Exercises Hartshorne

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1 Sheaves

Exercise 1.1. *Let A be an abelian group, and define the constant presheaf associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.*

Solution. Let sF denote the constant presheaf. Let's first see that each stalk $\mathscr{F}_P$ is a copy of A. Indeed, the elements of $\mathscr{F}_P$ are represented by pairs $\langle U, s\rangle$, with U open neighbourhood of P and $s \in A$. As the restriction maps are the identity, two pairs $\langle U, s\rangle$ and $\langle V, t\rangle$ represent the same element if and only if $s = t$, so $\mathscr{F}_P = A$.

Let s be an application from U to $\bigcup_{P\in U} \mathscr{F}_P$ satisfying properties (1) and (2) from the definition of associated sheaf. By (1), $s(P) \in \mathscr{F}_P$ is an element of A, and therefore s can be regarded as an application from U to A (that we will denote s'). In addition, let $B \subseteq A$. For each $P \in s'^{-1}(B)$, $\exists V_P$ neighbourhood of P such that $s'(V_P) = t \in B$. Then $s'^{-1}(B) = \bigcup_{P\in s'^{-1}(B)} V_P$ which is open. We have proved that the antiimage of every subset is open and therefore s' is continuos with A being given the discrete topology.

Reciprocally, any countinuous application s' from U to A can be regarded as an application s from U to $\bigcup_{P\in U} \mathscr{F}_P$, defining $s(P) = s'(P) \in \mathscr{F}_P$. This assignation guarantees that s satisfies (1). In addition, for each $P \in U$, the set $V = s'^{-1}(s'(P))$ is an open neighbourhood of P (by continuity of s'), and every $Q \in V$ has the same image $s'(P)$, which proves that s satisfies (2).

In conclusion, $\mathscr{F}^+(U)$ is the group of continuous maps from U into A, and therefore $\mathscr{F}^+$ is indeed the sheaf $\mathscr{A}$ defined in the text.

Exercise 1.2. *a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ show that for each point P, $(\ker\varphi)_P = \ker(\varphi_P)$ and $(\mathrm{im}\varphi)_P = \mathrm{im}(\varphi_P)$.*

b) Show that φ is injective (respectively surjective) if and only if the induced map on the stalks φ_P is injective (respectilevy surjective) for all P.

c) Show that a sequence $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Solution. a) $(\ker\varphi)_P = \{(U,s),\, s \in \ker(\varphi(U))\}$, modulo the usual germ equivalence relationship. Thus it can be regarded as a subset of $\mathscr{F}_P$ as $\ker(\varphi(U)) \subseteq \mathscr{F}(U)$. On the other side $\ker(\varphi_P)$ is a subset of $\mathscr{F}_P$. To see that the two sets are equal it's enough to check the double inclusion. Let $\overline{(U,s)} \in (\ker\varphi)_P$. Then, $\varphi_P(\overline{(U,s)}) = \overline{(U,(\varphi(U))(s))} = \overline{(U,0)} = 0 \Rightarrow \overline{(U,s)} \in \ker(\varphi_P)$. Reciprocally, given $\overline{(U,s)} \in \ker(\varphi_P) \Rightarrow \exists V \subset U$ such that $\varphi(U)(s)|_V = 0$. As restrictions commute with morphisms of sheaves, $\varphi(V)(s|_V) = \varphi(U)(s)|_V = 0$. Then, $\overline{(U,s)} = \overline{(V,s|_V)} \in (\ker\varphi)_P$. In conclusion, $(\ker\varphi)_P = \ker(\varphi_P)$.

As $\mathscr{F}_P = \mathscr{F}^+_P$, $(\mathrm{im}\varphi)_P$ is equal to the stack of the presheaf image at point P. $\mathrm{im}(\varphi_P) = \{\overline{(U,s)} \in \mathscr{G}_P \text{ such that } \exists\overline{(V,t)} \in \mathscr{F}_P | \varphi_P(\overline{(V,t)} = \overline{(U,s)}\}$. But as $\varphi_P(\overline{(V,t)}) = \overline{(V,\varphi(V)(t))}$ then $\overline{(U,s)} \in \mathrm{im}(\varphi_P) \iff \exists W$ neighbourhood of P, $W \subseteq V \cap U$ such that $\varphi(V)(t)|_W = s_W \iff \varphi(W)(t|_W) = s|_W \iff \overline{(U,s)} = \overline{(W,\varphi(W)(t|_W))} \iff \overline{(U,s)} \in (\mathrm{im}\varphi)_P$.

b) φ injective $\Rightarrow \ker\varphi = 0 \Rightarrow (\ker\phi)_P = 0\,\forall P$. Using part a) of the problem, $\ker\varphi_P = (\ker\varphi)_P = 0 \Rightarrow \varphi_P$ is injective $\forall P$. Reciprocally, let $x \in \ker\varphi(U)$. $\forall P \in U, (\ker\phi)_P = 0$ so the image of x in the stalk $(\ker\phi)_P$ is zero, which means that $\exists W_P \subseteq U$ neighbourhood of P such that $x|_{W_P} = 0$. But open sets W_P cover U and therefore, by property (3) of the definition of shieves, $x = 0$. In conclusion, $\ker\varphi(U) = 0\ \forall U \Rightarrow \ker\varphi = 0 \Rightarrow \varphi$ injective.

We proceed similarly with the surjectivity. $\mathrm{im}\varphi = \mathscr{G} \Rightarrow (\mathrm{im}\varphi)_P = \mathscr{G}_P \Rightarrow \mathrm{im}(\varphi_P) = \mathscr{G}_P \Rightarrow \varphi_P$ surjective. To prove the other implication, First we will prove a fact that is stated but not proved in the text: $\mathscr{F}^+ \cong \mathscr{F}$ if $\mathscr{F}$ is already a sheaf. Given an open set U, let V_P be the neighbourhood of P contained in U such that $\exists t \in \mathscr{F}(V_P)$ such that $t_Q = s(Q)\,\forall Q \in V_P$. The sets V_P cover U, and given two of these sets, V, V' and the respective elements t, t' we have that $\overline{(V', t')} = \overline{(V, t)}$ in every stalk $\mathscr{F}_Q \Rightarrow \exists W_Q$ such that $t'|_{W_Q} = t|_{W_Q} \forall Q \in V \cap V'$. Then these W_Q cover $V \cap V'$, and by property (3) applied to $t'|_{V\cap V'} - t_{V\cap V'}$, we have that $t_{V\cap V'} = t'_{V\cap V'}$. Then, by property (4) applied to the sets V_P, $\exists t \in \mathscr{F}(U)$ such that $t_Q = s(Q)\ \forall Q \in U$, which means that each application s is uniquely determined by $t \in \mathscr{F}(U)$, and then $\mathscr{F}^+(U) \cong \mathscr{F}(U)$. Now it's easy to check that φ is surjective. We have that $(\mathrm{im}\varphi)_P = \mathrm{im}(\varphi_P) = \mathscr{G}_P$ and so we have that $\mathfrak{m}\varphi(U)$ is the set of functions s from U to $\bigcup_{P \in U} \mathscr{G}_P$, which means that $\mathrm{im}\varphi$ is in fact $\mathscr{G}^+ \cong \mathscr{G}$ as $\mathscr{G}$ is already a sheaf.

c) Given a sequence of sheaves and morphisms $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$, it is exact $\iff \ker\varphi^i = \mathrm{im}\varphi^{i-1}$. If the sequence is exact, taking direct limits at both sides and using section a) we have that $\ker(\varphi^i_P) = (\ker\varphi^i)_P = (\mathrm{im}\varphi^{i-1})_P = \mathrm{im}(\varphi^{i-1}_P)$, and so the sequence of stalks at each point P is exact.

The other implicaton is consequence of a more general result that we will prove now: Two subsheaves are equal $\iff$ the corresponding stalks at each point are equal. Let $\mathscr{F}_1, \mathscr{F}_2$ be two subsheaves of $\mathscr{F}$, such that $(\mathscr{F}_1)_P = (\mathscr{F}_2)_P$. Let $t \in \mathscr{F}_1(U)$. For every $P \in U \exists V_P$ neighbourhood of P and $s \in \mathscr{F}_2(V)$ such that $s_P = t_P$. The sets $V_P \cap U$ cover U, and given two of these sets, V, V' and the respective elements s, s' we have that $\overline{(V', s')} = \overline{(V, s)} = \overline{(U, t)}$ in every stalk $\mathscr{F}_Q \Rightarrow \exists W_Q$ such that $s'|_{W_Q} = s|_{W_Q} \forall Q \in V \cap V'$. Then these W_Q cover $V \cap V'$, and by property (3) applied to $s'|_{V\cap V'} - s|_{V\cap V'}$, we have that $s|_{V\cap V'} = s'|_{V\cap V'}$. Then, by property (4) applied to the sets $V_P \cap U$, $\exists r \in \mathscr{F}_2(U)$ such that $t_P = r_P \forall P$. By property (3) applied to $r - t$ we get $s = t$ and so $t \in \mathscr{F}_2(U)$. So $\mathscr{F}_1(U) \subseteq \mathscr{F}_2(U)$ and by symmetry $\mathscr{F}_1(U) = \mathscr{F}_2(U)$.

Exercise 1.3. *a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$, for all i.*

b) Give an example of a surjective morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ and an open set U such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not injective.

Solution. a) From problem 1.2, φ is surjective $\iff$ the induced morphism on every stalk is. Suppose φ_P is surjective, then given $U \subseteq X$ open set, $s \in \mathcal{G}(U)$, $\forall P \in U \exists V$ neighbourhood of P and $t \in \mathcal{F}(V)$ such that $\overline{(U,s)} = \overline{(V,\varphi(t))} \Rightarrow \exists W_P \subseteq U \cap V$ such that $\varphi(t|_{W_P}) = s|_{W_P}$, and so $\{W_P\}$ is the covering that satisfies the desired property. Reciprocally, let $\overline{(U,s)} \in \mathcal{G}_P$. Then $\forall P \in U \exists i$ such that $P \in U_i \Rightarrow \overline{(U,s)} = \overline{(U_i, \varphi(U_i)(t_i))} = \varphi_P(\overline{(U_i,t_i)}) \Rightarrow \varphi_P$ is surjective.

b)

Exercise 1.4. *a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective foreach U. Show that the induced map $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ of associated sheaves is injective.*

b) Use part (a) to show that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then imφ *can be naturally identified with a subsheaf of $\mathcal{G}$, as mentioned in the text.*

Solution. a) Using 1.2 b) and the fact that $\mathcal{F}_P^+ = \mathcal{F}_P$, the map φ^+ is injective $\iff$ the maps $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ on the stalks are injective. Let $\overline{(U,s)} \in \mathcal{F}_P$ such that $\varphi_P(\overline{(U,s)}) = 0 \Rightarrow \exists W \subset U$ such that $\varphi(U)(s)|_W = 0 \Rightarrow \varphi(W)(s|_W) = 0$. But as $\varphi(U)$ is injective $\forall U$, then $s|_W = 0$ and therefore $\overline{(U,s)} = \overline{(W, s|_W)} = 0$ and thus φ_P is injective $\Rightarrow \varphi^+$ is injective.

b) Let's consider the presheaf image of a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, $U \mapsto \text{im}(\varphi(U))$. Then for each U, $\text{im}(\varphi(U)) \subseteq \mathcal{G}(U)$, and so the inclusion $i(U) : \text{im}(\varphi(U)) \to \mathcal{G}(U)$ is an injective morphism of abelian groups $\forall U$. Then, by section a), the induced map $i^+ : \text{im}\varphi \to \mathcal{G}^+ = \mathcal{G}$ is injective.

Exercise 1.5. *Show that a morphism of sheaves is an isomphism if and only if it is both injective and surjective.*

Solution. We know from Proposition 1.1 that a morphism of sheaves φ is an isomorphism $\iff$ the induced morphism on every stalk φ_P is an isomorphism. But the induced morphisms on stalks are morphisms of abelian groups, so they're isomorphisms if and only if they're surjective and injective. Now using Exercise 1.2 b) this is equivalent to φ being surjective and injective.

Exercise 1.6. *a) Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$$

b) Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$ is an exact sequence, show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$ and that $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.

Solution. Observation: First we will prove the equivalent of Exercise 1.4 a) for surjectivity. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is surjective for each U. Then the induced morphisms on stalks are also surjective: Given $\overline{(U,s)} \in \mathscr{G}_P \exists t \in \mathscr{F}(U)$ such that $\varphi(U)(t) = s \Rightarrow \overline{(U,s)} = \overline{(U,\varphi(U)(t))} = \varphi_P(\overline{(U,t)})$. By Exercise 1.2 b) and the fact that the stalks of the associated shief are equal to the stalks of the preshief ($\mathscr{F}_P = \mathscr{F}_P^+$), the induced morphism of shieves $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ is surjective.

a) The morphisms of abelian groups $\mathscr{F}(U) \to \mathscr{F}(U)/\mathscr{F}'(U)$ are surjective $\forall U$. So by the observation above, the morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is surjective. The fact that $\ker\varphi = \mathscr{F}'$ is a consequence of the result proved in Exercise 1.2c. Indeed, $(\ker\varphi)_P = \ker(\varphi_P)$. As $\varphi_P : \mathscr{F}_P \to (\mathscr{F}/\mathscr{F}')_P = \mathscr{F}_P/\mathscr{F}'_P$ are morphisms of abelian groups, their kernel is $\mathscr{F}'_P$. So $\ker\varphi$ and $\mathscr{F}'$ are 2 subssheaves of $\mathscr{F}$ and their stalks at each point P are equal so $\ker\varphi = \mathscr{F}'$. In conclusion there is an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$.

b) Let's name the applications of the sequence φ, ψ: $0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$. The morphisms of abelian groups $\phi(U) : \mathscr{F}'(U) \xrightarrow{\varphi} \mathrm{im}\varphi(U)$ define a morphism of presheaves. As $\phi(U)$ is surjective $\forall U$, using the observation above we have that $\mathscr{F}' \to \mathrm{im}\varphi$ is a surjective morphism of sheaves. Moreover, as φ is injective, $\varphi(U) : \mathscr{F}'(U) \to \mathscr{F}(U)$ is injective $\forall U \Rightarrow \phi(U) : \mathscr{F}'(U) \xrightarrow{\varphi} \mathrm{im}\varphi(U)$ is injective $\forall U$, and by Exercise 1.4 a) $\mathscr{F}' \to \mathrm{im}\varphi$ is an injective morphism of sheaves. So $\mathscr{F}' \to \mathrm{im}\varphi$ is surjective and injective $\Rightarrow$ is an isomorphism, and, in conclusion, $\mathrm{im}\varphi$ is the subsheaf of $\mathscr{F}$ isomorphic to $\mathscr{F}'$.

The surjective morphism of sheaves $\psi : \mathscr{F} \to \mathscr{F}''$ induces surjective morphisms of abelian groups on stacks $\psi_P : \mathscr{F}_P \to \mathscr{F}''_P$, which induce isomorphisms $\overline{\psi_P} : \mathscr{F}_P / \ker(\psi_P) \cong \mathscr{F}''_P \; \forall P$ sending the class of an element s_P to its image $\psi_P(s_P)$.

In addition, the morphisms of abelian groups $\psi(U) : \mathscr{F}(U) \to \mathscr{F}''(U)$ also induce the morphism of presheaves $\psi(U) : \mathscr{F}(U)/\ker\psi(U) \to \mathscr{F}''(U)$. To show that the map os associated sheaves $\mathscr{F}/\ker\psi \to \mathscr{F}''$ is an isomorphism, it is enoush to show that the corresponding morphisms on stalks $(\mathscr{F}/\ker\psi)_P \to \mathscr{F}''_P$ are isomorphisms. But, taking into account that $(\mathscr{F}/\ker\psi)_P = \mathscr{F}_P/(\ker\psi)_P = \mathscr{F}_P/\ker(\psi_P)$, the corresponding morphisms on stalks are in fact the $\overline{\psi_P}$ and we already know that these are isomorphisms, so in conclusion $\mathscr{F}/\ker\psi \cong \mathscr{F}''$. Finally as the sequence $0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is exact, $\mathrm{im}\phi = \ker\psi$ and we are done.

Exercise 1.7. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*

a) Show that $\mathrm{im}\varphi \cong \mathscr{F}/\ker\varphi$.

b) Show that coker$\varphi \cong \mathscr{G}/\mathrm{im}\varphi$.

Solution. a) Given $\varphi : \mathscr{F} \to \mathscr{G}$ the observation on Exercise 1.6 shows that $\mathscr{F} \to \mathrm{im}\varphi$ is surjective. Therefore we have an exact sequence $0 \to \ker\varphi \to \mathscr{F} \to \mathrm{im}\varphi \to 0$ and by Exercise 1.6 b) $\mathrm{im}\varphi \cong \mathscr{F}/\ker\varphi$.

b) The identity map $\mathrm{coker}\varphi(U) \to \mathscr{G}(U)/\mathrm{im}\varphi(U)$ is surjective and injective (it is in fact the definition of the cokernel), and it defines a morphism of presheaves. Then, by Exercise 1.4a) and Observation on 1.6 the induced map of associated sheaves $\mathrm{coker}\varphi \to \mathscr{G}/\mathrm{im}\varphi$ is surjective and injective, $\Rightarrow \mathrm{coker}\varphi \cong \mathscr{G}/\mathrm{im}\varphi$.

Exercise 1.8. *For any open subset $U \subseteq X$ show that the functor $\Gamma(U,)$ from sheaves on X to abelian groups is a left exact functor, i.e. if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U,\mathscr{F}') \to \Gamma(U,\mathscr{F}) \to \Gamma(U,\mathscr{F}'')$ is an exact sequence of groups. The functor $\Gamma(U,)$ need not be exact; see (Ex. 1.21) below.*

Solution. Let's note $\varphi : \mathscr{F}' \to \mathscr{F}$ and $\psi : \mathscr{F} \to \mathscr{F}''$. To show that the sequence $0 \to \Gamma(U,\mathscr{F}') \to \Gamma(U,\mathscr{F}) \to \Gamma(U,\mathscr{F}'')$ is exact we need to prove: a) That $\varphi(U) : \mathscr{F}'(U) \to \mathscr{F}(U)$ is injective and b) That $\ker\psi(U) = \mathrm{im}\varphi(U)$. a) is a consequence of the fact that a morphism of sheaves is injective $\iff$ the induced morphism on every section is injective. Let's proceed to prove

b) showing both inclusions. First note that by Exercise 1.2 c) the induced sequence $0 \to \mathscr{F}'_P \to \mathscr{F}_P \to \mathscr{F}''_P$ on stalks is exact, and therefore $\mathrm{im}(\varphi_P) = \ker(\varphi_P)\ \forall P$.

Let $y \in \mathrm{im}\varphi(U) \Rightarrow y = \varphi(U)(x)$. Then its image on the stalk $\overline{(U,y)} = \overline{(U,\varphi(U)(x))} = \varphi_P(\overline{(U,x)}) \in \mathrm{im}(\varphi_P) = \ker(\varphi_P)$. That means that $\exists W_P \subseteq U$ neighbourhood of P such that $\psi(W_P)(y|_{W_P}) = \psi(U)(y)|_{W_P} = 0 \Rightarrow y|_{W_P} \in \ker\psi(W_P)$. The sets $\{W_P\}$ are an open covering of U, and $(y|_{W_P})|_{W_P\cap W_Q} = (y|_{W_P\cap W_Q}) = (y|_{W_Q})|_{W_P\cap W_Q}$. So as $\ker\psi$ is a sheaf, $\exists y' \in \ker\psi(U)$ such that $y'|_{W_P} = y|_{W_P}$. As $\mathscr{F}$ is a sheaf, applying property (3) to $y - y'$ we get that $y = y'$ and therefore $y \in \ker\psi(U)$. This proves $\mathrm{im}\varphi(U) \subseteq \ker\psi(U)$. Reciprocally, let $y \in \ker\psi(U)$. The same argument on the stalks we did before proves that $\overline{(U,y)} \in \mathrm{im}(\varphi_P) \Rightarrow \exists W_P \subseteq U$ and $x_{W_P} \in \mathscr{F}'(W_P)$ such that $y|_{W_P} = \varphi(W)(x_{W_P})$. As $\varphi(W_P \cap W_Q)$ is injective, and sends $x_{W_P}|_{W_P\cap W_Q}$ and $x_{W_P}|_{W_P\cap W_Q}$ to the same element $y|_{W_P\cap W_Q}$, they must be equal, and therefore $\exists x \in \mathscr{F}'(U)$ such that $x|_{W_P} = x_{W_P}\ \forall P$. By property (3) of sheaf $\mathscr{F}$ applied to $\varphi(x) - y$ we get that $y = \varphi(x)$ and therefore $\mathrm{im}\varphi(U) \supseteq \ker\psi(U)$.

Exercise 1.9. *Direct Sum. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf. It is called the direct sum of $\mathscr{F}$ and $\mathscr{G}$ and is denoted by $\mathscr{F} \oplus \mathscr{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X.*

Solution. We have defined an abelian group for each open set U, but the problem doesn't specify the restriction morphisms. However, they're induced naturally by the restriction morphisms of $\mathscr{F}$ and $\mathscr{G}$: $\rho_{UV}^{\mathscr{F}\oplus\mathscr{G}} = \rho_{UV}^{\mathscr{F}} \oplus \rho_{UV}^{\mathscr{G}}$ that maps $(x,y) \in \mathscr{F}(U) \oplus \mathscr{G}(U) \mapsto (\rho_{UV}^{\mathscr{F}}(x), \rho_{UV}^{\mathscr{G}}(y))$. Now we can check that this presheaf satisfies (3) and (4) and therefore it is actually a sheaf.

- (3) Let U be an open set, and $\{V_i\}$ be an open covering of U. Let $(x,y) \in \mathscr{F}(U) \oplus \mathscr{G}(U)$ such that $(0,0) = 0 = (x,y)|_{V_i} = (x|_{V_i}, y_{V_i})\forall V_i$. This implies that $x|_{V_i} = 0$ and $y|_{V_i} = 0$ and therefore by sheaf property (3) of $\mathscr{F}$ and $\mathscr{G}$ we have $x = 0$ and $y = 0 \Rightarrow (x,y) = (0,0) = 0$.

- (4) Let U be an open set, and $\{V_i\}$ be an open covering of U. Let $(x_i, y_i) = s_i \in \mathscr{F}(V_i) \oplus \mathscr{G}(V_i)$ such that $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$, which means that $(x_i|_{V_i\cap V_j}, y_i|_{V_i\cap V_j}) = (x_i,y_i)|_{V_i\cap V_j} = s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j} = (x_j,y_j)|_{V_i\cap V_j} = (x_j|_{V_i\cap V_j}, y_j|_{V_i\cap V_j})$. Then, by property (4) of shieves $\mathscr{F}$ and $\mathscr{G}$, $\exists x \in \mathscr{F}(U), y \in \mathscr{G}(U)$ such that $x|_{V_i} = x_i$ and $y|_{V_i} = y_i$. Then, $(x,y) = s \in \mathscr{F}(U) \oplus \mathscr{G}(U)$ satisfies $s|_{V_i} = s_i$.

Then the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is indeed a sheaf. Now let's check that it plays the role of direct sum and direct product in category theory. We will use the notation $\mathscr{F}_1, \mathscr{F}_2$ instead of $\mathscr{F}, \mathscr{G}$ to simplify.

Direct product: The object $\mathscr{F}_1 \oplus \mathscr{F}_2$ along with two morphisms $\pi_i : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{F}_i$ is the direct product if it satisfies that $\forall$ sheaf $\mathscr{G}$ and sheaf morphisms $f_i : \mathscr{G} \to \mathscr{F}_i \exists! f : \mathscr{G} \to \mathscr{F}_1 \oplus \mathscr{F}_2$ such that $\pi_i \circ f = f_i$. Indeed, let the morphisms π_i be given by $\pi_i(U) : \mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \to \mathscr{F}_i(U)$ such that $\pi_i(U)((x_1, x_2)) \mapsto x_i$. Then, the morphism $\mathscr{G}(U) \to \mathscr{F}_1(U) \oplus \mathscr{F}_2(U)$ that maps $x \mapsto (f_1(x), f_2(x))$ satisfies the desired property, as $(\pi_i \circ f)(x) = \pi_i(f_1(x), f_2(x)) = f_i(x)$. In addition, suppose that $\exists f'$ morphism satisfying thsi property. Then $f'(U)(x) = (s_1, s_2) \in \mathscr{F}_1(U) \oplus \mathscr{F}_2(U)$, and as $\pi_i(U)(s_1, s_2) = f_i(x) \Rightarrow f'(U)(x) = (f_1(x), f_2(x)) \Rightarrow f' = f$.

Direct sum: The element $\mathscr{F}_1 \oplus \mathscr{F}_2$ is the categorical direct sum of $\mathscr{F}_1$ and $\mathscr{F}_2$ if there exist morphisms $i_j : \mathscr{F}_j \mathscr{F}_1 \oplus \mathscr{F}_2$ such that $\forall$ shief $\mathscr{G}$ and morphisms $f_i \mathscr{F}_i \to \mathscr{G}$, $\exists! f : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{G}$ such that $f \circ i_j = f_j$. Indeed, let's define $i_1(U)(x) = (x, 0)$ and $i_2(U)(x) = (0, x)$. Then, we define $f(U) : \mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \to \mathscr{G}$ such that $f(U)(x, y) = f_1(x) + f_2(y)$. It's clear that $f \circ i_j = f_j$. In addition, suppose that $\exists f'$ satisfying this property. Then, $f'(U)(x, 0) = f_1(U)(x)$, $f'(U)(0, y) = f_2(U)(x)$ and therefore $f'(U)(x, y) = f'(U)(x, 0) + f'(U)(0, y) = f_1(U)(x) + f_2(U)(y)$, which proves the uniqueness of f.

Exercise 1.10. *Direct Limit. Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the direct limit of the system $\{\mathscr{F}_i\}$, denoted $\varinjlim \mathscr{F}_i$ to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathscr{G}$ and a collection of morphisms $\mathscr{F}_i \to \mathscr{G}$ compatible with the maps of the direct syst, there exists a unique map $\varinjlim \mathscr{F}_i \to \mathscr{G}$ such that for each i, the original map $\mathscr{F}_i \to \mathscr{G}$ is obtained by composing the maps $\mathscr{F}_i \to \varinjlim \mathscr{F}_i \mathscr{G}$.*

Solution. Again we have the definition of a presheaf without specifying the restriction morphisms, so we need to figure out which is the natural way to define them. Let f_{ij} denote the morphisms of the direct system, and ρ^i_{UV} the restriction morphisms of each $\mathscr{F}_i$. Then, by definition of morphisms of sheaves we have the following commutative diagram
$$\begin{array}{ccc} \mathscr{F}_i(U) & \xrightarrow{f_{ij}(U)} & \mathscr{F}_j(U) \\ \downarrow{\rho^i_{UV}} & & \downarrow{\rho^j_{UV}} \\ \mathscr{F}_i(V) & \xrightarrow{f_{ij}(V)} & \mathscr{F}_j(V) \end{array}$$

Every element in $\varinjlim \mathscr{F}_i(U)$ can be expressed as the equivalence class of an element $s \in \mathscr{F}_i(U)$ for a certain i. So it seems natural to define $\rho_{UV} : \varinjlim \mathscr{F}_i(U) \to \varinjlim \mathscr{F}_i(V)$ mapping $\overline{s} \mapsto \overline{\rho^i_{UV}(s)}$. We only have to check that this application is well defined, i.e., that it doesn't depend on the chosen representative. Let $t \in \mathscr{F}_i(U), s \in \mathscr{F}_j(U)$ such that $\overline{t} = \overline{s}$. Then $\exists k$ such that $k \geq i, k \geq j$ and $f_{ik}(U)(t) = f_{jk}(U)(s)$ Then, $f_{ik}(V)(\rho^i_{UV}(t)) = \rho^k_{UV}(f_{ik}(U)(t)) = \rho^k_{UV}(f_{jk}(U)(s)) = f_{ik}(V)(\rho^j_{UV}(s))$, and therefore $\overline{\rho^i_{UV}(t)} = \overline{\rho^j_{UV}(s)}$ and the restriction morphisms (as we defined them) are well defined.

Let $\varphi_i : \mathscr{F}_i \to \mathscr{G}$ morphisms of sheaves compatible with the maps of the direct system. Then, we define the morphism of presheaves φ by $\varphi(U) : \varinjlim \mathscr{F}_i(U) \to \mathscr{G}$ that sends the class of an element $s \in \mathscr{F}_i$, $\overline{s} \in \varinjlim \mathscr{F}_i(U) \mapsto \varphi_i(U)(s) \in \mathscr{G}(U)$. This application is well defined as the morphisms of sheaves φ_i are compatible with the morphisms of the direct system. It is clearly unique satisfying the required property (it must send the class of an element s to its image by the initial morphism φ_i). Then, by proposition 1.2 the induced map $\varphi^+ : \varinjlim \mathscr{F} \to \mathscr{G}$ in unique and satisfies the requirements of the problem.

Exercise 1.11. *Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is already a sheaf. In particuar $\Gamma(X, \varinjlim \mathscr{F}_i) = \varinjlim \Gamma(X, \mathscr{F}_i)$.*

Solution. If X is a Noetherian topological space, then every subspace of X is quasi compact, and, in particular, every open subset is quasi-compact, i.e.,for every $\{V_i\}$ open covering of U we can extract a finite subcovering (Atiyah-Macdonald Exercise 6.6). Then, it is enough to check that properties (3) and (4) that define a sheaf are satisfied for finite coverings $\{V_i\}_{i=1}^n$ of U. We will use the same notation of last problem.

- (3) Let $\overline{s} \in \varinjlim \mathscr{F}_i(U)$, with $s \in \mathscr{F}_i(U)$. Suppose that $0 = \overline{s}|_{V_j} = \overline{\rho^i_{UV_j}(s)}$ That means that $\exists k_j \geq i$ such that $f_{ik_j}(V_j)(\rho^i_{UV_j}(s)) = 0$. As restrictions commute with morphisms of the direct system $\rho^{k_j}_{UV_j}(f_{ik_j}(U)(s)) = f_{ik_j}(V_j)(\rho^i_{UV_j}(s)) = 0$. As the covering is finite, we can define $m := \max_j\{k_j\}$ and we have that

$$\rho^m_{UV_j}(f_{im}(U)(s)) = f_{im}(V_j)(\rho^i_{UV_j}(s)) = f_{k_jm}(V_j)(f_{ik_j}(V_j)(\rho^i_{UV_j}(s)) = 0$$

  Now, by shief property (3) applied to $\mathscr{F}_m$, we have that $f_{im}(U)(s) = 0 \Rightarrow \overline{s} = 0$.

- (4) Suppose that we have a covering $\{V_i\}$ of a open set U, and $\overline{s_i} \in \varinjlim \mathscr{F}_j(V_i)$, with $s_i \in \mathscr{F}_{k_i}(V_i)$ such that $\overline{s_i}|_{V_i \cap V_j} = \overline{s_j}|_{V_i \cap V_j}$. Using the definition of restriction morphisms written in Exercise 10, that is equivalent to $\overline{\rho^{k_i}_{V_i V_i \cap V_j}(s_i)} = \overline{\rho^{k_j}_{V_j V_i \cap V_j}(s_j)}$. That implies $\exists l$ such that $l \geq k_i, l \geq k_j$ and $f_{k_i l}(V_i \cap V_j)(\rho^{k_i}_{V_i V_i \cap V_j}(s_i)) = f_{k_j l}(V_i \cap V_j)(\rho^{k_j}_{V_j V_i \cap V_j}(s_j))$. As morphisms of the direct system and restrictions commute, we have $\rho^l_{V_i V_i \cap V_j}(f_{K_i l}(V_i)(s_i)) = \rho^l_{V_j V_i \cap V_j}(f_{k_j l}(V_j)(s_j))$, which can be rewritten as $f_{k_i l}(V_i)(s_i)|_{V_i \cap V_j} = f_{k_j l}(V_j)(s_j)|_{V_i \cap V_j}$. Then by property (3) of shieves applied to $\mathscr{F}_l$ $\exists t \in \mathscr{F}_l(U)$ such that $t|_{V_i} = f_{k_i l}(V_i)(s_i)$. Then, this last equality implies that $\overline{t} \in \varinjlim \mathscr{F}_j(U)$ satisfies $\overline{t}|_{V_i} = \overline{t|_{V_i}} = \overline{s_i}$.

Then, as this is already a sheaf $\varinjlim \Gamma(U, \mathscr{F}_i) = \varinjlim \mathscr{F}_i(U) = \Gamma(U, \varinjlim \mathscr{F}_i)$, and the equality is true in particular when we take $U = X$.

Exercise 1.12. *Inverse limit. Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on X. Show that the presheaf $U \mapsto \varprojlim \mathscr{F}_i(U)$ is a sheaf. It is called the inverse limit of the system $\{\mathscr{F}_i\}$, and it is denoted by $\varprojlim \mathscr{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.*

Solution. $\varprojlim \mathscr{F}_i(U)$ is the group of coherent sequences over the abelian groups $\mathscr{F}_i(U)$, that is sequences (a_n) with $a_n \in \mathscr{F}_n(U)$ and satisfying $\theta_{n+1}(U)(a_{n+1}) = a_n$, where θ_i are the morphisms of sheaves of the inverse system. (Atiyah-MacDonald, chapter 10). The restriction morphisms of $\varprojlim \mathscr{F}_i$ are naturally $\rho_{UV}((a_n)) = (\rho^n_{UV}(a_n))$, which are well defined (it is indeed an element of $\varprojlim \mathscr{F}(V)$) because restrictions commute with morphisms θ_i. As usual, we will check the 2 properties of the sheaf definition.

- (3) Let (a_n) be a coherent sequence. Suppose that $(a_n)|_{V_i} = 0 \ \forall i \Rightarrow \rho_{UV_i}((a_n)) = 0 \Rightarrow \rho^n_{UV_i}(a_n) = 0 \ \forall n$ and $\forall i$. Then, by property (3) on each $\mathscr{F}_n$ it implies that $a_n = 0 \ \forall n$ and in conclusion $((a_n)) = 0$.

- (4) Let $(a^i_n) \in \varprojlim_j \mathscr{F}_j(V_i)$ such that $((a^i_n))|_{V_i \cap V_j} = ((a^j_n))|_{V_i \cap V_j} \Rightarrow \rho^n_{V_i V_i \cap V_j}(a^i_n) = \rho^n_{V_j V_i \cap V_j}(a^j_n) \ \forall n$. Now applying property (4) at each $\mathscr{F}_n$ we have that $\exists a_n \in \mathscr{F}_n(U)$ such that $a_n|_{V_i} = a^i_n \ \forall n \ \forall i$. Now the sequence (a_n) is coherent as $\theta_{n+1}(U)(a_{n+1})|_{V_i} = \theta_{n+1}(V_i)(a^i_{n+1}) = a^i_n$ by the coherence of each sequence (a^i_n). Then applying property (3) of $\mathscr{F}_n$ to the element $\theta_{n+1}(U)(a_{n+1}) - a_n$ we get that $\theta_{n+1}(U)(a_{n+1}) = a_n$ and the sequence is therefore coherent. In consequence, we have $\exists (a_n) \in \varprojlim \mathscr{F}_i(U)$ such that $(a_n)|_{V_i} = (a^i_n)$.

Now let's check that this sheaf satisfies the universal property of categorical inverse limits. An object X is the categorical inverse limit of objects X_i if there exist morphisms $\varphi_i : X \to X_i$ such that $\forall$ object Y and morphisms $\psi_i : Y \to X_i$ compatible with morphisms of the inverse system $\exists!$ morphism $\psi : Y \to X$ such that $\psi_i = \varphi_i \circ \psi$. In our situation, let's define $\varphi_i : \varprojlim \mathscr{F}_i \to \mathscr{F}_i$ such that $\varphi_i(U)((a_n)) = a_i$. Given a sheaf Y and morphisms compatible with the inverse system (that is, $\psi_i : Y \to \mathscr{F}_i$) such that $\psi_i = \theta_{i+1} \circ \psi_{i+1}$) we define the morphism of sheaves ψ by $\psi(U) : Y \to \varprojlim \mathscr{F}_i(U)$, $x \mapsto (\psi_i(U)(x))_i$. The morphism is well defined, because the resulting sequence is coherent as the morphisms ψ_i are compatible with the inverse system. It clearly satisfies $\psi_i = \varphi_i \circ \psi$. Moreover, the morphism ψ is unique: Indeed, suppose $\exists \psi'$ satisfying the same properties. Given $x \in Y(U)$, let $(b_n) = \psi'(x)$. Then as $\psi_i = \varphi_i \circ \psi$ we have $b_i = \psi_i(x) \Rightarrow \psi' = \psi$.

Exercise 1.13.

Solution.

Exercise 1.14. *Support. Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The support of s, denoted* $\operatorname{Supp} s$ *is defined to be $\{P \in U | s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that* $\operatorname{Supp} s$ *is a closed subset of U. We define the support of $\mathscr{F}$,* $\operatorname{Supp} \mathscr{F}$ *to be $\{P \in X | \mathscr{F}_P \neq 0\}$. It need not be a closed subset.*

Solution. Let $P \in \overline{\operatorname{Supp} s}$, that is, $\forall V \subseteq U$ such that $P \in V$ we have $\operatorname{Supp} s \cap V \neq \varnothing$. Suppose that $P \notin \operatorname{Supp} s$. Then, $\exists V \subseteq U$ such that $s|_V = 0$. Now let $Q \in V \cap \operatorname{Supp} s$. Then V is also a neighbourhood of Q and $s|_V = 0$, which implies that $Q \notin \operatorname{Supp} s$, a contradiction. So we must have $P \in \operatorname{Supp} s$, which means that $\overline{\operatorname{Supp} s} = \operatorname{Supp} s$, and so it is a closed subset.

Exercise 1.15. *Sheaf Hom. Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf $U \mapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf. It is called the sheaf of local morphisms of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and it is denoted $Hom(\mathscr{F}, \mathscr{G})$.*

Solution. First let's make some observations about $\mathscr{F}|_U$. Open sets of the subspace U are just open sets $V \subseteq U$, as U is open. So, in that case the limit $\varinjlim_{W \supseteq V} \mathscr{F}(W) = \mathscr{F}(V)$ so in that case the restriction presheaf is $V \mapsto \mathscr{F}(V)$ and therefore it is already a sheaf, and its restriction morphisms are the same as the ones of $\mathscr{F}$, but only on open sets $V \subseteq U$. Then, morphisms of sheaves $\varphi : \mathscr{F}|_U \to \mathscr{G}|_U$ are just morphisms of abelian groups $\varphi(V) : \mathscr{F}(V) \to \mathscr{G}(V)$ $\forall V \subseteq U$, that commute with restriction morphisms. We can define the sum of two morphisms $\varphi + \psi$ such that $(\varphi + \psi)(V)(x) = \varphi(V)(x) + \psi(V)(y)$. The sum is commutative and associative, as $\mathscr{G}(V)$ are abelian groups. The morphism that sends each element to zero is the neutral element and every morphism has an inverse $-\varphi$ such that $(-\varphi)(V)(x) = -\varphi(V)(x)$ $\forall V, \forall x$. Moreover, we can define the restrictions of the sheaf Hom naturally as ρ_{UV} : $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U) \to \mathrm{Hom}(\mathscr{F}|_V, \mathscr{G}|_V)$ as $\forall W \subseteq V \subseteq U, \varphi|_V(W) = \varphi(W)$. Now let's check that the two sheaf properties are satisfied. Let U be an open set and$\{V_i\}$ an open covering of U.

- (3) Suppose that we have $\varphi \in \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$, and $\varphi|_{V_i} = 0$, that is, $\varphi(W)(x) = 0$, $\forall W \subseteq V_i$, $\forall x \in \mathscr{F}(W)$. Now let $U' \subseteq U$ and $x \in \mathscr{F}(U')$. Let's note $V_i' = V_i \cap U'$. Then $\{V_i'\}$ is an open covering of U'. Then, $\varphi(U')(x)|_{V_i'} = \varphi(V_i')(x|_{V_i'}) = 0$, as $V_i' \subseteq V_i$. Therefore, by property (3) of $\mathscr{G}$, $\varphi(U')(x) = 0$, and this holds $\forall U' \subseteq U, x \in \mathscr{F}(U')$, so $\varphi = 0$.

- (4) Let's suppose that we have $\varphi_i \in \mathrm{Hom}(\mathscr{F}|_{V_i}, \mathscr{G}_{V_i})$ such that $\varphi_i|_{V_i \cap V_j} = \varphi_j|_{V_i \cap V_j}$, that is $\phi_i(V)(x) = \phi_j(V)(x)$, $\forall W \subseteq V_i \cap V_j$. Now, let $W \subseteq U$, and $x \in \mathscr{F}(W)$. Then, let $V_i' = W \cap V_i$, and $x_i = x|_{V_i'}$. $\{V_i'\}$ is an open covering of W and we have $\varphi_i(V_i')(x_i) = y_i$, and $y_i|_{V_i' \cap V_j'} = y_j|_{V_i' \cap V_j'}$ so we have by property (4) of $\mathscr{G}$ that $\exists y \in \mathscr{G}(W)$ such that $y|_{V_i'} = y_i$. Then we define $\varphi(W)(x) = y$. In this way, we have defined an image for every element of W, that is, an application $\mathscr{F}(W) \to \mathscr{G}(W)$. Now we have to check that this application is a morphism of abelian groups and commutes with restrictions, and we'll be done. It's clear that $\varphi(W)(0) = 0$, by property (3) of $\mathscr{G}$. Moreover, given $x_1, x_2 \in \mathscr{F}(W)$ we have $\varphi(W')(x_1|_{V_i'} + x_2|_{V_i'}) = y_{i1} + y_{i2}$. Then, applying property (3) of $\mathscr{G}$ on $\varphi(W)(x_1 + x_2) - y_1 - y_2$ we have indeed that $\varphi(W)(x_1 + x_2) = y_1 + y_2$. On the other hand, let $W \subseteq V \subseteq U$, and given $x \in \mathscr{F}(V)$. Let's name $y_{iW} := \varphi_i(V_i \cap W)(x|_{V_i \cap W}), y_{iV} := \varphi_i(V_i \cap V)(x|_{V_i \cap V})$, which induce images $\varphi(W)(x|_W) =: y_W, \varphi(V)(x) =: y_V$ by the construction we have done. As φ_i are morphisms of sheaves, $y_V|_{V_i \cap W} = y_{iW}$, and therefore applying once more property (3) on $\mathscr{G}$ we get $y_V|_W = y_W$, so $\varphi(W)(x|_W) = \varphi(V)(x)|_W$. So we have defined morphisms of abelian groups on each open subset of U, that commute with restrictions. This defines a morphism of sheaves $\varphi \in \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}_U)$ whose restriction on V_i is φ_i and we are done.

Exercise 1.16. *Flasque sheaves. A sheaf $\mathscr{F}$ on a topological space X is flasque if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.*

a) Show that a constant sheaf on an irreducible topological space is flasque.

b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ is flasque, then for any open set U the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ of abelian groups is also exact.

c) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flasque, then $\mathscr{F}''$ is flasque.

d) If $f : X \to Y$ is a continuous map, and if $\mathscr{F}$ is a flasque sheaf on X, then $f_*\mathscr{F}$ is a flasque sheaf on Y.

e) Let $\mathscr{F}$ be any sheaf on Y. We define a new sheaf $\mathscr{G}$, called the sheaf of discontinuous sections of $\mathscr{F}$ as follows. For each open set $U \subseteq X, \mathscr{G}(U)$ is the set of maps $s : U \to \bigcup_{P \in U} \mathscr{F}_P$ such that for each $P \in U, s(P) \in \mathscr{F}_P$. Show that $\mathscr{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathscr{F}$ to $\mathscr{G}$.

Solution. a) If X is an irreducible topological space, every nonempty set is dense and therefore every nonempty open set is connected $\Rightarrow \mathscr{F}(U) \cong A,\ \forall U$. In consequence, $\forall V \subseteq U$ the restriction map ρ_{UV} is the identity map $A \to A$ which is in particular surjective.

b) Let's name the morphisms of the exact sequence $0 \to \mathscr{F}' \xrightarrow{\psi} \mathscr{F} \xrightarrow{\varphi} \mathscr{F}'' \to 0$. By Exercise 1.8 we only need to check that $\varphi(U)$ is a surjective morphism of abelian groups $\forall U$. Let's fix U. By Exercise 1.3, φ surjective $\Rightarrow \exists \{U_i\}$ open covering of U, and $t_i \in \mathscr{F}(U_i)$ such that $\varphi(U_i)(t_i) = s|_{U_i}$.

c) If $\mathscr{F}'$ is flasque, then by section b) of this problem the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \xrightarrow{\varphi(U)} 0$ is exact, and in particular, $\varphi(U)$ is surjective $\forall U$. Then given $V \subseteq U$ and $s \in \mathscr{F}''(V) \Rightarrow \exists t \in \mathscr{F}(V)$ such that $\varphi(V)(t) = s$. As $\mathscr{F}$ is flasque, $\exists t' \in \mathscr{F}(U)$ with $t'|_V = t$. Then, $s' := \varphi(U)(t')$ satisfies $s'|_V = \varphi(U)(t')|_V = \varphi(V)(t) = s$. In consequence, the restriction $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective, and that means $\mathscr{F}''$ is flasque.

d) Let $V \subseteq U$ be open sets of Y. Then $f^{-1}(V) \subseteq f^{-1}(U)$ are open subsets of X, and as $\mathscr{F}$ is flasque, given $y \in (f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))\exists x \in \mathscr{F}(f^{-1}(U)) = (f_*\mathscr{F})(U)$ such that $x|_V = y$, and therefore $f_*\mathscr{F}$ is also flasque.

e) First let's note that the sections $\mathscr{G}(U)$ defined are indeed abelian groups: The sum of two applications s, t is defined as the application $P \mapsto s(P) + t(P)$. The neutral element is the application that sends each P to the class of 0 in $\mathscr{F}_P$. The restriction morphisms are well defined, as every applictaion acting on a set U naturally restricts to an application acting on $V \subseteq U$. Let's see that it is a sheaf. We have to check that properties (3) and (4) are satisfied. (3): If Given $s \in \mathscr{G}(U)$, and $\{U_i\}$ is an open covering of U, then$s|_{U_i}(P) = s(P) = 0$ for each $P \in U_i \Rightarrow s(P) = 0 \forall P \in U \Rightarrow s = 0$. (4): Given $s_i \in \mathscr{G}(U_i), and s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ we can define $s(P) = s_i(P)$ if $P \in U_i$. It is clearly well defined and satisfies $s|_{U_i} = s_i$. Moreover that $\mathscr{G}$ is a flasque sheaf is also easy to check, as given $U \supseteq V$ we can extend any $s \in \mathscr{G}(V)$ to an $s' \in \mathscr{G}(U)$, for example defining $s'(P) = 0 \in \mathscr{F}_P$ for each $P \notin V$ and $s'(P) = s(P)$ if $P \in V$.

We have a natural morphism from $\mathscr{F}$ to $\mathscr{G}$ that sends $\mathscr{F}(U) \ni x \to s_x \in \mathscr{G}(U)$ such that $s_x(P) = \overline{x} \in \mathscr{F}_P$. It is indeed injective, because if we have $x, y \in \mathscr{F}(U)$ such that $\overline{x} = \overline{y} \in \mathscr{F}_P$, $\Rightarrow \exists W_P \subseteq U | x_{W_P} = y_{W_P}$ and as W_P is an open covering of U so we must have $x = y$.

Exercise 1.17. *Skyscrapper sheaves. Let X be a topological space, let P be a point and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $I_P(A)(U) = A$ if $P \in U, 0$ otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \overline{\{P\}}$, and 0 elsewhere, where $\overline{\{P\}}$ denotes the closure of the set consisting of the point P. Hence the name "skyscrapper sheaf". Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\overline{\{P\}}$, and $i : \overline{\{P\}} \to X$ is the inclusion.*

Solution. First, let's check that $i_P(A)$ is a sheaf. Given $V \subseteq U$, if $P \in\notin V$ the restriction morphism is zero, and if $P \in V$ the restriction morphism if the identity on the group A. It's clear that property (3) and (4) are satisfied if $P \notin U$. So let U be an open set such that $P \in U$ and $\{V_i\}$ be an open covering of U. Let $s \in i_P(A)(U) = A$ Then, $P \in V_i$ for a certain i as $\{V_i\}$ is a covering. Then, $i_P(A)(V_i) = A$ and $0 = s|_{V_i} = s$. Now let $s_i \in i_P(A)(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. Then, $\forall P \in V_i, V_j$ we have that $s_i = s_j \in A$. So then there is a unique element $s \in A$ such that $s = s_i \forall i$ such that $P \in V_i$. If $P \notin V_i, s_i = 0 = s|_{V_i}$. In conclusion, $s \in i_P(A)(U)$ satisfies $s|_{V_i} = s_i \forall i$.

Let $Q \in \overline{\{P\}}$. Then $P \in U\ \forall U$ neighbourhood of Q, and therefore $i_P(A)(U) = A$. Then given $\overline{(U,s)}, \overline{V,t} \in (i_P(A))_P$, we have that $\forall W$ neighbourhood of Q, $W \subseteq U \cap V$, W is also a neighbourhood of P, and so $s|_W = s$, $t|_W = t$. In consequence, $\overline{(U,s)} = \overline{V,t} \iff t = s$, so $(i_P(A))_P = A$. On the other hand, if $Q \notin \overline{\{P\}}$, then $\exists U$ neighbourhood of Q such that $P \notin U$. Then, $\forall \overline{(V,t)} \in (i_P(A))_P$, $i_P(A)(V \cap U) = 0$ and then $t|_{U \cap V} = 0 \Rightarrow \overline{(V,t)} = 0$, so $(i_P(A))_P = 0$.

Now let's check that $i_*(A)$ is an equivalent definition of this sheaf. First, let's note that every open set of $\overline{\{P\}}$ contains P, and therefore every open set is dense and so $\overline{\{P\}}$ is an irreducible space, so by the proof of Exercise 1.16a) the constant sheaf A is in fact the sheaf $U \mapsto A$, $\forall U$. Then, given any U open set of X, we have two options.

- $P \in U$. In that case, $P \in \overline{\{P\}} \cap U = i^{-1}(U)$ and therefore $i_*(A)(U) = \mathscr{A}(i^{-1}(U)) = \mathscr{A}(\overline{\{P\}} \cap U) = A$
- $P \notin U$. In that case $i^{-1}(U) = \overline{\{P\}} \cap U = \varnothing$, and so we have $i_*(A)(U) = \mathscr{A}(i^{-1}(U)) = \mathscr{A}(\overline{\{P\}} \cap U) = \mathscr{A}(\varnothing) = 0$.

In conclusion, for each open set the sections of $i_*(A)$ and $i_P(A)$ are the same, as well as the restriction morphisms ($\rho_{UV} = Id_A$ if $P \in V$ and $\rho_{UV} = 0$ if $P \notin V$. Indeed, both definitions are equal $i_*(A) = i_P(A)$.

Exercise 1.18. *Adjoint property of f^{-1}. Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$ and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y*

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

Hence we say that f^{-1} is a left adjoint of f_* and that f_* is a right adjoint of f^{-1}.

Solution. Given $f : X \to Y$ continuous and $\mathscr{F}$ sheaf on X, the sheaf $f^{-1}f_*\mathscr{F}$ is the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \supseteq f(U)} \mathscr{F}(f^{-1}(V))$. The elements of this abelian group are equivalence classes of pairs (V, x) such that $\overline{(V,x)} = \overline{(V',x')} \iff \exists W,\ f(U) \subseteq W \subseteq V \cap V'$ such that $x|_{f^{-1}(W)} =$

$x'|_{f^{-1}(W)}$. Let's observe that $V \supseteq f(U) \Rightarrow f^{-1}(V) \supseteq f^{-1}(f(U)) \supseteq U$. Hence we have the natural map $\varinjlim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)) \ni \overline{(V,x)} \mapsto x|_U$. By the observation we have just made, it is well defined, as we can restrict first to $f^{-1}(W)$ and then to U as $f^{-1}(W) \supseteq U$. Then this defines a map $\varphi(U) : \varinjlim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ for every open set U, and this induces then a morphism of presheaves φ. By Proposition 1.2 we have then a unique morphism of sheaves $\varphi^+ : f^{-1}f_*\mathcal{F} \to \mathcal{F}$.

On the other hand, let $\mathcal{G}$ be a sheaf on Y. Then, $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf $\varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$. As $\varinjlim_{P \in U} \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \mathcal{G}_{f(P)}$, the sections of the sheaf $f^{-1}\mathcal{G}$ are groups of applications $s : U \to \bigcup_P \in U\mathcal{G}_{f(P)}$ satisfying that $\forall P \in U s(P) \in \mathcal{G}_{f(P)}$ and $\forall P \in U \exists U' \subseteq U$ such that $\forall Q \in U', t_f(Q) = s(Q)$. Then, the section $f_*f^{-1}\mathcal{G}(V)$ of the open set V are the group of applications $f^{-1}(V) \to \bigcup_{P \in f^{-1}(V)} \mathcal{G}_{f(P)}$ satisfying that $\forall P \in f^{-1}V, s(P) \in \mathcal{G}_{f(P)}$ and $\forall P \in f^{-1}(V) \exists V' \subseteq V$ such that $\forall Q \in f^{-1}(V'), t_f(Q) = s(Q)$. Then, there is a natural map from $\mathcal{G}(V) \to f_*f^{-1}\mathcal{G}(V)$ that assigns each element $s \in \mathcal{G}(V)$ the application $s : f^{-1}(V) \to \bigcup_{P \in f^{-1}(V)} \mathcal{G}_{f(P)}$ such that $s(P) = t_{f(P)}$. We will name ψ this morphism of sheaves.

Given a morphism of sheaves $\phi : f^{-1}\mathcal{G} \to \mathcal{F}$ it induces a morphism $\phi_* : f_*f^{-1}\mathcal{G} \to f_*\mathcal{F}$. Then, $\phi_* \circ \psi : \mathcal{G} \to f_*\mathcal{F}$. Similarly, given $\phi : \mathcal{G} \to f_*\mathcal{F}$ it induces a morphism $\phi' : f^{-1}\mathcal{G} \to f^{-1}f_*\mathcal{F}$. Then, $\varphi^+ \circ \phi' : f^{-1}\mathcal{G} \to \mathcal{F}$. We have to check that this correspondence is bijective.

Exercise 1.19. *Extending a sheaf by Zero. Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.*

a) *Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathcal{F})_P$ of the direct image sheaf on X is $\mathcal{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathcal{F}$ instead of $i_*\mathcal{F}$ and say "consider $\mathcal{F}$ as a sheaf on X" when we mean "consider $i_*\mathcal{F}$."*

b) *Now let $\mathcal{F}$ be a sheaf on U. Let $j_!(\mathcal{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathcal{F}))_P$ is equal to $\mathcal{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathcal{F}$ is the only sheaf on X which has this property, and whose restriction to*

U is $\mathscr{F}$. We call $j_!\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside U.

c) *Now let $\mathscr{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,*

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$$

Solution. a) $(i_*\mathscr{F})_P = \varinjlim_{P\in U} \mathscr{F}(i^{-1}(U)) = \varinjlim_{P\in U} \mathscr{F}(U\cap Z)$. If $P \in Z$, $U \cap Z \neq \varnothing$, $\forall U$ neighbourhood of P. Then, $U \cap Z$ are nonempty open neighbourhoods of P in Z, and every neighbourhood of P in Z is of that form, so

$$(i_*\mathscr{F})_P = \varinjlim_{P\in U} \mathscr{F}(U\cap Z) = \varinjlim_{P\in U',\ U' \text{ open set of } Z} U' = \mathscr{F}_P$$

On the other hand, if $P \notin U$, as Z is closed $\exists V$ neighbourhood of P such that $V \cap Z = \varnothing \Rightarrow \mathscr{F}(i^{-1}(V)) = 0$. Then, $\forall \overline{(U,x)} \in (i_*\mathscr{F})_P$, we have $\overline{(U,x)} = \overline{(U\cap V, x|_{U\cap V})} = 0$.

b) If $P \notin U$ then any neighbourhood V of P satisfies $V \not\subseteq U$ so $\mathscr{F}(V) = 0$ and $\varinjlim_{P\in V} \mathscr{F}(V) = 0$. If $P \in U$, the elements of $(j_!(\mathscr{F}))_P$ are equivalence classes $\overline{(V,x)}$. If $V \not\subseteq U$, then $\overline{(V,x)} = \overline{(U\cap V, x|_{U\cap V}}$, so every equivalence class of $(j_!(\mathscr{F}))_P$ has a representative $\overline{(V,x)}$ with $V \subseteq U$, so it is the same set as $\mathscr{F}_P$, and in consequence, $(j_!\mathscr{F})_P = \mathscr{F}_P$.

Suppose that we have another sheaf $\mathscr{G}$ on X with this property, and whose restriction to U is $\mathscr{F}$. We will name $\mathscr{H}$ the presheaf that defines $j_!(\mathscr{F})$. Then, let's see that we have an isomorphism between $\mathscr{G}(V)$ and $j_!(\mathscr{F})(V)$. Given an element $s \in \mathscr{G}(V)$, it induces an element of $j_!(\mathscr{F})(V)$, that is, an application $s : V \to \bigcap_{P\in V} \mathscr{G}_P = \bigcap_{P\in V} \mathscr{H}_P$, that maps $P \mapsto s_P$. It clearly satisfies $s(P) \in \mathscr{H}_P$ (Property 1). If $P \in U$, then $\exists t = s|_{V\cap U} \in \mathscr{G}(U\cap V) = \mathscr{F}(U\cap V) = \mathscr{H}(U\cap V)$ such that $\forall Q \in U\cap V, t_Q = s_Q = s(Q)$. If $P \notin U$, $s_P = 0$ so $\exists W_P \subseteq V$ such that $s|_{W_P} = 0$. Then, $\exists t = 0 \in \mathscr{H}(W_P) = 0$ such that $t_Q = s(Q) = 0$, $\forall Q \in W_P$. This proves that the application s we have defined also satisfies Property 2, and therefore it is an element of $j_!(\mathscr{F})(V)$. Reciprocally, let $s \in j_!(\mathscr{F})(V)$. Remember that while we are working in $U \cap V$ we can use the properties of a sheaf, as

$j_!(\mathscr{F})|_U = \mathscr{F}$ is a sheaf. Then, if $P \in U \cap V$ we have that $\exists W_P$ neighbourhood of P such that $\forall Q \in W_P \exists t \in \mathscr{H}(W_P) = \mathscr{F}(W_P)$ such that $t_Q = s(Q)$. Suppose that we have two points $P, P' \in U \cap V$ and $Q \in W_P \cap W_{P'}$, t_1 and t_2 the respective elements. Then, $(t_1)_Q = (t_2)_Q$ so $\exists V_Q \subseteq W_P \cap W_{P'}$ where $t_1|_{V_Q} = t_2|_{V_Q}$. Then the sets $\{V_Q\}$ are an open covering of $W_P \cap W_Q$ and therefore by property (3) $t_1|_{W_{P'}} = t_2|_{W_P}$. Now the sets W_P are an open covering of $U \cap V$ and they have $t_P \in \mathscr{F}(W_P)$ that agree in the intersections. So by property (4) there exists an element $t \in \mathscr{F}(U \cap V) = \mathscr{G}(U \cap V)$ such that $s(P) = t_P \; \forall P \in U \cap V$. We want to see this property but in V instead of in $U \cap V$. We will define an open neighbourhood for each $P \in V, P \notin U$ in the following way. If it exists an open neighbourhood of P that doesn't meet $U \cap V$, we choose this one. Otherwise, we know that $t_P = 0$ so it must exist $W_P \subseteq U$ (take an arbitrary W_P and intersect it with U, the intersection will be non-empty) neighbourhood of P such that $t|_{W_P} = 0$. Then we choose this W_P as an open neighbourhood of P. Then, the sets $\{\{W_P\}, U \cap V\}$ are an open cover of V, and let $t_P = 0 \in \mathscr{G}(W_P)$ and $t_{U \cap V} = t$. These elements agree on the intersections by the way we have defined the open cover. So therefore, as $\mathscr{G}$ is a sheaf there must exist an element $t' \in \mathscr{G}(V)$ such that $t'|_{W_P} = 0$ and $t'|_{U \cap V} = t$. In conclusion, t' satisfies that $s(P) = t'_P$ for every $P \in V$. In conclusion, we have finally proven that $j_!(\mathscr{F})(V) \cong \mathscr{G}(V) \; \forall V$ so the sheaf $j_!(\mathscr{F})$ is unique (up to isomorphism).

c) A sequence is exact $\iff$ it is exact on the stalks. Then we will define the sequence $0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$ in the natural way.

   Then we can have two cases. If $P \in Z$ (that is $\iff$ $P \notin U$), then $(j_!(\mathscr{F}|_U))_P = 0$, and $(i_*(\mathscr{F}|_Z))_P = \mathscr{F}_P$ and therefore the sequence becomes $0 \to\to \mathscr{F}_P \to \mathscr{F}_P \to 0$ which is cleary exact. On the other hand, if $P \in U$ (that is $\iff$ $P \notin Z$), then $(j_!(\mathscr{F}|_U))_P = \mathscr{F}_P$, and $(i_*(\mathscr{F}|_Z))_P = 0$ and then the exact sequence becomes $0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0 \to 0$, which is also exact.

Exercise 1.20. *Subsheaf with supports. Let Z be a closed subset of X, and let $\mathscr{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathscr{F})$ to be the subgroup of $\Gamma(X, \mathscr{F})$ consisting of all sections whose support (Ex. 1.14) is contained in Z.*

a) *Show that the presheaf $V \mapsto \Gamma_{Z \cap V}(V, \mathscr{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathscr{F}$ with supports in Z and it is denoted by $\mathscr{H}^0_Z(\mathscr{F})$.*

b) Let $U = X - Z$ and let $j : U \to X$ be the inclusion. Show that there is an exact sequence of sheaves on X

$$0 \to \mathscr{H}_Z^0(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$$

Furthermore, if $\mathscr{F}$ is flasque, then the map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is surjective.