Atiyah-MacDonald Introdution to Commutative Algebra Exercises

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September 25, 2020

Contents

1 Rings and Ideals

Exercise 1.1. *Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nipotent element and a unit is a unit.*

Solution. If x is a nilpotent element, then $\exists n \in \mathbb{N}$ such that $x^n = 0$. Let's consider the element $y = 1 - x + x^2 - \cdots \pm x^{n-1} \in A$. $(1+x)y = 1 \pm x^n = 1$, and therefore $1+x$ is a unit of A.

Let now $x \in A$ be a nilpotent element, and $y \in A$ a unit. Then $\exists n \in \mathbb{N} \mid x^n = 0$ and $\exists a \in A \mid ya = 1$. Then $a(y+x) = ay + ax = 1 + ax$ and we can reduce to the first case, as ax is also a nilpotent element. Therefore $a(y+x)$ is a unit and $\exists b \in A \mid ba(x+y) = 1$ which implies $(x+y)$ is also a unit.

Exercise 1.2. *Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that*

(i) f is a unit in $A[x]$ $\iff$ a_0 is a unit in A and $a_1, \dots, a_n$ are nilpotent. [If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Ex.1]

(ii) f is nilpotent $\iff$ $a_0, a_1, \dots, a_n$ are nilpotent.

(iii) f is a zero-divisor $\iff$ there exists $a \neq 0$ in A such that $af = 0$. [Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ $(0 \leq r \leq n)$.]

(iv) f is said to be primitive if $(a_0, a_1, \dots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\iff$ f and g are primitive.

Solution. (i) $\boxed{\Leftarrow}$ We will proceed by induction on n. If $n = 0$ then $f = a_0$ which is a unit in A and therefore also in $A[x]$, because $A \subset A[x]$. So let's suppose that the statement holds for $n = k$. In the case $n = k+1$ we have $f = a_0 + a_1x + \cdots + a_kx^k + a_{k+1}x^{k+1}$, where a_{k+1} is nilpotent by hypothesis and $a_0 + a_1x + \cdots + a_kx^k$ is a unit by induction hypothesis. Then $a_{k+1}x^{k+1}$ is also nilpotent and f is a sum of a unit and a nilpotent element, which implies that f is a unit by Exercise 1.1.

$\boxed{\Rightarrow}$ If $f = a_0 \in A$ the statement is clearly true. We can supose then that $n > 0$. Let $f^{-1} = b_0 + b_1x + \cdots + b_mx^m$. We will first see by induction on r that $a_n^{r+1}b_{m-r} = 0 \; \forall r \in \{0, \dots, m\}$.

$$1 = ff^{-1} = \sum_{i=0}^{n+m} \left(\sum_{j=0}^{i} a_j b^{i-j} \right) x^i$$

Therefore, the term x^{n+m} has coefficient 0 and $a_n b_m = 0$, which proves the base case $r = 0$. Let's assume that the statement holds $\forall k < r$. The coefficient of x^{n+m-r} can be expressed as $\sum_{i=0}^{n} a_{n-i} b_{m-r+i}$. As we have $n > 0$, and $r \leq m$ then $n + m - r \neq 0$ and we have that

$$\sum_{i=0}^{n} a_{n-i} b_{m-r+i} = 0$$

Multiplying by a_n^r, all terms of the sumatory except from $i = 0$ vanish by induction hypotesis, which results in

$$a_n^r a_n b_{m-r} = 0 \Rightarrow a_n^{r+1} b_{m-r} = 0$$

In particular, setting $r = m$ we have that $a_n^m f^{-1} = 0 \Rightarrow a_n^m ff^{-1} = 0 \Rightarrow a_n^m = 0$. Then a_n is nilpotent and $-a_n x^n$ is also nilpotent. By exercise 1, $f - a_n x^n = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ is a unit and we can repeat the same argument and conclude that a_{n-1} is nilpotent. Repeating that procedure n times we show that $a_1, \dots, a_n$ are nilpotent, and a_0 is a unit.

(ii) $\boxed{\Leftarrow}$ a_i nilpotent $\Rightarrow a_i x^i$ nilpotent, and the sum of nilpotent elements is also nilpotent by proposition 1.7. Then $f = \sum_{i=0}^{n} a_i x^i$ is nilpotent.

$\boxed{\Rightarrow}$ f nilpotent $\Rightarrow \exists m \in \mathbb{N}$ such that $f^m = 0$. In consequence, all the coefficients of th polynomial f^m are zero, and in particular, for the term of lowest degree we have $a_0^m = 0 \Rightarrow a_0$ is nilpotent. As the sum of nilpotents is nilpotent, then $f - a_0 = x(a_1 + \cdots + a_n x^{n-1})$ is also nilpotent. As x is not a zero divisor, then $a_1 + \cdots + a_n x^{n-1}$ is nilpotent and we can apply the same argument, which allows us to conclude that a_1 is nilpotent. Applying the same reasoning, we find that $a_0, \dots, a_n$ are nilpotent in A.

(iii) $\boxed{\Leftarrow}$ By the definition of zero-divisor.

$\boxed{\Rightarrow}$ If f is a zero-divisor then $\exists g \neq 0 \in A[x] \mid fg = 0$. Let's choose g a polynomial of minimum degree sugh that $fg = 0$, $g = b_0 + \cdots + b_m x^m$. We will see by induction that $a_{n-r} g = 0 \; \forall r \in \{0, \ldots, n\}$. The base case $r = 0$ is deduced directly from $fg = 0 \Rightarrow a_n b_m = 0 \Rightarrow a_n g = 0$, as $a_n g$ anihilates f and has degree lower than g. Let's suppose that the statement holds for $r < k - 1$ and prove it for $r = k$.

$$fg = 0 = a_0 g + \cdots + a_{n-r} g x^{n-r} + a_{n-(r-1)} g x^{n-(r-1)} + \cdots + a_n g x^n$$

By induction hypothesis all the terms $a_{n-(r-i)} g$ vanish and we have

$$0 = a_0 g + \cdots + a_{n-r} g x^{n-r}$$

And using the same argument than in base case, we conclude that $a_{n-r} g = 0$.

Then $a_i g = 0 \; \forall i \in \{0, \ldots, n\}$ and therefore $a_i b_m = 0 \; \forall i$. If $m > 0$ that means

$$fg = f(b_0 + \cdots + b_m x^m) = f(b_0 + \cdots + b_{m-1} x^{m-1}) = 0$$

That is a contradiction as we have chosen g with lowest degree, so $m = 0$ and $a = b_0$ is the element we were looking for.

(iv) Let $fg = c_0 + c_1 x + \cdots + c_{n+m} x^{n+m}$, with $c_k = \sum_{i=0}^{k} a_i b_{k-i}$.

$\boxed{\Rightarrow}$ $(a_0, \ldots, a_n) \supseteq (c_0, \ldots, c_{n+m})$ and $(b_0, \ldots, b_m) \supseteq (c_0, \ldots, c_{n+m})$. Then, if fg is primitive, $(c_0, \ldots, c_{n+m}) = (1)$ and therefore f and g are also primitive.

$\boxed{\Leftarrow}$ Suppose that fg is not primitive. Then let $(c_0, \ldots, c_{n+m}) = I \subseteq \mathfrak{m}$, with $I \neq A$ an ideal of A, and $\mathfrak{m}$ a maximal ideal of A containing I. The extension of $\mathfrak{m}$ in $A[x]$ is $\mathfrak{m}^e = \mathfrak{m}[x]$ which implies that $A[x]/\mathfrak{m}^e \cong A/\mathfrak{m}[x]$, which is a domain. In $A/\mathfrak{m}[x]$, $\overline{fg} = 0$, because $c_i \in \mathfrak{m} \; \forall i$. Then, either f or g must be zero. Let's say $\overline{f} = 0$. Then, $(1) = (a_0, \ldots, a_n) \subseteq \mathfrak{m}$ which is a contradiction. Then fg must be primitive.

Exercise 1.3. *Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_n]$ in several indeterminates.*

Solution. Let $f = \sum \lambda_I x_1^{i_1} \cdots x_n^{i_n}$, where I is the multiindex $I = (i_1, \dots, i_n)$.

(i) f is a unit in $A[x_1, \dots, x_n] \iff \lambda_{0,\dots,0}$ is a unit in A and λ_I is nilpotent $\forall I \neq (0, \dots, 0)$.

We will prove this by induction on the number of indeterminates n. The base case ($n = 1$) is the statement of Exercise 1.2 i). Let's assume that the statement is true for $n - 1$ indeterminates. Then f can be written as $f = \sum_{i=0}^{m} a_i x_n^i$, with $a_i \in A[x_1, \dots, x_{n-1}]$. By Exercise 1.2, f is a unit $\iff$ a_0 is a unit in $A[x_1, \dots, x_{n-1}]$ and a_i is nilpotent $\forall i \neq 0$. We complete the proof applying induction hypothesis on a_0.

(ii) f is nilpotent $\iff$ λ_I is nilpotent $\forall I$.

Let's proceed by induction on n. The base case ($n = 1$) is the statement of Exercise 1.2 ii). Let's assume that the statement is true for $n - 1$ indeterminates. In the case of n indeterminates, $f = \sum_{i=0}^{m} a_i x_n^i$, with $a_i \in A[x_1, \dots, x_{n-1}]$. By Exercise 1.2, f is nilpotent $\iff$ a_i is nilpotent $\forall i$. We complete the proof applying induction hypothesis on each a_i.

(iii) f is a zero-divisor $\iff$ there exists $b \neq 0$ in A such that $bf = 0$.

The inverse implication is obvoius. We will use induction again for the direct one. The base case ($n = 1$) is the statement of Exercise 1.2 iii). Let's assume that the statement is true for $n-1$ indeterminates. In the case of n indeterminates, $f = \sum_{i=0}^{m} a_i x_n^i$, with $a_i \in A[x_1, \dots, x_{n-1}]$. By Exercise 1.2, f is a zero divisor $\iff$ there exists $a \in A[x_1, \dots, x_{n-1}]$ such that $af = 0$. If we consider af as a polynomial on x_n over the ring $A[x_1, \dots, x_{n-1}]$ we conclude that $af = 0 \iff aa_i = 0 \; \forall i$ is a zero divisor $\forall i$. By induction hypothesis, that is equivalent to $\exists b_i \in A$ such that $b_i a_i = 0 \; \forall i$, and the argument used in the proof of 1.2 iii) guarantees that we can take the same element $b = b_i \forall i$.

(iv) fg is primitive $\iff$ f and g are primitive.

The proof is the same as Exercise 1.2 iv).

Exercise 1.4. *In the ring $A[x]$, the Jacobson radical is equal to the nilradical.*

Solution. Let $\mathcal{N}, \mathcal{J}$ denote the nilradical and the Jacobson radical of $A[x]$, respectively. We already know that $\mathcal{N} \subseteq \mathcal{J}$, as $\mathcal{N} = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$ and $\mathcal{J} = \bigcap_{\mathfrak{m} \text{ maximal}} \mathfrak{m}$, and every maximal ideal is prime.

Let's now prove the other inclusion. Let $f(x) \in \mathcal{J} \Rightarrow 1 - fg$ is a unit $\forall g \in A[x] \Rightarrow 1 - xf(x)$ is a unit $\Rightarrow a_0, \ldots, a_n$ are nilpotent (we have used Exercise 1.2 i)). Using now 1.2 ii) $\Rightarrow f$ is nilpotent $\Rightarrow f \in \mathcal{N}$.

Exercise 1.5. *Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that*

(i) f is a unit in $A[[x]] \iff a_0$ is a unit in A.

(ii) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true? (See Chapter 7, Exercise 2.)

(iii) f belongs to the Jacobson radical of $A[[x]] \iff a_0$ belongs to the Jacobson radical of A.

(iv) The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.

(v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.

Solution. (i) $\boxed{\Rightarrow}$ f is a unit in $A[[x]] \Rightarrow \exists g = \sum_{n=0}^{\infty} b_n x^n \in A[[x]]$ such that $fg = 1 \Rightarrow a_0 b_0 = 1$, which means that a_0 is a unit in A.

$\boxed{\Leftarrow}$ Let $a_0 b_0 = 1$. We want to define $\{b_n\}_{n \geq 0}$ such that

$$\sum_{n=0}^{\infty} b_n x^n \sum_{m=0}^{\infty} a_m x^m = 0$$

Let $\{b_n\}_{n \geq 0}$ be an arbitrary sequence that defines an element $g \in A[[x]]$. Then the term of fg corresponding to exponent k is $c_k = \sum_{i+j=k} a_i b_j$. That allows us to define the desired sequence $\{b_n\}_{n \geq 0}$ in a recursive way.

b_0 is already defined as the inverse of a_0. We want b_1 satisfying that $a_0 b_1 + b_0 a_1 = 0$, so $a_0 b_1 = -b_0 a_1$, and we can isolate b_1 as a_0 is invertible.

$$b_1 = -b_0^2 a_1$$

The same procediment allows to find b_{k+1} as a function of b_i, $0 \leq i \leq k$.

$$b_k = -b_0 \sum_{i+j=k,\, i>0} a_i b_j$$

That gives an explicit construction of $g = \sum_{n=0}^{\infty} b_n x^n \in A[[x]]$ such that $fg = 1$, and therefore proves that f is invertible.

(ii) Let's suppose that $\exists n \in \mathbb{N}$ such that $f^n = 0$. We will prove by induction that a_i is nilpotent $\forall i \in \mathbb{N}$.

$$0 = f^n = \left(\sum_{i=0}^{\infty} a_i x^i \right)^n$$

Then, each coefficient of f^n has to be zero. This condition for the independent term of f^n implies that $a_0^n = 0$ and therefore a_0 is nilpotent, which proves the base case. Let's assume that a_i, $0 \leq i \leq k$ are nilpotent. Then, by Proposition 1.7, $f - \sum_{i=0}^{k} a_i x^i = x^k \sum_{i=1}^{\infty} a_{i+k} x^k$ is also nilpotent. The same argument used in the base case suffices to prove that a_{k+1} is nilpotent, and the proof is complete. The converse is shown in exercise 7.2.

(iii) Using Proposition 1.9, f belongs to the Jakobson radical of $A[[x]] \iff 1 - fg$ is a unit $\forall g \in A[[x]]$. By part i) of the exercise, that happens $\iff 1 - a_0 b_0$ is a unit $\forall b_0 \in A$, which happens $\iff a_0$ belongs to the Jakobson radical of A.

(iv) First, we observe that $\mathfrak{m}$ is a maximal ideal of $A[[x]] \Rightarrow x \in \mathfrak{m}$: Otherwise, $\mathfrak{m} \subset \mathfrak{m} + (x)$, and $\mathfrak{m} + (x) \neq A[[x]]$ as it doesn't contain units: If $1 = f + g$, with $g \in (x)$ and $f \in \mathfrak{m}$, then by part i) of the exercise a_0 is a unit in A, which implies f is a unit and therefore $\mathfrak{m} = A[[x]]$, which is a contradiction.

Let $\mathfrak{m} \in \mathrm{Max}\{A[[x]]\}$. Let's suppose that $\mathfrak{m}^c = \mathfrak{m} \cap A$ is not maximal. Then $\exists a \in A$, $a \notin \mathfrak{m}^c$ such that $\mathfrak{m}^c + (a) \neq (1)$. But $a \notin \mathfrak{m}^c \Rightarrow a \notin \mathfrak{m}$ and therefore $\exists f \in \mathfrak{m}$, $g \in A[[x]]$ such that

$$f + ag = 1$$

That implies $f_0 + ag_0 = 1$. And, recalling that $x \in \mathfrak{m} \Rightarrow a_0 \in \mathfrak{m}^c$, and therefore $\mathfrak{m}^c + (a) = (1)$ which is a contradiction.

Moreover, $\mathfrak{m}^c + (x) \subseteq \mathfrak{m}$, and given $f = a_0 + x\sum_{i=0}^{\infty} a_{i+1}x^i \in \mathfrak{m}$ it's clear that $a_0 \in \mathfrak{m}^c$ and the set equality holds.

(v) Let $\mathfrak{p} \in \operatorname{Spec}\{A[[x]]\}$, and let's consider the ideal of $A[[x]]$ $\mathfrak{p}' = \mathfrak{p} + (x)$. The same argument used in iv) guarantees that $\mathfrak{p}' \neq A[[x]]$, and it's clear that $(\mathfrak{p}')^c = \mathfrak{p}$. We only need to check that $\mathfrak{p}'$ is prime, which follows from

$$\frac{A[[x]]}{\mathfrak{p}'} \cong \frac{A}{\mathfrak{p}}$$

Exercise 1.6. *A ring A is such that every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.*

Solution. Let's prove the inclusion $\mathcal{J} \subseteq \mathcal{N}$, as the other inclusion is true in general. Let $x \in \mathcal{J}$ and suppose that $x \notin \mathcal{N}$. Then $(x) \notin \mathcal{N} \Rightarrow \exists e \in (x)$, $e = ax \neq 0$ such that $e^2 = e \Rightarrow (ax)^2 = ax$. That implies that $(1-ax)ax = 0$, but $1-ax$ is a unit as $x \in \mathcal{J} \Rightarrow 1-ax$ is a unit $\forall a \in A$. Therefore $ax = 0$, which is a contradiction comming from the supposition that $x \notin \mathcal{N}$. In conclusion, $\mathcal{N} = \mathcal{J}$.

Exercise 1.7. *Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.*

Solution. Let $\mathfrak{p}$ be a prime ideal. $\forall x \in A \exists n > 0 | x^n = x \Rightarrow x(x^{n-1} - 1) = 0$. Let's reduce this equality modulo $\mathfrak{p}$. As $A/\mathfrak{p}$ is a domain, every non-zero element satisfies $x^{n-1} = 1$, and in particular every non-zero element is invertible, which means that $A/\mathfrak{p}$ is a field and therefore $\mathfrak{p}$ is maximal.

Exercise 1.8. *Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.*

Solution. Let's consider the set $\operatorname{Spec}\{A\}$ partially ordered by the relation $\geq = \subseteq$. By Theorem 1.3, $A \neq 0 \Rightarrow \operatorname{Spec}\{A\} \neq \emptyset$. Moreover, given a chain of prime ideals

$$\mathfrak{p}_1 \supseteq \mathfrak{p}_2 \supseteq \mathfrak{p}_3 \supseteq \dots$$

the ideal $\bigcap_{i=0}^{\infty} \mathfrak{p}_i$ is an upper bound of the chain. Therefore, Spec$\{A\}$ with the given order relation satisfies the conditions of Zorn's Lemma, which proves that Spec$\{A\}$ has minimal elements with respect to inclusion.

Exercise 1.9. *Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \iff \mathfrak{a}$ is an intersection of prime ideals.*

Solution. $\boxed{\Rightarrow}$ We know from Proposition 1.14 that

$$r(\mathfrak{a}) = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p} \text{ prime}} \mathfrak{p}$$

Therefore, $\mathfrak{a} = r(\mathfrak{a}) \Rightarrow \mathfrak{a}$ is intersection of prime ideals.

$\boxed{\Leftarrow}$ Let $\mathfrak{a} = \bigcap_i \mathfrak{p}_i$. Let $x \in r(\mathfrak{a})$. Then $\exists n > 0$ such that $x^n \in \mathfrak{a} \Rightarrow x^n \in \mathfrak{p}_i \ \forall i \Rightarrow x \in \mathfrak{p}_i \ \forall i \Rightarrow x \in \mathfrak{a}$. In conclusion, $r(\mathfrak{a}) = \mathfrak{a}$.

Exercise 1.10. *Let A be a ring, $\mathfrak{R}$ its nilradical. Show that the following are equivalent:*

(i) A has exactly one prime ideal;

(ii) Every element of A is either a unit or nilpotent;

(iii) $A/\mathfrak{R}$ is a field.

Solution. $\boxed{i) \Rightarrow ii)}$ Let $\mathfrak{p}$ be the only prime ideal. Then $\mathfrak{p}$ is also maximal and $\mathfrak{R} = \bigcap_{\mathfrak{p}_i \text{ prime}} \mathfrak{p}_i = \mathfrak{p}$. As every non-unit element is contained in a maximal, and $\mathfrak{R}$ is the only maximal ideal, then every non-unit is a nilpotent.

$\boxed{ii) \Rightarrow iii)}$ Given $x \in A$, if $x \in \mathfrak{R}$ then $0 = \overline{x} \in A/\mathfrak{R}$. Otherwise x is invertible. Therefore, every non-zero element in $A/\mathfrak{R}$ is invertible, which means that $A/\mathfrak{R}$ is a field.

$\boxed{iii) \Rightarrow i)}$ $A/\mathfrak{R}$ is a field $\Rightarrow \mathfrak{R} = \bigcap_{\mathfrak{p}_i \text{ prime}} \mathfrak{p}_i$ is maximal. Therefore, there can only be one prime ideal, as otherwise $\mathfrak{R} \subset \mathfrak{p}$, which would contradict $\mathfrak{R}$ maximal.

Exercise 1.11. *A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that*

(i) $2x = 0$ for all $x \in A$;

(ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements;

(iii) every finitely generated ideal in A is principal.

Solution. (i) As the ring is Boolean, $x + 1 = (x + 1)^2 = x^2 + 2x + 1^2 = x + 1 + 2x \Rightarrow 2x = 0$.

(ii) Let's reduce the equation $x^2 = x$ modulo a prime ideal $\mathfrak{p}$. $\overline{x}^2 = \overline{x} \Rightarrow \overline{x}(\overline{x} - 1) = 0$. As $A/\mathfrak{p}$ is a domain, then either $\overline{x} = 0$ or $\overline{x} = 1$. In consequence, $A/\mathfrak{p}$ is a field with only two elements.

(iii) It's enough to prove it for an ideal generated by two elements, and the general finitely generated case follows from induction. Let $\mathfrak{a} = (x, y)$, and let's consider the element $z = x + y + xy \in \mathfrak{a}$. We observe that $zx = x^2 + xy + x^2y = x + xy + xy = x$. Analogously, $zy = y$, which implies that $x \in (z)$ and $y \in (z)$, and therefore $\mathfrak{a} = (z)$.

Exercise 1.12. *A local ring contains no idempotent $\neq 0, 1$.*

Solution. Let x be an idempotent element of a local ring, $x^2 = x \Rightarrow x(x - 1) = 0$. If x is a unit, that is $\exists y$ such that $xy = 1$, then $1 = xy = x^2y = x$ and $x = 1$. Otherwise, Proposition 1.9 forces $1 - x$ to be a unit. Given that $(1 - x)^2 = 1 - 2x + x^2 = 1 - x$ the same argument used with x holds for $1 - x$, which means that $1 - x = 1$ and therefore $x = 0$.

Exercise 1.13. *Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.*

Solution.

Exercise 1.14. *In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.*

Solution. Let's consider the inclusion order relation in the set Σ. Σ is a non-empty set, as $(0) \in \Sigma$. Then, given a chain of ideals $\mathfrak{a}_i \in \Sigma$

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$$

The set $\bigcup_{i=1}^{\infty} \mathfrak{a}_i$ is an ideal that contains only divisors of zero, and therefore it's an upper bound of the chain. Then, $(\Sigma, \subseteq)$ satisfy the conditions of Zorn's Lemma, which guarantees the existence of maximal elements of Σ.

Let $\mathfrak{p}$ be a maximal element of Σ. Suppose $\mathfrak{p}$ is not prime. Then, $\exists x, y \notin \mathfrak{p}$ such that $xy \in \mathfrak{p}$. As xy is a zero divisor, $\exists z \neq 0$ such that $xyz = 0$, which implies that either x, y or both are zero divisors. Without loss of generality we suppose that x is a zero divisor, and we consider the ideal $\mathfrak{p}' = (x) + \mathfrak{p}$. Every element $a \in \mathfrak{p}'$ is a zero divisor, as $ay \in \mathfrak{p}$, and as $\mathfrak{p} \subset \mathfrak{p}'$, that is a contradiction with the maximality of $\mathfrak{p}$ in Σ.

Therefore, every maximal element of Σ is a prime ideal and hence the set of zero-divisors in A is a union of prime ideals (the maximal elements of Σ).

Exercise 1.15. *Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that*

(i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

(ii) $V(0) = X$, $V(1) = \emptyset$.

(iii) if $(E_i)_{i \in I}$ is any family of subsets of A, then

$$V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i \in I} V(E_i)$$

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{a})$ for any ideals $\mathfrak{a}$, $\mathfrak{b}$ of A.

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the Zariski topology. The topological space X is called the prime spectrum of A, and is written Spec(A).

Solution. (i) If $\mathfrak{p}$ is a prime ideal such that $E \subseteq \mathfrak{p} \Rightarrow \mathfrak{a} \subseteq \mathfrak{p}$, as $\mathfrak{a}$ is the smallest ideal containing E. On the other hand, it's clear that $\mathfrak{a} \supseteq E$, and then $\mathfrak{p} \supseteq \mathfrak{a} \Rightarrow \mathfrak{p} \supseteq E$. In conclusion, $V(E) = V(\mathfrak{a})$.

Let's now prove the other equality. On one hand, from $r(\mathfrak{a}) \supseteq \mathfrak{a}$ follows the inclusion $V(\mathfrak{a}) \supseteq V(r(\mathfrak{a}))$. On the other hand, as

$$r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}$$

it's clear that every prime ideal containing $\mathfrak{a}$ will also contain $r(\mathfrak{a})$, which proves the inclusion $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$.

(ii) 0 is contained in every ideal $\Rightarrow V(0) = X$, and 1 is not contained in any prime ideal $\Rightarrow V(1) = \emptyset$.

(iii) $\mathfrak{p} \in V(\bigcup_{i\in I} E_i \iff \bigcup_{i\in I} E_i \subseteq \mathfrak{p} \iff E_i \subseteq \mathfrak{p}\ \forall i \iff \mathfrak{p} \in V(E_i)\ \forall i \iff \mathfrak{p} \in \bigcap_{i\in I} V(E_i)$.

(iv) $\mathfrak{p} \in V(\mathfrak{a}\cap\mathfrak{b}) \Rightarrow \mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{p}$. As $\mathfrak{p}$ is prime, by Preposition 1.11 either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, which implies that $\mathfrak{p} \in V(\mathfrak{ab})$ and $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. That proves the inclusions $V(\mathfrak{a}\cap\mathfrak{b}) \subseteq V(\mathfrak{ab})$ and $V(\mathfrak{a}\cap\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.

Let $\mathfrak{ab} \subseteq \mathfrak{p}$. $x \in \mathfrak{a}\cap\mathfrak{b} \Rightarrow x^2 \in \mathfrak{ab} \subset \mathfrak{p} \Rightarrow x \in \mathfrak{p}$. Therefore, $\mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{p}$, which proves the inclusion $V(\mathfrak{ab}) \subseteq V(\mathfrak{a}\cap\mathfrak{b})$.

Let $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. Without loss of generality let's suppose that $\mathfrak{p} \supseteq \mathfrak{a} \Rightarrow \mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{a} \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} \in V(\mathfrak{a}\cap\mathfrak{b})$. This proves the inclusion $V(\mathfrak{a}\cap\mathfrak{b}) \supseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.

Exercise 1.16. *Draw pictures of Spec*($\mathbb{Z}$)*, Spec*($\mathbb{R}$)*, Spec*($\mathbb{C}[x]$)*, Spec*($\mathbb{R}[x]$)*, Spec*($\mathbb{Z}[x]$)*.*

Solution. (i) The prime ideals of $\mathbb{Z}$ are (0) and (p), with p prime. Any finite set of primes $p_1, \ldots, p_n$ correspond to a set in Spec($\mathbb{Z}$), $Y = \{(p_1), \ldots, (p_n)\}$. Let's consider the ideal $\mathfrak{a} = (p_1 p_2 \cdots p_n)$, and clearly $\mathfrak{a} \subseteq (p_i)\ \forall i$, and by the uniqueness of prime descomposition $\nexists \mathfrak{q} \neq (p_i)$ prime such that $\mathfrak{a} \subseteq \mathfrak{q}$, which implies that $V(\mathfrak{a}) = Y$. As every ideal is principal the inverse reasoning also holds. In conclusion, the closed sets of the topology are Spec($\mathbb{Z}$), $\emptyset$ and all finite subsets of Spec($\mathbb{Z}$) not containing (0).

(ii) $\mathbb{R}$ has only one prime ideal (0), so Spec($\mathbb{R}$) is a topological space with only one point.

(iii) In $\mathbb{C}[x]$ every element factorizes as a product of linear factors. Therefore, the prime ideals are $(x - \alpha)$, $\forall \alpha \in \mathbb{C}$, and (0). Then, Spec($\mathbb{C}[x]$) can be identified with the complex plane and an extra point corresponding to (0). By the same reasoning used for the case of $\mathbb{Z}$, the closed sets of the topology are all finite subsets of $\mathbb{C}$, Spec($\mathbb{C}[x]$) and $\emptyset$.

(iv) $\mathbb{R}[x]$ is a principal ideal domain, and therefore the prime ideals are those generated by an irreducible element. Irreducible polynomials are (0), $x-a$, $\forall a \in \mathbb{R}$ and x^2+bx+c, with $b^2-4c<0$. The closed subsets are again all finite sets of prime ideals not containing (0).

(v) The prime ideals of $\mathbb{Z}[x]$ are (0), (p), with $p \in \mathbb{Z}$ prime, (f), with f an irreducible polynomial and (p, f), with $p \in \mathbb{Z}$ prime and $f \in \mathbb{Z}[x]$ irreducible.

Exercise 1.17. *For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = Spec(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that*

(i) $X_f \cap X_g = X_{fg}$

(ii) $X_f = \emptyset \iff f$ *is nilpotent*

(iii) $X_f = X \iff f$ *is a unit*

(iv) $X_f = X_g \iff r((f)) = r((g))$

(v) X is quasi-compact (every open covering of X has a finite subcovering)

(vi) More generally, each X_f is quasi-compact

(vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f

The sets X_f are called basic open sets of $X = Spec(A)$.

Solution. The sets X_f are open because their complementaries are closed sets. Let's first prove that X_f are a basis of the topogoly, that is, $\forall U \in \mathcal{T}$, $U = \bigcup_i X_{f_i}$. Indeed, $U \in \mathcal{T} \Rightarrow U = X\ V(E)$ for a certain E. Then $U = \{\mathfrak{p}$ such that $E \not\subset \mathfrak{p}\} = \{\mathfrak{p}$ such that $\exists f \in E$, $f \notin \mathfrak{p}\}$.

$$U = \bigcup_{f\in E} X\ V(f) = \bigcup_{f \in E} X_f$$

(i) $\mathfrak{p} \in X_f \cap X_g \Rightarrow f \notin \mathfrak{p}$, $g \notin \mathfrak{p}$. As $\mathfrak{p}$ is prime, that implies $fg \notin \mathfrak{p} \Rightarrow \mathfrak{p} \in X_{fg}$, which proves the inclusion $X_f \cap X_g \subseteq X_{fg}$.

Conversely, $\mathfrak{p} \in X_{fg} \Rightarrow f \notin \mathfrak{p}$, $g \notin \mathfrak{p}$, and therefore $\mathfrak{p} \in X_f$, $\mathfrak{p} \in X_g$, which proves the inclusion $X_f \cap X_g \supseteq X_{fg}$.

(ii) $X_f = \emptyset \iff V(f) = \mathrm{Spec}(A) \iff f \in \mathfrak{p}\ \forall \mathfrak{p}$ prime $\iff f \in \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p} = \mathcal{N} \iff f$ is nilpotent.

(iii) $X_f = X \iff V(f) = \emptyset \iff \nexists \mathfrak{p}$ prime such that $f \in \mathfrak{p}$. It's clear that units satisfy this propiety, as f unit $\Rightarrow (f) = A$. Conversely, if f is not unit, $f \in \mathfrak{m}$ maximal (and in particular prime), by Theorem 1.3. In conclusion, $X_f = X \iff f$ is a unit.

(iv) It is immediate from the characterization $r((f)) = \bigcap_{f \in \mathfrak{p} \text{ prime}} \mathfrak{p}$.

(v) Let's consider an open covering of X. As we have shown that X_f are a basis of the topology, without loss of generality we can consider a basic covering

$$X = \bigcup_{i\in I} X_{f_i} = \bigcup_{i\in I} X \setminus V(f_i) = X \setminus \bigcap_{i\in I} V(f_i) \iff$$

$$\iff \bigcap_{i\in I} V(f_i) = \emptyset \iff V(\bigcup_{i\in I} f_i) = \emptyset \iff (f_i)_i = A$$

That means that $1 = \sum_{i\in J} g_i f_i$, for a certain J with $\#J < \infty$. Following the implications in opposite direction, we conclude that $\{X_{f_i}\}_{i\in J}$ covers X, that is, we have found a finite subcovering and we can conclude that X is quasi-compact.

(vi) The reasoning is the same as in the previous part.

$$X_f = \bigcup_{i\in I} X_{f_i} = X \setminus \bigcap_{i\in I} V(f_i) \Rightarrow V(\bigcup_{i\in I} f_i) = V(f)$$

which means that $f = \sum_{i\in J} g_i f_i$, for a certain J with $\#J < \infty$, and therefore $\{X_{f_i}\}_{i\in J}$ covers X_f.

(vii) $\boxed{\Rightarrow}$ It follows from the definition of quasi-compatness.

$\boxed{\Leftarrow}$ $U = \bigcup_{i=1}^{n} X_{f_i}$. As X_f are a basis of the topology, given any open covering of $U = \bigcup_{j\in J} V_j$ each V_j will be the union of some X_{f_i}, and therefore we can extract a finite covering.

Exercise 1.18. *For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = Spec(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that*

(i) The set $\{x\}$ is closed in Spec(A) $\iff$ $\mathfrak{p}_x$ is maximal.

(ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$.

(iii) $y \in \overline{\{x\}} \iff \mathfrak{p}_x \subseteq \mathfrak{p}_y$

(iv) X is a T_0 space (this means that if x, y are distinct points of X, then either there is a neighborhood of x which does not contain y, or else there is a neighborhood of y which does not contain x).

Solution. (i) $\{x\}$ is closed in Spec(A) $\iff$ $x = V(\mathfrak{p}_x)$ $\iff$ $\nexists \mathfrak{q}$ prime such that $\mathfrak{q} \supset \mathfrak{p}_x \Rightarrow \mathfrak{p}_x$ is maximal.

(ii) $(\mathfrak{p}_x)$ is closed and contains x. Given $V(\mathfrak{p})$ a closed set containing x, $\mathfrak{p}_x \supseteq \mathfrak{p} \Rightarrow V(\mathfrak{p}_x) \subseteq V(\mathfrak{p})$, so $V(\mathfrak{p})$ also contains $V(\mathfrak{p}_x)$, and therefore $\overline{\{x\}} = V(\mathfrak{p}_x)$.

(iii) $y \in \overline{\{x\}} \iff y \in V(\mathfrak{p}_x) \iff \mathfrak{p}_x \subseteq \mathfrak{p}_y$.

(iv) Let x, y be distinct points of X, then either $\exists f \in \mathfrak{p}_x$ such that $f \notin \mathfrak{p}_y$, or $\exists f \in \mathfrak{p}_y$ such that $f \notin \mathfrak{p}_x$. Let's suppose we have the first case (the other is symmetric). Therefore, $x \notin X_f$ but $y \in X_f$, and therefore X_f is the neighborhood we were looking for.

Exercise 1.19. *A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that Spec(A) is irreducible if and only if the nilradical of A is a prime ideal.*

Solution. We already know by problem 1.17 that $X_f = 0 \iff f$ belongs to the nilradical.
$\boxed{\Leftarrow}$ Let's suppose that X is not irreducible, that is $\exists U = \bigcup_i X_{f_i} \neq \emptyset\, V = \bigcup_j X_{g_j} \neq \emptyset$ two open sets such that $U \cap V = \emptyset$. Without loss of generality, we can take $X_{f_i}, X_{g_j} \neq \emptyset\ \forall i,j$. Then, $\bigcup_{i,j} X_{f_i} \cap X_{g_j} = \emptyset \Rightarrow X_{f_i} \cap X_{g_j} = \emptyset\ \forall i,j$. As the nilradical $\mathfrak{p}_{\mathcal{N}}$ is prime, $f \notin \mathfrak{p}_{\mathcal{N}} \Rightarrow \mathcal{N} \in X_f$. Therefore, either f_i or g_j belong to the nilradical $\Rightarrow X_{f_i}$ or $X_{g_j} = \emptyset$, which is a contradiction.

$\boxed{\Rightarrow}$ $fg \in \mathcal{N} \Rightarrow X_{fg} = \emptyset$, and $X_{fg} = X_f \cap X_g$. As Spec(A) is irreducible $\Rightarrow$ either $X_f = \emptyset$ or $X_g = \emptyset$, that is, either $f \in \mathcal{N}$ or $g \in \mathcal{N}$, which proves that $\mathcal{N}$ is prime.

Exercise 1.20. *Let X be a topological space.*

1. *If Y is an irreducible (Exercise 19) subspace of X, then the closure $\overline{Y}$ of Y in X is irreducible.*

2. *Every irreducible subspace of X is contained in a maximal irreducible subspace.*

3. *The maximal irreducible subspaces of X are closed and cover X. They are called the irreducible components of X. What are the irreducible components of a Hausdorff space?*

4. *If A is a ring and $X = Spec(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A (Exercise 8).*

Solution. Let X be a topological space.

(i) Let $U_1, U_2 \neq \emptyset$ be open subsets of $\overline{Y}$. By definition of closure, $U_i \cap Y \neq \emptyset$. Therefore, as Y is irreductible, $(U_1 \cap Y) \cap (U_2 \cap Y) \neq \emptyset$. In particular, $U_1 \cap U_2 \neq \emptyset$, which proves that $\overline{Y}$ is irreducible.

(ii) Let Σ be the set of all irreducible subspaces. We observe that the singletons $\{x\}$ are irreducible subspaces, and therefore $\Sigma \neq \emptyset$. Suppose we have a chain of subspaces $X_i \in \Sigma$, $X_1 \subseteq X_2 \subseteq \dots$. We will prove that $\bigcup_i X_i$ is also irreducible, and therefore the conditions of Zorn's Lemma apply to Σ and the existence of maximal elements is proven.

Let $U_1, U_2 \neq \emptyset$ be open subsets of $\bigcup_i X_i$, and $x_i \in U_i$. Therefore, $\exists k, l$ such that $x_1 \in X_k$ and $x_2 \in X_l$. Therefore, $(U_i \cap X_{max\{k,l\}}) \neq \emptyset$ and are open subsets of $X_{max\{k,l\}}$, and therefore $(U_1 \cap X_{max\{k,l\}}) \cap (U_2 \cap X_{max\{k,l\}}) \neq \emptyset$. In particular, $U_1 \cap U_2 \neq \emptyset$, which proves that $\bigcup_i X_i$ is irreducible.

(iii) Let Y be a maximal irreducible subspace. As $Y \subseteq \overline{Y}$ which is also irreducible by i), we must have $Y = \overline{Y}$, or otherwise we would reach a contradiction with the maximality of Y. This proves that Y is closed.

Now, let's consider the case of a Hausdorff space, where $\forall x, y \exists U_x, U_y$ with $x \in U_x, y \in U_y$ such that $U_x \cap U_y = \emptyset$, which means that $x, y \notin$ the same irreducible component. Therefore, the maximal irreducible components are the points of the space.

(iv) First, the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A are irreducible, because all open sets will be $U = V(\mathfrak{p}) \backslash V(E)$, and therefore $\mathfrak{p} \in$ all non-empty open sets. They're also maximal, as otherwise, $\exists \mathfrak{a}$ such that $V(\mathfrak{p}) \subseteq V(\mathfrak{a}) \iff \mathfrak{a} \subseteq \mathfrak{p}$ and $\exists \mathfrak{p}_1 \in V(\mathfrak{a})$, $\mathfrak{p}_2 \notin V(\mathfrak{p})$, which implies that $\mathfrak{p}_1 \subset \mathfrak{p}$, a contradiction with the minimality of $\mathfrak{p}$.

On the other hand, given an irreducible component Y, Y is closed by ii), and therefore $Y = V(\mathfrak{a}) = V(r(\mathfrak{a}))$. Let's prove that $r(\mathfrak{a})$ must be a minimal prime ideal. First of all, let's see that if $r(\mathfrak{a})$ is prime, it must be a minimal prime. Otherwise, $\exists \mathfrak{p}_{min} \subset r(\mathfrak{a})$ minimal prime ideal, and therefore $V(\mathfrak{p}_m in) \supset V(\mathfrak{a})$, which contradicts the maximality of $V(\mathfrak{a})$.

Now let's prove that $r(\mathfrak{a})$ must be prime. Let $fg \in r(\mathfrak{a}) \Rightarrow fg \in \mathfrak{p}_x \; \forall x \in V(\mathfrak{a})$, and therefore $X_{fg} = X_f \cap X_g = \emptyset$ by exericise 17. The irreductibility of $V(\mathfrak{a})$ implies that either X_f or X_g are empty, which ensures that either f or $g \in r(\mathfrak{a})$, and therefore $r(\mathfrak{a})$ is prime.

Exercise 1.21. *Let $\phi : A \to B$ be a ring homomorphism. Let $X = Spec(A)$ and $Y = Spec(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e., a point of X. Hence ϕ induces a mapping $\phi^* : Y \to X$. Show that*

(i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous.

(ii) If $\mathfrak{a}$ is an ideal of A, then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

(iii) If $\mathfrak{b}$ is an ideal of B, then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

(iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker(\phi))$ of X. (In particular, $Spec(A)$ and $Spec(A/\mathfrak{R})$ (where $\mathfrak{R}$ is the nilradical of A) are naturally homeomorphic.)

(v) If ϕ is injective, then $\phi^*(Y)$ is dense in X. More precisely, $\phi^*(Y)$ is dense in $X \iff Ker(\phi) \subseteq \mathfrak{R}$.

(vi) Let $\psi : B \to C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

(vii) *Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \to B$ by $\phi(x) = (\bar{x}, x)$, where $\bar{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ^* is bijective but not a homeomorphism.*

Solution. (i) $y \in \phi^{*-1}(X_f) \iff \phi^*(y) \in X_f \iff f \notin \phi^{-1}(\mathfrak{p}_y) \iff \phi(f) \notin \mathfrak{p}_y \iff y \in Y_{\phi(f)}$. Therefore, $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and as X_f is a base of the topology, the antiimage of an arbitraty open subset is open, which proves that ϕ^* is countinuous.

(ii) $y \in \phi^{*-1}(V(\mathfrak{a})) \iff \phi^*(y) \in V(\mathfrak{a}) \iff \phi^{-1}(\mathfrak{p}_y) \supseteq \mathfrak{a} \iff \mathfrak{p}_y \supseteq \mathfrak{a}^e \iff y \in V(\mathfrak{a}^e)$. Per tant, $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

(iii) $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{a})$, for some ideal $\mathfrak{a}$ yet to determine. We observe that $x \in \phi^*(V(\mathfrak{b})) \iff \exists y \in \operatorname{Spec}(B)$ such that $\mathfrak{p}_y \supseteq \mathfrak{b}$ and $\mathfrak{p}_x = \mathfrak{p}_y^c$. Then, it's clear that $\mathfrak{a} \subseteq \mathfrak{p}^c$, $\forall \mathfrak{p} \supseteq \mathfrak{b}$, which implies that $\mathfrak{a} \subseteq \bigcap_{\mathfrak{p} \supseteq \mathfrak{b}} \mathfrak{p}^c$. As the closure is the smallest closed subset containing $\phi^*(V(\mathfrak{b}))$, we have the equality

$$\overline{\phi^*(V(\mathfrak{b}))} = V\left(\bigcap_{\mathfrak{q} \supseteq \mathfrak{b}} \mathfrak{q}^c\right)$$

Using now Exercise 1.18 of the theory section, and Problem 1.15 i) we have that

$$V\left(\bigcap_{\mathfrak{q} \supseteq \mathfrak{b}} \mathfrak{q}^c\right) = V\left(\left(\bigcap_{\mathfrak{q} \supseteq \mathfrak{b}} \mathfrak{q}\right)^c\right) = V(r(\mathfrak{b})^c) = V(r(\mathfrak{b}^c)) = V(\mathfrak{b}^c)$$

(iv) If ϕ is surjective, then $A/\ker(\phi) \cong B$, and therefore $\exists$ a bijective correspondence between prime ideals of B prime ideals of $A/\ker(\phi)$, which correspond to prime ideals of A containing $\ker(\phi)$. That proves $Y \overset{\phi^*}{\cong} V(\ker(\phi))$.

Therefore, we already know that $\phi^* : Y \to V(\ker(\phi))$ is bijective and continuous. Now we have to prove that the inverse $\phi^{*-1} : V(\ker(\phi)) \to Y$ is also continuous. Let $V(\mathfrak{b})$ be an arbitrary closed set of Y. Let's check that its antiimage by ϕ^{*-1} is also closed. Indeed, $\mathfrak{p} \in V(\mathfrak{b}) \iff \mathfrak{p} \supseteq \mathfrak{b} \iff (\phi^{*-1})^{-1}(\mathfrak{p}) \supseteq (\phi^{*-1})^{-1}(\mathfrak{b}) \Rightarrow \phi^{*-1}(\mathfrak{p}) \supseteq \mathfrak{b}^c \iff \mathfrak{p}^c \in V(\mathfrak{b}^c) \Rightarrow (\phi^{*-1})^{-1}(V(\mathfrak{b})) = V(\mathfrak{b}^c)$.

(v) $\phi^*(Y)$ is dense in $X \iff \overline{\phi^*(Y)} = X$. We also know that

$$\overline{\phi^*(Y)} = \overline{\phi^*(V((0)))} = V((0)^c) = V(\ker(\phi))$$

Therefore, we only have to show that $V(\ker(\phi)) = X \iff \ker(\phi) \subseteq \mathfrak{R}$. That is true because $V(\ker(\phi)) = X \iff \mathfrak{p} \supseteq \ker(\phi)\ \forall \mathfrak{p}$ prime $\iff \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p} \supseteq \ker(\phi) \iff \ker(\phi) \subseteq \mathfrak{R}$.

(vi) Let $\mathfrak{p} \in \mathrm{Spec}(C)$. Then $(\psi \circ \phi)^*(\mathfrak{p}) = (\psi \circ \phi)^{-1}(\mathfrak{p}) = \phi^{-1}(\psi^{-1}(\mathfrak{p})) = \phi^*(\psi^*(\mathfrak{p})) = (\phi^* \circ \psi^*)(\mathfrak{p})$.

(vii) The prime ideals of $\mathrm{Spec}(B)$ are $\mathfrak{p}_1 = (0, K)$ and $\mathfrak{p}_2 = (A/\mathfrak{p}, 0)$. We have that $\phi^*(\mathfrak{p}_1) = \mathfrak{p}$ and $\phi^*(\mathfrak{p}_2) = (0)$, and therefore ϕ^* is clearly bijective. However, we will prove that ϕ^{*-1} is not continuous, as $(\phi^{*-1})^{-1}(\mathfrak{p}_2) = (0)$ which is not a closed subset, while $\mathfrak{p}_2$ is closed.

2 Modules

Exercise 2.1. *Show that* $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$ *if* m, n *are coprime.*

Solution. By Bézout's Identity, we know that, as m, n are coprime, $\exists\ a, b \in \mathbb{Z}$ such that $am + bn = 1$. Then, $\forall x \otimes y \in (\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$ we have that $x \otimes y = 1(x \otimes y) = (am + bn)(x \otimes y) = am(x \otimes y) + bn(x \otimes y) = amx \otimes y + x \otimes bny = 0$.

Exercise 2.2. *Let* A *be a ring,* $\mathfrak{a}$ *an ideal,* M *an* A*-module. Show that* $(A/\mathfrak{a}) \otimes_A M$ *is isomorphic to* $M/\mathfrak{a}M$*.*

Solution. Let's tensor the exact sequence $\mathfrak{a} \xrightarrow{incl} A \xrightarrow{\pi} A/\mathfrak{a} \to 0$, and we obtain the following sequence, which is exact by Proposition 2.18.

$$\mathfrak{a} \otimes M \xrightarrow{incl \otimes 1} A \otimes M \xrightarrow{\pi \otimes 1} A/\mathfrak{a} \otimes M \to 0$$

Therefore, that induces an isomorphism $A \otimes M/\mathfrak{a} \otimes M \cong A/\mathfrak{a} \otimes M$. On the other hand, let's consider the application $\varphi : \mathfrak{a} \otimes M \to \mathfrak{a}M$ defined by $\varphi(a \otimes m) = am$. It's clear that φ is exhaustive. Let's check the injectivity: $ax = 0 \Rightarrow 0 = 1 \otimes ax = a(1 \otimes x) = a \otimes x$, and therefore $\ker(\varphi) = (0)$ and that induces an isomorphism $\mathfrak{a} \otimes M \cong \mathfrak{a}M$. We also know that $A \otimes M \cong M$ by Proposition 2.14. Taking into account all these isomorphisms we finally get

$$A/\mathfrak{a} \otimes M \cong A \otimes M/\mathfrak{a} \otimes M \cong M/\mathfrak{a}M$$

Exercise 2.3. *Let* A *be a local ring,* M *and* N *finitely generated* A*-modules. Prove that if* $M \otimes N = 0$*, then* $M = 0$ *or* $N = 0$*.*

Solution. Let $k = A/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of A. Then, let's consider $M_k = k \otimes M$ and $N_k = k \otimes N$, which have an structure of k-vector spaces by extension of scalars. We observe that

$$M \otimes_A N = 0 \Rightarrow (M \otimes_A N)_k = 0 \Rightarrow M_k \otimes_k N_k = 0$$

As M_k and N_k are vector spaces, then $M_k \otimes_k N_k = 0 \Rightarrow M_k$ or $N_k = 0$, as otherwise $\exists u \neq 0 \in M_k$, $v \neq 0 \in N_k$. Then $\exists f, g$ linear applications such that $f : M_k \to k$, $g : N_k \to k$ such that $f(u) = 1$ and $g(v) = 1$. Then the application $\phi : M_k \times N_k \to k$ defined by $\phi(a, b) = f(a)g(b)$ is bilinear, and by the fundamental property of tensor product (Proposition 2.12), $\exists \phi' : M_k \times N_k$

linear such that $\phi'(u \otimes v) = f(u)g(v) = 1$, and by linearity of ϕ' that implies that $u \otimes v \neq 0$.

So we have proven that either M_k or $N_k = 0$. Without loss of generality, let's suppose $M_k = 0 \Rightarrow M \otimes k = 0$ as an A-module. By exercise 2, that implies that $M/\mathfrak{m}M = 0$ which means that $\mathfrak{m}M = M$. But $\mathfrak{m}$ is in fact the Jakobson radical, as A is local, and therefore by Nakayama's Lemma we have that $M = 0$.

Exercise 2.4. *Let M_i $(i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\iff$ each M_i is flat.*

Solution. M is flat $\iff$ $\forall$ exact sequence $0 \to N' \to N \to N'' \to 0$ the exact sequence $0 \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$ is exact. But it's clear that the sequence

$$0 \to \bigoplus_{i \in I} N'_i \to \bigoplus_{i \in I} N_i \to \bigoplus_{i \in I} N''_i \to 0$$

is exact $\iff$ each sequence $0 \to N'_i \to N_i \to N''_i \to 0$ is exact. Then it's enough to prove that $N \otimes (\bigoplus_{i \in I} M_i) \cong \bigoplus_{i \in I}(N \otimes M_i)$. That is the infinite case of Proposition 2.14*iii*), and a proof can be found here[1].

Exercise 2.5. *Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra.*

Solution. The elements of $A[x]$ are finite sums of elements of the form $a_i x^i$, with $a_i \in A$. Therefore we have that $A[x] \cong \bigoplus_{i \geq 0} A$. Taking into account that A is a flat A-module, because $A \otimes M \cong M$, and applying Exercise 4 we get that also $\bigoplus_{i \geq 0} A \cong A[x]$ is a flat A-module.

Exercise 2.6. *For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form*

$$m_0 + m_1 x + \cdots + m_r x^r \quad m_i \in M$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]$-module.

Show that $M[x] \cong A[x] \otimes_A M$.

[1] https://math.stackexchange.com/questions/563727/tensor-product-and-direct-sum

Solution. Defining the product of polynomials in the usual way, it's clear that $M[x]$ is an $A[x]$ module, because M is closed under product by elements of A and therefore $M[x]$ is closed under product of elements of $A[x]$.

Now let's consider the application $f : A[x] \times M \to M[x]$ defined by $f(\sum_i a_i x^i, m) = \sum_i a_i m x^i$, which is bilinear. Then that induces an A-module homomorphism $f' : A[x] \otimes M \to M[x]$ such that $f'((\sum_i a_i x^i) \otimes m) = \sum_i a_i m x^i$. Let's prove that it is in fact an isomorphism:

- **Surjectivity:** Given any element $\sum_i m_i x^i \in M[x]$, we consider the element $\sum_i (x^i \otimes m_i) \in A[x] \otimes M$ as it is a module and therefore closed under linear combinations. Then $f'(\sum_i (x^i \otimes m_i)) = \sum_i (f'(x_i \otimes m_i)) = \sum_i m_i x^i$, and therefore f' is surjective.

- **Injectivity:** Suppose that $\sum_i (a_i m) x^i = 0 \Rightarrow a_i m = 0 \; \forall i$. Therefore, $(\sum_i a_i x^i) \otimes m = \sum_i (a_i x^i \otimes m) = \sum_i (x^i \otimes a_i m) = 0$. That proves the injectivity of f'.

In conclusion, we have shown that $M[x] \cong A[x] \otimes_A M$.

Exercise 2.7. *Let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, is $m[x]$ a maximal ideal in $A[x]$?*

Solution. Let $p = \sum_{i=1}^n a_i x^i$ and $q = \sum_{j=1}^m b_j x^j$, with $pq \in \mathfrak{p}[x]$. We have to show that either p or $q \in \mathfrak{p}[x]$. We will proceed by induction on the degree of $pq = m + n$. If $m + n = 0$ then $p = a_0$, $q = b_0$ and $a_0 b_0 \in \mathfrak{p} \Rightarrow$ either $p = a_0$ or $q = b_0 \in \mathfrak{p}$. Now assuming that the statement is true for polynomials of degree $n + m - 1$ we will prove it for polynomials of degree $n + m$.

Let $p = a_n x^n + p'$ and $q = b_m x^m + q'$. As $pq \in \mathfrak{p}[x]$ all the coefficients of pq must belong to $\mathfrak{p}$. In particular, $a_n b_m \in \mathfrak{p} \Rightarrow a_n$ or $b_m \in \mathfrak{p}$. Without loss of generality we can suppose that $a_n \in \mathfrak{p}$. Then $a_n x^n (b_m x^m + q') \in \mathfrak{p}[x]$, and $pq - a_n x^n q = p'q \in \mathfrak{p}[x]$. As $\deg p'q < n + m$ we can apply induction hypothesis and either $q \in \mathfrak{p}[x]$ or $p' \in \mathfrak{p}[x] \Rightarrow p \in \mathfrak{p}[x]$ as $a_n \in \mathfrak{p}$.

However, maximal ideals are not so well behaved. For example, in the ring $\mathbb{Z}$ the ideal (2) is maximal but $(2)[x]$ is not maximal in $\mathbb{Z}[x]$ because $\mathbb{Z}[x]/(2)[x] \cong (\mathbb{Z}/(2))[x]$ which is not a field as for example the element $x \in (\mathbb{Z}/(2))[x]$ is not a unit (by Exercise 1.2).

Exercise 2.8. *(i) If M and N are flat A-modules, then so is $M \otimes_A N$.*

(ii) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Solution. (i) Let $0 \to M_0' \to M_0 \to M_0'' \to 0$ be an exact sequence of A-modules. Then, as M is flat, the sequence $0 \to M_0' \otimes_A M \to M_0 \otimes_A M \to M_0'' \otimes_A M \to 0$ is also exact. As N is exact, tensoring again the exact sequence with N remains exact, so $0 \to M_0' \otimes_A M \otimes_A N \to M_0 \otimes_A M \otimes_A N \to M_0'' \otimes_A M \otimes_A N \to 0$ is exact. And therefore tensoring an exact sequence with $M \otimes_A N$ mantains the exactness, which means that $M \otimes_A N$ is a flat A-module.

(ii) Given $0 \to M_0' \to M_0 \to M_0'' \to 0$ an exact sequence of A-modules, the sequence $0 \to M_0' \otimes_A B \to M \otimes_A B_0 \to M_0'' \otimes_A B \to 0$ is an exact sequence of A-modules, then it will also be an exact sequence of B-modules when considered by extension of scalars (restriction and extension of scalars when considered as functors between A-modules and B-modules are exact functors, as the sets and applications envolved in the sequence are still the same ones). Then tensoring with N also mantains the exactness of the sequence, ie

$$0 \to M_0' \otimes_A B \otimes_B N \to M_0 \otimes_A B \otimes_B N \to M_0'' \otimes_A B \otimes_B N \to 0$$

is exact. As $B \otimes_B N \cong N$ the sequence $0 \to M_0' \otimes_A N \to M_0 \otimes_A N \to M_0'' \otimes_A N \to 0$ is exact as a B-module sequence, and therefore also as an A-module sequence.

In conclusion, N is a flat A-module.

Exercise 2.9. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M" are finitely generated, then so is M.*

Solution. Let's name f, g the morphisms between $M' \to M$ and $M \to M''$, respectively. Then, M' is finitely generated $\Rightarrow \mathrm{Im}(f) \subseteq M$ is finitely generated, and let $x_1, \dots, x_n$ be generators of $\mathrm{Im}(f)$. On the other hand, $M/M' \cong M''$ which is also finitely generated. Let $y_1, \dots y_m \in M$ such that their projections in M/M' are generators. Then $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ generate M.

Exercise 2.10. *Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.*

Solution. By corollary 2.7, given N finitely generated, N' sumbodule of N and $\mathfrak{a}$ an ideal contained in the Jacobson radical, we have $N = \mathfrak{a}N + N' \Rightarrow N' = N$. In the situation of this exercise, $u' : M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective and therefore $\forall x \in N$, $x = x_1 + x_2$, with $x_1 \in \mathfrak{a}N$ and $x_2 \in \mathrm{Im}(u)$. So we have that $N = \mathfrak{a}N + \mathrm{Im}(u) \Rightarrow \mathrm{Im}(u) = N \Rightarrow u$ is surjective.

Exercise 2.11. *Let A be a ring $\neq 0$. Show that $A^m \cong A^n \Rightarrow m = n$.*

(i) If $\phi : A^m \to A^n$ is surjective, then $m \geq n$.

(ii) If $\phi : A^m \to A^n$ is injective, is it always the case that $m \leq n$?

Solution. Let $\mathfrak{m}$ be a maximal ideal of A, $k = A/\mathfrak{m}$ and let $\phi : A^m \to A^n$ be an isomorphism. As tensor product commutes with direct sums, $A/\mathfrak{m} \otimes A^m \cong \bigoplus_{i=1}^m A/\mathfrak{m} \otimes A \cong k^m$.

In conclusion $A/\mathfrak{m} \otimes A^m$ is a m-dimensional k-vector space and similarly, $A/\mathfrak{m} \otimes A^n$ is a n-dimensional k-vector space. The isomorphism between A^m and A^n is equivalent to the exactness of the sequence $0 \to A^m \to A^n \to 0$. Then, by 2.18, tensoring the sequence preserves the exactness, and therefore $0 \to A/\mathfrak{m} \otimes A^m \to A/\mathfrak{m} \otimes A^n \to 0$ is exact, which implies that $1 \otimes \phi : A/\mathfrak{m} \otimes A^m \to A/\mathfrak{m} \otimes A^n$ is an isomorphism of vector spaces. So the dimensions of the spaces must be the same $\Rightarrow n = m$.

(i) The same proof works for surjectivity, as it is preserved by tensoring (Proposition 2.18).

Exercise 2.12. *Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\ker(\phi)$ is finitely generated.*

Solution. Let $\{e_i\}_{i=1}^n$ be the natural generator set of A^n. As ϕ is surjective, $\exists u_i$ such that $\phi(u_i) = e_i$. Let $N \subseteq M$ be the submodule of M generated by $\{u_i\}_{i=1}^n$. The restriction of ϕ to N gives an isomorphism $N \cong A^n$, as $\phi(\sum_i a_i u_i) = 0 \Rightarrow \sum_i a_i e_i = 0 \Rightarrow a_i = 0 \forall i$.

Given $x \in M$, let $\phi(x) = (a_1, \dots, a_n)$. Then, let $y = x = x - \sum_{i=1}^n a_i u_i$, and we have that $x = x - y + y$ with $y \in N$ and $\phi(x - y) = \phi(x) - \phi(y) = 0 \Rightarrow x - y \in \ker(\phi)$. This expression of x as a sum of elements from N and $\ker(\phi)$ is unique, as it is completely determined by $\phi(x)$. In consequence, $M = N \oplus \ker(\phi)$. If $\ker(\phi)$ was not finitely generated, we would have that also M is not finitely generated, which is a contradiction.

Exercise 2.13. *Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.*

Solution. $\forall b \otimes y \in N_B$, we can write $b \otimes y = (1 \otimes by) + (b \otimes y - 1 \otimes by)$. It's clear that $1 \otimes by \in \mathrm{Im}(g)$. On the other hand, $p(b \otimes y - 1 \otimes by) = p(b \otimes y) - p(1 \otimes by) = 0 \Rightarrow b \otimes y - 1 \otimes by \in \ker(p)$. That expression of an element of N_B as a sum of elements of $\mathrm{Im}(g)$ and $\ker(p)$ is unique as it's completely determined by $p(b \otimes y)$. Therefore, $N_B = \mathrm{Im}(g) \oplus \ker(p)$.

In addition, $p \circ g = Id_N$. Then, $1 \otimes y = 0 \Rightarrow y = p(1 \otimes y) = 0$, which proves that g is injective.

Exercise 2.14. *A partially ordered set I is said to be a directed set if for each pair i, j in I there exists $k \in I$ such that $i \leq k$ and $j \leq k$.*

Let A be a ring, let I be a directed set and let $(M_i)_{i \in I}$ be a family of A-modules indexed by I. For each pair i, j in I such that $i \leq j$, let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism, and suppose that the following axioms are satisfied:

1. *μ_{ii} is the identity mapping of M_i, $\forall i \in I$.*
2. *$\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.*

Then the modules M_i and homomorphisms μ_{ij} are said to form a direct system $\boldsymbol{M} = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the direct limit of the direct system $\boldsymbol{M}$. Let C be the direct sum of the M_i and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i. The module M, or more correctly the pair consisting of M and the family of homomorphisms $\mu_i : M_i \to M$, is called the direct limit of the direct system M, and is written $\varinjlim M_i$. From the construction it is clear that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$.

3 Rings and Modules of Fractions

Exercise 3.1. *Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.*

Solution. Let $m_1, \dots, m_n$ generators of M.

$\boxed{\Leftarrow}$ $S^{-1}M = 0 \Rightarrow m_i/s = 0 \Rightarrow \exists t_i \in S$ such that $m_i t_i = 0$. Let $s = \prod_{i=1}^n t_i$, and we have that $sm_i = 0\ \forall i$. Then $\forall m \in M$, $m = \sum_{i=1}^n a_i m_i \Rightarrow sm = 0$. Then s satisfies $sM = 0$.

$\boxed{\Rightarrow}$ Let $m/t \in S^{-1}M$. We know that $\exists s \in S$ such that $sm = 0 \Rightarrow (m * 1 - 0 * t)s = 0 \Rightarrow m/t = 0/1$, and therefore $m/t = 0\ \forall m/t \in S^{-1}M \Rightarrow S^{-1}M = 0$.

Exercise 3.2. *Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1 + \mathfrak{a}$. Show that $S{-}1\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants.*

Solution. We will use the characterization of Jacobson radical given by Proposition 1.9. Let $\frac{a}{s_1} \in S^{-1}\mathfrak{a}$ and $\frac{x}{s_2} \in S^{-1}A$. $s_i = 1 + b_i$, with $b_i \in \mathfrak{a}$. Then,

$$1 - \frac{a}{s_1}\frac{x}{s_2} = \frac{s_1 s_2 - ax}{s_1 s_2} = \frac{1 + b_1 + b_2 + b_1 b_2 - ax}{s_1 s_2}$$

As $y = b_1 + b_2 + b_1 b_2 - ax \in \mathfrak{a}$ then $1 + y \in S$ and therefore $1 - \frac{a}{s_1}\frac{x}{s_2}$ is a unit $\Rightarrow S{-}1\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Now let's prove 2.5. Let $\mathfrak{a}$ such that $\mathfrak{a}M = M$. Then $(S^{-1}\mathfrak{a})(S^{-1}M) = S^{-1}M$. Let $S = 1 + \mathfrak{a}$. Then $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical, and as M is finitely generated, $S^{-1}M$ is also finitely generated. We can apply Nakayama's Lemma $\Rightarrow S^{-1}M = 0$. Now, by Exercise 3.1, $\exists s = 1 (mod \mathfrak{a})$ such that $sM = 0$.

Exercise 3.3. *Let A be a ring, let S and T be two multiplicatively closed subsets of A, and let U be the image of T in $S^{-1}A$. Show that the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic.*

Solution. Let's define $U^{-1}(S^{-1}A) \to (ST)^{-1}A$ that maps $\frac{y/s}{t/s'} \mapsto \frac{ys'}{ts}$. We will prove that it is an isomorphism. Fist, let's see that it is well defined.

$$\frac{y_1/s_1}{t_1/s_1'} = \frac{y_2/s_2}{t_2/s_2'} \iff \exists \frac{u}{v} \in U \text{ such that } \frac{u}{v}\left(\frac{y_1}{s_1}\frac{t_2}{s_2'} - \frac{y_2}{s_2}\frac{t_1}{s_1'}\right) = 0$$

$$u\left(\frac{y_1t_2}{s_1s_2'v}-\frac{y_2t_2}{s_2s_1'v}\right)=0\Rightarrow uv(y_1t_2s_2s_1'-y_2t_1s_1s_2')=0$$

In conclusion, $\frac{y_1s_1'}{t_1s_1}=\frac{y_2s_2'}{t_2s_2}\in(ST)^{-1}A$, and the application is well defined (the image doesn't depend on the representative).

- **Surjective:** Every $\frac{y}{ts}\in(ST)^{-1}A$ is image of $\frac{y/s}{t/1}$.

- **Injective:** Suppose $\frac{ys'}{ts}=0\Rightarrow\exists u\in S$, $v\in T$ such that $uvs'y=0\Rightarrow$ $(v(us')y-0)=0\Rightarrow\frac{vy}{1}=0\in U^{-1}(S^{-1}A)\Rightarrow\frac{y/s}{t/s'}=0$.

Exercise 3.4. *Let $f:A\to B$ be a homomorphism of rings and let S be a multiplicatively closed subset of A. Let $T=f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.*

Solution. First we note that the product of elements of A by elements of B can be defined by restriction of scalars. We define $\phi:S^{-1}B\to T^{-1}B$ that maps $\frac{b}{s}\mapsto\frac{b}{f(s)}$. ϕ is injective, as $\frac{b}{f(s)}=0\Rightarrow\exists t=f(s')\in f(S)$ such that $bt=bf(s')=bs'=0$. Then, $s'(b-0*s)=0\Rightarrow\frac{b}{s}=0$. ϕ is exhaustive because $\forall t\in T$, $t=f(s)$ for a certain s; that is, $\frac{b}{t}=\frac{b}{f(s)}\in\text{Im}(\phi)$.

Exercise 3.5. *Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain?*

Solution. Let $x\in A$ is nilpotent ($\iff x\in\mathfrak{R}$). Now let's consider $\text{Ann}(x)$ which is an ideal of A and therefore it's contained in a maximal ideal (in particular prime) $\mathfrak{p}_x$. Let $S=A-\mathfrak{p}_x$. We know that $S^{-1}\mathfrak{R}$ is the nilradical of $S^{-1}A$ by Corollary 3.12. $S^{-1}\mathfrak{R}=0\Rightarrow\forall y\in\mathcal{R}$, $\exists s\in S$ such that $ys=0$. In particular, take $y=x$ and, as $\text{Ann}(x)\cap S=\emptyset$, then $x=0$. This argument is valid $\forall x\in\mathfrak{R}\Rightarrow$ A has no nilpotent element.

On the contrary, if $A_\mathfrak{p}$ is an integral domain $\forall\mathfrak{p}$, A is not necessarily an integral domain. Let's show it with a counter-example. Let $A=k^2$, with k a field. It's clear that A is not integral, as $(1,0)(0,1)=0$. As showed in Exercise 1.22, the only prime ideals of A are the ideals generated by $(0,1)$ and $(1,0)$. Let $\mathfrak{p}=((0,1))$ and let's consider the ring $A_\mathfrak{p}$. Let $x/s\neq 0$ and $y/t\neq 0\Rightarrow\nexists s\in A-\mathfrak{p}$ such that $sx=0$ or $sy=0$, which means that the first coordinate of x and $y \neq 0$. Then, $\frac{x}{s}\frac{y}{t} \neq 0$, and we have showed that $A_{\mathfrak{p}}$ is integral. The same argument works with the other prime ideal $\mathfrak{q} = ((1,0))$ by symmetry and therefore $A_{\mathfrak{p}}$ is an integral domain $\forall \mathfrak{p}$.