Exercises Hartshorne

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1 Sheaves

Exercise 1.1. *Let A be an abelian group, and define the constant presheaf associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.*

Solution. Let $\mathscr{F}$ denote the constant presheaf. Let's first see that each stalk $\mathscr{F}_P$ is a copy of A. Indeed, the elements of $\mathscr{F}_P$ are represented by pairs $\langle U, s\rangle$, with U open neighbourhood of P and $s \in A$. As the restriction maps are the identity, two pairs $\langle U, s\rangle$ and $\langle V, t\rangle$ represent the same element if and only if $s = t$, so $\mathscr{F}_P = A$.

Let s be an application from U to $\bigcup_{P\in U} \mathscr{F}_P$ satisfying properties (1) and (2) from the definition of associated sheaf. By (1), $s(P) \in \mathscr{F}_P$ is an element of A, and therefore s can be regarded as an application from U to A. In addition, let $B \subseteq A$. For each $P \in s^{-1}(B)$, $\exists V_P$ neighbourhood of P such that $s(V_P) = t \in B$. Then $s^{-1}(B) = \bigcup_{P\in s^{-1}(B)} V_P$ which is open. We have proved that the antiimage of every subset is open and therefore s is continuous with A being given the discrete topology.

Reciprocally, any countinuous application s from U to A can be regarded as an application from U to $\bigcup_{P\in U} \mathscr{F}_P$, and as $\mathscr{F}_P = A$, s satisfies (1). In addition, for each $P \in U$, the set $V = s^{-1}(s(P))$ is an open neighbourhood of P (by continuity of s), and every $Q \in V$ has the same image $s(P)$, which proves that s satisfies (2).

In conclusion, $\mathscr{F}^+(U)$ is the group of continuous maps from U into A, and therefore $\mathscr{F}^+$ is indeed the sheaf $\mathscr{A}$ defined in the text.

Exercise 1.2. *a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ show that for each point P, $(\ker\varphi)_P = \ker(\varphi_P)$ and $(\mathrm{im}\varphi)_P = \mathrm{im}(\varphi_P)$.*

b) Show that φ is injective (respectively surjective) if and only if the induced map on the stalks φ_P is injective (respectilevy surjective) for all P.

c) Show that a sequence $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Solution. a) $(\ker\varphi)_P = \{(U, s),\ s \in \ker(\varphi(U))\}$, modulo the usual germ equivalence relationship. Thus it can be regarded as a subset of $\mathscr{F}_P$ as $\ker(\varphi(U)) \subseteq \mathscr{F}(U)$. On the other side $\ker(\varphi_P)$ is a subset of $\mathscr{F}_P$. To see that the two sets are equal it's enough to check the double inclusion. Let $\overline{(U, s)} \in (\ker\varphi)_P$. Then, $\varphi_P(\overline{(U, s)}) = \overline{(U, (\varphi(U))(s))} = \overline{(U, 0)} = 0 \Rightarrow \overline{(U, s)} \in \ker(\varphi_P)$. Reciprocally, given $\overline{(U, s)} \in \ker(\varphi_P) \Rightarrow \exists V \subset U$ such that $\varphi(U)(s)|_V = 0$. As restrictions commute with morphisms of sheaves, $\varphi(V)(s|_V) = \varphi(U)(s)|_V = 0$. Then, $\overline{(U, s)} = \overline{(V, s|_V)} \in (\ker\varphi)_P$. In conclusion, $(\ker\varphi)_P = \ker(\varphi_P)$.

As $\mathscr{F}_P = \mathscr{F}_P^+$, $(\mathrm{im}\varphi)_P$ is equal to the stack of the presheaf image at point P. $\mathrm{im}(\varphi_P) = \{\overline{(U,s)} \in \mathscr{G}_P$ such that $\exists \overline{(V,t)} \in \mathscr{F}_P | \varphi_P(\overline{(V,t)}) = \overline{(U,s)}\}$. But as $\varphi_P(\overline{(V,t)}) = \overline{(V, \varphi(V)(t))}$ then $\overline{(U,s)} \in \mathrm{im}(\varphi_P) \iff \exists W$ neighbourhood of P, $W \subseteq V \cap U$ such that $\varphi(V)(t)|_W = s_W \iff \varphi(W)(t|_W) = s|_W \iff \overline{(U,s)} = \overline{(W, \varphi(W)(t|_W))} \iff \overline{(U,s)} \in (\mathrm{im}\varphi)_P$.

b) φ injective $\Rightarrow \ker\varphi = 0 \Rightarrow (\ker\phi)_P = 0\, \forall P$. Using part a) of the problem, $\ker\varphi_P = (\ker\varphi)_P = 0 \Rightarrow \varphi_P$ is injective $\forall P$. Reciprocally, let $x \in \ker\varphi(U)$. $\forall P \in U, (\ker\phi)_P = 0$ so the image of x in the stalk $(\ker\phi)_P$ is zero, which means that $\exists W_P \subseteq U$ neighbourhood of P such that $x|_{W_P} = 0$. But open sets W_P cover U and therefore, by property (3) of the definition of shieves, $x = 0$. In conclusion, $\ker\varphi(U) = 0\ \forall U \Rightarrow \ker\varphi = 0 \Rightarrow \varphi$ injective.

We proceed similarly with the surjectivity. $\mathrm{im}\varphi = \mathscr{G} \Rightarrow (\mathrm{im}\varphi)_P = \mathscr{G}_P \Rightarrow \mathrm{im}(\varphi_P) = \mathscr{G}_P \Rightarrow \varphi_P$ surjective. To prove the other implication, first we will prove a fact that is stated but not proved in the text: $\mathscr{F}^+ \cong \mathscr{F}$ if $\mathscr{F}$ is already a sheaf. Given an open set U, let V_P be the neighbourhood of P contained in U such that $\exists t \in \mathscr{F}(V_P)$ such that $t_Q = s(Q)\, \forall Q \in V_P$. The sets V_P cover U, and given two of these sets, V, V' and the respective elements t, t' we have that $\overline{(V',t')} = \overline{(V,t)}$ in every stalk $\mathscr{F}_Q \Rightarrow \exists W_Q$ such that $t'|_{W_Q} = t|_{W_Q} \forall Q \in V \cap V'$. Then these W_Q cover $V \cap V'$, and by property (3) applied to $t'|_{V\cap V'} - t_{V\cap V'}$, we have that $t_{V\cap V'} = t'_{V\cap V'}$. Then, by property (4) applied to the sets V_P, $\exists t \in \mathscr{F}(U)$ such that $t_Q = s(Q)\ \forall Q \in U$, which means that each application s is uniquely determined by $t \in \mathscr{F}(U)$, and then $\mathscr{F}^+(U) \cong \mathscr{F}(U)$. Now it's easy to check that φ is surjective. We have that $(\mathrm{im}\varphi)_P = \mathrm{im}(\varphi_P) = \mathscr{G}_P$ and so we have that $\mathrm{im}\varphi(U)$ is the set of functions s from U to $\bigcup_{P\in U} \mathscr{G}_P$, which means that $\mathrm{im}\varphi$ is in fact $\mathscr{G}^+ \cong \mathscr{G}$ as $\mathscr{G}$ is already a sheaf.

c) Given a sequence of sheaves and morphisms $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$, it is exact $\iff \ker\varphi^i = \mathrm{im}\varphi^{i-1}$. If the sequence is exact, taking direct limits at both sides and using section a) we have that $\ker(\varphi^i_P) = (\ker\varphi^i)_P = (\mathrm{im}\varphi^{i-1})_P = \mathrm{im}(\varphi^{i-1}_P)$, and so the sequence of stalks at each point P is exact.

The other implicaton is consequence of a more general result that we will prove now: Two subsheaves are equal $\iff$ the corresponding stalks at each point are equal. Let $\mathscr{F}_1, \mathscr{F}_2$ be two subsheaves of $\mathscr{F}$, such that $(\mathscr{F}_1)_P = (\mathscr{F}_2)_P$. Let $t \in \mathscr{F}_1(U)$. For every $P \in U \exists V_P$ neighbourhood of P and $s \in \mathscr{F}_2(V)$ such that $s_P = t_P$. The sets $V_P \cap U$ cover U, and given two of these sets, V, V' and the respective elements s, s' we have that $\overline{(V',s')} = \overline{(V,s)} = \overline{(U,t)}$ in every stalk $\mathscr{F}_Q \Rightarrow \exists W_Q$ such that $s'|_{W_Q} = s|_{W_Q} \forall Q \in V \cap V'$. Then these W_Q cover $V \cap V'$, and by property (3) applied to $s'|_{V\cap V'} - s|_{V\cap V'}$, we have that $s|_{V\cap V'} = s'|_{V\cap V'}$. Then, by property (4) applied to the sets $V_P \cap U$, $\exists r \in \mathscr{F}_2(U)$ such that $t_P = r_P \forall P$. By property (3) applied to $r - t$ we get $s = t$ and so $t \in \mathscr{F}_2(U)$. So $\mathscr{F}_1(U) \subseteq \mathscr{F}_2(U)$ and by symmetry $\mathscr{F}_1(U) = \mathscr{F}_2(U)$.

Exercise 1.3. *a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$, for all i.*

b) Give an example of a surjective morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ and an open set U such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

Solution. a) From problem 1.2, φ is surjective $\iff$ the induced morphism on every stalk is. Suppose φ_P is surjective, then given $U \subseteq X$ open set, $s \in \mathscr{G}(U)$, $\forall P \in U \exists V$ neighbourhood of P and $t \in \mathscr{F}(V)$ such that $\overline{(U,s)} = \overline{(V, \varphi(t))} \Rightarrow \exists W_P \subseteq U \cap V$ such that

$\varphi(t|_{W_P}) = s|_{W_P}$, and so $\{W_P\}$ is the covering that satisfies the desired property. Reciprocally, let $\overline{(U,s)} \in \mathcal{G}_P$. Then $\forall P \in U \exists i$ such that $P \in U_i \Rightarrow \overline{(U,s)} = \overline{(U_i, \varphi(U_i)(t_i))} = \varphi_P(\overline{(U_i, t_i)}) \Rightarrow \varphi_P$ is surjective.

b) An example of this situation is given on Exercise 1.21c), where there is a surjective morphism $\mathcal{O}_X \to i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$, but the map on global sections is not surjective.

Exercise 1.4. *a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective foreach U. Show that the induced map $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ of associated sheaves is injective.*

b) Use part (a) to show that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then imφ *can be naturally identified with a subsheaf of $\mathcal{G}$, as mentioned in the text.*

Solution. a) Using 1.2 b) and the fact that $\mathcal{F}_P^+ = \mathcal{F}_P$, the map φ^+ is injective $\iff$ the maps $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ on the stalks are injective. Let $\overline{(U,s)} \in \mathcal{F}_P$ such that $\varphi_P(\overline{(U,s)}) = 0 \Rightarrow \exists W \subset U$ such that $\varphi(U)(s)|_W = 0 \Rightarrow \varphi(W)(s|_W) = 0$. But as $\varphi(U)$ is injective $\forall U$, then $s|_W = 0$ and therefore $\overline{(U,s)} = \overline{(W, s|_W)} = 0$ and thus φ_P is injective $\Rightarrow \varphi^+$ is injective.

b) Let's consider the presheaf image of a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, $U \mapsto \mathrm{im}(\varphi(U))$. Then for each U, $\mathrm{im}(\varphi(U)) \subseteq \mathcal{G}(U)$, and so the inclusion $i(U) : \mathrm{im}(\varphi(U)) \to \mathcal{G}(U)$ is an injective morphism of abelian groups $\forall U$. Then, by section a), the induced map $i^+ : \mathrm{im}\varphi \to \mathcal{G}^+ = \mathcal{G}$ is injective.

Exercise 1.5. *Show that a morphism of sheaves is an isomphism if and only if it is both injective and surjective.*

Solution. We know from Proposition 1.1 that a morphism of sheaves φ is an isomorphism $\iff$ the induced morphism on every stalk φ_P is an isomorphism. But the induced morphisms on stalks are morphisms of abelian groups, so they're isomorphisms if and only if they're surjective and injective. Now using Exercise 1.2 b) this is equivalent to φ being surjective and injective.

Exercise 1.6. *a) Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Show that the natural map of $\mathcal{F}$ to the quotient sheaf $\mathcal{F}/\mathcal{F}'$ is surjective, and has kernel $\mathcal{F}'$. Thus there is an exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$$

b) Conversely, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$ is an exact sequence, show that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$ and that $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.

Solution. Observation: First we will prove the equivalent of Exercise 1.4 a) for surjectivity. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for each U. Then the induced morphisms on stalks are also surjective: Given $\overline{(U,s)} \in \mathcal{G}_P \exists t \in \mathcal{F}(U)$ such that $\varphi(U)(t) = s \Rightarrow \overline{(U,s)} = \overline{(U, \varphi(U)(t))} = \varphi_P(\overline{(U,t)})$. By Exercise 1.2 b) and the fact that the stalks of the associated shief are equal to the stalks of the preshief ($\mathcal{F}_P = \mathcal{F}_P^+$), the induced morphism of shieves $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ is surjective.

a) The morphisms of abelian groups $\mathcal{F}(U) \to \mathcal{F}(U)/\mathcal{F}'(U)$ are surjective $\forall U$. So by the observation above, the morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{F}/\mathcal{F}'$ is surjective. The fact that $\ker\varphi = \mathcal{F}'$ is a consequence of the result proved in Exercise 1.2c. Indeed, $(\ker\varphi)_P = \ker(\varphi_P)$. As $\varphi_P : \mathcal{F}_P \to (\mathcal{F}/\mathcal{F}')_P = \mathcal{F}_P/\mathcal{F}'_P$ are morphisms of abelian groups, their kernel is $\mathcal{F}'_P$. So $\ker\varphi$ and $\mathcal{F}'$ are 2 subsheaves of $\mathcal{F}$ and their stalks at each point P are equal so $\ker\varphi = \mathcal{F}'$. In conclusion there is an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$.

b) Let's name the applications of the sequence φ, ψ: $0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$. The morphisms of abelian groups $\phi(U) : \mathscr{F}'(U) \xrightarrow{\varphi} \text{im}\varphi(U)$ define a morphism of presheaves. As $\phi(U)$ is surjective $\forall U$, using the observation above we have that $\mathscr{F}' \to \text{im}\varphi$ is a surjective morphism of sheaves. Moreover, as φ is injective, $\varphi(U) : \mathscr{F}'(U) \to \mathscr{F}(U)$ is injective $\forall U \Rightarrow \phi(U) : \mathscr{F}'(U) \xrightarrow{\varphi} \text{im}\varphi(U)$ is injective $\forall U$, and by Exercise 1.4 a) $\mathscr{F}' \to \text{im}\varphi$ is an injective morphism of sheaves. So $\mathscr{F}' \to \text{im}\varphi$ is surjective and injective $\Rightarrow$ is an isomorphism, and, in conclusion, $\text{im}\varphi$ is the subsheaf of $\mathscr{F}$ isomorphic to $\mathscr{F}'$.

The surjective morphism of sheaves $\psi : \mathscr{F} \to \mathscr{F}''$ induces surjective morphisms of abelian groups on stacks $\psi_P : \mathscr{F}_P \to \mathscr{F}''_P$, which induce isomorphisms $\overline{\psi_P} : \mathscr{F}_P/\ker(\psi_P) \cong \mathscr{F}''_P \ \forall P$ sending the class of an element s_P to its image $\psi_P(s_P)$.

In addition, the morphisms of abelian groups $\psi(U) : \mathscr{F}(U) \to \mathscr{F}''(U)$ also induce the morphism of presheaves $\psi(U) : \mathscr{F}(U)/\ker\psi(U) \to \mathscr{F}''(U)$. To show that the map os associated sheaves $\mathscr{F}/\ker\psi \to \mathscr{F}''$ is an isomorphism, it is enoush to show that the corresponding morphisms on stalks $(\mathscr{F}/\ker\psi)_P \to \mathscr{F}''_P$ are isomorphisms. But, taking into account that $(\mathscr{F}/\ker\psi)_P = \mathscr{F}_P/(\ker\psi)_P = \mathscr{F}_P/\ker(\psi_P)$, the corresponding morphisms on stalks are in fact the $\overline{\psi_P}$, and we already know that these are isomorphisms. So in conclusion $\mathscr{F}/\ker\psi \cong \mathscr{F}''$. Finally as the sequence $0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is exact, $\text{im}\phi = \ker\psi$ and we are done.

Exercise 1.7. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*

a) Show that $\text{im}\varphi \cong \mathscr{F}/\ker\varphi$.

b) Show that $\text{coker}\varphi \cong \mathscr{G}/\text{im}\varphi$.

Solution. a) Given $\varphi : \mathscr{F} \to \mathscr{G}$ the observation on Exercise 1.6 shows that $\mathscr{F} \to \text{im}\varphi$ is surjective. Therefore we have an exact sequence $0 \to \ker\varphi \to \mathscr{F} \to \text{im}\varphi \to 0$ and by Exercise 1.6 b) $\text{im}\varphi \cong \mathscr{F}/\ker\varphi$.

b) The identity map $\text{coker}\varphi(U) \to \mathscr{G}(U)/\text{im}\varphi(U)$ is surjective and injective (it is in fact the definition of the cokernel), and it defines a morphism of presheaves. Then, by Exercise 1.4a) and Observation on 1.6 the induced map of associated sheaves $\text{coker}\varphi \to \mathscr{G}/\text{im}\varphi$ is surjective and injective, $\Rightarrow \text{coker}\varphi \cong \mathscr{G}/\text{im}\varphi$.

Exercise 1.8. *For any open subset $U \subseteq X$ show that the functor $\Gamma(U,)$ from sheaves on X to abelian groups is a left exact functor, i.e. if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U, \mathscr{F}') \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$ is an exact sequence of groups. The functor $\Gamma(U, \cdot)$ need not be exact; see (Ex. 1.21) below.*

Solution. Let's note $\varphi : \mathscr{F}' \to \mathscr{F}$ and $\psi : \mathscr{F} \to \mathscr{F}''$. To show that the sequence $0 \to \Gamma(U, \mathscr{F}') \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$ is exact we need to prove: a) That $\varphi(U) : \mathscr{F}'(U) \to \mathscr{F}(U)$ is injective and b) That $\ker\psi(U) = \text{im}\varphi(U)$. a) is a consequence of the fact that a morphism of sheaves is injective $\iff$ the induced morphism on every section is injective. Let's proceed to prove b) showing both inclusions. First note that by Exercise 1.2 c) the induced sequence $0 \to \mathscr{F}'_P \to \mathscr{F}_P \to \mathscr{F}''_P$ on stalks is exact, and therefore $\text{im}(\varphi_P) = \ker(\varphi_P) \ \forall P$.

Let $y \in \text{im}\varphi(U) \Rightarrow y = \varphi(U)(x)$. Then its image on the stalk $\overline{(U, y)} = \overline{(U, \varphi(U)(x))} = \varphi_P(\overline{(U, x)}) \in \text{im}(\varphi_P) = \ker(\varphi_P)$. That means that $\exists W_P \subseteq U$ neighbourhood of P such that $\psi(W_P)(y|_{W_P}) = \psi(U)(y)|_{W_P} = 0 \Rightarrow y|_{W_P} \in \ker\psi(W_P)$. The sets $\{W_P\}$ are an open covering of U, and $(y|_{W_P})|_{W_P \cap W_Q} = (y|_{W_P \cap W_Q}) = (y|_{W_Q})|_{W_P \cap W_Q}$. So as $\ker\psi$ is a sheaf, $\exists y' \in \ker\psi(U)$ such that $y'|_{W_P} = y|_{W_P}$. As $\mathscr{F}$ is a sheaf, applying property (3) to $y - y'$ we get that $y = y'$ and therefore $y \in \ker\psi(U)$. This proves $\text{im}\varphi(U) \subseteq \ker\psi(U)$. Reciprocally, let $y \in \ker\psi(U)$. The same argument on the stalks we did before proves that $\overline{(U, y)} \in \text{im}(\varphi_P) \Rightarrow \exists W_P \subseteq U$

and $x_{W_P} \in \mathcal{F}'(W_P)$ such that $y|_{W_P} = \varphi(W)(x_{W_P})$. As $\varphi(W_P \cap W_Q)$ is injective, and sends $x_{W_P}|_{W_P \cap W_Q}$ and $x_{W_P}|_{W_P \cap W_Q}$ to the same element $y|_{W_P \cap W_Q}$, they must be equal, and therefore $\exists x \in \mathcal{F}'(U)$ such that $x|_{W_P} = x_{W_P}$ $\forall P$. By property (3) of sheaf $\mathcal{F}$ applied to $\varphi(x) - y$ we get that $y = \varphi(x)$ and therefore $\text{im}\varphi(U) \supseteq \ker \psi(U)$.

Exercise 1.9. *Direct Sum. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf. It is called the direct sum of $\mathcal{F}$ and $\mathcal{G}$ and is denoted by $\mathcal{F} \oplus \mathcal{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X.*

Solution. We have defined an abelian group for each open set U, but the problem doesn't specify the restriction morphisms. However, they're induced naturally by the restriction morphisms of $\mathcal{F}$ and $\mathcal{G}$: $\rho_{UV}^{\mathcal{F}\oplus\mathcal{G}} = \rho_{UV}^{\mathcal{F}} \oplus \rho_{UV}^{\mathcal{G}}$ that maps $(x, y) \in \mathcal{F}(U) \oplus \mathcal{G}(U) \mapsto (\rho_{UV}^{\mathcal{F}}(x), \rho_{UV}^{\mathcal{G}}(y))$. Now we can check that this presheaf satisfies (3) and (4) and therefore it is actually a sheaf.

- (3) Let U be an open set, and $\{V_i\}$ be an open covering of U. Let $(x, y) \in \mathcal{F}(U) \oplus \mathcal{G}(U)$ such that $(0, 0) = 0 = (x, y)|_{V_i} = (x|_{V_i}, y|_{V_i}) \forall V_i$. This implies that $x|_{V_i} = 0$ and $y|_{V_i} = 0$ and therefore by sheaf property (3) of $\mathcal{F}$ and $\mathcal{G}$ we have $x = 0$ and $y = 0 \Rightarrow (x, y) = (0, 0) = 0$.

- (4) Let U be an open set, and $\{V_i\}$ be an open covering of U. Let $(x_i, y_i) = s_i \in \mathcal{F}(V_i) \oplus \mathcal{G}(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, which means that $(x_i|_{V_i \cap V_j}, y_i|_{V_i \cap V_j}) = (x_i, y_i)|_{V_i \cap V_j} = s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j} = (x_j, y_j)|_{V_i \cap V_j} = (x_j|_{V_i \cap V_j}, y_j|_{V_i \cap V_j})$. Then, by property (4) of shieves $\mathcal{F}$ and $\mathcal{G}$, $\exists x \in \mathcal{F}(U), y \in \mathcal{G}(U)$ such that $x|_{V_i} = x_i$ and $y|_{V_i} = y_i$. Then, $(x, y) = s \in \mathcal{F}(U) \oplus \mathcal{G}(U)$ satisfies $s|_{V_i} = s_i$.

Then the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is indeed a sheaf. Now let's check that it plays the role of direct sum and direct product in category theory. We will use the notation $\mathcal{F}_1, \mathcal{F}_2$ instead of $\mathcal{F}, \mathcal{G}$ to simplify.

Direct product: The object $\mathcal{F}_1 \oplus \mathcal{F}_2$ along with two morphisms $\pi_i : \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_i$ is the direct product if it satisfies that $\forall$ sheaf $\mathcal{G}$ and sheaf morphisms $f_i : \mathcal{G} \to \mathcal{F}_i$ $\exists! f : \mathcal{G} \to \mathcal{F}_1 \oplus \mathcal{F}_2$ such that $\pi_i \circ f = f_i$. Indeed, let the morphisms π_i be given by $\pi_i(U) : \mathcal{F}_1(U) \oplus \mathcal{F}_2(U) \to \mathcal{F}_i(U)$ such that $\pi_i(U)((x_1, x_2)) \mapsto x_i$. Then, the morphism $\mathcal{G}(U) \to \mathcal{F}_1(U) \oplus \mathcal{F}_2(U)$ that maps $x \mapsto (f_1(x), f_2(x))$ satisfies the desired property, as $(\pi_i \circ f)(x) = \pi_i(f_1(x), f_2(x)) = f_i(x)$. In addition, suppose that $\exists f'$ morphism satisfying this property. Then $f'(U)(x) = (s_1, s_2) \in \mathcal{F}_1(U) \oplus \mathcal{F}_2(U)$, and as $\pi_i(U)(s_1, s_2) = f_i(x) \Rightarrow f'(U)(x) = (f_1(x), f_2(x)) \Rightarrow f' = f$.

Direct sum: The element $\mathcal{F}_1 \oplus \mathcal{F}_2$ is the categorical direct sum of $\mathcal{F}_1$ and $\mathcal{F}_2$ if there exist morphisms $i_j : \mathcal{F}_j \to \mathcal{F}_1 \oplus \mathcal{F}_2$ such that $\forall$ shief $\mathcal{G}$ and morphisms $f_i : \mathcal{F}_i \to \mathcal{G}$, $\exists! f : \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{G}$ such that $f \circ i_j = f_j$. Indeed, let's define $i_1(U)(x) = (x, 0)$ and $i_2(U)(x) = (0, x)$. Then, we define $f(U) : \mathcal{F}_1(U) \oplus \mathcal{F}_2(U) \to \mathcal{G}$ such that $f(U)(x, y) = f_1(U)(x) + f_2(U)(y)$. It's clear that $f \circ i_j = f_j$. In addition, suppose that $\exists f'$ satisfying this property. Then, $f'(U)(x, 0) = f_1(U)(x)$, $f'(U)(0, y) = f_2(U)(x)$ and therefore $f'(U)(x, y) = f'(U)(x, 0) + f'(U)(0, y) = f_1(U)(x) + f_2(U)(y)$, which proves the uniqueness of f.

Exercise 1.10. *Direct Limit. Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the direct limit of the system $\{\mathcal{F}_i\}$, denoted $\varinjlim \mathcal{F}_i$ to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathcal{G}$ and a collection of morphisms $\mathcal{F}_i \to \mathcal{G}$ compatible with the maps of the direct system, there exists a unique map $\varinjlim \mathcal{F}_i \to \mathcal{G}$ such that for each i, the original map $\mathcal{F}_i \to \mathcal{G}$ is obtained by composing the maps $\mathcal{F}_i \to \varinjlim \mathcal{F}_i \mathcal{G}$.*

Solution. Again we have the definition of a presheaf without specifying the restriction morphisms, so we need to figure out which is the natural way to define them. Let f_{ij} denote the

morphisms of the direct system, and ρ^i_{UV} the restriction morphisms of each $\mathcal{F}_i$. Then, by definition of morphisms of sheaves we have the following commutative diagram
$$\begin{array}{ccc} \mathcal{F}_i(U) & \xrightarrow{f_{ij}(U)} & \mathcal{F}_j(U) \\ \downarrow \rho^i_{UV} & & \downarrow \rho^j_{UV} \\ \mathcal{F}_i(V) & \xrightarrow{f_{ij}(V)} & \mathcal{F}_j(V) \end{array}$$

Every element in $\varinjlim \mathcal{F}_i(U)$ can be expressed as the equivalence class of an element $s \in \mathcal{F}_i(U)$ for a certain i. So it seems natural to define $\rho_{UV} : \varinjlim \mathcal{F}_i(U) \to \varinjlim \mathcal{F}_i(V)$ mapping $\overline{s} \mapsto \overline{\rho^i_{UV}(s)}$. We only have to check that this application is well defined, i.e., that it doesn't depend on the chosen representative. Let $t \in \mathcal{F}_i(U), s \in \mathcal{F}_j(U)$ such that $\overline{t} = \overline{s}$. Then $\exists k$ such that $k \geq i, k \geq j$ and $f_{ik}(U)(t) = f_{jk}(U)(s)$ Then, $f_{ik}(V)(\rho^i_{UV}(t)) = \rho^k_{UV}(f_{ik}(U)(t)) = \rho^k_{UV}(f_{jk}(U)(s)) = f_{ik}(V)(\rho^j_{UV}(s))$, and therefore $\overline{\rho^i_{UV}(t)} = \overline{\rho^j_{UV}(s)}$ and the restriction morphisms are well defined.

Let $\varphi_i : \mathcal{F}_i \to \mathcal{G}$ morphisms of sheaves compatible with the maps of the direct system. Then, we define the morphism of presheaves φ by $\varphi(U) : \varinjlim \mathcal{F}_i(U) \to \mathcal{G}$ that sends the class of an element $s \in \mathcal{F}_i$, $\overline{s} \in \varinjlim \mathcal{F}_i(U) \mapsto \varphi_i(U)(s) \in \mathcal{G}(U)$. This application is well defined as the morphisms of sheaves φ_i are compatible with the morphisms of the direct system. It is clearly unique satisfying the required property (it must send the class of an element s to its image by the initial morphism φ_i). Then, by proposition 1.2 the induced map $\varphi^+ : \varinjlim \mathcal{F} \to \mathcal{G}$ is unique and satisfies the requirements of the problem.

Exercise 1.11. *Let $\{\mathcal{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ is already a sheaf. In particuar $\Gamma(X, \varinjlim \mathcal{F}_i) = \varinjlim \Gamma(X, \mathcal{F}_i)$.*

Solution. If X is a Noetherian topological space, then every subspace of X is quasi compact, and, in particular, every open subset is quasi-compact, i.e.,for every $\{V_i\}$ open covering of U we can extract a finite subcovering (Atiyah-Macdonald Exercise 6.6). Then, it is enough to check that properties (3) and (4) that define a sheaf are satisfied for finite coverings $\{V_i\}_{i=1}^n$ of U. We will use the same notation of last problem.

- (3) Let $\overline{s} \in \varinjlim \mathcal{F}_i(U)$, with $s \in \mathcal{F}_i(U)$. Suppose that $0 = \overline{s}|_{V_j} = \overline{\rho^i_{UV_j}(s)}$ That means that $\exists k_j \geq i$ such that $f_{ik_j}(V_j)(\rho^i_{UV_j}(s)) = 0$. As restrictions commute with morphisms of the direct system $\rho^{k_j}_{UV_j}(f_{ik_j}(U)(s)) = f_{ik_j}(V_j)(\rho^i_{UV_j}(s)) = 0$. As the covering is finite, we can define $m := \max_j\{k_j\}$ and we have that
$$\rho^m_{UV_j}(f_{im}(U)(s)) = f_{im}(V_j)(\rho^i_{UV_j}(s)) = f_{k_jm}(V_j)(f_{ik_j}(V_j)(\rho^i_{UV_j}(s)) = 0$$
Now, by shief property (3) applied to $\mathcal{F}_m$, we have that $f_{im}(U)(s) = 0 \Rightarrow \overline{s} = 0$.

- (4) Suppose that we have a covering $\{V_i\}$ of a open set U, and $\overline{s_i} \in \varinjlim \mathcal{F}_j(V_i)$, with $s_i \in \mathcal{F}_{k_i}(V_i)$ such that $\overline{s_i}|_{V_i \cap V_j} = \overline{s_j}|_{V_i \cap V_j}$. Using the definition of restriction morphisms written in Exercise 10, that is equivalent to $\overline{\rho^{k_i}_{V_iV_i\cap V_j}(s_i)} = \overline{\rho^{k_j}_{V_jV_i\cap V_j}(s_j)}$. That implies $\exists l$ such that $l \geq k_i, l \geq k_j$ and $f_{k_il}(V_i \cap V_j)(\rho^{k_i}_{V_iV_i\cap V_j}(s_i)) = f_{k_jl}(V_i \cap V_j)(\rho^{k_j}_{V_jV_i\cap V_j}(s_j))$. As morphisms of the direct system and restrictions commute, we have $\rho^l_{V_iV_i\cap V_j}(f_{K_il}(V_i)(s_i)) = \rho^l_{V_jV_i\cap V_j}(f_{k_jl}(V_j)(s_j))$, which can be rewritten as $f_{k_il}(V_i)(s_i)|_{V_i\cap V_j} = f_{k_jl}(V_j)(s_j)|_{V_i\cap V_j}$. Then by property (3) of shieves applied to $\mathcal{F}_l$ $\exists t \in \mathcal{F}_l(U)$ such that $t|_{V_i} = f_{k_il}(V_i)(s_i)$. Then, this last equality implies that $\overline{t} \in \varinjlim \mathcal{F}_j(U)$ satisfies $\overline{t}|_{V_i} = \overline{t|_{V_i}} = \overline{s_i}$.

Then, as this is already a sheaf $\varinjlim \Gamma(U, \mathcal{F}_i) = \varinjlim \mathcal{F}_i(U) = \Gamma(U, \varinjlim \mathcal{F}_i)$, and the equality is true in particular when we take $U = X$.

Exercise 1.12. *Inverse limit. Let $\{\mathcal{F}_i\}$ be a direct system of sheaves on X. Show that the presheaf $U \mapsto \varprojlim \mathcal{F}_i(U)$ is a sheaf. It is called the inverse limit of the system $\{\mathcal{F}_i\}$, and it is denoted by $\varprojlim \mathcal{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.*

Solution. $\varprojlim \mathcal{F}_i(U)$ is the group of coherent sequences over the abelian groups $\mathcal{F}_i(U)$, that is sequences (a_n) with $a_n \in \mathcal{F}_n(U)$ and satisfying $\theta_{n+1}(U)(a_{n+1}) = a_n$, where θ_i are the morphisms of sheaves of the inverse system. (Atiyah-MacDonald, chapter 10). The restriction morphisms of $\varprojlim \mathcal{F}_i$ are naturally $\rho_{UV}((a_n)) = (\rho^n_{UV}(a_n))$, which are well defined (it is indeed an element of $\varprojlim \mathcal{F}(V)$) because restrictions commute with morphisms θ_i. As usual, we will check the 2 properties of the sheaf definition.

- (3) Let (a_n) be a coherent sequence. Suppose that $(a_n)|_{V_i} = 0\ \forall i \Rightarrow \rho_{UV_i}((a_n)) = 0 \Rightarrow \rho^n_{UV_i}(a_n) = 0\ \forall n$ and $\forall i$. Then, by property (3) on each $\mathcal{F}_n$ it implies that $a_n = 0\ \forall n$ and in conclusion $((a_n)) = 0$.

- (4) Let $(a^i_n) \in \varprojlim_j \mathcal{F}_j(V_i)$ such that $((a^i_n))|_{V_i\cap V_j} = ((a^j_n))|_{V_i\cap V_j} \Rightarrow \rho^n_{V_iV_i\cap V_j}(a^i_n) = \rho^n_{V_jV_i\cap V_j}(a^j_n)\ \forall n$. Now applying property (4) at each $\mathcal{F}_n$ we have that $\exists a_n \in \mathcal{F}_n(U)$ such that $a_n|_{V_i} = a^i_n\ \forall n\ \forall i$. Now the sequence (a_n) is coherent as $\theta_{n+1}(U)(a_{n+1})|_{V_i} = \theta_{n+1}(V_i)(a^i_{n+1}) = a^i_n$ by the coherence of each sequence (a^i_n). Then applying property (3) of $\mathcal{F}_n$ to the element $\theta_{n+1}(U)(a_{n+1}) - a_n$ we get that $\theta_{n+1}(U)(a_{n+1}) = a_n$ and the sequence is therefore coherent. In consequence, we have $\exists (a_n) \in \varprojlim \mathcal{F}_i(U)$ such that $(a_n)|_{V_i} = (a^i_n)$.

Now let's check that this sheaf satisfies the universal property of categorical inverse limits. An object X is the categorical inverse limit of objects X_i if there exist morphisms $\varphi_i : X \to X_i$ such that $\forall$ object Y and morphisms $\psi_i : Y \to X_i$ compatible with morphisms of the inverse system $\exists!$ morphism $\psi : Y \to X$ such that $\psi_i = \varphi_i \circ \psi$. In our situation, let's define $\varphi_i : \varprojlim \mathcal{F}_i \to \mathcal{F}_i$ such that $\varphi_i(U)((a_n)) = a_i$. Given a sheaf Y and morphisms compatible with the inverse system (that is, $\psi_i : Y \to \mathcal{F}_i$ such that $\psi_i = \theta_{i+1} \circ \psi_{i+1}$) we define the morphism of sheaves ψ by $\psi(U) : Y \to \varprojlim \mathcal{F}_i(U)$, $x \mapsto (\psi_i(U)(x))_i$. The morphism is well defined, because the resulting sequence is coherent as the morphisms ψ_i are compatible with the inverse system. It clearly satisfies $\psi_i = \varphi_i \circ \psi$. Moreover, the morphism ψ is unique: Indeed, suppose $\exists \psi'$ satisfying the same properties. Given $x \in Y(U)$, let $(b_n) = \psi'(x)$. Then as $\psi_i = \varphi_i \circ \psi$ we have $b_i = \psi_i(x) \Rightarrow \psi' = \psi$.

Exercise 1.13. *Espace Étalé of a Presheaf. Given a presheaf $\mathcal{F}$ on X, we define a topological space Spé($\mathcal{F}$), called the espace étalé of $\mathcal{F}$ as follows. As a set,* $\text{Spé}(\mathcal{F}) = \bigcup_{P\in X} \mathcal{F}_P$. *We define a projection map $\pi : \text{Spé}(\mathcal{F}) \to X$ by sending $s \in \mathcal{F}_P$ to P. For each open set $U \subseteq X$ and each section $s \in \mathcal{F}(U)$, we obtain a map $\overline{s} : U \to \text{Spé}(\mathcal{F})$ by sending $P \mapsto s_P$. This map has the property that $\pi \circ \overline{s} = id_U$, in other words, it is a section of π over U. We now make* Spé($\mathcal{F}$) *into a topological space by giving it the strongest topology such that all the maps $\overline{s} : U \to \text{Spé}(\mathcal{F})$ for all U, and all $s \in \mathcal{F}(U)$ are continuous. Now show that the sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ can be described as follows: for any open set $U \subseteq X$, $\mathcal{F}^+(U)$ is the set of continuous sections of* Spé($\mathcal{F}$) *over U. In particular the original presheaf $\mathcal{F}$ was a sheaf if and only if $\mathcal{F}(U)$ is equal to the set of all continuous sectiobs of* Spé($\mathcal{F}$) *over U.*

Solution. Let $s \in \mathcal{F}^+(U)$, that is, an application $s : U \to \bigcup_{P\in U} \mathcal{F}_P \subseteq \text{Spé}(\mathcal{F})$ satisfying (1) and (2) of the definition 1.2. Let's check that it is a continuous section when regarded as an application from U to Spé($\mathcal{F}$). First by Property (1), $s(P) \in \mathcal{F}_P$, which implies that $\pi \circ s = id_U$, so s is a section. Now let V be an open subset of Spé($\mathcal{F}$). Then $s^{-1}(V) = \{P \in U$ such that $s(P) \in V\}$. But by property (2), for each $P \in U$ $\exists W_P \subseteq U$ and $t^{W_P} \in \mathcal{F}(W_P)$ such that $s(P) = (t^{W_P})_P$. Then, $s^{-1}(V) = \bigcup_{P\in U}\{Q \in W_P$ such that $(t^{W_P})_P \in V\} =$

$\bigcup_{P\in U}(\overline{t^{W_P}})^{-1}(V)$, where $(\overline{t^{W_P}})^{-1} : W_P \to \text{Spé}(\mathscr{F})$ is the application that sends $W_P \ni P \mapsto (t^{W_P})_P$, and therefore it is continuous by the definition of the topology in Spé($\mathscr{F}$). So $(\overline{t^{W_P}})^{-1}(V)$ is an open set and therefore $s^{-1}(V)$ is open, which means that s is continuous.

Before proving the reverse inclusion, let's observe that, with the topology defined on Spé($\mathscr{F}$), the sets $\{t_Q\}_{Q\in V}$ (where $t \in \mathscr{F}(V)$) are open. Indeed, given one of the maps $\overline{s} : U \to \text{Spé}(\mathscr{F})$, $\overline{s}^{-1}(\{t_Q\}_{Q\in V}) = \{P \in U \text{ such that } \overline{s}(P) = t_P\} = \{P \in V\cap U \text{ such that } s_P = t_P\}$. But we know that $s_P = t_P \iff \exists W_P \subseteq U \cap V$ neighbourhood of P such that $s|_{W_P} = t|_{W_P}$. Then

$$\overline{s}^{-1}(\{t_Q\}_{Q\in V}) = \bigcup_{P\in\overline{s}^{-1}(\{t_Q\}_{Q\in V})} W_P$$

Which is open. So the sets $\{t_Q\}_{Q\in V}$ are open in Spé($\mathscr{F}$). Now let $s : U \to \text{Spé}(\mathscr{F})$ be a continuous section. $\pi \circ s = id_U$, which means that $s(P) \in \mathscr{F}_P$ and therefore property (1) is satisfied. Given any $P \in U$, we have $s(P) = t_P$ for a certain $t \in \mathscr{F}(V)$, $V \subseteq U$ neighbourhood of P. Therefore we know that $s^{-1}(\{t_Q\}_{Q\in V})$ is an open set W_P such $s(Q) = t_Q\ \forall Q \in W_P$. In conclusion, s satisfies property (2) and we have shown that each continuous section is an element of $\mathscr{F}^+(U)$.

Exercise 1.14. *Support. Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The support of s, denoted* Supp s *is defined to be $\{P \in U | s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that* Supp s *is a closed subset of U. We define the support of $\mathscr{F}$,* Supp $\mathscr{F}$ *to be $\{P \in X | \mathscr{F}_P \neq 0\}$. It need not be a closed subset.*

Solution. Let $P \in \overline{\text{Supp}\, s}$, that is, $\forall V \subseteq U$ such that $P \in V$ we have $\text{Supp}\, s \cap V \neq \varnothing$. Suppose that $P \notin \text{Supp}\, s$. Then, $\exists V \subseteq U$ such that $s|_V = 0$. Now let $Q \in V \cap \text{Supp}\, s$. Then V is also a neighbourhood of Q and $s|_V = 0$, which implies that $Q \notin \text{Supp}\, s$, a contradiction. So we must have $P \in \text{Supp}\, s$, which means that $\overline{\text{Supp}\, s} = \text{Supp}\, s$, and so it is a closed subset.

Exercise 1.15. *Sheaf Hom. Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set* $\text{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ *of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf $U \mapsto \text{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf. It is called the sheaf of local morphisms of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and it is denoted $\mathcal{H}om(\mathscr{F}, \mathscr{G})$.*

Solution. First let's make some observations about $\mathscr{F}|_U$. Open sets of the subspace U are just open sets $V \subseteq U$, as U is open. So, in that case the limit $\varinjlim_{W\supseteq V} \mathscr{F}(W) = \mathscr{F}(V)$. Then, the restriction presheaf is $V \mapsto \mathscr{F}(V)$ and therefore it is already a sheaf, and its restriction morphisms are the same as the ones of $\mathscr{F}$, but only on open sets $V \subseteq U$. Then, morphisms of sheaves $\varphi : \mathscr{F}|_U \to \mathscr{G}|_U$ are just morphisms of abelian groups $\varphi(V) : \mathscr{F}(V) \to \mathscr{G}(V)\ \forall V \subseteq U$, that commute with restriction morphisms. We can define the sum of two morphisms $\varphi + \psi$ such that $(\varphi+\psi)(V)(x) = \varphi(V)(x) + \psi(V)(y)$. The sum is commutative and associative, as $\mathscr{G}(V)$ are abelian groups. The morphism that sends each element to zero is the neutral element and every morphism has an inverse $-\varphi$ such that $(-\varphi)(V)(x) = -\varphi(V)(x)\ \forall V, \forall x$. Moreover, we can define the restrictions of the sheaf Hom naturally as $\rho_{UV} : \text{Hom}(\mathscr{F}|_U, \mathscr{G}|_U) \to \text{Hom}(\mathscr{F}|_V, \mathscr{G}|_V)$ as $\forall W \subseteq V \subseteq U, \varphi|_V(W) = \varphi(W)$. Now let's check that the two sheaf properties are satisfied. Let U be an open set and$\{V_i\}$ an open covering of U.

- (3) Suppose that we have $\varphi \in \text{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$, and $\varphi|_{V_i} = 0$, that is, $\varphi(W)(x) = 0$, $\forall W \subseteq V_i$, $\forall x \in \mathscr{F}(W)$. Now let $U' \subseteq U$ and $x \in \mathscr{F}(U')$. Let's note $V_i' = V_i \cap U'$. Then $\{V_i'\}$ is an open covering of U'. Then, $\varphi(U')(x)|_{V_i'} = \varphi(V_i')(x|_{V_i'}) = 0$, as $V_i' \subseteq V_i$. Therefore, by property (3) of $\mathscr{G}$, $\varphi(U')(x) = 0$, and this holds $\forall U' \subseteq U, x \in \mathscr{F}(U')$, so $\varphi = 0$.

- (4) Let's suppose that we have $\varphi_i \in \mathrm{Hom}(\mathcal{F}|_{V_i}, \mathcal{G}|_{V_i})$ such that $\varphi_i|_{V_i \cap V_j} = \varphi_j|_{V_i \cap V_j}$, that is $\varphi_i(W)(x) = \varphi_j(W)(x)$, $\forall W \subseteq V_i \cap V_j$, $\forall x \in \mathcal{F}(W)$. Now, let $W \subseteq U$, and $x \in \mathcal{F}(W)$. Then, let $V_i' = W \cap V_i$, and $x_i = x|_{V_i'}$. $\{V_i'\}$ is an open covering of W and we have $\varphi_i(V_i')(x_i) = y_i$, and $y_i|_{V_i' \cap V_j'} = y_j|_{V_i' \cap V_j'}$ so we have by property (4) of $\mathcal{G}$ that $\exists y \in \mathcal{G}(W)$ such that $y|_{V_i'} = y_i$. Then we define $\varphi(W)(x) = y$. This way we have defined an image for every element of W, that is, an application $\mathcal{F}(W) \to \mathcal{G}(W)$. Now we have to check that this application is a morphism of abelian groups and commutes with restrictions, and we'll be done. It's clear that $\varphi(W)(0) = 0$, by property (3) of $\mathcal{G}$. Moreover, given $x_1, x_2 \in \mathcal{F}(W)$ we have $\varphi(W)(x_1|_{V_i'} + x_2|_{V_i'}) = y_{i1} + y_{i2}$. Then, applying property (3) of $\mathcal{G}$ on $\varphi(W)(x_1+x_2) - y_1 - y_2$ we have indeed that $\varphi(W)(x_1+x_2) = y_1 + y_2$. On the other hand, let $W \subseteq V \subseteq U$, and let $x \in \mathcal{F}(V)$. Let's name $y_{iW} := \varphi_i(V_i \cap W)(x|_{V_i \cap W}), y_{iV} := \varphi_i(V_i \cap V)(x|_{V_i \cap V})$, which induce images $\varphi(W)(x|_W) =: y_W, \varphi(V)(x) =: y_V$ by the construction we have done. As φ_i are morphisms of sheaves, $y_V|_{V_i \cap W} = y_{iW}$, and therefore applying once more property (3) on $\mathcal{G}$ we get $y_V|_W = y_W$, so $\varphi(W)(x|_W) = \varphi(V)(x)|_W$. So we have defined morphisms of abelian groups on each open subset of U, that commute with restrictions. This defines a morphism of sheaves $\varphi \in \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}_U)$ whose restriction on V_i is φ_i, which means that Property (4) is satisfied.

Exercise 1.16. *Flasque sheaves. A sheaf $\mathcal{F}$ on a topological space X is flasque if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.*

a) Show that a constant sheaf on an irreducible topological space is flasque.

b) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ is flasque, then for any open set U the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ of abelian groups is also exact.

c) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.

d) If $f : X \to Y$ is a continuous map, and if $\mathcal{F}$ is a flasque sheaf on X, then $f_*\mathcal{F}$ is a flasque sheaf on Y.

e) Let $\mathcal{F}$ be any sheaf on Y. We define a new sheaf $\mathcal{G}$, called the sheaf of discontinuous sections of $\mathcal{F}$ as follows. For each open set $U \subseteq X, \mathcal{G}(U)$ is the set of maps $s : U \to \bigcup_{P \in U} \mathcal{F}_P$ such that for each $P \in U, s(P) \in \mathcal{F}_P$. Show that $\mathcal{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathcal{F}$ to $\mathcal{G}$.

Solution. a) If X is an irreducible topological space, every nonempty set is dense and therefore every nonempty open set is connected $\Rightarrow \mathcal{F}(U) \cong A$, $\forall U$. In consequence, $\forall V \subseteq U$ the restriction map ρ_{UV} is the identity map $A \to A$ which is in particular surjective.

b)

c) If $\mathcal{F}'$ is flasque, then by section b) of this problem the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \xrightarrow{\varphi(U)} 0$ is exact, and in particular, $\varphi(U)$ is surjective $\forall U$. Then given $V \subseteq U$ and $s \in \mathcal{F}''(V) \Rightarrow \exists t \in \mathcal{F}(V)$ such that $\varphi(V)(t) = s$. As $\mathcal{F}$ is flasque, $\exists t' \in \mathcal{F}(U)$ with $t'|_V = t$. Then, $s' := \varphi(U)(t')$ satisfies $s'|_V = \varphi(U)(t')|_V = \varphi(V)(t) = s$. In consequence, the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective, and that means $\mathcal{F}''$ is flasque.

d) Let $V \subseteq U$ be open sets of Y. Then $f^{-1}(V) \subseteq f^{-1}(U)$ are open subsets of X, and as $\mathcal{F}$ is flasque, given $y \in (f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V)) \exists x \in \mathcal{F}(f^{-1}(U)) = (f_*\mathcal{F})(U)$ such that $x|_V = y$, and therefore $f_*\mathcal{F}$ is also flasque.

e) First let's note that the sections $\mathscr{G}(U)$ defined are indeed abelian groups: The sum of two applications s, t is defined as the application $P \mapsto s(P)+t(P)$. The neutral element is the application that sends each P to the class of 0 in $\mathscr{F}_P$. The restriction morphisms are well defined, as every applictaion acting on a set U naturally restricts to an application acting on $V \subseteq U$. Let's see that it is a sheaf. We have to check that properties (3) and (4) are satisfied. (3): Let $s \in \mathscr{G}(U)$, and $\{U_i\}$ an open covering of U. Then $s|_{U_i}(P) = s(P) = 0$ for each $P \in U_i \Rightarrow s(P) = 0 \forall P \in U \Rightarrow s = 0$. (4): Given $s_i \in \mathscr{G}(U_i)$, and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ we can define $s(P) = s_i(P)$ if $P \in U_i$. It is clearly well defined and satisfies $s|_{U_i} = s_i$. Moreover that $\mathscr{G}$ is a flasque sheaf is also easy to check, as given $U \supseteq V$ we can extend any $s \in \mathscr{G}(V)$ to an $s' \in \mathscr{G}(U)$, for example defining $s'(P) = 0 \in \mathscr{F}_P$ for each $P \notin V$ and $s'(P) = s(P)$ if $P \in V$.

   We have a natural morphism from $\mathscr{F}$ to $\mathscr{G}$ that sends $\mathscr{F}(U) \ni x \to s_x \in \mathscr{G}(U)$ such that $s_x(P) = \overline{x} \in \mathscr{F}_P$. It is indeed injective, because if we have $x, y \in \mathscr{F}(U)$ such that $\overline{x} = \overline{y} \in \mathscr{F}_P, \Rightarrow \exists W_P \subseteq U$ such that $x|_{W_P} = y|_{W_P}$ and as W_P is an open covering of U so we must have $x = y$.

Exercise 1.17. *Skyscrapper sheaves. Let X be a topological space, let P be a point and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U, 0$ otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \overline{\{P\}}$, and 0 elsewhere, where $\overline{\{P\}}$ denotes the closure of the set consisting of the point P. Hence the name "skyscrapper sheaf". Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\overline{\{P\}}$, and $i : \overline{\{P\}} \to X$ is the inclusion.*

Solution. First, let's check that $i_P(A)$ is a sheaf. Given $V \subseteq U$, if $P \in\notin V$ the restriction morphism is zero, and if $P \in V$ the restriction morphism is the identity on the group A. It's clear that property (3) and (4) are satisfied if $P \notin U$. So let U be an open set such that $P \in U$ and $\{V_i\}$ be an open covering of U. Let $s \in i_P(A)(U) = A$ Then, $P \in V_i$ for a certain i as $\{V_i\}$ is a covering. Then, $i_P(A)(V_i) = A$ and $0 = s|_{V_i} = s$. Now let $s_i \in i_P(A)(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. Then, if $P \in V_i, V_j$ we have that $s_i = s_j \in A$. So then there is a unique element $s \in A$ such that $s = s_i \forall i$ such that $P \in V_i$. If $P \notin V_i, s_i = 0 = s|_{V_i}$. In conclusion, $s \in i_P(A)(U)$ satisfies $s|_{V_i} = s_i \forall i$.

Let $Q \in \overline{\{P\}}$. Then $P \in U$ $\forall U$ neighbourhood of Q, and therefore $i_P(A)(U) = A$. Then given $\overline{(U,s)}, \overline{(V,t)} \in (i_P(A))_P$, we have that $\forall W$ neighbourhood of Q, $W \subseteq U \cap V$, W is also a neighbourhood of P, and so $s|_W = s$, $t|_W = t$. In consequence, $\overline{(U,s)} = \overline{(V,t)} \iff t = s$, so $(i_P(A))_P = A$. On the other hand, if $Q \notin \overline{\{P\}}$, then $\exists U$ neighbourhood of Q such that $P \notin U$. Then, $\forall \overline{(V,t)} \in (i_P(A))_P$, $i_P(A)(V \cap U) = 0$ and then $t|_{U \cap V} = 0 \Rightarrow \overline{(V,t)} = 0$, so $(i_P(A))_P = 0$.

Now let's check that $i_*(A)$ is an equivalent definition of this sheaf. First, let's note that every open set of $\overline{\{P\}}$ contains P, and therefore every open set is dense and so $\overline{\{P\}}$ is an irreducible space, so by the proof of Exercise 1.16a) the constant sheaf A is in fact the sheaf $U \mapsto A$, $\forall U$. Then, given any U open set of X, we have two options.

- $P \in U$. In that case, $P \in \overline{\{P\}} \cap U = i^{-1}(U)$ and therefore $i_*(A)(U) = \mathscr{A}(i^{-1}(U)) = \mathscr{A}(\overline{\{P\}} \cap U) = A$
- $P \notin U$. In that case $i^{-1}(U) = \overline{\{P\}} \cap U = \varnothing$, and so we have $i_*(A)(U) = \mathscr{A}(i^{-1}(U)) = \mathscr{A}(\overline{\{P\}} \cap U) = \mathscr{A}(\varnothing) = 0$.

In conclusion, for each open set the sections of $i_*(A)$ and $i_P(A)$ are the same, as well as the restriction morphisms ($\rho_{UV} = Id_A$ if $P \in V$ and $\rho_{UV} = 0$ if $P \notin V$). Indeed, both definitions are equal so $i_*(A) = i_P(A)$.

Exercise 1.18. *Adjoint property of f^{-1}. Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$ and for any sheaf*

$\mathcal{G}$ on Y there is a natural map $\mathcal{G} \to f_* f^{-1}\mathcal{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y

$$\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$$

Hence we say that f^{-1} is a left adjoint of f_* and that f_* is a right adjoint of f^{-1}.

Solution. Given $f : X \to Y$ continuous and $\mathcal{F}$ sheaf on X, the sheaf $f^{-1}f_*\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V))$. The elements of this abelian group are equivalence classes of pairs (V, x) such that $\overline{(V, x)} = \overline{(V', x')} \iff \exists W,\ f(U) \subseteq W \subseteq V \cap V'$ such that $x|_{f^{-1}(W)} = x'|_{f^{-1}(W)}$. Let's observe that $V \supseteq f(U) \Rightarrow f^{-1}(V) \supseteq f^{-1}(f(U)) \supseteq U$. Hence we have the natural map $\varinjlim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)) \ni \overline{(V, x)} \mapsto x|_U$. By the observation we have just made, it is well defined, as we can restrict first to $f^{-1}(W)$ and then to U as $f^{-1}(W) \supseteq U$. Then this defines a map $\varphi(U) : \varinjlim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ for every open set U, and this induces then a morphism of presheaves ϵ'. By Proposition 1.2 we have then a unique morphism of sheaves $\epsilon : f^{-1}f_*\mathcal{F} \to \mathcal{F}$.

On the other hand, let $\mathcal{G}$ be a sheaf on Y. Then, $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf $\varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$ (let $(f^{-1}\mathcal{G})_{pre}$ denote this presheaf). As $\varinjlim_{P \in U} \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \mathcal{G}_{f(P)}$, the sections of the sheaf $f^{-1}\mathcal{G}$ are applications $s : U \to \bigcup_{P \in U} \mathcal{G}_{f(P)}$ satisfying that $\forall P \in U, s(P) \in \mathcal{G}_{f(P)}$ and $\forall P \in U \exists U' \subseteq U$ and $t \in \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$ such that $\forall Q \in U', t_{f(Q)} = s(Q)$. Then, the sections $f_*f^{-1}\mathcal{G}(V)$ are the group of applications $s : f^{-1}(V) \to \bigcup_{P \in f^{-1}(V)} \mathcal{G}_{f(P)}$ satisfying that $s(P) \in \mathcal{G}_{f(P)}$ and $\forall P \in f^{-1}(V) \exists U' \subseteq f^{-1}(V)$ and $t \in \varinjlim_{W \supseteq f(f^{-1}(V))} \mathcal{G}(W)$ such that $\forall Q \in U', t_{f(Q)} = s(Q)$. Then, as $V \supseteq f(f^{-1}(V))$, there is a natural map from $\mathcal{G}(V) \to f_*f^{-1}\mathcal{G}(V)$ that assigns each element $s \in \mathcal{G}(V)$ the application $s : f^{-1}(V) \to \bigcup_{P \in f^{-1}(V)} \mathcal{G}_{f(P)}$ such that $s(P) = \overline{(V, s)}_{f(P)}$. We will name ν this morphism of sheaves.

Now let's prove that there is a bijection between the sets of morphisms. Let $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$ be a morphism of sheaves, such that $\varphi(U)$ maps aplications $s : U \to \bigcup_{P \in U} \mathcal{G}_{f(P)}$ satisfying properties (1), (2) to elements $\varphi(U)(s) \in \mathcal{F}(U)$. φ induces a morphism $\varphi_* : f_*f^{-1}\mathcal{G} \to f_*\mathcal{F}$ that maps applications $s : f^{-1}(V) \to \bigcup_{P \in f^{-1}(V)} \mathcal{G}_{f(P)}$ satisfying (1) and (2) to elements $\varphi(f^{-1}(V))(s) \in \mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V)$. Now let $\psi := \varphi_* \circ \nu : \mathcal{G} \to f_*\mathcal{F}$, that sends an element $s \in \mathcal{G}(V)$ to the image of the aplication $s(P) = \overline{(V, s)}|_{f(P)}$. By abuse of notation we will name s both the application in $f_*f^{-1}\mathcal{G}(V)$ and the element $s \in \mathcal{G}(V)$ that induces it. Then we can write $\psi(V)(s) = \varphi(f^{-1}(V))(s)$. Now ψ induces an application $\psi_{-1} : f^{-1}\mathcal{G}_{pre} \to f^{-1}f_*\mathcal{F}$, that sends $\overline{(V, t)} \in \varinjlim_{W \supseteq f(U)} \mathcal{G}(W)$ to the class of the element $(V, \psi(V)(t)) = (V, \varphi(f^{-1}(V))(t))$ in $\varinjlim_{W \supseteq f(U)} \mathcal{F}(f^{-1}(V))$. Finally, $\epsilon \circ \psi_{-1} : f^{-1}\mathcal{G}_{pre} \to \mathcal{F}$ sends the class of an element (V, t) to $\varphi(f^{-1}(V))(t)$, and therefore it is an aplication which is equal to $\varphi \circ \theta$ (where θ is the morphism $f^{-1}\mathcal{G}_{pre} \to f^{-1}\mathcal{G}$). In conclusion, the application induced by $\epsilon \circ \psi_{-1}$ that maps from $f^{-1}\mathcal{G}$ to $\mathcal{F}$ is the application φ.

Now let $\psi : \mathcal{G} \to f_*\mathcal{F}$, $\psi(V)(x) \in \mathcal{F}(f^{-1}(V))$. This induces an application between direct limits $\varinjlim_{W \supseteq f(U)} \mathcal{G}(W) \ni \overline{(V, x)} \mapsto \overline{(V, \psi(V)(x))} \in \varinjlim_{W \supseteq f(U)} \mathcal{F}(f^{-1}(W))$, that in turn induces an application $\psi_{-1} : f^{-1}\mathcal{G} \to f^{-1}f_*\mathcal{F}$. Given an application $s : U \to \bigcup_{P \in U} \mathcal{G}_{f(P)}$, $s(P) \in \mathcal{G}_{f(P)}$ and $\exists U' \subseteq U$ such that $s(Q) = \overline{(V, t)}_{f(Q)}$ $\forall Q \in U'$ for a certain $t \in \varinjlim_{W \supseteq f(U')} \mathcal{G}(W)$, we have $\psi_{-1}(U)(s) : U \to \bigcup_{P \in U} \mathcal{F}_P$ takes for every P the corresponding neighbourhood U' already defined and makes $s(Q) = \overline{(V, \psi(V)(t))}_Q$ $\forall Q \in U'$. Then we have finally induced an application $\varphi := \epsilon \circ \psi_{-1} : f^{-1}\mathcal{G} \to \mathcal{F}$. From this φ we can induce an application $\varphi_* : f_*f^{-1}\mathcal{G} \to f_*\mathcal{F}$ that sends applications $s : f^{-1}(V) \to \bigcup_{P \in f^{-1}(V)} \mathcal{G}_{f(P)}$ such that $\forall P\ \exists V'$ with $f^{-1}(V')$ a neighbourhood of P and $s(Q) = \overline{(V'', t)}_{f(Q)}$ for a certain $\overline{(V'', t)} \in \varinjlim_{W \supseteq f(f^{-1}(V'))} \mathcal{G}(W)$ to

$\varphi_*(V)(s) : f^{-1}(V) \to \bigcup_{P\in f^{-1}(V)} \mathscr{F}_P$, $\varphi_*(s)(P) = (\psi(V'')(t)|_f^{-1}(V'))_P$. Finally, $\varphi_* \circ \nu$ sends an element $t \in \mathscr{G}(V)$ to the application $s : f^{-1}(V) \to \bigcup_{P\in f^{-1}(V)} \mathscr{F}_P$ such that $s(P) = (\psi(V)(t))_P$, which corresponds to the element $\psi(V)(t) \in \mathscr{F}(f^{-1}(V))$. Therefore, $\varphi_* \circ \nu = \psi$.

In conclusion, we have shown that given $\psi : \mathscr{G} \to f_*\mathscr{F}$ we can induce a unique application $\varphi : f^{-1}\mathscr{G} \to \mathscr{F}$ and the other way round. In conclusion, we have a set bijection

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

Exercise 1.19. *Extending a sheaf by Zero. Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.*

a) Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathscr{F}$ instead of $i_*\mathscr{F}$ and say "consider $\mathscr{F}$ as a sheaf on X" when we mean "consider $i_*\mathscr{F}$."

b) Now let $\mathscr{F}$ be a sheaf on U. Let $j_!(\mathscr{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathscr{F}))_P$ is equal to $\mathscr{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathscr{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$. We call $j_!\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside U.

c) Now let $\mathscr{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$$

Solution. a) $(i_*\mathscr{F})_P = \varinjlim_{P\in U} \mathscr{F}(i^{-1}(U)) = \varinjlim_{P\in U} \mathscr{F}(U \cap Z)$. If $P \in Z$, $U \cap Z \neq \varnothing$, $\forall U$ neighbourhood of P. Then, $U \cap Z$ are nonempty open neighbourhoods of P in Z, and every neighbourhood of P in Z is of that form, so

$$(i_*\mathscr{F})_P = \varinjlim_{P\in U} \mathscr{F}(U \cap Z) = \varinjlim_{P\in U',\ U' \text{ open set of } Z} U' = \mathscr{F}_P$$

On the other hand, if $P \notin U$, as Z is closed $\exists V$ neighbourhood of P such that $V \cap Z = \varnothing \Rightarrow \mathscr{F}(i^{-1}(V)) = 0$. Then, $\forall \overline{(U,x)} \in (i_*\mathscr{F})_P$, we have $\overline{(U,x)} = \overline{(U \cap V, x|_{U\cap V})} = 0$.

b) If $P \notin U$ then any neighbourhood V of P satisfies $V \not\subseteq U$ so $\mathscr{F}(V) = 0$ and $\varinjlim_{P\in V} \mathscr{F}(V) = 0$. If $P \in U$, the elements of $(j_!(\mathscr{F}))_P$ are equivalence classes $\overline{(V,x)}$. If $V \not\subseteq U$, then $\overline{(V,x)} = \overline{(U \cap V, x|_{U\cap V})}$, so every equivalence class of $(j_!(\mathscr{F}))_P$ has a representative $\overline{(V,x)}$ with $V \subseteq U$, so it is the same set as $\mathscr{F}_P$, and in consequence, $(j_!\mathscr{F})_P = \mathscr{F}_P$.

Suppose that we have another sheaf $\mathscr{G}$ on X with this property, and whose restriction to U is $\mathscr{F}$. We will name $\mathscr{H}$ the presheaf that defines $j_!(\mathscr{F})$. Then, let's see that we have an isomorphism between $\mathscr{G}(V)$ and $j_!(\mathscr{F})(V)$. Given an element $s \in \mathscr{G}(V)$, it induces an element of $j_!(\mathscr{F})(V)$, that is, an application $s : V \to \bigcap_{P\in V} \mathscr{G}_P = \bigcap_{P\in V} \mathscr{H}_P$, that maps $P \mapsto s_P$. It clearly satisfies $s(P) \in \mathscr{H}_P$ (Property 1). If $P \in U$, then $\exists t = s|_{V\cap U} \in \mathscr{G}(U \cap V) = \mathscr{F}(U \cap V) = \mathscr{H}(U \cap V)$ such that $\forall Q \in U \cap V$, $t_Q = s_Q = s(Q)$. If $P \notin U$, $s_P = 0$ so $\exists W_P \subseteq V$ such that $s|_{W_P} = 0$. Then, $\exists t = 0 \in \mathscr{H}(W_P) = 0$ such that $t_Q = s(Q) = 0$, $\forall Q \in W_P$. This proves that the application s we have defined also satisfies Property 2, and therefore it is an element of $j_!(\mathscr{F})(V)$. Reciprocally, let $s \in j_!(\mathscr{F})(V)$. Remember that while we are working in $U \cap V$ we can use the properties of a sheaf, as $j_!(\mathscr{F})|_U = \mathscr{F}$ is a sheaf. Then, if $P \in U \cap V$ we have that $\exists W_P$ neighbourhood of P such that $\forall Q \in W_P \exists t \in \mathscr{H}(W_P) = \mathscr{F}(W_P)$ such that $t_Q = s(Q)$. Suppose that we have two points $P, P' \in U \cap V$ and $Q \in W_P \cap W_{P'}$, t_1 and t_2 the respective elements. Then, $(t_1)_Q = (t_2)_Q$ so $\exists V_Q \subseteq W_P \cap W_{P'}$ where $t_1|_{V_Q} = t_2|_{V_Q}$. Then the sets $\{V_Q\}$ are an open covering of $W_P \cap W_Q$ and therefore by property (3) $t_1|_{W_{P'}} = t_2|_{W_P}$. Now the sets W_P are an open covering of $U \cap V$ and they have $t_P \in \mathscr{F}(W_P)$ that agree in the intersections. So by property (4) there exists an element $t \in \mathscr{F}(U \cap V) = \mathscr{G}(U \cap V)$ such that $s(P) = t_P \; \forall P \in U \cap V$. We want to see this property but in V instead of in $U \cap V$. We will define an open neighbourhood for each $P \in V, P \notin U$ in the following way. If it exists an open neighbourhood of P that doesn't meet $U \cap V$, we choose this one. Otherwise, we know that $t_P = 0$ so it must exist $W_P \subseteq U$ neighbourhood of P such that $t|_{W_P} = 0$. Then we choose this W_P as an open neighbourhood of P. Then, the sets $\{\{W_P\}, U \cap V\}$ are an open cover of V, and let $t_{W_P} = 0 \in \mathscr{G}(W_P)$ and $t_{U \cap V} = t$. These elements agree on the intersections by the way we have defined the open cover. So therefore, as $\mathscr{G}$ is a sheaf there must exist an element $t' \in \mathscr{G}(V)$ such that $t'|_{W_P} = 0$ and $t'|_{U \cap V} = t$. In conclusion, t' satisfies that $s(P) = t'_P$ for every $P \in V$. In conclusion, we have finally proven that $j_!(\mathscr{F})(V) \cong \mathscr{G}(V) \; \forall V$ so the sheaf $j_!(\mathscr{F})$ is unique (up to isomorphism).

c) Let's define an application from the presheaf that defines $j_!(\mathscr{F}|_U) \to \mathscr{F}$ as the inclusion. On the other hand, let's define an application from $\mathscr{F}(U) \to \mathscr{F}|_Z(U \cap Z)$ that maps a section s to its image in $\overline{(V, s)} \in \varinjlim_{V \supseteq V \cap Z} \mathscr{F}(V)$. These aplications induce a sequence on the associated sheaves $0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$. We know that a sequence is exact $\iff$ it is exact on the stalks.

   Then we can have two cases. If $P \in Z$ (that is $\iff$ $P \notin U$), then $(j_!(\mathscr{F}|_U))_P = 0$, and $(i_*(\mathscr{F}|_Z))_P = \mathscr{F}_P$ and therefore the sequence becomes $0 \to 0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0$ which is cleary exact. On the other hand, if $P \in U$ (that is $\iff$ $P \notin Z$), then $(j_!(\mathscr{F}|_U))_P = \mathscr{F}_P$, and $(i_*(\mathscr{F}|_Z))_P = 0$ and then the exact sequence becomes $0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0 \to 0$, which is also exact.

Exercise 1.20. *Subsheaf with supports. Let Z be a closed subset of X, and let $\mathscr{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathscr{F})$ to be the subgroup of $\Gamma(X, \mathscr{F})$ consisting of all sections whose support (Ex. 1.14) is contained in Z.*

a) Show that the presheaf $V \mapsto \Gamma_{Z \cap V}(V, \mathscr{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathscr{F}$ with supports in Z and it is denoted by $\mathscr{H}^0_Z(\mathscr{F})$.

b) Let $U = X - Z$ and let $j : U \to X$ be the inclusion. Show that there is an exact sequence of sheaves on X

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$$

Furthermore, if $\mathscr{F}$ is flasque, then the map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is surjective.

Solution. a) $\Gamma_{Z \cap V}(V, \mathscr{F}|_V) = \{s \in \mathscr{F}(V)$ such that $s_P = 0 \; \forall P \in V, P \notin Z \cap V\}$, and it is a subgroup of $\mathscr{F}(V)$. The restrictions of this presehaf are naturally the same morphisms of the sheaf $\mathscr{F}$: Given $W \subset V$, and $s \in \Gamma_{Z \cap V}(V, \mathscr{F}|_V)$ then we have $s|_W \in \mathscr{F}(W)$ and given $P \in W, P \notin Z \cap W \Rightarrow P \in V, P \notin Z \cap V$ so $s_P = 0$ and therefore $(s|_W)_P = 0$, so we have also $s|_W \in \Gamma_{Z \cap W}(W, \mathscr{F}|_W)$, and so the restriction morphisms are well defined.

As usual, to see that the presehaf $V \mapsto \Gamma_{Z \cap V}(V, \mathscr{F}|_V)$ we have to check the two properties of the definition. Let $\{V_i\}$ be an open cover of V. Property (3) is clear, as given $s \in \Gamma_{Z \cap V}(V, \mathscr{F}|_V) \Rightarrow s \in \mathscr{F}(V)$, so if $s|_{V_i} = 0$ by property (3) of $\mathscr{F}$ $s = 0$. Now let $s_i \in \Gamma_{Z \cap V_i}(V_i, \mathscr{F}|_{V_i})$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. Then $\exists s \in \mathscr{F}(V)$ such that $s|_{V_i} = s_i$ by property (4) of $\mathscr{F}$. We only have to check that this $s \in \Gamma_{Z \cap V}(V, \mathscr{F}|_V)$. Let $P \in V, P \notin Z \cap V$. As the V_i are a covering of V, there must exist i such that $P \in V_i$. Then, $s_P = \overline{(V, s)} = \overline{(V_i, s_i)} = (s_i)_P = 0$, and so $s \in \Gamma_{Z \cap V}(V, \mathscr{F}|_V)$.

b) Let's define a sequence $0 \to \mathcal{H}_Z^0(\mathcal{F}) \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} j_*(\mathcal{F}|_U) \to 0$ where the applications $\varphi(V) : \mathcal{H}_Z^0(\mathcal{F})(V) \to \mathcal{F}(V)$ are the inclusion (as $\mathcal{H}_Z^0(\mathcal{F})(V)$ is a subgroup of $\mathcal{F}(V)$), and $\mathcal{F}(V) \to j_*(\mathcal{F}|_U)(V)$ is the restriction, as $j_*(\mathcal{F}|_U)(V) = \mathcal{F}(U \cap V)$. Let's see that the sequence is exact as a sequence of abelian groups foreach open set V (that implies exactness on the stalks and therefore exactness of the sequence of morphisms of sheaves). It is clear that $\varphi(V)$ is injective, as it is the inclusion. Now given $s \in \ker \psi(V)$, that is $s \in \mathcal{F}(V)$ such that $s|_{U\cap V} = 0$, we have that $\forall P \in U \cap V$, $U \cap V$ is a neighbourhood of P and therefore $s|_{U\cap V} = 0 \Rightarrow s_P = 0$ for every $P \in U \cap V$, that is, the support of s is contained in $Z \cap V$ and therefore $s \in \mathcal{H}_Z^0(\mathcal{F})(V) = \mathrm{im}\varphi(V)$. Reciprocally, if $s \in \mathcal{F}(V)$ such that $s_P = 0$, $\forall P \in U \cap V$, $\exists W_P$ neighbourhood of P such that $s|_{W_P} = 0$. Then $\{W_P\}_{P\in U\cap V}$ are an pen covering of $U \cap V$, and therefore $s|_{U\cap V} = 0$. In conclusion, $\ker \psi(V) = \mathrm{im}\varphi(V)$ and the sequence is exact. Moreover, if $\mathcal{F}$ is flasque, every restriction morphism is surjective, and in particular ψ is surjective.

Exercise 1.21. *Some Examples of Sheaves on Varieties. Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathcal{O}_X$ be the sheaf of regular functions on X.*

a) Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathcal{I}_Y(U)$ is a sheaf. It is caled the sheaf of ideals $\mathcal{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

b) If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

c) Now let $X = \mathbb{P}$ and let Y be the union of two dictinct points $P, Q \in X$. Then there is an exact sequence of sheaves on X, where $\mathcal{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$,

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{F} \to 0$$

Show however that the induced map on global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{F})$ is not surjective. This shows that the global section functor $\Gamma(X,)$ is not exact.

d) Let again $X = \mathbb{P}$ and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathcal{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathcal{K}$. Show that the quotient sheaf $\mathcal{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P\in X} i_P(I_P)$, where I_P is the group $K/\mathcal{O}_P$ and $i_P(I_P)$ denotes the skyscraper sheaf given by I_P at point P.

e) Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O}) \to 0$$

is exact.

Solution. a) The restriction morphisms of this presheaf are naturally the same morphisms as in the sheaf of regular functions, that is $\forall V \subseteq U$, and $f \in \mathcal{I}_Y(U)$, we define the restriction $f|_V$ as the restriction of the domain to the open set V: As $f(x) = 0\ \forall x \in Y \cap U$, in particular $f(x) = 0\ \forall x \in V \cap Y$, which means that $f|_V \in \mathcal{I}_Y(V)$. Now let's check that the presheaf defined is indeed a sheaf. Let U be an open set of X and $\{V_i\}$ be an open covering of U. Let $f \in \mathcal{O}_X(U)$ such that $f|_{V_i} = 0$, which means that $f(x) = 0\ \forall x \in V_i$. But as $\{V_i\}$ cover U, then we have $f(x) = 0\ \forall x \in U \Rightarrow f = 0$, and property (3) is satisfied. On the other hand, given $f_i \in \mathcal{I}_Y(V_i)$, such that $f_i|_{V_i\cap V_j} = f_j|_{U_i\cap U_j}$, let's define

$f : U \to k$, as $f(x) = f_i(x)$ if $x \in V_i$. This is then a function from U to k, and it is well defined, as the image of the points of $V_i \cap V_j$ by f_i is the same of the image by f_j. It is a regular function (as regularity is a local property). Moreover, given $x \in U \cap Y$, $x \in U \cap V_i$ for a certain V_i, and therefore $f(x) = f_i(x) = 0$. In conclusion, $f \in \mathscr{I}_Y(U)$ and clearly $f|_{V_i} = f_i$. This proves property (4) and therefore $\mathscr{I}_Y$ is a sheaf.

b) If $i : Y \to X$ is the inclusion, then $i_*(\mathcal{O}_Y)(U) = \mathcal{O}_Y(U \cap Y)$ is the ring of regular functions from $U \cap Y$ to k. Let's define a sequence $0 \to \mathscr{I}_Y \to \mathcal{O}_X \to i_*(\mathcal{O}_Y) \to 0$ by defining a sequence of abelian groups for each open set U:

$$0 \to \mathscr{I}_Y(U) \xrightarrow{\varphi(U)} \mathcal{O}_X(U) \xrightarrow{\psi(U)} i_*(\mathcal{O}_Y)(U) \to 0$$

We define $\varphi(U)$ to be the inclusion (remember that $\mathscr{I}_Y(U)$ is a subgroup of $\mathcal{O}_X(U)$), so then $\varphi(U)$ is injective. We define $\psi(U)$ to be the domain restriction to $U \cap Y$. Note that if we restrict the domain of a regular function it is still a regular function, and therefore $\psi(U)$ is well defined. It's also clear that $\ker \psi(U) = \mathrm{im}\varphi(U)$ as both are exactly the elements of $\mathcal{O}_X$ that vanish at all points of $Y \cap U$.

Now let's prove that the morphism of sheaves ψ is surjective. Indeed, given any point of U, we have either that $P \in Y \cap U$ or $\exists$ an open neighbourhood $W'_P \subseteq U$ such that $W'_P \cap Y = \varnothing$, because Y is closed. Then, given f a regular function on $U \cap Y$, we have that $\forall P \in U \cap Y$ there exists a neighbourhood of P, W_P where $f_P = g_P/h_P$, with g_P, h_P polynomials and $h(x) \neq 0$, $\forall x \in W_P$. Then $W_P = Y \cap V_P$ for a certain V_P open set of X. Then the function g_P/h_P is regular on V_P and $\psi(V_P)(f/g) = f/g \in \mathcal{O}_Y(Y \cap V_P) = \mathcal{O}_Y(W_P)$. For the points $P \notin Y \cap U$ we define $f = 0$ on W'_P. Then the sets $\{W'_P\} \cup \{V_P\}$ are an open cover of U and so ψ is surjective as it satisfies the characterization of Exercise 1.3a).

c) First let's observe that here $\mathcal{O}_P$ denotes $\mathcal{O}_{P,Y}$, that is, the germs of regular functions of Y near P. As $Y = \{P, Q\}$, and the singletons are closed sets of the Zariski topology (if $P = (x_1, y_1)$, then $P = Z(x_1 Y - y_1 X)$), then in that case (as the number of points of Y is finite) the singletons are also open sets, and therefore two regular functions are equal in $\mathcal{O}_P$ if and only if their value in P is the same. In consequence, $\mathcal{O}_P = k$, $\mathcal{O}_Q = k$. Then, $i_*\mathcal{O}_P(U) = \mathcal{O}_P(U \cap Y) = k$ if $P \in U$ or 0 otherwise. Then, let's define a sequence $0 \to \mathscr{I}_Y \xrightarrow{\varphi} \mathcal{O}_X \xrightarrow{\psi} \mathscr{F} \to 0$ in the following way. Let $\varphi(U)$ be the inclusion for each U. It is well defined and injective as we know that $\mathscr{I}_Y(U)$ is a subgroup of $\mathcal{O}_X(U)$. $\psi(U)$ maps a regular function f to the pair of values $(f(P), f(Q))$ in case both P and Q belong to U. If the point P or Q is not in U it maps the corresponding coordinate to zero. It is clear that $\ker \psi(U) = \mathrm{im}\varphi(U)$ because both sets are the regular functions on U that vanish both at P and Q. We can define $\mathscr{I}_Y(U) = \mathcal{O}_X(U)$ when $Y \cap U = \varnothing$ and that way we still have $\ker \psi(U) = \mathrm{im}\varphi(U)$ when both $P, Q \notin U$. Finally, let's see that ψ is surjective. Given any open set U, there is an open covering $U_1 = U \setminus \{P\}$, $U_2 = U \setminus \{Q\}$, such that $\forall s = (s_1, s_2) \in \mathscr{F}(U)$, with $s_i \in k$, we can take $s_i \in \mathcal{O}_X(U_i)$ and we have $\psi(U_i)(s_i) = s|_{U_i}$, as $\psi(U_1)(t_1) = (t_1, 0) = s|_{U_1}$ and $\psi(U_2)(t_2) = (0, t_2) = s|_{U_2}$. Then ψ satisfies the charectarization of Exercise 1.3a) and therefore it is surjective.

However, the map on global sections is not surjective. It is easy to see this if we use the result of Theorem 3.4 of Chapter I, that states that $\mathcal{O}_X(X) = k$. Therefore, $\mathrm{im}\psi(X)$ is the set of pairs $\{(t, t)\}$, $t \in k$ which is not equal to the direct sum $k \oplus k$.

d) For each open set U, let $\varphi(U) : \mathcal{O}(U) \to K$ mapping $f \mapsto \overline{(U, f)}$. This application is injective by remark 3.1.1 of Chapter I (if two regular functions are equal in an open set $V \subseteq U$ then they're equal in U). Then, by Exercise 1.4a) the induced morphism of sheaves

$\varphi : \mathcal{O} \to \mathcal{K}$ is injective. Now let's define an application $\psi(U) : K \to \sum_{P\in X} i_P(I_P)(U)$. We will index each coordinate of $\sum_{P\in X} i_P(I_P)(U)$ with the corresponding point P. Then, we define the coordinate of point P to be 0 if $P \notin U$ or $\overline{f} \in K/\mathcal{O}_P$ if $P \in U$. Then, it's clear that $\ker\psi(U) = \{f \in K \text{ such that } f \in \mathcal{O}_P \forall P \in U\} = \mathcal{O}_X(U) = \mathrm{im}\varphi(U)$. Now we have to check that ψ is surjective. Let $s \in \sum_{P\in X} i_P(I_P)(U)$. Then, the coordinates of all points must be zero except for a finite number of them, let $P_1, \ldots, P_n$ be the indices of these coordinates. As points are closed sets in the Zariski topology, then $W_i = X\ \{P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n\}$ are open sets. Then, $s|_{W_i\cap U}$ is zero in all coordinates except in P_i. Now, as the application $K \to K/\mathcal{O}_{P_i}$ is surjective $\exists f_i \in K$ such that $\psi(W_i \cap U)(f_i) = s|_{U\cap W_i}$. Now $W_i \cap U$ are an open cover of U and therefore the application on the stalks is surjective and in consequence $\psi : \mathcal{K} \to \sum_{P\in X} i_P(I_P)$ is surjective. Finally, we have proven that we have an exact sequence

$$0 \to \mathcal{O} \to \mathcal{K} \to \sum_{P\in X} i_P(I_P) \to 0$$

And using now exercise 1.6 we have an isomorphism $\mathcal{K}/\mathcal{O} \cong \sum_{P\in X} i_P(I_P)$.

e) By Exercise 1.8 we only need to prove that $\Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O})$ is surjective. $\Gamma(X, \mathcal{K})$ are aplications $s : X \to \bigcup_{P\in X} K(X)$ such that $\forall P \exists U$ neighbourhood of P such that $s(Q) = f\ \forall Q \in U$. Similarly, $\Gamma(X, \mathcal{K}/\mathcal{O})$ are aplications $s : X \to \bigcup_{P\in X} K(X)/\mathcal{O}_P$ such that $\forall P \in X$, $\exists U$ neighbourhood of P and $t \in K(X)/\mathcal{O}(U)$ such that $s(Q) = t_Q\ \forall Q \in U$. Then, the application $\Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O})$ is surjective, because $\pi : K(X) \to K(X)/\mathcal{O}(U)$ is surjective for every U, and then from an application $s : X \to \bigcup_{P\in X} K(X)/\mathcal{O}_P$, we choose at each open set, a $f \in K(X)$ such that $\pi(f) = t$.

Exercise 1.22. *Glueing sheaves. Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i and for each i, j an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i\cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$ such that (1) for each i, $\varphi_{ii} = id$ and (2) for each i, j, k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathcal{F}$ on X togheather with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathcal{F}$ is obtained by glueing the sheaves $\mathcal{F}_i$ via the isomorphism φ_{ij}.*

Solution. For every open set $V \subseteq X$ let's define $\mathcal{F}(V)$ as the subgroup of $\prod_i \mathcal{F}_i(U_i \cap V)$ where the elements satisfy $\varphi_{ij}(s_i|_{U_i\cap U_j\cap V}) = s_j|_{U_i\cap U_j\cap V}$. It is clearly a subgroup of the direct product ($0 \in \mathcal{F}(V)$ and it is closed under sum because $\varphi_{ij}(U_i \cap U_j \cap V)$ is a morphism of groups). We define the restriction morphisms as the restriction of each coordinate i by the restrcition morphisms of the sheaf $\mathcal{F}_i$. Now let's prove that this is a sheaf showing that it satisfies the additional properties.

- (3) Let $\{V_i\}$ be an open covering of V. Note that $\{U_i \cap V_j\}_j$ is now an open covering of $U_i \cap V$. Let $s = (s_i) \in \mathcal{F}(V)$ such that $s|_{V_j} = 0\ \forall j$. Taking into account how we have defined the restriction morphisms, this implies that $s_i|_{U_i\cap V_j} = 0\ \forall i, j$. But this implies that $s_i = 0$ using property (3) of the sheaf $\mathcal{F}_i$. So $s_i = 0\ \forall i$, and so $s = (s_i) = 0$ and property (3) is satisfied on $\mathcal{F}$.

- (4) Similarly, let $s^j = (s_i^j) \in \mathcal{F}(V_j)$ such that $s^j|_{V_k\cap V_j} = s^k|_{V_k\cap V_j}$. That implies that we have on each coordinate $s_i^j|_{V_k\cap V_j} = s_i^k|_{V_j\cap V_k}$. This induces elements $s_i \in \mathcal{F}(U_i \cap V)$ such that $s_i|_{V_j} = s_i^j$, by property (4) of each sheaf $\mathcal{F}_i$, and so we can form an element of the direct product $s = (s_i) \in \prod_i \mathcal{F}(U_i \cap V)$. In addiction applying property (3) on $\varphi_{ij}(U_i \cap U_j \cap V)(s_i|_{U_i\cap U_j\cap V}) - s_j|_{U_i\cap U_j\cap V}$ (they agree when we restrict to each V_j) we have that the element s we constructed belongs indeed to $\mathcal{F}(V)$ and therefore property (4) is satisfied.

Now we can consider ψ_i as the projection map on coordinate i. This projection is clearly surjective and it is also injective if we restrict to U_i. So $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$ are isomorphisms and clearly satisfy $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$ because the coordinates satisfy $\varphi_{ij}(s_i|_{U_i \cap U_j \cap V}) = s_j|_{U_i \cap U_j \cap V}$.

Let $\mathscr{G}$ another sheaf on X satisfying these properties. Let $\psi_i' : \mathscr{G}|_{U_i} \to \mathscr{F}_i$. Then there is an isomorphism $\phi : \mathscr{G} \to \mathscr{F}$ defined by $\phi(V) : \mathscr{G}(V) \to \mathscr{F}(V)$, $\phi(V)(x) = \prod_i \psi_i'(x|_{U_i \cap V})$. ϕ is clearly injective on each V open set, using property (3) of $\mathscr{G}$ and the fact that ψ_i' are isomorphisms. Moreover, ψ_i' are surjective, so for each $y_i \in \mathscr{F}_i(U_i \cap V) \exists \{V_{ij}\}_j$ covering of $U_i \cap V$ and $x_{ij} \in V_j$ such that $\psi_i'(x_{ij}) = y_i|_{V_{ij}}$. Then let $y = (y_i) \in \mathscr{F}(V)$ and let's fix a certain i,

$$\phi(V_{ij})(x_{ij}) = (\psi_k'(x_{ij}|_{Vij \cap U_k}))_k = (\varphi_{ik} \circ \psi_i')(x_{ij}|_{Vij \cap U_k}))_k = (y_k|_{V_{ij} \cap U_k})_k = y|_{V_{ij}}$$

So the morphism is also surjective (satisfies charectarization of Exercise 1.3a) with V_{ij} as the open covering. In conclusion, the sheaf $\mathscr{F}$ is unique up to isomorphism.

2 Schemes

Exercise 2.1. *Let A be a ring, let $X = \mathrm{Spec}A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to* $\mathrm{Spec}A_f$*.*

Solution. We have to build an isomorphism of locally ringed spaces between $\mathrm{Spec}(A_f)$ and $(D(f), \mathcal{O}_X|_{D(f)})$. Let $S = \{f^n\}_{n\geq 0}$, so $A_f = S^{-1}A$. Then, we know that there is a bijective correspondence between prime ideals of A_f and prime ideals of A that don't cut S, which in this case is equivalent to ideals of A that don't contain f (as $f^n \in \mathfrak{p} \iff f \in \mathfrak{p}$) (Atiyah-MacDonald, Proposition 3.11).

We have a morphism of rings $\varphi : A \to A_f$ that maps $a \mapsto a/1$. It induces an application $f : \mathrm{Spec}(A_f) \to D(f)$, $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ which is bijective and whose inverse is $f^{-1} : D(f) \to \mathrm{Spec}(A_f)$, and $f^{-1}(\mathfrak{p}) = S^{-1}\mathfrak{p}$. f is continuous (as it was seen at Proposition 2.3). As every ideal in A_f is of the form $S^{-1}\mathfrak{a}$ for a certain $\mathfrak{a}$ ideal of A (also a result in Proposition 3.11 of Atiyah-MacDonald), then the closed sets of $\mathrm{Spec}(A_f)$ are of the form $V(S^{-1}\mathfrak{a}) = \{S^{-1}\mathfrak{p}, \text{ with } \mathfrak{p} \text{ prime ideals of A such that} f \notin \mathfrak{p}, \mathfrak{p} \supseteq \mathfrak{a}\}$. Then $(f^{-1})^{-1}(V(S^{-1}\mathfrak{a})) = V(\mathfrak{a}) \cap D(f)$, which is a closed subset of $D(f)$. In conclusion, f^{-1} is also continuous and so f is an homeomorphism.

Now we need to build an isomorphism of sheaves $f^\# : \mathcal{O}_X|_{D(f)} \to f_*\mathcal{O}_{Spec(A)}$. Let $f^\#(V) : \mathcal{O}_X|_{D(f)}(V) \to f_*\mathcal{O}_{Spec(A)}(V)$ map the section $\mathcal{O}_X|_{D(f)}(V) \ni s : V \to \bigcup_{\mathfrak{p}\in V} A_\mathfrak{p}$ to the section $s' : \mathcal{O}_{\mathrm{Spec}A_f}(f^{-1}(V))$ that maps a prime ideal $S^{-1}\mathfrak{p}$ to $s'(S^{-1}\mathfrak{p}) = \frac{a/1}{g/1} \in (S^{-1}A)_{S^{-1}\mathfrak{p}}$ if $s(\mathfrak{p}) = a/g \in A_\mathfrak{p}$. To check that this is an isomorphism it's enough to check that the induced application on stalks is an isomorphism (By Exercise 1.2). But we already know by Proposition 2.2 that the stalk at point $\mathfrak{p}$ of $\mathcal{O}_{\mathrm{Spec}(A)}$ is $A_\mathfrak{p}$ and the stalk at point $S^{-1}\mathfrak{p}$ of $\mathcal{O}_{\mathrm{Spec}(A_f)}$ is $(S^{-1}A)_{S^{-1}\mathfrak{p}}$.

Then, it is enough to check that $\phi : A_\mathfrak{p} \to (S^{-1}A)_{S^{-1}\mathfrak{p}}$ such that $\phi(\frac{a}{g}) = \frac{a/1}{g/1}$ is an isomorphism of rings.

- **Surjectivity:** Let $\frac{a/f^n}{g/f^m} \in (S^{-1}A)_{S^{-1}\mathfrak{p}}$. Then if $n \geq m$ we have that

$$\phi\left(\frac{a}{gf^{n-m}}\right) = \frac{a/1}{gf^(n-m)/1} = \frac{af^n/f^n}{gf^n/f^m} = \frac{a/f^n}{g/f^m}$$

  because $(af^n/f^n)(g/f^m) - (a/f^n)(gf^n/f^m) = 0$.

  Similarly, if $m \geq n$ and then

$$\phi\left(\frac{af^{m-n}}{g}\right) = \frac{af^{m-n}/1}{g/1} = \frac{af^m/f^n}{gf^m/f^m} = \frac{a/f^n}{g/f^m}$$

  because $(af^m/f^n)(g/f^m) - (a/f^n)(gf^m/f^m) = 0$.

- **Injectivity:** $\frac{a/1}{g/1} = 0 \iff \exists h/f^k \in S^{-1}A$ such that $(h/f^k)(a/1) = 0 \iff ha = 0 \iff \frac{a}{g} = 0$

So we have only left to check that we have indeed a local homeomorphism. Again, it will be enough to check that $\phi^{-1}(V)(S^{-1}\mathfrak{p}(S^{-1}A)_{S^{-1}\mathfrak{p}}) = \mathfrak{p}A_\mathfrak{p}$. But this is clear as $\frac{a/1}{g/1} \in S^{-1}(\mathfrak{p})(S^{-1}(A))_{S^{-1}\mathfrak{p}} \iff a/1 \in S^{-1}\mathfrak{p} \iff a \in \mathfrak{p} \iff a/g \in \mathfrak{p}A_\mathfrak{p}$. With all these results we have finally that $(D(f), \mathcal{O}_X|_{D(f)}) \cong \mathrm{Spec}(A_f)$

Exercise 2.2. *Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. We call this the induced scheme structure on the open set U, and we refer to $(U, \mathcal{O}_X|_U)$ as an open subscheme of X.*

Solution. Observation: Let's observe that if we have an isomorphism of ringed spaces, the restriction to open sets is still an isomorphism. That is, if $(X, \mathcal{O}_X) \cong (Y, \mathcal{O}_Y)$ via the isomorphism of locally ringed spaces $(f, f^\#)$, then given U any open set of X, $(U, \mathcal{O}_X|_U) \cong (U, \mathcal{O}_Y|_{f(U)})$. It is clear that the restriction of the homeomorphism f to U, $f|_U : U \to f(U)$ is still an homeomorphism. In addition, the stalks at $P \in U$ are still the same, and so are the induced morphisms on stalks $f_P^\#$, so they're isomorphisms and local and therefore the morphism of sheaves $f^\#|_U$ is an is a local isomorphism, and so we have indeed that $(U, \mathcal{O}_X|_U) \cong (U, \mathcal{O}_Y|_{f(U)})$.

Let $P \in U$. As $(X, \mathcal{O}_X)$ is a scheme, then $\exists V$ an open neighbourhood of P such that $(V, \mathcal{O}_X|_V) \cong (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ for a certain ring A. Let $(f, f^\#)$ denote this isomorphism of locally ringed spaces. As $U \cap V$ is an open set of V, then $f(U \cap V)$ is an open set of $\mathrm{Spec}(A)$, so $f(U \cap V) = \bigcup_{i \in I} D(f_i)$, so in particular $f(P) \in D(f_i)$ for a certain i. Then, $P \in f^{-1}(D(f)) \subseteq V \cap U \subset U$, so taking into account the observation we have that $(f^{-1}(D(f)), \mathcal{O}_X|_{f^{-1}(D(f))}) \cong (D(f), \mathcal{O}_{\mathrm{Spec}(A)}|_{D(f)})$. Finally, using Exercise 2.1 we have the isomorphism $(D(f), \mathcal{O}_{\mathrm{Spec}(A)}|_{D(f)}) \cong \mathrm{Spec}(A_f)$. So in conclusion for each $P \in U$ there is an open neighbourhood of P which is isomorphic as a locally ringed space to the spectrum of a ring, that is, $(U, \mathcal{O}_X|_U)$ is a scheme.

Exercise 2.3. *Reduced Schemes. A scheme $(X, \mathcal{O}_X)$ is reduced if for every open set $U \subseteq X$ the ring $\mathcal{O}_X(U)$ has no niloptent elements.*

a) Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$ the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.

b) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_red$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{red}$ where for any ring A, we denote A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{red})$ is a scheme. We call it the reduced scheme associated to X and we denote it by X_{red}. Show that there is a morphism of schemes $X_{red} \to X$ which is an homeomorphism on the underlying topological spaces.

c) Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{red}$ such that f is obtained by composing g with the natural map $Y_{red} \to Y$.

Solution. a) $\boxed{\Rightarrow}$ Suppose that $0 \neq \overline{(U, f)} \in \mathcal{O}_{X,P}$ is a nilpotent element. This implies that $\exists n$ such that $\overline{(U, f^n)} = \overline{(U, f)}^n = 0$, that is, $\exists W \subseteq U$ such that $f^n|_W = 0$ But restriction morphisms are ring morphisms so $(f|_W)^n = 0$. Moreover, $f|_W$ is not zero, because that would imply that $\overline{(U, f)} = 0$. So in conclusion $f|_W \in \mathcal{O}_X(W)$ is nilpotent which is a contradiction.

$\boxed{\Leftarrow}$ Now suppose that $\mathcal{O}_X(U)$ has niloptent elements for a certain U. That means that $\exists f \neq 0 \in \mathcal{O}_X(U)$, $\exists n \geq 1$ such that $f^n = 0$. Now $\exists P \in U$ such that $\overline{(U, f)} \in \mathcal{O}_{X,P}$ is not zero, as otherwise we would have an open covering of U, $\{W_Q\}_{Q \in U}$ such that $f|_{W_Q} = 0$, which would imply $f = 0$. Then, we have $0 \neq \overline{(U, f)}$, but $0 = \overline{(U, f^n)} = \overline{(U, f)}^n$, so $\mathcal{O}_{X,P}$ has nilpotent elements, which is a contradiction.

b) First let's make some observations that will be useful later.

Obs 1: Given a morphism of rings $\varphi : A \to B$, it induces a morphism between the reduced rings $\varphi_{red} : A_{red} \to B_{red}$, defined by $\varphi_{red}(\overline{a}) = \overline{\varphi(a)}$. The application is well defined, as the image of a nilpotent element is also nilpotent (let $f \neq 0, n \geq 0$ such that $f^n = 0$; then $(\varphi(f))^n = \varphi(f^n) = 0$). Moreover if φ is injective φ_{red} is also injective: Indeed, $\varphi_{red}(\overline{f}) = 0 \Rightarrow \overline{\varphi(f)} = 0 \Rightarrow \exists n$ such that $\varphi(f^n) = (\varphi(f))^n = 0$, and, by injectivity of φ, $f^n = 0 \Rightarrow \overline{f} = 0$. Finally, if φ is surjective, φ_{red} is also surjective: Given any $\overline{b} \in B_{red}$, $\exists a \in A$ such that $\varphi(a) = b$ so $\varphi_{red}(\overline{a}) = \overline{b}$.

Obs 2: We will prove that $((\mathcal{O}_X)_{red})_P = (\mathcal{O}_{X,P})_{red}$. Indeed, both sets are pairs of elements (U,f) under a certain equivalence relation. Two pairs (U,f) and (V,g) are equal on $((\mathcal{O}_X)_{red})_P$ if and only if $\exists W \subseteq U \cap V$ such that $\overline{f}|_W = \overline{g}|_W \iff \overline{f|_W} = \overline{g|_W} \iff f|_W = g|_W + r$ with r a nilpotent element of $\mathcal{O}_X(W)$. Replacing W by a smaller neighbourhood if necessary, this happens if and only if the element $(W,r) \in \mathcal{O}_{X,P}$ is nilpotent, so if and only if the elements (U,f) and (V,g) are related in $(\mathcal{O}_{X,P})_{red}$.

Obs 3: Let's prove that $(\mathrm{Spec}(A), (\mathcal{O}_{\mathrm{Spec}(A)})_{red}) \cong (\mathrm{Spec}(A_{red}), \mathcal{O}_{\mathrm{Spec}(A_{red})})$. First note that the natural application $A \to A_{red}$ induces an application $f : \mathrm{Spec}(A_{red}) \to \mathrm{Spec}(A)$, which is an homeomorphism (Atiyah-Macdonald, Exercise 1.21 iv.). Now for each open set U, $(\mathcal{O}_{\mathrm{Spec}(A)}(U))_{red}$ is the ring consisting on the applications $s : U \to \bigcup_{\mathfrak{p}\in U} A_{\mathfrak{p}}$ such that $\exists U' \subseteq U$ and $a, f \in A$, $f \notin \mathfrak{q}$, $\forall \mathfrak{q} \in U'$ such that $s(\mathfrak{q}) = a/f \in \mathfrak{q}$, for every $\mathfrak{q} \in U'$, module the equivalence relation that two applications are equal if their difference is a nilpotent application (the image of each $\mathfrak{p}$ is nilpotent in $A_{\mathfrak{p}}$). Then $(\mathcal{O}_{\mathrm{Spec}(A)})_{red}(U)$ is the ring of applications $\phi : U \to \bigcup_{P\in U}((\mathcal{O}_{\mathrm{Spec}(A)})_{red})_P$ such that $\forall P \in U$ exists $U' \subseteq U$ neighbourhood of P and $s \in \mathcal{O}_{\mathrm{Spec}(A)}(U')_{red}$ such that $\forall \mathfrak{q} \in U' \phi(\mathfrak{q}) = s_{\mathfrak{q}}$. Now by Observation 2 we can identify the stalk $((\mathcal{O}_{\mathrm{Spec}(A)})_{red})_P$ with $(A_{\mathfrak{p}})_{red}$. Therefore, this induces a morphism of sheaves $f^{\#} : (\mathcal{O}_{\mathrm{Spec}(A)})_{red} \to \mathcal{O}_{\mathrm{Spec}(A_{red})}$ by the natural local isomorphism $(A_{\mathfrak{p}})_{red} \cong (A_{red})_{\overline{\mathfrak{p}}}$. As these are the stalks and they're isomorphic, then $(f, f^{\#})$ is in fact an isomorphism of locally ringed spaces, so in conclusion

$$(\mathrm{Spec}(A), (\mathcal{O}_{\mathrm{Spec}(A)})_{red}) \cong (\mathrm{Spec}(A_{red}), \mathcal{O}_{\mathrm{Spec}(A_{red})})$$

Now let's proceed to prove that $(X, (\mathcal{O}_X)_{red})$ is a scheme. Ldt $P \in X$. As $(X, \mathcal{O}_X)$ is a scheme, then $\exists V$ a neighbourhood of P such that $(V, \mathcal{O}_X|_V) \cong (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$, so we have an isomorphism of locally ringed spaces $f : \mathrm{Spec}(A) \to V$, $f^{\#} : \mathcal{O}_X|_V \to f_*\mathcal{O}_{\mathrm{Spec}(A)}$. As a morphism of sheaves, $f^{\#}$ induces a morphism of rings on every open set: $f^{\#}(W) : \mathcal{O}_X|_V(W) \to f_*\mathcal{O}_{Spec(A)}(W)$. By Obs 1, we can induce morphisms $f^{\#}(W)_{red} : \mathcal{O}_X|_V(W)_{red} \to f_*\mathcal{O}_{\mathrm{Spec}(A)}(W)_{red}$, which are injective by Observation 1. In turn, this induces a morphism on the associated sheaves: $f^{\#}_{red} : (\mathcal{O}_X|_V)_{red} \to (f_*\mathcal{O}_{Spec(A)})_{red}$, which is injective by Exercise 1.4a. Let's check that it is also surjective. Indeed, given $\overline{x} \in f_*\mathcal{O}_{\mathrm{Spec}(A)}(W)$ we know that $\exists\{W_i\}$ open cover of W, and $x_i \in \mathcal{O}_X|_V(W)$ such that $f^{\#}(W_i)(x_i) = x|_{W_i}$. Then, again using observation 1, $f^{\#}_{red}(W_i)(\overline{x_i}) = \overline{x|_{W_i}} = \overline{x}|_{W_i}$, and so $f^{\#}_{red}$ is surjective. Moreover, using observation 2, the induced applications on stalks are in fact the reductions of the applications on stalks, that is $(f^{\#}_{red})_P = (f^{\#}_P)_{red}$ so they're local morphisms because $f^{\#}_P$ are local morphisms and the reduction preserves the correspondence between ideals. Finally let's observe that $(f_*\mathcal{O}_{\mathrm{Spec}(A)})_{red} = f_*(\mathcal{O}_{\mathrm{Spec}(A)})_{red}$ and $(\mathcal{O}_X|_V)_{red} = (\mathcal{O}_X)_{red}|_V$, (it's just a matter of checking that the corresponding rings on any open set are equal). So we have proven that we have a morphism of sheaves $(V, (\mathcal{O}_X)_{red}|_V) \cong (\mathrm{Spec}(A), (\mathcal{O}_{\mathrm{Spec}(A)})_{red})$ and using observation 3 and composing morphisms, we have finally

$$(V, (\mathcal{O}_X)_{red}|_V) \cong (\mathrm{Spec}(A_{red}), \mathcal{O}_{\mathrm{Spec}(A_{red})})$$

Indeed, we have proven that every point P has an open neighbourhood that is isomorphic to the spectra of a certain ring, i.e. $(X, (\mathcal{O}_X)_{red})$ is a scheme.

Now let's build a mosphim of schemes between X_{red} and X. That is, let $f : X \to X$ be the identity application, which is an homeomorphism. Now let $f^{\#}(U) : \mathcal{O}_X(U) \to (\mathcal{O}_X)_{red}(U)$ be the application that maps $x \in \mathcal{O}_X(U)$ to the constant application $s : U \to ((\mathcal{O}_X)_{red})_P$, $s(P) = \overline{x}_P$, where $\overline{x}$ is the image of x in $(\mathcal{O}_X(U))_{red}$. This induces a morphism of sheaves $f^{\#} : \mathcal{O}_X \to (\mathcal{O}_X)_{red}$, with induced applications on stalks $f^{\#}_P : (\mathcal{O}_X)_P \to ((\mathcal{O}_X)_{red})_P$.

$f_P^\#$ maps an element to its reduction modulo the nilpotent element of the stalk (we are using the definition of the morphism $f^\#$ and observation 2 to identify $((\mathcal{O}_X)_{red})_P$ and $(\mathcal{O}_{X,P})_{red})$, and therefore its a local morphism of rings. Then $(f, f^\#)$ is a morphism of locally ringed spaces, with f homeomorphism, as desired.

c)

Exercise 2.4. *Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $F : X \to \operatorname{Spec}(A)$, we have an associated map on sheaves $f^\# : \mathcal{O}_{\operatorname{Spec}(A)} \to f_*\mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map*

$$\alpha : \operatorname{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec}(A)) \to \operatorname{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathcal{O}_X))$$

Show that α is bijective (cf. (I, 3.5) for an analogous statement about varieties).

Solution. We will define an map $\beta : \operatorname{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathcal{O}_X)) \to \operatorname{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec}(A))$. Given $\varphi : A \to \Gamma(X, \mathcal{O}_X)$ morphism of rings, composing with the maps $\pi_P : \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,P}$, $x \mapsto \overline{(X, x)}$ we obtain morphisms of rings $\varphi_P : A \to \mathcal{O}_{X,P}$ for each $P \in X$.

As $(X, \mathcal{O}_X)$ is a scheme, then it is in particular a locally ringed space, so $\mathcal{O}_{X,P}$ is a local ring for all P. Let $\mathfrak{m}_P$ be the maximal ideal of $\mathcal{O}_{X,P}$. Then, we can define an application $f : X \to \operatorname{Spec}(A)$ as $f(P) = \varphi_P^{-1}(\mathfrak{m}_P)$. The application is clearly well defined, as the antiimage of a prime ideal is prime. Now we will check that it is a continuous application. It is enough to prove that the antiimage of a basic open subset is open. So let $h \in A$, $D(h)$ the set of prime ideals of A that don't contain h. Then, $f^{-1}(D(h)) = \{P \in X \text{ such that } h \notin \varphi_P^{-1}(\mathfrak{m}_P)\}$. But $h \notin \varphi_P^{-1}(\mathfrak{m}_P) \iff \varphi_P(h) \notin \mathfrak{m}_P \iff \overline{(X, \varphi(h))}$ is a unit in $\mathcal{O}_{X,P} \iff \exists U_P$ neighbourhood of P and $k \in \Gamma(U_P, \mathcal{O}_X)$ such that $\varphi(h)|_{U_P} k = 1$. Then, $\varphi(h)_Q \in \mathcal{O}_{X,Q}$ is a unit $\forall Q \in U_P \Rightarrow h \notin \varphi_Q(\mathfrak{m}_Q)$ $\forall Q \in U_P$. That means that $f^{-1}(D(h)) = \bigcup_{P \in f^{-1}(D(h))} U_P$ is an open set, and therefore f is continuous.

Note that the applications φ_P can induce naturally applications $\varphi'_P : A_{\varphi_P^{-1}(\mathfrak{m}_P)} \to \mathcal{O}_{X,P}$ mapping $\frac{a_1}{a_2} \mapsto \frac{\varphi_P(a_1)}{\varphi_P(a_2)}$. These are well defined local ring morphisms. Now let's define a morphism of sheaves $f^\#$ between $\mathcal{O}_{\operatorname{Spec}(A)}$ and $f_*\mathcal{O}_X$ as follows: Given $s \in \mathcal{O}_{\operatorname{Spec}(A)}(U)$, $s : U \to \bigcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$, the composition $\varphi'_P \circ s \circ f : f^{-1}(U) \to \bigcup_{P \in f^{-1}(U)} \mathcal{O}_{X,P}$ is an element of $\Gamma(f^{-1}(U), \mathcal{O}_X)$. Moreover, the induced morphisms on stalks are just the applications φ'_P, which we already know that are local morphisms. In conclusion, we have defined a morphism of schemes

$$\beta(\varphi) = (f, f^\#) : (X, \mathcal{O}_X) \to (\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$$

Now let $(f, f^\#) := \beta(\varphi)$. Taking global sections of $f^\#$ we obtain a map $\psi : \Gamma(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}) \to \Gamma(X, \mathcal{O}_X)$. As we saw on Proposition 2.2, $\Gamma(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$ is isomorphic to A via the map $A \ni a \mapsto s : \operatorname{Spec}(A) \to \bigcup_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}}, s(\mathfrak{p}) = \frac{a}{1}$. So given an element of A, $f^\#(\operatorname{Spec}(A))(a)$ is the application $s : X \to \bigcup_{P \in X} \mathcal{O}_{X,P}$ that maps each P to $\overline{(X, \varphi(a))} \in \mathcal{O}_{X,P}$, which can be identified with the element $\varphi(a) \in \Gamma(X, \mathcal{O}_X)$. This proves that $\alpha(\beta(\varphi)) = \varphi$, so the application α is surjective.

Reciprocally, let $\varphi := \alpha((f, f^\#))$ a morphism of rings between A and $\Gamma(X, \mathcal{O}_X)$, that is, $\alpha = f^\#(\operatorname{Spec}(A))$. The morphism of schemes $f^\#$ induces morphisms on stalks: $f_P^\# : (\mathcal{O}_{\operatorname{Spec}(A)})_{f(p)} \to \mathcal{O}_{X,P}$ given by $\overline{(U, x)} \mapsto \overline{(f^{-1}(U), f^\#(U)(x))}$. Composing with the morphism $i_P : A \mapsto (\mathcal{O}_{\operatorname{Spec}(A)})_{f(P)}$, $i(a) = \overline{(\operatorname{Spec}(A), a)}$ we obtain a morphism $f_P^\# \circ i : A \to \mathcal{O}_{X,P}$ mapping $a \mapsto \overline{(X, \varphi(a))}$. On the other hand, we have already seen that there are morphisms $A \to \mathcal{O}_{X,P}$ induced by φ, that we have named $\varphi_P = \pi_P \circ \varphi$, that are equal to $f_P^\# \circ i$. Therefore, we have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & A_{f(P)} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle f_P^\#} \\ \Gamma(X, \mathcal{O}_X) & \xrightarrow{\pi_P} & \mathcal{O}_{X,P} \end{array}$$

In particular, the antiimage of the maximal ideal $\mathfrak{m}_P$ by the two paths must be equal, so $f(p) = (f_P^{\#} \circ i)^{-1}(\mathfrak{m}_P) = (\pi_P \circ \varphi)^{-1} = \varphi_P^{-1}(\mathfrak{m}_P)$. This means that the application between topological spaces $\beta(\varphi)$ is the same as the original application f. Moreover, it is also a consequence of the commutative diagram that the morphisms induced by β on stalks are the same as the originals, that is, $\beta(\varphi)_P^{\#} = f_P^{\#}$. Then, given an open set $U \subseteq \mathrm{Spec}(A)$, $x \in \Gamma(U, \mathcal{O}_{\mathrm{Spec}(A)}$ we have that $\beta(\varphi)^{\#}(U)(x)_P = f^{\#}(U)(x)_P$ for every $P \in U$. Then, $\beta(\varphi)^{\#}(U)(x)$ and $f^{\#}(U)(x)$ agree when restricted to an open neighbourhood W_P of P. These open neighbourhoods $\{W_P\}$ form an open cover of U, and by Property 3 of sheaves we have the equality in U. In conclusion, we have proven that $\beta(\alpha(f, f^{\#})) = (f, f^{\#})$, so α is injective. This completes the proof, and so we have a bijective correspondence

$$\alpha : \mathrm{Hom}_{\mathfrak{Sch}}(X, \mathrm{Spec}(A)) \to \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathcal{O}_X))$$

Exercise 2.5. *Describe* $\mathrm{Spec}(\mathbb{Z})$ *and show that it is a final object for the category of schemes, i.e., each scheme* X *admits a unique morphism to* $\mathrm{Spec}(\mathbb{Z})$.

Solution. The prime ideals of Z are $\mathfrak{p} = (p)$ with p a prime integer, and (0). All the primes of the first type are maximal ideals, so they're closed points in the ring spectrum. Moreover, every ideal contains the zero ideal, so (0) is a generic point in the ring spectrum, such that its closure is the whole space.

Given any commutative ring with unity A, there exists a unique morphism of rings $\varphi : \mathbb{Z} \to A$, defined by $\varphi(n) = n1_A$. In particular, given a scheme $(X, \mathcal{O}_X)$, we take $A = \Gamma(X, \mathcal{O}_X)$, and it exists a unique ring morphism $\mathbb{Z} \to \Gamma(X, \mathcal{O}_X)$. So, by the bijective correspondence between ring morphisms $B \to \Gamma(X, \mathcal{O}_X)$ and morphisms of schemes $X \to \mathrm{Spec}(B)$ proven in Exercise 2.4, there exists a unique morphism of schemes $X \to \mathrm{Spec}(\mathbb{Z})$, so $\mathrm{Spec}(\mathbb{Z})$ is a final object in the category of schemes.

Exercise 2.6. *Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes.*

Solution. The zero ring $R = 0$, has only one element, so the only ideal of R is the whole ring and therefore it has no prime ideals. Therefore, $\mathrm{Spec}(R) = \varnothing$, the spectrum of the zero ring is the empty set.

Now, given any scheme $(X, \mathcal{O}_X)$ there is a unique application $\varnothing : \mathrm{Spec}(R) \to X$, which is the empty function, and it is continuous (the antiimage of every set is the empty set which is open). Moreover, the only open set in $\mathrm{Spec}(R)$ is the empty set $U = \varnothing$ so by definition of sheaf $\mathcal{O}_{\mathrm{Spec}(R)}(U) = 0$ and so for each $U \subseteq X$ we have a unique application $\mathcal{O}_X(U) \to \mathcal{O}_{\mathrm{Spec}(R)}(f^{-1}(U)) = 0$ which is the zero application (maps every element to zero). In conclusion, we have proven that there is a unique morphism of schemes $\mathrm{Spec}(R) \to X$, so $\mathrm{Spec}(R)$ is an initial object for the category of schemes.

Exercise 2.7. *Let* X *be a scheme. For any* $x \in X$*, let* $\mathcal{O}_X$ *be the local ring at* x *and* $\mathfrak{m}_x$ *its maximal ideal. We define the residual field of* x *on* X *to be the field* $k(x) = \mathcal{O}_X/\mathfrak{m}_x$*. Now let* K *be any field. Show that to give a morphism of* $\mathrm{Spec}K$ *to* X *is it equivalent to give a point* $x \in X$ *and an inclusion map* $k(x) \to K$.

Solution. Let $P \in \mathrm{Spec}K$ be the only point of this topological space. Given a morphism of schemes, $(f, f^{\#}) : (\mathrm{Spec}(K), \mathcal{O}_{\mathrm{Spec}(K)}) \to (X, \mathcal{O}_X)$, f is completely determined by a point $x \in X$, the image of the only point $P \in \mathrm{Spec}(K)$. In addition, $f^{\#}$ induces a morphism on the stalks $f_P^{\#} : \mathcal{O}_{X,x} \to K$. It is a local morphism because $(f, f^{\#})$ is a morphism of locally ringed spaces, and so $\ker(f_P^{\#}) = (f_P^{\#})^{-1}(0) = \mathfrak{m}_x$ and therefore we can induce an injective morphism $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \to k$. Reciprocally, given an injection $k(x) \to K$ we can induce a unique local morphism $\mathcal{O}_{X,x} \to K$ sending an element to its class in $k(x)$ and then applying φ. We will see now that any morphism $f^{\#}$ on the structure sheafs it totally determined by the induced application on the stalk at P.

Indeed, given an open set $U \subseteq X$, if $x \notin U$ then $f^{-1}(U) = \varnothing$, so the only possible application $\mathcal{O}_X(U) \to \mathcal{O}_{\mathrm{Spec}(K)}(f^{-1}(U))$ is the zero application. Now let's observe that if $x \in U, V$ and $V \subseteq U$, then the restriction morphisms from $f^{-1}(U)$ to $f^{-1}(V)$ is the identity, and therefore $f^{\#}(U)(s) = f^{\#}(U)(s)|_{f^{-1}(V)} = f^{\#}(V)(s|_V)$. Than means that two elements $s, t \in \mathcal{O}_X(U)$ have the same image by $f^{\#}(U)$ if and only if they're equal in a neighbourhood of x. In conclusion, the image of an element s by $f^{\#}(U)$ is equal to the image of its stalk at x by the application $f_P^{\#}$.

In conclusion, a morphism of schemes $(f, f^{\#}) : (\mathrm{Spec}(K), \mathcal{O}_{\mathrm{Spec}(K)}) \to (X, \mathcal{O}_X)$ is completely determined by a point $x \in X$ and an injection $k(x) \to K$.

Exercise 2.8. *Let X be a scheme. For any point $x \in X$ we define the Zariski tangent space T_X to X at x to be the dual space of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\epsilon]/\epsilon^2$ be the ring of dual numbers over k. Show that to give a k-morphism of* Spec$k[\epsilon]/\epsilon^2$ *to X is equivalent to giving a point $x \in X$ rational over k (i.e., such that $k(x) = k$), and an element of T_x.*

Exercise 2.9. *If X is a topological space, and Z an irreducible closed subset of X, a generic point for Z is a point such that $Z = \overline{\{\zeta\}}$. If X is a scheme, show that every (nonempty) closed subset has a unique generic point.*

Solution. Foreach $P \in Z$, let's consider V_P the open neighbourhood of P such that $(V_P, \mathcal{O}_X|_{V_P})$ is isomorphic to the spectrum of a given ring. Let's fix $P \in Z$ and let A such that $(V_P, \mathcal{O}_X|_{V_P}) \cong (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$. Let f be the homeomorphism $f : V_P \to \mathrm{Spec}(A)$.

Let's observe that $Z \cap V_P$ is irreducible (as an open set of $Z \cap V_P$ is of the form $U \cap Z \cap V_P$ and then $(U_1 \cap Z \cap V_P) \cap (U_2 \cap Z \cap V_P) = (Z \cap V_P) \cap (Z \cap U_1) \cap (Z \cap U_2)$ which is non empty because it's the intersection of nonempty open sets of Z, which is irreducible.

As f is an homeomorphism, then $f(Z \cap V_P)$ is irreducible and closed (as a subset of $\mathrm{Spec}(A)$). As it is closed, $\exists \mathfrak{a}$ ideal of A such that $f(Z \cap V_P) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$. Now let $fg \in r(\mathfrak{a}) \Rightarrow fg \in \mathfrak{q}\ \forall \mathfrak{q} \in V(\mathfrak{a}) \Rightarrow D(fg) \cap V(\mathfrak{a}) = \varnothing$. As $D(fg) = D(f) \cap D(g)$ (Atiyah-MacDonald, exercise 1.17), then $(D(f) \cap V(\mathfrak{a})) \cap (D(g) \cap V(\mathfrak{a})) = \varnothing$. By irreducibility of $V(\mathfrak{a})$, either $D(f) \cap V(\mathfrak{a})$ or $D(g) \cap V(\mathfrak{a})$ must be empty, so either f or g belong to $\mathfrak{q}$, $\forall \mathfrak{q} \in V(\mathfrak{a})$. In conclusion, $r(\mathfrak{a})$ is a prime ideal, that we will name $\mathfrak{p}$, and $f(Z \cap V_P) = V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, and $\mathfrak{p}$ is the only point of $\mathrm{Spec}(A)$ with this property. Then, $f^{-1}(\mathfrak{p}) = Q_P \in V_P \cap Z$, and as the closure of image is the image of the closure under an homeomorphism, then $\overline{\{Q_P\}} = Z \cap V_P$, where the closure here is the closure in V_P. Then, the closure of Q_P in Z is $\overline{\{Q_P\}} = Z \cap \overline{V_P}$.

Now note that $Z \setminus (Z \cap \overline{V_P})$ and $Z \cap V_P$ are open sets of Z, and their intersection is empty. As Z is irreducible, one of them must be empty. $Z \cap V_P$ is not empty, as P belongs to this subset, so we must have $Z \setminus (Z \cap \overline{V_P}) = \varnothing \Rightarrow Z \cap \overline{V_P} = Z$. So, in conclusion, $\overline{\{Q_P\}} = Z$ and we have proved the existence of a generic point of Z. Now let's prove the uniqueness. Note that the point Q_P is unique with this property in $V_P \cap Z$ (as $\mathfrak{p}$ was unique, as we already observed), but we could have a different point Q_P for each open set V_P. However, as $\overline{\{Q_P\}} = Z$, $\forall P' \in Z$ and $\forall U$ open neighbourhood of P', $Q \in U$. In particular, taking $U = V_{P'}$ we have that $Q_P \in V_{P'}$, and therefore $Q_P = Q_{P'}$ (via the composition of homeomorphisms from V_P and $V_{P'}$ to the corresponding ring spectrums) and so the generic point is unique, $Q_P = \zeta\ \forall P$.

Exercise 2.10. *Describe* $\mathrm{Spec}(\mathbb{R}[x])$. *How does it compare to the set* $\mathbb{R}$*? To* $\mathbb{C}$?

Solution. $\mathbb{R}$ is a field, and so $\mathbb{R}[x]$ is a principal ideal domain, so $\mathrm{Spec}(\mathbb{R}[x]) = (0) \cup \mathrm{Max}(\mathbb{R}[x])$. Then, the prime ideals of $\mathbb{R}[x]$ are the zero ideal and ideals of the form (f), with f an irreducible polynomial (which are all maximal ideals). Then $\mathrm{Spec}(\mathbb{R}[x])$ has a generic point (0) and a closed point for each irreducible polynomial. We know that irreducible polynomials in $\mathbb{R}[x]$ can be of two types: $f = x - a$, with $a \in \mathbb{R}$, or $f = (x-(a+bi))(x-(a-bi))$, $a, b \in \mathbb{R}, b \neq 0$. Therefore, in $\mathrm{Spec}(\mathbb{R}[x])$ we have a closed point for each $a \in \mathbb{R}$ and a closed point for each pair of complex conjugates. Then, $\mathrm{Spec}(\mathbb{R}[x])$ can be identified with the closed upper half plane plus a generic point (0).

Exercise 2.11. *Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe* $\mathrm{Spec}k[x]$. *What are the residue fields of its points? How many points are there with a given residue field?*

Solution. As k is a field, following the same reasoning of last exercise, the spectrum of $k[x]$ will consist of a closed point for each irreducible polynomial in $k[x]$ plus a generic point corresponding to the ideal (0). There are irreducible polynomials in $k[x]$ of arbitrary degree n: Indeed, we can consider an extension of degree n of $\mathbb{F}_p$, $\mathbb{F}_{p^n}$. As the multiplicative group of $\mathbb{F}_{p^n}$ is finite, it is cyclic, so $\mathbb{F}_{p^n} = \{0, \alpha, \ldots, \alpha^{p^{n-1}(p-1)}\}$, and then $\mathbb{F}_{p^n} = \mathbb{F}_p(\alpha)$. As $[\mathbb{F}_{p^n} : \mathbb{F}_p] = n$, then $\mathrm{Irr}(\alpha, \mathbb{F}_p, x)$ has degree n.

Then, given $f(x) \in k[x]$ irreducible of degree n, the residue field at the point corresponding to (f) is $k[x]_{(f)}/(f)k[x]_{(f)}$. As localization and quotient commute, this field is the same as $(k[x]/(f))_{(f)}$, but $k[x]/(f)$ is already a field, $\mathbb{F}_{p^n}$ so every element is already invertible, and when we localize we are not adding any elements. In conclusion, the residue field at the point corresponding to the ideal (f) is $\mathbb{F}_{p^n}$, where n is the degree of the polynomial. The residue field at the generic point (0) is the field of fractions of the ring $\mathbb{F}_p$, that is, $\mathbb{F}_p(x)$.

Then, the number of points with a given residue field is exacly the same as the number of irreducible polynomials of $k[x]$ with a given degree n. This number is given by the expression $\frac{1}{n}\sum_{d|n}\mu(n/d)p^d$, where μ is the Möbius function: $\mu(n) = 0$ if $p^2|n$ for a certain prime p, and $\mu(n) = (-1)^k$ otherwise, where k is the number of different prime factors of n.

Exercise 2.12. *Glueing Lemma. Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possible infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$, and let it have the induced scheme structure (Ex. 2.2). Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ such that (1) for each i, j $\varphi_{ij} = \varphi_{ji}^{-1}$ and (2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$. Then show that there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}. We say that X is obtained by glueing the schemes X_i along the isomorphisms φ_{ij}. An interesting special case is when the family X_i is arbitrary, but the U_{ij} and φ_{ij} are all empty. Then the scheme X is called the disjoint union of the X_i and its denoted $\bigsqcup X_i$.*

Solution. We will proceed as in 2.3.5. Let's consider X the topological space $\bigsqcup X_i$ modulo the equivalence relation $\varphi_{ij}(x)\ x, \forall x \in U_{ij},\ \forall i, j$. We have maps ψ_i which send each element $x \in X_i$ to its equivalence class in X. Then, we define a topology on X as follows: A set $V \subseteq X$ is open if and only if $\psi_i^{-1}(V)$ is open for each i. Its clear that this topology makes the applications ψ_i continuous.

Again following the developement of 2.3.5 we can define a sheaf of rings on X as

$$\mathcal{O}_X(V) = \{(s_i) \in \prod_i \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \text{ such that } \varphi_{ij}(s_i|_{\psi_i^{-1}(V)\cap U_{ij}}) = s_j|_{\psi_j^{-1}(V)\cap U_{ji}}\}$$

Now, to avoid checking step by step that it is indeed a sheaf of rings, we will make use of Exercise 1.22. Note that $\psi_i(X_i)$ are open sets of X, as $\psi_j(\psi_i(X_i)) = X_i$ if $i = j$, and U_{ji} if $j \neq i$. Then, we can define on $\psi_i(X_i)$ a sheaf by direct image, $(\psi_i)_*\mathcal{O}_{X_i}$. Moreover,

$\psi_i(X_i) \cap \psi_j(X_j)$ are the elements of X that admit representatives both in X_i and X_j, so $\psi_i(X_i) \cap \psi_j(X_j) = \psi_i(U_{ij}) = \psi_j(U_{ji})$. Therefore, we have isomorphisms of sheaves

$$\phi_{ij} := \varphi_{ij_*}^{\#} : (\psi_j)_* \mathcal{O}_{X_j}|_{\psi_i(X_i)\cap\psi_j(X_j)} \to (\psi_i)_* \mathcal{O}_{X_j}|_{\psi_i(X_i)\cap\psi_j(X_j)}$$

And those morphisms satisfy $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $\psi_i(X_i) \cap \psi_j(X_j) \cap \psi_k(X_k)$. Then, we are under the situation of Exercise 1.22, and the sheaf we have defined on X is the same as the sheaf defined by glueing the sheafs $(\psi_i)_*\mathcal{O}_{X_i}$ defined in the open cover $\psi_i(X_i)$. So we already know that $\mathcal{O}_X$ is a sheaf.

In addition, let's observe that $(\psi_i)_*\mathcal{O}_{X_i}$ and $\mathcal{O}_{X_i}$ are isomorphic sheaves, as ψ_i is injective (the equivalence relation that defines X only identifies points from different X_i, so each element of X has at most one representative in X_i), so ψ_i is an homeomorphism when we restrict the image to $\psi(X_i)$ instaed of X.

Moreover, it is clear that for each point $P \in X$, $P = \psi_i(Q)$ for a certain i and $Q \in X_i$. Then, the ring $\mathcal{O}_{X,P}$ is isomorphic to the local ring $\mathcal{O}_{X_i,Q}$ and so it is local. Moreover, every point has an open set isomorphic to the spectrum of a ring (as the image of an open set of X_i by ψ_i is an open set of X). This proves that $(X, \mathcal{O}_X)$ is a scheme.

Now let's consider the morphisms of shemes $(\psi_i, \psi_i^{\#}) : (X, \mathcal{O}_X) \to (X_i, \mathcal{O}_{X_i})$, where $\psi_i^{\#}$ is the projection on the $i - th$ coordinate. We have already commented that $(X_i, \mathcal{O}_{X_i})$ and $(\psi(X_i), (\psi_i)_*\mathcal{O}_{X_i})$ are isomorphic schemes. We also know that $\mathcal{O}_X|_{\psi_i(X_i)} \to (\psi_i)_*\mathcal{O}_{X_i}$ is an isomorphism of sheaves by Exercise 1.22. Then the composition, which is $\psi_i^{\#}|_{\psi(X_i)}$ is an isomorphism of sheaves, and so we have that $(\psi_i, \psi_i^{\#})$ is an isomorphism of schemes from X_i to an open subscheme of X, corresponding to $\psi_i(X_i)$. This proves property (1). We have already proven properties (2) and (3) earlier. With respect to (4), it follows from the fact that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$, which is proved in 1.22, toghether with the definition of the sheaf morphisms ϕ_{ij}.

Exercise 2.13. *A topological space is quasi-compact if every open cover has a finite subcover.*

a) Show that a topological space is noetherian if and only if every open subset is quasi-compact.

b) If X is an affine scheme, show that $\mathrm{sp}(X)$ *is quasi-compact, but not in general noetherian. We say that a scheme X is quasi-compact if* $\mathrm{sp}(X)$ *is.*

c) If A is a noetherian ring, show that $\mathrm{sp}(\mathrm{Spec}(A))$ *is a noetherian topological space.*

d) Give an example to show that $\mathrm{sp}(\mathrm{Spec}(A))$ *can be noetherian even when A is not.*

Solution. a) $\boxed{\Rightarrow}$ Let X be a noetherian topological space. Every open subset U of X is noetherian, as open sets of U are also open sets of X, so every ascending chain of open sets of U must stabilize, and so U is indeed noetherian. Then, it is enough to prove that a noetherian topological space is quasi-compact. Now let $\{U_\alpha\}$ be an open cover of X. Now let's build an ascending open chain: We pick an open set $U_1 \in \{U_\alpha\}$. If $U_1 \neq X$, there mus exist another open set $U_2 \in \{U_\alpha\}$ such that $U_2 \not\subseteq U_1$. Then $U_1 \subset U_1 \cup U_2$. Again, if $U_1 \cup U_2 \neq X$, $\exists U_3 \not\subseteq U_1 \cap U_2$. Repeating this steps inductively we build an ascending chain

$$U_1 \subset U_1 \cup U_2 \subset \cdots \subset \bigcup_{i=1}^{n} U_i \subset \ldots$$

As X is noetherian, the chain must stabilize at some point m and so we must have $\bigcup_{i=1}^{m} U_i = X$, so X is quasi-compact.

$\boxed{\Leftarrow}$ Consider a chain of open sets of X, $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \dots$. Then $U = \bigcup_{i=1}^{\infty} U_i$ is an open set, and $\{U_i\}_{i=1}^{\infty}$ is an open cover of U. U is quasi-compact as it is an open subset, so $\exists U_{i_1}, \dots U_{i_m}$ that cover U. Therefore, the chain stabilizes at point i_m, and therefore X is noetherian.

b) If X is an affine scheme, then $X = \operatorname{Spec}(A)$ for a certain ring A. As the sets $D(f)$ form a base of the topology, every open covering of X can be regarded as a basic open covering. So let $X = \bigcup_{i\in I} D(f_i) = \bigcup_{i\in I} X \setminus V(f_i) = X \setminus \bigcap_{i\in I} V(f_i)$. This happens if and only if $V(\bigcup_{i\in I} f_i = \bigcap_{i\in I} V(f_i) = \varnothing \iff$ the ideal generated by the elements f_i is the whole ring A. That means that $\exists J$ a finite subset of I such that $1 = \sum i \in J g_i f_i$. Then, following the implications in the opposite direction, we have $\bigcap_{i\in J} V(f_i) = \varnothing \Rightarrow X \setminus \bigcap_{i\in J} V(f_i) = \bigcup_{i\in J} X \setminus V(f_i) = \bigcup_{i\in J} D(f_i) = X$. So the set $\{D(f_i)\}_{i\in J}$ is a finite open subcovering of $\{D(f_i)\}_{i\in I}$. That shows that $\operatorname{sp}(X)$ is quasi-compact.

Now consider the ring $A = k[x_1, \dots, x_n \dots]$, that is a ring of polynomials with infinite indeterminates over a field k. The ideals $(x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, x_2, \dots, x_n) \subset \dots$ are all prime and form an ascending chain that doesn't stabilize. Then, $V(x_1) \supset V(x_1, x_2) \supset \cdots \supset V(x_1, x_2, \dots, x_n) \supset \dots$ is a descending chain of closed sets of $\operatorname{sp}(X)$ which doesn't stabilize, and so the topological space is not noetherian.

c) Let $V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \supseteq \dots$ be a descending chain of closed sets in $\operatorname{sp}(\operatorname{Spec}(A))$. That means that we have a corresponding chain on ideals $r(\mathfrak{a}_1) \subseteq r(\mathfrak{a}_2) \subseteq \dots$. As the ring is noetherian, this chain must stabilize. Then, taking into account that $V(\mathfrak{a}) = V(r(\mathfrak{a}))$, the initial descending chain of closed sets also stabilizes, so the topological space $\operatorname{Spec}(A)$ $\operatorname{sp}(\operatorname{Spec}(A))$ is noetherian.

d) Let's consider the ring $A = k[x_1, \dots, x_n \dots]/(x_1, x_2^2, \dots, x_n^n \dots)$. These ring has only one prime ideal, $(x_1, x_2, \dots, x_n, \dots)$, so $\operatorname{sp}(\operatorname{Spec}(A))$ has only one point, and therefore every descending chain of closed sets must stabilize and $\operatorname{sp}(\operatorname{Spec}(A))$ is noetherian. However, the chain of ideals of A $(x_1) \subset (x_1, x_2) \subset \dots$ doesn't stabilize, and so A is not a noetherian ring.

Exercise 2.14. *a) Let S be a graded ring. Show that $\operatorname{Proj}(S) = \varnothing$ if and only if every element of S_+ is nilpotent.*

b) Let $\varphi : S \to T$ be a morphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \operatorname{Proj}(T) | \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of $\operatorname{Proj}(T)$, and show that φ determines a natural morphism $f : U \to \operatorname{Proj}(S)$.

c) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj}(T)$ and the morphism $f : \operatorname{Proj}(T) \to \operatorname{Proj}(S)$ is an isomorphism.

d) Let V be a projective variety with homogeneous coordinate ring S. Show that $t(V) \cong \operatorname{Proj}(S)$.

Solution. a) Suppose that every element of S_+ is nilpotent, and let $\mathfrak{p}$ be an homogeneous prime ideal. Given $x \in S_+$, $x^n = 0 \in \mathfrak{p}$ for a certain integer n, so we must have $x \in \mathfrak{p}$. This means that $\mathfrak{p} \supseteq S_+$ and therefore $\operatorname{Proj}(S) = \varnothing$, as it is the set of homogeneous prime ideals that don't contain S_+. Reciprocally, suppose that $\operatorname{Proj}(S) = \varnothing$. This means that every homogeneous prime ideal of S contains S_+.

Reciprocally, let now $\operatorname{Proj}(S) = \varnothing$. Let's first prove that every prime ideal $\mathfrak{p}$ contains an homogeneous prime ideal $\mathfrak{p}_h$, the one generated by the homogeneous elements of $\mathfrak{p}$. It is clear that $\mathfrak{p}_h$ is homogeneous and that $\mathfrak{p}_h \subseteq \mathfrak{p}$. We need to prove that $\mathfrak{p}_h$ is prime. Let f, g such that $fg \in \mathfrak{p}_h$. As S is a graded ring, then f and g admit a unique descomposition as a sum of homogeneous elements, $f = \sum_{i=0}^{n} f_i$ and $g = \sum_{i=0}^{m} g_i$. We have to prove that either f or g belong to $\mathfrak{p}_h$. We will proceed by induction on $n+m$. If $n = m = 0$, then, $fg \in \mathfrak{p}_h \subseteq \mathfrak{p} \Rightarrow f$ or $g \in \mathfrak{p}$, and as they're homogeneous elements, they belong to $\mathfrak{p}_h$. In the general case, $fg = f_n g_m + (\sum_{i=0}^{n-1} f_i)(\sum_{i=0}^{m-1} m_i) + f_n(\sum_{i=0}^{m-1} g_i) + g_m(\sum_{i=0}^{n-1} f_i)$. As the ideal $\mathfrak{p}_h$ is homogeneous, then the component of degree i of fg belongs to $\mathfrak{p}_h$ for each i. In particular, the component of degree nm, which is $f_n g_m$ belongs to $\mathfrak{p}_h \subseteq \mathfrak{p}$, so either f_n or g_m belong to $\mathfrak{p}$, and therefore $\in \mathfrak{p}_h$ as they're homogeneous. Suppose that $f_n \in \mathfrak{p}_h$. Then $\mathfrak{p}_h \ni fg - f_n g = (\sum_{i=0}^{n-1} f_i)(\sum_{i=0}^{m} g_i)$. Applying induction hypothesis, either g or $f - f_n$ belong to $\mathfrak{p}_h$. As we already know that $f_n \in \mathfrak{p}_h$, then we have indeed that f or g belong to $\mathfrak{p}_h$, so it is a prime ideal.

The fact that every prime ideal contains an homogeneous prime ideal implies that the nilradical of S van be calculated as the intersection of all homogeneous prime ideals. As $\mathrm{Proj}(S) = \varnothing$, every homogeneous prime ideal contains S_+, and therefore the nilradical contains S_+, so every element of S_+ is nilpotent.

b) If φ is a graded morphism, the image of an homogeneous element is homgeneous, and therefore, if $\mathfrak{a}$ is an homogeneous ideal of S, $\varphi(\mathfrak{p})^e$ is an homogeneous ideal of T. Note also that $\mathfrak{p} \supseteq \varphi(\mathfrak{a}) \iff \mathfrak{p} \supseteq \varphi(\mathfrak{a})$. Then, $V(\varphi(S_+)^e)$ is a closed subset of $\mathrm{Proj}(T)$ and therefore its complementary, which is U, is open.

It is clear that φ defines an application between prime ideals of T and prime ideals of S, that maps $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. The contraction of an homogeneous ideal $\mathfrak{p}$ is homogeneous: Indeed, let $f = f_0 + \cdots + f_n$ such that $\varphi(f) = x$, with x homogeneous of degree d, then $\varphi(f_i) = 0$ if $i \neq d$ and $\varphi(f_i) = x$ if $i = d$. So f_i, $i \neq d \in \varphi^{-1}(\mathfrak{p})$ as they belong to the kernel of φ and $f_d \in \varphi^{-1}(x)$. So the components of each element of $\varphi^{-1}(\mathfrak{p})$ belong to the ideal, and therefore $\varphi^{-1}(\mathfrak{p})$ is homogeneous. This means that we can restrict the application between prime ideals of T and prime ideals of S to homogeneous prime ideals. In addition, $\mathfrak{p} \not\supseteq \varphi(S_+) \Rightarrow\Rightarrow \varphi^{-1}(\mathfrak{p}) \not\supseteq \varphi^{-1}(\varphi(S_+))$. We know that $\varphi^{-1}(\varphi(S_+)) \supseteq S_+$. But $\varphi(S_+)$ doesn't have elements of degree 0, so its antiimage doesn't have elements of degree 0, which means that $\varphi^{-1}(\varphi(S_+)) = S_+$. In conclusion, $\mathfrak{p} \not\supseteq \varphi(S_+) \Rightarrow \varphi^{-1}(\mathfrak{p}) \not\supseteq S_+$. Then, the images of elements of U belong to $\mathrm{Proj}(S)$, so when we restrict the application φ^{-1} to U we obtain an application $f : U \to \mathrm{Proj}(S)$.

Now let's check that this is a continuous map between the corresponding topological spaces (with U having the induced topology by $\mathrm{Proj}(T)$). It is enough to check that the antiimage of a basic open set is open. Let $D_+(h)$ be a basic open subset of $\mathrm{Proj}(S)$, (with $h \in S$). Then, $f^{-1}(D_+(h)) = \{\mathfrak{p} \in U \text{ such that } h \notin \varphi^{-1}(\mathfrak{p})\}$ As $h \notin \varphi^{-1}(\mathfrak{p}) \iff \varphi(h) \notin \mathfrak{p}$, then $f^{-1}(D_+(h))$ is equal to $D_+(\varphi(h)) \cap U$, which is open in U. Then the application f is indeed continuous.

Let's define an associated morphism of sheaves as follows: $f^{\#} : \mathcal{O}_{\mathrm{Proj}(S)} \to f_* \mathcal{O}_{\mathrm{Proj}(T)}|_U$. Given an application $s : V \to \bigsqcup_{\mathfrak{p} \in V} S_{(\mathfrak{p})} \in \mathcal{O}_{\mathrm{Proj}(S)}$ we map it to the application $f^{\#}(s) = \varphi_{\mathfrak{p}} \circ s \circ f : f^{-1}(V) \to \bigsqcup_{\mathfrak{p} \in f^{-1}(V)} T_{(\mathfrak{p})}$, where $\varphi_{\mathfrak{p}} : S_{\varphi^{-1}(\mathfrak{p})} \to T_{\mathfrak{p}}$ is the localization of φ (as φ preserves degrees, $\varphi_{\mathfrak{p}}$ also does, and therefore takes elements of degree zero to elements of degree zero, and so $f^{\#}$ is well defined). $f^{\#}$ is in addition a morphism of locally ringed spaces, as the induced applictions on stalks are the local ring morphisms $\varphi_{\mathfrak{p}}$. In conclusion, $(f, f^{\#})$ is a morphism of schemes between U and $\mathrm{Proj}(S)$.

c) If $\mathfrak{p} \supseteq \varphi(S_+)$ and φ_d is an isomorphism, then $\mathfrak{p} \supseteq T_d$, $\forall d \geq d_0$. Now, given $x \in T_+$, $\exists n$ such that $x^n \in \mathfrak{p}$ (n such that $n\deg(x) \geq d_0$). As $\mathfrak{p}$ is prime, $x^n \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$. So $\mathfrak{p} \supseteq \varphi(S_+) \iff \mathfrak{p} \supseteq T_+$, and therefore $U = \mathrm{Proj}(T)$.

Now let's make 2 observations that will be useful to prove that f is a morphism. **Obs 1:** Every $\mathfrak{p}$ homogeneous prime ideal of S that doesn't contain S_+ cannot contain all T_d for arbitrary degree $d \geq d_0$, as therefore, $\forall x \in T_d, d > 0, \exists n$ such that the degree of x^n is $\geq d_0$, and thereore $x^n \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$, and all S_+ is contained in $\mathfrak{p}$, which is a contradiction.

Obs 2: As φ_d is an isomorphism, $\varphi(a^n) \in \varphi(\mathfrak{p}) \cap T_d$, with $d \geq d_0$ implies that $a^n \in \mathfrak{p}$, as φ is injective. Then $a^n \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$.

- f is injective: Let $\mathfrak{p}, \mathfrak{q}$ homogeneous prime ideals of T such that $f(\mathfrak{p}) = f(\mathfrak{q}) \iff \varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q})$. Let $x \in \mathfrak{p}$ homogeneous of degree > 0. Then, $\exists n$ such that $d := \deg(x^n) \geq d_0$. Then, as φ_d is an isomorphism, $\exists! a \in \varphi_d^{-1}(x^n) \subseteq \varphi_d^{-1}(\mathfrak{p} \cap T_d) = \varphi^{-1}(\mathfrak{q} \cap T_d)$. Therefore, $\varphi_d(a) = x^n \in \mathfrak{q} \Rightarrow x \in \mathfrak{q}$ as $\mathfrak{q}$ is prime. Now let $x \in \mathfrak{p}$ of degree $d = 0$. Now let's take y of homogeneous of degree $\geq d_0$, $y \notin \mathfrak{q}$, which exists by Observation 1. Then, $yx \in \mathfrak{p}$ has degree $\geq d_0$ and therefore $\exists! a \in S_d$ such that $a \in \varphi^{-1}(xy) \in \varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q})$. That implies that $xy \in \mathfrak{q}$, but as $y \notin \mathfrak{q}$ and $\mathfrak{q}$ is prime, then $x \in \mathfrak{q}$. This proves $\mathfrak{p} \subseteq \mathfrak{q}$ and by the symmetric argument, the two ideals are equal so f is injective.
- f is surjective: Let $\mathfrak{p}$ be a prime ideal of $\mathrm{Proj} S$. Let $\mathfrak{q}$ be the ideal generated by the homogeneous elements of T such that $x^n \in \varphi(\mathfrak{p})$ for a certain n, or homogeneous elements of degree zero of T such that $xy \in \varphi(\mathfrak{p})$ for a certain y such that $y \notin \varphi(\mathfrak{p})$. The ideal $\mathfrak{q}$ is homogeneous (because it's generated by homogeneous elements). Let's check that it is a prime ideal: Given x, y homogeneous elements of degree ≥ 1, such that $xy \in \mathfrak{q} \Rightarrow \exists n$ such that $(xy)^n \in \varphi(\mathfrak{p})$. We can assume that $n \geq d_0$ as $a^n \in \varphi(\mathfrak{p}) \Rightarrow a^{nm} \in \varphi(\mathfrak{p})\ \forall m \geq 1$. Then, as x^n and y^n are elements of T_d, $d \geq 0$, and φ_d is surjective $\Rightarrow \exists! a, b \in S$ such that $\varphi(a) = x^n$, $\varphi(b) = y^n$. Then $\varphi(ab) \in \varphi(\mathfrak{p})$ and as φ_d is injective for $d \geq d_0$, then $ab \in \mathfrak{p}$. As $\mathfrak{p}$ is prime either a or $b \in \mathfrak{p} \Rightarrow x^n$ or y^n belong to $\varphi(\mathfrak{p})$ and therefore x or y belong to $\mathfrak{q}$. Now let x have degree zero, and $xy \in \mathfrak{q}$. If y has degree 0, then this means that $\exists z$ such that $xyz \in \varphi(\mathfrak{p})$. Then yz has degree d and $x(yz) \in \varphi(\mathfrak{p})$, which means that $x \in \mathfrak{q}$. If y has degree different from 0, then $(xy)^n \in \varphi(\mathfrak{p})$. If $y^n \notin \varphi(\mathfrak{p})$ then $x \in \mathfrak{q}$. If $y^n \in \varphi(\mathfrak{p})$, then $y \in \mathfrak{q}$. In conclusion, $\mathfrak{q}$ is a prime ideal.

  Now let $x \in \mathfrak{q}$. If $\deg(x) \geq 1$ then $\exists n$ such that $x^n \in \varphi(\mathfrak{p}) \Rightarrow x^n = \varphi(a)$, with $a \in \mathfrak{p}$. Let $b \in \varphi^{-1}(x)$. Then, $\varphi(b^n) \in \varphi(\mathfrak{p}) \Rightarrow b \in \mathfrak{p}$ (by observation 2). If $\deg(x) = 0$, then $\exists y \notin \varphi(\mathfrak{p})$, $y \in T_d$, $d \geq d_0$ such that $xy \in \varphi(\mathfrak{p})$. As φ_d is an isomorphism, then $\exists! a \in S_d \cap \mathfrak{p}$ such that $\varphi(a) = xy$, and $\exists! b \in S_d, b \notin \mathfrak{p}$ such that $\varphi(b) = y$. Let $c \in \varphi^{-1}(x)$. Then $\varphi(cb) = \varphi(a) \Rightarrow cb = a \in \mathfrak{p}$ but $b \notin \mathfrak{p} \Rightarrow c \in \mathfrak{p}$. With this reasoning we have proved that $\varphi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$. Reciprocally, given x an homogeneous element of $\mathfrak{p}$, $\varphi(x) \in \varphi(\mathfrak{p}) \Rightarrow \varphi(x) \in \mathfrak{q}$ and this means that $x \in \varphi^{-1}(\varphi(x)) \in \varphi^{-1}(\mathfrak{q})$. This proves that $\mathfrak{p} \subseteq \varphi^{-1}(\mathfrak{q})$. In conclusion, we have found an homogeneous ideal $\mathfrak{q}$ such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, so f is surjective.
- f^{-1} is continuous: It is enough to check that the antiimage of a basic open subset is open. Let $h \in T$ and consider the basic open subset $D_+(h)$. Note that $D_+(h) = D_+(h^n)$, so if h has degree $\neq 0$, for a given n, h^n has enough degree such that $h^n \notin \mathfrak{p} \iff \varphi^{-1}(h^n) \notin \varphi^{-1}(\mathfrak{p})$, and therefore, $(f^{-1})^{-1}(D_+(h)) = D(\varphi^{-1}(h^n))$. If $\deg(h) = 0$, then $h \in \mathfrak{p} \iff \exists y \notin \mathfrak{p}$ such that $yh \in \mathfrak{p}$. Then we can choose an element y of degree atlesat d_0 by Observation 1, and therefore $(f^{-1})^{-1}(V(h)) = (f^{-1})^{-1}(V(yh)) = V(\varphi^{-1}(yh))$. Then, $(f^{-1})^{-1}(D_+(h)) = D_+(\varphi(yh))$. In conclusion, the antiimage of an open set is open, and f^{-1} is continuous. Since now we have proven that f is an homeomorphism.
- $f^{\#}$ is an isomorphism of sheaves: First let's prove injectivity. Suppose that we have

two applications $s, s' \in \mathcal{O}_{\mathrm{Proj}(S)}(V)$ such that $f^{\#}(V)(s) = f^{\#}(V)(s')$. Now let $\mathfrak{p}$ be a prime of $f^{-1}(V)$ and $f(\mathfrak{p}) \in V$. Let $s(f(\mathfrak{p})) = \frac{a}{t}$ and $s'(f(\mathfrak{p})) = \frac{a'}{t'}$. Then, $(f^{\#}(V)(s))(\mathfrak{p}) = (f^{\#}(V)(s'))(\mathfrak{p}) \Rightarrow \frac{\varphi(a)}{\varphi(t)} = \frac{\varphi(a')}{\varphi(t')}$. That happens if and only if $\exists y \notin \mathfrak{p}$ such that $y(\varphi(at') - \varphi(a't)) = 0$. Note that we can multiply this expression by elements of higher degree ans it is still zero. Then, by observation 1, we can suppose that $\deg(y) = d \geq d_0$ and therefore $\exists x \in S_d$, $x \notin f(\mathfrak{p})$ such that $\varphi(at'x) - \varphi(a'tx) = 0$. As $a'tx$ and $at'x$ are homogeneous elements with degrees higher that d_0, then φ is injective and $at'x = a'tx \Rightarrow x(at' - a't) = 0 \Rightarrow \frac{a}{t} = \frac{a'}{t'}$ in $S_(f(\mathfrak{p}))$. As this is valid for every prime $\mathfrak{p}$, $s = s'$, and $f^{\#}$ is injective.

To prove surjectivity we will prove surjectivity of the induced applications on stalks, that are $\varphi_{\mathfrak{p}}$. Indeed, given $\frac{a}{t} \in T_{(\mathfrak{p})}$, $\exists y \notin \mathfrak{p}$ with $\deg(y) = d \geq d_0$. Then, $\frac{a}{t} = \frac{ay}{ty}$. As φ_d is surjective $\forall d \geq d_0$ and $ay, ty \in T_d$, with $d \geq d_0$, then $\exists a', t' \in S_d$, $t' \notin f(\mathfrak{p})$, such that $ay = \varphi(a')$ and $ty = \varphi(t')$. Then $\frac{a}{t} = \varphi_{\mathfrak{p}}(\frac{a'}{t'})$ and so $\varphi_{\mathfrak{p}}$ is surjective.