On Progressive Tax Systems with Heterogeneous Preferences

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Abstract

The properties of progressive income tax systems vis-à-vis standard measures of inequality and polarization have been studied elsewhere, both for economies with exogenous and endogenous income. In the case of endogenous income, preferences are assumed to be identical across consumers. This paper relaxes the preference homogeneity assumption. Using the relative Lorenz inequality order and the relative Foster-Wolfson bipolarization order, we show that income tax systems are inequality and bipolarization reducing—regardless of the economy they are applied to—only if they are progressive; and we identify conditions on heterogeneous consumer preferences under which progressive tax systems are inequality and bipolarization reducing.

Keywords: Progressive income taxation, income inequality, income bipolarization, endogenous income.
JEL classifications: D63, D71.
Practitioner point: The interplay between progressive income taxation and the reduction of income inequality and bipolarization in economies with heterogeneous preferences.

1. Introduction

Normatively, progressive income tax systems can be viewed as essential mechanisms for the reduction of 'market-driven' income inequality. The theoretical literature on the foundations of progressive taxation goes back to the seminal result on the equivalence between tax progressivity—in the sense of increasing average tax rates on income—and the inequality reducing property (see Jakobsson, 1976; Fellman, 1976; Kakwani, 1977).

This result, which is couched in terms of exogenous income, has been extended in several directions (see, e.g., Hemming and Keen, 1983; Eichhorn et al., 1984; Liu, 1985; Formby et al., 1986; Thon, 1987; Latham, 1988; Thistle, 1988; Moyes, 1988, 1989, 1994; Le Breton et al., 1996; Ebert and Moyes, 2000; Ju and Moreno-Ternero, 2008; Carbonell-Nicolau, 2019). For the most part, these extensions maintain the exogenous income framework. The case of endogenous income presents some subtleties. In fact, there

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are consumer preferences for which progressive tax schedules increase inequality (see Allingham, 1979; Ebert and Moyes, 2003, 2007).

An extension of the Jakobsson-Fellman-Kakwani result to the case of endogenous income is furnished in Carbonell-Nicolau and Llavador (2018, 2021b). Using the relative Lorenz inequality order, these works show that inequality reducing tax schedules—i.e., tax schedules effecting a more equal post-tax income distribution, regardless of the wage/ability distribution—are necessarily progressive, in the sense of increasing marginal tax rates on income; and identify necessary and sufficient conditions on consumer preferences for various sets of progressive tax schedules to be inequality reducing.

A further paper, Carbonell-Nicolau and Llavador (2021a), establishes the equivalence between the inequality reducing property and an analogous property formulated in terms of a different metric, the Foster-Wolfson relative bipolarization order (Foster and Wolfson, 2010; Chakravarty, 2009, 2015), which has been used in the literature as a measure of the size of the middle class (see, e.g., Foster and Wolfson, 2010; Wolfson, 1994). Thus, tax schedules are inequality reducing if and only if they are bipolarization reducing, an equivalence that extends the scope of the Jakobsson-Fellman-Kakwani result and its variants.

The analysis of behavioral labor responses to income taxation is based on the standard Mirrlees model (Mirrlees, 1971), which assumes away any differences in preferences among consumers. This paper relaxes the preference homogeneity assumption. The framework of analysis is the Mirrlees model, augmented with a second source of heterogeneity, namely preference heterogeneity, in addition to the standard variation in wages/abilities. Thus, an economy consists of a distribution of wages/abilities and a distribution of preferences.

We confine attention to economies for which consumption is nondecreasing with the wage rate, a property that extends the standard agent monotonicity condition (Mirrlees, 1971; Seade, 1982) to the case of heterogeneous preferences. For such economies, an extension of the results in Carbonell-Nicolau and Llavador (2018, 2021b) is proven. First, inequality reducing tax schedules—i.e., tax schedules effecting a more equal post-tax income distribution, regardless of the economy they are applied to—are necessarily progressive. Second, we identify necessary and sufficient conditions on the distribution of preferences for various subclasses of progressive tax schedules to be inequality reducing.

A version of the equivalence between inequality reducing and bipolarization reducing tax schedules can be proven for families of preference distributions that are 'sufficiently rich.' Specifically, if $\boldsymbol{u} = (u_1, \ldots, u_n)$ is a distribution of utility functions describing the preferences of n individuals, the distribution $\boldsymbol{u}' = (u'_1, \ldots, u'_n)$ is called a *simple transformation* of $\boldsymbol{u}$ if $\boldsymbol{u}'$ takes the form $(u_i, \ldots, u_i, u_{i+1}, \ldots, u_{i+1})$ for some $i < n$, i.e., if the first (resp., last) i (resp, $n - i$) individuals are endowed with the utility function u_i (resp., u_{i+1}). A set of preference distributions is *closed under simple transformations* if it contains simple transformations of its elements. For such closed subsets of utility vectors, tax schedules are inequality reducing if and only if they are bipolarization reducing.

For families of preference distributions that are closed under simple transformations, a full characterization of progressive tax schedules, in terms of their inequality and bipolarization reducing properties, can be stated. We identify one such family of preference distributions and illustrate particular subclasses defined in terms of two standard families of utility functions: the Greenwood, Hercowitz, and Huffman utility functions (Greenwood et al., 1988) and the constant elasticity of substitution utility functions.

2. Characterizing income tax progressivity

We extend the model in Carbonell-Nicolau and Llavador (2018, 2021a,b) by allowing for heterogeneity of preferences across individuals.

Individual preferences are described by means of a utility function. All utility functions share the following properties. First, all utility functions are assumed to be real-valued functions defined on consumption-labor pairs (x, l) in the product set $\mathbb{R}_+ \times [0, L)$, where $0 < L \leq \infty$. For an individual endowed with a utility function u, $u(x, l)$ represents the individual's utility from x units of consumption and l units of labor. Throughout the sequel, all utility functions u are assumed to satisfy the following conditions:

(i) u is continuous.

(ii) $u(\cdot, l)$ strictly increasing in x for each $l \in [0, L)$ and $u(x, \cdot)$ strictly decreasing in l for each $x > 0$.

(iii) u is strictly quasiconcave on $\mathbb{R}_{++} \times [0, L)$ and twice continuously differentiable on $\mathbb{R}_{++} \times (0, L)$.

(iv) For any $\overline{u}$ and any sequence (x^n, l^n) in $\mathbb{R}_{++} \times (0, L)$ such that $u(x^n, l^n) = \overline{u}$ for each n and $l^n \to L$,

$$MRS(x^n, l^n) \to \infty, \tag{1}$$

where, for $(x, l) \in \mathbb{R}_{++} \times (0, L)$, $MRS(x, l)$ represents the marginal rate of substitution of labor for consumption, i.e.,

$$MRS(x, l) = -\frac{\partial u(x, l)}{\partial l} \bigg/ \frac{\partial u(x, l)}{\partial x}.$$

(v) For each $a > 0$, there exists $l > 0$ such that $u(al, l) > u(0, 0)$.

The condition in (1) states that the compensation required by an individual for an extra unit of labor tends to infinity along any indifference curve as the individual's leisure time approaches zero. The last condition, (v), implies that, in the absence of taxes, an individual whose wage rate is $a > 0$ always consumes a positive amount.

A *tax schedule* is a continuous and nondecreasing map $T : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

- $T(y) \leq y$ for each $y \in \mathbb{R}_+$.
- The map $y \mapsto y - T(y)$ is nondecreasing (i.e., T is order-preserving).

For every pre-tax income level $y \in \mathbb{R}_+$, $T(y)$ represents the associated tax liability ($T(y)$ being a subsidy if $T(y) < 0$).

A tax schedule T is *piecewise linear* if $\mathbb{R}_+$ can be partitioned into finitely many intervals $I_1, \ldots, I_K$ satisfying the following: for each k, there exist $\beta \in \mathbb{R}$ and $t \in [0, 1)$ such that $T(y) = \beta + ty$ for all $y \in I_k$.

The set of all piecewise linear tax schedules is denoted by $\mathscr{T}$.

The set of all utility functions satisfying the conditions (i)-(v) is denoted by $\mathscr{U}$.

An individual of ability $a > 0$ whose utility function is $u \in \mathscr{U}$ and who chooses $l \in [0, L)$ units of labor and faces a tax schedule $T \in \mathscr{T}$, affords $al - T(al)$ units of consumption and derives a utility of $u(al - T(al), l)$. Thus, the individual's problem is

$$\max_{l \in [0,L)} u(al - T(al), l). \tag{2}$$

A solution to (2) is denoted by $l^u(a, T)$.[1] It expresses the utility maximizing units of labor as a function of the "wage rate" a and the tax schedule T. Corresponding *pre-tax* and *post-tax income functions* are denoted by

$$y^u(a, T) = al^u(a, T) \quad \text{and} \quad x^u(a, T) = al^u(a, T) - T(al^u(a, T)),$$

respectively.[2] In the special case when T is identically zero (no taxation), we write $y^u(a, T) = y^u(a, 0)$ and $x^u(a, T) = x^u(a, 0)$. Note that $x^u(a, 0) = y^u(a, 0)$.

We now define the class of utility vectors for which our main characterizations of progressive income tax systems are valid. To this end, we first define the wage elasticity of income induced by a utility function.

Consider the following problem:

$$\max_{l \in [0,L)} u(al + b, l), \tag{5}$$

where $a > 0$ and $b \in \mathbb{R}_+$. Note that (5) can be viewed as the problem solved by an individual of ability a who receives a fixed subsidy $b \geq 0$, and whose utility function is $u \in \mathscr{U}$.

Since u is strictly quasiconcave on $\mathbb{R}_{++} \times [0, 1)$, for each $(a, b) \in \mathbb{R}_{++} \times \mathbb{R}_+$, there is a unique solution $l^u(a, b)$ to (5), with associated pre-tax and post-tax incomes $y^u(a, b)$ and $x^u(a, b)$, respectively.

For $(a, b) \in \mathbb{R}_{++} \times \mathbb{R}_+$, the *wage elasticity of income* for u at (a, b) is defined by

$$\zeta^u(a, b) := \frac{\partial(al^u(a, b) + b)}{\partial a} \cdot \frac{a}{al^u(a, b) + b}.$$[3,4]

[1]To see that (2) has a solution, fix L^* in $(0, L)$. The problem

$$\max_{l \in [0,L^*]} u(al - T(al), l) \tag{3}$$

has a solution, since the objective function is continuous and the feasible set is compact. Let l^* be a solution to (3). Let $u^* = u(al^* - T(al^*), l^*)$. Note that, by the condition (iv), there exists $\bar{l} < L$ such that $u(al - T(al), l) < u^*$ for all $l \in [\bar{l}, L)$. Consequently, any solution to the problem

$$\max_{l \in [0,\bar{l}]} u(al - T(al), l) \tag{4}$$

is also a solution to (2). Since (4) has a solution, so does (2).

[2]A solution to (2) exists, but need not be unique, and so pre-tax and post-tax solution functions are not uniquely defined.

[3]The condition (v) guarantees that $al^u(a, b) + b$ is positive.

[4]For each $b \geq 0$, the derivative of the map $a \mapsto l^u(a, b)$ exists for all but at most one $a > 0$. When this derivative does not exist, it is because the right and left derivatives, which exist, differ from one another. For a such that the derivative of the map $a \mapsto l^u(a, b)$ does not exist, $\zeta^u(a, b)$ is defined in terms of the right derivative of the map $a \mapsto l^u(a, b)$.

Throughout the sequel, the number of individuals is fixed at n.

We consider groups of individuals of size n and describe their preferences by means of utility functions in $\mathscr{U}$. Thus, a vector $(u_1, \ldots, u_n) \in \mathscr{U}^n$ of utility functions lists the individual preferences for each member of the group, where u_i represents individual i's utility function ($i \in \{1, \ldots, n\}$).

An *ability distribution* is a vector of ability parameters, $(a_1, \ldots, a_n) \in \mathbb{R}^n_{++}$, with its coordinates arranged in increasing order, i.e., $a_1 \leq \cdots \leq a_n$.

Let $\mathbb{U}$ be the set of all utility vectors $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathscr{U}^n$ satisfying the following conditions:

(a) For each i, $a' \geq a > 0$, and $b \in \mathbb{R}_+$,

$$x^{u_i}(a, b) \leq x^{u_i}(a', b).$$

(b) For each $i < n$, $a' \geq a > 0$, and $b \in \mathbb{R}_+$,

$$x^{u_i}(a, b) \leq x^{u_{i+1}}(a', b).$$

Note that $x^u(a, b)$ represents the post-tax income for an 'a-individual' whose utility is u and who receives a subsidy of \$$b$.

The first condition, (a), states that consumption is (weakly) increasing with the wage rate.

The second condition, (b), is similar. It compares consumption levels across utility functions. Note that, given an ability distribution $(a_1, \ldots, a_n)$ with $a_1 \leq \cdots \leq a_n$, u_i represents the utility function of individual i, and so higher order coordinates in the vector $(u_1, \ldots, u_n)$ describe the preferences of higher ability individuals. Thus, a higher order coordinate in $\boldsymbol{u}$, together with higher ability, implies higher consumption.

The above conditions extend the standard agent monotonicity condition introduced in Mirrlees (1971) (see also Seade (1982)) to vectors of heterogeneous preferences. Indeed, if the utility function is common across individuals, we obtain the condition in (a), which is equivalent to the original single crossing condition in Mirrlees (1971) (see Mirrlees, 1971, p. 182).

We now define two families of utility vectors that will be used in the formulation of our main results.

For $B \subseteq \mathbb{R}_+$ and $R \subseteq [0, 1)$, let $\mathbb{U}(B, R)$ be the class of all utility vectors $(u_1, \ldots, u_n)$ in $\mathbb{U}$ satisfying the following condition:

- For each $i < n$,

$$\frac{x^{u_i}((1-t)a, b)}{x^{u_i}(a, 0)} \geq \frac{x^{u_{i+1}}((1-t)a', b)}{x^{u_{i+1}}(a', 0)}, \quad \text{for all } (a, a', b, t) \in \mathbb{R}^2_{++} \times B \times R \text{ with } a' \geq a.$$

Note that $x^u((1-t)a, b)$ represents the post-tax income for an a-individual whose utility is u and who receives a tax subsidy of \$$b$ and whose wage income is taxed at a constant marginal tax rate t; and $x^u(a, 0)$ represents the individual's income in the absence of taxes or subsidies. Thus, the above inequality states that the post-tax income $x^u((1-t)a, b)$ as a fraction of $x^u(a, 0)$ decreases as a increases and the order rank for the vector of utility functions $(u_1, \ldots, u_n)$ increases.

For $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$, let $\mathbb{U}^*(B,R)$ be the class of all utility vectors $(u_1,\ldots,u_n)$ in $\mathbb{U}$ satisfying the following conditions:

- For each $i < n$,

$$\frac{x^{u_i}((1-t)a,b)}{x^{u_i}(a,0)} \geq \frac{x^{u_{i+1}}((1-t)a,b)}{x^{u_{i+1}}(a,0)}, \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R.$$

- For each i,

$$\zeta^{u_i}((1-t)a,b) \leq \zeta^{u_i}(a,0), \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R.$$

The inequality in the first condition states that the post-tax income $x^u((1-t)a,b)$ as a fraction of $x^u(a,0)$ decreases as the order rank for the vector of utility functions $(u_1,\ldots,u_n)$ increases.

The elasticity condition in the second bullet point states that, for a fixed utility function u and ability level a, the combined effect of the tax subsidy b and the proportional tax t decreases the wage elasticity of income.

The proof of the following lemma is relegated to Section 4.1.

Lemma 1. *For $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$, $\mathbb{U}^*(B,R) \subseteq \mathbb{U}(B,R)$.*

Given a utility vector $\boldsymbol{u} = (u_1,\ldots,u_n) \in \mathscr{U}^n$, an ability distribution $0 < a_1 \leq \cdots \leq a_n$, and income functions $x^{u_1},\ldots,x^{u_n}$, a tax schedule T in $\mathscr{T}$ gives rise to a post-tax income distribution

$$(x^{u_1}(a_1,T),\ldots,x^{u_n}(a_n,T)).$$

Similarly,

$$(x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0))$$

represents the income distribution in the absence of taxes.

Given two distributions $\boldsymbol{x} = (x_1,\ldots,x_n)$ and $\boldsymbol{y} = (y_1,\ldots,y_n)$ with positive total income, we say that $\boldsymbol{x}$ *Lorenz dominates* $\boldsymbol{y}$, a dominance relation denoted by '$\boldsymbol{x} \succcurlyeq_L \boldsymbol{y}$,' if

$$\frac{\sum_{i=1}^{l} x_{[i]}}{\sum_{i=1}^{n} x_{[i]}} \geq \frac{\sum_{i=1}^{l} y_{[i]}}{\sum_{i=1}^{n} y_{[i]}}, \quad \text{for all } l \in \{1,\ldots,n\},$$

where $(x_{[1]},\ldots,x_{[n]})$ (resp., $(y_{[1]},\ldots,y_{[n]})$) is a rearrangement of the coordinates in $\boldsymbol{x}$ (resp., $\boldsymbol{y}$) in increasing order: $x_{[1]} \leq \cdots \leq x_{[n]}$ and $y_{[1]} \leq \cdots \leq y_{[n]}$.

A tax schedule $T \in \mathscr{T}$ is said to be *inequality reducing with respect to* $\mathbb{U}' \subseteq \mathbb{U}$, or $\mathbb{U}'$-ir, if

$$(x^{u_1}(a_1,T),\ldots,x^{u_n}(a_n,T)) \succcurlyeq_L (x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0))$$

for each ability distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of income functions $(x^{u_1},\ldots,x^{u_n})$, and every vector of utility functions $(u_1,\ldots,u_n) \in \mathbb{U}'$.[5]

The subset of all $\mathbb{U}'$-ir tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{\mathbb{U}'\text{-ir}}$.

[5]Since marginal tax rates are less than unity for the tax schedules in $\mathscr{T}$, condition (v) ensures that income levels $x^{u_i}(a_i,T)$ and $x^{u_i}(a_i,0)$ are positive.

A *marginal-rate progressive tax schedule* is a convex tax schedule in $\mathscr{T}$, which exhibits marginal tax rates that increase with income. The set of marginal-rate progressive tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{\text{m-prog}}$.

For any tax schedule T in $\mathscr{T}$, $T(0) = -b$ for some $b \geq 0$. The parameter b can be viewed as a subsidy.

Given a set of subsidies $B \subseteq \mathbb{R}_+$ and a set of marginal tax rates $R \subseteq [0, 1)$, let $\mathscr{T}_{\text{m-prog}}(B, R)$ represent the set of all marginal-rate progressive tax schedules whose associated subsidies are members of the set B and whose marginal tax rates are taken from R.

The following is the first main result of this paper. Its proof is relegated to Section 4.2.

Theorem 1. *For* $\mathbb{U}' \subseteq \mathbb{U}$, $B \subseteq \mathbb{R}_+$, *and* $R \subseteq [0, 1)$,

$$\mathscr{T}_{\mathbb{U}(B,R)\text{-ir}} \subseteq \mathscr{T}_{\mathbb{U}^*(B,R)\text{-ir}} \subseteq \mathscr{T}_{\text{m-prog}} \quad \textit{and} \quad \left[\mathscr{T}_{\text{m-prog}}(B, R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}} \Leftrightarrow \mathbb{U}' \subseteq \mathbb{U}(B, R)\right].$$

Theorem 1 can be extended to a second characterization of progressivity in terms of inequality *and* bipolarization reducing tax schedules.

The Foster-Wolfson bipolarization order (Foster and Wolfson, 2010; Chakravarty, 2009, 2015) is a measure of the degree of income polarization between two income groups, taking median income as the demarcation point.

For two income distributions $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ with the same median income, m, we write $\boldsymbol{y} \succcurlyeq_{\text{FW}} \boldsymbol{x}$ to indicate that $\boldsymbol{y}$ is *more bipolarized than* $\boldsymbol{x}$, if

$$\begin{aligned}
\sum_{k \leq i < \frac{n+1}{2}} (m - x_i) &\leq \sum_{k \leq i < \frac{n+1}{2}} (m - y_i), \quad \forall k : 1 \leq k < \frac{n+1}{2}, \\
\sum_{\frac{n+1}{2} < i \leq k} (x_i - m) &\leq \sum_{\frac{n+1}{2} < i \leq k} (y_i - m), \quad \forall k : \frac{n+1}{2} < k \leq n.
\end{aligned}$$

The Foster-Wolfson bipolarization order compares income distributions on the basis of an aggregate measure of the deviation of income levels from median income, with lower aggregate deviations corresponding to less bipolarized distributions.

Assuming that proportional changes in income do not alter the degree of bipolarization, $\succcurlyeq_{\text{FW}}$ can be extended to pairs of income distributions with different median incomes as follows.

Let $m(\boldsymbol{x})$ (resp., $m(\boldsymbol{y})$) denote the median income of $\boldsymbol{x}$ (resp., $\boldsymbol{y}$), and suppose that $m(\boldsymbol{x}) > 0$ and $m(\boldsymbol{y}) > 0$. Then the transformation

$$\boldsymbol{y}' = \frac{m(\boldsymbol{x})}{m(\boldsymbol{y})}(y_1, \ldots, y_n)$$

of $\boldsymbol{y}$ has the same median as $\boldsymbol{x}$ and we write

$$\boldsymbol{y} \succcurlyeq_{\text{FW}} \boldsymbol{x} \Leftrightarrow \boldsymbol{y}' \succcurlyeq_{\text{FW}} \boldsymbol{x}.$$

A tax schedule $T \in \mathscr{T}$ is said to be *bipolarization reducing with respect to* $\boldsymbol{u}$, or $\boldsymbol{u}$-bpr, if

$$(x^{u_1}(a_1, 0), \ldots, x^{u_n}(a_n, 0)) \succcurlyeq_{\text{FW}} (x^{u_1}(a_1, T), \ldots, x^{u_n}(a_n, T))$$

for each ability distribution $0 < a_1 \leq \cdots \leq a_n$ and every vector of income functions $(x^{u_1}, \ldots, x^{u_n})$.

The subset of all $\mathbb{U}$-bpr tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{\mathbb{U}\text{-bpr}}$.

The equivalence between the criteria of inequality reduction and bipolarization reduction was first established in Carbonell-Nicolau and Llavador (2021a, Theorem 4) for economies with a single utility function. The following result extends the equivalence to economies with heterogeneous preferences. The proof is given in Section 4.4.

Given $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{U}$, $\boldsymbol{u}' = (u_1', \ldots, u_n')$ is called a *simple transformation of* $\boldsymbol{u}$ if there exists $i < n$ such that

$$\begin{aligned} u_j' &= u_i, && \text{for each } j \leq i, \\ u_j' &= u_{i+1}, && \text{for each } j \geq i+1. \end{aligned}$$

A subset $\mathbb{U}' \subseteq \mathbb{U}$ is *closed under simple transformations* if $\boldsymbol{u} \in \mathbb{U}'$ implies that $\boldsymbol{u}' \in \mathbb{U}'$ for every simple transformation $\boldsymbol{u}'$ of $\boldsymbol{u}$.

Theorem 2. *If $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations, then $\mathscr{T}_{\mathbb{U}'\text{-ir}} = \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.*

For $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$, the set $\mathbb{U}^*(B,R)$ is closed under simple transformations.

Lemma 2. *For $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$, $\mathbb{U}^*(B,R)$ is closed under simple transformations.*

For the proof of Lemma 2, the reader is referred to Section 4.3.

Combining Theorem 1, Theorem 2, and Lemma 2 gives the following result.

Theorem 3. *For $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$, if $\mathbb{U}^*(B,R) \subseteq \mathbb{U}' \subseteq \mathbb{U}$ and $\mathbb{U}'$ is closed under simple transformations, then*

$$\mathscr{T}_{\mathbb{U}'\text{-ir}} = \mathscr{T}_{\mathbb{U}'\text{-bpr}} \subseteq \mathscr{T}_{\mathbb{U}^*(B,R)\text{-ir}} = \mathscr{T}_{\mathbb{U}^*(B,R)\text{-bpr}} \subseteq \mathscr{T}_{\text{m-prog}}$$

and

$$\mathscr{T}_{\text{m-prog}}(B,R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}} = \mathscr{T}_{\mathbb{U}'\text{-bpr}} \Leftrightarrow \mathbb{U}' \subseteq \mathbb{U}(B,R).$$

3. Examples

By Theorem 3 and Lemma 2, for $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$, the progressive tax schedules in $\mathscr{T}_{\text{m-prog}}(B,R)$ are inequality and bipolarization reducing with respect to the class of utility vectors $\mathbb{U}^*(B,R)$.

In this section, we consider two standard families of utility functions and identify subclasses of $\mathbb{U}^*(B,R)$ within these families.

3.1. Greenwood, Hercowitz, and Huffman (GHH) preferences

Consider the family of Greenwood, Hercowitz, and Huffman (GHH) (Greenwood et al., 1988) utility functions:

$$u_{(\sigma,\chi)}(x,l) := \frac{1}{1-\sigma}\left(x - \frac{l^{1+\chi}}{1+\chi}\right)^{1-\sigma},$$

where $1 \neq \sigma > 0$ and $\chi > 0$.

Here, the upper bound on labor, L, is ∞. It is easy to verify that $u_{(\sigma,\chi)} \in \mathcal{U}$.

For the GHH utility function, we have

$$x^{u_{(\sigma,\chi)}}(a,b) = al^{u_{(\sigma,\chi)}}(a,b) + b = a^{1+\frac{1}{\chi}} + b,$$

$$\zeta^{u_{(\sigma,\chi)}}(a,0) = 1 + \frac{1}{\chi} \tag{6}$$

and

$$\zeta^{u_{(\sigma,\chi)}}((1-t)a,b) = \frac{(1+\frac{1}{\chi})((1-t)a)^{1+\frac{1}{\chi}}}{((1-t)a)^{1+\frac{1}{\chi}} + b}. \tag{7}$$

Since $\frac{\partial x^{u_{(\sigma,\chi)}}(a,b)}{\partial a} \geq 0$ and

$$\frac{\partial x^{u_{(\sigma,\chi)}}(a,b)}{\partial \chi} = -\frac{1}{\chi^2}\ln(a)a^{1+\frac{1}{\chi}},$$

the following is a sufficient condition for a vector of utilities $(u_{(\sigma_1,\chi_1)},\ldots,u_{(\sigma_n,\chi_n)})$ to be a member of $\mathbb{U}$:

$$\chi_1 \geq \cdots \geq \chi_n. \tag{8}$$

For $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$, the class $\mathbb{U}^*(B,R)$ within the family of GHH utility functions is the set of all vectors of GHH utility functions $(u_{(\sigma_1,\chi_1)},\ldots,u_{(\sigma_n,\chi_n)})$ in $\mathbb{U}$ satisfying the following conditions:

{1} For each $i < n$,

$$\frac{x^{u_{(\sigma_i,\chi_i)}}((1-t)a,b)}{x^{u_{(\sigma_i,\chi_i)}}(a,0)} \geq \frac{x^{u_{(\sigma_{i+1},\chi_{i+1})}}((1-t)a,b)}{x^{u_{(\sigma_{i+1},\chi_{i+1})}}(a,0)}, \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R.$$

{2} For each i,

$$\zeta^{u_{(\sigma_i,\chi_i)}}((1-t)a,b) \leq \zeta^{u_{(\sigma_i,\chi_i)}}(a,0), \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R. \tag{9}$$

First, note that (6)-(7) give (9), and so {2} holds.

Next, note that, given (8), a sufficient condition for {1} to hold is:

$$\frac{\partial\left(\frac{x^{u_{(\sigma,\chi)}}((1-t)a,b)}{x^{u_{(\sigma,\chi)}}(a,0)}\right)}{\partial \chi} \geq 0, \tag{10}$$
$$\text{for all } (a,b,t,\sigma,\chi) \in \mathbb{R}_{++} \times B \times R \times [\mathbb{R}_{++} \setminus \{1\}] \times \mathbb{R}_{++}.$$

This inequality holds if and only if

$$\frac{x^{u_{(\sigma,\chi)}}((1-t)a,b)}{\partial \chi} x^{(\sigma,\chi)}(a,0) \geq \frac{x^{u_{(\sigma,\chi)}}(a,0)}{\partial \chi} x^{u_{(\sigma,\chi)}}((1-t)a,b),$$

i.e., if and only if

$$\xi^{\chi}((1-t)a,b) \geq \xi^{\chi}(a,0), \tag{11}$$

where $\xi^{\chi}(a,b) = \frac{x^{u_{(\sigma,\chi)}}(a,b)}{\partial\chi} \cdot \frac{\chi}{x^{u_{(\sigma,\chi)}}(a,b)}$ is the elasticity of income with respect to χ.

Now, since

$$\xi^{\chi}((1-t)a,b) = -\frac{\ln((1-t)a)((1-t)a)^{1+\frac{1}{\chi}}}{\chi\left(((1-t)a)^{1+\frac{1}{\chi}}+b\right)},$$

and

$$\xi^{\chi}(a,0) = -\frac{\ln(a)}{\chi},$$

and since

$$-\frac{\ln((1-t)a)((1-t)a)^{1+\frac{1}{\chi}}}{\chi\left(((1-t)a)^{1+\frac{1}{\chi}}+b\right)} \geq -\frac{\ln(a)}{\chi}$$
$$\Leftrightarrow \ln((1-t)a)((1-t)a)^{1+\frac{1}{\chi}} \leq \ln(a)(((1-t)a)^{1+\frac{1}{\chi}}+b),$$

it follows that (11) (and hence (10)) holds. Consequently, {1} holds.

We have seen that any vector of utility functions $(u_{(\sigma_1,\chi_1)},\ldots,u_{(\sigma_n,\chi_n)})$ satisfying (8) is a member of $\mathbb{U}^*(B,R)$—and hence a member of $\mathbb{U}(B,R)$ (Lemma 1)—for any $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$.

By Theorem 3 (and Lemma 1 and Lemma 2), the progressive tax schedules in $\mathcal{T}_{\text{m-prog}}(B,R)$ are inequality and bipolarization reducing with respect to $\mathbb{U}^*(B,R) \supseteq \mathbb{U}'$, where $\mathbb{U}'$ denotes the set of utility vectors $(u_{(\sigma_1,\chi_1)},\ldots,u_{(\sigma_n,\chi_n)})$ satisfying (8).

3.2. Constant elasticity of substitution (CES) preferences

In this subsection, we illustrate Theorem 3 for the family of constant elasticity of substitution (CES) utility functions:

$$u_{(\beta,\gamma)}(x,l) := \begin{cases} x^{\gamma}+\beta(1-l)^{\gamma} & \text{if } \gamma \in (0,1), \\ -x^{\gamma}-\beta(1-l)^{\gamma} & \text{if } \gamma < 0. \end{cases}$$

Here, $\beta > 0$ and $\frac{1}{1-\gamma}$ determines the elasticity of substitution between consumption and leisure.

The upper bound on labor, L, is 1. It is easy to verify that $u_{(\beta,\gamma)} \in \mathcal{U}$.

For the CES utility function, we have

$$x^{u_{(\beta,\gamma)}}(a,b) = al^{u_{(\beta,\gamma)}}(a,b)+b = \begin{cases} \frac{\left(\frac{a}{\beta}\right)^{\frac{1}{1-\gamma}}(a+b)}{a+\left(\frac{a}{\beta}\right)^{\frac{1}{1-\gamma}}} & \text{if } \left(\frac{a}{\beta}\right)^{\frac{1}{1-\gamma}} \geq b, \\ b & \text{otherwise,} \end{cases}$$

$$\zeta^{u_{(\beta,\gamma)}}(a,0) = \frac{(1-\gamma)a^{\frac{1}{1-\gamma}}+a\beta^{\frac{1}{1-\gamma}}}{(1-\gamma)a\left(a^{\frac{\gamma}{1-\gamma}}+\beta^{\frac{1}{1-\gamma}}\right)}$$

and

$$\zeta^{u_{(\beta,\gamma)}}((1-t)a,b)=\begin{cases}\frac{(1-\gamma)((1-t)a)^{\frac{1}{1-\gamma}}+(1-t)a\beta^{\frac{1}{1-\gamma}}+b\gamma\beta^{\frac{1}{1-\gamma}}}{(1-\gamma)((1-t)a+b)\left(((1-t)a)^{\frac{\gamma}{1-\gamma}}+\beta^{\frac{1}{1-\gamma}}\right)} & \text{if } \left(\frac{(1-t)a}{\beta}\right)^{\frac{1}{1-\gamma}}\geq b,\\ 0 & \text{otherwise.}\end{cases}$$

Since

$$\frac{\partial x^{u_{(\beta,\gamma)}}(a,b)}{\partial a}=\frac{\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{a}{\beta}\right)^{1/(1-\gamma)}(a+b)}{\left(a+\left(\frac{a}{\beta}\right)^{1/(1-\gamma)}\right)^2}\geq 0 \quad \text{if } \left(\frac{a}{\beta}\right)^{1/(1-\gamma)}>b,$$

$$\frac{\partial x^{u_{(\beta,\gamma)}}(a,b)}{\partial \gamma}=\frac{\frac{a(a+b)}{(1-\gamma)^2}\ln\left(\frac{a}{\beta}\right)\left(\frac{a}{\beta}\right)^{1/(1-\gamma)}}{\left(a+\left(\frac{a}{\beta}\right)^{1/(1-\gamma)}\right)^2}\geq 0 \quad \text{if } \left(\frac{a}{\beta}\right)^{1/(1-\gamma)}>b,$$

and

$$\frac{\partial x^{u_{(\beta,\gamma)}}(a,b)}{\partial \beta}=-\frac{\frac{a(a+b)}{\beta^2(1-\gamma)}\left(\frac{a}{\beta}\right)^{\gamma/(1-\gamma)}}{\left(a+\left(\frac{a}{\beta}\right)^{1/(1-\gamma)}\right)^2}\leq 0 \quad \text{if } \left(\frac{a}{\beta}\right)^{1/(1-\gamma)}>b,$$

it follows that the following is a sufficient condition for a vector of utilities $(u_{(\beta_1,\gamma_1)},\ldots,u_{(\beta_n,\gamma_n)})$ to be a member of $\mathbb{U}$:

$$\gamma_1\leq\cdots\leq\gamma_n \quad \text{and} \quad \beta_1\geq\cdots\geq\beta_n. \tag{12}$$

For $B\subseteq\mathbb{R}_+$ and $R\subseteq[0,1)$, the members of $\mathbb{U}^*(B,R)$ within the CES class are the utility vectors $(u_{(\beta_1,\gamma_1)},\ldots,u_{(\beta_n,\gamma_n)})$ in $\mathbb{U}$ satisfying the following conditions:

{1} For each $i<n$,

$$\frac{x^{u_{(\beta_i,\gamma_i)}}((1-t)a,b)}{x^{u_{(\beta_i,\gamma_i)}}(a,0)}\geq\frac{x^{u_{(\beta_{i+1},\gamma_{i+1})}}((1-t)a,b)}{x^{u_{(\beta_{i+1},\gamma_{i+1})}}(a,0)}, \quad \text{for all } (a,b,t)\in\mathbb{R}_{++}\times B\times R.$$

{2} For each i,

$$\zeta^{u_{(\beta_i,\gamma_i)}}((1-t)a,b)\leq\zeta^{u_{(\beta_i,\gamma_i)}}(a,0), \quad \text{for all } (a,b,t)\in\mathbb{R}_{++}\times B\times R.$$

Note that, given (12), the following is a sufficient condition for {1} to hold:

$$\frac{\partial\left(\frac{x^{u_{(\beta,\gamma)}}((1-t)a,b)}{x^{u_{(\beta,\gamma)}}(a,0)}\right)}{\partial\gamma}\leq 0 \quad \text{and} \quad \frac{\partial\left(\frac{x^{u_{(\beta,\gamma)}}((1-t)a,b)}{x^{(\beta,\gamma)}(a,0)}\right)}{\partial\beta}\geq 0, \tag{13}$$
$$\text{for all } (a,b,t,\beta,\gamma)\in\mathbb{R}_{++}\times B\times R\times\mathbb{R}_{++}\times[(-\infty,1)\setminus\{0\}].$$

The first inequality holds if and only if

$$\frac{x^{u_{(\beta,\gamma)}}((1-t)a,b)}{\partial\gamma}x^{(\beta,\gamma)}(a,0)\leq\frac{x^{u_{(\beta,\gamma)}}(a,0)}{\partial\gamma}x^{u_{(\beta,\gamma)}}((1-t)a,b),$$

i.e., if and only if

$$\xi^{\gamma}((1-t)a,b) \leq \xi^{\gamma}(a,0) \quad \text{if } \gamma \in (0,1), \tag{14}$$
$$\xi^{\gamma}((1-t)a,b) \geq \xi^{\gamma}(a,0) \quad \text{if } \gamma < 0, \tag{15}$$

where $\xi^{\gamma}(a,b) = \frac{x^{u(\beta,\gamma)}(a,b)}{\partial\gamma} \cdot \frac{\gamma}{x^{u(\beta,\gamma)}(a,b)}$ is the elasticity of income with respect to γ.

Similarly, the second inequality in (13) holds if and only if

$$\xi^{\beta}((1-t)a,b) \geq \xi^{\beta}(a,0), \tag{16}$$

where $\xi^{\beta}(a,b) = \frac{x^{u(\beta,\gamma)}(a,b)}{\partial\beta} \cdot \frac{\beta}{x^{u(\beta,\gamma)}(a,b)}$ is the elasticity of income with respect to β.

Now, since

$$\xi^{\gamma}(a,b) = \begin{cases} \gamma a \ln(a/\beta) \left[a + (a/\beta)^{1/(1-\gamma)}\right] & \text{if } (a/\beta)^{1/(1-\gamma)} \geq b, \\ 0 & \text{otherwise,} \end{cases}$$

and since

$$\begin{aligned}
\xi^{\gamma}((1-t)a,b) &= 0 \leq \gamma a \ln(a/\beta) \left[a + (a/\beta)^{1/(1-\gamma)}\right] = \xi^{\gamma}(a,0) \\
&\qquad \text{if } ((1-t)a/\beta)^{1/(1-\gamma)} < b \text{ and } \gamma \in (0,1), \\
\xi^{\gamma}((1-t)a,b) &= \gamma(1-t)a \ln((1-t)a/\beta) \left[(1-t)a + ((1-t)a/\beta)^{1/(1-\gamma)}\right] \\
&\leq \gamma a \ln(a/\beta) \left[a + (a/\beta)^{1/(1-\gamma)}\right] = \xi^{\gamma}(a,0) \\
&\qquad \text{if } ((1-t)a/\beta)^{1/(1-\gamma)} \geq b \text{ and } \gamma \in (0,1), \\
\xi^{\gamma}((1-t)a,b) &= 0 \geq \gamma a \ln(a/\beta) \left[a + (a/\beta)^{1/(1-\gamma)}\right] = \xi^{\gamma}(a,0) \\
&\qquad \text{if } ((1-t)a/\beta)^{1/(1-\gamma)} < b \text{ and } \gamma < 0, \\
\xi^{\gamma}((1-t)a,b) &= \gamma(1-t)a \ln((1-t)a/\beta) \left[(1-t)a + ((1-t)a/\beta)^{1/(1-\gamma)}\right] \\
&\geq \gamma a \ln(a/\beta) \left[a + (a/\beta)^{1/(1-\gamma)}\right] = \xi^{\gamma}(a,0) \\
&\qquad \text{if } ((1-t)a/\beta)^{1/(1-\gamma)} \geq b \text{ and } \gamma < 0,
\end{aligned}$$

it follows that (14)-(15) hold.

Moreover, since

$$\xi^{\beta}(a,b) = \begin{cases} -\frac{1}{1-\gamma} \left[a + (a/\beta)^{1/(1-\gamma)}\right] & \text{if } (a/\beta)^{1/(1-\gamma)} \geq b, \\ 0 & \text{otherwise,} \end{cases}$$

and since

$$\xi^{\beta}((1-t)a,b)=0\geq-\frac{1}{1-\gamma}\left[a+(a/\beta)^{1/(1-\gamma)}\right]=\xi^{\beta}(a,0)$$
$$\text{if } ((1-t)a/\beta)^{1/(1-\gamma)}<b,$$
$$\xi^{\beta}((1-t)a,b)=-\frac{1}{1-\gamma}\left[(1-t)a+((1-t)a/\beta)^{1/(1-\gamma)}\right]$$
$$\geq-\frac{1}{1-\gamma}\left[a+(a/\beta)^{1/(1-\gamma)}\right]=\xi^{\beta}(a,0)$$
$$\text{if } ((1-t)a/\beta)^{1/(1-\gamma)}\geq b,$$

it follows that (16) holds.

Thus, (12) implies that {1} holds.

Now, to illustrate Theorem 3, let's consider the distributional properties of a progressive, linear tax $T(y)=-b+ty$, where $b>0$ and $t\in[0,1)$.

Theorem 3 (together with Lemma 1 and Lemma 2) implies that the tax schedule T is inequality and bipolarization reducing with respect to the class $\mathbb{U}^*(\{b\},\{t\})$.

Can we identify a CES subclass contained in $\mathbb{U}^*(\{b\},\{t\})$? Given that {1} holds under (12), a vector of utilities $\boldsymbol{u}=(u_{(\beta_1,\gamma_1)},\dots,u_{(\beta_n,\gamma_n)})$ satisfying (12) belongs to $\mathbb{U}^*(\{b\},\{t\})$ if

$$\zeta^{u_{(\beta_i,\gamma_i)}}((1-t)a,b)\leq\zeta^{u_{(\beta_i,\gamma_i)}}(a,0),\quad \text{for each } i \text{ and } a>0.$$

Fix i and $a>0$. If $((1-t)a/\beta_i)^{1/(1-\gamma_i)}<b$, then

$$\zeta^{u_{(\beta_i,\gamma_i)}}((1-t)a,b)=0\leq\frac{(1-\gamma_i)a^{\frac{1}{1-\gamma_i}}+a\beta_i^{\frac{1}{1-\gamma_i}}}{(1-\gamma_i)a\left(a^{\frac{\gamma_i}{1-\gamma_i}}+\beta_i^{\frac{1}{1-\gamma_i}}\right)}=\zeta^{u_{(\beta_i,\gamma_i)}}(a,0).$$

If $((1-t)a/\beta_i)^{1/(1-\gamma_i)}\geq b$, then $\zeta^{u_{(\beta_i,\gamma_i)}}((1-t)a,b)\leq\zeta^{u_{(\beta_i,\gamma_i)}}(a,0)$ if and only if

$$\frac{(1-\gamma_i)((1-t)a)^{\frac{1}{1-\gamma_i}}+(1-t)a\beta_i^{\frac{1}{1-\gamma_i}}+b\gamma_i\beta_i^{\frac{1}{1-\gamma_i}}}{(1-\gamma_i)((1-t)a+b)(((1-t)a)^{\frac{\gamma_i}{1-\gamma_i}}+\beta_i^{\frac{1}{1-\gamma_i}})}\leq\frac{(1-\gamma_i)a^{\frac{1}{1-\gamma_i}}+a\beta_i^{\frac{1}{1-\gamma_i}}}{(1-\gamma_i)a(a^{\frac{\gamma_i}{1-\gamma_i}}+\beta_i^{\frac{1}{1-\gamma_i}})}.$$

Manipulation of this inequality yields

$$\left[(1-t)^{\frac{\gamma_i}{1-\gamma_i}}(1-\gamma_i)a^{\frac{1+\gamma_i}{1-\gamma_i}}+a^{\frac{1}{1-\gamma_i}}\beta_i^{\frac{1}{1-\gamma_i}}((1-t)^{\frac{\gamma_i}{1-\gamma_i}}+1-2\gamma_i)+(1-\gamma_i)a\beta_i^{\frac{2}{1-\gamma_i}}\right]b$$
$$\geq a^{\frac{2-\gamma_i}{1-\gamma_i}}\beta_i^{\frac{1}{1-\gamma_i}}\gamma_i\left(1-t-(1-t)^{\frac{1}{1-\gamma_i}}\right).$$

This inequality must hold for all $a>0$ such that $((1-t)a/\beta_i)^{1/(1-\gamma_i)}\geq b$. Note that if $\gamma_i\in(\frac{1}{2},1)$, as $a\to\infty$, the bracketed term on the left-hand side converges to infinity faster than the right-hand side. Thus, as $a\to\infty$ (for $\gamma_i\in(\frac{1}{2},1)$), the lower bound on b tends to zero. Consequently, there is a large enough $a_i^*>0$ such that, if $b_i\geq((1-t)a_i^*/\beta_i)^{1/(1-\gamma_i)}$, then, for any $a>0$ with $((1-t)a/\beta_i)^{1/(1-\gamma_i)}\geq b_i$, we have $\zeta^{u_{(\beta_i,\gamma_i)}}((1-t)a,b_i)\leq\zeta^{u_{(\beta_i,\gamma_i)}}(a,0)$.

Thus, there exists $b^* \geq 0$ such that, for $b \geq b^*$, any vector of utilities $(u_{(\beta_1,\gamma_1)}, \ldots, u_{(\beta_n,\gamma_n)})$ satisfying (12) belongs to $\mathbb{U}^*(\{b\},\{t\})$.

4. Proofs

4.1. Proof of Lemma 1

Lemma 1 is restated here for the convenience of the reader.

Lemma 1. *For $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$, $\mathbb{U}^*(B,R) \subseteq \mathbb{U}(B,R)$.*

Proof. Fix $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$. Suppose that $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{U}^*(B,R)$. Then, the following conditions are satisfied:

- For each $i < n$,
$$\frac{x^{u_i}((1-t)a,b)}{x^{u_i}(a,0)} \geq \frac{x^{u_{i+1}}((1-t)a,b)}{x^{u_{i+1}}(a,0)}, \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R.$$

- For each i,
$$\zeta^{u_i}((1-t)a,b) \leq \zeta^{u_i}(a,0), \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R.$$

The second inequality can be expressed as

$$\frac{\partial x^{u_i}((1-t)a,b)}{\partial a} \cdot \frac{a}{x^{u_i}((1-t)a,b)} \leq \frac{\partial x^{u_i}(a,0)}{\partial a} \cdot \frac{a}{x^{u_i}(a,0)},$$

or, equivalently, as

$$\frac{\partial x^{u_i}((1-t)a,b)}{\partial a} x^{u_i}(a,0) - \frac{\partial x^{u_i}(a,0)}{\partial a} x^{u_i}((1-t)a,b) \leq 0.$$

Consequently, for each i,

$$\frac{\partial (x^{u_i}((1-t)a,b)/x^{u_i}(a,0))}{\partial a} \leq 0, \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R,$$

implying that

$$\frac{x^{u_i}((1-t)a,b)}{x^{u_i}(a,0)} \geq \frac{x^{u_i}((1-t)a',b)}{x^{u_i}(a',0)}, \quad \text{for all } (a,a',b,t) \in \mathbb{R}^2_{++} \times B \times R \text{ with } a' \geq a.$$

This condition, combined with that in the first bullet point above, yields, for each $i < n$,

$$\frac{x^{u_i}((1-t)a,b)}{x^{u_i}(a,0)} \geq \frac{x^{u_{i+1}}((1-t)a',b)}{x^{u_{i+1}}(a',0)}, \quad \text{for all } (a,a',b,t) \in \mathbb{R}^2_{++} \times B \times R \text{ with } a' \geq a,$$

implying that $\boldsymbol{u} \in \mathbb{U}(B,R)$. ∎

4.2. Proof of Theorem 1

The following lemmas will be used in the proof of Theorem 1.

Lemma 3. *For $\mathbb{U}' \subseteq \mathbb{U}$, a tax schedule $T \in \mathscr{T}$ is $\mathbb{U}'$-ir if and only if*

$$\frac{x^{u_1}(a_1, T)}{x^{u_1}(a_1, 0)} \geq \cdots \geq \frac{x^{u_n}(a_n, T)}{x^{u_n}(a_n, 0)}$$

for each ability distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$, and every vector of income functions $(x^{u_1}, \ldots, x^{u_n})$.

Lemma 3 follows from Lemma 3 in Carbonell-Nicolau and Llavador (2018).

Recall that, given $B \subseteq \mathbb{R}_+$ and $R \subseteq [0, 1)$, $\mathscr{T}_{\text{m-prog}}(B, R)$ represents the set of all marginal-rate progressive—i.e., convex—tax schedules whose corresponding subsidies are members of the set B and whose marginal tax rates are taken from R.

The liner tax schedules in $\mathscr{T}_{\text{m-prog}}(B, R)$ take the form $T(y) = -b + ty$ for $b \in B$ and $t \in R$. The set of all linear tax schedules in $\mathscr{T}_{\text{m-prog}}(B, R)$ is denoted by $\mathscr{T}_{\text{lin}}(B, R)$.

Lemma 4. *For $\mathbb{U}' \subseteq \mathbb{U}$,*

$$\mathscr{T}_{\text{m-prog}}(B, R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}} \Leftrightarrow \mathscr{T}_{\text{lin}}(B, R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}}.$$

Lemma 4 was first proven for a single utility function in Carbonell-Nicolau and Llavador (2021b, Theorem 2). The argument used there easily extends to the case of heterogeneous preferences and is therefore omitted.

We are now ready to prove Theorem 1.

Theorem 1. *For $\mathbb{U}' \subseteq \mathbb{U}$, $B \subseteq \mathbb{R}_+$, and $R \subseteq [0, 1)$,*

$$\mathscr{T}_{\mathbb{U}(B,R)\text{-ir}} \subseteq \mathscr{T}_{\mathbb{U}^*(B,R)\text{-ir}} \subseteq \mathscr{T}_{\text{m-prog}} \quad \textit{and} \quad \left[\mathscr{T}_{\text{m-prog}}(B, R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}} \Leftrightarrow \mathbb{U}' \subseteq \mathbb{U}(B, R)\right].$$

Proof. Fix $\mathbb{U}' \subseteq \mathbb{U}$, $B \subseteq \mathbb{R}_+$, and $R \subseteq [0, 1)$. The containment $\mathscr{T}_{\mathbb{U}(B,R)\text{-ir}} \subseteq \mathscr{T}_{\mathbb{U}^*(B,R)\text{-ir}}$ follows from Lemma 1.

To establish the containment $\mathscr{T}_{\mathbb{U}^*(B,R)\text{-ir}} \subseteq \mathscr{T}_{\text{m-prog}}$, we show that $T \notin \mathscr{T}_{\text{m-prog}}$ implies that $T \notin \mathscr{T}_{\mathbb{U}^*(B,R)\text{-ir}}$.

Suppose that $T \notin \mathscr{T}_{\text{m-prog}}$. Let $u(x, l) = x(L - l)$, for $L < \infty$ (for the case when $L = \infty$, see Section 3.1). It is easy to verify that $u \in \mathscr{U}$. Moreover, since, for $a > 0$,

$$\zeta^u((1 - t)a, b) = \begin{cases} 0 & \text{if } aL < b, \\ \frac{aL}{aL+b} & \text{if } aL \geq b, \end{cases}$$

and $\zeta^u(a, 0) = 1$, it follows that

$$\zeta^u((1 - t)a, b) \leq \zeta^u(a, 0), \quad \text{for all } (a, b, t) \in \mathbb{R}_{++} \times B \times R.$$

Now let $\boldsymbol{u} = (u_1, \ldots, u_n)$ be a vector of utility functions with $u_i = u$ for each i. Recall that $\mathbb{U}^*(B, R)$ is the class of all utility vectors $(u_1, \ldots, u_n)$ in $\mathbb{U}$ satisfying the following conditions:

- For each $i < n$,

$$\frac{x^{u_i}((1-t)a,b)}{x^{u_i}(a,0)} \geq \frac{x^{u_{i+1}}((1-t)a,b)}{x^{u_{i+1}}(a,0)}, \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R.$$

- For each i,

$$\zeta^{u_i}((1-t)a,b) \leq \zeta^{u_i}(a,0), \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R.$$

Clearly, the vector $\boldsymbol{u} = (u,\ldots,u)$ satisfies the above two conditions. Moreover, $\boldsymbol{u} \in \mathbb{U}$. Indeed,

$$x^u(a,b) = \begin{cases} b & \text{if } aL < b, \\ \frac{aL+b}{2} & \text{if } aL \geq b, \end{cases}$$

and so, for $a' \geq a > 0$ and $b \in \mathbb{R}_+$,

$$x^u(a,b) \leq x^u(a',b).$$

Hence, $\boldsymbol{u} \in \mathbb{U}$, and so $\boldsymbol{u} \in \mathbb{U}^*(B,R)$.

Next, we show that there exist $a^* > 0$ and $0 < \underline{x} < \overline{x}$ such that

$$x^u(a,T) \leq \underline{x} \quad \text{for all } a < a^*, \tag{17}$$
$$x^u(a,T) \geq \overline{x} \quad \text{for all } a > a^*. \tag{18}$$

Let $I_1,\ldots,I_K$ be the partition of $\mathbb{R}_+$ into intervals such that T is linear on I_k for each k. Because T is not convex, there exist two contiguous intervals,

$$I_k = [\underline{y}, y^*] \quad \text{and} \quad I_{k'} = [y^*, \overline{y}],$$

such that T is concave on $I_k \cup I_{k'}$. Therefore, the restriction of the map

$$y \in \mathbb{R}_+ \mapsto x(y)$$

from pre-tax income y to post-tax income $x(y) = y - T(y)$ to the set $I_k \cup I_{k'}$ can be expressed as follows:

$$x(y) = \begin{cases} \alpha + \beta y & \text{if } y \in I_k, \\ \alpha' + \beta' y & \text{if } y \in I_{k'}, \end{cases}$$

where $\alpha, \alpha' \in \mathbb{R}$, $\alpha > \alpha'$, and $\beta' > \beta > 0$. Note that $x(y) > 0$ if $y > 0$ (since marginal tax rates are less than unity).

The marginal rate of substitution of x for y for an 'a-individual' whose utility function is u is given by

$$MRS_a(x,y) = -\frac{\partial u(x,y/a)/\partial y}{\partial u(x,y/a)/\partial x} = \frac{x}{aL-y}.$$

Note that y is defined only on $[0,aL)$.

Choose a post-tax income function x^u. Because

$$\lim_{a \searrow \underline{y}/L} MRS_a(x(\underline{y}),\underline{y}) = \infty,$$

it follows that there exists $\underline{a} > 0$ such that $x^u(a,T) \leq x(\underline{y}) < x(y^*)$ for all $a \leq \underline{a}$. Let

$$a^* = \inf\{a > 0 : x^u(a,T) > x(y^*)\}.$$

Then $x^u(a,T) > x(y^*)$ for all $a > a^*$ and $x^u(a,t) \leq x(y^*)$ for all $a < a^*$.

We claim that

$$\sup\{x^u(a,T) : a < a^*\} < x(y^*). \tag{19}$$

To see this, note that $\sup\{x^u(a,T) : a < a^*\} = x(y^*)$ implies that there are sequences (a_n) and (y_n) such that $a_n \nearrow a^*$, each y_n is a solution to

$$\max_{y \in [0, a_n L)} u(x(y), y/a_n), \tag{20}$$

$y_n \nearrow y^*$, and

$$x^u(a_n, T) = x(y_n) \to x(y^*).$$

A necessary condition for y_n to solve (20) is

$$MRS_{a_n}(x(y_n), y_n) = \frac{x(y_n)}{a_n L - y_n} = \beta. \tag{21}$$

Let $a^\circ > 0$ be defined by

$$\frac{x(y^*)}{a^\circ L - y^*} = \beta. \tag{22}$$

Since

$$MRS_{a^\circ}(x(y^*), y^*) = \beta < \beta' = \frac{dx(y^*)}{dy},$$

there exists $\hat{y} \in I_{k'}$ such that

$$u(x(y^*), y^*/a^\circ) < u(x(\hat{y}), \hat{y}/a^\circ). \tag{23}$$

From (21) and (22), we see that

$$a_n = \frac{1}{L}\left(\frac{x(y_n)}{\beta} + y_n\right) \to \frac{1}{L}\left(\frac{x(y^*)}{\beta} + y^*\right) = a^\circ$$

(since $y_n \to y^*$ and $x(y^n) \to x(y^*)$). Consequently, since the map

$$(a, y) \in \mathbb{R}_{++} \times \mathbb{R}_+ \mapsto u(x(y), y/a)$$

is continuous and $a_n \to a^\circ$, $y_n \to y^*$, and $x(y_n) \to x(y^*)$,

$$u(x(y_n), y_n/a_n) \to u(x(y^*), y^*/a^\circ) < u(x(\hat{y}), \hat{y}/a^\circ), \tag{24}$$

where the last inequality follows from (23).

Since $u(x(y^*), y^*/a^\circ) < u(x(\hat{y}), \hat{y}/a^\circ)$ (see (24)), we may choose $\varepsilon > 0$ such that

$$u(x(y^*), y^*/a^\circ) + 2\varepsilon < u(x(\hat{y}), \hat{y}/a^\circ).$$

Since $u(x(y_n), y_n/a_n) \to u(x(y^*), y^*/a^\circ)$ (see (24)), there exists N such that

$$u(x(y_n), y_n/a_n) < u(x(y^*), y^*/a^\circ) + \varepsilon, \quad \text{for all } n \geq N.$$

Moreover, since $a_n \to a^\circ$, there exists M such that

$$u(x(\hat{y}), \hat{y}/a^\circ) - \varepsilon < u(x(\hat{y}), \hat{y}/a_n), \quad \text{for all } n \geq M.$$

Consequently, for $n \geq \max\{M, N\}$,

$$u(x(y_n), y_n/a_n) < u(x(\hat{y}), \hat{y}/a_n),$$

implying that y_n does not solve (20), a contradiction.

Thus, (19) holds, and so, because $x^u(a, T) > x(y^*)$ for all $a > a^*$ and $x^u(a, t) \leq x(y^*)$ for all $a < a^*$, there exist $a^* > 0$ and $0 < \underline{x} < \overline{x}$ such that (17)-(18) hold.

Because the map $a \mapsto x^u(a, 0)$ is continuous, for $0 < a < a^* < a'$ with a and a' close enough to a^*, we have

$$\frac{x^u(a, T)}{x^u(a, 0)} < \frac{x^u(a', T)}{x^u(a', 0)}.$$

Hence, since $\boldsymbol{u} \in \mathbb{U}^*(B, R)$, Lemma 3 implies that $T \notin \mathscr{T}_{\mathbb{U}^*(B,R)\text{-ir}}$, as we sought.

It remains to show that

$$\mathscr{T}_{\text{m-prog}}(B, R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}} \Leftrightarrow \mathbb{U}' \subseteq \mathbb{U}(B, R).$$

By Lemma 4, $\mathscr{T}_{\text{m-prog}}(B, R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}}$ is equivalent to $\mathscr{T}_{\text{lin}}(B, R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}}$. By Lemma 3, $\mathscr{T}_{\text{lin}}(B, R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}}$ holds if and only if, for each $T \in \mathscr{T}_{\text{lin}}(B, R)$,

$$\frac{x^{u_i}(a_i, T)}{x^{u_i}(a_i, 0)} \geq \frac{x^{u_{i+1}}(a_{i+1}, T)}{x^{u_{i+1}}(a_{i+1}, 0)}$$

for each $i < n$ and each ability distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of income functions $(x^{u_1}, \ldots, x^{u_n})$, and every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$. Equivalently, $\mathscr{T}_{\text{lin}}(B, R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}}$ holds if and only if

$$\frac{x^{u_i}((1-t)a_i, b)}{x^{u_i}(a_i, 0)} \geq \frac{x^{u_{i+1}}((1-t)a_{i+1}, b)}{x^{u_{i+1}}(a_{i+1}, 0)}, \quad \text{for all } (b, t) \in B \times R,$$

for each $i < n$, each ability distribution $0 < a_1 \leq \cdots \leq a_n$, and every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$. Equivalently, $\mathscr{T}_{\text{lin}}(B, R) \subseteq \mathscr{T}_{\mathbb{U}'\text{-ir}}$ holds if and only if, for each $i < n$,

$$\frac{x^{u_i}((1-t)a_i, b)}{x^{u_i}(a_i, 0)} \geq \frac{x^{u_{i+1}}((1-t)a_{i+1}, b)}{x^{u_{i+1}}(a_{i+1}, 0)}, \quad \text{for all } (a, a', b, t) \in \mathbb{R}^2_{++} \times B \times R \text{ with } a' \geq a,$$

for every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$, i.e., if and only if $\mathbb{U}' \subseteq \mathbb{U}(B, R)$. ∎

4.3. Proof of Lemma 2

Lemma 2. *For $B \subseteq \mathbb{R}_+$ and $R \subseteq [0, 1)$, $\mathbb{U}^*(B, R)$ is closed under simple transformations.*

Proof. Fix $B \subseteq \mathbb{R}_+$ and $R \subseteq [0,1)$. Recall that $\mathbb{U}^*(B,R)$ is the class of all utility vectors $(u_1,\ldots,u_n)$ in $\mathbb{U}$ satisfying the following conditions:

- For each $i < n$,

$$\frac{x^{u_i}((1-t)a,b)}{x^{u_i}(a,0)} \geq \frac{x^{u_{i+1}}((1-t)a,b)}{x^{u_{i+1}}(a,0)}, \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R. \tag{25}$$

- For each i,

$$\zeta^{u_i}((1-t)a,b) \leq \zeta^{u_i}(a,0), \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R. \tag{26}$$

Pick $\boldsymbol{u} = (u_1,\ldots,u_n)$ in $\mathbb{U}^*(B,R)$. Let $\boldsymbol{u}' = (u'_1,\ldots,u'_n)$ be a simple transformation of $\boldsymbol{u}$, i.e., suppose that there exists $i^* < n$ such that

$$\begin{aligned} u'_i &= u_{i^*}, && \text{for each } i \leq i^*, \\ u'_i &= u_{i^*+1}, && \text{for each } i \geq i^*+1. \end{aligned}$$

It is easy to see that $\boldsymbol{u} \in \mathbb{U}$ implies that $\boldsymbol{u}' \in \mathbb{U}$. Moreover, since $u'_i \in \{u_{i^*}, u_{i^*+1}\}$ for each i, and since (26) holds for every i, it follows that

$$\zeta^{u'_i}((1-t)a,b) \leq \zeta^{u'_i}(a,0), \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R.$$

Finally, for $i < i^*$ and $i^*+1 \leq i < n$, we have

$$\frac{x^{u'_i}((1-t)a,b)}{x^{u'_i}(a,0)} = \frac{x^{u'_{i+1}}((1-t)a,b)}{x^{u'_{i+1}}(a,0)}, \quad \text{for all } (a,b,t) \in \mathbb{R}_{++} \times B \times R,$$

and, since (25) holds for every $i < n$,

$$\frac{x^{u'_{i^*}}((1-t)a,b)}{x^{u'_{i^*}}(a,0)} = \frac{x^{u_{i^*}}((1-t)a,b)}{x^{u_{i^*}}(a,0)} \geq \frac{x^{u_{i^*+1}}((1-t)a,b)}{x^{u_{i^*+1}}(a,0)} = \frac{x^{u'_{i^*+1}}((1-t)a,b)}{x^{u'_{i^*+1}}(a,0)}$$

for all $(a,b,t) \in \mathbb{R}_{++} \times B \times R$. Hence, $\boldsymbol{u}' \in \mathbb{U}^*(B,R)$. ■

4.4. Proof of Theorem 2

Theorem 2. *If $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations, then $\mathcal{T}_{\mathbb{U}'\text{-ir}} = \mathcal{T}_{\mathbb{U}'\text{-bpr}}$.*

Proof. Suppose that $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations. First, we prove the containment $\mathcal{T}_{\mathbb{U}'\text{-ir}} \subseteq \mathcal{T}_{\mathbb{U}'\text{-bpr}}$.

Pick $T \in \mathcal{T}_{\mathbb{U}'\text{-ir}}$, $0 < a_1 \leq \cdots \leq a_n$, $\boldsymbol{u} = (u_1,\ldots,u_n) \in \mathbb{U}'$, and a vector of income functions $(x^{u_1},\ldots,x^{u_n})$.

First, suppose that n is odd. Let a_m denote the median ability level. For $i < m$, we have

$$\begin{aligned}\frac{1}{x^{u_m}(a_m,T)}&(x^{u_m}(a_m,T) - x^{u_i}(a_i,T))\\ &= \frac{1}{x^{u_m}(a_m,0)}\left[x^{u_m}(a_m,0) - \left(\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)}\frac{x^{u_m}(a_m,0)}{x^{u_m}(a_m,T)}\right)x^{u_i}(a_i,0)\right]\\ &\leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_m}(a_m,0) - x^{u_i}(a_i,0)),\end{aligned}$$

where the last inequality follows from the inequality

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \geq \frac{x^{u_m}(a_m,T)}{x^{u_m}(a_m,0)},$$

which holds because $a_i \leq a_m$ and T is $\mathbb{U}'$-ir (see Lemma 3).

Similarly, for $i > m$, we have

$$\begin{aligned}\frac{1}{x^{u_m}(a_m,T)}&(x^{u_i}(a_i,T) - x^{u_m}(a_m,T))\\ &= \frac{1}{x^{u_m}(a_m,0)}\left[\left(\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)}\frac{x^{u_m}(a_m,0)}{x^{u_m}(a_m,T)}\right)x^{u_i}(a_i,0) - x^{u_m}(a_m,0)\right]\\ &\leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_i}(a_i,0) - x^{u_m}(a_m,0)),\end{aligned}$$

where the last inequality follows from the inequality

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \leq \frac{x^{u_m}(a_m,T)}{x^{u_m}(a_m,0)},$$

which holds because $a_m \leq a_i$ and T is $\mathbb{U}'$-ir (see Lemma 3).

Because

$$\begin{aligned}\frac{1}{x^{u_m}(a_m,T)}(x^{u_m}(a_m,T) - x^{u_i}(a_i,T)) &\leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_m}(a_m,0) - x^{u_i}(a_i,0)), \quad \text{for } i < n,\\ \frac{1}{x^{u_m}(a_m,T)}(x^{u_i}(a_i,T) - x^{u_m}(a_m,T)) &\leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_i}(a_i,0) - x^{u_m}(a_m,0)), \quad \text{for } i > n,\end{aligned}$$

we see that

$$\frac{x^{u_m}(a_m,T)}{x^{u_m}(a_m,0)}(x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0)) \succcurlyeq_{\mathrm{FW}} (x^{u_1}(a_1,T),\ldots,x^{u_n}(a_n,T)),$$

whence

$$(x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0)) \succcurlyeq_{\mathrm{FW}} (x^{u_1}(a_1,T),\ldots,x^{u_n}(a_n,T)).$$

Consequently, $T \in \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

Next, suppose that n is even. Let

$$m = m(x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0)) = \frac{x^{u_{n/2}}(a_{n/2},0) + x^{u_{(n/2)+1}}(a_{(n/2)+1},0)}{2}$$

and

$$m' = m(x^{u_1}(a_1, T), \ldots, x^{u_n}(a_n, T)) = \frac{x^{u_{n/2}}(a_{n/2}, T) + x^{u_{(n/2)+1}}(a_{(n/2)+1}, T)}{2}.$$

For $i \leq n/2$, we have

$$\begin{aligned}\frac{1}{m'}(m' - x^{u_i}(a_i, T)) &= \frac{1}{m}\left[m - \left(\frac{x^{u_i}(a_i, T)}{x^{u_i}(a_i, 0)}\frac{m}{m'}\right)x^{u_i}(a_i, 0)\right] \\ &\leq \frac{1}{m}(m - x^{u_i}(a_i, 0)),\end{aligned}$$

where the last inequality follows from the inequalities

$$\frac{x^{u_i}(a_i, T)}{x^{u_i}(a_i, 0)} \geq \frac{x^{u_{n/2}}(a_{n/2}, T)}{x^{u_{n/2}}(a_{n/2}, 0)} \geq \frac{m'}{m};$$

the first inequality holds because $a_i \leq a_{n/2}$ and T is $\mathbb{U}'$-ir (see Lemma 3); the second inequality is expressible as

$$\frac{x^{u_{n/2}}(a_{n/2}, T)}{x^{u_{n/2}}(a_{n/2}, 0)} \geq \frac{x^{u_{(n/2)+1}}(a_{(n/2)+1}, T)}{x^{u_{(n/2)+1}}(a_{(n/2)+1}, 0)},$$

which holds because $a_{n/2} \leq a_{(n/2)+1}$ and T is $\mathbb{U}'$-ir (see Lemma 3).

Because

$$\begin{aligned}\frac{1}{m'}(m' - x^{u_i}(a_i, T)) &\leq \frac{1}{m}(m - x^{u_i}(a_i, 0)), &&\text{for } i \leq n/2, \\ \frac{1}{m'}(x^{u_i}(a_i, T) - m') &\leq \frac{1}{m}(x^{u_i}(a_i, 0) - m), &&\text{for } i \geq (n/2) + 1,\end{aligned}$$

we have

$$(x^{u_1}(a_1, 0), \ldots, x^{u_n}(a_n, 0)) \succcurlyeq_{\mathrm{FW}} (x^{u_1}(a_1, T), \ldots, x^{u_n}(a_n, T)).$$

Consequently, $T \in \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

It remains to prove the containment $\mathscr{T}_{\mathbb{U}'\text{-ir}} \supseteq \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

Choose $T \in \mathscr{T}_{\mathbb{U}'\text{-bpr}}$, $0 < a_1 \leq \cdots \leq a_n$, $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{U}'$, and a vector of income functions $(x^{u_1}, \ldots, x^{u_n})$.

First, suppose that n is odd. Pick $i < n$ and a_i, and define an ability distribution $0 < a'_1 \leq \cdots \leq a'_n$ satisfying

$$a'_j = a_i \text{ for each } j \leq i \quad \text{and} \quad a'_j = a_{i+1} \text{ for each } j \geq i + 1.$$

Note that

$$a'_{m-1} = a_i \leq a'_m \leq a_{i+1} = a'_{m+1},$$

where a'_m represents the median ability level. Moreover, either $a'_m = a_i$ or $a'_m = a_{i+1}$. Suppose that $a'_m = a_{i+1}$ (the other case can be handled similarly).

Because $\mathbb{U}'$ is closed under simple transformations, the utility vector $\boldsymbol{u}' = (u'_1, \ldots, u'_n)$, where

$$\begin{aligned} u'_j &= u_i, && \text{for each } j \le i, \\ u'_j &= u_{i+1}, && \text{for each } j \ge i+1, \end{aligned}$$

is a member of $\mathbb{U}'$.

Because T is $\mathbb{U}'$-bpr,

$$\frac{1}{x^{u'_m}(a'_m, T)} \sum_{j=i}^{m} (x^{u'_m}(a'_m, T) - x^{u'_j}(a'_j, T)) \le \frac{1}{x^{u'_m}(a'_m, 0)} \sum_{j=i}^{m} (x^{u'_m}(a'_m, 0) - x^{u'_j}(a'_j, 0)). \tag{27}$$

Since $a'_m = a_{i+1}$ and $a'_j = a_{i+1}$ and $u'_j = u_{i+1}$ for $j \ge i+1$, we have

$$\sum_{j=i+1}^{m} (x^{u'_m}(a'_m, T) - x^{u'_j}(a'_j, T)) = 0 \quad \text{and} \quad \sum_{j=i+1}^{m} (x^{u'_m}(a'_m, 0) - x^{u'_j}(a'_j, 0) = 0.$$

Consequently, (27) can be expressed as

$$\frac{1}{x^{u'_m}(a'_m, T)} (x^{u'_m}(a'_m, T) - x^{u'_i}(a'_i, T)) \le \frac{1}{x^{u'_m}(a'_m, 0)} (x^{u'_m}(a'_m, 0) - x^{u'_i}(a'_i, 0)),$$

whence

$$\frac{x^{u'_i}(a'_i, T)}{x^{u'_i}(a'_i, 0)} \ge \frac{x^{u'_m}(a'_m, T)}{x^{u'_m}(a'_m, 0)}.$$

Now since $u'_i = u_i$, $u'_m = u_{i+1}$, $a'_i = a_i$, and $a'_m = a_{i+1}$, it follows that

$$\frac{x^{u_i}(a_i, T)}{x^{u_i}(a_i, 0)} \ge \frac{x^{u_{i+1}}(a_{i+1}, T)}{x^{u_{i+1}}(a_{i+1}, 0)}.$$

Since $i < n$ was arbitrary, we see that

$$\frac{x^{u_1}(a_1, T)}{x^{u_1}(a_1, 0)} \ge \cdots \ge \frac{x^{u_n}(a_n, T)}{x^{u_n}(a_n, 0)}.$$

Since $0 < a_1 \le \cdots \le a_n$, $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{U}'$, and $(x^{u_1}, \ldots, x^{u_n})$ were arbitrary, Lemma 3 implies that $T \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$.

Next, suppose that n is even. Pick $i < n$ and a_i, and define an ability distribution $0 < a'_1 \le \cdots \le a'_n$ satisfying

$$a'_j = a_i \text{ for each } j \le i \quad \text{and} \quad a'_j = a_{i+1} \text{ for each } j \ge i+1.$$

Because $\mathbb{U}'$ is closed under simple transformations, the utility vector $\boldsymbol{u}' = (u'_1, \ldots, u'_n)$, where

$$\begin{aligned} u'_j &= u_i, && \text{for each } j \le i, \\ u'_j &= u_{i+1}, && \text{for each } j \ge i+1, \end{aligned}$$

is a member of $\mathbb{U}'$.

Note that the income distributions

$$(x^{u'_1}(a'_1, T), \ldots, x^{u'_n}(a'_n, T)) \quad \text{and} \quad (x^{u'_1}(a'_1, 0), \ldots, x^{u'_n}(a'_n, 0))$$

satisfy

$$x^{u'_1}(a'_1, T) = \cdots = x^{u'_i}(a'_i, T) \leq m' \leq x^{u'_{i+1}}(a'_{i+1}, T) = \cdots = x^{u'_n}(a'_n, T), \tag{28}$$

where m' represents the median income for $(x^{u'_1}(a'_1, T), \ldots, x^{u'_n}(a'_n, T))$, and

$$x^{u'_1}(a'_1, 0) = \cdots = x^{u'_i}(a'_i, 0) \leq m \leq x^{u'_{i+1}}(a'_{i+1}, 0) = \cdots = x^{u'_n}(a'_n, 0), \tag{29}$$

where m represents the median income for $(x^{u'_1}(a'_1, 0), \ldots, x^{u'_n}(a'_n, 0))$.

As in the previous case, it suffices to show that

$$\frac{x^{u_i}(a_i, T)}{x^{u_i}(a_i, 0)} \geq \frac{x^{u_{i+1}}(a_{i+1}, T)}{x^{u_{i+1}}(a_{i+1}, 0)}.$$

If

$$x^{u'_1}(a'_1, T) = \cdots = x^{u'_i}(a'_i, T) = m' = x^{u'_{i+1}}(a'_{i+1}, T) = \cdots = x^{u'_n}(a'_n, T), \tag{30}$$

then

$$x^{u'_1}(a'_1, 0) = \cdots = x^{u'_i}(a'_i, 0) = m = x^{u'_{i+1}}(a'_{i+1}, 0) = \cdots = x^{u'_n}(a'_n, 0). \tag{31}$$

Indeed, $x^{u'_i}(a'_i, 0) < x^{u'_{i+1}}(a'_{i+1}, 0)$ implies that $x^{u'_i}(a'_i, T) < x^{u'_{i+1}}(a'_{i+1}, T)$, since marginal tax rates are less than unity. Under (30)-(31), we have

$$\frac{x^{u'_i}(a'_i, T)}{x^{u'_i}(a'_i, 0)} = \frac{x^{u_i}(a_i, T)}{x^{u_i}(a_i, 0)} = \frac{m'}{m} = \frac{x^{u_{i+1}}(a_{i+1}, T)}{x^{u_{i+1}}(a_{i+1}, 0)} = \frac{x^{u'_{i+1}}(a'_{i+1}, T)}{x^{u'_{i+1}}(a'_{i+1}, 0}.$$

If $x^{u'_i}(a'_i, T) < x^{u'_{i+1}}(a'_{i+1}, T)$ then $x^{u'_i}(a'_i, 0) < x^{u'_{i+1}}(a'_{i+1}, 0)$. Thus, if $m' = x^{u'_i}(a'_i, T)$ (resp., $m' = x^{u'_{i+1}}(a'_{i+1}, T)$), then $m = x^{u'_i}(a'_i, 0)$ (resp., $m = x^{u'_{i+1}}(a'_{i+1}, 0)$). We consider the case when $m' = x^{u'_i}(a'_i, T)$ and $m = x^{u'_i}(a'_i, 0)$ and omit the other case, which can be handled similarly.

Suppose that $m' = x^{u'_i}(a'_i, T)$ and $m = x^{u'_i}(a'_i, 0)$. Because T is $\mathbb{U}'$-bpr,

$$\frac{1}{m'} \sum_{\frac{n+1}{2} < j \leq i+1} (x^{u'_j}(a'_j, T) - m') \leq \frac{1}{m} \sum_{\frac{n+1}{2} < j \leq i+1} (x^{u'_j}(a'_j, 0) - m). \tag{32}$$

Given (28)-(29), and since $m' = x^{u'_i}(a'_i, T)$ and $m = x^{u'_i}(a'_i, 0)$, we see that (32) can be expressed as

$$\frac{1}{m'}(x^{u'_{i+1}}(a'_{i+1}, T) - m') \leq \frac{1}{m}(x^{u'_{i+1}}(a'_{i+1}, 0) - m),$$

whence

$$\frac{x^{u'_i}(a'_i, T)}{x^{u'_i}(a'_i, 0)} = \frac{m'}{m} \geq \frac{x^{u'_{i+1}}(a'_{i+1}, T)}{x^{u'_{i+1}}(a'_{i+1}, 0)}.$$

Now since $u_i' = u_i$ and $u_{i+1}' = u_{i+1}$, it follows that

$$\frac{x^{u_i}(a_i, T)}{x^{u_i}(a_i, 0)} \geq \frac{x^{u_{i+1}}(a_{i+1}, T)}{x^{u_{i+1}}(a_{i+1}, 0)},$$

as we sought. ■

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