# Hierarchy Measurement Revisited

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**Abstract**

Hierarchy permeates all aspects of human behavior and societal outcomes, making precise measurements fundamental to understanding its ramifications. However, existing hierarchical measures fail to capture key structural dimensions of hierarchy. This work addresses these limitations by significantly expanding the axiomatic foundation for hierarchical measurement. We develop precise measures of hierarchical depth that enhance comparability across different organizational structures. Our expanded axiom set captures dimensions that traditional measures overlook, revealing systematic violations in existing approaches. As a robust alternative, we propose the *average depth index*—a hierarchical measure that satisfies all axioms introduced in this work. This approach provides a more reliable and theoretically grounded tool for comparing organizational hierarchies.

*Keywords*: hierarchy measurement, hierarchical pre-order, hierarchical index, directed acyclic graph.
*JEL classifications*: D23, L22.

## 1. Introduction

Hierarchy is fundamental in shaping behavior across animal and human societies, serving as a powerful determinant of social outcomes. In animal groups, hierarchical structures regulate resource access, mating opportunities, and social alliances. Similarly, hierarchy influences behavior in human societies through power dynamics, social status, and cultural norms. Whether expressed through formal organizational charts, informal networks, or socially constructed status, hierarchical structures permeate nearly every aspect of human interaction.

High-status individuals often display greater confidence and assertiveness, while those lower in the hierarchy may conform to norms or strive for upward mobility—frequently driven by social comparison and competition, such as the phenomenon of "keeping up with the Joneses."

Ultimately, hierarchy-induced behavior shapes outcomes in complex ways. Hierarchies can promote stability, adaptability, coordination, and efficiency in a complex and uncertain world (Simon, 1962, 1973; Holland, 1998; Lawrence and Lorsch, 1967; Thompson, 1967;

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Lincoln and Kalleberg, 1985). Informal hierarchical networks further contribute to organizational resilience by facilitating crisis management (Krackhardt and Stern, 1988), improving profitability beyond traditional financial metrics (Sarkar et al., 2010), and compensating for dysfunctional formal systems (Culen, 2017).

However, these same structures can generate unintended consequences that undermine their benefits. Hierarchies produce emergent collective behaviors and group dynamics that influence outcomes in unpredictable ways (Anderson, 1972). They can enable power abuse, entrench social and economic inequalities (Fix, 2017, 2018, 2019, 2021), and stifle innovation (Wright, 2024).

Hierarchy, therefore, is not merely a structural feature—it is an active force that shapes behavior and determines the consequences of social interaction.

To fully understand the impact of hierarchy on behavior and outcomes, precise and consistent measurement is essential. Without rigorous methods to quantify hierarchy, comparing structures across institutions or analyzing their dynamic evolution becomes difficult.

To address this need for precise measurement, this paper extends the axiomatic theory for hierarchy measurement developed in Carbonell-Nicolau (2025a,b) through a significant expansion of its axiom set, thereby enhancing our ability to compare hierarchical structures both across organizations and over time.

We model hierarchies as directed acyclic graphs in which nodes represent individuals (employees, managers, executives) and edges capture subordination relationships that flow from supervisor to subordinate.

Our theoretical framework rests on hierarchical pre-orders—reflexive and transitive binary relations defined across the universe of possible hierarchical structures. This axiomatic approach offers a rigorous foundation for measuring and comparing firm hierarchies, introducing mathematical precision and analytical flexibility to Organizational Economics. By establishing consistency through formal axioms, our framework addresses the fundamental limitations inherent in ad-hoc measurement approaches that have dominated the field.

The expanded axiom set we develop focuses on two fundamental dimensions of organizational structure: hierarchical depth and structural transformation dynamics. We formalize the relationship between structural modifications and overall hierarchical measurement, developing precise analytical tools for understanding organizational changes, including management layer additions, various forms of organigram integration, and real-world scenarios such as partial subunit mergers and acquisitions.

Our framework builds on a core hierarchical pre-order established in Carbonell-Nicolau (2025a,b). This pre-order is the only ordering relationship fully characterized by three fundamental axioms: Anonymity, the Replication Principle, and Subordination Removal.

Anonymity ensures that the essential hierarchical structure remains unchanged when we relabel the nodes within a hierarchy. In other words, the specific names or identifiers assigned to positions do not affect the underlying organizational relationships—what matters is the pattern of connections, not the labels themselves.

The Replication Principle addresses how we evaluate multiple copies of hierarchical structures. When we have several identical copies of a given hierarchy—essentially creating a structure with multiple disconnected components——this principle treats these replicated hierarchies as having the same degree of hierarchical organization as the original single hierarchy.

Subordination Removal captures a fundamental principle about how hierarchical relationships affect organizational structure. This axiom states that when we eliminate a direct subordination relationship—specifically by removing the link between a subordinate $i$ and their immediate supervisor $j$, and instead connecting $i$ directly to $j$'s supervisor—we necessarily reduce the overall "hierarchical degree" of the system. By shortening the chain of command and removing $j$'s direct control over $i$, we create a flatter, less hierarchical structure.

While the present work retains these fundamental principles of hierarchy measurement, we recognize their insufficiency when studying organizational transformations that affect structural depth, such as the addition of management layers and the integration of separate hierarchies.

To illustrate, consider a simple example: hierarchy $h$ consists of two disconnected nodes, while hierarchy $h'$ introduces a "boss" node that directly supervises both nodes from above. This new structure cannot be derived from $h$ by removing subordination links. Removing supervisory links from $h'$ would result in either three unsupervised nodes or one subordinate node and two unsupervised nodes, depending on the extent of the removal. In contrast, $h'$ incorporates a new top-management node, which intuitively deepens the hierarchical structure. However, under the core measure from Carbonell-Nicolau (2025a,b), $h$ and $h'$ remain incomparable.

To address this limitation, we propose several new axioms that formalize how hierarchical structures can be extended while increasing their hierarchical depth.

The Upward Extension axiom formalizes the intuitive notion that adding top managers to an existing hierarchy increases its hierarchical depth. When a hierarchy $h$ is appended above another hierarchy $h_o$, the resulting compound hierarchy exhibits a greater hierarchical structure than $h$ alone. For upward extension to be valid, each leaf node in the upper hierarchy $h_o$ must connect to at least one root node in the lower hierarchy $h$. This connection requirement ensures a consistent increase in structural depth. It prevents scenarios where a weakly connected upper hierarchy introduces excessive "flatness" that could overshadow the lower hierarchy and reduce their connection to a negligible detail.

The Downward Extension axiom operates in reverse, stating that appending a hierarchy below another hierarchy results in a greater hierarchical structure when every root node in the lower hierarchy connects to one or more leaf nodes at maximum depth in the upper hierarchy. Here, the "depth" of a node refers to the number of its direct or indirect supervisors.

The notion of "appending" differs between these two axioms: upward extension requires every leaf node in the upper hierarchy to link to at least one root node in the lower hierarchy, creating mandatory connections from top to bottom. In contrast, downward extension allows more flexibility by requiring that every root node in the lower hierarchy become a direct subordinate to a maximum-depth leaf node in the upper hierarchy while not requiring all upper leaf nodes to connect downward.

Beyond these axioms, additional variants capture different aspects of hierarchical extension under weaker connection requirements. The Upward Extension* and Downward Extension* axioms are formulated using a more relaxed notion of hierarchy appending, where at least one leaf node in the upper hierarchy connects to at least one root node in the lower hierarchy rather than requiring comprehensive connections.

Upward Extension* shares the conceptual foundation of standard Upward Extension but operates under different conditions. It asserts that adding a hierarchy $h_o$ on top of $h$ increases the hierarchical degree when two conditions are met: first, $h_0$ must be weakly more hierarchical than $h$, and second, only partial linkage is required between the hierarchies. This weaker connection requirement is compensated by the stronger hierarchical structure of the appended hierarchy, ensuring that the compound structure achieves greater depth.

Similarly, Downward Extension* stipulates that adding $h_0$ below $h$ increases hierarchical degree when $h_0$ is weakly more hierarchical than $h$. As with Upward Extension*, the connection between hierarchies is partial, requiring only that at least one leaf node in the upper hierarchy connects to at least one root node in the lower hierarchy.

These axioms with partial connection requirements, working in conjunction with those requiring strict connections, allow for more comprehensive comparisons between hierarchical structures while capturing different aspects of hierarchical composition.

Each extension principle gives rise to a canonical hierarchical pre-order—a reflexive and transitive binary relation on the set of all hierarchies—that fully characterizes the underlying extension axiom when combined with the Anonymity axiom and the Replication Principle. These individual hierarchical pre-orders can be combined into a composite pre-order by considering the transitive closure of their union. This approach creates a more complete measure by combining all extension pre-orders, though the composite framework can consider any subset of measures.

Every composite hierarchical pre-order defined in this manner represents the coarsest pre-order that aligns with the underlying axioms, providing a comprehensive framework for comparing hierarchical structures across multiple extension and connection requirements.

With our axiomatic groundwork in place, we shift our focus to the study of hierarchical indices, which, unlike our pre-order measures, give rise to complete rankings of hierarchies. Hierarchical indices present clear computational advantages due to their simplicity, contrasting with the complexity of composite hierarchical pre-orders defined as the transitive closure of a union of canonical measures. However, this computational convenience comes at a cost: every comparison between hierarchies receives a definitive ranking, even in ambiguous cases where none of the established axioms applies or provides clear guidance.

Our analysis of hierarchical indices begins with the *average depth index*, introduced in prior work (Carbonell-Nicolau, 2025a), which calculates the mean depth—the number of direct or indirect supervisors—across all nodes in a given hierarchy. We demonstrate that this index fulfills all the axioms considered in this paper, ensuring consistency with our hierarchical pre-orders whenever they produce a definitive ranking between two hierarchies. This alignment makes the average depth index a reliable computational tool that preserves the theoretical foundations established by our axiomatic framework.

We also examine other hierarchical indices proposed in the literature, contextualizing them within our axiomatic framework. Our analysis reveals that, unlike the average depth index, these alternative indices systematically violate our axioms. This finding highlights the importance of axiomatic validation in developing hierarchical measures and demonstrates why the average depth index stands out as both computationally efficient and theoretically sound.

The *global reaching centrality (grc) index*, introduced by Mones et al. (2012), is based on the average concentration of "reaching centrality," which measures a node's ability to reach other nodes in the hierarchy. While this index satisfies the Anonymity axiom, it violates all other axioms in our framework.

The failure of the Subordination Removal axiom has been previously illustrated in Carbonell-Nicolau (2025a). Additionally, the replication operation does not leave the grc index unchanged, thus violating the Replication Principle. This violation occurs because replication proportionally increases the number of nodes without proportionally increasing the concentration of reaching centrality across those nodes, resulting in a lower average reaching centrality for the replicated structure.

We further demonstrate that global reaching centrality fails to capture the notion of vertical hierarchical growth underlying the various extension principles considered in this paper. The fundamental issue with the grc index's violations of these principles stems from the fact that deeper hierarchies may be associated with a less concentrated distribution of reaching centrality. This relationship between depth and centrality concentration illuminates the grc index's systematic failure to align with our axiomatic framework.

Krackhardt (1994) studied informal organizations—networks of relationships between individuals that are not necessarily captured by official organizational charts but are formed to accomplish tasks quickly and effectively. He identified four dimensions along which analysts can measure the similarities between an informal organization and what he termed formal organizational structures. Krackhardt assumed these formal structures take the form of traditional hierarchies, which he modeled as out-trees or arborescences, particular instances of directed acyclic graphs representing well-delineated hierarchical relationships.

For each dimension, Krackhardt defined an index quantifying a network's "hierarchical degree". We evaluate these measures, defined on the domain of directed acyclic graphs, within our axiomatic framework.

Krackhardt's *connectedness index* measures the extent of information flows between nodes in a hierarchy without regard for directionality, effectively quantifying linkages in the underlying undirected graph. Our analysis shows that the connectedness index violates all of our axioms except Anonymity.

The violation of the Replication Principle occurs because replication reduces connectedness by creating one or more copies of the original hierarchy, with each copy disconnected from its counterparts. The Subordination Removal axiom is violated because connectedness does not depend on the specific structure of subordination relations—these relations can be eliminated without breaking the linkages between nodes measured by the index.

Similarly, the connectedness index violates the various extension axioms because appending operations between hierarchies can be conducted without altering existing linkages between nodes within each hierarchy.

Krackhardt's *graph hierarchy index* proves trivial within the domain of directed acyclic graphs, as it assumes a constant value across all acyclic hierarchical structures. Consequently, this measure lacks discriminatory power when comparing different hierarchical configurations, making it unsuitable for our analytical purposes.

Krackhardt's *graph efficiency index* quantifies the gap between a hierarchy and its minimally connected counterpart, effectively measuring linkage "redundancy." Since Krackhardt evaluates efficiency within weakly connected components rather than across disconnected components, the graph efficiency index remains consistent with the Replication Principle—replication creates separate components without affecting internal efficiency measures.

However, this measure violates both the Subordination Removal axiom and the extension axioms. The Subordination Removal violation occurs because subordination operations can be conducted while maintaining minimal connections across nodes, leaving the efficiency measure unaltered despite fundamental changes to the hierarchical structure. Similarly, the extension axioms are violated because hierarchy appending can be achieved without adding redundant links, keeping the graph efficiency index constant even when the overall hierarchical structure has demonstrably increased in complexity and depth.

Krackhardt's final measure, the *least upper boundedness index*, assesses how closely a hierarchy approximates a pure arborescence. Like the efficiency index, it is calculated within weakly connected components rather than across disconnected ones, ensuring consistency with the Replication Principle.

However, the measure exhibits a fundamental limitation: it assigns the same value to all pure trees, regardless of the complexity or depth of their hierarchical structure. In other words, it is insensitive to differences in hierarchical depth or the specific patterns of subordination. As a result, the index cannot distinguish between shallow and deep hierarchies, thereby violating the Subordination Removal and extension axioms. While the index effectively identifies tree-like structures, it fails to capture the nuanced variations in hierarchical organization that our axiomatic framework is designed to measure.

Other hierarchical indices have been introduced in the literature, such as those proposed by Trusina et al. (2004), Luo and Magee (2011), Corominas-Murtra et al. (2013), and Czégel and Palla (2015).

The indices developed by Corominas-Murtra et al. (2013) and Trusina et al. (2004) are specifically designed to detect deviations from hierarchical structure that do not occur in directed acyclic graphs. As a result, these indices reduce to constant values when applied to directed acyclic graphs, making them ineffective for distinguishing between different topologies within this class of graphs.

A similar limitation applies to the measure introduced by Luo and Magee (2011). The flow hierarchy metric discussed in that work counts the number of links in a network that are not part of loops. Since directed acyclic graphs, by definition, contain no loops, every link is counted, and the metric always yields its maximum value. Consequently, this measure cannot differentiate between distinct directed acyclic graphs or provide meaningful comparisons of their hierarchical structures.

Czégel and Palla (2015) propose an algorithmic approach to measuring hierarchy based on random walks within a network. In their method, the stationary distribution of random walkers reflects each node's position in the hierarchy: a greater concentration of walkers at higher levels indicates a more pronounced hierarchical structure. The measure itself quantifies the inequality in this distribution.

An important feature of this approach is its dependence on a decay parameter, which the analyst must choose. The form of the stationary distribution—and thus the value of the hierarchy index—varies according to this parameter. While the measure intuitively captures hierarchy in multi-level structures, its reliance on the decay parameter can violate the extension axioms if it is not optimally set. This dependence limits the measure's ability to consistently reflect hierarchical growth through upward or downward extensions.

The paper is organized as follows. Section 2 formally defines hierarchies as directed acyclic graphs and introduces key concepts such as root nodes, leaf nodes, and node depth. Section 3 presents the axiomatic framework for hierarchy measurement, detailing the core axioms: Anonymity, Replication Principle, Subordination Removal, Upward Extension, Downward Extension, and their relaxed variants. Section 4 examines hierarchical indices, focusing on the average depth index and its compatibility with our axioms while discussing alternative measures from the literature. Section 5 concludes with a summary of our contributions and suggestions for future research. Technical results are proven in the appendices.

## 2. Hierarchies

A *hierarchy* is formally defined as a *directed acyclic graph*, consisting of a set of nodes connected by directed edges. In this context, nodes represent individuals, while edges represent subordination relationships, with each edge originating from a supervisor and terminating at the supervisor's direct subordinate. The acyclicity of the graph ensures that traversing any path along edge directions will never return to the starting node, meaning there are no directed cycles.

This structure enforces a unidirectional flow of authority. Individuals may have both direct supervisors (immediate predecessors) and indirect supervisors (supervisors of their supervisors), creating a transitive chain of command. Authority propagates downward from higher-level supervisors through intermediate managers to subordinates, ensuring no ambiguity in reporting lines.

The acyclicity property imposes a strict hierarchical constraint: For any individual with $k$ total supervisors (direct and indirect), all their supervisors must necessarily have fewer than $k$ supervisors themselves. This guarantees a monotonically decreasing authority structure, where supervisory roles are strictly ordered.

A composite hierarchy, denoted as $(h_1, \dots, h_k)$ where $k \geq 1$, consists of $k$ independent hierarchies. In this context, independence means each $h_i$ is a directed acyclic graph, and no directed edges exist between nodes of distinct hierarchies $h_i$ and $h_j$ for $i \neq j$.

We define the *size* of a hierarchy as the cardinality of its node set, which corresponds to the total number of individuals within the hierarchical structure.

A *root node* is a node with no supervisors, representing an apex of authority in the hierarchy. A node with no subordinates is called a *leaf node*.

The *depth* of a node is the total count of its supervisors (direct or indirect), measuring its distance from the hierarchy's topmost level. Nodes closer to the root have lower depth values, while those with longer supervisory chains have greater depth.

## 3. Axioms for hierarchy measurement

Let $\mathscr{H}_n$ denote the set of all possible hierarchies comprising exactly $n$ individuals, representing all valid organizational configurations adhering to our directed acyclic graph structure.

The universal set of all possible hierarchies is then defined as:

$$\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n.$$

**Definition 1.** A *hierarchical pre-order* on $\mathscr{H}$ is a reflexive and transitive binary relation on $\mathscr{H}$.

A hierarchical pre-order $\succcurlyeq$ is a binary relation that systematically ranks hierarchies by their "degree of hierarchy," where $h' \succcurlyeq h$ signifies "$h'$ is no less hierarchical than $h$."

This relation splits into two components: its symmetric part ($\sim$), where $h' \sim h$ denotes mutual comparability (equivalent hierarchies in authority structure), and its asymmetric part ($\succ$), where $h' \succ h$ indicates strict dominance (one hierarchy is unambiguously superior to another).

A hierarchy $h'$ is called a *relabeling* of $h$ if it is obtained by reassigning labels (identifiers) to individuals in $h$ while preserving the structure of authority relationships.

This can be formally stated as follows. There exists a bijection $\phi$ from the nodes in $h$ to those in $h'$ such that for all nodes $i, j$ in $h$,

$$j \text{ is an immediate supervisor of } i \text{ in } h \Leftrightarrow \phi(j) \text{ is an immediate supervisor of } \phi(i) \text{ in } h'.$$

This ensures $h$ and $h'$ are structurally identical, differing only in individual labels.

**Anonymity (A).** A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies A if for any two hierarchies $h$ and $h'$ in $\mathscr{H}$, $h \sim h'$ whenever $h'$ is a relabeling of $h$.

Thus, $h$ and $h'$ are indistinguishable in their hierarchical degree when labels are irrelevant.

A *replication* of a hierarchy $h \in \mathscr{H}$ is a hierarchy in $\mathscr{H}$ of the form $(h, \dots, h)$. By convention, $h$ is a replication of itself.

The Replication Principle asserts that replicating a hierarchy does not change its hierarchical degree.

**Replication Principle (RP).** A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies RP if for any two hierarchies $h$ and $h'$ in $\mathscr{H}$, $h' \sim h$ whenever $h'$ is a replication of $h$.

We now introduce an axiom concerning hierarchical extensions that occur when one hierarchy is positioned above another by connecting the leaf nodes of the upper hierarchy to the root nodes of the lower hierarchy.

The operation of *appending a hierarchy $h'$ to another hierarchy $h$*, denoted $h \oplus h'$, involves positioning $h'$ beneath $h$ such that each leaf node in $h$ connects to one or more root nodes in $h'$. This connection establishes a supervisory relationship in the resulting compound hierarchy: the leaf nodes in $h$ become direct supervisors of their corresponding root nodes in $h'$.

Two important properties of this “addition” operator should be noted.

First, the operation $h \oplus h'$ is order-dependent and generally non-commutative—the sequence of hierarchies determines the outcome.

Second, since there isn’t a uniquely defined way to append one hierarchy to another, $h \oplus h'$ actually defines a *set* of hierarchies that can be obtained through the appending procedure.

**Upward Extension (UE).** A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies UE if for any $h, h' \in \mathscr{H}$, we have $h' \succ h$ whenever $h' \in h_o \oplus h$ for some $h_o \in \mathscr{H}$.

This axiom states that when a hierarchy $h$ is appended to another hierarchy $h_o$, the resulting compound hierarchy—a superstructure where $h_o$ is positioned “on top” of $h$—is more hierarchical than $h$ alone. The terminology “Upward Extension” emphasizes the upward expansion of the authority structure through the addition of higher-level nodes, consistent with the axiom’s focus on increasing hierarchical depth through vertical, upward extension.

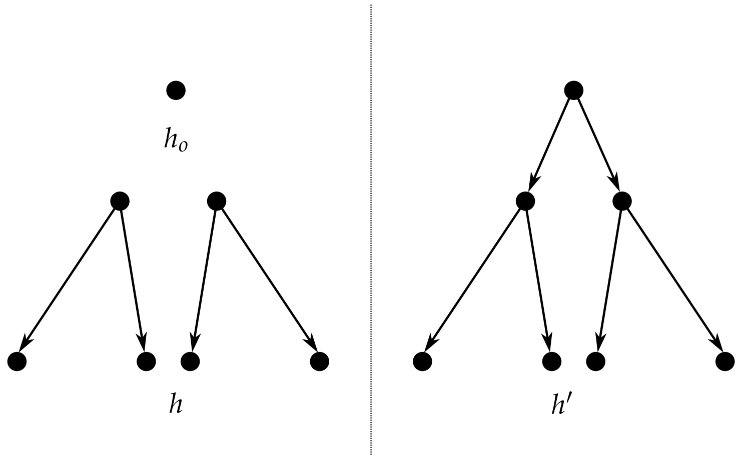


Figure 1: Illustrating UE.

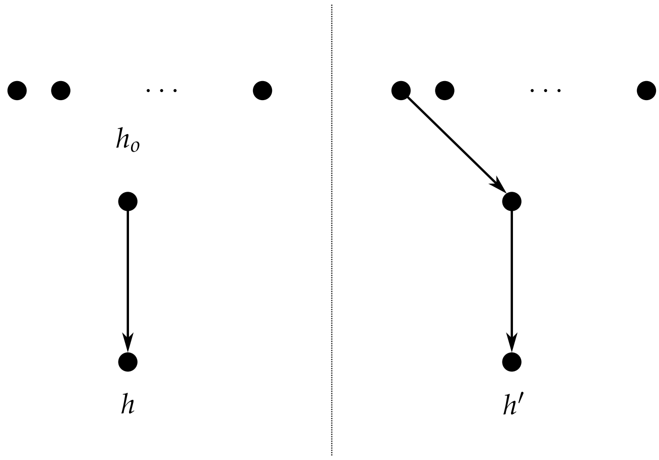


Figure 2: Illustrating UE.

A simple illustration of UE is provided in Figure 1, where the right panel demonstrates the result of appending hierarchy $h$ to hierarchy $h_o$, which consists of a single isolated node. Note that $h'$ belongs to the compound set $h_o \oplus h$. According to UE, $h'$ is more hierarchical than $h$.

A critical requirement of this hierarchical structure is that each leaf node in $h_o$ must connect to at least one root node in $h$. This constraint prevents configurations like the one shown in Figure 2, where hierarchy $h$ is positioned below a predominantly flat hierarchy $h_o$, but connections between leaf nodes in $h_o$ and root nodes in $h$ are incomplete. Such incomplete connections create structural ambiguity, especially when $h_o$ is extensive, as the resulting compound hierarchy $h'$ maintains an essentially flat structure with only two isolated "anecdotal" supervisory levels at one boundary.

The proper method of appending hierarchy $h$ to $h_o$ is demonstrated in Figure 3, where $h''$ represents the resulting compound hierarchy in $h_o \oplus h$, with every leaf node in $h_o$ connected to the (unique) root node in $h$. According to UE, $h''$ possesses a greater hierarchical degree than $h$ alone—an intuitive conclusion given the increased depth of $h''$.

Next, we establish a "more hierarchical than" pre-order based on the UE axiom. As a first step, we introduce a symmetric binary relation $\sim^*$ on the set of $n$-person hierarchies, $\mathscr{H}_n$, defined as follows:

$$h \sim^* h' \Leftrightarrow h \text{ is a relabeling of } h'.$$

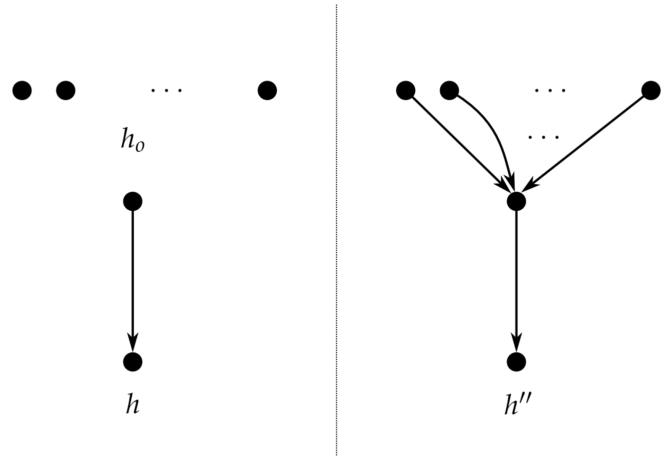


Figure 3: Illustrating UE.

This relation can be extended to the domain $\mathscr{H}$ of all hierarchies as follows: for $h, h' \in \mathscr{H}$,

$$h \sim^* h' \Leftrightarrow h_r \sim^* h'_r$$

for some equally-sized replicas $h_r$ and $h'_r$ of $h$ and $h'$, respectively.

The proof of the following lemma is relegated to Appendix A.1.

**Lemma 1.** *The binary relation $\sim^*$ defined on $\mathscr{H}$ is symmetric, reflexive, and transitive.*

Next, define the following binary relation $\succcurlyeq_{UE}$ on $\mathscr{H}$:

$$h' \succcurlyeq_{UE} h \Leftrightarrow \begin{cases} \exists h_o, H' \sim^* h', H \sim^* h : H' \in h_o \oplus H \\ \text{or} \\ h \sim^* h'. \end{cases}$$

The symmetric and asymmetric parts of $\succcurlyeq_{UE}$ are denoted by $\sim_{UE}$ and $\succ_{UE}$, respectively.

We first verify that $\succcurlyeq_{UE}$ is a hierarchical pre-order satisfying A, RP, and UE.

**Proposition 1.** *The binary relation $\succcurlyeq_{UE}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying A, RP, and UE.*

The proof of Proposition 1 is presented in Appendix A.2.

Given two hierarchical pre-orders $\succcurlyeq$ and $\succcurlyeq'$ on $\mathscr{H}$, $\succcurlyeq$ is $\succcurlyeq'$*-consistent* if it satisfies the following conditions for all hierarchies $h, h' \in \mathscr{H}$:

- $h \succ' h' \Rightarrow h \succ h'$.
- $h \sim' h' \Rightarrow h \sim h'$.

The hierarchical pre-order $\succcurlyeq_{UE}$ stands as the unique coarsest hierarchical pre-order that satisfies the A, RP, and UE axioms.

**Theorem 1.** *A hierarchical pre-order on $\mathscr{H}$ satisfies A, RP, and UE if and only if it is $\succcurlyeq_{UE}$-consistent.*

*Proof.* (*Sufficiency.*) Assume that $\succcurlyeq$ is a $\succcurlyeq_{UE}$-consistent hierarchical pre-order on $\mathscr{H}$. According to Proposition 1, $\succcurlyeq_{UE}$ satisfies the axioms A, RP, and UE. Since $\succcurlyeq$ is $\succcurlyeq_{UE}$-consistent, it follows that $\succcurlyeq$ must also satisfy these three axioms.

(*Necessity*.) Suppose that $\succcurlyeq$ is a hierarchical pre-order on $\mathscr{H}$ satisfying axioms A, RP, and UE. We need to show that $\succcurlyeq$ is $\succcurlyeq_{UE}$-consistent, which requires proving that for every pair of hierarchies $h, h' \in \mathscr{H}$:

(a) $h \succ_{UE} h' \Rightarrow h \succ h'$.

(b) $h \sim_{UE} h' \Rightarrow h \sim h'$.

Suppose first that $h \sim_{UE} h'$. Then there are two possibilities: either $h \sim^* h'$ or

$$\exists h_o, H' \sim^* h', H \sim^* h : H' \in h_o \oplus H \quad \text{and} \quad \exists h_o', H''' \sim^* h', H'' \sim^* h : H'' \in h_o' \oplus H'''.$$

However, the second case is impossible. Indeed, $H' \in h_o \oplus H$ implies that the depth of $H'$ exceeds that of $H$, while $H'' \in h_o' \oplus H'''$ implies that the depth of $H''$ exceeds that of $H'''$. Since

$$H' \sim^* h' \sim^* H''' \quad \text{and} \quad H \sim^* h \sim^* H'',$$

we have

$$H' \sim^* H''' \quad \text{and} \quad H \sim^* H''$$

(Lemma 1). Consequently, because replication and relabeling preserves depth, it follows that the depth of $H'$ and $H'''$ exceeds the depth of $H$ and $H''$, a contradiction.

Thus, $h \sim_{UE} h'$ implies $h \sim^* h'$, so there exist replicas $h_r$ and $h_r'$ of $h$ and $h'$, respectively, such that $h_r$ is a relabeling of $h_r'$.

Given that $\succcurlyeq$ satisfies axioms A and RP, we have:

$$h \sim h_r \sim h_r' \sim h'. \tag{1}$$

Since $\succcurlyeq$ is reflexive and transitive, $\sim$ is transitive (Sen, 2017, Lemma 1*a, p. 56). Therefore, (1) implies $h \sim h'$, which establishes (b).

Now suppose that $h \succ_{UE} h'$. Then

$$\exists h_o, H' \sim^* h', H \sim^* h : H \in h_o \oplus H'.$$

Consequently, because $\succcurlyeq$ satisfies the UE axiom, we see that

$$H \succ H'. \tag{2}$$

Now, since $h \sim^* H$, there are replicas $h_r$ and $H_r$ of $h$ and $H$, respectively, such that $h_r$ is a relabeling of $H_r$. Similarly, there are replicas $H_r'$ and $h_r'$ of $H'$ and $h'$, respectively, such that $H_r'$ is a relabeling of $h_r'$.

Hence, since $\succcurlyeq$ satisfies the A and RP axioms, it follows that

$$h \sim h_r \sim H_r \sim H \quad \text{and} \quad h' \sim h_r' \sim H_r' \sim H'.$$

Combining these relations with (2) yields

$$h \sim h_r \sim H_r \sim H \succ H' \sim H_r' \sim h_r' \sim h'.$$

Since $\succcurlyeq$ is reflexive and transitive, this chain of relations implies $h \succ h'$ (Sen, 2017, Lemma 1*a, p. 56). This establishes (a) and completes the proof. ∎

The hierarchical pre-order $\succcurlyeq_{UE}$ can be compared to the core pre-order introduced in Carbonell-Nicolau (2025b), defined as follows.

For any two hierarchies $h$ and $h'$ in $\mathscr{H}_n$, $h \succcurlyeq_H h'$ if and only if there exists a bijection $\phi$ from the set of individuals in $h$ to the set of individuals in $h'$ satisfying the following condition:

- For every individual $i$ in $h$ such that $\phi(i)$ has at least one supervisor in $h'$, any immediate supervisor $j'$ of $\phi(i)$ in $h'$ must satisfy one of the following:
  (a) If there exists an alternative indirect path from $j'$ to $\phi(i)$ in $h'$ besides the direct edge from $j'$ to $\phi(i)$, then $\phi^{-1}(j')$ must be a direct supervisor of $i$ in $h$.
  (b) Otherwise, $\phi^{-1}(j')$ must be either a direct or indirect supervisor of $i$ in $h$.

This relation can be extended to the domain $\mathscr{H}$ as follows: for $h, h' \in \mathscr{H}$, $h' \succcurlyeq_H h$ if and only if there exist equally-sized replications $h_r$ and $h'_r$ of $h$ and $h'$, respectively, such that $h'_r \succcurlyeq_H h_r$. This extension constitutes a properly defined hierarchical pre-order on $\mathscr{H}$ (Carbonell-Nicolau, 2025b).

As shown in Carbonell-Nicolau (2025b), $\succcurlyeq_H$ is intimately related to the notion of subordination removal, a concept first formalized in the context of hierarchical trees by Carbonell-Nicolau (2025a) and subsequently generalized by Carbonell-Nicolau (2025b), who recasts the mechanism within the broader mathematical structure of directed acyclic graphs.

**Definition 2.** We say that hierarchy $h'$ is obtained from hierarchy $h$ by *removing a subordination relation* if there exist a subordinate $i$ in $h$ and an immediate supervisor $j$ of $i$ satisfying one of the following conditions:

- If $j$ has no supervisors in $h$, then $h'$ is the hierarchy obtained by removing the directed edge from $j$ to $i$ while preserving all other relationships. Specifically:
  - The sub-hierarchy rooted at node $i$, $h(i)$, is no longer under $j$'s direct supervision.
  - Individual $i$ loses exactly one direct supervisor (namely, $j$).
  - The structure of sub-hierarchy $h(i)$ remains unchanged.
  - All other supervisory relationships in $h$ are preserved in $h'$.
- If $j$ has at least one supervisor in $h$, then $h'$ is the hierarchy obtained by:
  - Removing the directed edge from $j$ to $i$.
  - For each direct supervisor $j'$ of $j$ in $h$, either
    * adding a new directed edge from $j'$ to $i$ if no (direct or indirect) path from $j'$ to $i$ exists in $h$ besides the path through $j$; or
    * making no change if there already exists an alternative path (besides the path through $j$) from $j'$ to $i$ in $h$.
  - Preserving the structure of sub-hierarchy $h(i)$.
  - Preserving all other supervisory relationships in $h$.

This definition formalizes the process of subordination relation removal in two scenarios:

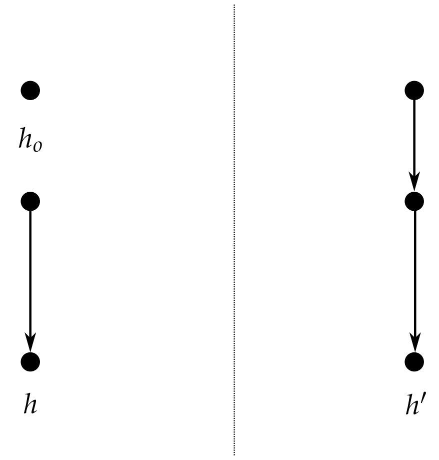


Figure 4: $\succcurlyeq_{UE}$ versus $\succcurlyeq_{H}$.

- If supervisor $j$ has no superiors in the hierarchy, the subordination link $j \to i$ is completely eliminated.
- When $j$ has superiors, then, for each immediate superior $j'$ of $i$, a direct link $j' \to i$ is established (bypassing $j$ entirely), only if there is no alternative path (besides the path through $j$) from $j'$ to $i$ in $h$.

**Subordination Removal (SR).** A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies SR if for any two hierarchies $h$ and $h'$ in $\mathscr{H}$, $h \succ h'$ whenever $h'$ is obtained from $h$ by removing a subordination relation.

Carbonell-Nicolau (2025b) demonstrates that the strict dominance relation $h \succ_H h'$ precisely characterizes hierarchies obtainable through successive removals of subordination relationships. Additionally, $\succcurlyeq_H$-consistent hierarchical pre-orders can be fully characterized by the A, RP, and SR axioms.

The core measures $\succcurlyeq_{UE}$ and $\succcurlyeq_H$ capture different aspects of hierarchical structure. While $\succcurlyeq_H$ focuses on the fundamental concept of subordination removal, $\succcurlyeq_{UE}$ specifically responds to the superposition of hierarchies through the appending operation, which enhances a structure's vertical dimension.

This complementarity between the two measures is illustrated in Figure 4, where $h'$ results from appending $h$ to $h_o$, establishing that $h' \succ_{UE} h$. However, $h'$ and $h$ are not comparable under $\succcurlyeq_H$. To understand why, consider Figure 5, which shows a 3-times replica of $h$ in the left panel and a 2-times replica of $h'$ in the right panel.

Observe that it is impossible to construct a bijection between the nodes of these two replicas that would make the 2-times replica more hierarchical than the 3-times replica under $\succcurlyeq_H$. This is because at least one root node in the latter hierarchy must connect to a subordinate in the former hierarchy whose immediate supervisor corresponds to a non-supervisor of that root node.

Conversely, situations exist where $h \succ_H h'$ while $h$ and $h'$ remain non-comparable under $\succcurlyeq_{UE}$. For a simple illustration, consider the hierarchies depicted in Figure 6. Using the bijection indicated by the double-arrowed correspondence between nodes across hierarchies shown in the figure, we can easily verify that $h \succ_H h'$. However, $h$ cannot be derived from $h'$ through composition of measures in $h_o \oplus h'$ for any $h_o$, since such a compound measure would necessarily contain at least four nodes.

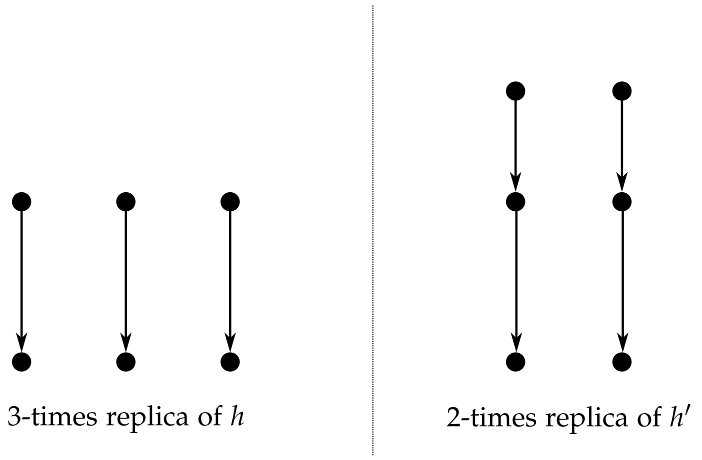


Figure 5: $\succcurlyeq_{UE}$ versus $\succcurlyeq_H$.

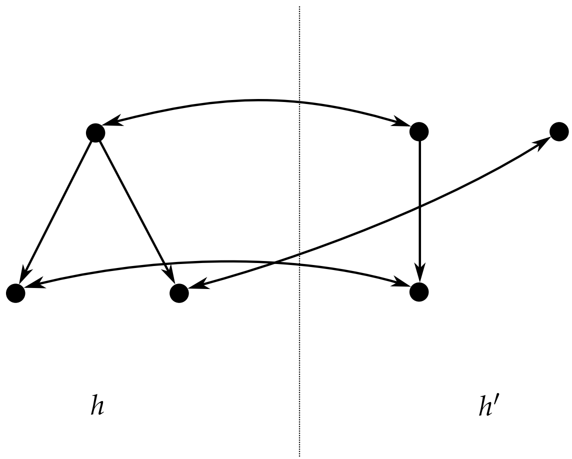


Figure 6: $\succcurlyeq_{UE}$ versus $\succcurlyeq_H$.

Since these two measures of hierarchy capture distinct aspects of hierarchical structure, combining them—by considering the transitive closure of their union—yields a more comprehensive metric.

Let $R = \succcurlyeq_H \cup \succcurlyeq_{UE}$ denote the union of the two relations, defined as:

$$hRh' \Leftrightarrow h \succcurlyeq_H h' \text{ or } h \succcurlyeq_{UE} h'.$$

Let $I$ and $P$ represent the symmetric and asymmetric parts of $R$, respectively.

**Lemma 2.** *$R$ has no cycles except those formed entirely by indifference ($I$).*

*Proof.* Suppose that

$$h_1 R \cdots R h_m R h_1.$$

Observe that

$$h_1 R \cdots R h_m$$

implies $I_s(h_1) \geq I_s(h_m)$, where $I_s(h'')$ represents the average number of (direct or indirect) supervisors in $h''$. This is clearly the case if $R = \succcurlyeq_{UE}$. If $R = \succcurlyeq_H$, the inequality follows from the fact (proven in Carbonell-Nicolau (2025b)) that the hierarchical order $\succcurlyeq_{I_s}$ on $\mathscr{H}$ defined by

$$h \succcurlyeq_{I_s} h' \Leftrightarrow I_s(h) \geq I_s(h'),$$

is $\succcurlyeq_H$-consistent, so that $h \succcurlyeq_H h'$ implies $I_s(h) \geq I_s(h')$.

Similarly,

$$h_m R h_1$$

implies $I_s(h_m) \geq I_s(h_1)$.

Consequently, $h_1$ and $h_m$ have the same average depth, $I_s(h_1) = I_s(h_m)$, implying that

$$h_1 I \cdots I h_m I h_1.$$

Indeed, if at least one of the relations were strict, one of the following would hold, since strict dominance in either $\succcurlyeq_H$ or $\succcurlyeq_{UE}$ forces a strict increase in $I_s$:[1]

- $I_s(h_m) < I_s(h_1)$ and $I_s(h_1) \leq I_s(h_m)$. This is a contradiction.
- $I_s(h_m) \leq I_s(h_1)$ and $I_s(h_1) < I_s(h_m)$. This is a also contradiction. ∎

Although both $\succcurlyeq_H$ and $\succcurlyeq_{UE}$ are individually transitive, their union $R$ need not be. To restore transitivity, we take the transitive closure $R^*$, which is the smallest (coarsest) transitive relation containing $R$.

To see that the union relation $R$ does not generally inherit transitivity from its transitive components, consider the three hierarchies illustrated in Figure 7. We can observe that:

$$h \succ_{UE} h' \succ_H h''.$$

However, it can be verified that neither $h \succcurlyeq_H h''$ nor $h \succcurlyeq_{UE} h''$ holds.

The transitive closure $R^*$ can be defined as follows: $hR^*h'$ if and only if there exists a sequence of hierarchies

$$h = h_0, h_1, \ldots, h_m = h'$$

[1] In the case of $\succcurlyeq_H$, this follows from $\succcurlyeq_H$-consistency of $\succcurlyeq_{I_s}$.

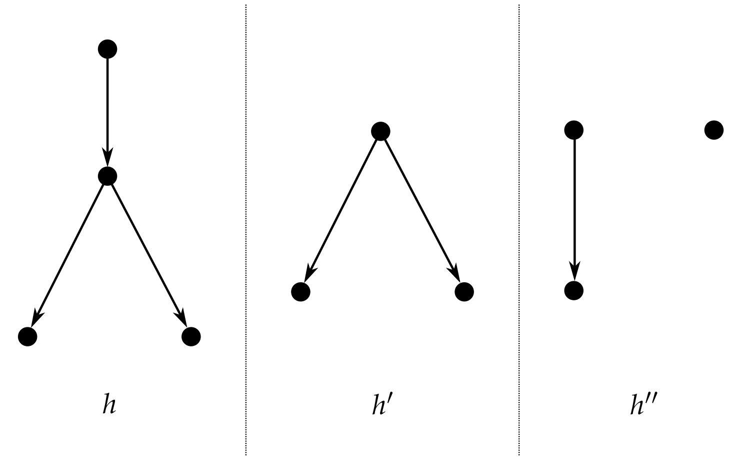


Figure 7: Non-transitivity of $R$.

such that $h_k R h_{k+1}$ for each $k \in \{0, \ldots, m-1\}$.

Since both $\succcurlyeq_H$ and $\succcurlyeq_{UE}$ are transitive relations, $R^*$ can be equivalently defined as: $hR^*h'$ if and only if there exists a sequence of hierarchies

$$h = h_0, h_1, \ldots, h_m = h'$$

where consecutive pairs in the sequence alternate between the relations $\succcurlyeq_H$ and $\succcurlyeq_{UE}$, following either pattern:

$$h_0 \succcurlyeq_H h_1 \succcurlyeq_{UE} h_2 \succcurlyeq_H \cdots$$

or

$$h_0 \succcurlyeq_{UE} h_1 \succcurlyeq_H h_2 \succcurlyeq_{UE} \cdots .$$

The symmetric and asymmetric parts of $R^*$ are denoted by $I^*$ and $P^*$ respectively.

The transitive closure $R^*$ is reflexive and transitive and satisfies the four axioms A, RP, UE, and SR.

**Proposition 2.** *The binary relation $R^*$ is a hierarchical pre-order on $\mathscr{H}$ satisfying* A, RP, UE, *and* SR.

*Proof.* $R^*$ is transitive by definition of transitive closure and it is reflexive by reflexivity of $\succcurlyeq_H$ and $\succcurlyeq_{UE}$. It satisfies A, RP, UE, and SR because it contains the union $R = \succcurlyeq_H \cup \succcurlyeq_{UE}$ and $\succcurlyeq_H$ satisfies A, RP, and SR (Carbonell-Nicolau, 2025b) and $\succcurlyeq_{UE}$ satisfies A, RP, and UE (Proposition 1). ■

$R^*$-consistency is necessary and sufficient for a hierarchical pre-order on $\mathscr{H}$ to satisfy A, RP, UE, and SR.

**Theorem 2.** *A hierarchical pre-order on $\mathscr{H}$ satisfies* A, RP, UE, *and* SR *if and only if it is $R^*$-consistent.*

The proof of this result is relegated to Appendix A.3

We now introduce a new axiom concerning hierarchical extensions that occur when one hierarchy is positioned below another by connecting certain leaf nodes of the upper hierarchy to the root nodes of the lower hierarchy.

This axiom requires a concept of "appending" that differs from the "⊕" operator introduced earlier. A hierarchy obtained by positioning $h'$ beneath $h$ such that every root

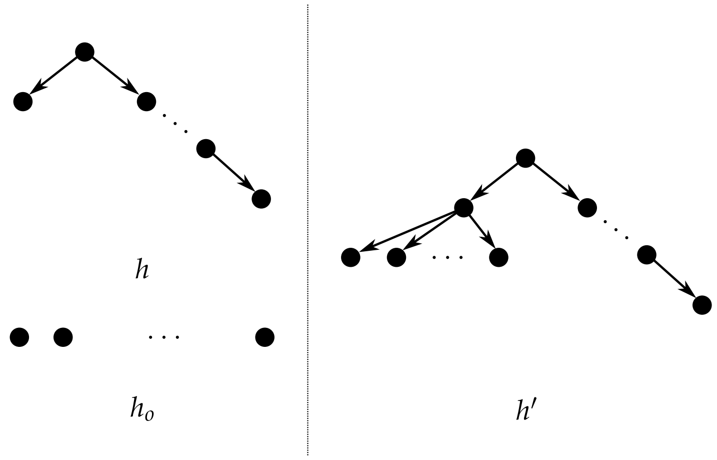


Figure 8: $h' \notin h \boxplus h_o$.

node in $h'$ is connected to one or more leave modes in $h$ with maximum depth within $h$, in such a way that the resulting compound structure is a directed acyclic graph, is said to result from the "addition" operation "$h \boxplus h'$."

Like the "$\oplus$" operator, this appending operation is order-dependent and generally non-commutative. Additionally, $h \boxplus h'$ defines a set of possible hierarchies that can result from this appending procedure, as there are multiple ways to connect the hierarchies.

**Downward Extension (DE).** A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies DE if for any $h, h' \in \mathscr{H}$, we have $h' \succ h$ whenever $h' \in h \boxplus h_o$ for some $h_o \in \mathscr{H}$.

This axiom states that when a hierarchy $h_o$ is appended to another hierarchy $h$ via the operator $\boxplus$, the resulting compound hierarchy—a structure where $h_o$ is positioned below $h$—is more hierarchical than $h$ alone. The term "Downward Extension" reflects how the authority structure expands downward through the addition of lower-level nodes.

We illustrate the operation $\boxplus$ in Figure 8 and Figure 9.

Figure 8 demonstrates a case where a purely horizontal hierarchy, $h_o$, is appended to hierarchy $h$ by connecting to a leaf node in $h$ that does not possess maximum depth within $h$. The resulting compound hierarchy, $h'$, is not a member of $h \boxplus h_o$, as it violates the maximum depth requirement.

In contrast, Figure 9 shows the proper application of $\boxplus$, where hierarchy $h_o$ is appended specifically to a leaf node in $h$ that has maximum depth within $h$. This creates a compound structure that is a valid member of $h \boxplus h_o$, satisfying the condition required by our axiom.

According to DE, the hierarchy $h'$ from Figure 9 is definitively more hierarchical than $h$ (i.e., $h' \succ h$), while the axiom makes no determination about the relative hierarchical nature of the structures in Figure 8. This distinction highlights the specific scope of the Downward Extension axiom, which only applies when appending occurs at maximum-depth leaf nodes.

The binary relation associated with the DE axiom, denoted $\succcurlyeq_{DE}$, can be formally defined on $\mathscr{H}$ in a manner analogous to $\succcurlyeq_{UE}$:

$$h' \succcurlyeq_{DE} h \Leftrightarrow \begin{cases} \exists h_o, H' \sim^* h', H \sim^* h \text{ such that } H' \in H \boxplus h_o \\ \text{or} \\ h \sim^* h'. \end{cases}$$

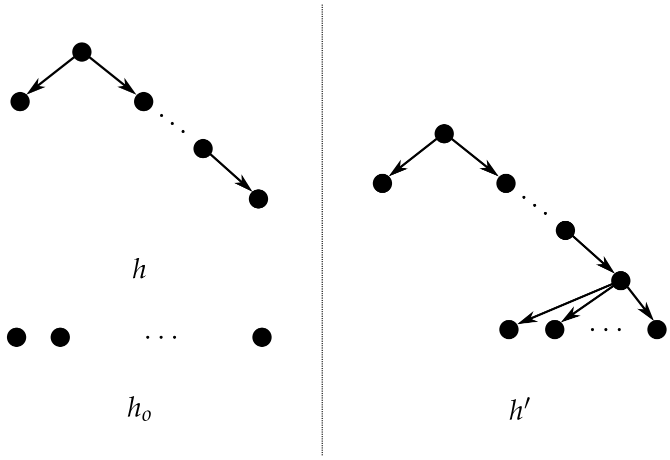


Figure 9: $h' \in h \boxplus h_o$.

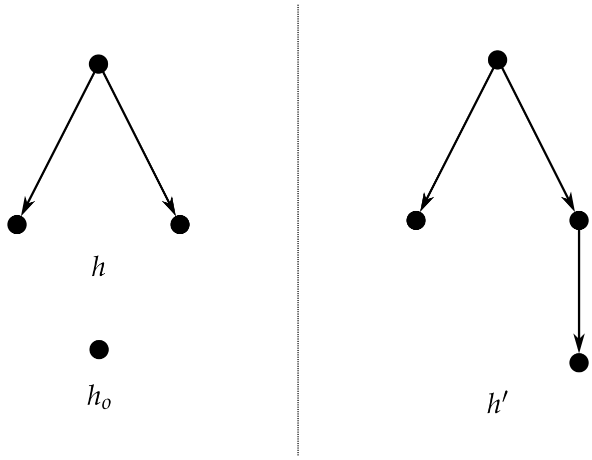


Figure 10: $h' \succ_{DE} h$ but $h' \not\succcurlyeq_{UE} h$ and $h' \not\succcurlyeq_{H} h$.

We denote the symmetric part of $\succcurlyeq_{DE}$ by $\sim_{DE}$ and its asymmetric part by $\succ_{DE}$.

We note that there exist pairs of hierarchies in $\mathscr{H}$ that are comparable under $\succcurlyeq_{DE}$ but not under either $\succcurlyeq_{UE}$ or $\succcurlyeq_{H}$.

To demonstrate this, consider the hierarchies depicted in Figure 10. We can observe that $h' \in h \boxplus h_o$, which by definition implies $h' \succ_{DE} h$. However, there exists no hierarchy $h^*$ such that $h'$ belongs to $h^* \oplus h$, meaning $h' \not\succcurlyeq_{UE} h$.

Furthermore, examining all possible replicas of $h$ and $h'$ (denoted $H$ and $H'$ respectively), we can verify that $H'$ cannot be obtained from $H$ through any sequence of subordination removals. This implies $h' \not\succcurlyeq_{H} h$.

Similarly, both hierarchies are uncomparable under $R^*$.

This example illustrates how the Downward Extension relation captures hierarchical comparisons that remain undetected by both $\succcurlyeq_{UE}$ and $\succcurlyeq_{H}$.

The analysis of the properties of $\succcurlyeq_{DE}$ follows a parallel structure to our earlier examination of $\succcurlyeq_{UE}$. We establish that $\succcurlyeq_{DE}$ is a hierarchical pre-order satisfying A, RP, and DE.

**Proposition 3.** *The binary relation $\succcurlyeq_{DE}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying* A, RP, *and* DE.

The complete proof of this proposition can be found in Appendix A.4.

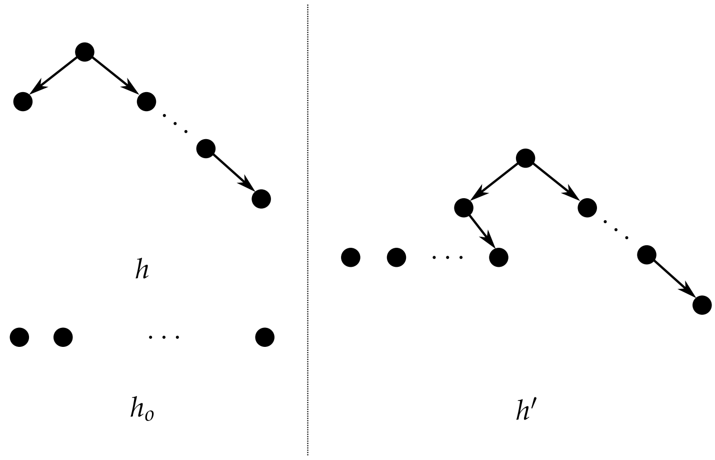


Figure 11: $h' \in h \uplus h_o$.

Furthermore, we can characterize $\succcurlyeq_{DE}$ as the unique coarsest hierarchical pre-order that simultaneously satisfies the A, RP, and DE axioms.

**Theorem 3.** *A hierarchical pre-order on $\mathscr{H}$ satisfies A, RP, and DE if and only if it is $\succcurlyeq_{DE}$-consistent.*

The detailed proof of Theorem 3 is provided in Appendix A.5.

Next, we introduce two additional axioms that serve as variants of UE and DE. These new formulations capture distinct aspects of hierarchical structure that were not addressed by the original axioms.

Before presenting these axioms, we must define a more generalized notion of "appending." For any two hierarchies $h, h' \in \mathscr{H}$, we denote by $h \uplus h'$ the set of all possible hierarchies in $\mathscr{H}$ that can be formed by positioning $h'$ beneath $h$ such that at least one leaf node in $h$ connects to at least one root node in $h'$.

This operation can be illustrated by referring to Figure 8, which demonstrates an instance where hierarchy $h_o$ is appended to hierarchy $h$ via the $\uplus$ operation, yielding $h' \in h \uplus h_o$. Similarly, Figure 11 provides another example where $h_o$ is positioned below $h$, resulting in a compound hierarchy that belongs to the set $h \uplus h_o$.

It is important to observe that in neither of the examples mentioned does the resulting hierarchy $h'$ belong to either $h \oplus h_o$ or $h \boxplus h_o$.

To clarify this distinction, consider Figure 8. Here, $h' \notin h \oplus h_o$ because the $\oplus$ operator requires *every* leaf node in $h$ to connect to a root node in $h_o$—a condition clearly violated in this figure, where only a subset of leaf nodes from $h$ establish connections with $h_o$.

Similarly, $h' \notin h \boxplus h_o$ since the connecting leaf node in $h$ does not possess maximum depth within $h$, which is a defining requirement of the $\boxplus$ operation.

We are now ready to formulate the last two axioms.

**Upward Extension* (UE*).** A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies UE* if for any $h, h' \in \mathscr{H}$, we have $h' \succ h$ whenever $h' \in h_o \uplus h$ and $h_o \succcurlyeq h$ for some $h_o \in \mathscr{H}$.

**Downward Extension* (DE*).** A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies DE* if for any $h, h' \in \mathscr{H}$, we have $h' \succ h$ whenever $h' \in h \uplus h_o$ and $h_o \succcurlyeq h$ for some $h_o \in \mathscr{H}$.

It is instructive to compare UE* with the earlier UE axiom, as they represent different perspectives on hierarchical growth.

UE establishes that adding any hierarchy $h_o$ on top of $h$ increases the hierarchical degree, provided that all leaf nodes of $h_o$ are connected to the roots of $h$. This can be conceptualized as stacking a new layer of authority (e.g., a new management tier) on top of an existing organization. Because every leaf of $h_o$ is linked to $h$, the resulting structure $h'$ has greater depth—more levels of authority—making it more hierarchical. Importantly, no condition is placed on the hierarchical nature of $h_o$ itself; it can be flat, deep, or anything in between. The key requirement is the complete connection through all leaves of $h_o$.

In contrast, UE* asserts that adding a hierarchy $h_o$ on top of $h$ increases the hierarchical degree under the following two conditions:

- $h_o$ must itself be (weakly) more hierarchical than $h$.
- The connection between $h_o$ and $h$ requires only partial linkage (not all leaves of $h_o$ need to connect to roots of $h$).

This can be visualized as attaching a highly hierarchical structure (e.g., a tall, well-organized management team) to an existing organization, even if the attachment is partial (e.g., only one manager oversees the original group). Because $h_o$ is already more hierarchical than $h$, even a loose connection elevates the overall hierarchical nature of the resulting structure $h'$.

Thus, while UE focuses on vertical growth through complete attachment—where adding any new layer on top, fully connected, deepens the hierarchy—UE* emphasizes the quality of the added structure, allowing a sufficiently hierarchical $h_o$ to enhance $h$ even with partial linkage.

Similarly, we can compare DE with DE*, which mirror the relationship between UE and UE* but in the downward direction.

DE states that extending a hierarchy $h$ downward by adding any hierarchy $h_o$ increases the hierarchical degree, provided that every root node of $h_o$ connects to leaf nodes of $h$ having maximum depth. This represents a downward extension of authority by attaching subordinates specifically to the lowest-level positions in the existing structure. The resulting hierarchy $h'$ gains additional levels of subordination, thus becoming more hierarchical. As with UE, no restrictions are placed on the hierarchical nature of $h_o$ itself—the critical factor is the complete connection to the deepest leaves of $h$.

Conversely, DE* takes a different approach, stipulating that adding $h_o$ below $h$ increases hierarchical degree when:

- $h_o$ is (weakly) more hierarchical than $h$.
- Only partial connection is required between $h$ and $h_o$ (neither all roots of $h_o$ need to connect, nor must connections be to maximum-depth leaves).

This corresponds to integrating a highly structured group of subordinates into an organization, even through limited points of attachment. The superior hierarchical quality of $h_o$ ensures that even with minimal connection, the resulting structure $h'$ becomes more hierarchical than $h$ alone.

In essence, while DE prioritizes complete integration at the deepest level to achieve vertical growth, DE* focuses on the hierarchical quality of what is being appended, requiring only minimal attachment to enhance the overall structure.

Our next task is to develop essential hierarchical pre-orders that align with the A, RP, and UE* (respectively, DE*) axioms.

Consider the binary relations on $\mathscr{H}$ defined recursively as follows:

- $R_0 = \sim^*$.
- For each $k \in \{1, 2, \ldots\}$,

$$R_k = R_{k-1} \cup \{(h', h) : \exists h_0, \ldots, h_m = h : \forall l \in \{0, \ldots, m-1\}, \\ h_l R_{k-1} h_{l+1} \;\&\; \exists H' \sim^* h', H \sim^* h : H' \in h_0 \uplus H\}.$$

We now define $\succcurlyeq_{UE^*}$ as the transitive closure of the union

$$\bigcup_{k=0}^{\infty} R_k.$$

Equivalently, $h \succcurlyeq_{UE^*} h'$ if and only if there exists a finite sequence

$$h = h_0,\ h_1,\ h_2,\ \ldots,\ h_m = h',$$

where each consecutive pair $(h_l, h_{l+1})$ satisfies $h_l R_k h_{l+1}$ for some $k \geq 0$.

As usual, the symmetric part of $\succcurlyeq_{UE^*}$ is denoted by $\sim_{UE^*}$ and its asymmetric part by $\succ_{UE^*}$.

The relation $\succcurlyeq_{UE^*}$ is a proper hierarchical pre-order satisfying A, RP, and UE*.

**Proposition 4.** *The binary relation $\succcurlyeq_{UE^*}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying A, RP, and UE*.*

The reader is referred to Appendix A.6 for the proof of Proposition 4.

The relation $\succcurlyeq_{UE^*}$ is the unique coarsest hierarchical pre-order that satisfies A, RP, and UE*.

**Theorem 4.** *A hierarchical pre-order on $\mathscr{H}$ satisfies A, RP, and UE* if and only if it is $\succcurlyeq_{UE^*}$-consistent.*

The proof of this result is relegated to Appendix A.7.

A similar analysis can be extended to the DE* axiom: we can construct the unique coarsest hierarchical pre-order that simultaneously satisfies the A, RP, and DE* axioms.

Define a sequence of binary relations on $\mathscr{H}$ as follows:

- $S_0 = \sim^*$,
- For each $k \in \{1, 2, \ldots\}$,

$$S_k = S_{k-1} \cup \{(h', h) : \exists h_0, \ldots, h_m = h : \forall l \in \{0, \ldots, m-1\}, \\ h_l S_{k-1} h_{l+1} \;\&\; \exists H' \sim^* h', H \sim^* h : H' \in H \uplus h_0\}.$$

Now define $\succcurlyeq_{DE^*}$ as the transitive closure of the union

$$\bigcup_{k=0}^{\infty} S_k.$$

The characterizations of $\succcurlyeq_{DE^*}$ presented below are analogous to those of $\succcurlyeq_{UE^*}$ established in Proposition 4 and Theorem 4. The proof of Proposition 5 appears in Appendix A.8, while the proof of Theorem 5 is provided in Appendix A.9.

**Proposition 5.** *The binary relation $\succcurlyeq_{DE^*}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying A, RP, and DE*.*

**Theorem 5.** *A hierarchical pre-order on $\mathscr{H}$ satisfies A, RP, and DE* if and only if it is $\succcurlyeq_{DE^*}$-consistent.*

In the context of hierarchical structures, various axioms can be employed to capture different facets of hierarchy. We now develop a theorem that characterizes hierarchical pre-orders satisfying a collection of such axioms by relating them to a fundamental pre-order constructed as the transitive closure of the union of individual pre-orders, each corresponding to a specific axiom set. This result generalizes Theorem 2.

A collection $\{\succcurlyeq_i\}_{i\in I}$ of hierarchical pre-orders in $\mathscr{H}$ is called a *composite axiomatic collection of hierarchical pre-orders* if it satisfies the following conditions:

1. For each index $i$ in some index set $I$, the pre-order $\succcurlyeq_i$ satisfies a corresponding set of axioms $A_i$. Each axiom is formally defined as a conditional statement of the following form:

$$\forall h, h' \in \mathscr{H}, P(h, h', \mathscr{H}', \succcurlyeq_i) \Rightarrow Q(h, h', \succcurlyeq_i)$$

   where:

   - $P(h, h', \mathscr{H}', \succcurlyeq_i)$ is a predicate describing a relationship or transformation between $h$ and $h'$. This predicate may depend not only on the pair $(h, h')$, but also on some subset $\mathscr{H}'$ of $\mathscr{H}$ and on the pre-order $\succcurlyeq_i$ itself.
   - $Q(h, h', \succcurlyeq_i)$ represents one of the following relations: $h \succcurlyeq_i h'$, $h \sim_i h'$, $h' \succcurlyeq_i h$, $h \succ_i h'$, or $h' \succ_i h$.

2. Any cycle in the union $\bigcup_{i\in I} \succcurlyeq_i$ must be formed entirely by indifference in each individual $\succcurlyeq_i$.

3. For some $i \in I$, the set $A_i$ contains the A and RP axioms.

4. For each $i \in I$:

   - If $h \sim_i h'$, then $h \sim^* h'$.
   - If $h \succ_i h'$, then there exists a finite sequence of hierarchies in $\mathscr{H}$,

$$h = h_0, h_1, \ldots, h_n = h',$$

   such that for each $l \in \{0, \ldots, n-1\}$, there exists some axiom $a \in A_i$ whose conditional statement for the tuple $(h_A, h'_A, \mathscr{H}', \succcurlyeq_i)$ is

$$P(h_A, h'_A, \mathscr{H}', \succcurlyeq_i) \Rightarrow Q(h_A, h'_A, \succcurlyeq_i),$$

the predicate $P(H_l, H_{l+1}, \mathscr{H}', \succcurlyeq_i)$ holds for some $H_l \sim_i h_l$ and $H_{l+1} \sim_i h_{l+1}$, and

$$Q(H_l, H_{l+1}, \succcurlyeq_i) = Q(H_l, H_{l+1}, \succcurlyeq)$$

for every hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfying axiom $a$, A, and RP.

Now define $\tilde{R}$ as the transitive closure of the union of all $\succcurlyeq_i$, $\bigcup_{i \in I} \succcurlyeq_i$.

The following two results are proven in Appendix A.10 and Appendix A.11, respectively.

**Proposition 6.** *The binary relation $\tilde{R}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying all the axioms in $\bigcup_{i \in I} A_i$.*

**Theorem 6.** *A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies all $A_i$ for $i \in I$ if and only if it is $\tilde{R}$-consistent.*

As an example of a composite axiomatic collection of hierarchical pre-orders, consider

$$\{\succcurlyeq_H, \succcurlyeq_{UE}, \succcurlyeq_{DE}, \succcurlyeq_{UE^*}, \succcurlyeq_{DE^*}\}.$$

**Proposition 7.** *The collection*

$$\{\succcurlyeq_H, \succcurlyeq_{UE}, \succcurlyeq_{DE}, \succcurlyeq_{UE^*}, \succcurlyeq_{DE^*}\}$$

*is a composite axiomatic collection of hierarchical pre-orders.*

We prove this proposition in Appendix A.12.

As a consequence of Proposition 7 and Theorem 6, we can identify the coarsest hierarchical pre-order on $\mathscr{H}$ that satisfies the seven axioms A, RP, SR, UE, DE, UE*, and DE*. This pre-order, denoted by $\hat{R}$, is defined as the transitive closure of the union

$$\succcurlyeq_H \cup \succcurlyeq_{UE} \cup \succcurlyeq_{DE} \cup \succcurlyeq_{UE^*} \cup \succcurlyeq_{DE^*}.$$

The pre-order $\hat{R}$ is thus the minimal hierarchical relation that encompasses all ordering principles represented by these axioms.

**Corollary 1.** *A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies A, RP, SR, UE, DE, UE*, and DE* if and only if it is $\hat{R}$-consistent.*

*Proof.* The statement follows from Theorem 6 and Proposition 7. ∎

## 4. Hierarchical indices

Having established our axiomatic theory of core pre-orders, we now extend our investigation to examine complete hierarchical orders, with a focus on one measure that quantifies average hierarchy depth. We demonstrate its compliance with all hierarchy measurement axioms introduced in this work, establishing its theoretical robustness.

We then analyze hierarchical pre-orders from prior research, situating these established measures within our theoretical framework.

Hierarchical orders offer computational advantages through simple formulae, avoiding the complexities of pre-orders that rely on transitive closures of binary relation unions. This makes them readily applicable to real-world datasets. However, by inducing a complete order on hierarchies, these measures necessarily make definitive comparisons between any two hierarchical structures—even in ambiguous cases where our axioms alone would not single out a more hierarchical structure.

A *hierarchical index* is a mapping $I : \mathscr{H} \to \mathbb{R}$ that assigns a "hierarchical degree" $I(h)$ to every hierarchy $h$ in $\mathscr{H}$.

In the following subsections, we examine various hierarchical indices through the lens of our axiomatic theory of hierarchical measurement.

## 4.1. Average depth

A hierarchical index measuring the average depth of each hierarchy is formally defined as follows:

$$I_s(h) = \frac{1}{n} \sum_{i \in h} s_h(i),$$

where $s_h(i)$ denotes the number of (direct or indirect) supervisors of individual $i$ in hierarchy $h$. This index represents the average number of supervisors per individual within the hierarchy $h$.

The average depth index induces a hierarchical order $\succcurlyeq_{I_s}$ defined by

$$h \succcurlyeq_{I_s} h' \Leftrightarrow I_s(h) > I_s(h'),$$

which satisfies the A, RP, and SR axioms (Carbonell-Nicolau, 2025b).

The UE axiom applied to $\succcurlyeq_{I_s}$ states that for any $h, h' \in \mathscr{H}$, $h' \succ_{I_s} h$ whenever $h' \in h_o \oplus h$ for some $h_o \in \mathscr{H}$.

Observe that appending a hierarchy $h$ to another hierarchy $h_o$ (i.e., forming $h' \in h_o \oplus h$) strictly increases the average depth $I_s$. Indeed, when $h$ is appended to $h_o$, every root node in $h$ gains additional supervisors from the leaves of $h_o$. The depth of each node in $h$ increases by the depth of its new supervisors in $h_o$ plus one (for the direct edge). The total depth of $h'$ equals the total depth of $h_o$ plus the total depth of $h$ adjusted upward due to the new supervisory relationships. Even if $h_o$ is a single node, appending it increases the average depth. Hence, for any non-trivial $h_o$, the added supervisors increase the total depth of $h'$, ensuring $I_s(h') > I_s(h)$. Thus $\succcurlyeq_{I_s}$ satisfies UE.

The DE axioms asserts that if $h'$ is obtained from a hierarchy $h$ by attaching another hierarchy $h_o$ below one of $h$'s deepest leaves, then $h'$ is strictly more hierarchical than $h$ (i.e. $h' \succ_{I_s} h$).

The index $I_s$ measures the average number of supervisors per person, i.e., the average distance up the chart from each individual to the top.

Let $D$ be the maximum depth in $h$. By definition, the average depth $I_s(h)$ cannot exceed $D$. When you attach a node of $h_o$ under a leaf at depth $D$, each new node ends up at depth at least $D + 1$. Since $D + 1 > I_s(h)$, adding these nodes pulls the overall average up.

Thus any valid downward extension strictly increases the average-depth index, so $\succcurlyeq_{I_s}$ satisfies DE.

The UE* axiom requires $h' \succ_{I_s} h$ whenever $h, h' \in \mathscr{H}$ and $h' \in h_o \uplus h$ and $h_o \succcurlyeq_{I_s} h$ for some $h_o \in \mathscr{H}$.

Let $h' \in h_o \uplus h$, where $h_o \succcurlyeq_{I_s} h$. This means $I_s(h_o) \geq I_s(h)$, and $h'$ is formed by appending $h$ to $h_o$ via $\uplus$ (connecting at least one leaf of $h_o$ to a root of $h$).

Let $d$ be the depth of the attaching leaf node in $h_o$. Each node in $h$ (originally with depth $k$ in $h$) now has depth $d+1+k$ in $h'$.

Let $n_o$ and $n_h$ be the sizes of $h_o$ and $h$. The total depth of $h'$ is:

$$\sum_{i\in h'} s_{h'}(i) = \sum_{i\in h_o} s_{h_o}(i) + \sum_{i\in h} s_h(i) + (d+1)\cdot n_h.$$

The average depth of $h'$ is:

$$I_s(h') = \frac{1}{n_o+n_h}\sum_{i\in h'} s_{h'}(i) = \frac{1}{n_o+n_h}\left(\sum_{i\in h_o} s_{h_o}(i) + \sum_{i\in h} s_h(i) + (d+1)\cdot n_h\right).$$

Subtract $I_s(h)$ from $I_s(h')$:

$$I_s(h') - I_s(h) = \frac{I_s(h_o)\cdot n_o - I_s(h)\cdot n_o + (d+1)\cdot n_h}{n_o+n_h}.$$

Since $I_s(h_o) \geq I_s(h)$, the term $I_s(h_o) - I_s(h) \geq 0$. Additionally, $(d+1)\cdot n_h > 0$ (as $d \geq 0$ and $n_h \geq 1$). Thus, $I_s(h') > I_s(h)$, implying $h' \succ_{I_s} h$.

Thus, the average depth index $\succcurlyeq_{I_s}$ satisfies UE* because appending $h$ to $h_o$ (with $h_o \succcurlyeq_{I_s} h$) strictly increases the average depth, ensuring $h' \succ_{I_s} h$.

One can similarly show that $\succcurlyeq_{I_s}$ satisfies DE*.

**Proposition 8.** *The hierarchical order $\succcurlyeq_{I_s}$ satisfies* A, RP, SR, UE, DE, UE*, *and* DE*.

An immediate consequence of Corollary 1 and Proposition 8 is that $\succcurlyeq_{I_s}$ is $\hat{R}$-consistent, where $\hat{R}$ is the transitive closure of the union

$$\succcurlyeq_H \cup \succcurlyeq_{UE} \cup \succcurlyeq_{DE} \cup \succcurlyeq_{UE^*} \cup \succcurlyeq_{DE^*}.$$

## 4.2. Global reaching centrality

In this section, we situate the global reaching centrality index proposed by Mones et al. (2012) within our axiomatic framework, examining how this established measure aligns with our theoretical principles for hierarchy quantification.

For a hierarchy $h \in \mathscr{H}$ with a set of nodes $V(h)$ and total number of nodes $N = |V(h)|$, the *global reaching centrality* (grc) is defined as:

$$\mathrm{grc}(h) = \frac{1}{N-1}\sum_{v\in V(h)} \left(C_R^{\max} - C_R(v)\right),$$

where:

- $C_R(v)$ represents the *reaching centrality* of node $v$, defined as follows:

$$C_R(v) = \frac{1}{N-1}|\{w \in V(h)\setminus\{v\} : \text{there exists a directed path from } v \text{ to } w \text{ in } h\}|.$$

  It represents the fraction of other nodes that can be reached from $v$ via directed paths.

- $C_R^{\max} = \max_{u\in V(h)} C_R(u)$ is the maximum reaching centrality among all nodes in $h$.

The grc index measures the degree of hierarchy within a directed acyclic graph by evaluating how concentrated the ability to reach others is among its nodes. In a highly hierarchical structure, such as an organization with a clear chain of command, a few nodes (e.g., those at the top) can reach many others through directed paths, while most nodes have limited or no reach. This results in a high grc value, as the differences $C_R^{\max} - C_R(v)$ are large for most nodes due to the significant disparity in reaching centralities. Conversely, in a flat structure where no node can reach others (e.g., a graph with no edges), all reaching centralities are zero, leading to a grc of zero, indicating the absence of hierarchy. Thus, the grc effectively quantifies the extent to which influence or control is centralized within the graph, providing a clear metric for the hierarchical nature of the system it represents.

As demonstrated in Carbonell-Nicolau (2025a), the hierarchical order $\succcurlyeq_{\text{grc}}$ induced by the grc index violates the SR axiom. While $\succcurlyeq_{\text{grc}}$ clearly satisfies the A axiom, the grc index, as defined, contradicts the Replication Principle, RP.

To illustrate this violation, consider hierarchy $h$ with two nodes $v \to w$. We have:

$$\begin{aligned} N &= 2. \\ C_R(v) &= 1. \\ C_R(w) &= 0. \\ C_R^{\max} &= 1. \end{aligned}$$

Thus:

$$\text{grc}(h) = \frac{1}{1}(0+1) = 1$$

Now consider $h'$, a one-time replica of $h$ with two copies: $v_1 \to w_1$ and $v_2 \to w_2$. For this hierarchy:

$$\begin{aligned} N' &= 4. \\ C_R(v_1) = C_R(v_2) &= \frac{1}{3}. \\ C_R(w_1) = C_R(w_2) &= 0. \\ C_R^{\max} &= \frac{1}{3}. \end{aligned}$$

Thus:

$$\text{grc}(h') = \frac{1}{3}\left(0 + \frac{1}{3} + 0 + \frac{1}{3}\right) = \frac{2}{9}.$$

Therefore, $\text{grc}(h') < \text{grc}(h)$. The RP axiom requires $\text{grc}(h') = \text{grc}(h)$, but the standard grc decreases with replication due to its normalization by the larger denominator.

Next, we show that $\succcurlyeq_{\text{grc}}$ also violates UE. Consider the hierarchy $h$ with two nodes $v$ and $w$ and a single directed edge $v \to w$. Here $N = 2$. The reaching centralities are

$$C_R(v) = \frac{|\{w\}|}{N-1} = \frac{1}{1} = 1 \quad \text{and} \quad C_R(w) = 0,$$

so

$$C_R^{\max} = 1,$$

and hence

$$\mathrm{grc}(h) = \frac{1}{N-1}\sum_{u\in h}\big(C_R^{\max} - C_R(u)\big) = \frac{1}{1}\big((1-1)+(1-0)\big) = 1.$$

Form $h''$ by adding a new root $x$ on top of $h$: $x \longrightarrow v \longrightarrow w$. Now $N' = 3$. Compute

$$C_R(x) = \frac{2}{2} = 1, \quad C_R(v) = \frac{1}{2} = 0.5, \quad C_R(w) = 0,$$

so again $C_R^{\max} = 1$, but

$$\mathrm{grc}(h'') = \frac{1}{N'-1}\sum_{u\in h''}\big(C_R^{\max} - C_R(u)\big) = \frac{1}{2}\big((1-1)+(1-0.5)+(1-0)\big) = \frac{1.5}{2} = 0.75.$$

Since

$$\mathrm{grc}(h'') = 0.75 < 1 = \mathrm{grc}(h),$$

the grc-induced order $\succcurlyeq_{\mathrm{grc}}$ violates the UE axiom.

This example reveals a fundamental limitation of the grc measure when assessing vertical hierarchical growth. While $h''$ contains a deeper vertical structure with three levels ($x \longrightarrow v \longrightarrow w$) compared to the simpler two-level hierarchy in $h$ ($v \longrightarrow w$), the grc measure counterintuitively decreases.

The issue stems from how grc quantifies hierarchical inequality through the distribution of reaching centrality. In hierarchy $h$, node $v$ holds all possible reaching centrality (perfect concentration), reaching 100% of other nodes, while $w$ reaches none. In contrast, in $h''$, the reaching power becomes more distributed: $x$ reaches all other nodes, but $v$ still reaches 50% of the network. This distribution of reaching power is interpreted by grc as a decrease in hierarchical inequality, even though the hierarchy has actually become deeper and more stratified.

This counterintuitive behavior demonstrates why grc fails to consistently capture vertical hierarchical growth, prioritizing concentration of reaching power over the development of multi-level command chains.

A similar example can be constructed to demonstrate that $\succcurlyeq_{\mathrm{grc}}$ also violates the DE axiom. When adding subordinates to lower-level nodes, grc can increase or decrease depending on how the addition affects the overall distribution of reaching centrality, rather than consistently capturing the deepening of hierarchical relationships.

Furthermore, slight modifications of these examples can be used to prove that $\succcurlyeq_{\mathrm{grc}}$ fails to satisfy the UE* and DE* axioms.

Specifically for the case of UE*, consider the following example.

For the hierarchy $h$, $v \longrightarrow w$, we have $\mathrm{grc}(h) = 1$.

Now form $h'''$ by appending $h$ to $h_o = x \to y$, resulting in $x \to y \to v \to w$. Reaching centralities:

$$\begin{aligned} C_R(x) &= 1 \text{ (reaches all nodes)}. \\ C_R(y) &= \frac{2}{3} \text{ (reaches } v, w). \\ C_R(v) &= \frac{1}{3} \text{ (reaches } w). \\ C_R(w) &= 0. \end{aligned}$$

Hence,

$$\text{grc}(h''') = \frac{(1-1)+(1-\frac{2}{3})+(1-\frac{1}{3})+(1-0)}{4-1} = \frac{0+\frac{1}{3}+\frac{2}{3}+1}{3} = \frac{2}{3},$$

and so

$$\text{grc}(h''') = \frac{2}{3} < \text{grc}(h) = 1.$$

Since appending $h_o$ (which is equally hierarchical to $h$) reduces grc, $\succcurlyeq_{\text{grc}}$ violates UE*.

**Proposition 9.** *The hierarchical order $\succcurlyeq_{\text{grc}}$ satisfies* A *and violates* RP, SR, UE, DE, UE*, *and* DE*.

## 4.3. Krackhardt's graph theoretical dimensions

Krackhardt (1994) identified four distinct dimensions for measuring the "hierarchical degree" in networks of informal organizations. In the following sections, we examine each of these dimensions in detail.

### 4.3.1. Connectedness

The first dimension, connectedness, quantifies the extent to which the network is connected as a whole.

In Krackhardt's original definition, connectedness measures the degree to which all nodes in the network are connected through paths, regardless of direction. Specifically, it is the fraction of pairs of nodes that are mutually reachable when the directed graph is treated as undirected.[2]

Formally, within our framework, the *connectedness index* for a hierarchy $h \in \mathscr{H}$ can be formulated as follows:

$$C(h) = \frac{|\{(i,j) \in V \times V : i \neq j \text{ and there exists an undirected path between } i \text{ and } j\}|}{n(n-1)},$$

where:

[2]Treating the graph as undirected for the connectedness index overlooks the directionality of information flows, leading to potential information leaks.

For example, consider a simple chain where $j \to i$. Information flows unidirectionally from $j$ to $i$ (e.g., from supervisor to subordinate), but not vice versa, reflecting the hierarchical nature of supervision. In directed acyclic graphs, information moves solely from supervisors to subordinates along defined paths. In contrast, networks with cycles allow bidirectional information flows. A refined connectedness measure should account for the directionality of these information pathways to better capture the network's structure.

- $V$ is the set of all nodes in $h$.
- $n = |V|$ is the total number of nodes in $h$.
- The numerator counts all ordered pairs $(i, j)$ where $j$ is reachable from $i$ via at least one path when direction is ignored.
- The denominator $n(n-1)$ represents the total number of possible ordered pairs (excluding self-loops).

When applying this definition to directed acyclic graphs (which represent hierarchies in our framework), we observe that connectedness measures the degree to which the hierarchy forms a connected structure rather than separate disconnected components. For non-trivial connected hierarchies (i.e., $n > 1$), $C(h) = 1$, while for disconnected hierarchies, $0 < C(h) < 1$.

The hierarchical order induced by the connectedness index is denoted by $\succcurlyeq_C$.

The connectedness index $C(h)$ measures the proportion of node pairs $(i, j)$ that are mutually reachable when treating the directed graph as undirected. Relabeling nodes does not affect the existence or count of such paths—it only changes node labels. Thus, relabeling preserves the connectedness index, so $\succcurlyeq_C$ treats structurally identical hierarchies as equivalent, fulfilling the A axiom.

To demonstrate that the connectedness-based hierarchical order $\succcurlyeq_C$ violates the RP axiom, we present the following counterexample.

Consider hierarchy $h$, a simple two-person structure with a single directed edge $j \longrightarrow i$. With $n = 2$ nodes, these two nodes are connected when direction is ignored, giving us one unordered pair with an undirected path. The connectedness index of $h$ is therefore:

$$C(h) = \frac{|\{(i, j) \in V \times V : i \neq j \text{ and there exists an undirected path between } i \text{ and } j\}|}{n(n-1)}$$
$$= \frac{2}{2(2-1)} = \frac{2}{2} = 1.$$

Now consider hierarchy $h'$, formed by creating two independent copies of $h$: $j_1 \longrightarrow i_1$ and $j_2 \longrightarrow i_2$. This replicated hierarchy contains $n' = 4$ nodes. When treated as an undirected graph, $h'$ has two connected components. The node pairs with undirected paths are $(i_1, j_1)$, $(j_1, i_1)$, $(i_2, j_2)$ and $(j_2, i_2)$, giving a connectedness index of:

$$C(h') = \frac{4}{n'(n'-1)} = \frac{4}{4(4-1)} = \frac{4}{12} = \frac{1}{3}.$$

Since $C(h) = 1 > \frac{1}{3} = C(h')$, we must conclude that $h \succ_C h'$ according to the connectedness-based ordering. However, the RP axiom explicitly requires that $h \sim_C h'$ (the original hierarchy and its replication should be equivalent). This contradiction demonstrates that $\succcurlyeq_C$ violates the RP axiom.

The SR axiom stipulates that whenever a hierarchy $h'$ is derived from $h$ by removing a subordination relation, then $h$ must be considered more hierarchical than $h'$.

To demonstrate that the hierarchical order $\succcurlyeq_C$ based on Krackhardt's connectedness index violates the SR axiom, consider the following example:

- Hierarchy $\hat{h}$: A chain structure with nodes $j \longrightarrow i \longrightarrow k$.

- Hierarchy $\tilde{h}$: A star structure with edges $j \rightarrow i$ and $j \rightarrow k$.

Hierarchy $\tilde{h}$ is obtained from $\hat{h}$ by removing the subordination relation $i \rightarrow k$ and adding the relation $j \rightarrow k$. Since this transformation involves removing a subordination relation, the SR axiom requires that $\hat{h} \succ_C \tilde{h}$.

Let us calculate the connectedness index for both hierarchies:

For hierarchy $\hat{h}$ (chain structure):

- When treated as an undirected graph, all three nodes form a single connected component.
- All 6 possible ordered pairs have undirected paths between them: $(j, i)$, $(i, j)$, $(i, k)$, $(k, i)$, $(j, k)$, and $(k, j)$.
- Therefore, $C(\hat{h}) = \frac{6}{3(3-1)} = \frac{6}{6} = 1$.

For hierarchy $\tilde{h}$ (star structure):

- When treated as an undirected graph, all three nodes also form a single connected component.
- All 6 possible ordered pairs have undirected paths between them: $(j, i)$, $(i, j)$, $(j, k)$, $(k, j)$, $(i, k)$, and $(k, i)$.
- Therefore, $C(\tilde{h}) = \frac{6}{3(3-1)} = \frac{6}{6} = 1$.

Since $C(\hat{h}) = C(\tilde{h}) = 1$, we have $\hat{h} \sim_C \tilde{h}$ according to the connectedness-based ordering. However, the SR axiom requires $\hat{h} \succ_C \tilde{h}$. This contradiction demonstrates that $\succcurlyeq_C$ violates the SR axiom.

The fundamental issue is that Krackhardt's connectedness index measures only whether nodes can reach each other when direction is ignored, not the specific structure of subordination relationships. In both hierarchies, all nodes can reach all other nodes when direction is ignored, resulting in identical connectedness values despite their different hierarchical structures.

This violation makes intuitive sense: connectedness, as defined by Krackhardt, primarily captures whether a network forms a cohesive whole rather than the degree of hierarchical subordination within that network. The example clearly shows that significant changes to subordination relations can occur without affecting the overall connectedness of the structure.

Let us now examine why the connectedness-based hierarchical order $\succcurlyeq_C$ may violate the UE axiom.

Consider hierarchy $h$, a simple two-person structure with one directed edge $j \rightarrow i$. When treated as an undirected graph, the connectedness index of $h$ is:

$$C(h) = \frac{2}{2(2-1)} = \frac{2}{2} = 1.$$

Now consider hierarchy $h'$ formed by appending $h$ to a single-node hierarchy $k$, resulting in a three-node chain $k \rightarrow j \rightarrow i$. In $h'$, when treated as an undirected graph, all three nodes are connected. The total number of ordered pairs with undirected paths is 6: $(i, j)$, $(j, i)$, $(i, k)$, $(k, i)$, $(j, k)$ and $(k, j)$. This gives us:

$$C(h') = \frac{6}{3(3-1)} = \frac{6}{6} = 1.$$

According to the UE axiom, appending $h$ to $k$ should make $h'$ *strictly* more hierarchical than $h$. However, we observe that $C(h') = C(h) = 1$, which implies $h' \sim_C h$ (the hierarchies are equivalent under the connectedness-based ordering).

This equality contradicts the requirement of the UE axiom, which demands a strict increase in hierarchical measure after upward extension. Krackhardt's original connectedness index fails to capture the increased depth of hierarchy in $h'$ compared to $h$, demonstrating that $\succcurlyeq_C$ violates the UE axiom.

Similar examples can be used to demonstrate that $\succcurlyeq_C$ fails to satisfy DE, UE*, and DE*.

**Proposition 10.** *The hierarchical order $\succcurlyeq_C$ satisfies* A *and violates* RP, SR, UE, DE, UE*, *and* DE*.

#### 4.3.2. Graph hierarchy

We now briefly discuss Krackhardt's graph hierarchy measure, noting that this measure proves uninformative in our framework of directed acyclic graphs because it remains constant across this structural domain.

Krackhardt's measure quantifies how closely a structure adheres to a pure hierarchical pattern with clear top-down authority. A high hierarchy score indicates an unambiguous chain of command where influence flows from upper nodes to subordinates without cycles or lateral connections.

Formally, Krackhardt's *graph hierarchy index $H(h)$* is defined as:

$$H(h) = 1 - \frac{P}{\max P},$$

where

- $P$ denotes number of *unordered pairs* $\{i, j\}$ with mutual reachability (i.e., paths exist both $i \to j$ *and* $j \to i$),
- $\max P$ represents the total *unordered pairs* $\{i, j\}$ with at least one directed path (either $i \to j$ *or* $j \to i$).

The interpretation of this index is straightforward:

- $H(h) = 1$ indicates "perfect hierarchy" with no mutual reachability between any node pairs.
- $H(h) = 0$ indicates "maximal cyclicity" where every reachable pair is mutually reachable.

For directed acyclic graphs—which by definition contain no cycles—mutual reachability is impossible, meaning $P = 0$. Consequently, in any directed acyclic graph with at least one directed path ($\max P > 0$), we always have $H(h) = 1$.

This invariance renders Krackhardt's graph hierarchy index constant across our domain of hierarchical structures, making it unsuitable for distinguishing between different acyclic hierarchical configurations.

### 4.3.3. Graph efficiency

Krackhardt's *graph efficiency index* $E(h)$ measures the structural efficiency of a hierarchy by quantifying the redundancy of supervisory links relative to the theoretical maximum possible redundancy. This index evaluates how densely the hierarchy deviates from a minimally connected structure.

For a hierarchy $h$ with $k$ weakly connected components (where the $i$-th component has $n_i$ nodes) and a total of $m = \sum_{i=1}^{k} m_i$ edges, the graph efficiency index is defined as:[3]

$$E(h) = 1 - \frac{\text{actual redundancy}}{\text{maximum possible redundancy}} = 1 - \frac{\sum\limits_{i=1}^{k} \big(m_i - (n_i - 1)\big)}{\sum\limits_{i=1}^{k} \frac{(n_i-1)(n_i-2)}{2}},$$

where:

- actual redundancy: $\sum_{i=1}^{k} \big(m_i - (n_i - 1)\big)$ represents the total number of supervisory links exceeding the minimum required to keep all components weakly connected ($n_i - 1$ links per component);
- maximum possible redundancy: $\sum_{i=1}^{k} \frac{(n_i-1)(n_i-2)}{2}$ represents the theoretical maximum number of redundant links allowed in each component before it becomes a complete graph.

This measure can be given the following interpretation:

- $E(h) = 1$: A perfectly efficient hierarchy with no redundant links (i.e., an outtree).
- $E(h) < 1$: Increasing redundancy in supervisory relationships, deviating from the minimally connected structure.
- $E(h) = 0$: A maximally redundant hierarchy where every possible supervisory link exists.

Let $\succcurlyeq_E$ represent the hierarchical order induced by the graph efficiency index $E$.

The graph efficiency index $E(h)$ is defined entirely by structural properties of the hierarchy—specifically, the number of nodes and edges within each weakly connected component. These parameters, $n_i$ (the size of the $i$-th component) and $m_i$ (its edge count), remain invariant under any relabeling of nodes.

When renaming nodes to create hierarchy $h'$ from $h$, the component sizes $n_i$ and edge counts $m_i$ are preserved. This ensures that structurally identical hierarchies receive identical efficiency scores, regardless of node labels.

Consequently, the efficiency-based hierarchical ordering $\succcurlyeq_E$ satisfies the A axiom.

We now show that $\succcurlyeq_E$ satisfies the RP axiom.

Let $h'$ be a $k$-times replication of $h$. This means each component in $h$ is replicated into $k$ identical copies in $h'$. If $h$ has $c$ components, then $h'$ will have $k \cdot c$ components. For each original component $i$ in $h$ with $n_i$ nodes and $m_i$ edges, we now have $k$ identical copies in $h'$.

Therefore, for $h'$:

[3] A directed acyclic graph is weakly connected if, when all directed edges are converted to undirected edges, the resulting undirected graph is connected. Equivalently, for every pair of vertices $i$ and $j$, there exists an undirected path connecting them.

- The total number of edges becomes $\sum_{i=1}^{k\cdot c} m'_i = k \cdot \sum_{i=1}^{c} m_i$.
- The total number of nodes becomes $\sum_{i=1}^{k\cdot c} n'_i = k \cdot \sum_{i=1}^{c} n_i$.

The actual redundancy in $h'$ is:

$$\text{actual redundancy}(h') = \sum_{i=1}^{k\cdot c} \left(m'_i - (n'_i - 1)\right) = k \cdot \sum_{i=1}^{c} \left(m_i - (n_i - 1)\right).$$

And the maximum possible redundancy becomes:

$$\text{maximum possible redundancy}(h') = \sum_{i=1}^{k\cdot c} \frac{(n'_i - 1)(n'_i - 2)}{2} = k \cdot \sum_{i=1}^{c} \frac{(n_i - 1)(n_i - 2)}{2}.$$

Thus, the graph efficiency measure $E$ for $h'$ is:

$$\begin{aligned} E(h') &= 1 - \frac{\text{actual redundancy}(h')}{\text{maximum possible redundancy}(h')} \\ &= 1 - \frac{k \cdot \sum_{i=1}^{c} \left(m_i - (n_i - 1)\right)}{k \cdot \sum_{i=1}^{c} \frac{(n_i-1)(n_i-2)}{2}} \\ &= 1 - \frac{\sum_{i=1}^{c} \left(m_i - (n_i - 1)\right)}{\sum_{i=1}^{c} \frac{(n_i-1)(n_i-2)}{2}} \\ &= E(h). \end{aligned}$$

The scaling factor $k$ cancels out in the calculation, demonstrating that $E(h') = E(h)$. Since $E(h') = E(h)$, it follows that $h' \sim_E h$. Therefore, $\succcurlyeq_E$ satisfies RP.

Next, we show that $\succcurlyeq_E$ violates the SR axiom.

Consider hierarchy $h$ with a chain structure $j \to i \to k$. By eliminating the subordination relation between $i$ and $k$, we obtain hierarchy $h'$ with a star structure where $j \to i$ and $j \to k$.

For hierarchy $h$:

- Number of nodes: $n = 3$.
- Number of edges: $m = 2$.
- Minimum edges required for weak connectivity: $n - 1 = 2$.
- Actual redundancy: $m - (n - 1) = 2 - 2 = 0$.
- Maximum possible redundancy: $\frac{(n-1)(n-2)}{2} = \frac{2\cdot 1}{2} = 1$.

Therefore:

$$E(h) = 1 - \frac{\text{actual redundancy}}{\text{maximum possible redundancy}} = 1 - \frac{0}{1} = 1.$$

For hierarchy $h'$:

- Number of nodes: $n = 3$.

- Number of edges: $m = 2$.
- Minimum edges required for weak connectivity: $n - 1 = 2$.
- Actual redundancy: $m - (n - 1) = 2 - 2 = 0$.
- Maximum possible redundancy: $\frac{(n-1)(n-2)}{2} = \frac{2 \cdot 1}{2} = 1$.

Therefore:

$$E(h') = 1 - \frac{\text{actual redundancy}}{\text{maximum possible redundancy}} = 1 - \frac{0}{1} = 1.$$

Consequently:

$$h \sim_E h'.$$

This demonstrates that the graph efficiency index $E$ fails to distinguish between hierarchies that differ solely by subordination removal. Although $h'$ is obtained from $h$ by removing a subordination relation, we have $E(h) = E(h')$, which means $h \sim_E h'$. This contradicts the requirement of the SR axiom, proving that $\succcurlyeq_E$ violates SR.

Next, we show that $\succcurlyeq_E$ fails to satisfy the UE axiom.

Consider the following hierarchies:

- $h$: a chain structure with $j \to i$.
- $h_o$: an isolated node $k$.
- $h'$: a chain structure with $k \to j \to i$.

Note that $h'$ can be obtained by extending hierarchy $h$ with hierarchy $h_o$, formally expressed as:

$$h' \in h_o \oplus h.$$

Because neither $h$ nor $h'$ have any redundant links,

$$E(h) = 1 \quad \text{and} \quad E(h') = 1.$$

Hence,

$$h \sim_E h'.$$

However, according to the UE axiom, we should have $h' \succ_E h$ whenever $h' \in h_o \oplus h$. This contradiction demonstrates that $\succcurlyeq_E$ violates the UE axiom.

One can similarly show that $\succcurlyeq_E$ also violates DE, UE*, and DE*.

**Proposition 11.** *The hierarchical order $\succcurlyeq_E$ satisfies A and RP and violates SR, UE, DE, UE*, and DE*.*

### 4.3.4. Least upper boundedness

The least upper boundedness index measures the extent to which a hierarchy approximates a pure tree structure by quantifying the presence of a unique common boss for each pair of nodes within the same component.

For a hierarchy $h \in \mathscr{H}$ with node set $V$ and $k$ weakly connected components, where each component $i$ contains $n_i$ nodes, the *least upper boundedness index* $\mathrm{LUB}(h)$ is defined as:

$$\mathrm{LUB}(h) = \frac{\left|\left\{(i,j) \in V \times V : i \neq j,\ i \text{ and } j \text{ are in the same component, and } \exists!\ \mathrm{LUB}(i,j)\right\}\right|}{\sum_{i=1}^{k} n_i(n_i - 1)},$$

where:

- The numerator counts ordered pairs of distinct nodes within the same component that have a unique least upper bound.
- The denominator represents the total number of ordered pairs of distinct nodes within all components.

To establish whether a unique least upper bound exists for a pair of nodes $(i,j)$ (where $i$ and $j$ belong to the same component):

1. Identify the set of common bosses:

$$B(i,j) = \{i' \in V : \text{both } i \text{ and } j \text{ have directed paths from } i'\}.$$

   By convention, each $i'$ has a directed path to itself.

2. Determine the subset of minimal elements $M(i,j) \subseteq B(i,j)$: A node $i' \in B(i,j)$ is minimal if there is no other node $j' \in B(i,j)$ such that there exists a directed path from $i'$ to $j'$.

3. A unique least upper bound exists if and only if $M(i,j)$ is a singleton.

The LUB index represents the proportion of node pairs within the same component that have a unique least upper bound in the hierarchy. This measures how closely each component resembles a pure tree structure:

- $\mathrm{LUB}(h) = 1$: Each component has a perfect tree-like structure where every pair of nodes has exactly one least common boss.
- $\mathrm{LUB}(h) = 0$: No pair of nodes in any component has a unique least common boss.
- $0 < \mathrm{LUB}(h) < 1$: Components exhibit varying degrees of tree-like characteristics.

The LUB index thus captures the unity of command principle in organizational theory for each component of the hierarchy.

Let $\succcurlyeq_L$ represent the hierarchical order induced by the LUB index.

Observe that $\mathrm{LUB}(h)$ depends solely on the structure of $h$, not on the specific labels of the nodes. If $h'$ is a relabeling of $h$, then:

- For every pair of nodes $(i,j)$ in $h$, all structural relationships (paths, reachability, boss relationships) are preserved in the corresponding pair in $h'$.
- The set of pairs with unique least upper bounds remains unchanged under relabeling.

- The total number of such pairs is preserved.

- The denominator (total number of pairs within components) remains constant.

Therefore, $\mathrm{LUB}(h) = \mathrm{LUB}(h')$, which implies $h \sim_L h'$. This demonstrates that $\succcurlyeq_L$ satisfies the A axiom.

Next, we argue that the order $\succcurlyeq_L$ satisfies RP.

Let $h'$ be a $k$-times replication of $h$. By definition, $h'$ consists of $k$ disjoint copies of $h$. For the LUB index:

- Numerator: Each copy of $h$ in $h'$ has $x(h)$ pairs with a unique least upper bound. The total is $k \cdot x(h)$.

- Denominator: Each copy contributes $y(h)$ pairs. The total is $k \cdot y(h)$.

Thus,

$$\mathrm{LUB}(h') = \frac{k \cdot x(h)}{k \cdot y(h)} = \mathrm{LUB}(h).$$

Since $\mathrm{LUB}(h') = \mathrm{LUB}(h)$, we have $h' \sim_L h$. Hence, $\succcurlyeq_L$ satisfies RP.

Next, we demonstrate that $\succcurlyeq_L$ violates the SR axiom.

Consider hierarchy $h$, a chain structure $j \to i \to k$. By removing the subordination relation between $i$ and $k$, we obtain hierarchy $h'$, a star structure with relations $j \to i$ and $j \to k$.

For hierarchy $h$ (the chain structure), with $n = 3$ nodes, there are $n(n-1) = 3 \cdot 2 = 6$ ordered pairs of distinct nodes. For each pair, a unique least upper bound (LUB) exists:

| Node Pair | Unique LUB |
|---|---|
| $(j, i)$ | $\mathrm{LUB} = j$ |
| $(j, k)$ | $\mathrm{LUB} = j$ |
| $(i, j)$ | $\mathrm{LUB} = j$ |
| $(i, k)$ | $\mathrm{LUB} = i$ |
| $(k, j)$ | $\mathrm{LUB} = j$ |
| $(k, i)$ | $\mathrm{LUB} = i$ |

For hierarchy $h'$ (the star structure), there are also 6 ordered pairs of distinct nodes. For each pair, a unique LUB exists, but in this case, the unique LUB for all pairs is node $j$:

| Node Pair | Unique LUB |
|---|---|
| $(j, i)$ | $\mathrm{LUB} = j$ |
| $(j, k)$ | $\mathrm{LUB} = j$ |
| $(i, j)$ | $\mathrm{LUB} = j$ |
| $(i, k)$ | $\mathrm{LUB} = j$ |
| $(k, j)$ | $\mathrm{LUB} = j$ |
| $(k, i)$ | $\mathrm{LUB} = j$ |

Therefore:

$$\mathrm{LUB}(h) = \frac{6}{6} = 1 = \frac{6}{6} = \mathrm{LUB}(h').$$

This implies $h \sim_L h'$, even though $h'$ is obtained from $h$ by removing the subordination relation $i \to k$. According to the SR axiom, we should have $h \succ_L h'$, which contradicts our finding. Hence, $\succcurlyeq_L$ violates the SR axiom.

Similar examples can be used to illustrate that $\succcurlyeq_L$ violates UE, DE, UE*, and DE*.

**Proposition 12.** *The hierarchical order $\succcurlyeq_L$ satisfies* A *and* RP *and violates* SR, UE, DE, UE*, *and* DE*.

## 5. Concluding remarks

This paper establishes a robust axiomatic foundation for hierarchy measurement by introducing novel axioms centered on structural depth. These new axioms complement existing principles—the Anonymity axiom, Replication Principle, and Subordination Removal axiom from Carbonell-Nicolau (2025a,b)—to create a more comprehensive theoretical framework.

Our expanded axiom set enables the development of hierarchical measures with significantly enhanced discriminatory power for ranking pairs of hierarchies. We formulate canonical hierarchical pre-orders uniquely characterized by these underlying principles and identify the average depth hierarchical index as a computationally efficient, full completion of these pre-orders. This index measures hierarchy by averaging each node's number of (direct or indirect) supervisors, providing both axiom consistency and practical applicability.

The theoretical framework developed here strengthens the conceptual basis for measuring hierarchies and identifies several avenues for future research.

A natural avenue for investigation involves the empirical study of hierarchical transformations during organizational processes such as mergers and acquisitions. The extension axioms developed here, which address upward and downward structural integration, capture stylized outcomes commonly observed in such contexts, including the addition of management layers and integration of distinct organizational charts. This connection suggests a testable hypothesis: real-world mergers and acquisitions tend to produce deeper, more hierarchical structures. This claim can be assessed using the measures introduced in this study, thereby linking axiomatic theory with organizational data. The average depth index provides a particularly promising tool for such inquiry due to its computational simplicity and complete alignment with our expanded axiom set.

Two important caveats warrant attention in pursuing this empirical agenda. First, the scarcity of publicly available data on internal organizational hierarchies may pose a significant barrier to large-scale validation. Second, while the average depth index provides a complete ordering of hierarchies, it may deliver definitive comparisons even in cases where our axioms yield no such conclusion. This mismatch can lead to potentially overstated or non-compelling rankings in edge cases not covered by the axiomatic structure.

To address these limitations, one may employ the composite axiomatic collections constructed in this paper—specifically, the transitive closures of the union of canonical hierarchical pre-orders. These offer a more principled approach by ensuring that every comparison is grounded in specific axioms. However, such methods are computationally intensive, particularly for large organizational networks.

This computational challenge suggests another direction for future research: developing scalable algorithms that can efficiently compute composite pre-orders. Such advances would enable the application of theoretically sound methods to real-world data without compromising either computational feasibility or axiomatic consistency.

# A. Proofs

## A.1. Proof of Lemma 1

**Lemma 1.** *The binary relation $\sim^*$ defined on $\mathscr{H}$ is symmetric, reflexive, and transitive.*

*Proof.* Symmetry is implied by the symmetry of the relabeling relation itself. If $h' \sim^* h$, then there exist replicas of $h'$ and $h$ that are relabelings of each other, which implies $h \sim^* h'$.

Reflexivity follows from the fact that each hierarchy is trivially a relabeling of itself. Therefore, $h \sim^* h$ for any hierarchy $h$.

To prove transitivity, suppose that

$$h'' \sim^* h' \sim^* h.$$

By definition of $\sim^*$, there exist replicas $h''_r$ and $h'_r$ of $h''$ and $h'$, respectively, such that $h''_r$ is a relabeling of $h'_r$. Similarly, there exist replicas $h'_{rr}$ and $h_{rr}$ of $h'$ and $h$, respectively, such that $h'_{rr}$ is a relabeling of $h_{rr}$.

Suppose that $h'_r$ and $h'_{rr}$ are $l$ and $m$-dimensional replicas of $h'$, respectively. Then the $m$-replica of $h'_r$, $h'_{rrr}$, equals the $l$-replica of $h'_{rr}$. Therefore, we have:

- The $m$-replica of $h''_r$ is a relabeling of $h'_{rrr}$.
- $h'_{rrr}$ is a relabeling of the $l$-replica of $h_{rr}$.

Consequently, by transitivity of the relabeling relation, the $m$-replica of $h''_r$ is a relabeling of the $l$-replica of $h_{rr}$. Since the $m$-replica of $h''_r$ is a replica of $h''$ and the $l$-replica of $h_{rr}$ is a replica of $h$, it follows that $h'' \sim^* h$. ∎

## A.2. Proof of Proposition 1

**Proposition 1.** *The binary relation $\succcurlyeq_{UE}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying A, RP, and UE.*

*Proof.* Reflexivity follows from the fact that each hierarchy is a relabeling of itself.

To prove transitivity, suppose that

$$h'' \succcurlyeq_{UE} h' \succcurlyeq_{UE} h.$$

We consider four cases.

*Case 1. We have*

$$H'' \in h'_o \oplus H' \quad \textit{and} \quad \tilde{H} \in h_o \oplus H, \tag{3}$$

*for $H'' \sim^* h''$, $H' \sim^* h' \sim^* \tilde{H}$, and $H \sim^* h$.* By Lemma 1, $H' \sim^* \tilde{H}$, so there exist replicas $H'_r$ and $\tilde{H}_r$ of $H'$ and $\tilde{H}$, respectively, such that $H'_r$ is a relabeling of $\tilde{H}_r$. Therefore, there exist $h_a$ and $h_b$ such that $h_a$ is a relabeling of $h_b$ and

$$H'_r = (h_a, \ldots, h_a) \quad \text{and} \quad \tilde{H}_r = (h_b, \ldots, h_b).$$

By (3), we have

$$\begin{aligned}(H'', \ldots, H'') &\in (h'_o \oplus H') \times \cdots \times (h'_o \oplus H') \\ &\subseteq (h'_o, \ldots, h'_o) \oplus (H', \ldots, H') = (h'_o, \ldots, h'_o) \oplus H'_r = (h'_o, \ldots, h'_o) \oplus (h_a, \ldots, h_a)\end{aligned}$$

and

$$(h_b, \ldots, h_b) = \tilde{H}_r = (\tilde{H}, \ldots, \tilde{H}) \in (h_o \oplus H) \times \cdots \times (h_o \oplus H) \subseteq (h_o, \ldots, h_o) \oplus (H, \ldots, H).$$

Therefore, since $(h_a, \ldots, h_a)$ is a relabeling of $(h_b, \ldots, h_b)$ and $h_a$ is a relabeling of $h_b$, there exists a relabeling $(H''', \ldots, H''')$ of $(H'', \ldots, H'')$ such that

$$(H''', \ldots, H''') \in (h'_o, \ldots, h'_o) \oplus (h_o, \ldots, h_o) \oplus (H, \ldots, H).$$

Since

$$(H''', \ldots, H''') \sim^* (H'', \ldots, H'') \sim^* h'' \quad \text{and} \quad (H, \ldots, H) \sim^* h,$$

it follows that $h'' \succcurlyeq_{UE} h$.

*Case 2. $h' \sim^* h$ and*

$$H'' \in h'_o \oplus H' \tag{4}$$

*for $H'' \sim^* h''$ and $H' \sim^* h'$.* By Lemma 1, $H' \sim^* h$, so there exist replicas $H'_r$ and $h_r$ of $H'$ and $h$, respectively, such that $H'_r$ is a relabeling of $h_r$. Hence,

$$H'_r = (H', \ldots, H') = (h_a, \ldots, h_a) = (h_b, \ldots, h_b) = (h, \ldots, h) = h_r$$

for some $h_a$ and $h_b$ such that $h_a$ is a relabeling of $h_b$, and (4) implies

$$\begin{aligned} (H'', \ldots, H'') &\in (h'_o \oplus H') \times \cdots \times (h'_o \oplus H') \\ &\subseteq (h'_o, \ldots, h'_o) \oplus (H', \ldots, H') = (h'_o, \ldots, h'_o) \oplus (h_a, \ldots, h_a). \end{aligned}$$

Since $(h_a, \ldots, h_a)$ is a relabeling of $(h_b, \ldots, h_b)$ and $h_a$ is a relabeling of $h_b$, it follows that there exists a relabeling $(H''', \ldots, H''')$ of $(H'', \ldots, H'')$ such that

$$(H''', \ldots, H''') \in (h'_o, \ldots, h'_o) \oplus (h_b, \ldots, h_b) = (h'_o, \ldots, h'_o) \oplus (h, \ldots, h).$$

Consequently, since

$$(H''', \ldots, H''') \sim^* (H'', \ldots, H'') \sim^* h'' \quad \text{and} \quad (h, \ldots, h) \sim^* h,$$

we have $h'' \succcurlyeq_{UE} h$.

*Case 3. $h'' \sim^* h'$ and*

$$\tilde{H} \in h_o \oplus H$$

*for $\tilde{H} \sim^* h'$, and $H \sim^* h$.* This case can be handled as Case 2.

*Case 4. $h'' \sim^* h' \sim^* h$.* By Lemma 1, $h'' \sim^* h$, implying $h'' \succcurlyeq_{UE} h$.

To see that $\succcurlyeq_{UE}$ satisfies A, pick any two hierarchies $h$ and $h'$. Suppose that $h$ is a relabeling of $h'$. Then $h \sim^* h'$ and $h' \sim^* h$, implying $h \sim_{UE} h'$.

The relation $\succcurlyeq_{UE}$ also satisfies the RP axiom, since, $(h, \ldots, h) \sim^* h$ and $h \sim^* (h, \ldots, h)$ for any hierarchy $h$, implying $h' \sim_{UE} h$ whenever $h'$ is a replication of $h$.

The relation $\succcurlyeq_{UE}$ also adheres to the UE axiom. Indeed, suppose that $h' \in h_o \oplus h$ for some $h_o \in \mathscr{H}$. Then, the definition of $\succcurlyeq_{UE}$ implies $h' \succcurlyeq_{UE} h$.

To ensure that $h' \succ_{UE} h$, we must confirm that $h \succcurlyeq_{UE} h'$ does not hold. Suppose, for contradiction, that $h \succcurlyeq_{UE} h'$. This implies either $h \sim^* h'$ or

$$\exists h'_o, H \sim^* h, H' \sim^* h' : H \in h'_o \oplus H'.$$

*Case A.* $h \sim^* h'$. If $h \sim^* h'$, there exist equally-sized replicas $h_r$ of $h$ and $h'_r$ of $h'$ such that $h_r$ is a relabeling of $h'_r$. However, since $h' \in h_o \oplus h$, the depth of $h'$ exceeds that of $h$. Since replicas preserve depth, it follows that $h_r$ and $h'_r$ have different depths. Relabeling preserves structure, including depth, so no relabeling can make $h_r$ and $h'_r$ equivalent. Thus, $h \nsim^* h'$.

*Case B.* $H \in h'_o \oplus H'$ *with* $H \sim^* h$, $H' \sim^* h'$. This case is impossible due to depth differences. Since $h' \in h_o \oplus h$, $h'$'s depth exceeds $h$'s depth. Replicas and relabelings preserve depth so $H \sim^* h$ and $H' \sim^* h'$ imply $H'$'s depth exceeds $H$'s depth. But if $H \in h'_o \oplus H'$, $H$'s depth would need to exceed that of $H'$, a contradiction.

Therefore, neither case holds, confirming $h \not\succcurlyeq_{UE} h'$. ■

## A.3. Proof of Theorem 2

**Theorem 2.** *A hierarchical pre-order on $\mathscr{H}$ satisfies A, RP, UE, and SR if and only if it is $R^*$-consistent.*

*Proof.* (*Sufficiency.*) Assume that $\succcurlyeq$ is $R^*$-consistent. By Proposition 2, $R^*$ satisfies the axioms A, RP, UE, and SR. Since $\succcurlyeq$ is $R^*$-consistent, it follows that $\succcurlyeq$ must also satisfy these axioms.

(*Necessity.*) Suppose that $\succcurlyeq$ is a hierarchical pre-order on $\mathscr{H}$ satisfying axioms A, RP, UE, and SR. We must show that $\succcurlyeq$ is $R^*$-consistent, i.e., that for every pair of hierarchies $h, h' \in \mathscr{H}$:

(a) $hP^*h' \Rightarrow h \succ h'$.

(b) $hI^*h' \Rightarrow h \sim h'$.

Suppose first that $hI^*h'$. Then there are finitely many hierarchies in $\mathscr{H}$,

$$h_0, \dots, h_k, h'_0, \dots, h'_l,$$

such that

$$(h = h_0)Rh_1R \cdots R(h_k = h' = h'_0)Rh'_1R \cdots R(h'_l = h).$$

Since this represents a cycle, Lemma 2 implies

$$(h = h_0)Ih_1I \cdots I(h_k = h').$$

We claim that for each $\kappa \in \{0, \dots, k-1\}$, there exist replicas $H_\kappa$ and $H_{\kappa+1}$ of $h_\kappa$ and $h_{\kappa+1}$ respectively such that $H_\kappa$ is a relabeling of $H_{\kappa+1}$. To see this, fix $\kappa \in \{0, \dots, k-1\}$ and note that $h_\kappa I h_{\kappa+1}$ implies

$$[h_\kappa \succcurlyeq_H h_{\kappa+1} \text{ or } h_\kappa \succcurlyeq_{UE} h_{\kappa+1}] \quad \text{and} \quad [h_{\kappa+1} \succcurlyeq_H h_\kappa \text{ or } h_{\kappa+1} \succcurlyeq_{UE} h_\kappa]. \tag{5}$$

There are four exhaustive cases to consider.

*Case 1.* $h_\kappa \sim_H h_{\kappa+1}$. Then there exist equally-sized replicas $H_\kappa$ and $H_{\kappa+1}$ of $h_\kappa$ and $h_{\kappa+1}$, respectively, such that $H_\kappa \sim_H H_{\kappa+1}$. By Lemma 4 in Carbonell-Nicolau (2025b), it follows that $H_\kappa$ is a relabeling of $H_{\kappa+1}$.

*Case 2.* $h_\kappa \sim_{UE} h_{\kappa+1}$. Using the argument from the proof of Theorem 1, we can conclude that $h_\kappa \sim^* h_{\kappa+1}$, which gives the desired conclusion.

*Case 3.* $h_\kappa \succ_H h_{\kappa+1}$. In this case, (5) implies $h_{\kappa+1} \not\succ_{UE} h_\kappa$. The case where $h_{\kappa+1} \succ_{UE} h_\kappa$ is impossible, since $h_\kappa \succ_H h_{\kappa+1}$ implies that the average depth in $h_\kappa$ exceeds that of $h_{\kappa+1}$, contradicting $h_{\kappa+1} \succ_{UE} h_\kappa$. Indeed, the order $\succcurlyeq_{I_s}$ on $\mathscr{H}$ defined by

$$h \succcurlyeq_{I_s} h' \Leftrightarrow I_s(h) \geq I_s(h'),$$

where $I_s(h'')$ represents the average number of (direct or indirect) supervisors in $h''$, is $\succcurlyeq_H$-consistent (Carbonell-Nicolau, 2025b). Consequently, $h_\kappa \succ_H h_{\kappa+1}$ implies $I_s(h_\kappa) > I_s(h_{\kappa+1})$.

Thus, we have $h_\kappa \succ_H h_{\kappa+1}$ and $h_{\kappa+1} \sim_{UE} h_\kappa$, so we can proceed as in Case 2.

*Case 4.* $h_\kappa \succ_{UE} h_{\kappa+1}$. In this case, (5) implies $h_{\kappa+1} \not\succ_H h_\kappa$. The case where $h_{\kappa+1} \succ_H h_\kappa$ is impossible, since $h_\kappa \succ_{UE} h_{\kappa+1}$ implies that the average depth in $h_\kappa$ exceeds that of $h_{\kappa+1}$, contradicting $h_{\kappa+1} \succ_{UE} h_\kappa$.

Thus, we have $h_\kappa \succ_{UE} h_{\kappa+1}$ and $h_{\kappa+1} \sim_H h_\kappa$, so we can proceed as in Case 1.

We have seen that for each $\kappa \in \{0, \ldots, k-1\}$, there exist replicas $H_\kappa$ and $H_{\kappa+1}$ of $h_\kappa$ and $h_{\kappa+1}$ respectively such that $H_\kappa$ is a relabeling of $H_{\kappa+1}$. Consequently, because $\succcurlyeq$ satisfies A and RP, we have

$$h \sim H_0 \sim \cdots \sim H_k \sim h'. \tag{6}$$

By transitivity of $\succcurlyeq$, $\sim$ is transitive, so (6) implies $h \sim h'$. This establishes (b).

It remains to prove (a). Suppose that $hP^*h'$. Then there are finitely many hierarchies in $\mathscr{H}$,

$$h_0'', \ldots, h_l'',$$

such that

$$(h = h_0'')Rh_1''R \cdots R(h_l'' = h'), \tag{7}$$

with at least one strict relation.

Suppose that $h_\kappa'' P h_{\kappa+1}''$, where $\kappa \in \{0, \ldots, l-1\}$. Then

$$h_\kappa'' \succ_H h_{\kappa+1}'' \quad \text{or} \quad h_\kappa'' \succ_{UE} h_{\kappa+1}''.$$

Suppose first that $h_\kappa'' \succ_H h_{\kappa+1}''$. Then by Theorem 1 in Carbonell-Nicolau (2025b), $h_{\kappa+1}''$ can be obtained from some relabeling of $h_\kappa''$ by successive removals of subordination relations. Consequently, since $\succcurlyeq$ satisfies SR, there are finitely many hierarchies in $\mathscr{H}$,

$$\tilde{h}_0, \ldots, \tilde{h}_\ell,$$

such that

$$\tilde{h}_0 \succ \tilde{h}_1 \succ \cdots \succ (\tilde{h}_\ell = h_{\kappa+1}''),$$

where $\tilde{h}_0$ is a relabeling of $h_\kappa''$. Hence, because $\succcurlyeq$ satisfies A,

$$h_\kappa'' \sim \tilde{h}_0 \succ \tilde{h}_1 \succ \cdots \succ (\tilde{h}_\ell = h_{\kappa+1}'').$$

Since $\succcurlyeq$ is reflexive and transitive, it follows that $h_\kappa'' \succ h_{\kappa+1}''$ (Sen, 2017, Lemma 1*a, p. 56).

Next, suppose that $h''_\kappa \succ_{UE} h''_{\kappa+1}$. Then

$$\exists h_o, H''_\kappa \sim^* h''_\kappa, H''_{\kappa+1} \sim^* h''_{\kappa+1} : H''_\kappa \in h_o \oplus H''_{\kappa+1}.$$

Consequently, because $\succcurlyeq$ satisfies the UE axiom, we see that

$$H''_\kappa \succ H''_{\kappa+1},$$

and, since $\succcurlyeq$ satisfies A and RP, it follows that $h''_\kappa \succ h''_{\kappa+1}$.

We conclude that $h''_\kappa P h''_{\kappa+1}$, where $\kappa \in \{0, \dots, l-1\}$, implies $h''_\kappa \succ h''_{\kappa+1}$.

Furthermore, if $h''_\kappa I h''_{\kappa+1}$, where $\kappa \in \{0, \dots, l-1\}$, then, applying the argument used in the proof of (b), we see that there exist replicas $\tilde{H}_\kappa$ and $\tilde{H}_{\kappa+1}$ of $h''_\kappa$ and $h''_{\kappa+1}$, respectively, such that $\tilde{H}_\kappa$ is a relabeling of $\tilde{H}_{\kappa+1}$. Therefore, since $\succcurlyeq$ satisfies A and RP, we have

$$h''_\kappa \sim \tilde{H}_\kappa \sim \tilde{H}_{\kappa+1} \sim h''_{\kappa+1},$$

implying $h''_\kappa \sim h''_{\kappa+1}$ by transitivity of $\sim$.

Thus, $h''_\kappa I h''_{\kappa+1}$, where $\kappa \in \{0, \dots, l-1\}$, implies $h''_\kappa \sim h''_{\kappa+1}$. Moreover, we have also established that $h''_\kappa P h''_{\kappa+1}$, where $\kappa \in \{0, \dots, l-1\}$, implies $h''_\kappa \succ h''_{\kappa+1}$. Consequently, (7), which holds with at least one strict dominance, implies

$$(h = h''_0) \succcurlyeq h''_1 \succcurlyeq \cdots \succcurlyeq (h''_l = h'),$$

with at least one strict dominance. Since $\succcurlyeq$ is reflexive and transitive, Lemma 1*a in Sen (2017, p. 56) implies $h \succ h'$. This establishes (a). ■

## A.4. Proof of Proposition 3

**Proposition 3.** *The binary relation $\succcurlyeq_{DE}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying A, RP, and DE.*

*Proof.* Reflexivity follows from the fact that each hierarchy is a relabeling of itself.

To prove transitivity, suppose that

$$h'' \succcurlyeq_{DE} h' \succcurlyeq_{DE} h.$$

We consider four cases.

*Case 1. We have*

$$H'' \in H' \boxplus h'_o \quad \text{and} \quad \tilde{H} \in H \boxplus h_o, \tag{8}$$

*for $H'' \sim^* h''$, $H' \sim^* h' \sim^* \tilde{H}$, and $H \sim^* h$.* By Lemma 1, $H' \sim^* \tilde{H}$, so there exist replicas $H'_r$ and $\tilde{H}_r$ of $H'$ and $\tilde{H}$, respectively, such that $H'_r$ is a relabeling of $\tilde{H}_r$. Therefore, there exist $h_a$ and $h_b$ such that $h_a$ is a relabeling of $h_b$ and

$$H'_r = (h_a, \dots, h_a) \quad \text{and} \quad \tilde{H}_r = (h_b, \dots, h_b).$$

By (8), we have

$$\begin{aligned}(H'', \dots, H'') &\in (H' \boxplus h'_o) \times \cdots \times (H' \boxplus h'_o) \\ &\subseteq (H', \dots, H') \boxplus (h'_o, \dots, h'_o) = H'_r \boxplus (h'_o, \dots, h'_o) = (h_a, \dots, h_a) \boxplus (h'_o, \dots, h'_o)\end{aligned}$$

and

$$(h_b, \dots, h_b) = \tilde{H}_r = (\tilde{H}, \dots, \tilde{H}) \in (H \boxplus h_o) \times \dots \times (H \boxplus h_o) \subseteq (H, \dots, H) \boxplus (h_o, \dots, h_o).$$

Therefore, since $(h_a, \dots, h_a)$ is a relabeling of $(h_b, \dots, h_b)$ and $h_a$ is a relabeling of $h_b$, there exists a relabeling $(H''', \dots, H''')$ of $(H'', \dots, H'')$ such that

$$(H''', \dots, H''') \in (H, \dots, H) \boxplus (h_o, \dots, h_o) \boxplus (h'_o, \dots, h'_o).$$

Since

$$(H''', \dots, H''') \sim^* (H'', \dots, H'') \sim^* h'' \quad \text{and} \quad (H, \dots, H) \sim^* h,$$

it follows that $h'' \succcurlyeq_{DE} h$.

*Case 2. $h' \sim^* h$ and*

$$H'' \in H' \boxplus h'_o \tag{9}$$

*for $H'' \sim^* h''$ and $H' \sim^* h'$.* By Lemma 1, $H' \sim^* h$, so there exist replicas $H'_r$ and $h_r$ of $H'$ and $h$, respectively, such that $H'_r$ is a relabeling of $h_r$. Hence,

$$H'_r = (H', \dots, H') = (h_a, \dots, h_a) = (h_b, \dots, h_b) = (h, \dots, h) = h_r$$

for some $h_a$ and $h_b$ such that $h_a$ is a relabeling of $h_b$, and (9) implies

$$\begin{aligned}(H'', \dots, H'') &\in (H' \boxplus h'_o) \times \dots \times (H' \boxplus h'_o) \\ &\subseteq (H', \dots, H') \boxplus (h'_o, \dots, h'_o) = (h_a, \dots, h_a) \boxplus (h'_o, \dots, h'_o).\end{aligned}$$

Since $(h_a, \dots, h_a)$ is a relabeling of $(h_b, \dots, h_b)$ and $h_a$ is a relabeling of $h_b$, it follows that there exists a relabeling $(H''', \dots, H''')$ of $(H'', \dots, H'')$ such that

$$(H''', \dots, H''') \in (h_b, \dots, h_b) \boxplus (h'_o, \dots, h'_o) = (h, \dots, h) \boxplus (h'_o, \dots, h'_o).$$

Consequently, since

$$(H''', \dots, H''') \sim^* (H'', \dots, H'') \sim^* h'' \quad \text{and} \quad (h, \dots, h) \sim^* h,$$

we have $h'' \succcurlyeq_{DE} h$.

*Case 3. $h'' \sim^* h'$ and*

$$\tilde{H} \in H \boxplus h_o$$

*for $\tilde{H} \sim^* h'$, and $H \sim^* h$.* This case can be handled as Case 2.

*Case 4. $h'' \sim^* h' \sim^* h$.* By Lemma 1, $h'' \sim^* h$, implying $h'' \succcurlyeq_{DE} h$.

To see that $\succcurlyeq_{DE}$ satisfies A, pick any two hierarchies $h$ and $h'$. Suppose that $h$ is a relabeling of $h'$. Then $h \sim^* h'$ and $h' \sim^* h$, implying $h \sim_{DE} h'$.

The relation $\succcurlyeq_{DE}$ also satisfies the RP axiom, since, $(h, \dots, h) \sim^* h$ and $h \sim^* (h, \dots, h)$ for any hierarchy $h$, implying $h' \sim_{DE} h$ whenever $h'$ is a replication of $h$.

The relation $\succcurlyeq_{DE}$ also adheres to the DE axiom. Indeed, suppose that $h' \in h \boxplus h_o$ for some $h_o \in \mathscr{H}$. Then, the definition of $\succcurlyeq_{DE}$ implies $h' \succcurlyeq_{DE} h$.

To ensure that $h' \succ_{DE} h$, we must confirm that $h \succcurlyeq_{DE} h'$ does not hold. Suppose, for contradiction, that $h \succcurlyeq_{DE} h'$. This implies either $h \sim^* h'$ or

$$\exists h'_o, H \sim^* h, H' \sim^* h' : H \in H' \boxplus h'_o.$$

*Case A.* $h \sim^* h'$. If $h \sim^* h'$, there exist equally-sized replicas $h_r$ of $h$ and $h'_r$ of $h'$ such that $h_r$ is a relabeling of $h'_r$. However, since $h' \in h \boxplus h_o$, the depth of $h'$ exceeds that of $h$. Since replicas preserve depth, it follows that $h_r$ and $h'_r$ have different depths. Relabeling preserves structure, including depth, so no relabeling can make $h_r$ and $h'_r$ equivalent. Thus, $h \nsim^* h'$.

*Case B.* $H \in H' \boxplus h'_o$ *with* $H \sim^* h$, $H' \sim^* h'$. This case is impossible due to depth differences. Since $h' \in h \boxplus h_o$, $h'$'s depth exceeds $h$'s depth. Replicas and relabelings preserve depth so $H \sim^* h$ and $H' \sim^* h'$ imply $H'$'s depth exceeds $H$'s depth. But if $H \in h'_o \oplus H'$, $H$'s depth would need to exceed that of $H'$, a contradiction.

Therefore, neither case holds, confirming $h \not\succcurlyeq_{DE} h'$. ∎

## A.5. Proof of Theorem 3

**Theorem 3.** *A hierarchical pre-order on $\mathscr{H}$ satisfies A, RP, and DE if and only if it is $\succcurlyeq_{DE}$-consistent.*

*Proof.* (*Sufficiency.*) Assume that $\succcurlyeq$ is a $\succcurlyeq_{DE}$-consistent hierarchical pre-order on $\mathscr{H}$. According to Proposition 3, $\succcurlyeq_{DE}$ satisfies the axioms A, RP, and DE. Since $\succcurlyeq$ is $\succcurlyeq_{DE}$-consistent, it follows that $\succcurlyeq$ must also satisfy these three axioms.

(*Necessity.*) Suppose that $\succcurlyeq$ is a hierarchical pre-order on $\mathscr{H}$ satisfying axioms A, RP, and DE. We need to show that $\succcurlyeq$ is $\succcurlyeq_{DE}$-consistent, which requires proving that for every pair of hierarchies $h, h' \in \mathscr{H}$:

(a) $h \succ_{DE} h' \Rightarrow h \succ h'$.

(b) $h \sim_{DE} h' \Rightarrow h \sim h'$.

Suppose first that $h \sim_{DE} h'$. Then there are two possibilities: either $h \sim^* h'$ or

$$\exists h_o, H' \sim^* h', H \sim^* h : H' \in H \boxplus h_o \quad \text{and} \quad \exists h'_o, H''' \sim^* h', H'' \sim^* h : H'' \in H''' \boxplus h'_o.$$

However, the second case is impossible. Indeed, $H' \in H \boxplus h_o$ implies that the depth of $H'$ exceeds that of $H$, while $H'' \in H''' \boxplus h'_o$ implies that the depth of $H''$ exceeds that of $H'''$. Since

$$H' \sim^* h' \sim^* H''' \quad \text{and} \quad H \sim^* h \sim^* H'',$$

we have

$$H' \sim^* H''' \quad \text{and} \quad H \sim^* H''$$

(Lemma 1). Consequently, because replication and relabeling preserves depth, it follows that the depth of $H'$ and $H'''$ exceeds the depth of $H$ and $H''$, a contradiction.

Thus, $h \sim_{DE} h'$ implies $h \sim^* h'$, so there exist replicas $h_r$ and $h'_r$ of $h$ and $h'$, respectively, such that $h_r$ is a relabeling of $h'_r$.

Given that $\succcurlyeq$ satisfies axioms A and RP, we have:

$$h \sim h_r \sim h'_r \sim h'. \tag{10}$$

Since $\succcurlyeq$ is reflexive and transitive, $\sim$ is transitive (Sen, 2017, Lemma 1*a, p. 56). Therefore, (10) implies $h \sim h'$, which establishes (b).

Now suppose that $h \succ_{DE} h'$. Then

$$\exists h_o, H' \sim^* h', H \sim^* h : H \in H' \boxplus h_o.$$

Consequently, because $\succcurlyeq$ satisfies the DE axiom, we see that

$$H \succ H'. \tag{11}$$

Now, since $h \sim^* H$, there are replicas $h_r$ and $H_r$ of $h$ and $H$, respectively, such that $h_r$ is a relabeling of $H_r$. Similarly, there are replicas $H'_r$ and $h'_r$ of $H'$ and $h'$, respectively, such that $H'_r$ is a relabeling of $h'_r$.

Hence, since $\succcurlyeq$ satisfies the A and RP axioms, it follows that

$$h \sim h_r \sim H_r \sim H \quad \text{and} \quad h' \sim h'_r \sim H'_r \sim H'.$$

Combining these relations with (11) yields

$$h \sim h_r \sim H_r \sim H \succ H' \sim H'_r \sim h'_r \sim h'.$$

Since $\succcurlyeq$ is reflexive and transitive, this chain of relations implies $h \succ h'$ (Sen, 2017, Lemma 1*a, p. 56). This establishes (a) and completes the proof. ∎

## A.6. Proof of Proposition 4

**Proposition 4.** *The binary relation $\succcurlyeq_{UE^*}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying A, RP, and UE*.*

*Proof.* Reflexivity and transitivity are immediate.

If $h'$ is a relabeling or a replica of $h$, then $h' \sim^* h$, implying $h' \sim_{UE^*} h$. Thus, $\succcurlyeq_{UE^*}$ satisfies A and RP.

Suppose that $h' \in h_o \uplus h$ and $h_o \succcurlyeq_{UE^*} h$ for some $h_o \in \mathscr{H}$. Then there exists a finite sequence

$$h_o = h_0, \ldots, h_m = h$$

such that

$$h_0 R_{k_0} h_1 R_{k_1} \cdots R_{k_{m-1}} h_m,$$

implying

$$(h_o = h_0) R_{k^*} h_1 R_{k^*} \cdots R_{k^*} (h_m = h), \tag{12}$$

where $k^* = \max\{k_0, \ldots, k_{m-1}\}$.

Since $h' \in h_o \uplus h$ and (12) holds, we have $h' R_{k^*+1} h$, implying $h' \succcurlyeq_{UE^*} h$.

Now, since $h' \in h_o \uplus h$, $h'$ has a higher maximum depth than $h$, so $h \not\succcurlyeq_{UE^*} h'$.

Consequently, $h' \succ_{UE^*} h$. We conclude that $\succcurlyeq_{UE^*}$ satisfies UE*. ∎

## A.7. Proof of Theorem 4

The proof of Theorem 4 relies on the following intermediate result.

Let $I_k$ and $P_k$ denote the symmetric and asymmetric parts of $R_k$, respectively.

**Lemma 3.** *Suppose that*

$$h_0 R_{k_1} \cdots R_{k_{l+1}} h_l R_k h_0.$$

*Then*

$$h_0 I_{k_1} \cdots I_{k_{l+1}} h_l I_k h_0.$$

*Proof.* Suppose that

$$h_0 R_{k_1} \cdots R_{k_{l+1}} h_l R_k h_0.$$

Observe that

$$h_0 R_{k_1} \cdots R_{k_{l+1}} h_l$$

implies $I_s(h_0) \geq I_s(h_l)$, where $I_s(h'')$ represents the average number of (direct or indirect) supervisors in $h''$.

Similarly, $h_l R_k h_0$ implies $I_s(h_l) \geq I_s(h_0)$.

Consequently, $h_0$ and $h_l$ have the same average depth, $I_s(h_0) = I_s(h_l)$, implying that

$$h_0 I_{k_1} \cdots I_{k_{l+1}} h_l I_k h_0.$$

Indeed, if at least one of the relations were strict, one of the following would hold, since strict dominance in $R_\kappa$ forces a strict increase in $I_s$:

- $I_s(h_l) < I_s(h_0)$ and $I_s(h_0) \leq I_s(h_l)$. This is a contradiction.
- $I_s(h_l) \leq I_s(h_0)$ and $I_s(h_0) < I_s(h_l)$. This is a also contradiction. ∎

We are now ready to prove Theorem 4

**Theorem 4.** *A hierarchical pre-order on $\mathscr{H}$ satisfies A, RP, and UE* if and only if it is $\succcurlyeq_{UE^*}$-consistent.*

*Proof.* (*Sufficiency.*) Assume that $\succcurlyeq$ is $\succcurlyeq_{UE^*}$-consistent. By Proposition 4, $\succcurlyeq_{UE^*}$ satisfies the axioms A, RP, and UE*. Since $\succcurlyeq$ is $\succcurlyeq_{UE^*}$-consistent, it follows that $\succcurlyeq$ must also satisfy these axioms.

(*Necessity.*) Suppose that $\succcurlyeq$ is a hierarchical pre-order on $\mathscr{H}$ satisfying axioms A, RP, and UE*. We must show that $\succcurlyeq$ is $\succcurlyeq_{UE^*}$-consistent, i.e., that for every pair of hierarchies $h, h' \in \mathscr{H}$:

(a) $h \succ_{UE^*} h' \Rightarrow h \succ h'$.

(b) $h \sim_{UE^*} h' \Rightarrow h \sim h'$.

Suppose first that $h \sim_{UE^*} h'$. Then there are finitely many hierarchies in $\mathscr{H}$,

$$h_0, \ldots, h_k, h'_0, \ldots, h'_L,$$

such that

$$(h = h_0) R_{k_1} h_1 R_{k_2} \cdots R_{k_k} (h_k = h' = h'_0) R_{k'_1} h'_1 R_{k'_2} \cdots R_{k'_L} (h'_L = h).$$

Since this represents a cycle, Lemma 3 implies

$$(h = h_0) I_{k_1} h_1 I_{k_2} \cdots I_{k_k} (h_k = h').$$

Hence,

$$(h = h_0)I_{k^*}h_1I_{k^*}\cdots I_{k^*}(h_k = h'),$$

where $k^* = \max\{k_1, \ldots, k_k\}$. Note that this relational chain implies

$$(h = h_0) \sim^* h_1 \sim^* \cdots \sim^* (h_k = h').$$

Consequently, because $\succcurlyeq$ satisfies A and RP, we have

$$h \sim h_0 \sim \cdots \sim h_k \sim h'. \tag{13}$$

By transitivity of $\succcurlyeq$, $\sim$ is transitive, so (13) implies $h \sim h'$. This establishes (b).

It remains to prove (a). First, we show that if $(h, h')$ belongs to the transitive closure of $R_k$ then $h \succcurlyeq h'$. We proceed by induction on $k$.

If $(h, h')$ belongs to the transitive closure of $R_0$ then $hR_0h'$ (Lemma 1), implying $h \sim h'$ because $\succcurlyeq$ satisfies A and RP.

Now suppose that, for each $k \in \{0, \ldots, \kappa\}$, if $(h, h')$ belongs to the transitive closure of $R_k$ then $h \succcurlyeq h'$. We will show that if $(h, h')$ belongs to the transitive closure of $R_{\kappa+1}$ then $h \succcurlyeq h'$.

If $(h, h')$ belongs to the transitive closure of $R_{\kappa+1}$, then there exist finitely many hierarchies

$$\bar{h}_0, \ldots, \bar{h}_\ell$$

in $\mathscr{H}$ such that

$$\bar{h}_0 = hR_{\kappa+1}\cdots R_{\kappa+1}\bar{h}_\ell = h'. \tag{14}$$

Observe that $\bar{h}_lR_{\kappa+1}\bar{h}_{l+1}$ implies that either $\bar{h}_l \sim^* \bar{h}_{l+1}$ or

$$\exists \tilde{h}_0R_\kappa\cdots R_\kappa\tilde{h}_m = \bar{h}_{l+1}, H_l \sim^* \bar{h}_l, H_{l+1} \sim^* \bar{h}_{l+1} : H_l \in \tilde{h}_0 \uplus H_{l+1}. \tag{15}$$

In the first case, we see that $\bar{h}_l \sim \bar{h}_{l+1}$ because $\succcurlyeq$ satisfies A and RP. In the second case, (15) implies that $(\tilde{h}_0, \bar{h}_{l+1})$ belongs to the transitive closure of $R_\kappa$, so the induction hypothesis gives $\tilde{h}_0 \succcurlyeq \bar{h}_{l+1}$. Because

$$H_l \in \tilde{h}_0 \uplus H_{l+1} \quad \text{and} \quad \tilde{h}_0 \succcurlyeq \bar{h}_{l+1} \sim^* H_{l+1},$$

we have

$$H_l \in \tilde{h}_0 \uplus H_{l+1} \quad \text{and} \quad \tilde{h}_0 \succcurlyeq \bar{h}_{l+1} \sim H_{l+1}$$

(since $\succcurlyeq$ satisfies A and RP). Therefore, by transitivity of $\succcurlyeq$, we have

$$H_l \in \tilde{h}_0 \uplus H_{l+1} \quad \text{and} \quad \tilde{h}_0 \succcurlyeq H_{l+1},$$

implying $H_l \succ H_{l+1}$, since $\succcurlyeq$ satisfies the UE* axiom. Thus,

$$\bar{h}_l \sim^* H_l \succ H_{l+1} \sim^* \bar{h}_{l+1},$$

whence

$$\bar{h}_l \sim H_l \succ H_{l+1} \sim \bar{h}_{l+1} \tag{16}$$

(since $\succcurlyeq$ satisfies A and RP). Because $\succcurlyeq$ is reflexive and transitive, (16) yields $\bar{h}_l \succ \bar{h}_{l+1}$ (Sen, 2017, Lemma 1*a, p. 56).

We have seen that $\bar{h}_l R_{\kappa+1} \bar{h}_{l+1}$ implies

$$\bar{h}_l \sim \bar{h}_{l+1} \quad \text{or} \quad \bar{h}_l \succ \bar{h}_{l+1}.$$

Consequently, (14) implies

$$\bar{h}_0 = h \succcurlyeq \cdots \succcurlyeq \bar{h}_\ell = h'.$$

By transitivity of $\succcurlyeq$, we obtain $h \succcurlyeq h'$, as we sought.

We are now ready to prove (a). Suppose that $h \succ_{UE^*} h'$. Then there are finitely many hierarchies in $\mathscr{H}$,

$$h''_0, \ldots, h''_l,$$

such that

$$(h = h''_0) R_{k_1} h''_1 R_{k_2} \cdots R_{k_l} (h''_l = h'), \tag{17}$$

with at least one strict relation.

Suppose that $h''_\kappa P_{k_{\kappa+1}} h''_{\kappa+1}$, where $\kappa \in \{0, \ldots, l-1\}$. Then

$$\exists \hat{h}_0 R_{k_{\kappa+1}-1} \cdots R_{k_{\kappa+1}-1} \hat{h}_M = h''_{\kappa+1}, H''_\kappa \sim^* h''_\kappa, H''_{\kappa+1} \sim^* h''_{\kappa+1} : H''_\kappa \in \hat{h}_0 \uplus H''_{\kappa+1}.$$

Since $(\hat{h}_0, h''_{\kappa+1})$ belongs to the transitive closure of $R_{k_{\kappa+1}-1}$, our previous result yields $\hat{h}_0 \succcurlyeq h''_{\kappa+1}$. Consequently, because $\succcurlyeq$ satisfies the UE* axiom, we see that

$$H''_\kappa \succ H''_{\kappa+1},$$

and, since $\succcurlyeq$ satisfies A and RP, it follows that $h''_\kappa \succ h''_{\kappa+1}$.

We conclude that $h''_\kappa P_{k_{\kappa+1}} h''_{\kappa+1}$, where $\kappa \in \{0, \ldots, l-1\}$, implies $h''_\kappa \succ h''_{\kappa+1}$.

Furthermore, if $h''_\kappa I_{k_{\kappa+1}} h''_{\kappa+1}$, where $\kappa \in \{0, \ldots, l-1\}$, then, applying the argument used in the proof of (b), we see that $h''_\kappa \sim h''_{\kappa+1}$.

Consequently, (17), which holds with at least one relation being strict, implies

$$(h = h''_0) \succcurlyeq h''_1 \succcurlyeq \cdots \succcurlyeq (h''_l = h'),$$

with at least one strict dominance. Since $\succcurlyeq$ is reflexive and transitive, Lemma 1*a in Sen (2017, p. 56) implies $h \succ h'$. This establishes (a). ∎

## A.8. Proof of Proposition 5

**Proposition 5.** *The binary relation $\succcurlyeq_{DE^*}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying A, RP, and DE*.*

*Proof.* Reflexivity and transitivity are immediate by the definition of $\succcurlyeq_{DE^*}$: $\succcurlyeq_{DE^*}$ is defined as the transitive closure of $\bigcup_{k=0}^{\infty} S_k$, where $S_0 = \sim^*$ ensures $h \succcurlyeq_{DE^*} h$ for all $h \in \mathscr{H}$, and the transitive closure guarantees transitivity.

To verify A and RP, consider any two hierarchies $h, h' \in \mathscr{H}$. If $h'$ is a relabeling or a replica of $h$, then $h' \sim^* h$. Since $S_0 = \sim^*$ is included in $\succcurlyeq_{DE^*}$, it follows that $h' S_0 h$ and $h S_0 h'$, implying $h' \succcurlyeq_{DE^*} h$ and $h \succcurlyeq_{DE^*} h'$. Thus, $h' \sim_{DE^*} h$, so $\succcurlyeq_{DE^*}$ satisfies A and RP.

Suppose that $h' \in h \uplus h_0$ and $h_0 \succcurlyeq_{DE^*} h$ for some $h_0 \in \mathscr{H}$. Since $h_0 \succcurlyeq_{DE^*} h$, there exists a finite sequence of hierarchies

$$h_0 = h'_0, h'_1, \ldots, h'_m = h$$

such that

$$h'_0 S_{k_0} h'_1 S_{k_1} \cdots S_{k_{m-1}} h'_m,$$

where each $S_{k_l}$ is from the sequence defining $\succcurlyeq_{DE^*}$. Define

$$k^* = \max\{k_0, \ldots, k_{m-1}\},$$

so the sequence can be rewritten as

$$(h_0 = h'_0) S_{k^*} h'_1 S_{k^*} \cdots S_{k^*} (h'_m = h).$$

Given $h' \in h \uplus h_0$ and $h_0 S_{k^*} h$, the definition of $S_{k^*+1}$ implies

$$(h', h) \in S_{k^*+1},$$

since there exist $H' \sim^* h'$, $H \sim^* h$ such that $H' \in H \uplus h_0$, and $h_0 S_{k^*} h$. Thus, $h' \succcurlyeq_{DE^*} h$.

To confirm strict dominance, note that since $h' \in h \uplus h_0$, $h'$ has a higher maximum depth than $h$. Suppose, for contradiction, that $h \succcurlyeq_{DE^*} h'$. Then there exists a sequence

$$h = h''_0, h''_1, \ldots, h''_n = h'$$

with $h''_l S_{k_l} h''_{l+1}$. However, each step in the sequence via $S_k$ either preserves depth (via $\sim^*$) or increases maximum depth (via the $\uplus$ operation), making it impossible for $h$ (with lower depth) to be obtained from $h'$ (with higher depth). Thus, $h \not\succcurlyeq_{DE^*} h'$, and hence $h' \succ_{DE^*} h$.

Consequently, $\succcurlyeq_{DE^*}$ satisfies DE*. Therefore, $\succcurlyeq_{DE^*}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying A, RP, and DE*. ■

## A.9. Proof of Theorem 5

**Theorem 5.** *A hierarchical pre-order on $\mathscr{H}$ satisfies A, RP, and DE* if and only if it is $\succcurlyeq_{DE^*}$-consistent.*

*Proof.* (*Sufficiency.*) Assume that $\succcurlyeq$ is $\succcurlyeq_{DE^*}$-consistent. By Proposition 5, $\succcurlyeq_{DE^*}$ satisfies A, RP, and DE*. Since $\succcurlyeq$ is $\succcurlyeq_{DE^*}$-consistent, it follows that $\succcurlyeq$ also satisfies these axioms.

(*Necessity.*) Suppose that $\succcurlyeq$ satisfies A, RP, and DE*. We need to show that $\succcurlyeq$ is $\succcurlyeq_{DE^*}$-consistent, i.e., for all $h, h' \in \mathscr{H}$:

- If $h \succ_{DE^*} h'$, then $h \succ h'$.
- If $h \sim_{DE^*} h'$, then $h \sim h'$.

First, suppose $h \sim_{DE^*} h'$. Then there exists a cycle in $\succcurlyeq_{DE^*}$, which, by a lemma analogous to Lemma 3, implies that all relations in the cycle are indifferences. Specifically, there are hierarchies such that

$$(h = h_0) S_{k_1} h_1 S_{k_2} \cdots S_{k_m} (h_m = h' = h'_0) S_{k'_1} h'_1 S_{k'_2} \cdots S_{k'_n} (h'_n = h),$$

and thus

$$(h = h_0) I_{k_1} h_1 I_{k_2} \cdots I_{k_m} (h_m = h'),$$

where $I_k$ is the symmetric part of $S_k$. Since $I_k$ implies $\sim^*$, we have

$$h \sim^* h_1 \sim^* \cdots \sim^* h',$$

and thus $h \sim h'$ because $\succcurlyeq$ satisfies A and RP.

Next, suppose $h \succ_{DE^*} h'$. Then there exists a sequence

$$(h = h_0'') S_{k_1} h_1'' S_{k_2} \cdots S_{k_l} (h_l'' = h'), \tag{18}$$

with at least one strict relation. We need to show that this implies $h \succ h'$.

First, we show that if $(h, h')$ is in the transitive closure of $S_k$, then $h \succcurlyeq h'$. We proceed by induction on $k$.

For $k = 0$, since $S_0 = \sim^*$, and $\succcurlyeq$ satisfies A and RP, $hS_0h'$ implies $h \sim h'$.

Assume that for all $k \leq \kappa$, if $(h, h')$ is in the transitive closure of $S_k$, then $h \succcurlyeq h'$.

For $k = \kappa + 1$, if $(h, h')$ is in the transitive closure of $S_{\kappa+1}$, there exists a sequence

$$(h = \bar{h}_0) S_{\kappa+1} \bar{h}_1 S_{\kappa+1} \cdots S_{\kappa+1} (\bar{h}_\ell = h'). \tag{19}$$

For each $\bar{h}_{l'} S_{\kappa+1} \bar{h}_{l'+1}$, either $\bar{h}_{l'} S_\kappa \bar{h}_{l'+1}$, which by induction implies $\bar{h}_{l'} \succcurlyeq \bar{h}_{l'+1}$, or there exist a sequence

$$\tilde{h}_0 S_\kappa \cdots S_\kappa (\tilde{h}_{m'} = \bar{h}_{l'+1})$$

and $H_{l'} \sim^* \bar{h}_{l'}$, $H_{l'+1} \sim^* \bar{h}_{l'+1}$ such that $H_{l'} \in H_{l'+1} \uplus \tilde{h}_0$.

Since $(\tilde{h}_0, \tilde{h}_{m'} = \bar{h}_{l'+1})$ is in the transitive closure of $S_\kappa$, by induction, $\tilde{h}_0 \succcurlyeq \bar{h}_{l'+1}$.

Because $\tilde{h}_0 \succcurlyeq \bar{h}_{l'+1} \sim^* H_{l'+1}$, we have $\tilde{h}_0 \succcurlyeq \bar{h}_{l'+1} \sim H_{l'+1}$, since $\succcurlyeq$ satisfies A and RP.

The relations $\tilde{h}_0 \succcurlyeq \bar{h}_{l'+1} \sim H_{l'+1}$ imply $\tilde{h}_0 \succcurlyeq H_{l'+1}$. This follows from the reflexivity and transitivity of $\succcurlyeq$ (Sen, 2017, Lemma 1*a, p. 56).

Then, since $H_{l'} \in H_{l'+1} \uplus \tilde{h}_0$ and $\tilde{h}_0 \succcurlyeq H_{l'+1}$, and $\succcurlyeq$ satisfies DE*, we have $H_{l'} \succ H_{l'+1}$.

Thus,

$$\bar{h}_{l'} \sim^* H_{l'} \succ H_{l'+1} \sim^* \bar{h}_{l'+1},$$

so

$$\bar{h}_{l'} \sim H_{l'} \succ H_{l'+1} \sim \bar{h}_{l'+1},$$

and this chain implies $\bar{h}_{l'} \succ \bar{h}_{l'+1}$.

We have shown that $\bar{h}_{l'} S_{\kappa+1} \bar{h}_{l'+1}$ implies $\bar{h}_{l'} \succcurlyeq \bar{h}_{l'+1}$. Consequently, (19) implies

$$h = \bar{h}_0 \succcurlyeq \bar{h}_1 \succcurlyeq \cdots \succcurlyeq \bar{h}_\ell = h',$$

whence $h \succcurlyeq h'$ by transitivity of $\succcurlyeq$.

We conclude that if $(h, h')$ is in the transitive closure of $S_k$, then $h \succcurlyeq h'$. Hence, using (18), which holds with at least one strict relation, we see that

$$(h = h_0'') \succcurlyeq h_1'' \succcurlyeq \cdots \succcurlyeq (h_l'' = h'),$$

with at least one $\succ$. Hence, $h \succ h'$, as required. ∎

## A.10. Proof of Proposition 6

**Proposition 6.** *The binary relation $\tilde{R}$ is a hierarchical pre-order on $\mathscr{H}$ satisfying all the axioms in $\bigcup_{i\in I} A_i$.*

*Proof.* For any hierarchy $h \in \mathscr{H}$, each pre-order $\succcurlyeq_i$ is reflexive. Thus, $h \succcurlyeq_i h$ for all $i \in I$, implying $h\tilde{R}h$. Hence, $\tilde{R}$ is reflexive.

By construction, $\tilde{R}$ is the transitive closure of $\bigcup_{i\in I} \succcurlyeq_i$, so $\tilde{R}$ is transitive.

Let $A \in \bigcup_{i\in I} A_i$ be an arbitrary axiom. By assumption, each $\succcurlyeq_i$ satisfies its corresponding axioms in $A_i$. For any $h, h' \in \mathscr{H}$, $A$ imposes a condition $P(h, h') \Rightarrow Q(h, h', \succcurlyeq_i)$ for some $i \in I$. Since $\tilde{R}$ contains $\succcurlyeq_i$, the condition $Q(h, h', \tilde{R})$ holds. Thus, $\tilde{R}$ inherits all axioms in $\bigcup_{i\in I} A_i$.

Therefore, $\tilde{R}$ is a reflexive and transitive hierarchical pre-order satisfying all axioms in $\bigcup_{i\in I} A_i$. ∎

## A.11. Proof of Theorem 6

**Theorem 6.** *A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies all $A_i$ for $i \in I$ if and only if it is $\tilde{R}$-consistent.*

*Proof.* (*Sufficiency.*) Assume that $\succcurlyeq$ is $\tilde{R}$-consistent. By Proposition 6, $\tilde{R}$ satisfies all axioms in $\bigcup_i A_i$. Pick any axiom $A \in \bigcup_i A_i$ and suppose that $P(h, h') \Rightarrow Q(h, h', \tilde{R})$ is its conditional statement for $\tilde{R}$. To see that $\succcurlyeq$ satisfies $A$, assume $P(h, h')$. Then $Q(h, h', \tilde{R})$ holds and so $Q(h, h', \succcurlyeq)$ must also hold by $\tilde{R}$-consistency. For example, if $Q(h, h', \tilde{R})$ represents the statement "$h\tilde{R}h'$," then either $h\tilde{P}h'$ or $h\tilde{I}h'$, where $\tilde{P}$ and $\tilde{I}$ denote the asymetric and symmetric parts of $\tilde{R}$, respectively. If $h\tilde{P}h'$, then $h \succ_i h'$ by $\tilde{R}$-consistency, and if $h\tilde{I}h'$, then $h \sim_i h'$ by $\tilde{R}$-consistency.

(*Necessity.*) Suppose that $\succcurlyeq$ is a hierarchical pre-order on $\mathscr{H}$ satisfying all the axioms in $\bigcup_i A_i$. We must show that $\succcurlyeq$ is $\tilde{R}$-consistent, i.e., that for every pair of hierarchies $h, h' \in \mathscr{H}$:

(a) $h\tilde{P}h' \Rightarrow h \succ h'$.

(b) $h\tilde{I}h' \Rightarrow h \sim h'$.

Suppose first that $h\tilde{I}h'$. Then there are finitely many hierarchies in $\mathscr{H}$,

$$h_0, \dots, h_k, h'_0, \dots, h'_L,$$

such that

$$(h = h_0)(\cup_i \succcurlyeq_i)h_1(\cup_i \succcurlyeq_i)\cdots(\cup_i \succcurlyeq_i)(h_k = h' = h'_0)(\cup_i \succcurlyeq_i)h'_1(\cup_i \succcurlyeq_i)\cdots(\cup_i \succcurlyeq_i)(h'_L = h).$$

Since this chain forms a cycle, by the second condition in the definition of a composite axiomatic collection of hierarchical pre-orders, the cycle must be formed entirely by indifference in each individual $\succcurlyeq_i$:

$$(h = h_0) \sim_{i_1} h_1 \sim_{i_2} \cdots \sim_{i_k} (h_k = h' = h'_0) \sim_{i'_1} h'_1 \sim_{i'_2} \cdots \sim_{i'_L} (h'_L = h).$$

By assumption, this implies

$$(h = h_0) \sim^* h_1 \sim^* \cdots \sim^* (h_k = h')$$

(see the first bullet point in the fourth condition of the definition of a composite axiomatic collection of hierarchical pre-orders). Since $\succcurlyeq$ satisfies all the axioms in $\bigcup_i A_i$, and since $A_i$ includes the A and RP axioms for some $i$ (see the third condition in the definition of a composite axiomatic collection of hierarchical pre-orders), we see that $\succcurlyeq$ satisfies the A and RP axioms. Hence,

$$(h = h_0) \sim h_1 \sim \cdots \sim (h_k = h'),$$

which yields $h \sim h'$ by transitivity of $\sim$. This establishes (b).

It remains to prove (a). Suppose that $h\tilde{P}h'$. Then there are finitely many hierarchies in $\mathscr{H}$,

$$h_0'', \ldots, h_l'',$$

such that

$$(h = h_0'')(\cup_i \succcurlyeq_i)h_1''(\cup_i \succcurlyeq_i)\cdots(\cup_i \succcurlyeq_i)(h_l'' = h'), \tag{20}$$

with at least one strict relation.

Suppose that $h_\kappa''(\cup_i \succcurlyeq_i)h_{\kappa+1}''$ and not $h_{\kappa+1}''(\cup_i \succcurlyeq_i)h_\kappa''$, where $\kappa \in \{0, \ldots, l-1\}$. Then

$$h_\kappa'' \succ_j h_{\kappa+1}'', \quad \text{some } j. \tag{21}$$

Hence, there exist finitely many hierarchies in $\mathscr{H}$,

$$h_\kappa'' = \tilde{h}_0, \ldots, \tilde{h}_m = h_{\kappa+1}'', \tag{22}$$

such that

$$P_\ell(\tilde{H}_\ell, \tilde{H}_{\ell+1}, \mathscr{H}_\ell, \succcurlyeq_j) \quad \text{holds for each } \ell \in \{0, \ldots, m-1\} \text{ and some } \tilde{H}_\ell \sim_j \tilde{h}_\ell \text{ and } \tilde{H}_{\ell+1} \sim_j \tilde{h}_{\ell+1}, \tag{23}$$

where each $P_\ell(\tilde{H}_\ell, \tilde{H}_{\ell+1}, \mathscr{H}_\ell, \succcurlyeq_j)$ is a predicate corresponding to axiom $a_\ell \in A_j$.

Since $\succcurlyeq_j$ satisfies the axiom $a_\ell$, we have

$$P_\ell(\tilde{H}_\ell, \tilde{H}_{\ell+1}, \mathscr{H}_\ell, \succcurlyeq_j) \Rightarrow Q_\ell(\tilde{H}_\ell, \tilde{H}_{\ell+1}, \succcurlyeq_j). \tag{24}$$

By (23) and (24), we see that $Q_\ell(\tilde{H}_\ell, \tilde{H}_{\ell+1}, \succcurlyeq_j)$ holds for each $\ell \in \{0, \ldots, m-1\}$. This, together with (21) and (22), implies

$$h_\kappa'' \sim_j \tilde{H}_0 \succcurlyeq_j \cdots \succcurlyeq_j \tilde{H}_m \sim_j h_{\kappa+1}'',$$

with at least one strict dominance. Since $\succcurlyeq$ satisfies axiom $a_\ell$ for each $\ell$, we have

$$Q_\ell(\tilde{H}_\ell, \tilde{H}_{\ell+1}, \succcurlyeq_j) = Q_\ell(\tilde{H}_\ell, \tilde{H}_{\ell+1}, \succcurlyeq), \quad \text{for each } \ell.$$

Consequently,

$$h_\kappa'' \sim_j \tilde{H}_0 \succcurlyeq \cdots \succcurlyeq \tilde{H}_m \sim_j h_{\kappa+1}'',$$

with at least one strict dominance. Hence, from the fourth condition in the definition of a composite axiomatic collection of hierarchical pre-orders, we see that

$$h_\kappa'' \sim^* \tilde{H}_0 \succcurlyeq \cdots \succcurlyeq \tilde{H}_m \sim^* h_{\kappa+1}'',$$

with at least one strict dominance. Since $\succcurlyeq$ satisfies A and RP, it follows that

$$h''_\kappa \sim \tilde{H}_0 \succcurlyeq \cdots \succcurlyeq \tilde{H}_m \sim h''_{\kappa+1},$$

with at least one strict dominance. Consequently, because $\succcurlyeq$ is reflexive and transitive, we obtain $h''_\kappa \succ h''_{\kappa+1}$ (Sen, 2017, Lemma 1*a, p. 56).

We conclude that $h''_\kappa(\cup_i \succcurlyeq_i)h''_{\kappa+1}$ and not $h''_{\kappa+1}(\cup_i \succcurlyeq_i)h''_\kappa$, where $\kappa \in \{0, \ldots, l-1\}$, implies $h''_\kappa \succ h''_{\kappa+1}$.

Furthermore, if $h''_\kappa(\cup_i \succcurlyeq_i)h''_{\kappa+1}$ and $h''_{\kappa+1}(\cup_i \succcurlyeq_i)h''_\kappa$, where $\kappa \in \{0, \ldots, l-1\}$, then, applying the argument used in the proof of (b), we see that $h''_\kappa \sim h''_{\kappa+1}$.

Hence, (20), which holds with at least one strict dominance, implies

$$(h = h''_0) \succcurlyeq h''_1 \succcurlyeq \cdots \succcurlyeq (h''_l = h'),$$

with at least one strict dominance. Since $\succcurlyeq$ is reflexive and transitive, Lemma 1*a in Sen (2017, p. 56) implies $h \succ h'$. This establishes (a). ■

## A.12. Proof of Proposition 7

**Proposition 7.** *The collection*

$$\{\succcurlyeq_H, \succcurlyeq_{UE}, \succcurlyeq_{DE}, \succcurlyeq_{UE^*}, \succcurlyeq_{DE^*}\}$$

*is a composite axiomatic collection of hierarchical pre-orders.*

*Proof.* We verify that the given collection of hierarchical pre-orders satisfies the conditions required by the definition of a composite axiomatic collection of hierarchical pre-orders.

1. Each hierarchical pre-order in the collection satisfies the following axioms:
   - $\succcurlyeq_H$: A, RP, and SR.
   - $\succcurlyeq_{UE}$: A, RP, and UE.
   - $\succcurlyeq_{DE}$: A, RP, and DE.
   - $\succcurlyeq_{UE^*}$: A, RP, and UE*.
   - $\succcurlyeq_{DE^*}$: A, RP, and DE*.

2. Let us denote the collection of hierarchical pre-orders by $\{\succcurlyeq_i\}_{i \in I}$. Suppose there exists a cycle in the union of these pre-orders:

   $$h_0(\cup_i \succcurlyeq_i)h_1(\cup_i \succcurlyeq_i)\cdots(\cup_i \succcurlyeq_i)h_n(\cup_i \succcurlyeq_i)h_0.$$

   For each pre-order in the collection, the average depth weakly decreases along this chain, and strictly decreases for any strict relation. Therefore, this cycle can only be consistent with indifference in each individual pre-order:

   $$h_0 \sim_{i_1} h_1 \sim_{i_2} \cdots \sim_{i_n} h_n \sim_{i_{n+1}} h_0.$$

3. Each member of the collection satisfies A and RP.

4. For each $i \in I$:
   - If $h \sim_i h'$ then $h \sim^* h'$. This follows from:
     - Lemma 4 in Carbonell-Nicolau (2025b) if $\sim_i = \sim_H$.
     - The proof of Theorem 1 if $\sim_i = \sim_{UE}$.
     - The proof of Theorem 3 if $\sim_i = \sim_{DE}$.
     - The proof of Theorem 4 if $\sim_i = \sim_{UE^*}$.
     - The proof of Theorem 5 if $\sim_i = \sim_{DE^*}$.
   - Suppose that $h \succ_i h'$. Then:
     - If $\succ_i = \succ_H$, then $h'$ can be obtained from some relabeling $H$ of $h$ by successive removals of subordination relations (Carbonell-Nicolau, 2025b, Theorem 1). Hence, there is a finite sequence

$$H = h_0, h_1, \ldots, h_K = h'$$

       such that $h_l \succ h_{l+1}$ for each $l \in \{0, \ldots, K-1\}$ and every hierarchical pre-order $\succcurlyeq$ satisfying SR.
     - If $\succ_i = \succ_{UE}$, then

$$\exists h_o, H' \sim^* h', H \sim^* h : H \in h_o \oplus H'.$$

       Therefore, $H \succ H'$ for every hierarchical pre-order $\succcurlyeq$ satisfying UE.
     - If $\succ_i = \succ_{DE}$, then

$$\exists h_o, H' \sim^* h', H \sim^* h : H \in H' \boxplus h_o.$$

       Therefore, $H \succ H'$ for every hierarchical pre-order $\succcurlyeq$ satisfying DE.
     - If $\succ_i = \succ_{UE^*}$, then there are finitely many hierarchies in $\mathscr{H}$,

$$h''_0, \ldots, h''_l,$$

       such that

$$(h = h''_0) R_{k_1} h''_1 R_{k_2} \cdots R_{k_l} (h''_l = h'),$$

       with at least one strict relation.
       From the proof of Theorem 4, we know that the following holds:
       * If $h''_\kappa P_{k_{\kappa+1}} h''_{\kappa+1}$, where $\kappa \in \{0, \ldots, l-1\}$, then

$$\exists \hat{h}_0 R_{k_{\kappa+1}-1} \cdots R_{k_{\kappa+1}-1} \hat{h}_M = h''_{\kappa+1}, H''_\kappa \sim^* h''_\kappa, H''_{\kappa+1} \sim^* h''_{\kappa+1} \\ : H''_\kappa \in \hat{h}_0 \uplus H''_{\kappa+1},$$

         so that $H''_\kappa \succ_{UE^*} H''_{\kappa+1}$, and $H''_\kappa \succ H''_{\kappa+1}$ for every hierarchical pre-order $\succcurlyeq$ satisfying A, RP, and UE*.
       * If $h''_\kappa I_{k_{\kappa+1}} h''_{\kappa+1}$, where $\kappa \in \{0, \ldots, l-1\}$, then $h''_\kappa \sim^* h''_{\kappa+1}$, so that $h''_\kappa \sim_{UE^*} h''_{\kappa+1}$, and $h''_\kappa \sim h''_{\kappa+1}$ for every $\succcurlyeq$ satisfying A and RP.

– If $\succ_i = \succ_{DE^*}$, then there exists a sequence

$$(h = h''_0) S_{k_1} h''_1 S_{k_2} \cdots S_{k_l} (h''_l = h'),$$

with at least one strict relation.
From the proof of Theorem 5, we know that the following holds:

* If $h''_\kappa S_{k_{\kappa+1}} h''_{\kappa+1}$ and not $h''_{\kappa+1} S_{k_{\kappa+1}} h''_\kappa$, where $\kappa \in \{0, \ldots, l-1\}$, then

$$\exists \hat{h}_0 S_{k_{\kappa+1}-1} \cdots S_{k_{\kappa+1}-1} \hat{h}_M = h''_{\kappa+1}, H''_\kappa \sim^* h''_\kappa, H''_{\kappa+1} \sim^* h''_{\kappa+1} \\ : H''_\kappa \in H''_{\kappa+1} \uplus \hat{h}_0,$$

so that $H''_\kappa \succ_{DE^*} H''_{\kappa+1}$, and $H''_\kappa \succ H''_{\kappa+1}$ for every hierarchical pre-order $\succcurlyeq$ satisfying A, RP, and DE*.

* If $h''_\kappa S_{k_{\kappa+1}} h''_{\kappa+1}$ and $h''_{\kappa+1} S_{k_{\kappa+1}} h''_\kappa$, where $\kappa \in \{0, \ldots, l-1\}$, then $h''_\kappa \sim^* h''_{\kappa+1}$, so that $h''_\kappa \sim_{DE^*} h''_{\kappa+1}$, and $h''_\kappa \sim h''_{\kappa+1}$ for every $\succcurlyeq$ satisfying A and RP. ■

# References


Anderson, P. W. (1972) "More is different: Broken symmetry and the hierarchical structure of science," *Science*, 177 (4047), 393–396.

Carbonell-Nicolau, Oriol (2025a) "Measuring Hierarchy," *Social Choice and Welfare*, 10.1007/s00355-025-01582-1.

——— (2025b) "Measuring Hierarchy with Multiple Supervisors," https://oriolcn.github.io/papers/ghierarchy.pdf, Mimeograph.

Corominas-Murtra, Bernat, Joaquín Goñi, Ricard V Solé, and Carlos Rodríguez-Caso (2013) "On the origins of hierarchy in complex networks," *Proc. Natl. Acad. Sci. U. S. A.*, 110 (33), 13316–13321.

Culen, Julia (2017) "Hierarchy, Informal Networks and How to Navigate the Full Mess," https://juliaculen.medium.com/hierarchy-informal-networks-and-how-to-navigate-the-full-mess-89d442b096a2, Accessed: 2025-05-26.

Czégel, Dániel and Gergely Palla (2015) "Random walk hierarchy measure: What is more hierarchical, a chain, a tree or a star?" *Sci. Rep.*, 5 (1), 17994.

Fix, Blair (2017) "Energy and institution size," *PLOS ONE*, 12 (2), e0171823, 10.1371/journal.pone.0171823.

——— (2018) "Hierarchy and the power-law income distribution tail," *Journal of Computational Social Science*, 1 (2), 471–491, 10.1007/s42001-018-0019-8.

——— (2019) "Energy, hierarchy and the origin of inequality," *PLOS ONE*, 14 (4), 1–24, 10.1371/journal.pone.0215692.

——— (2021) "Redistributing Income Through Hierarchy," *Real-World Economics Review* (98), 58–86, https://rwer.wordpress.com/2021/12/16/rwer-issue-no-98.

Holland, John H. (1998) *Emergence: From chaos to order*: Addison-Wesley.

Krackhardt, D. (1994) "Graph theoretical dimensions of informal organizations," in *Computational Organization Theory*, 89–111: Taylor & Francis.

Krackhardt, David and Robert N Stern (1988) "Informal networks and organizational crises: An experimental simulation," *Social Psychology Quarterly*, 51 (2), 123–140.

Lawrence, Paul R. and Jay W. Lorsch (1967) *Organization and Environment: Managing Differentiation and Integration*, Boston, MA: Harvard University Press.

Lincoln, James R and Arne L Kalleberg (1985) "Work organization and work-force commitment: A study of plants and employees in the U.S. and Japan," *American Sociological Review*, 50, 738–760.

Luo, Jianxi and Christopher L Magee (2011) "Detecting evolving patterns of self-organizing networks by flow hierarchy measurement," *Complexity*, 16 (6), 53–61.

Mones, Enys, Lilla Vicsek, and Tamás Vicsek (2012) "Hierarchy Measure for Complex Networks," *PLOS ONE*, 7 (3), 1–10, 10.1371/journal.pone.0033799.

Sarkar, Abhra, Stephen E Fienberg, and David Krackhardt (2010) "Predicting profitability using advice branch bank networks," *Statistical Methodology*, 7 (4), 429–444.

Sen, Amartya (2017) *Collective Choice and Social Welfare*, Cambridge, MA: Harvard University Press.

Simon, Herbert A. (1962) "The architecture of complexity," *Proceedings of the American Philosophical Society*, 106 (6), 467–482.

——— (1973) "The organization of complex systems," in Pattee, Howard H. ed. *Hierarchy theory: The challenge of complex systems*, 1–27: George Braziller.

Thompson, James D. (1967) *Organizations in Action: Social Science Bases of Administrative Theory*, New York: McGraw-Hill.

Trusina, Ala, Sergei Maslov, Petter Minnhagen, and Kim Sneppen (2004) "Hierarchy measures in complex networks," *Phys. Rev. Lett.*, 92 (17), 178702.

Wright, Jack (2024) "The hierarchy in economics and its implications," *Econ. Philos.*, 40 (2), 257–278.