_n,0)) \quad \text{and} \quad \boldsymbol{x} = (x^u(a_1,T), \ldots, x^u(a_n,T)),$$

we have

$$P_R(\boldsymbol{y}) = K\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{\boldsymbol{y}}(y^u(a_i,0))^{\alpha}|\ln y^u(a_i,0)-\ln y^u(a_j,0)|$$
$$\geq K\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{\boldsymbol{x}}(x^u(a_i,T))^{\alpha}|\ln x^u(a_i,T)-\ln x^u(a_j,T)| = P_R(\boldsymbol{x}),$$

implying that

$$(y^u(a_1,0),\ldots,y^u(a_n,0)) \succcurlyeq_{RP} (x^u(a_1,T),\ldots,x^u(a_n,T)).$$

Hence, $T\in\mathcal{T}_{(\succcurlyeq_{RP},u)\text{-}r}$. ■

Therefore, a linear tax schedule is polarization-reducing if and only if it is inequality-reducing with respect to the relative Lorenz criterion.

Corollary 9. *For $u\in\mathcal{U}$, $\mathcal{T}_{(\succcurlyeq_{RP},u)\text{-}r}\cap\mathcal{T}_{lin} = \mathcal{T}_{(\succcurlyeq_{RL},u)\text{-}r}\cap\mathcal{T}_{lin}$.*

Proof. It is a direct implication of Theorem 6 and Theorem 7. ■

However, the reverse inclusion is not true in general:

$$\mathcal{T}_{(\succcurlyeq_{RP},u)\text{-}r}\nsubseteq\mathcal{T}_{lin}\cap\mathcal{T}_{(\succcurlyeq_{RL},u)\text{-}r}.$$

Consider, for example, the case when $n=2$, which yields

$$P_R(z_1,z_2) = K\,|\ln(z_1)-\ln(z_2)|\,.$$

For any $\boldsymbol{a}=(a_1,a_2)$ with $a_1\leq a_2$,

$$P_R(y(\boldsymbol{a},0))\geq P_R(x(\boldsymbol{a},T)) \Leftrightarrow \ln y(a_2,0)-\ln y(a_1,0)\geq \ln x(a_2,T)-\ln x(a_1,T),$$

which is equivalent to $\frac{x(a,T)}{y(a,0)}$ being non-increasing in a. In combination with Lemma 2 in Carbonell-Nicolau and Llavador (2021b), this implies that, for $n=2$,

$$\mathcal{T}_{(\succcurlyeq_{RP},u)\text{-}r} = \mathcal{T}_{(\succcurlyeq_{RL},u)\text{-}r}.$$

On the other hand, as per Propositions 1 and 2 in Carbonell-Nicolau and Llavador (2021a), there exist non-linear, $(\succcurlyeq_{RL},u)$-r (hence $(\succcurlyeq_{RP},u)$-r) tax schedules for certain CES and quasi-linear utility functions.[15]

The following result provides a full characterization of various subclasses of relative polarization-reducing, linear tax schedules.

Theorem 8. *For $u\in\mathcal{U}$, $b\geq 0$, and $R\subseteq[0,1)$,*

$$\mathcal{T}_{(\succcurlyeq_{RP},u)\text{-}r}\supseteq\mathcal{T}_{lin}\cap\mathcal{T}_{(\succcurlyeq_{RL},u)\text{-}r} = \mathcal{T}_{lin}\cap\mathcal{T}_{(\succcurlyeq_{RB},u)\text{-}r}\subseteq\mathcal{T}_{m\text{-}prog} \tag{18}$$

and

$$\mathcal{T}_{(\succcurlyeq_{RP},u)\text{-}r}\supseteq\mathcal{T}_{lin}(b,R)\Leftrightarrow u\in\mathcal{U}_{SI}(b,R). \tag{19}$$

[15]While $\mathcal{T}_{(\succcurlyeq_{RP},u)\text{-}r}\nsubseteq\mathcal{T}_{lin}$, as we have illustrated, if one requires that tax schedules be $(\succcurlyeq_{RP},u)$-r for *any*—rather than for a fixed—population size n, then the inclusion holds. In this case, a tax schedule is relative polarization-reducing if and only if it is linear and relative-Lorenz inequality-reducing.

Proof. The equality in (18) follows from Theorem 4 in Carbonell-Nicolau and Llavador (2021b). The first (resp., second) inclusion in (18) is a consequence of Theorem 7 (resp., Theorem 6).

To see that the equivalence in (19) holds, suppose that $\mathscr{T}_{(\succcurlyeq_{RP},u)\text{-}r} \supseteq \mathscr{T}_{lin}(b,R)$. Then Theorem 6 gives $\mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r} \supseteq \mathscr{T}_{lin}(b,R)$, which implies $u \in \mathscr{U}_{SI}(b,R)$ by Theorem 3 in Carbonell-Nicolau and Llavador (2021a).

Conversely, suppose that $u \in \mathscr{U}_{SI}(b,R)$. Then Theorem 3 in Carbonell-Nicolau and Llavador (2021a) gives $\mathscr{T}_{(\succcurlyeq_{RL},u)\text{-}r} \supseteq \mathscr{T}_{lin}(b,R)$, which implies $\mathscr{T}_{(\succcurlyeq_{RP},u)\text{-}r} \supseteq \mathscr{T}_{lin}(b,R)$ by the first containment in (18). ■

Analogous results can be obtained for the absolute version of the polarization order induced by the Esteban-Ray index. The analysis pertaining to the absolute polarization order is presented as a series of results and examples in Appendix F.

7 Concluding remarks

This paper delves into the theoretical foundations of tax progressivity, with a specific focus on inequality and (bi)polarization. Our findings can be organized into three groups.

First, we show that marginal-rate progressivity is a necessary condition for "spread-reducing" tax schedules, consistent with a fundamental and intuitive notion of variance (Axiom 1), which encompasses a wide array of variance-sensitive normative criteria for the evaluation of income distributions, including the standard measures of income inequality and (bi)polarization. Specifically, for any social preorder satisfying Axiom 1 and either scale or translation invariance, order-reducing taxation must necessarily be marginal-rate progressive (Theorem 1). Since the standard measures of inequality and (bi)polarization are all particular instances of social preorders satisfying said invariance properties, inequality and (bi)polarization-reducing tax structures must exhibit increasing marginal tax rates on income (theorems 2-6, and 9).

A second group of results highlights the relationship between inequality and (bi)polarization. First, we establish the equivalence between inequality and bipolarization-reducing tax schedules: under scale or translation invariance, a tax schedule reduces inequality if and only if it reduces bipolarization (theorems 2 to 5). Secondly, we show that any polarization-reducing tax schedule necessarily reduces inequality according to the Lorenz criterion (theorems 6 and 9). Furthermore, for linear taxes, there is no tension between polarization and inequality-reducing taxation: any linear tax schedule reduces polarization if and only if it reduces inequality with respect to the Lorenz criterion (corollaries 9 and 10).

Finally, we identify the families of utility functions for which marginal-rate progressivity is not only necessary but also sufficient for inequality reduction and depolarization. If we assume scale invariance, the family of utility functions is characterized by elasticities of income satisfying condition (6).[16] On the other hand, under the translation invariance assumption, a tax schedule is inequality and bipolarization-reducing if and only if the underlying preferences satisfy condition (7).[17] As illustrated in Carbonell-Nicolau and Llavador (2018, 2021a), these elasticity conditions are often satisfied by standard families of utility functions.

This paper investigates the relationship between equality and depolarization and the progressivity of tax schedules. By examining the theoretical foundations of inequality and polarization measurement, we characterize the conditions under which tax schedules can

[16] For the Esteban-Ray polarization order, this class of preferences characterizes the polarization-reducing properties of linear, progressive tax schedules (Theorem 8).

[17] For the Esteban-Ray absolute polarization measure, this class of preferences characterizes the polarization-reducing properties of linear, progressive tax schedules (Theorem 11).

effectively reduce income inequality and polarization, providing a normative foundation for progressive income taxes.

Having established the relationship between progressivity, equality, and depolarization, a natural line of inquiry concerns the relationship between progressivity and variance-sensitive normative criteria for the evaluation of income distributions. One may inquire whether a monotonic relation exists between tax progressivity and order-reducing measures based on social preorders, i.e., whether a higher degree of progressivity consistently leads to higher inequality and (bi)polarization. Le Breton et al. (1996) offers promising results for the specific case of exogenous income and the relative Lorenz preorder. This extension constitutes a natural avenue for future research.

We conclude with a remark. While we focus on social preorders defined on tax-mediated income distributions, some authors have suggested alternative domains for the various normative criteria, such as the set of *welfare* (as opposed to *income*) distributions, as measured by vectors of utility indices. As pointed out in Carbonell-Nicolau and Llavador (2018), this idea poses difficulties in that measures of dispersion and/or (bi)polarization have a marked cardinality component and are generally not invariant to order-preserving utility transformations.

Appendix

A List of preorders

$\succsim$: social preorder = reflexive and transitive binary relation + Axiom 1.

$\succsim_I$: inequality preorder = reflexive and transitive binary relation + TP.

$\succsim_{SI}$: scale-invariant inequality preorder = inequality preorder + SI.

$\succsim_{TI}$: translation-invariant inequality preorder = inequality preorder + TI.

$\succsim_{RL}$: relative-Lorenz preorder = binary relation induced by relative Lorenz criterion.

$\succsim_{AL}$: absolute-Lorenz preorder = binary relation induced by absolute Lorenz criterion.

$\succsim_L$: Lorenz preorder = binary relation induced by either relative or absolute Lorenz criterion.

$\succsim_B$: bipolarization preorder = reflexive and transitive binary relation + IS + IB.

$\succsim_{RB}$: relative-bipolarization preorder = binary relation induced by the curve RB.

$\succsim_{AB}$: absolute-bipolarization preorder = binary relation induced by the curve AB.

$\succsim_{SB}$: scale-invariant bipolarization preorder = bipolarization preorder + SI.

$\succsim_{TB}$: translation-invariant bipolarization preorder = bipolarization preorder + TI.

$\succsim_{RP}$: relative-polarization order = binary relation induced by the polarization index in Esteban and Ray (1994).

$\succsim_{AP}$: absolute-polarization order = binary relation induced by the absolute counterpart of the polarization index in Esteban and Ray (1994).

B Proofs of Lemma 1, Lemma 2 and Lemma 3

Lemma 1. *For $u \in \mathscr{U}$ and a social preorder $\succsim$ satisfying SI, $\mathscr{T}_{(\succsim,u)\text{-}r} \subseteq \mathscr{T}_{(\succsim_{RL},u)\text{-}r}$.*

Proof. Suppose that $T \in \mathscr{T}$ is $(\succsim,u)$-r. Let $\boldsymbol{a} = (a_1,\ldots,a_n) \in \mathscr{A}_n$. Choose any pair a_i and a_j with $a_i < a_j$. Let $\boldsymbol{a}' = (a'_1,\ldots,a'_n)$ satisfy

$$a'_1 = \cdots = a'_{\frac{n}{2}} = a_i < a_j = a'_{\frac{n}{2}+1} = \cdots = a'_n.$$

Note that

$$\underline{y} := y^u(a'_1,0) = \cdots = y^u(a'_{\frac{n}{2}},0) < y^u(a'_{\frac{n}{2}+1},0) = \cdots = y^u(a'_n,0) =: \overline{y}$$

and

$$\underline{x} := x^u(a'_1,T) = \cdots = x^u(a'_{\frac{n}{2}},T) \leq x^u(a'_{\frac{n}{2}+1},T) = \cdots = x^u(a'_n,T) =: \overline{x}.$$

Consequently, the two distributions $(\alpha\underline{y},\alpha\overline{y})$ and $(\underline{x},\overline{x})$ satisfy

$$\alpha\underline{y} + \alpha\overline{y} = \underline{x} + \overline{x},$$

where $\alpha := \frac{\underline{x}+\overline{x}}{\underline{y}+\overline{y}}$. Since $\succcurlyeq$ is scale invariant, we have $(\alpha\underline{y},\alpha\overline{y}) \sim (\underline{y},\overline{y})$. Because T is $(\succcurlyeq,u)$-r, we have $(\underline{y},\overline{y}) \succcurlyeq (\underline{x},\overline{x})$. Consequently,

$$(\alpha\underline{y},\alpha\overline{y}) \sim (\underline{y},\overline{y}) \succcurlyeq (\underline{x},\overline{x}),$$

implying that $(\alpha\underline{y},\alpha\overline{y}) \succcurlyeq (\underline{x},\overline{x})$ (Sen, 2017, Lemma 1*a, p. 56). By Axiom 1, we have

$$\alpha\overline{y} - \alpha\underline{y} \geq \overline{x} - \underline{x}.$$

Since $\alpha\underline{y} + \alpha\overline{y} = \underline{x} + \overline{x}$, it follows that

$$\alpha\underline{y} \leq \underline{x} \quad \text{and} \quad \alpha\overline{y} \geq \overline{x},$$

implying that

$$\frac{\underline{x}}{\alpha\underline{y}} \geq 1 \geq \frac{\overline{x}}{\alpha\overline{y}}.$$

Consequently,

$$\frac{x^u(a_i,T)}{y^u(a_i,0)} = \frac{\underline{x}}{\underline{y}} \geq \frac{\overline{x}}{\overline{y}} = \frac{x^u(a_j,T)}{y^u(a_j,0)}.$$

Since a_i and a_j were arbitrary coordinates in $\boldsymbol{a}$ with $a_i < a_j$, it follows from Lemma 2 in Carbonell-Nicolau and Llavador (2021b) that T is $(\succcurlyeq_{RL},u)$-r.

We have established that $\mathcal{T}_{(\succcurlyeq,u)\text{-}r} \subseteq \mathcal{T}_{(\succcurlyeq_{RL},u)\text{-}r}$. ■

Lemma 2. *For $u \in \mathscr{U}$ and a social preorder $\succcurlyeq$ satisfying TI, $\mathcal{T}_{(\succcurlyeq,u)\text{-}r} \subseteq \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r}$.*

Proof. Suppose that $T \in \mathcal{T}$ is $(\succcurlyeq,u)$-r. Let $\boldsymbol{a} = (a_1,\ldots,a_n) \in \mathscr{A}_n$ (n even). Choose any pair a_i and a_j with $a_i < a_j$. Let $\boldsymbol{a}' = (a'_1,\ldots,a'_n)$ satisfy

$$a'_1 = \cdots = a'_{\frac{n}{2}} = a_i < a_j = a'_{\frac{n}{2}+1} = \cdots = a'_n.$$

Note that

$$\underline{y} := y^u(a'_1,0) = \cdots = y^u(a'_{\frac{n}{2}},0) < y^u(a'_{\frac{n}{2}+1},0) = \cdots = y^u(a'_n,0) =: \overline{y}$$

and

$$\underline{x} := x^u(a'_1,T) = \cdots = x^u(a'_{\frac{n}{2}},T) \leq x^u(a'_{\frac{n}{2}+1},T) = \cdots = x^u(a'_n,T) =: \overline{x}.$$

Suppose that $\underline{y} + \overline{y} \geq \underline{x} + \overline{x}$ (the case when $\underline{y} + \overline{y} < \underline{x} + \overline{x}$ can be handled similarly). Define $\varepsilon \geq 0$ by

$$\underline{y} + \overline{y} = \underline{x} + \varepsilon + \overline{x} + \varepsilon. \tag{20}$$

Since $\succcurlyeq$ is translation invariant, we have $(\underline{x}+\varepsilon,\overline{x}+\varepsilon) \sim (\underline{x},\overline{x})$. Because T is $(\succcurlyeq,u)$-r, we have $(\underline{y},\overline{y}) \succcurlyeq (\underline{x},\overline{x})$. Consequently,

$$(\underline{y},\overline{y}) \succcurlyeq (\underline{x},\overline{x}) \sim (\underline{x}+\varepsilon,\overline{x}+\varepsilon),$$

implying that $(\underline{y},\overline{y}) \succcurlyeq (\underline{x}+\varepsilon,\overline{x}+\varepsilon)$ (Sen, 2017, Lemma 1*a, p. 56). By (20) and Axiom 1, we have

$$\overline{y} - \underline{y} \geq (\overline{x}+\varepsilon) - (\underline{x}+\varepsilon) = \overline{x} - \underline{x}.$$

Consequently,

$$y^u(a_j,0) - y^u(a_i,0) \geq x^u(a_j,T) - x^u(a_i,T).$$

Since a_i and a_j were arbitrary coordinates in $\boldsymbol{a}$ with $a_i < a_j$, it follows from Lemma 3 that T is $(\succcurlyeq_{AL},u)$-r.

We have established the containment $\mathcal{T}_{(\succcurlyeq,u)\text{-}r} \subseteq \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r}$. ■

Lemma 3. *For $u \in \mathscr{U}$, a tax schedule $T \in \mathscr{T}$ is $(\succcurlyeq_{AL}, u)$-r if and only if, for any pre-tax and post-tax income functions x^u and y^u,*

$$y^u(a',0) - y^u(a,0) \geq x^u(a',T) - x^u(a,T), \quad \text{whenever } a' > a > 0. \tag{4}$$

Proof. [$\Leftarrow$] Fix $u \in \mathscr{U}$ and $T \in \mathscr{T}$ and suppose that (4) holds for any x^u and y^u. We must show that T is $(\succcurlyeq_{AL}, u)$-r, i.e.,

$$(y^u(a_1,0),\ldots,y^u(a_n,0)) \succcurlyeq_{AL} (x^u(a_1,T),\ldots,x^u(a_n,T))$$

for every ability distribution $(a_1,\ldots,a_n) \in \mathscr{A}_n$ and every pre-tax and post-tax income functions y^u and x^u.

Fix $(a_1,\ldots,a_n) \in \mathscr{A}_n$, y^u, and x^u. By (4), we have

$$y^u(a_{i+1},0) - x^u(a_{i+1},T) \geq y^u(a_i,0) - x^u(a_i,T), \quad i \in \{1,\ldots,n-1\}. \tag{21}$$

Let

$$\begin{aligned} \boldsymbol{y} = (y_1,\ldots,y_n) &:= (y^u(a_1,0),\ldots,y^u(a_n,0)), \\ \boldsymbol{x} = (x_1,\ldots,x_n) &:= (x^u(a_1,T),\ldots,x^u(a_n,T)), \end{aligned}$$

and

$$\mu(\boldsymbol{y}) = \frac{\sum_{i=1}^n y_i}{n}, \quad \mu(\boldsymbol{x}) = \frac{\sum_{i=1}^n x_i}{n}.$$

Note that (21) can be rewritten as

$$\hat{y}_{i+1} - \hat{x}_{i+1} \geq \hat{y}_i - \hat{x}_i, \quad i \in \{1,\ldots,n-1\}, \tag{22}$$

where

$$\hat{y}_i := y_i - \mu(\boldsymbol{y}) \quad \text{and} \quad \hat{x}_i := x_i - \mu(\boldsymbol{x}), \quad i \in \{1,\ldots,n\}.$$

Since

$$\sum_{i=1}^n (\hat{x}_i - \hat{y}_i) = 0 \tag{23}$$

and (by (22)) $\hat{x}_i - \hat{y}_i$ is nonincreasing in i, there exists I such that

$$\begin{aligned} \hat{x}_i - \hat{y}_i &\geq 0, \quad i \in \{1,\ldots,I\}, \\ \hat{x}_i - \hat{y}_i &\leq 0, \quad i \in \{I+1,\ldots,n\}. \end{aligned}$$

Consequently,

$$\sum_{i=1}^k (\hat{x}_i - \hat{y}_i) \geq 0, \quad k \in \{1,\ldots,I\}, \tag{24}$$

$$\sum_{i=k}^n (\hat{x}_i - \hat{y}_i) \leq 0, \quad k \in \{I+1,\ldots,n\}. \tag{25}$$

Note that

$$\sum_{i=1}^k (\hat{x}_i - \hat{y}_i) = \sum_{i=1}^n (\hat{x}_i - \hat{y}_i) - \sum_{i=k+1}^n (\hat{x}_i - \hat{y}_i) \geq 0, \quad k \in \{I+1,\ldots,n\}, \tag{26}$$

where the inequality follows from (23) and (25). Combining (24) and (26) yields

$$\sum_{i=1}^k (\hat{x}_i - \hat{y}_i) \geq 0, \quad k \in \{1,\ldots,n\},$$

implying that $\boldsymbol{y} \succcurlyeq_{AL} \boldsymbol{x}$, as we sought.

[$\Rightarrow$] Suppose that there exist $a' > a > 0$, x^u, and y^u such that

$$y^u(a',0) - y^u(a,0) < x^u(a',T) - x^u(a,T). \tag{27}$$

It suffices to show that T is not $(\succcurlyeq_{AL}, u)$-r, i.e.,

$$(y^u(a_1,0),\dots,y^u(a_n,0)) \not\succcurlyeq (x^u(a_1,T),\dots,x^u(a_n,T)) \tag{28}$$

for some $(a_1,\dots,a_n) \in \mathscr{A}_n$, y^u, and x^u.

Let $\boldsymbol{a} = (a_1,\dots,a_n)$ satisfy

$$a_1 = \cdots = a_{n-1} = a < a' = a_n.$$

Then,

$$\begin{aligned} y^u(a_1,0) - \frac{\sum_{i=1}^n y^u(a_i,0)}{n} &= \frac{y^u(a,0) - y^u(a',0)}{n} \\ &> \frac{x^u(a,T) - x^u(a',T)}{n} = x^u(a_1,T) - \frac{\sum_{i=1}^n x^u(a_i,T)}{n} \end{aligned}$$

(where the inequality follows from (27)), implying that

$$y^u(a_1,0) - \frac{\sum_{i=1}^n y^u(a_i,0)}{n} > x^u(a_1,T) - \frac{\sum_{i=1}^n x^u(a_i,T)}{n},$$

which gives (28), as desired. ∎

C Proofs of Lemma 4 and Lemma 5

Lemma 4. *An inequality preorder $\succcurlyeq_I$ satisfies SI if and only if it is $\succcurlyeq_{RL}$-consistent.*

Proof. If $\succcurlyeq$ is $\succcurlyeq_{RL}$-consistent, then it satisfies SI and TP because $\succcurlyeq_{RL}$ satisfies these axioms. Conversely, suppose that $\succcurlyeq$ satisfies SI and TP. Fix $\boldsymbol{x}$ and $\boldsymbol{y}$. Note that

$$\boldsymbol{x}' \sim_{RL} \boldsymbol{x} \quad \text{and} \quad \boldsymbol{y}' \sim_{RL} \boldsymbol{y},$$

where

$$\boldsymbol{x}' := \left(\frac{1}{\sum_i x_i}\right)\boldsymbol{x} \quad \text{and} \quad \boldsymbol{y}' := \left(\frac{1}{\sum_i y_i}\right)\boldsymbol{y}.$$

If $\boldsymbol{x} \sim_{RL} \boldsymbol{y}$, then

$$\boldsymbol{x}' \sim_{RL} \boldsymbol{x} \sim_{RL} \boldsymbol{y} \sim_{RL} \boldsymbol{y}',$$

implying $\boldsymbol{x}' \sim_{RL} \boldsymbol{y}'$. Hence, since $\boldsymbol{x}'$ and $\boldsymbol{y}'$ have the same income, we have $\boldsymbol{x}' = \boldsymbol{y}'$, implying $\boldsymbol{x}' \sim \boldsymbol{y}'$ (by reflexivity of $\succcurlyeq$). Therefore, by SI, we have

$$\boldsymbol{x} \sim \boldsymbol{x}' \sim \boldsymbol{y}' \sim \boldsymbol{y},$$

implying $\boldsymbol{x} \sim \boldsymbol{y}$.[18]

If $\boldsymbol{x} \succ_{RL} \boldsymbol{y}$, then

$$\boldsymbol{x}' \sim_{RL} \boldsymbol{x} \succ_{RL} \boldsymbol{y} \sim_{RL} \boldsymbol{y}',$$

implying $\boldsymbol{x}' \succ_{RL} \boldsymbol{y}'$. Consequently, because $\boldsymbol{x}'$ and $\boldsymbol{y}'$ have the same income, Lemma B.1 in Marshall et al. (2011, chapter 2) implies that $\boldsymbol{x}'$ can be obtained from $\boldsymbol{y}'$ by a finite sequence of progressive transfers, and so (by TP) $\boldsymbol{x}' \succ \boldsymbol{y}'$. Therefore, by SI, we have

$$\boldsymbol{x} \sim \boldsymbol{x}' \succ \boldsymbol{y}' \sim \boldsymbol{y}.$$

Since $\succcurlyeq$ is reflexive and transitive, this yields $\boldsymbol{x} \succ \boldsymbol{y}$. ∎

[18] The relation $\sim$ is transitive because $\succcurlyeq$ is reflexive and transitive.

Lemma 5. *An inequality preorder satisfies TI if and only if it is $\succcurlyeq_{AL}$-consistent.*

Proof. (ii) Secondly, we prove the result for TI.
If $\succcurlyeq$ is $\succcurlyeq_{AL}$-consistent, then it satisfies TI and TP because $\succcurlyeq_{AL}$ satisfies these axioms. Conversely, suppose that $\succcurlyeq$ satisfies TI and TP. Fix $\boldsymbol{x}$ and $\boldsymbol{y}$. Suppose that

$$\sum_i x_i \leq \sum_i y_i$$

(the case when $\sum_i x_i > \sum_i y_i$ can be handled similarly). Define ε by

$$n\varepsilon + \sum_i x_i = \sum_i y_i.$$

Note that

$$\boldsymbol{x}' \sim_{AL} \boldsymbol{x},$$

where

$$\boldsymbol{x}' := (x_1 + \varepsilon, \ldots, x_n + \varepsilon).$$

If $\boldsymbol{x} \sim_{AL} \boldsymbol{y}$, then

$$\boldsymbol{x}' \sim_{AL} \boldsymbol{x} \sim_{AL} \boldsymbol{y},$$

implying $\boldsymbol{x}' \sim_{AL} \boldsymbol{y}$. Hence, since $\boldsymbol{x}'$ and $\boldsymbol{y}$ have the same income, we have $\boldsymbol{x}' = \boldsymbol{y}$, implying $\boldsymbol{x}' \sim \boldsymbol{y}$ (by reflexivity of $\succcurlyeq$). Therefore, by TI, we have

$$\boldsymbol{x} \sim \boldsymbol{x}' \sim \boldsymbol{y},$$

implying $\boldsymbol{x} \sim \boldsymbol{y}$.[19]
If $\boldsymbol{x} \succ_{AL} \boldsymbol{y}$, then

$$\boldsymbol{x}' \sim_{AL} \boldsymbol{x} \succ_{AL} \boldsymbol{y},$$

implying $\boldsymbol{x}' \succ_{AL} \boldsymbol{y}$. Consequently, because $\boldsymbol{x}'$ and $\boldsymbol{y}$ have the same income, Lemma B.1 in Marshall et al. (2011, chapter 1) implies that $\boldsymbol{x}'$ can be obtained from $\boldsymbol{y}$ by a finite sequence of progressive transfers, and so (by TP) $\boldsymbol{x}' \succ \boldsymbol{y}$. Therefore, by TI, we have

$$\boldsymbol{x} \sim \boldsymbol{x}' \succ \boldsymbol{y}.$$

Since $\succcurlyeq$ is reflexive and transitive, this yields $\boldsymbol{x} \succ \boldsymbol{y}$. ■

D Proofs of Lemma 6 and Lemma 7

Lemma 6. *For $u \in \mathscr{U}$, $b \geq 0$, and $R \subseteq [0,1)$, $\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r} \Leftrightarrow \mathscr{T}_{lin}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$.*

Proof. The 'only if' part is trivial, since $\mathscr{T}_{lin}(b,R) \subseteq \mathscr{T}_{m\text{-}prog}(b,R)$.

To see that $\mathscr{T}_{lin}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$ implies $\mathscr{T}_{m\text{-}prog}(b,R) \subseteq \mathscr{T}_{(\succcurlyeq_{AL},u)\text{-}r}$, fix $T \in \mathscr{T}_{m\text{-}prog}(b,R)$. By Lemma 3, it suffices to show that

$$y^u(a',0) - x^u(a',T) \geq y^u(a,0) - x^u(a,T), \quad \text{whenever } a' > a > 0,$$

for any pre-tax and post-tax income functions x^u and y^u.

Because $T \in \mathscr{T}_{m\text{-}prog}(b,R)$, there exist a finite partition of $\mathbb{R}_{++}$,

$$A_1 := (0,a_1], A_2 := [a_1,a_2], \ldots, A_{K+1} := [a_K,\infty),$$

[19] The relation $\sim$ is transitive because $\succcurlyeq$ is reflexive and transitive.

and $K+1$ linear tax schedules in $\mathcal{T}_{lin}(b,R)$, $T_1,\ldots,T_{K+1}$, such that

$$x^u(a,T)=x^u(a,T_k),\quad a\in A_k,\quad k\in\{1,\ldots,K+1\}.$$

Since, by assumption, the tax schedules $T_1,\ldots,T_{K+1}$ are $(\succcurlyeq_{AL},u)$-r, Lemma 3 gives, for every $k\in\{1,\ldots,K+1\}$

$$y^u(a',0)-x^u(a',T_k)\geq y^u(a,0)-x^u(a,T_k),\quad \text{whenever } a'>a \text{ in } A_k,$$

for any pre-tax and post-tax income functions x^u and y^u.

For $a'>a>0$, we have $a'\in A_{k'}$ and $a\in A_k$ for some $k'>k$ and

$$\begin{aligned}
y^u(a',0)-x^u(a',T)&=y^u(a',0)-x^u(a',T_{k'})\\
&\geq y^u(a_{k'-1},0)-x^u(a_{k'-1},T_{k'})\\
&=y^u(a_{k'-1},0)-x^u(a_{k'-1},T_{k'-1})\\
&\geq y^u(a_{k'-2},0)-x^u(a_{k'-2},T_{k'-1})\\
&\;\;\vdots\\
&\geq y^u(a_k,0)-x^u(a_k,T_k)\\
&\geq y^u(a,0)-x^u(a,T_k)\\
&=y^u(a,0)-x^u(a,T),
\end{aligned}$$

as we sought. ■

Lemma 7. *For $u\in\mathcal{U}$, $b\geq 0$, and $R\subseteq[0,1)$, $\mathcal{T}_{lin}(b,R)\subseteq\mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r}\Leftrightarrow u\in\mathcal{U}_{TI}(b,R)$.*

Proof. By Lemma 3, the linear tax schedules $T(y)=-b+ry$ in $\mathcal{T}_{lin}(b,R)$ are $(\succcurlyeq_{AL},u)$-r if and only if the map

$$a\mapsto y^u(a,0)-x^u(a,T)=al^u(a,0)-\left[a(1-r)l^u((1-r)a,b)+b\right]$$

defined on $\mathbb{R}_{++}$ is nondecreasing for every $(b,r)\in[b,\infty)\times R$. Equivalently,

$$l^u(a',0)+a'\frac{\partial l^u(a',0)}{\partial a}\geq(1-r)\left[l^u((1-r)a',b')+a'(1-r)\frac{\partial l^u((1-r)a',b')}{\partial a}\right]$$

for every $(a',b',r)\in\mathbb{R}_{++}\times[b,\infty)\times R$, or

$$al^u(a,0)\big(1+\xi^u(a,0)\big)\geq(1-r)al^u((1-r)a,b')\big(1+\xi^u\left((1-r)a,b'\right)\big)$$

for every $(a,b',r)\in\mathbb{R}_{++}\times[b,\infty)\times R$. Letting

$$y^u(a,0)=al^u(a,0)\quad\text{and}\quad y^u\left((1-r)a,b'\right)=(1-r)al^u\left((1-r)a,b'\right)$$

yields (7), and hence $u\in\mathcal{U}_{TI}(b,R)$. ■

E Proofs of Lemma 11 and Lemma 12

Lemma 11. *A bipolarization preorder $\succcurlyeq_B$ satisfies TI if and only if it is $\succcurlyeq_{AB}$-consistent.*

Proof. If $\succcurlyeq_B$ is $\succcurlyeq_{AB}$-consistent, then it satisfies TI, IS, and IB because $\succcurlyeq_{AB}$ satisfies these axioms.

Conversely, suppose that $\succcurlyeq_B$ satisfies TI, IS, and IB. Fix $\boldsymbol{x}$ and $\boldsymbol{y}$, with respective median incomes $m(\boldsymbol{x})$ and $m(\boldsymbol{y})$. Suppose that $m(\boldsymbol{x}) \leq m(\boldsymbol{y})$ (the case when $m(\boldsymbol{x}) > m(\boldsymbol{y})$ can be handled similarly). Let ε be defined by

$$m(\boldsymbol{x}) + \varepsilon = m(\boldsymbol{y}).$$

Then $m(\boldsymbol{x}') = m(\boldsymbol{y})$, where

$$\boldsymbol{x}' := (x_1 + \varepsilon, \dots, x_n + \varepsilon).$$

If $\boldsymbol{x} \sim_{AB} \boldsymbol{y}$, then

$$\boldsymbol{x}' \sim_{AB} \boldsymbol{x} \sim_{AB} \boldsymbol{y},$$

implying $\boldsymbol{x}' \sim_{AB} \boldsymbol{y}$. Hence, since $\boldsymbol{x}'$ and $\boldsymbol{y}$ have the same median, we have $\boldsymbol{x}' = \boldsymbol{y}$, implying $\boldsymbol{x}' \sim \boldsymbol{y}$ (by reflexivity of $\succcurlyeq_B$).[20] Therefore, by TI, we have

$$\boldsymbol{x} \sim \boldsymbol{x}' \sim \boldsymbol{y},$$

implying $\boldsymbol{x} \sim \boldsymbol{y}$.

If $\boldsymbol{x} \succ_{AB} \boldsymbol{y}$, then

$$\boldsymbol{x}' \sim_{AB} \boldsymbol{x} \succ_{AB} \boldsymbol{y},$$

implying $\boldsymbol{x}' \succ_{AB} \boldsymbol{y}$. Consequently, because $\boldsymbol{x}'$ and $\boldsymbol{y}$ have the same median, Theorem 2.1 in Chakravarty (2015) implies that $\boldsymbol{x}'$ can be obtained from $\boldsymbol{y}$ by a finite sequence of IS and/or IB transformations, and so $\boldsymbol{x}' \succ_B \boldsymbol{y}$.[21] Therefore, by TI, we have

$$\boldsymbol{x} \sim_B \boldsymbol{x}' \succ_B \boldsymbol{y}.$$

Since $\succcurlyeq_B$ is reflexive and transitive, this yields $\boldsymbol{x} \succ_B \boldsymbol{y}$. ∎

Lemma 12. *For $u \in \mathscr{U}$, $\mathscr{T}_{(\succcurlyeq_{AB}, u)\text{-}r} = \mathscr{T}_{(\succcurlyeq_{AL}, u)\text{-}r}$.*

Proof. $[\mathscr{T}_{(\succcurlyeq_{AB}, u)\text{-}r} \subseteq \mathscr{T}_{(\succcurlyeq_{AL}, u)\text{-}r}.]$ Take $T \in \mathscr{T}_{(\succcurlyeq_{AB}, u)\text{-}r}$ and $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathscr{A}_n$. Choose any pair a_i and a_j with $a_i < a_j$. Let $\boldsymbol{a}' = (a'_1, \dots, a'_n) \in \mathscr{A}_n$ satisfy $a'_{m-1} := a_i < a'_m < a_j =: a'_{m+1}$. Because $T \in \mathscr{T}_{(\succcurlyeq_{AB}, u)\text{-}r}$, it follows from (13) that

$$\begin{aligned} AB((y^u(a'_1, 0), \dots, y^u(a'_n, 0)), m-1) &= y^u(a'_m, 0) - y^u(a'_{m-1}, 0)) \\ &\geq x^u(a'_m, T) - x^u(a'_{m-1}, T) = AB((x^u(a'_1, T), \dots, x^u(a'_n, T)), m-1) \end{aligned}$$

and

$$\begin{aligned} AB((y^u(a'_1, 0), \dots, y^u(a'_n, T)), m+1) &= y^u(a'_{m+1}, 0) - y^u(a'_m, 0) \\ &\geq x^u(a'_{m+1}, T) - x^u(a'_m, T) = AB((x^u(a'_1, T), \dots, x^u(a'_n, T)), m+1). \end{aligned}$$

Arranging terms and using $a'_{m-1} = a_i$ and $a'_{m+1} = a_j$ yields

$$y^u(a_j, 0) - x^u(a_j, T) \geq y^u(a'_m, 0) - x^u(a'_m, T) \geq y^u(a_i, 0) - x^u(a_i, T).$$

[20] Because $\boldsymbol{x}' \sim_{AB} \boldsymbol{y}$, $AB(\boldsymbol{x}', \alpha) = AB(\boldsymbol{y}, \alpha)$ for all $\alpha \in [0, 1]$. In particular, for $\alpha = (m-1)/n$,

$$1/n(x'_m - x'_{m-1}) = AB(\boldsymbol{x}', (m-1)/n) = AB(\boldsymbol{y}, (m-1)/n) = 1/n(y_m - y_{m-1})$$

and, because $x'_m = y_m$, $x'_{m-1} = y_{m-1}$. Now, for $\alpha = m-2$, we obtain $x'_{m-2} = y_{m-2}$, and we can unravel all the way to $x'_1 = y_1$. Proceed similarly for $\alpha = (m+1)/n, \dots, \alpha = 1$ to obtain $x'_i = y_i$ for $i = m+1, \dots, n$.

[21] An IS transformation is a (weakly) spread-increasing movement of income levels away from the median income, a movement consistent with the antecedent in IS. An IB transformation results from bipolarity-increasing progressive transfers on either side of the median income, a transformation consistent with the antecedent in IB.

Since a_i and a_j were arbitrary coordinates in $\boldsymbol{a}$, it follows from Lemma 3 that $T \in \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r}$, as we sought.

$[\mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r} \subseteq \mathcal{T}_{(\succcurlyeq_{AB},u)\text{-}r}.]$ Take $T \in \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r}$ and $\boldsymbol{a} = (a_1,\ldots,a_n) \in \mathscr{A}_n$. Define $b_i := y^u(a_i,0) - x^u(a_i,T)$. Because $T \in \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r}$, it follows from Lemma 3 that $b_i \le b_j$ for all $i < j$. Observe that we can write, for each i,

$$x^u(a_m,T) - x^u(a_i,T) = y^u(a_m,0) - b_m - y^u(a_i,0) + b_i = y^u(a_m,0) - y^u(a_i,0) + b_i - b_m. \quad (29)$$

Consider first $i < m$. Then $b_i - b_m \le 0$. Consequently, from (29) and the monotonicity of x^u and y^u in a (Lemma 1 in Carbonell-Nicolau and Llavador, 2021b), one obtains

$$x^u(a_m,T) - x^u(a_i,T) \le y^u(a_m,0) - y^u(a_i,0). \quad (30)$$

Similarly, for $i > m$, $b_i - b_m \ge 0$ and

$$x^u(a_i,T) - x^u(a_m,T) \le y^u(a_i,0) - y^u(a_m,0). \quad (31)$$

From (30) and (31), it follows that, for $\alpha = \frac{j}{n}$ and $j \in \{1,\ldots,n\}$,

$$AB((y^u(a_1,0),\ldots,y^u(a_n,0)),\alpha) \ge AB((x^u(a_1,T),\ldots,x^u(a_n,T)),\alpha),$$

and, consequently, for all $\alpha \in [0,1]$,

$$AB((y^u(a_1,0),\ldots,y^u(a_n,0)),\alpha) \ge AB((x^u(a_1,T),\ldots,x^u(a_n,T)),\alpha),$$

implying that

$$(y^u(a_1,0),\ldots,y^u(a_n,0)) \succcurlyeq_{AB} (x^u(a_1,T),\ldots,x^u(a_n,T)).$$

Since $\boldsymbol{a}$ was arbitrary in $\mathscr{A}_n$, we conclude that $T \in \mathcal{T}_{(\succcurlyeq_{AB},u)\text{-}r}$. ■

F Results for the absolute version of the polarization order induced by the Esteban-Ray index

Proposition 5. *The polarization order $\succcurlyeq_{AP}$ is translation-invariant and satisfies Axiom 1.*

Proof. First, we show that $\succcurlyeq_{AP}$ is translation-invariant. Let $\boldsymbol{z},\boldsymbol{z}' \in \mathcal{Z}_n$ such that $\boldsymbol{z}' = \boldsymbol{z} + \beta$. Then,

$$P_A(\boldsymbol{z}') = K\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{\boldsymbol{z}'}(z'_i)^{\alpha}|z'_i - z'_j| = K\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{\boldsymbol{z}}(z_i)^{\alpha}|z_i - z_j| = P_A(\boldsymbol{z}).$$

To see that Axiom 1 is satisfied, suppose that $(\underline{x},\overline{x})$ and $(\underline{y},\overline{y})$ are perfectly bimodal income distributions with the same total income. We have

$$(\underline{y},\overline{y}) \succcurlyeq_{AP} (\underline{x},\overline{x}) \Leftrightarrow P_A(\underline{y},\overline{y}) \ge P_A(\underline{x},\overline{x}) \Leftrightarrow K\left(\frac{n}{2}\right)^{2+\alpha} 2(\overline{y}-\underline{y}) \ge K\left(\frac{n}{2}\right)^{2+\alpha} 2(\overline{x}-\underline{x}) \Leftrightarrow \overline{y}-\underline{y} \ge \overline{x}-\underline{x}.$$

Hence, Axiom 1 holds. ■

Theorem 9. *For $u \in \mathcal{U}$,*

$$\mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r} \subseteq \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r} \subseteq \mathcal{T}_{m\text{-}prog}. \quad (32)$$

Proof. Because $\succcurlyeq_{AP}$ is translation-invariant and satisfies Axiom 1 (Proposition 5), Theorem 1 and Lemma 2 imply (32). ■

The following example shows that, in general, $\mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r} \not\supseteq \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r}$.

Example 2. Take the Cobb-Douglas utility function, the two-bracket tax schedule, and the vector of abilities from Example 1 in the relative polarization case. As before, $\boldsymbol{y}(\boldsymbol{a},0) = (0.2, 0.25, 3)$ and $\boldsymbol{x}(\boldsymbol{a},T) = (0.5, 0.5, 3.25)$. The corresponding absolute polarization index values (for $\alpha = 1.6$) are $P_A(\boldsymbol{y}(\boldsymbol{a},0)) = 11.2K < 22.2K = P_A(\boldsymbol{x}(\boldsymbol{a},T))$, and so $T \notin \mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r}$.

Theorem 10. *For $u \in \mathcal{U}$, $\mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r} \supseteq \mathcal{T}_{lin} \cap \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r}$.*

Proof. Suppose that $T \in \mathcal{T}_{lin} \cap \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r}$. Let $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathcal{A}_n$. Because T is $(\succcurlyeq_{AL}, u)$-r, Lemma 3 implies that for i, j with $j \geq i$, we have

$$y^u(a_j,0) - y^u(a_i,0) \geq x^u(a_j,T) - x^u(a_i,T).$$

In addition, because $u \in \mathcal{U}$, $a_j > a_i$ implies

$$y^u(a_i,0) < y^u(a_j,0) \quad \text{and} \quad x^u(a_i,T) < x^u(a_j,T),$$

where the last inequality uses the fact that T is linear. Consequently, letting

$$\boldsymbol{y} = (y^u(a_1,0), \dots, y^u(a_n,0)) \quad \text{and} \quad \boldsymbol{x} = (x^u(a_1,T), \dots, x^u(a_n,T)),$$

we have

$$\begin{aligned} P_A(\boldsymbol{y}) = K \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{\boldsymbol{y}} (y^u(a_i,0))^{\alpha} |y^u(a_i,0) - y^u(a_j,0)| \\ \geq K \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{\boldsymbol{x}} (x^u(a_i,T))^{\alpha} |x^u(a_i,T) - x^u(a_j,T)| = P_A(\boldsymbol{x}), \end{aligned}$$

implying that

$$(y^u(a_1,0), \dots, y^u(a_n,0)) \succcurlyeq_{AP} (x^u(a_1,T), \dots, x^u(a_n,T)).$$

Hence, $T \in \mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r}$. ∎

Corollary 10. *For $u \in \mathcal{U}$, $\mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r} \cap \mathcal{T}_{lin} = \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r} \cap \mathcal{T}_{lin}$.*

Proof. It is a direct implication of Theorem 9 and Theorem 10. ∎

In general, $\mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r} \nsubseteq \mathcal{T}_{lin} \cap \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r}$.

Example 3. Take the Cobb-Douglas utility from Example 1, $u(c,l) = c(1-l)$. Consider the linear tax $(t,b) = (0.5, 0.5)$, and the vector of abilities $\boldsymbol{a} = (0.4, 0.5, 150)$. Now, $\boldsymbol{y}(\boldsymbol{a}',0) = (0.2, 0.25, 75)$ and $\boldsymbol{x}(\boldsymbol{a},T) = (0.5, 0.5, 37.75)$. The corresponding absolute polarization index values (for $\alpha = 1.6$) are $P_A(\boldsymbol{y}(\boldsymbol{a}',0)) = 299.2K < 300.3K = P_A(\boldsymbol{x}(\boldsymbol{a}',T))$, and so the linear tax T is not a member of $\mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r}$.

Theorem 11. *For $u \in \mathcal{U}$, $b \geq 0$, and $R \subseteq [0,1)$,*

$$\mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r} \supseteq \mathcal{T}_{lin} \cap \mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r} = \mathcal{T}_{lin} \cap \mathcal{T}_{(\succcurlyeq_{AB},u)\text{-}r} \subseteq \mathcal{T}_{m\text{-}prog} \tag{33}$$

and

$$\mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r} \supseteq \mathcal{T}_{lin}(b,R) \Leftrightarrow u \in \mathcal{U}_{TI}(b,R). \tag{34}$$

Proof. The equality in (33) follows from Lemma 12. The first (resp., second) inclusion in (33) is a consequence of Theorem 10 (resp., Theorem 9).

To see that the equivalence in (34) holds, suppose that $\mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r} \supseteq \mathcal{T}_{lin}(b,R)$. Then Theorem 9 gives $\mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r} \supseteq \mathcal{T}_{lin}(b,R)$, which implies $u \in \mathcal{U}_{TI}(b,R)$ by Lemma 7.

Conversely, suppose that $u \in \mathcal{U}_{TI}(b,R)$. Then Lemma 7 gives $\mathcal{T}_{(\succcurlyeq_{AL},u)\text{-}r} \supseteq \mathcal{T}_{lin}(b,R)$, which implies $\mathcal{T}_{(\succcurlyeq_{AP},u)\text{-}r} \supseteq \mathcal{T}_{lin}(b,R)$ by the first containment in (33). ∎

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