

Exercises Hartshorne

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1 Sheaves

Exercise 1.1. *Let A be an abelian group, and define the constant presheaf associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \emptyset$, with restriction maps the identity. Show that the constant sheaf \mathcal{A} defined in the text is the sheaf associated to this presheaf.*

Solution. Let s_F denote the constant presheaf. Let's first see that each stalk \mathcal{F}_P is a copy of A . Indeed, the elements of \mathcal{F}_P are represented by pairs $\langle U, s \rangle$, with U open neighbourhood of P and $s \in A$. As the restriction maps are the identity, two pairs $\langle U, s \rangle$ and $\langle V, t \rangle$ represent the same element if and only if $s = t$, so $\mathcal{F}_P = A$.

Let s be an application from U to $\bigcup_{P \in U} \mathcal{F}_P$ satisfying properties (1) and (2) from the definition of associated sheaf. By (1), $s(P) \in \mathcal{F}_P$ is an element of A , and therefore s can be regarded as an application from U to A (that we will denote s'). In addition, let $B \subseteq A$. For each $P \in s'^{-1}(B)$, $\exists V_P$ neighbourhood of P such that $s'(V_P) = t \in B$. Then $s'^{-1}(B) = \bigcup_{P \in s'^{-1}(B)} V_P$ which is open. We have proved that the antiimage of every subset is open and therefore s' is continuous with A being given the discrete topology.

Reciprocally, any continuous application s' from U to A can be regarded as an application s from U to $\bigcup_{P \in U} \mathcal{F}_P$, defining $s(P) = s'(P) \in \mathcal{F}_P$. This assignation guarantees that s satisfies (1). In addition, for each $P \in U$, the set $V = s'^{-1}(s'(P))$ is an open neighbourhood of P (by continuity of s'), and every $Q \in V$ has the same image $s'(P)$, which proves that s satisfies (2).

In conclusion, $\mathcal{F}^+(U)$ is the group of continuous maps from U into A , and therefore \mathcal{F}^+ is indeed the sheaf \mathcal{A} defined in the text.

- Exercise 1.2.** a) For any morphism of sheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ show that for each point P , $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$.
- b) Show that φ is injective (respectively surjective) if and only if the induced map on the stalks φ_P is injective (respectively surjective) for all P .
- c) Show that a sequence $\dots \rightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \rightarrow \dots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Solution. a) $(\ker \varphi)_P = \{(U, s), s \in \ker(\varphi(U))\}$, modulo the usual germ equivalence relationship. Thus it can be regarded as a subset of \mathcal{F}_P as $\ker(\varphi(U)) \subseteq \mathcal{F}(U)$. On the other side $\ker(\varphi_P)$ is a subset of \mathcal{F}_P . To see that the two sets are equal it's enough to check the double inclusion. Let $(U, s) \in (\ker \varphi)_P$. Then, $\varphi_P((U, s)) = (U, (\varphi(U))(s)) = (U, 0) = 0 \Rightarrow (U, s) \in \ker(\varphi_P)$. Reciprocally, given $(U, s) \in \ker(\varphi_P) \Rightarrow \exists V \subset U$ such that $\varphi(U)(s)|_V = 0$. As restrictions commute with morphisms of sheaves, $\varphi(V)(s|_V) = \varphi(U)(s)|_V = 0$. Then, $(U, s) = (V, s|_V) \in (\ker \varphi)_P$. In conclusion, $(\ker \varphi)_P = \ker(\varphi_P)$.

As $\mathcal{F}_P = \mathcal{F}_P^+$, $(\operatorname{im} \varphi)_P$ is equal to the stack of the presheaf image at point P . $\operatorname{im}(\varphi_P) = \{(U, s) \in \mathcal{G}_P \text{ such that } \exists (V, t) \in \mathcal{F}_P | \varphi_P((V, t)) = (U, s)\}$. But as $\varphi_P((V, t)) = (V, \varphi(V)(t))$ then $(U, s) \in \operatorname{im}(\varphi_P) \iff \exists W$ neighbourhood of P , $W \subseteq V \cap U$ such that $\varphi(V)(t)|_W = s|_W \iff \varphi(W)(t|_W) = s|_W \iff (U, s) = (W, \varphi(W)(t|_W)) \iff (U, s) \in (\operatorname{im} \varphi)_P$.

- b) φ injective $\Rightarrow \ker \varphi = 0 \Rightarrow (\ker \varphi)_P = 0 \forall P$. Using part a) of the problem, $\ker \varphi_P = (\ker \varphi)_P = 0 \Rightarrow \varphi_P$ is injective $\forall P$. Reciprocally, let $x \in \ker \varphi(U)$. $\forall P \in U, (\ker \varphi)_P = 0$ so the image of x in the stalk $(\ker \varphi)_P$ is zero, which means that $\exists W_P \subseteq U$ neighbourhood of P such that $x|_{W_P} = 0$. But open sets W_P cover U and therefore, by property (3) of the definition of sheaves, $x = 0$. In conclusion, $\ker \varphi(U) = 0 \forall U \Rightarrow \ker \varphi = 0 \Rightarrow \varphi$ injective.

We proceed similarly with the surjectivity. $\operatorname{im} \varphi = \mathcal{G} \Rightarrow (\operatorname{im} \varphi)_P = \mathcal{G}_P \Rightarrow \operatorname{im}(\varphi_P) = \mathcal{G}_P \Rightarrow \varphi_P$ surjective. To prove the other implication, First we will prove a fact that is stated but not proved in the text: $\mathcal{F}^+ \cong \mathcal{F}$ if \mathcal{F} is already a sheaf. Given an open set U , let V_P be the

neighbourhood of P contained in U such that $\exists t \in \mathcal{F}(V_P)$ such that $t_Q = s(Q) \forall Q \in V_P$. The sets V_P cover U , and given two of these sets, V, V' and the respective elements t, t' we have that $\overline{(V', t')} = \overline{(V, t)}$ in every stalk $\mathcal{F}_Q \Rightarrow \exists W_Q$ such that $t'|_{W_Q} = t|_{W_Q} \forall Q \in V \cap V'$. Then these W_Q cover $V \cap V'$, and by property (3) applied to $t'|_{V \cap V'} - t|_{V \cap V'}$, we have that $t|_{V \cap V'} = t'|_{V \cap V'}$. Then, by property (4) applied to the sets V_P , $\exists t \in \mathcal{F}(U)$ such that $t_Q = s(Q) \forall Q \in U$, which means that each application s is uniquely determined by $t \in \mathcal{F}(U)$, and then $\mathcal{F}^+(U) \cong \mathcal{F}(U)$. Now it's easy to check that φ is surjective. We have that $(\text{im } \varphi)_P = \text{im}(\varphi_P) = \mathcal{G}_P$ and so we have that $\mathbf{m}\varphi(U)$ is the set of functions s from U to $\bigcup_{P \in U} \mathcal{G}_P$, which means that $\text{im } \varphi$ is in fact $\mathcal{G}^+ \cong \mathcal{G}$ as \mathcal{G} is already a sheaf.

- c) Given a sequence of sheaves and morphisms $\dots \rightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \rightarrow \dots$, it is exact $\iff \ker \varphi^i = \text{im } \varphi^{i-1}$. If the sequence is exact, taking direct limits at both sides and using section a) we have that $\ker(\varphi_P^i) = (\ker \varphi^i)_P = (\text{im } \varphi^{i-1})_P = \text{im}(\varphi_P^{i-1})$, and so the sequence of stalks at each point P is exact.

The other implication is consequence of a more general result that we will prove now: Two subsheaves are equal \iff the corresponding stalks at each point are equal. Let $\mathcal{F}_1, \mathcal{F}_2$ be two subsheaves of \mathcal{F} , such that $(\mathcal{F}_1)_P = (\mathcal{F}_2)_P$. Let $t \in \mathcal{F}_1(U)$. For every $P \in U \exists V_P$ neighbourhood of P and $s \in \mathcal{F}_2(V)$ such that $s_P = t_P$. The sets $V_P \cap U$ cover U , and given two of these sets, V, V' and the respective elements s, s' we have that $\overline{(V', s')} = \overline{(V, s)} = \overline{(U, t)}$ in every stalk $\mathcal{F}_Q \Rightarrow \exists W_Q$ such that $s'|_{W_Q} = s|_{W_Q} \forall Q \in V \cap V'$. Then these W_Q cover $V \cap V'$, and by property (3) applied to $s'|_{V \cap V'} - s|_{V \cap V'}$, we have that $s|_{V \cap V'} = s'|_{V \cap V'}$. Then, by property (4) applied to the sets $V_P \cap U$, $\exists r \in \mathcal{F}_2(U)$ such that $t_P = r_P \forall P$. By property (3) applied to $r - t$ we get $s = t$ and so $t \in \mathcal{F}_2(U)$. So $\mathcal{F}_1(U) \subseteq \mathcal{F}_2(U)$ and by symmetry $\mathcal{F}_1(U) = \mathcal{F}_2(U)$.

- Exercise 1.3.** a) Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on X . Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}$ of U , and there are elements $t_i \in \mathcal{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$, for all i .
- b) Give an example of a surjective morphism of sheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ and an open set U such that $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is not injective.

Solution. a) From problem 1.2, φ is surjective \iff the induced morphism on every stalk is. Suppose φ_P is surjective, then given $U \subseteq X$ open set, $s \in \mathcal{G}(U)$, $\forall P \in U \exists V$ neighbourhood of P and $t \in \mathcal{F}(V)$ such that $\overline{(U, s)} = \overline{(V, \varphi(t))} \Rightarrow \exists W_P \subseteq U \cap V$ such that $\varphi(t|_{W_P}) = s|_{W_P}$, and so $\{W_P\}$ is the covering that satisfies the desired property. Reciprocally, let $\overline{(U, s)} \in \mathcal{G}_P$. Then $\forall P \in U \exists i$ such that $P \in U_i \Rightarrow \overline{(U, s)} = \overline{(U_i, \varphi(U_i)(t_i))} = \varphi_P(\overline{(U_i, t_i)}) \Rightarrow \varphi_P$ is surjective.

b)

Exercise 1.4. a) Using 1.2 b) and the fact that $\mathcal{F}_P^+ = \mathcal{F}_P$, the map φ^+ is injective \iff the maps $\varphi_P : \mathcal{F}_P \rightarrow \mathcal{G}_P$ on the stalks are injective. Let $\overline{(U, s)} \in \mathcal{F}_P$ such that $\varphi_P(\overline{(U, s)}) = 0 \Rightarrow \exists W \subset U$ such that $\varphi(U)(s)|_W = 0 \Rightarrow \varphi(W)(s|_W) = 0$. But as $\varphi(U)$ is injective $\forall U$, then $s|_W = 0$ and therefore $\overline{(U, s)} = \overline{(W, s|_W)} = 0$ and thus φ_P is injective $\Rightarrow \varphi^+$ is injective.

b)

Solution.