# Exercises Hartshorne

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## 1 Sheaves

**Exercise 1.1.** *Let $A$ be an abelian group, and define the constant presheaf associated to $A$ on the topological space $X$ to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.*

**Solution.** Let $sF$ denote the constant presheaf. Let's first see that each stalk $\mathscr{F}_P$ is a copy of $A$. Indeed, the elements of $\mathscr{F}_P$ are represented by pairs $\langle U, s\rangle$, with $U$ open neighbourhood of $P$ and $s \in A$. As the restriction maps are the identity, two pairs $\langle U, s\rangle$ and $\langle V, t\rangle$ represent the same element if and only if $s = t$, so $\mathscr{F}_P = A$.

Let $s$ be an application from $U$ to $\bigcup_{P\in U} \mathscr{F}_P$ satisfying properties (1) and (2) from the definition of associated sheaf. By (1), $s(P) \in \mathscr{F}_P$ is an element of $A$, and therefore $s$ can be regarded as an application from $U$ to $A$ (that we will denote $s'$). In addition, let $B \subseteq A$. For each $P \in s'^{-1}(B)$, $\exists V_P$ neighbourhood of $P$ such that $s'(V_P) = t \in B$. Then $s'^{-1}(B) = \bigcup_{P\in s'^{-1}(B)} V_P$ which is open. We have proved that the antiimage of every subset is open and therefore $s'$ is continuos with $A$ being given the discrete topology.

Reciprocally, any countinuous application $s'$ from $U$ to $A$ can be regarded as an application $s$ from $U$ to $\bigcup_{P\in U} \mathscr{F}_P$, defining $s(P) = s'(P) \in \mathscr{F}_P$. This assignation guarantees that $s$ satisfies (1). In addition, for each $P \in U$, the set $V = s'^{-1}(s'(P))$ is an open neighbourhood of $P$ (by continuity of s'), and every $Q \in V$ has the same image $s'(P)$, which proves that $s$ satisfies (2).

In conclusion, $\mathscr{F}^+(U)$ is the group of continuous maps from $U$ into $A$, and therefore $\mathscr{F}^+$ is indeed the sheaf $\mathscr{A}$ defined in the text.

**Exercise 1.2.** *a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ show that for each point $P$, $(\ker\varphi)_P = \ker(\varphi_P)$ and $(im\varphi)_P = im(\varphi_P)$.*

*b) Show that $\varphi$ is injective (respectively surjective) if and only if the induced map on the stalks $\varphi_P$ is injective (respectilevy surjective) for all $P$.*

*c) Show that a sequence $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.*

**Solution.** a) $(\ker\varphi)_P = \{(U,s),\, s \in \ker(\varphi(U))\}$, modulo the usual germ equivalence relationship. Thus it can be regarded as a subset of $\mathscr{F}_P$ as $\ker(\varphi(U)) \subseteq \mathscr{F}(U)$. On the other side $\ker(\varphi_P)$ is a subset of $\mathscr{F}_P$. To see that the two sets are equal it's enough to check the double inclusion. Let $\overline{(U,s)} \in (\ker\varphi)_P$. Then, $\varphi_P(\overline{(U,s)}) = \overline{(U,(\varphi(U))(s))} = \overline{(U,0)} = 0 \Rightarrow \overline{(U,s)} \in \ker(\varphi_P)$. Reciprocally, given $\overline{(U,s)} \in \ker(\varphi_P) \Rightarrow \exists V \subset U$ such that $\varphi(U)(s)|_V = 0$. As restrictions commute with morphisms of sheaves, $\varphi(V)(s|_V) = \varphi(U)(s)|_V = 0$. Then, $\overline{(U,s)} = \overline{(V,s|_V)} \in (\ker\varphi)_P$. In conclusion, $(\ker\varphi)_P = \ker(\varphi_P)$.

As $\mathscr{F}_P = \mathscr{F}^+_P$, $(\mathrm{im}\varphi)_P$ is equal to the stack of the presheaf image at point P. $\mathrm{im}(\varphi_P) = \{\overline{(U,s)} \in \mathscr{G}_P$ such that $\exists\overline{(V,t)} \in \mathscr{F}_P | \varphi_P(\overline{(V,t)} = \overline{(U,s)}\}$. But as $\varphi_P(\overline{(V,t)}) = \overline{(V,\varphi(V)(t))}$ then $\overline{(U,s)} \in \mathrm{im}(\varphi_P) \iff \exists W$ neighbourhood of $P$, $W \subseteq V \cap U$ such that $\varphi(V)(t)|_W = s_W \iff \varphi(W)(t|_W) = s|_W \iff \overline{(U,s)} = \overline{(W,\varphi(W)(t|_W))} \iff \overline{(U,s)} \in (\mathrm{im}\varphi)_P$.

b) $\varphi$ injective $\Rightarrow \ker\varphi = 0 \Rightarrow (\ker\phi)_P = 0\,\forall P$. Using part a) of the problem, $\ker\varphi_P = (\ker\varphi)_P = 0 \Rightarrow \varphi_P$ is injective $\forall P$. Reciprocally, let $x \in \ker\varphi(U)$. $\forall P \in U, (\ker\phi)_P = 0$ so the image of $x$ in the stalk $(\ker\phi)_P$ is zero, which means that $\exists W_P \subseteq U$ neighbourhood of $P$ such that $x|_{W_P} = 0$. But open sets $W_P$ cover $U$ and therefore, by property (3) of the definition of shieves, $x = 0$. In conclusion, $\ker\varphi(U) = 0\ \forall U \Rightarrow \ker\varphi = 0 \Rightarrow \varphi$ injective.

We proceed similarly with the surjectivity. $\mathrm{im}\varphi = \mathscr{G} \Rightarrow (\mathrm{im}\varphi)_P = \mathscr{G}_P \Rightarrow \mathrm{im}(\varphi_P) = \mathscr{G}_P \Rightarrow \varphi_P$ surjective. To prove the other implication, First we will prove a fact that is stated but not proved in the text: $\mathscr{F}^+ \cong \mathscr{F}$ if $\mathscr{F}$ is already a sheaf. Given an open set $U$, let $V_P$ be the neighbourhood of $P$ contained in $U$ such that $\exists t \in \mathscr{F}(V_P)$ such that $t_Q = s(Q)\,\forall Q \in V_P$. The sets $V_P$ cover $U$, and given two of these sets, $V, V'$ and the respective elements $t, t'$ we have that $\overline{(V', t')} = \overline{(V, t)}$ in every stalk $\mathscr{F}_Q \Rightarrow \exists W_Q$ such that $t'|_{W_Q} = t|_{W_Q} \forall Q \in V \cap V'$. Then these $W_Q$ cover $V \cap V'$, and by property (3) applied to $t'|_{V\cap V'} - t_{V\cap V'}$, we have that $t_{V\cap V'} = t'_{V\cap V'}$. Then, by property (4) applied to the sets $V_P$, $\exists t \in \mathscr{F}(U)$ such that $t_Q = s(Q)\ \forall Q \in U$, which means that each application $s$ is uniquely determined by $t \in \mathscr{F}(U)$, and then $\mathscr{F}^+(U) \cong \mathscr{F}(U)$. Now it's easy to check that $\varphi$ is surjective. We have that $(\mathrm{im}\varphi)_P = \mathrm{im}(\varphi_P) = \mathscr{G}_P$ and so we have that $\mathfrak{m}\varphi(U)$ is the set of functions $s$ from $U$ to $\bigcup_{P\in U} \mathscr{G}_P$, which means that $\mathrm{im}\varphi$ is in fact $\mathscr{G}^+ \cong \mathscr{G}$ as $\mathscr{G}$ is already a sheaf.

c) Given a sequence of sheaves and morphisms $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$, it is exact $\iff \ker\varphi^i = \mathrm{im}\varphi^{i-1}$. If the sequence is exact, taking direct limits at both sides and using section a) we have that $\ker(\varphi^i_P) = (\ker\varphi^i)_P = (\mathrm{im}\varphi^{i-1})_P = \mathrm{im}(\varphi^{i-1}_P)$, and so the sequence of stalks at each point $P$ is exact.

   The other implicaton is consequence of a more general result that we will prove now: Two subsheaves are equal $\iff$ the corresponding stalks at each point are equal. Let $\mathscr{F}_1, \mathscr{F}_2$ be two subsheaves of $\mathscr{F}$, such that $(\mathscr{F}_1)_P = (\mathscr{F}_2)_P$. Let $t \in \mathscr{F}_1(U)$. For every $P \in U \exists V_P$ neighbourhood of $P$ and $s \in \mathscr{F}_2(V)$ such that $s_P = t_P$. The sets $V_P \cap U$ cover $U$, and given two of these sets, $V, V'$ and the respective elements $s, s'$ we have that $\overline{(V', s')} = \overline{(V, s)} = \overline{(U, t)}$ in every stalk $\mathscr{F}_Q \Rightarrow \exists W_Q$ such that $s'|_{W_Q} = s|_{W_Q} \forall Q \in V \cap V'$. Then these $W_Q$ cover $V \cap V'$, and by property (3) applied to $s'|_{V\cap V'} - s|_{V\cap V'}$, we have that $s|_{V\cap V'} = s'|_{V\cap V'}$. Then, by property (4) applied to the sets $V_P \cap U$, $\exists r \in \mathscr{F}_2(U)$ such that $t_P = r_P \forall P$. By property (3) applied to $r - t$ we get $s = t$ and so $t \in \mathscr{F}_2(U)$. So $\mathscr{F}_1(U) \subseteq \mathscr{F}_2(U)$ and by symmetry $\mathscr{F}_1(U) = \mathscr{F}_2(U)$.

**Exercise 1.3.** *a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on $X$. Show that $\varphi$ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of $U$, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$, for all $i$.*

*b) Give an example of a surjective morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ and an open set $U$ such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not injective.*

**Solution.** a) From problem 1.2, $\varphi$ is surjective $\iff$ the induced morphism on every stalk is. Suppose $\varphi_P$ is surjective, then given $U \subseteq X$ open set, $s \in \mathscr{G}(U)$, $\forall P \in U \exists V$ neighbourhood of $P$ and $t \in \mathscr{F}(V)$ such that $\overline{(U, s)} = \overline{(V, \varphi(t))} \Rightarrow \exists W_P \subseteq U \cap V$ such that $\varphi(t|_{W_P}) = s|_{W_P}$, and so $\{W_P\}$ is the covering that satisfies the desired property. Reciprocally, let $\overline{(U, s)} \in \mathscr{G}_P$. Then $\forall P \in U \exists i$ such that $P \in U_i \Rightarrow \overline{(U, s)} = \overline{(U_i, \varphi(U_i)(t_i))} = \varphi_P(\overline{(U_i, t_i)})$.

b)