Exercises Hartshorne

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1 Sheaves

Exercise 1.1. *Let A be an abelian group, and define the constant presheaf associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.*

Solution. Let sF denote the constant presheaf. Let's first see that each stalk $\mathscr{F}_P$ is a copy of A. Indeed, the elements of $\mathscr{F}_P$ are represented by pairs $\langle U, s\rangle$, with U open neighbourhood of P and $s \in A$. As the restriction maps are the identity, two pairs $\langle U, s\rangle$ and $\langle V, t\rangle$ represent the same element if and only if $s = t$, so $\mathscr{F}_P = A$.

Let s be an application from U to $\bigcup_{P\in U} \mathscr{F}_P$ satisfying properties (1) and (2) from the definition of associated sheaf. By (1), $s(P) \in \mathscr{F}_P$ is an element of A, and therefore s can be regarded as an application from U to A (that we will denote s'). In addition, let $B \subseteq A$. For each $P \in s'^{-1}(B)$, $\exists V_P$ neighbourhood of P such that $s'(V_P) = t \in B$. Then $s'^{-1}(B) = \bigcup_{P\in s'^{-1}(B)} V_P$ which is open. We have proved that the antiimage of every subset is open and therefore s' is continuos with A being given the discrete topology.

Reciprocally, any countinuous application s' from U to A can be regarded as an application s from U to $\bigcup_{P\in U} \mathscr{F}_P$, defining $s(P) = s'(P) \in \mathscr{F}_P$. This assignation guarantees that s satisfies (1). In addition, for each $P \in U$, the set $V = s'^{-1}(s'(P))$ is an open neighbourhood of P (by continuity of s'), and every $Q \in V$ has the same image $s'(P)$, which proves that s satisfies (2).

In conclusion, $\mathscr{F}^+(U)$ is the group of continuous maps from U into A, and therefore $\mathscr{F}^+$ is indeed the sheaf $\mathscr{A}$ defined in the text.

Exercise 1.2. *a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ show that for each point P, $(\ker\varphi)_P = \ker(\varphi_P)$ and $(im\varphi)_P = im(\varphi_P)$.*

b) Show that φ is injective (respectively surjective) if and only if the induced map on the stalks φ_P is injective (respectilevy surjective) for all P.

c) Show that a sequence $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Solution. a) $(\ker\varphi)_P = \{(U,s),\, s \in \ker(\varphi(U))\}$, modulo the usual germ equivalence relationship. Thus it can be regarded as a subset of $\mathscr{F}_P$ as $\ker(\varphi(U)) \subseteq \mathscr{F}(U)$. On the other side $\ker(\varphi_P)$ is a subset of $\mathscr{F}_P$. To see that the two sets are equal it's enough to check the double inclusion. Let $\overline{(U,s)} \in (\ker\varphi)_P$. Then, $\varphi_P(\overline{(U,s)}) = \overline{(U,(\varphi(U))(s))} = \overline{(U,0)} = 0 \Rightarrow \overline{(U,s)} \in \ker(\varphi_P)$. Reciprocally, given $\overline{(U,s)} \in \ker(\varphi_P) \Rightarrow \exists V \subset U$ such that $\varphi(U)(s)|_V = 0$. As restrictions commute with morphisms of sheaves, $\varphi(V)(s|_V) = \varphi(U)(s)|_V = 0$. Then, $\overline{(U,s)} = \overline{(V,s|_V)} \in (\ker\varphi)_P$. In conclusion, $(\ker\varphi)_P = \ker(\varphi_P)$.

As $\mathscr{F}_P = \mathscr{F}^+_P$, $(\mathrm{im}\varphi)_P$ is equal to the stack of the presheaf image at point P. $\mathrm{im}(\varphi_P) = \{\overline{(U,s)} \in \mathscr{G}_P$ such that $\exists\overline{(V,t)} \in \mathscr{F}_P | \varphi_P(\overline{(V,t)} = \overline{(U,s)}\}$. But as $\varphi_P(\overline{(V,t)}) = \overline{(V,\varphi(V)(t))}$ then $\overline{(U,s)} \in \mathrm{im}(\varphi_P) \iff \exists W$ neighbourhood of P, $W \subseteq V \cap U$ such that $\varphi(V)(t)|_W = s_W \iff \varphi(W)(t|_W) = s|_W \iff \overline{(U,s)} = \overline{(W,\varphi(W)(t|_W))} \iff \overline{(U,s)} \in (\mathrm{im}\varphi)_P$.

b) φ injective $\Rightarrow \ker\varphi = 0 \Rightarrow (\ker\phi)_P = 0\,\forall P$. Using part a) of the problem, $\ker\varphi_P = (\ker\varphi)_P = 0 \Rightarrow \varphi_P$ is injective $\forall P$. Reciprocally, let $x \in \ker\varphi(U)$. $\forall P \in U, (\ker\phi)_P = 0$ so the image of x in the stalk $(\ker\phi)_P$ is zero, which means that $\exists W_P \subseteq U$ neighbourhood of P such that $x|_{W_P} = 0$. But open sets W_P cover U and therefore, by property (3) of the definition of shieves, $x = 0$. In conclusion, $\ker\varphi(U) = 0\ \forall U \Rightarrow \ker\varphi = 0 \Rightarrow \varphi$ injective.

We proceed similarly with the surjectivity. $\mathrm{im}\varphi = \mathscr{G} \Rightarrow (\mathrm{im}\varphi)_P = \mathscr{G}_P \Rightarrow \mathrm{im}(\varphi_P) = \mathscr{G}_P \Rightarrow \varphi_P$ surjective. To prove the other implication, First we will prove a fact that is stated but not proved in the text: $\mathscr{F}^+ \cong \mathscr{F}$ if $\mathscr{F}$ is already a sheaf. Given an open set U, let V_P be the neighbourhood of P contained in U such that $\exists t \in \mathscr{F}(V_P)$ such that $t_Q = s(Q)\,\forall Q \in V_P$. The sets V_P cover U, and given two of these sets, V, V' and the respective elements t, t' we have that $\overline{(V', t')} = \overline{(V, t)}$ in every stalk $\mathscr{F}_Q \Rightarrow \exists W_Q$ such that $t'|_{W_Q} = t|_{W_Q} \forall Q \in V \cap V'$. Then these W_Q cover $V \cap V'$, and by property (3) applied to $t'|_{V\cap V'} - t_{V\cap V'}$, we have that $t_{V\cap V'} = t'_{V\cap V'}$. Then, by property (4) applied to the sets V_P, $\exists t \in \mathscr{F}(U)$ such that $t_Q = s(Q)\ \forall Q \in U$, which means that each application s is uniquely determined by $t \in \mathscr{F}(U)$, and then $\mathscr{F}^+(U) \cong \mathscr{F}(U)$. Now it's easy to check that φ is surjective. We have that $(\mathrm{im}\varphi)_P = \mathrm{im}(\varphi_P) = \mathscr{G}_P$ and so we have that $\mathfrak{m}\varphi(U)$ is the set of functions s from U to $\bigcup_{P\in U} \mathscr{G}_P$, which means that $\mathrm{im}\varphi$ is in fact $\mathscr{G}^+ \cong \mathscr{G}$ as $\mathscr{G}$ is already a sheaf.

c) Given a sequence of sheaves and morphisms $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$, it is exact $\iff \ker\varphi^i = \mathrm{im}\varphi^{i-1}$. If the sequence is exact, taking direct limits at both sides and using section a) we have that $\ker(\varphi^i_P) = (\ker\varphi^i)_P = (\mathrm{im}\varphi^{i-1})_P = \mathrm{im}(\varphi^{i-1}_P)$, and so the sequence of stalks at each point P is exact.

The other implicaton is consequence of a more general result that we will prove now: Two subsheaves are equal $\iff$ the corresponding stalks at each point are equal. Let $\mathscr{F}_1, \mathscr{F}_2$ be two subsheaves of $\mathscr{F}$, such that $(\mathscr{F}_1)_P = (\mathscr{F}_2)_P$. Let $t \in \mathscr{F}_1(U)$. For every $P \in U \exists V_P$ neighbourhood of P and $s \in \mathscr{F}_2(V)$ such that $s_P = t_P$. The sets $V_P \cap U$ cover U, and given two of these sets, V, V' and the respective elements s, s' we have that $\overline{(V', s')} = \overline{(V, s)} = \overline{(U, t)}$ in every stalk $\mathscr{F}_Q \Rightarrow \exists W_Q$ such that $s'|_{W_Q} = s|_{W_Q} \forall Q \in V \cap V'$. Then these W_Q cover $V \cap V'$, and by property (3) applied to $s'|_{V\cap V'} - s|_{V\cap V'}$, we have that $s|_{V\cap V'} = s'|_{V\cap V'}$. Then, by property (4) applied to the sets $V_P \cap U$, $\exists r \in \mathscr{F}_2(U)$ such that $t_P = r_P \forall P$. By property (3) applied to $r - t$ we get $s = t$ and so $t \in \mathscr{F}_2(U)$. So $\mathscr{F}_1(U) \subseteq \mathscr{F}_2(U)$ and by symmetry $\mathscr{F}_1(U) = \mathscr{F}_2(U)$.

Exercise 1.3. *a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$, for all i.*

b) Give an example of a surjective morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ and an open set U such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not injective.

Solution. a) From problem 1.2, φ is surjective $\iff$ the induced morphism on every stalk is. Suppose φ_P is surjective, then given $U \subseteq X$ open set, $s \in \mathscr{G}(U)$, $\forall P \in U \exists V$ neighbourhood of P and $t \in \mathscr{F}(V)$ such that $\overline{(U,s)} = \overline{(V,\varphi(t))} \Rightarrow \exists W_P \subseteq U \cap V$ such that $\varphi(t|_{W_P}) = s|_{W_P}$, and so $\{W_P\}$ is the covering that satisfies the desired property. Reciprocally, let $\overline{(U,s)} \in \mathscr{G}_P$. Then $\forall P \in U \exists i$ such that $P \in U_i \Rightarrow \overline{(U,s)} = \overline{(U_i, \varphi(U_i)(t_i))} = \varphi_P(\overline{(U_i,t_i)}) \Rightarrow \varphi_P$ is surjective.

b)

Exercise 1.4. *a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective foreach U. Show that the induced map $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.*

b) Use part (a) to show that if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then imφ can be naturally identified with a subsheaf of $\mathscr{G}$, as mentioned in the text.

Solution. a) Using 1.2 b) and the fact that $\mathscr{F}_P^+ = \mathscr{F}_P$, the map φ^+ is injective $\iff$ the maps $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ on the stalks are injective. Let $\overline{(U,s)} \in \mathscr{F}_P$ such that $\varphi_P(\overline{(U,s)}) = 0 \Rightarrow \exists W \subset U$ such that $\varphi(U)(s)|_W = 0 \Rightarrow \varphi(W)(s|_W) = 0$. But as $\varphi(U)$ is injective $\forall U$, then $s|_W = 0$ and therefore $\overline{(U,s)} = \overline{(W, s|_W)} = 0$ and thus φ_P is injective $\Rightarrow \varphi^+$ is injective.

b) Let's consider the presheaf image of a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, $U \mapsto \operatorname{im}(\varphi(U))$. Then for each U, $\operatorname{im}(\varphi(U)) \subseteq \mathscr{G}(U)$, and so the inclusion $i(U) : \operatorname{im}(\varphi(U)) \to \mathscr{G}(U)$ is an injective morphism of abelian groups $\forall U$. Then, by section a), the induced map $i^+ : \operatorname{im}\varphi \to \mathscr{G}^+ = \mathscr{G}$ is injective.

Exercise 1.5. *Show that a morphism of sheaves is an isomphism if and only if it is both injective and surjective.*

Solution.

Exercise 1.6. *a) Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$$

b) *Conversely, if* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$ *is an exact sequence, show that* $\mathscr{F}'$ *is isomorphic to a subsheaf of* $\mathscr{F}$ *and that* $\mathscr{F}''$ *is isomorphic to the quotient of* $\mathscr{F}$ *by this subsheaf.*