Exercises Hartshorne

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1 Sheaves

Exercise 1.1. *Let A be an abelian group, and define the constant presheaf associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathcal{A}$ defined in the text is the sheaf associated to this presheaf.*

Solution. Let sF denote the constant presheaf. Let's first see that each stalk $\mathcal{F}_P$ is a copy of A. Indeed, the elements of $\mathcal{F}_P$ are represented by pairs $\langle U, s\rangle$, with U open neighbourhood of P and $s \in A$. As the restriction maps are the identity, two pairs $\langle U, s\rangle$ and $\langle V, t\rangle$ represent the same element if and only if $s = t$, so $\mathcal{F}_P = A$.

Let s be an application from U to $\bigcup_{P\in U} \mathcal{F}_P$ satisfying properties (1) and (2) from the definition of associated sheaf. By (1), $s(P) \in \mathcal{F}_P$ is an element of A, and therefore s can be regarded as an application from U to A (that we will denote s'). In addition, let $B \subseteq A$. For each $P \in s'^{-1}(B)$, $\exists V_P$ neighbourhood of P such that $s'(V_P) = t \in B$. Then $s'^{-1}(B) = \bigcup_{P\in s'^{-1}(B)} V_P$ which is open. We have proved that the antiimage of every subset is open and therefore s' is continuos with A being given the discrete topology.

Reciprocally, any countinuous application s' from U to A can be regarded as an application s from U to $\bigcup_{P\in U} \mathcal{F}_P$, defining $s(P) = s'(P) \in \mathcal{F}_P$. This assignation guarantees that s satisfies (1). In addition, for each $P \in U$, the set $V = s'^{-1}(s'(P))$ is an open neighbourhood of P (by continuity of s'), and every $Q \in V$ has the same image $s'(P)$, which proves that s satisfies (2).

In conclusion, $\mathcal{F}^+(U)$ is the group of continuous maps from U into A, and therefore $\mathcal{F}^+$ is indeed the sheaf $\mathcal{A}$ defined in the text.