

Atiyah-MacDonald Introduction to Commutative Algebra Exercises

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1 Rings and Ideals

Exercise 1.1. Let x be a nilpotent element of a ring A . Show that $1 + x$ is a unit of A . Deduce that the sum of a nilpotent element and a unit is a unit.

Solution. If x is a nilpotent element, then $\exists n \in \mathbb{N}$ such that $x^n = 0$. Let's consider the element $y = 1 - x + x^2 - \cdots \pm x^{n-1} \in A$. $(1 + x)y = 1 \pm x^n = 1$, and therefore $1 + x$ is a unit of A .

Let now $x \in A$ be a nilpotent element, and $y \in A$ a unit. Then $\exists n \in \mathbb{N} \mid x^n = 0$ and $\exists a \in A \mid ya = 1$. Then $a(y + x) = ay + ax = 1 + ax$ and we can reduce to the first case, as ax is also a nilpotent element. Therefore $a(y + x)$ is a unit and $\exists b \in A \mid ba(x + y) = 1$ which implies $(x + y)$ is also a unit.

Exercise 1.2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x , with coefficients in A . Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

(i) f is a unit in $A[x] \iff a_0$ is a unit in A and a_1, \dots, a_n are nilpotent.
[If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f , prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Ex. 1]

(ii) f is nilpotent $\iff a_0, a_1, \dots, a_n$ are nilpotent.

(iii) f is a zero-divisor \iff there exists $a \neq 0$ in A such that $af = 0$.
[Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ ($0 \leq r \leq n$).]

(iv) f is said to be primitive if $(a_0, a_1, \dots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\iff f$ and g are primitive.

Solution. (i) $\boxed{\Leftarrow}$ We will proceed by induction on n . If $n = 0$ then $f = a_0$ which is a unit in A and therefore also in $A[x]$, because $A \subset A[x]$. So let's suppose that the statement holds for $n = k$. In the case $n = k + 1$ we have $f = a_0 + a_1x + \cdots + a_kx^k + a_{k+1}x^{k+1}$, where a_{k+1} is nilpotent by hypothesis and $a_0 + a_1x + \cdots + a_kx^k$ is a unit by induction hypothesis. Then $a_{k+1}x^{k+1}$ is also nilpotent and f is a sum of a unit and a nilpotent element, which implies that f is a unit by Exercise 1.1.

$\boxed{\Rightarrow}$ If $f = a_0 \in A$ the statement is clearly true. We can suppose then that $n > 0$. Let $f^{-1} = b_0 + b_1x + \cdots + b_mx^m$. We will first see by

induction on r that $a_n^{r+1}b_{m-r} = 0 \forall r \in \{0, \dots, m\}$.

$$1 = ff^{-1} = \sum_{i=0}^{n+m} \left(\sum_{j=0}^i a_j b^{i-j} \right) x^i$$

Therefore, the term x^{n+m} has coefficient 0 and $a_n b_m = 0$, which proves the base case $r = 0$. Let's assume that the statement holds $\forall k < r$. The coefficient of x^{n+m-r} can be expressed as $\sum_{i=0}^n a_{n-i} b_{m-r+i}$. As we have $n > 0$, and $r \leq m$ then $n + m - r \neq 0$ and we have that

$$\sum_{i=0}^n a_{n-i} b_{m-r+i} = 0$$

Multiplying by a_n^r , all terms of the sumatory except from $i = 0$ vanish by induction hypotesis, which results in

$$a_n^r a_n b_{m-r} = 0 \Rightarrow a_n^{r+1} b_{m-r} = 0$$

In particular, setting $r = m$ we have that $a_n^m f^{-1} = 0 \Rightarrow a_n^m f f^{-1} = 0 \Rightarrow a_n^m = 0$. Then a_n is nilpotent and $-a_n x^n$ is also nilpotent. By exercise 1, $f - a_n x^n = a_0 + a_1 x + \dots + a_{n-1} x^{n-1}$ is a unit and we can repeat the same argument and conclude that a_{n-1} is nilpotent. Repeating that procedure n times we show that a_1, \dots, a_n are nilpotent, and a_0 is a unit.

- (ii) $\boxed{\Leftarrow}$ a_i nilpotent $\Rightarrow a_i x^i$ nilpotent, and the sum of nilpotent elements is also nilpotent by proposition 1.7. Then $f = \sum_{i=0}^n a_i x^i$ is nilpotent.

$\boxed{\Rightarrow}$ f nilpotent $\Rightarrow \exists m \in \mathbb{N}$ such that $f^m = 0$. In consequence, all the coefficients of the polynomial f^m are zero, and in particular, for the term of lowest degree we have $a_0^m = 0 \Rightarrow a_0$ is nilpotent. As the sum of nilpotents is nilpotent, then $f - a_0 = x(a_1 + \dots + a_n x^{n-1})$ is also nilpotent. As x is not a zero divisor, then $a_1 + \dots + a_n x^{n-1}$ is nilpotent and we can apply the same argument, which allows us to conclude that a_1 is nilpotent. Applying the same reasoning, we find that a_0, \dots, a_n are nilpotent in A .

(iii) \Leftarrow By the definition of zero-divisor.

\Rightarrow If f is a zero-divisor then $\exists g \neq 0 \in A[x] \mid fg = 0$. Let's choose g a polynomial of minimum degree such that $fg = 0$, $g = b_0 + \dots + b_m x^m$. We will see by induction that $a_{n-r}g = 0 \forall r \in \{0, \dots, n\}$. The base case $r = 0$ is deduced directly from $fg = 0 \Rightarrow a_n b_m = 0 \Rightarrow a_n g = 0$, as $a_n g$ annihilates f and has degree lower than g . Let's suppose that the statement holds for $r < k - 1$ and prove it for $r = k$.

$$fg = 0 = a_0 g + \dots + a_{n-r} g x^{n-r} + a_{n-(r-1)} g x^{n-(r-1)} + \dots + a_n g x^n$$

By induction hypothesis all the terms $a_{n-(r-i)}g$ vanish and we have

$$0 = a_0 g + \dots + a_{n-r} g x^{n-r}$$

And using the same argument than in base case, we conclude that $a_{n-r}g = 0$.

Then $a_i g = 0 \forall i \in \{0, \dots, n\}$ and therefore $a_i b_m = 0 \forall i$. If $m > 0$ that means

$$fg = f(b_0 + \dots + b_m x^m) = f(b_0 + \dots + b_{m-1} x^{m-1}) = 0$$

That is a contradiction as we have chosen g with lowest degree, so $m = 0$ and $a = b_0$ is the element we were looking for.

(iv) Let $fg = c_0 + c_1 x + \dots + c_{n+m} x^{n+m}$, with $c_k = \sum_{i=0}^k a_i b_{k-i}$.

\Rightarrow $(a_0, \dots, a_n) \supseteq (c_0, \dots, c_{n+m})$ and $(b_0, \dots, b_m) \supseteq (c_0, \dots, c_{n+m})$. Then, if fg is primitive, $(c_0, \dots, c_{n+m}) = (1)$ and therefore f and g are also primitive.

\Leftarrow Suppose that fg is not primitive. Then let $(c_0, \dots, c_{n+m}) = I \subseteq \mathfrak{m}$, with $I \neq A$ an ideal of A , and \mathfrak{m} a maximal ideal of A containing I . The extension of \mathfrak{m} in $A[x]$ is $\mathfrak{m}^e = \mathfrak{m}[x]$ which implies that $A[x]/\mathfrak{m}^e \cong A/\mathfrak{m}[x]$, which is a domain. In $A/\mathfrak{m}[x]$, $\overline{fg} = 0$, because $c_i \in \mathfrak{m} \forall i$. Then, either f or g must be zero. Let's say $\overline{f} = 0$. Then, $(1) = (a_0, \dots, a_n) \subseteq \mathfrak{m}$ which is a contradiction. Then fg must be primitive.

Exercise 1.3. Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \dots, x_n]$ in several indeterminates.

Solution. Let $f = \sum \lambda_I x_1^{i_1} \cdots x_n^{i_n}$, where I is the multiindex $I = (i_1, \dots, i_n)$.

- (i) f is a unit in $A[x_1, \dots, x_n] \iff \lambda_{0, \dots, 0}$ is a unit in A and λ_I is nilpotent $\forall I \neq (0, \dots, 0)$.

We will prove this by induction on the number of indeterminates n . The base case ($n = 1$) is the statement of Exercise 1.2 i). Let's assume that the statement is true for $n - 1$ indeterminates. Then f can be written as $f = \sum_{i=0}^m a_i x_n^i$, with $a_i \in A[x_1, \dots, x_{n-1}]$. By Exercise 1.2, f is a unit $\iff a_0$ is a unit in $A[x_1, \dots, x_{n-1}]$ and a_i is nilpotent $\forall i \neq 0$. We complete the proof applying induction hypothesis on a_0 .

- (ii) f is nilpotent $\iff \lambda_I$ is nilpotent $\forall I$.

Let's proceed by induction on n . The base case ($n = 1$) is the statement of Exercise 1.2 ii). Let's assume that the statement is true for $n - 1$ indeterminates. In the case of n indeterminates, $f = \sum_{i=0}^m a_i x_n^i$, with $a_i \in A[x_1, \dots, x_{n-1}]$. By Exercise 1.2, f is nilpotent $\iff a_i$ is nilpotent $\forall i$. We complete the proof applying induction hypothesis on each a_i .

- (iii) f is a zero-divisor \iff there exists $b \neq 0$ in A such that $bf = 0$.

The inverse implication is obvious. We will use induction again for the direct one. The base case ($n = 1$) is the statement of Exercise 1.2 iii). Let's assume that the statement is true for $n - 1$ indeterminates. In the case of n indeterminates, $f = \sum_{i=0}^m a_i x_n^i$, with $a_i \in A[x_1, \dots, x_{n-1}]$. By Exercise 1.2, f is a zero divisor \iff there exists $a \in A[x_1, \dots, x_{n-1}]$ such that $af = 0$. If we consider af as a polynomial on x_n over the ring $A[x_1, \dots, x_{n-1}]$ we conclude that $af = 0 \iff aa_i = 0 \forall i$ is a zero divisor $\forall i$. By induction hypothesis, that is equivalent to $\exists b_i \in A$ such that $b_i a_i = 0 \forall i$, and the argument used in the proof of 1.2 iii) guarantees that we can take the same element $b = b_i \forall i$.

- (iv) fg is primitive $\iff f$ and g are primitive.

The proof is the same as Exercise 1.2 iv).

Exercise 1.4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution. Let \mathcal{N}, \mathcal{J} denote the nilradical and the Jacobson radical of $A[x]$, respectively. We already know that $\mathcal{N} \subseteq \mathcal{J}$, as $\mathcal{N} = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$ and $\mathcal{J} = \bigcap_{\mathfrak{m} \text{ maximal}} \mathfrak{m}$, and every maximal ideal is prime.

Let's now prove the other inclusion. Let $f(x) \in \mathcal{J} \Rightarrow 1 - fg$ is a unit $\forall g \in A[x] \Rightarrow 1 - xf(x)$ is a unit $\Rightarrow a_0, \dots, a_n$ are nilpotent (we have used Exercise 1.2 i)). Using now 1.2 ii) $\Rightarrow f$ is nilpotent $\Rightarrow f \in \mathcal{N}$.

Exercise 1.5. Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A . Show that

- (i) f is a unit in $A[[x]] \iff a_0$ is a unit in A .
- (ii) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true? (See Chapter 7, Exercise 2.)
- (iii) f belongs to the Jacobson radical of $A[[x]] \iff a_0$ belongs to the Jacobson radical of A .
- (iv) The contraction of a maximal ideal \mathfrak{m} of $A[[x]]$ is a maximal ideal of A , and \mathfrak{m} is generated by \mathfrak{m}^c and x .
- (v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.

Solution. (i) \Rightarrow f is a unit in $A[[x]] \Rightarrow \exists g = \sum_{n=0}^{\infty} b_n x^n \in A[[x]]$ such that $fg = 1 \Rightarrow a_0 b_0 = 1$, which means that a_0 is a unit in A .

\Leftarrow Let $a_0 b_0 = 1$. We want to define $\{b_n\}_{n \geq 0}$ such that

$$\sum_{n=0}^{\infty} b_n x^n \sum_{m=0}^{\infty} a_m x^m = 0$$

Let $\{b_n\}_{n \geq 0}$ be an arbitrary sequence that defines an element $g \in A[[x]]$. Then the term of fg corresponding to exponent k is $c_k = \sum_{i+j=k} a_i b_j$. That allows us to define the desired sequence $\{b_n\}_{n \geq 0}$ in a recursive way.

b_0 is already defined as the inverse of a_0 . We want b_1 satisfying that $a_0 b_1 + b_0 a_1 = 0$, so $a_0 b_1 = -b_0 a_1$, and we can isolate b_1 as a_0 is invertible.

$$b_1 = -b_0^2 a_1$$

The same procediment allows to find b_{k+1} as a function of b_i , $0 \leq i \leq k$.

$$b_k = -b_0 \sum_{i+j=k, i>0} a_i b_j$$

That gives an explicit construction of $g = \sum_{n=0}^{\infty} b_n x^n \in A[[x]]$ such that $fg = 1$, and therefore proves that f is invertible.

- (ii) Let's suppose that $\exists n \in \mathbb{N}$ such that $f^n = 0$. We will prove by induction that a_i is nilpotent $\forall i \in \mathbb{N}$.

$$0 = f^n = \left(\sum_{i=0}^{\infty} a_i x^i \right)^n$$

Then, each coefficient of f^n has to be zero. This condition for the independent term of f^n implies that $a_0^n = 0$ and therefore a_0 is nilpotent, which proves the base case. Let's assume that a_i , $0 \leq i \leq k$ are nilpotent. Then, by Proposition 1.7, $f - \sum_{i=0}^k a_i x^i = x^k \sum_{i=1}^{\infty} a_{i+k} x^i$ is also nilpotent. The same argument used in the base case suffices to prove that a_{k+1} is nilpotent, and the proof is complete. The converse is shown in exercise 7.2.

- (iii) Using Proposition 1.9, f belongs to the Jakobson radical of $A[[x]] \iff 1 - fg$ is a unit $\forall g \in A[[x]]$. By part i) of the exercise, that happens $\iff 1 - a_0 b_0$ is a unit $\forall b_0 \in A$, which happens $\iff a_0$ belongs to the Jakobson radical of A .
- (iv) First, we observe that \mathfrak{m} is a maximal ideal of $A[[x]] \Rightarrow x \in \mathfrak{m}$: Otherwise, $\mathfrak{m} \subset \mathfrak{m} + (x)$, and $\mathfrak{m} + (x) \neq A[[x]]$ as it doesn't contain units: If $1 = f + g$, with $g \in (x)$ and $f \in \mathfrak{m}$, then by part i) of the exercise a_0 is a unit in A , which implies f is a unit and therefore $\mathfrak{m} = A[[x]]$, which is a contradiction.

Let $\mathfrak{m} \in \text{Max}\{A[[x]]\}$. Let's suppose that $\mathfrak{m}^c = \mathfrak{m} \cap A$ is not maximal. Then $\exists a \in A$, $a \notin \mathfrak{m}^c$ such that $\mathfrak{m}^c + (a) \neq (1)$. But $a \notin \mathfrak{m}^c \Rightarrow a \notin \mathfrak{m}$ and therefore $\exists f \in \mathfrak{m}$, $g \in A[[x]]$ such that

$$f + ag = 1$$

That implies $f_0 + ag_0 = 1$. And, recalling that $x \in \mathfrak{m} \Rightarrow a_0 \in \mathfrak{m}^c$, and therefore $\mathfrak{m}^c + (a) = (1)$ which is a contradiction.

Moreover, $\mathfrak{m}^c + (x) \subseteq \mathfrak{m}$, and given $f = a_0 + x \sum_{i=0}^{\infty} a_{i+1}x^i \in \mathfrak{m}$ it's clear that $a_0 \in \mathfrak{m}^c$ and the set equality holds.

- (v) Let $\mathfrak{p} \in \text{Spec}\{A[[x]]\}$, and let's consider the ideal of $A[[x]]$ $\mathfrak{p}' = \mathfrak{p} + (x)$. The same argument used in iv) guarantees that $\mathfrak{p}' \neq A[[x]]$, and it's clear that $(\mathfrak{p}')^c = \mathfrak{p}$. We only need to check that \mathfrak{p}' is prime, which follows from

$$\frac{A[[x]]}{\mathfrak{p}'} \cong \frac{A}{\mathfrak{p}}$$

Exercise 1.6. *A ring A is such that every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.*

Solution. Let's prove the inclusion $\mathcal{J} \subseteq \mathcal{N}$, as the other inclusion is true in general. Let $x \in \mathcal{J}$ and suppose that $x \notin \mathcal{N}$. Then $(x) \not\subseteq \mathcal{N} \Rightarrow \exists e \in (x)$, $e = ax \neq 0$ such that $e^2 = e \Rightarrow (ax)^2 = ax$. That implies that $(1-ax)ax = 0$, but $1-ax$ is a unit as $x \in \mathcal{J} \Rightarrow 1-ax$ is a unit $\forall a \in A$. Therefore $ax = 0$, which is a contradiction coming from the supposition that $x \notin \mathcal{N}$. In conclusion, $\mathcal{N} = \mathcal{J}$.

Exercise 1.7. *Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.*

Solution. Let \mathfrak{p} be a prime ideal. $\forall x \in A \exists n > 0 | x^n = x \Rightarrow x(x^{n-1} - 1) = 0$. Let's reduce this equality modulo \mathfrak{p} . As A/\mathfrak{p} is a domain, every non-zero element satisfies $x^{n-1} = 1$, and in particular every non-zero element is invertible, which means that A/\mathfrak{p} is a field and therefore \mathfrak{p} is maximal.

Exercise 1.8. *Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.*

Solution. Let's consider the set $\text{Spec}\{A\}$ partially ordered by the relation \supseteq . By Theorem 1.3, $A \neq 0 \Rightarrow \text{Spec}\{A\} \neq \emptyset$. Moreover, given a chain of prime ideals

$$\mathfrak{p}_1 \supseteq \mathfrak{p}_2 \supseteq \mathfrak{p}_3 \supseteq \dots$$

the ideal $\bigcap_{i=0}^{\infty} \mathfrak{p}_i$ is an upper bound of the chain. Therefore, $\text{Spec}\{A\}$ with the given order relation satisfies the conditions of Zorn's Lemma, which proves that $\text{Spec}\{A\}$ has minimal elements with respect to inclusion.

Exercise 1.9. Let \mathfrak{a} be an ideal $\neq (1)$ in a ring A . Show that $\mathfrak{a} = r(\mathfrak{a}) \iff \mathfrak{a}$ is an intersection of prime ideals.

Solution. \Rightarrow We know from Proposition 1.14 that

$$r(\mathfrak{a}) = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p} \text{ prime}} \mathfrak{p}$$

Therefore, $\mathfrak{a} = r(\mathfrak{a}) \Rightarrow \mathfrak{a}$ is intersection of prime ideals.

\Leftarrow Let $\mathfrak{a} = \bigcap_i \mathfrak{p}_i$. Let $x \in r(\mathfrak{a})$. Then $\exists n > 0$ such that $x^n \in \mathfrak{a} \Rightarrow x^n \in \mathfrak{p}_i \forall i \Rightarrow x \in \mathfrak{p}_i \forall i \Rightarrow x \in \mathfrak{a}$. In conclusion, $r(\mathfrak{a}) = \mathfrak{a}$.

Exercise 1.10. Let A be a ring, \mathfrak{R} its nilradical. Show that the following are equivalent:

- (i) A has exactly one prime ideal;
- (ii) Every element of A is either a unit or nilpotent;
- (iii) A/\mathfrak{R} is a field.

Solution. $i) \Rightarrow ii)$ Let \mathfrak{p} be the only prime ideal. Then \mathfrak{p} is also maximal and $\mathfrak{R} = \bigcap_{\mathfrak{p}_i \text{ prime}} \mathfrak{p}_i = \mathfrak{p}$. As every non-unit element is contained in a maximal, and \mathfrak{R} is the only maximal ideal, then every non-unit is a nilpotent.

$ii) \Rightarrow iii)$ Given $x \in A$, if $x \in \mathfrak{R}$ then $0 = \bar{x} \in A/\mathfrak{R}$. Otherwise x is invertible. Therefore, every non-zero element in A/\mathfrak{R} is invertible, which means that A/\mathfrak{R} is a field.

$iii) \Rightarrow i)$ A/\mathfrak{R} is a field $\Rightarrow \mathfrak{R} = \bigcap_{\mathfrak{p}_i \text{ prime}} \mathfrak{p}_i$ is maximal. Therefore, there can only be one prime ideal, as otherwise $\mathfrak{R} \subset \mathfrak{p}$, which would contradict \mathfrak{R} maximal.

Exercise 1.11. A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A , show that

- (i) $2x = 0$ for all $x \in A$;

- (ii) every prime ideal \mathfrak{p} is maximal, and A/\mathfrak{p} is a field with two elements;
 (iii) every finitely generated ideal in A is principal.

Solution. (i) As the ring is Boolean, $x + 1 = (x + 1)^2 = x^2 + 2x + 1^2 = x + 1 + 2x \Rightarrow 2x = 0$.

(ii) Let's reduce the equation $x^2 = x$ modulo a prime ideal \mathfrak{p} . $\bar{x}^2 = \bar{x} \Rightarrow \bar{x}(\bar{x} - 1) = 0$. As A/\mathfrak{p} is a domain, then either $\bar{x} = 0$ or $\bar{x} = 1$. In consequence, A/\mathfrak{p} is a field with only two elements.

(iii) It's enough to prove it for an ideal generated by two elements, and the general finitely generated case follows from induction. Let $\mathfrak{a} = (x, y)$, and let's consider the element $z = x + y + xy \in \mathfrak{a}$. We observe that $zx = x^2 + xy + x^2y = x + xy + xy = x$. Analogously, $zy = y$, which implies that $x \in (z)$ and $y \in (z)$, and therefore $\mathfrak{a} = (z)$.

Exercise 1.12. A local ring contains no idempotent $\neq 0, 1$.

Solution. Let x be an idempotent element of a local ring, $x^2 = x \Rightarrow x(x - 1) = 0$. If x is a unit, that is $\exists y$ such that $xy = 1$, then $1 = xy = x^2y = x$ and $x = 1$. Otherwise, Proposition 1.9 forces $1 - x$ to be a unit. Given that $(1 - x)^2 = 1 - 2x + x^2 = 1 - x$ the same argument used with x holds for $1 - x$, which means that $1 - x = 1$ and therefore $x = 0$.

Exercise 1.13. Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K . Let A be the polynomial ring over K generated by indeterminates x_f , one for each $f \in \Sigma$. Let \mathfrak{a} be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Solution.

Exercise 1.14. In a ring A , let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Solution. Let's consider the inclusion order relation in the set Σ . Σ is a non-empty set, as $(0) \in \Sigma$. Then, given a chain of ideals $\mathfrak{a}_i \in \Sigma$

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$$

The set $\bigcup_{i=1}^{\infty} \mathfrak{a}_i$ is an ideal that contains only divisors of zero, and therefore it's an upper bound of the chain. Then, (Σ, \subseteq) satisfy the conditions of Zorn's Lemma, which guarantees the existence of maximal elements of Σ .

Let \mathfrak{p} be a maximal element of Σ . Suppose \mathfrak{p} is not prime. Then, $\exists x, y \notin \mathfrak{p}$ such that $xy \in \mathfrak{p}$. As xy is a zero divisor, $\exists z \neq 0$ such that $xyz = 0$, which implies that either x , y or both are zero divisors. Without loss of generality we suppose that x is a zero divisor, and we consider the ideal $\mathfrak{p}' = (x) + \mathfrak{p}$. Every element $a \in \mathfrak{p}'$ is a zero divisor, as $ay \in \mathfrak{p}$, and as $\mathfrak{p} \subset \mathfrak{p}'$, that is a contradiction with the maximality of \mathfrak{p} in Σ .

Therefore, every maximal element of Σ is a prime ideal and hence the set of zero-divisors in A is a union of prime ideals (the maximal elements of Σ).

Exercise 1.15. *Let A be a ring and let X be the set of all prime ideals of A . For each subset E of A , let $V(E)$ denote the set of all prime ideals of A which contain E . Prove that*

(i) *if \mathfrak{a} is the ideal generated by E , then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.*

(ii) $V(0) = X$, $V(1) = \emptyset$.

(iii) *if $(E_i)_{i \in I}$ is any family of subsets of A , then*

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i)$$

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals \mathfrak{a} , \mathfrak{b} of A .

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the Zariski topology. The topological space X is called the prime spectrum of A , and is written $\text{Spec}(A)$.

Solution. (i) If \mathfrak{p} is a prime ideal such that $E \subseteq \mathfrak{p} \Rightarrow \mathfrak{a} \subseteq \mathfrak{p}$, as \mathfrak{a} is the smallest ideal containing E . On the other hand, it's clear that $\mathfrak{a} \supseteq E$, and then $\mathfrak{p} \supseteq \mathfrak{a} \Rightarrow \mathfrak{p} \supseteq E$. In conclusion, $V(E) = V(\mathfrak{a})$.

Let's now prove the other equality. On one hand, from $r(\mathfrak{a}) \supseteq \mathfrak{a}$ follows the inclusion $V(\mathfrak{a}) \supseteq V(r(\mathfrak{a}))$. On the other hand, as

$$r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}$$

it's clear that every prime ideal containing \mathfrak{a} will also contain $r(\mathfrak{a})$, which proves the inclusion $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$.

(ii) 0 is contained in every ideal $\Rightarrow V(0) = X$, and 1 is not contained in any prime ideal $\Rightarrow V(1) = \emptyset$.

(iii) $\mathfrak{p} \in V(\bigcup_{i \in I} E_i) \iff \bigcup_{i \in I} E_i \subseteq \mathfrak{p} \iff E_i \subseteq \mathfrak{p} \forall i \iff \mathfrak{p} \in V(E_i) \forall i \iff \mathfrak{p} \in \bigcap_{i \in I} V(E_i)$.

(iv) $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b}) \Rightarrow \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$. As \mathfrak{p} is prime, by Proposition 1.11 either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, which implies that $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$ and $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. That proves the inclusions $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ and $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.

Let $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. $x \in \mathfrak{a} \cap \mathfrak{b} \Rightarrow x^2 \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p} \Rightarrow x \in \mathfrak{p}$. Therefore, $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$, which proves the inclusion $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$.

Let $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. Without loss of generality let's suppose that $\mathfrak{p} \supseteq \mathfrak{a} \Rightarrow \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a} \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$. This proves the inclusion $V(\mathfrak{a} \cap \mathfrak{b}) \supseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.

Exercise 1.16. Draw pictures of $\text{Spec}(\mathbb{Z})$, $\text{Spec}(\mathbb{R})$, $\text{Spec}(\mathbb{C}[x])$, $\text{Spec}(\mathbb{R}[x])$, $\text{Spec}(\mathbb{Z}[x])$.

Solution. (i) The prime ideals of \mathbb{Z} are (0) and (p) , with p prime. Any finite set of primes p_1, \dots, p_n correspond to a set in $\text{Spec}(\mathbb{Z})$, $Y = \{(p_1), \dots, (p_n)\}$. Let's consider the ideal $\mathfrak{a} = (p_1 p_2 \cdots p_n)$, and clearly $\mathfrak{a} \subseteq (p_i) \forall i$, and by the uniqueness of prime decomposition $\nexists \mathfrak{q} \neq (p_i)$ prime such that $\mathfrak{a} \subseteq \mathfrak{q}$, which implies that $V(\mathfrak{a}) = Y$. As every ideal is principal the inverse reasoning also holds. In conclusion, the closed sets of the topology are $\text{Spec}(\mathbb{Z})$, \emptyset and all finite subsets of $\text{Spec}(\mathbb{Z})$ not containing (0) .

(ii) \mathbb{R} has only one prime ideal (0) , so $\text{Spec}(\mathbb{R})$ is a topological space with only one point.

(iii) In $\mathbb{C}[x]$ every element factorizes as a product of linear factors. Therefore, the prime ideals are $(x - \alpha)$, $\forall \alpha \in \mathbb{C}$, and (0) . Then, $\text{Spec}(\mathbb{C}[x])$ can be identified with the complex plane and an extra point corresponding to (0) . By the same reasoning used for the case of \mathbb{Z} , the closed sets of the topology are all finite subsets of \mathbb{C} , $\text{Spec}(\mathbb{C}[x])$ and \emptyset .

- (iv) $\mathbb{R}[x]$ is a principal ideal domain, and therefore the prime ideals are those generated by an irreducible element. Irreducible polynomials are (0) , $x - a$, $\forall a \in \mathbb{R}$ and $x^2 + bx + c$, with $b^2 - 4c < 0$. The closed subsets are again all finite sets of prime ideals not containing (0) .
- (v) The prime ideals of $\mathbb{Z}[x]$ are (0) , (p) , with $p \in \mathbb{Z}$ prime, (f) , with f an irreducible polynomial and (p, f) , with $p \in \mathbb{Z}$ prime and $f \in \mathbb{Z}[x]$ irreducible.

Exercise 1.17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \text{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

- (i) $X_f \cap X_g = X_{fg}$
- (ii) $X_f = \emptyset \iff f$ is nilpotent
- (iii) $X_f = X \iff f$ is a unit
- (iv) $X_f = X_g \iff r((f)) = r((g))$
- (v) X is quasi-compact (every open covering of X has a finite subcovering)
- (vi) More generally, each X_f is quasi-compact
- (vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f

The sets X_f are called basic open sets of $X = \text{Spec}(A)$.

Solution. The sets X_f are open because their complementaries are closed sets. Let's first prove that X_f are a basis of the topology, that is, $\forall U \in \mathcal{T}$, $U = \bigcup_i X_{f_i}$. Indeed, $U \in \mathcal{T} \Rightarrow U = X \setminus V(E)$ for a certain E . Then $U = \{\mathfrak{p} \text{ such that } E \not\subseteq \mathfrak{p}\} = \{\mathfrak{p} \text{ such that } \exists f \in E, f \notin \mathfrak{p}\}$.

$$U = \bigcup_{f \in E} X \setminus V(f) = \bigcup_{f \in E} X_f$$

- (i) $\mathfrak{p} \in X_f \cap X_g \Rightarrow f \notin \mathfrak{p}, g \notin \mathfrak{p}$. As \mathfrak{p} is prime, that implies $fg \notin \mathfrak{p} \Rightarrow \mathfrak{p} \in X_{fg}$, which proves the inclusion $X_f \cap X_g \subseteq X_{fg}$.

Conversely, $\mathfrak{p} \in X_{fg} \Rightarrow fg \notin \mathfrak{p}$, and therefore $\mathfrak{p} \in X_f$, $\mathfrak{p} \in X_g$, which proves the inclusion $X_f \cap X_g \supseteq X_{fg}$.

- (ii) $X_f = \emptyset \iff V(f) = \text{Spec}(A) \iff f \in \mathfrak{p} \forall \mathfrak{p} \text{ prime} \iff f \in \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p} = \mathcal{N} \iff f \text{ is nilpotent.}$
- (iii) $X_f = X \iff V(f) = \emptyset \iff \nexists \mathfrak{p} \text{ prime such that } f \in \mathfrak{p}.$ It's clear that units satisfy this property, as $f \text{ unit} \Rightarrow (f) = A$. Conversely, if f is not unit, $f \in \mathfrak{m}$ maximal (and in particular prime), by Theorem 1.3. In conclusion, $X_f = X \iff f \text{ is a unit.}$
- (iv) It is immediate from the characterization $r((f)) = \bigcap_{f \in \mathfrak{p} \text{ prime}} \mathfrak{p}.$
- (v) Let's consider an open covering of X . As we have shown that X_f are a basis of the topology, without loss of generality we can consider a basic covering

$$\begin{aligned} X &= \bigcup_{i \in I} X_{f_i} = \bigcup_{i \in I} X \setminus V(f_i) = X \setminus \bigcap_{i \in I} V(f_i) \iff \\ &\iff \bigcap_{i \in I} V(f_i) = \emptyset \iff V\left(\bigcup_{i \in I} f_i\right) = \emptyset \iff (f_i)_i = A \end{aligned}$$

That means that $1 = \sum_{i \in J} g_i f_i$, for a certain J with $\#J < \infty$. Following the implications in opposite direction, we conclude that $\{X_{f_i}\}_{i \in J}$ covers X , that is, we have found a finite subcovering and we can conclude that X is quasi-compact.

- (vi) The reasoning is the same as in the previous part.

$$X_f = \bigcup_{i \in I} X_{f_i} = X \setminus \bigcap_{i \in I} V(f_i) \Rightarrow V\left(\bigcup_{i \in I} f_i\right) = V(f)$$

which means that $f = \sum_{i \in J} g_i f_i$, for a certain J with $\#J < \infty$, and therefore $\{X_{f_i}\}_{i \in J}$ covers X_f .

- (vii) \Rightarrow It follows from the definition of quasi-compactness.

\Leftarrow $U = \bigcup_{i=1}^n X_{f_i}$. As X_f are a basis of the topology, given any open covering of $U = \bigcup_{j \in J} V_j$ each V_j will be the union of some X_{f_i} , and therefore we can extract a finite covering.

Exercise 1.18. For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \text{Spec}(A)$. When thinking of x as a prime ideal of A , we denote it by \mathfrak{p}_x (logically, of course, it is the same thing). Show that

- (i) The set $\{x\}$ is closed in $\text{Spec}(A) \iff \mathfrak{p}_x$ is maximal.
- (ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$.
- (iii) $y \in \overline{\{x\}} \iff \mathfrak{p}_x \subseteq \mathfrak{p}_y$
- (iv) X is a T_0 space (this means that if x, y are distinct points of X , then either there is a neighborhood of x which does not contain y , or else there is a neighborhood of y which does not contain x).

Solution. (i) $\{x\}$ is closed in $\text{Spec}(A) \iff x = V(\mathfrak{p}_x) \iff \nexists \mathfrak{q}$ prime such that $\mathfrak{q} \supset \mathfrak{p}_x \Rightarrow \mathfrak{p}_x$ is maximal.

(ii) (\mathfrak{p}_x) is closed and contains x . Given $V(\mathfrak{p})$ a closed set containing x , $\mathfrak{p}_x \supseteq \mathfrak{p} \Rightarrow V(\mathfrak{p}_x) \subseteq V(\mathfrak{p})$, so $V(\mathfrak{p})$ also contains $V(\mathfrak{p}_x)$, and therefore $\{x\} = V(\mathfrak{p}_x)$.

(iii) $y \in \overline{\{x\}} \iff y \in V(\mathfrak{p}_x) \iff \mathfrak{p}_x \subseteq \mathfrak{p}_y$.

(iv) Let x, y be distinct points of X , then either $\exists f \in \mathfrak{p}_x$ such that $f \notin \mathfrak{p}_y$, or $\exists f \in \mathfrak{p}_y$ such that $f \notin \mathfrak{p}_x$. Let's suppose we have the first case (the other is symmetric). Therefore, $x \notin X_f$ but $y \in X_f$, and therefore X_f is the neighborhood we were looking for.

Exercise 1.19. A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X . Show that $\text{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Solution. We already know by problem 1.17 that $X_f = \emptyset \iff f$ belongs to the nilradical.

\Leftarrow Let's suppose that X is not irreducible, that is $\exists U = \bigcup_i X_{f_i} \neq \emptyset, V = \bigcup_j X_{g_j} \neq \emptyset$ two open sets such that $U \cap V = \emptyset$. Without loss of generality, we can take $X_{f_i}, X_{g_j} \neq \emptyset \forall i, j$. Then, $\bigcup_{i,j} X_{f_i} \cap X_{g_j} = \emptyset \Rightarrow X_{f_i} \cap X_{g_j} = \emptyset \forall i, j$. As the nilradical \mathfrak{p}_N is prime, $f \notin \mathfrak{p}_N \Rightarrow \mathcal{N} \in X_f$. Therefore, either f_i or g_j belong to the nilradical $\Rightarrow X_{f_i}$ or $X_{g_j} = \emptyset$, which is a contradiction.

\Rightarrow $fg \in \mathcal{N} \Rightarrow X_{fg} = \emptyset$, and $X_{fg} = X_f \cap X_g$. As $\text{Spec}(A)$ is irreducible \Rightarrow either $X_f = \emptyset$ or $X_g = \emptyset$, that is, either $f \in \mathcal{N}$ or $g \in \mathcal{N}$, which proves that \mathcal{N} is prime.

Exercise 1.20. Let X be a topological space.

1. If Y is an irreducible (Exercise 19) subspace of X , then the closure \overline{Y} of Y in X is irreducible.
2. Every irreducible subspace of X is contained in a maximal irreducible subspace.
3. The maximal irreducible subspaces of X are closed and cover X . They are called the irreducible components of X . What are the irreducible components of a Hausdorff space?
4. If A is a ring and $X = \text{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where \mathfrak{p} is a minimal prime ideal of A (Exercise 8).

Solution. Let X be a topological space.

- (i) Let $U_1, U_2 \neq \emptyset$ be open subsets of \overline{Y} . By definition of closure, $U_i \cap Y \neq \emptyset$. Therefore, as Y is irreducible, $(U_1 \cap Y) \cap (U_2 \cap Y) \neq \emptyset$. In particular, $U_1 \cap U_2 \neq \emptyset$, which proves that \overline{Y} is irreducible.
- (ii) Let Σ be the set of all irreducible subspaces. We observe that the singletons $\{x\}$ are irreducible subspaces, and therefore $\Sigma \neq \emptyset$. Suppose we have a chain of subspaces $X_i \in \Sigma$, $X_1 \subseteq X_2 \subseteq \dots$. We will prove that $\bigcup_i X_i$ is also irreducible, and therefore the conditions of Zorn's Lemma apply to Σ and the existence of maximal elements is proven.
Let $U_1, U_2 \neq \emptyset$ be open subsets of $\bigcup_i X_i$, and $x_i \in U_i$. Therefore, $\exists k, l$ such that $x_1 \in X_k$ and $x_2 \in X_l$. Therefore, $(U_i \cap X_{\max\{k,l\}}) \neq \emptyset$ and are open subsets of $X_{\max\{k,l\}}$, and therefore $(U_1 \cap X_{\max\{k,l\}}) \cap (U_2 \cap X_{\max\{k,l\}}) \neq \emptyset$. In particular, $U_1 \cap U_2 \neq \emptyset$, which proves that $\bigcup_i X_i$ is irreducible.
- (iii) Let Y be a maximal irreducible subspace. As $Y \subseteq \overline{Y}$ which is also irreducible by i), we must have $Y = \overline{Y}$, or otherwise we would reach a contradiction with the maximality of Y . This proves that Y is closed.

Now, let's consider the case of a Hausdorff space, where $\forall x, y \exists U_x, U_y$ with $x \in U_x, y \in U_y$ such that $U_x \cap U_y = \emptyset$, which means that $x, y \notin$ the same irreducible component. Therefore, the maximal irreducible components are the points of the space.

- (iv) First, the closed sets $V(\mathfrak{p})$, where \mathfrak{p} is a minimal prime ideal of A are irreducible, because all open sets will be $U = V(\mathfrak{p}) \setminus V(E)$, and therefore $\mathfrak{p} \in$ all non-empty open sets. They're also maximal, as otherwise, $\exists \mathfrak{a}$ such that $V(\mathfrak{p}) \subseteq V(\mathfrak{a}) \iff \mathfrak{a} \subseteq \mathfrak{p}$ and $\exists \mathfrak{p}_1 \in V(\mathfrak{a}), \mathfrak{p}_2 \notin V(\mathfrak{p})$, which implies that $\mathfrak{p}_1 \subset \mathfrak{p}$, a contradiction with the minimality of \mathfrak{p} .

On the other hand, given an irreducible component Y , Y is closed by ii), and therefore $Y = V(\mathfrak{a}) = V(r(\mathfrak{a}))$. Let's prove that $r(\mathfrak{a})$ must be a minimal prime ideal. First of all, let's see that if $r(\mathfrak{a})$ is prime, it must be a minimal prime. Otherwise, $\exists \mathfrak{p}_{min} \subset r(\mathfrak{a})$ minimal prime ideal, and therefore $V(\mathfrak{p}_{min}) \supset V(\mathfrak{a})$, which contradicts the maximality of $V(\mathfrak{a})$.

Now let's prove that $r(\mathfrak{a})$ must be prime. Let $fg \in r(\mathfrak{a}) \Rightarrow fg \in \mathfrak{p}_x \forall x \in V(\mathfrak{a})$, and therefore $X_{fg} = X_f \cap X_g = \emptyset$ by exercise 17. The irreducibility of $V(\mathfrak{a})$ implies that either X_f or X_g are empty, which ensures that either f or $g \in r(\mathfrak{a})$, and therefore $r(\mathfrak{a})$ is prime.

Exercise 1.21. Let $\phi : A \rightarrow B$ be a ring homomorphism. Let $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A , i.e., a point of X . Hence ϕ induces a mapping $\phi^* : Y \rightarrow X$. Show that

- (i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous.
- (ii) If \mathfrak{a} is an ideal of A , then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.
- (iii) If \mathfrak{b} is an ideal of B , then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^e)$.
- (iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker(\phi))$ of X . (In particular, $\text{Spec}(A)$ and $\text{Spec}(A/\mathfrak{R})$ (where \mathfrak{R} is the nilradical of A) are naturally homeomorphic.)
- (v) If ϕ is injective, then $\phi^*(Y)$ is dense in X . More precisely, $\phi^*(Y)$ is dense in $X \iff \text{Ker}(\phi) \subseteq \mathfrak{R}$.
- (vi) Let $\psi : B \rightarrow C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

(vii) Let A be an integral domain with just one non-zero prime ideal \mathfrak{p} , and let K be the field of fractions of A . Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \rightarrow B$ by $\phi(x) = (\bar{x}, x)$, where \bar{x} is the image of x in A/\mathfrak{p} . Show that ϕ^* is bijective but not a homeomorphism.

Solution. (i) $y \in \phi^{*-1}(X_f) \iff \phi^*(y) \in X_f \iff f \notin \phi^{-1}(\mathfrak{p}_y) \iff \phi(f) \notin \mathfrak{p}_y \iff y \in Y_{\phi(f)}$. Therefore, $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and as X_f is a base of the topology, the antiimage of an arbitrary open subset is open, which proves that ϕ^* is continuous.

(ii) $y \in \phi^{*-1}(V(\mathfrak{a})) \iff \phi^*(y) \in V(\mathfrak{a}) \iff \phi^{-1}(\mathfrak{p}_y) \supseteq \mathfrak{a} \iff \mathfrak{p}_y \supseteq \mathfrak{a}^e \iff y \in V(\mathfrak{a}^e)$. Per tant, $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

(iii) $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{a})$, for some ideal \mathfrak{a} yet to determine. We observe that $x \in \phi^*(V(\mathfrak{b})) \iff \exists y \in \text{Spec}(B)$ such that $\mathfrak{p}_y \supseteq \mathfrak{b}$ and $\mathfrak{p}_x = \mathfrak{p}_y^c$. Then, it's clear that $\mathfrak{a} \subseteq \mathfrak{p}^c$, $\forall \mathfrak{p} \supseteq \mathfrak{b}$, which implies that $\mathfrak{a} \subseteq \bigcap_{\mathfrak{p} \supseteq \mathfrak{b}} \mathfrak{p}^c$. As the closure is the smallest closed subset containing $\phi^*(V(\mathfrak{b}))$, we have the equality

$$\overline{\phi^*(V(\mathfrak{b}))} = V\left(\bigcap_{\mathfrak{q} \supseteq \mathfrak{b}} \mathfrak{q}^c\right)$$

Using now Exercise 1.18 of the theory section, and Problem 1.15 i) we have that

$$V\left(\bigcap_{\mathfrak{q} \supseteq \mathfrak{b}} \mathfrak{q}^c\right) = V\left(\left(\bigcap_{\mathfrak{q} \supseteq \mathfrak{b}} \mathfrak{q}\right)^c\right) = V(r(\mathfrak{b})^c) = V(r(\mathfrak{b}^c)) = V(\mathfrak{b}^c)$$

(iv) If ϕ is surjective, then $A/\ker(\phi) \cong B$, and therefore \exists a bijective correspondence between prime ideals of B and prime ideals of $A/\ker(\phi)$, which correspond to prime ideals of A containing $\ker(\phi)$. That proves $Y \xrightarrow{\phi^*} V(\ker(\phi))$.

Therefore, we already know that $\phi^* : Y \rightarrow V(\ker(\phi))$ is bijective and continuous. Now we have to prove that the inverse $\phi^{*-1} : V(\ker(\phi)) \rightarrow Y$ is also continuous. Let $V(\mathfrak{b})$ be an arbitrary closed set of Y . Let's check that its antiimage by ϕ^{*-1} is also closed. Indeed, $\mathfrak{p} \in V(\mathfrak{b}) \iff \mathfrak{p} \supseteq \mathfrak{b} \iff (\phi^{*-1})^{-1}(\mathfrak{p}) \supseteq (\phi^{*-1})^{-1}(\mathfrak{b}) \Rightarrow \phi^{*-1}(\mathfrak{p}) \supseteq \mathfrak{b}^c \iff \mathfrak{p}^c \in V(\mathfrak{b}^c) \Rightarrow (\phi^{*-1})^{-1}(V(\mathfrak{b})) = V(\mathfrak{b}^c)$.

(v) $\phi^*(Y)$ is dense in $X \iff \overline{\phi^*(Y)} = X$. We also know that

$$\overline{\phi^*(Y)} = \overline{\phi^*(V((0)))} = V((0)^c) = V(\ker(\phi))$$

Therefore, we only have to show that $V(\ker(\phi)) = X \iff \ker(\phi) \subseteq \mathfrak{R}$.
That is true because $V(\ker(\phi)) = X \iff \mathfrak{p} \supseteq \ker(\phi) \forall \mathfrak{p} \text{ prime} \iff \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p} \supseteq \ker(\phi) \iff \ker(\phi) \subseteq \mathfrak{R}$.

(vi) Let $\mathfrak{p} \in \text{Spec}(C)$. Then $(\psi \circ \phi)^*(\mathfrak{p}) = (\psi \circ \phi)^{-1}(\mathfrak{p}) = \phi^{-1}(\psi^{-1}(\mathfrak{p})) = \phi^*(\psi^*(\mathfrak{p})) = (\phi^* \circ \psi^*)(\mathfrak{p})$.

(vii) The prime ideals of $\text{Spec}(B)$ are $\mathfrak{p}_1 = (0, K)$ and $\mathfrak{p}_2 = (A/\mathfrak{p}, 0)$. We have that $\phi^*(\mathfrak{p}_1) = \mathfrak{p}$ and $\phi^*(\mathfrak{p}_2) = (0)$, and therefore ϕ^* is clearly bijective. However, we will prove that ϕ^{*-1} is not continuous, as $(\phi^{*-1})^{-1}(\mathfrak{p}_2) = (0)$ which is not a closed subset, while \mathfrak{p}_2 is closed.

2 Modules

Exercise 2.1. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Solution. By Bézout's Identity, we know that, as m, n are coprime, $\exists a, b \in \mathbb{Z}$ such that $am + bn = 1$. Then, $\forall x \otimes y \in (\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$ we have that $x \otimes y = 1(x \otimes y) = (am + bn)(x \otimes y) = am(x \otimes y) + bn(x \otimes y) = amx \otimes y + x \otimes bny = 0$.

Exercise 2.2. Let A be a ring, \mathfrak{a} an ideal, M an A -module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.

Solution. Let's tensorialize the exact sequence $0 \rightarrow \mathfrak{a} \xrightarrow{\text{incl}} A \xrightarrow{\pi} A/\mathfrak{a} \rightarrow 0$, and we obtain the sequence

$$0 \rightarrow \mathfrak{a} \otimes M \xrightarrow{\text{incl} \otimes 1} A \otimes M \xrightarrow{\pi \otimes 1} A/\mathfrak{a} \otimes M \rightarrow 0$$