Exercises Galois Theory for Schemes

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1 Galois Theory for fields

Exercise 1.1. *Let $K \subset L$ be a Galois extension of fields, and let I be a set of subfields $E \subset L$ with $K \subset E$ for which $[E : K] < \infty$ for every $E \in I$ and $\bigcup_{E\in I} E = L$. Prove that I, when partially ordered by inclusion, is directed.*

Solution. Let $E', E \in I$. Then EE' is a finite extension of K, because its generated by a the union of generators of E and E', which is a finite set of algebraic elements. Moreover, EE' is separable (L is Galois $\Rightarrow$ separable, and $EE' \subseteq L$ so all elements of EE' are separable). Then, by the primitive element theorem, $EE' = K(\alpha)$ for a certain element $\alpha \in L$. Then, as $\bigcup_{E\in I} E = L$, $\exists F \in I$ such that $\alpha \in F$. Then, $EE' \subseteq F$ and so $E, E' \subseteq F$. This proves that I is directed.

Exercise 1.2. *Let $K \subset L$ be a Galois extension of fields, and I any directed set of subfields $E \subset L$ with $K \subset E$ Galois for which $\bigcup_{E\in I} E = L$. Prove that there is an isomorphism of profinite groups* $\mathrm{Gal}(L/K) \cong \varprojlim_{E\in I} \mathrm{Gal}(E/K)$.

Solution. We will chech that the application $\phi : \mathrm{Gal}(L/K) \to \varprojlim_{E\in I} \mathrm{Gal}(E/K)$ defined by $\sigma \mapsto (\sigma|_E)_{E\in I}$ is the desired isomorphism of topological groups. First note that it is a well defined group morphism: As $K \subset E$ is Galois then $\sigma(E) = E$ for every $\sigma \in \mathrm{Gal}(L/K)$, so the restriction of σ to E is indeed an element of $\mathrm{Gal}(E/K)$. Moreover, if $F, E \in I$ with $F \subset E$, then $\phi(\sigma)_{E'} = \sigma|_{E'} = (\sigma|_E)|_{E'} = f_{EE'}(\phi(\sigma)_E)$.

Now let's prove the continuity. For that we need to know the topology of $\varprojlim_{E\in I} \mathrm{Gal}(E/K)$. The topology will be induced by the product topology. A basis of open sets for the topology of $\prod_{E\in I} \mathrm{Gal}(E/K)$ is then

$$\left\{ \prod_{E\notin J} \mathrm{Gal}(E/K) \times \prod_{E\in J} U_E \right\}$$

Where J denotes a finite subset of I, and U_E is an open set of $\mathrm{Gal}(E/K)$. Then, $U_E = \bigcup_{\sigma,F} U_{\sigma,F}$ for certain $\sigma \in \mathrm{Gal}(E/K)$, $K \subset F \subset E$, $[F : K] < \infty$. Therefore every basic open set of $\prod_{E\in I} \mathrm{Gal}(E/K)$ can be expressed as the union of sets $\prod_{E\notin J} \mathrm{Gal}(E/K) \times \prod_{E\in J} U_{\sigma^E, F^E}$, so those sets form a base of the topology of $\prod_{E\in I} \mathrm{Gal}(E/K)$. In conclusion, the following sets are a base of $\varprojlim_{E\in I} \mathrm{Gal}(E/K)$:

$$\mathcal{B} = \left\{ \varprojlim_{E\in I} \mathrm{Gal}(E/K) \cap \left(\prod_{E\notin J} \mathrm{Gal}(E/K) \times \prod_{E\in J} U_{\sigma^E, F^E} \right) \right\}$$

Note that given $\sigma : E \to E$ we can extend it to $\overline{\sigma} : \overline{L} \to \overline{L}$ and then restrict it to L to obtain $\sigma' : L \to L$ such that $\sigma'|_E = \sigma$. Then, the antiimage by ϕ of an open set $U \in \mathcal{B}$ is the set

$\tau \in \mathrm{Gal}(L/K)$ such that $\tau|_{F^E} = \sigma^E|_{F^E}\ \forall E \in J\} = \bigcap_{E\in J} U_{\sigma'^E, F^E}$, which is a finite intersection of open sets of $\mathrm{Gal}(L/K)$, and so it is open.

Now let's prove injectivity of ϕ: Let $\phi(\sigma) = \phi(\tau)$. It is enough to check that $\tau(\alpha) = \sigma(\alpha)$, $\forall \alpha \in L$. Indeed, let $\alpha \in L$. Then as $\bigcup_{E\in I} E = L$, $\alpha \in E$ for a certain $E \in I$, and $\sigma|_E = \tau|_E$, which means $\sigma(\alpha) = \tau(\alpha)$ as desired.

Let $(\sigma_E)_{E\in I} \in \varprojlim_{E\in I} \mathrm{Gal}(E/K)$. We will define $\sigma \in \mathrm{Gal}(L/K)$ as $\sigma(\alpha) = \sigma_E(\alpha)$ if $\alpha \in E$. It is clear that $\phi(\sigma) = (\sigma_E)_{E\in I}$, so we just have to check that σ is well defined, that is, if $\alpha \in E, E'$, with $E, E' \in I$, then $\sigma_E(\alpha) = \sigma'_E(\alpha)$. But as the set is directed, $\exists F \in I$ such that $F \supset E, E'$ and so clearly $\alpha \in F$. As $(\sigma_E) \in \varprojlim_{E\in I} \mathrm{Gal}(E/K)$, then $\sigma|_E = \sigma_F|_E$ and $\sigma|'_E = \sigma_F|'_E$, so $\sigma_E(\alpha) = \sigma_F(\alpha) = \sigma'_E(\alpha)$ as desired.

Finally note that $\mathrm{Gal}(L/K)$ is compact because it is profinite and $\varprojlim_{E\in I} \mathrm{Gal}(E/K)$ is Hausdorff, because each $\mathrm{Gal}(E/K)$ is Hausdorff and products and subspaces of Hausdorff are Hausdorff. Then ϕ is bijective and continuous group morphism, so it is an isomorphism of topological groups.

Exercise 1.3. *a) Let $K \subset L$ be a Galois extension of fields, with Galois group G. View G as a subset of the set L^L of all functions $L \to L$. Let L be given the discrete topology and L^L the product topology. Prove that the topology of the profinite group G coincides with the relative topology inside L^L.*

b) Conversely, let L be any field and $G \subset Aut(L)$ a subgroup that is compact when viewed as a subset of L^L (topologized as in (a)). Prove that $L^G \subset L$ is Galois with Galois group G.

c) Prove that any profinite group is isomorphic to the Galois group of a suitably chosen Galois extension of fields.

Solution. a) A basic open set of L^L is of the form $U = \prod_{\alpha\in J} U_\alpha \times \prod_{\alpha\notin J, \alpha\in L} L$, where J is a finite set of elements of L and U_α is a subset of L which is not the total. Then, a basic open set of G (with topology induced by L^L) will be of the form $G \cap U = \{\sigma \in G$ such that $\sigma(\alpha) \in U_\alpha, \forall \alpha \in J\}$. As J is a finite subset of L, then $K(J)$ is a finite extension. Let's note F the normal closure of $K(J)$, which will also be a finite extension of K, which in addition is Galois. Let's consider the set $V = \{\sigma \in \mathrm{Gal}(F/K)$ such that $\sigma(\alpha) \in U_\alpha, \forall \alpha \in J\}$. Then

$$G \cap U = G \cap \left(V \times \prod_{L \supset E \neq F, E/K \text{ finite Galois}} \mathrm{Gal}(E/K) \right)$$

And the right hand side is an open set of G as a profinite group. This proves that the topology of G as a profinite group is finer that that of G as a subset of L^L.

Reciprocally, let's take a basic open set of G as a profinite group, $U = G \cap (\prod_{E\in J} \mathrm{Gal}(E/K) \times \prod_{E\notin J} U_E)$. Each $E \in J$ can be expressed as $K(\alpha_E)$, by the primitive element theorem, so the action of σ on E is totally determined by the image of the element α_E. Then, let $U_{\alpha_E} = \bigcup_{\sigma\in U_E} \sigma(\alpha)$, and the open set U can now be described as

$$G \cap \left(\prod_{E\in J} U_{\alpha_E} \times \prod_{\alpha\in L, \alpha\neq\alpha_E} L \right)$$

which is an open set of G as a subspace of L^L. This proves that the topology of G as a subset of L^L is finer than the topology of G as a profinite group. In conclusion both topologies of G are the same one.

b) First let's prove that $L^G \subset L$ is algebraic: Indeed, let $\alpha \in L$ and let's take the cover of G given by

$$\left\{ \{\beta\} \times \prod_{\alpha' \neq \alpha} L \right\}_{\beta \in L}$$

As G is compact as a subset of L^L we can extract a finite subcovering from that covering, and that means that the orbit of α under G is finite. Let $f(x) = \prod_{\beta \in G\alpha}(x - \beta)$. That polynomial is invariant under the action of G, so it has coefficients in L^G. Clearly α is a root of f, so α is algebraic. Then by definition $L^G \subset L$ is Galois, and we have that $G \subseteq \mathrm{Gal}(L/L^G)$, as every element of G fixes L^G. As G is compact and L^L is Hausdorff, G is closed in L^L, and then it is closed as a subgroup of $\mathrm{Gal}(L/L^G)$, by (a). Then, by the correspondence between closed subgroups of $\mathrm{Gal}(L/L^G)$ and field extensions, given by 2.3, we have that $\mathrm{Gal}(L/L^G) = G$.

c) Let K be any field and let X be the set of conjugacy classes τH, where H is an open normal subgroup of G. Let G act on X as follows: Given $\sigma \in G$, $\sigma(\tau H) = (\sigma\tau)H$. Let $L = K(X)$. Then the action of G on X induces an automorphism of L for every element of G, so we have a natural map $G \to \mathrm{Aut}(L)$. This map is injective: Indeed, an element $\sigma \in G$ acts trivially on X if and only if $\sigma \in H$, $\forall H$ open normal subgroup. As G is profinite, then $G \cong \varprojlim G/H$, where H runs through all the open normal subgroups of G (this is a result in Cassels, Algebraic Number Theory, Chapter V, Corollary 1), and so the only $\sigma \in H$ $\forall H$ is the identity. Then the only element acting trivially on X is the identity, and so $G \to \mathrm{Aut}(L)$ is injective and we can view G as a subset of $\mathrm{Aut}(L)$ (1).

Note that every element of L can be expressed as a quotient of polynomials with indeterminates as elements of X, so α has a finite orbit, because if $\mathcal{H}_\alpha$ is the set of groups with conjugacy classes appearing in the expression of α, then $|G\alpha| \leq \prod_{H \in \mathcal{H}_\alpha}[G : H]$. Then, given a basic open set of L^L, $U = \prod_{\alpha \in J} U_\alpha \times \prod_{\alpha \notin J} L$, with J a finite subset of L. For $\alpha \in J$, $H \in \mathcal{H}$, consider all the conjugacy classes of H that appear in the expression of a certain β, with $\beta \in U_\alpha$, and note it U_H. Then, the open set U can be expressed as an open set of G as a profinite group as follows:

$$U = \prod_{\alpha \in J} \prod_{H \in \mathcal{H}_\alpha} U_H \times \prod_{H \notin \mathcal{H}_\alpha,\ \forall \alpha} G/H$$

This shows that the topoogy of G as a profinite group is finer that its topology as a subspace of L^L (that is more general than the proof done in (a) of the same fact, because in (a) we knew that G was a Galois group). Then, given an open cover of G in L^L it is also an open cover of G as a profinite group, and as all profinite groups are compact, we can extract a finite covering. This proves that G is compact when viewed as a subset of L^L. (2)

Now G satisfies the two conditions of (b), so L/L^G is then a Galois with Galois group G.

Exercise 1.4.

Exercise 1.5. *Let $K \subset L$ be a Galois extension of fields, $S \subset \mathrm{Gal}(L/K)$ any subset, and $E = \{x \in L : \forall \sigma \in S \sigma(x) = x\}$. Prove that $\mathrm{Gal}(L/E)$ is the closure of the subgroup of $\mathrm{Gal}(L/K)$ generated by S.*

Solution. $\forall \sigma \in S$, $x \in E$, we have $\sigma(x) = x$, so E is fixed by S and so $< S > \subset \mathrm{Gal}(L/E)$. To chech $\overline{< S >} = \mathrm{Gal}(L/E)$, it is enough to check that $U_{\sigma,F} \cap < S > \neq \varnothing$ for every $\sigma \in \mathrm{Gal}(L/E)$ and every F. We will proceed as in the proof of the main theorem: Given a finite extension

$K \subset F$, let M be a finite Galois extension such that $F \subset M$. Let's restrict $< S >$ to M to obtain H' a subgroup of $\mathrm{Gal}(M/K)$. We have that $M^{H'} = E \cap M$, as both sides of the equality are the elements of M fixed by S. We have $\sigma|_{M^{H'}} = Id$, so $\sigma|_M \in \mathrm{Gal}(M/M^{H'}) = H' =< S > |_M$. Then, $\exists \tau \in < S >$ such that $\tau|_M = \sigma_M$, and restricting further to F we have finally $U_{\sigma,F} \cap < S > \neq \varnothing$.

Exercise 1.6. *Let $K \subset L$ be a Galois extension of fields, $S \subset \mathrm{Gal}(L/K)$ and $H' \subset H \subset \mathrm{Gal}(L/K)$ closed subgroups with index$[H : H'] < \infty$. Prove that $L^H \subset L^{H'}$ is finite, and that $[L^{H'} : L^H] =$ index$[H : H']$. Which part of the conclusion is still true if H', H are not necessarily closed?*

Solution. $L^H \subset L^{H'}$ is a Galois extension. Indeed, it is algebraic, because L is algebraic over K so every element of $L^{H'}$ is algebraic over K and therefore also over L^H. Moreover, H is a subgroup of $\mathrm{Aut}(L^{H'})$ so $L^H \subset L^{H'}$ is Galois.

As H' is closed in $\mathrm{Gal}(L/K)$, it is also closed in H, and $H = \mathrm{Gal}(L/L^H)$. As H' corresponds to a closed subgroup of finite index of H (by hypothesis), then it is an open subgroup of H. Now, using the fundamental theorem 2.3(a), we have that $L^H \subset L^{H'}$ is finite, and $[L^{H'} : L^H] = \mathrm{index}[H : H']$.

If H, H' are not necessarily closed, then every coset of H' induces a morphism $\tau : L^{H'} \to L$ such that L^H is invariant (that is, an L^H-immersion). The number of such immersions is the separablility index $[L^{H'} : L^H]$, which we knoe that divides the degree of the extension for finite extensions. Then, it still holds that $[L^{H'} : L^H] \geq \mathrm{index}[H : H']$.

Exercise 1.7. *Let K, L, F be subfields of a field Ω, and suppose that $K \subset L$ is Galois and that $K \subset F$. Prove that $F \subset LF$ is Galois, and that* $\mathrm{Gal}(LF/F) \cong Gal(L/L\ F)$ *(as topological groups).*

Solution. Every element of L is normal algebraic and separable. As these properties are conserved by adjuntion, then $LF = F(L)$ is also algebraic, separable and normal, so it is Galois. Now consider the application $\Phi : \mathrm{Gal}(LF/F) \to \mathrm{Gal}(L/L \cap F)$ defined by restriction $\Phi(\sigma) = \sigma|_L$.

The application is well defined, as L is normal, so $\sigma(L) = L$ and $\Phi(\sigma)(\alpha) = \sigma(\alpha) = \alpha$ for every element $\alpha \in L \cap F$. This proves that $\Phi(\sigma)$ is a indeed an element of $\mathrm{Gal}(L/L \cap F)$. Now let's check that we have an isomorphism of topological groups.

First we prove injectivity: Let $\sigma \in \mathrm{Gal}(LF/F)$ such that $\sigma|_L = Id$. Then, $\forall \alpha \in L, \sigma(\alpha) = \alpha$. As $LF = F(L)$, then σ is completely determined by its image over the elements of L, so if $\sigma|_L = Id$, then $\sigma = Id$. A similar argument works to prove surjectivity: Given $\tau \in \mathrm{Gal}(L/L \cap F)$ we define $\sigma \in \mathrm{Gal}(LF/F)$ as $\sigma(\alpha) = \sigma(\sum_{i=1}^n a_i \alpha_i) = \sum_{i=1}^n a_i \sigma(\alpha_i)$, and we clearly have $\Phi(\sigma) = \tau$.

Now let's prove continuity. Let $U_{\sigma,E}$ be an open set of $\mathrm{Gal}(L/L \cap F)$. $U_{\sigma,E} = \{\tau \in \mathrm{Gal}(L/L \cap F)$ such that $\tau|_E = \sigma|_E\}$, with $[E : L \cap F] < \infty$. We have $\Phi^{-1}(U_{\sigma,E}) = \{\tau \in \mathrm{Gal}(LF/F)$ such that $\tau|_E = \sigma|_E\}$. The image of τ on E is completely determined by the image of a certain element α, by the primitive element theorem. So let $E' = F(\alpha)$ and we have that $[E' : F] < \infty$ and $\tau|'_E = \sigma'|'_E$, where σ' is the only element of $\mathrm{Gal}(LF/F)$ such that $\Phi(\sigma') = \sigma$. This proves that $\Phi^{-1}(U_{\sigma,E}) = U_{\sigma',E'}$ which is open, and this proves the continuity. So we have a continuous bijective map between profinite groups, and therefore it is an isomorphism of topological groups.

Exercise 1.8.

Exercise 1.9. *Let K be a field. Prove that for every Galois extension $K \subset L$ the group* $\mathrm{Gal}(L/K)$ *is isomorphic to a quotient of the absolute Galois group of K.*

Solution. Let $K \subset L$ be a Galois extension, and consider $\overline{L} = \overline{K}$ the algebraic closure. Then we have $K \subset L \subset K_s$ Galois extensions, and by the fundamental theorem (2.3 (d)) $\mathrm{Gal}(K_s/L)$ is a normal subgroup of $\mathrm{Gal}(K_s/K)$ and $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(K_s/K)/\mathrm{Gal}(K_s/L)$.

Exercise 1.10. *A Steinitz number or supernatural number is a formal expression $a = \prod_{p\ prime} p^{a(p)}$, where $a(p) \in \{0, 1, 2, \ldots, \infty\}$ for each number p. If $a = \prod_p p^{a(p)}$ is a Steinitz number, we denote by $a\hat{\mathbb{Z}}$ the subgroup of $\hat{\mathbb{Z}}$ corresponding to $\prod_p p^{a(p)}\mathbb{Z}_p$ (with $p^\infty \mathbb{Z}_p = \{0\}$) under the isomorphism $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$.*

a) Prove that the map $a \mapsto a\hat{\mathbb{Z}}$ from the set of Steinitz numbers to the set of closed subgroups of $\hat{\mathbb{Z}}$ is bijective. Prove also that $a\hat{Z}$ is open if and only if a is finite, i.e. $\sum_p a(p) < \infty$.

b) Let $\mathbb{F}_q$ be a finite field, with algebraic closure $\overline{\mathbb{F}_q}$. For a Steinitz number a, let $\mathbb{F}_{q^a}$ be the set of all $x \in \mathbb{F}_q$ for which $[\mathbb{F}_q(x) : \mathbb{F}_q]$ divides a (in an obvious sense). Prove that the map $a \mapsto \mathbb{F}_{q^a}$ is a bijection from the set of Steinitz numbers to the set of intermediate fields of $\mathbb{F}_q \subset \overline{\mathbb{F}_q}$.

Solution.

Injectivity of the application is clear. To show surjectivity, it is enough to prove that every closed subgroup of $\hat{\mathbb{Z}}$ is of the form $a\hat{\mathbb{Z}}$. Indeed, if G is a closed subgroup of $\hat{\mathbb{Z}}$, then $G = \hat{\mathbb{Z}} \cap \prod G_n$, where G_n is a subgroup of $\mathbb{Z}/n\mathbb{Z}$ for each n. When $n = p^m$, the only possibilities are $G_{p^m} = p^{k_m}\mathbb{Z}/p^m\mathbb{Z}$. Moreover, if $k_m \neq m$ for a certain m we will have $G_{p^{m'}} = p^{k_m}\mathbb{Z}/p^{m'}\mathbb{Z}$. Let's define $a(p)$ as this value of k_m. Then, under the isomorphism $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ the closed subgroup G corresponds to $\prod_p p^{a(p)}\mathbb{Z}_p$, which is by definition $a\hat{\mathbb{Z}}$.

It is clear that if a is finite, then $a\hat{\mathbb{Z}}$ has finite index and so it is open. Reciprocally, using again problem 1.11, we know that $a\hat{\mathbb{Z}}$ is open if and only if $\exists n$ such that $\ker f_n : \hat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$ is a subgroup of $a\hat{\mathbb{Z}}$. Then, if $n = \prod p_i^{k_i}$, $\ker f_n = \prod_{p_i|n} p_i^{k_i}\mathbb{Z}_{p_i} \times \prod p \nmid n\mathbb{Z}_p$. $\ker f_n \subset a\hat{\mathbb{Z}} \iff a(p) = 0$ for all $p \nmid n$ and $a(p_i) \leq k_i$ for all $p_i|n$. Then $\sum_p a(p) \leq \sum_{i=1}^m k_i \leq \infty$, and so a is finite.

By (a) we have a correspondence between Steinitz numbers and the set of closed subgroups of $\hat{\mathbb{Z}}$. As $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$, then theorem 2.3 gives a correspondence between Steinitz numbers and intermediate extensions $\mathbb{F}_q \subset E \subset \overline{\mathbb{F}_q}$ given by $a \mapsto \overline{\mathbb{F}_q}^{a\hat{\mathbb{Z}}}$. So we only need to check that $\overline{\mathbb{F}_q}^{a\hat{\mathbb{Z}}} = \mathbb{F}_{q^a}$.

Let $x \in \mathbb{F}_{q^a}$. Then $[\mathbb{F}_q(x) : \mathbb{F}_q] = n$, and $n|a$ so $\mathrm{Gal}(\mathbb{F}_q(x)/\mathbb{F}_q) = \mathbb{Z}/n\mathbb{Z}$, and so $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q(x)) = n\hat{\mathbb{Z}}$. Then as $n|a$ it is clear that $a\hat{\mathbb{Z}} \subset n\hat{\mathbb{Z}}$, and so $x \in \mathbb{F}_q^{a\hat{\mathbb{Z}}}$.

Reciprocally, given $x \in \mathbb{F}_q^{a\hat{\mathbb{Z}}}$ let's consider the extension $\mathbb{F}_q \subset \mathbb{F}_q(x)$, which is finite and of degree a certain n. Then, $\mathbb{F}_q(x) \subset \mathbb{F}_q^{a\hat{\mathbb{Z}}}$, and so $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q^{a\hat{\mathbb{Z}}}) \subset \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q(x))$, and $a\hat{\mathbb{Z}} \subset n\hat{\mathbb{Z}}$, which implies that $n|a$ and so $x \in \mathbb{F}_{q^a}$.

Exercise 1.11.

Exercise 1.12. *Let K be a field, K_s its separable closure, m a positive integer not divisible by char(K), and ω the number of m-th roots of unity in K.*

a) Let for $\tau \in \mathrm{Gal}(K_s/K)$ the integer $c(\tau)$ be such that $\tau(\varsigma_m) = \varsigma_m^{c(\tau)}$, where ς_m denotes a primitive m-th root of unity. Prove that ω is the greatest common divisor of m and all numbers $c(\tau) - 1$, $\tau \in \mathrm{Gal}(K_s/K)$.

*b) **Schienzel's Theorem**. Let $a \in K$. Prove that the splitting field of $X^m - a$ over K is abelian over K if and only if $a^\omega = b^m$ for some $b \in K$. [Hint for the only if part: if $\alpha^m = a$, prove that $\alpha^{c(\tau)}/\tau(\alpha) \in K^*$ for all τ.]*

Solution. a) Let d be the least exponent such that $\varsigma_m^d \in K$. Then, the subgroup of the m-th roots of unity generated by ς_m^d has order ω, and so $\omega d = m$. Then it is clear that $\omega | m$. Moreover, let $\tau \in \mathrm{Gal}(K_s/K)$ and as $\varsigma_m^d \in K$ we will have that $\tau(\varsigma_m^d) = \varsigma_m^{dc(\tau)} = \varsigma_m^d$. This means that $c(\tau)d \equiv d \pmod m \Rightarrow c(\tau)d \equiv d \pmod{d\omega} \Rightarrow c(\tau) \equiv 1 \pmod \omega \Rightarrow \omega | c(\tau) - 1$. This proves that ω is a common divisor of m and $c(\tau) - 1$, for all $\tau \in \mathrm{Gal}(K_s/K)$. Now suppose that the greatest common divisor is not ω, that is, that exists $k > 1$ such that $k\omega | m$ and $k\omega | c(\tau) - 1$, $\forall \tau$. Then let $k\omega d' = m$, where $d'k = d$. Note that $\forall \tau$ we will have $\tau(\varsigma_m^{d'}) = \varsigma_m^{c(\tau)d'}$. And as $c(\tau) - 1 \equiv 0 \pmod{k\omega}$, then $d'c(\tau) \equiv d' \pmod m$ so $\varsigma_m^{d'} \in K$, which is a contradiction as $d' < d$.

b) Let L be the splitting field of $X^m - a$. The roots of this polynomial are $\varsigma_m^k \alpha$, for a certain α such that $\alpha^m = a$. Then $L = K(\alpha, \varsigma_m)$, and every element $\tau \in \mathrm{Gal}(L/K)$ is totally determined by $\tau(\varsigma_m) = \varsigma_m^{c(\tau)}$ and $\tau(\alpha) = \varsigma_m^s \alpha$ for a certain s. Given an element $g \in \mathrm{Gal}(L/K)$ defined by $s, c(g)$, we have $g(\varsigma_m^k \alpha) = g(\varsigma_m^k) g(\alpha) = \varsigma_m^{kc(g)} \varsigma_m^s \alpha$. Then every element of $\mathrm{Gal}(L/K)$ can be expressed as $g = \sigma\tau$, with $\sigma \in \mathrm{Gal}(L/K(\varsigma_m))$ and $\tau \in \mathrm{Gal}(L/K(\alpha))$. These subgroups are abelian: The first one is a cyclic group of order a divisor of m, and the second one is a sugbroup of the multiplicative group $\mathbb{Z}/m\mathbb{Z}$. Note that then $\mathrm{Gal}(L/K)$ is abelian if and only if arbitrary σ, τ belonging to these subgroups commute.

Let $\sigma \in \mathrm{Gal}(L/K(\varsigma_m))$ such that $\sigma(\alpha) = \varsigma_m^s \alpha$ and $\tau \in \mathrm{Gal}(L/K(\alpha))$. Then, $\sigma\tau(\varsigma_m^k \alpha) = \varsigma_m^{kc(\tau)+s} \alpha$ and $\tau\sigma(\varsigma_m^k \alpha) = \varsigma_m^{c(\tau)(k+s)} \alpha$. Then the group is abelian if and only if $c(\tau)s \equiv s \pmod m$ for all the possible values of s and $c(\tau)$.

Now, if $a^\omega = b^m$, the cyclic group $\mathrm{Gal}(L/K(\varsigma_m))$ has order a divisor of ω, as the order is the least divisor of m such that $\alpha^k \in K$, and so $k | \omega$. Then, we have $\sigma^\omega = Id$ and so $s\omega \equiv 0 (mod m)$, which means that $d | s$. Then, we have $c(\tau) = 1 + a\omega$ and multiplying by $s = s'd$ we get $c(\tau)s = s + s'm$ so we have indeed that $c(\tau)s \equiv s \pmod m$ and so the group is abelian.

Reciprocally, let's follow the indication of the hint. Consider $\frac{\alpha^{c(\tau)}}{\tau(\alpha)}$ and apply $\sigma \in \mathrm{Gal}(L/K)$ to this number. Let $\sigma(\alpha) = \varsigma_m^t \alpha$ and so we have that

$$\sigma\left(\frac{\alpha^{c(\tau)}}{\tau(\alpha)}\right) = \frac{\sigma(\alpha)^{c(\tau)}}{\tau(\sigma(\alpha))} = \frac{(\varsigma_m^t \alpha)^{c(\tau)}}{\varsigma_m^{tc(\tau)} \tau(\alpha)} = \frac{\alpha^{c(\tau)}}{\tau(\alpha)}$$

So it is invariant by action of $\mathrm{Gal}(L/K)$ and therefore it is an element of K. Now let's choose an element τ such that $\tau \in \mathrm{Gal}(L/K(\alpha))$ and so we will have $\alpha^{c(\tau)-1} \in K$. Obviously we have alse $\alpha^m \in K$ so this leads $\alpha^w \in K$.

Exercise 1.13. *a) Prove that* $Q \cap M^{*m} = Q^{m/gcd(m,2)}$.

b) Let $L_m = M(\alpha \in \overline{\mathbb{Q}} : \alpha^m \in \mathbb{Q})$, *for* $m \in \mathbb{Z}_{>0}$. *Prove that* $M \subset L_m$ *is Galois, and that there is an isomorphism of topological groups*

$$\mathrm{Gal}(L_m/M) \to \mathrm{Hom}(Q, E_m^{gcd(m,2)})$$

mapping σ *to* $\sigma(\alpha^{1/m})/\alpha^{1/m}$.

c) Define $E_m \to E_n$ *by* $\varsigma \mapsto \varsigma^{m/n}$ *for* n *dividing* m, *and let* $\hat{E} = \varprojlim E_n$ *with respect to these maps. Prove that* $\hat{(E)} \cong \hat{\mathbb{Z}}$ *as topological groups.*

d) Prove that $M \subset L$ is Galois and that the isomorphisms in (b) combine to yield an isomorphism of topological groups

$$\mathrm{Gal}(L/M) \to \mathrm{Hom}(Q, \hat{E}^2)$$

here $\mathrm{Hom}(Q, \hat{E}^2)$ *has the relative topology in* $(\hat{E}^2)^Q$. *Prove also that this Galois group is isomorphic to the product of a countably infinite collection of copies of* $\hat{Z}$.

Exercise 1.14. *a) Let A be a local ring and $x \in A$ such that $x^2 = x$. Prove that $x = 0$ or $x = 1$.*

b) Prove that any ring isomorphism $\prod_{i=1}^{s} A_i \to \prod -j = 1^t B_j$, where the A_i and B_j are local rings and $t, s < \infty$, is induced by a bijection $\sigma : \{1, 2, \ldots, s\} \to \{1, 2, \ldots, t\}$ and isomorphisms $A_i \to B_{\sigma(i)}$.

Solution. a) If A is a local ring then $\forall x \in A$, either x is a unit or x is an element of the Jacobson radical. Then, if $x = x^2 \Rightarrow x(1-x) = 0$ we have two options: If x is a unit, then $x^{-1}x(1-x) = 0 \Rightarrow (1-x) = 0 \Rightarrow x = 1$. If x is not a unit then as an element of the Jacobson radical we have $1 - xy$ is a unit $\forall y \in A$. In particular, taking $y = 1$ we have that $1 - x$ is a unit, and so $x(1-x)(1-x)^{-1} = 0 \Rightarrow x = 0$.

b) Let's denote $e_i \in \prod_{i=1}^{s} A_i$ the element that has zeros at all positions except at position i, where it has a 1. Let $\phi : \prod_{i=1}^{s} A_i \to \prod_{j=1}^{t} B_j$ denote the isomorphism of the statement. Then, $\phi(e_i)^2 = \phi(e_i^2) = \phi(e_i)$, so we must have for each coordinate j that $\phi(e_i)_j$ equals either 0 or 1, by part (a) of this problem (note that B_j is a local ring.

Moreover, by injectivity of the application ϕ, it is impossible that all the coordinates equal 0, because we would have that $\phi(e_i) = \phi(0) \Rightarrow e_i = 0$, which is a contradiction. Then, there is at least one coordinate such that $\phi(e_i)_j = 1$. Let's fix that j. By surjectivity of ϕ, we have now that

2 Galois categories

Exercise 2.1. *A directed graph D consists of a set $V = V_D$ of vertices, a set $E = E_D$ of edges, a source map $s = s_D : E \to V$ and a target map $t = t_D : E \to V$; each $e \in E$ is to be thought as an arrow from $s(e)$ to $t(e)$. Let D be a directed graph and $\boldsymbol{C}$ a category. A D-diagram in $\boldsymbol{C}$ is a map that assigns to each $v \in V$ an object X_v of $\boldsymbol{C}$ and to each $e \in E$ a morphism f_e from $X_{s(e)}$ to $X_{t(e)}$ in $\boldsymbol{C}$. A morphism from a D-diagram $((X_v)_{v\in V}, (f_e)_{e\in E})$ to a D-diagram $((Y_v)_{v\in V}, (g_e)_{e\in E})$ is a collection of morphisms $(h_v : X_v \to Y_v)_{v\in V}$ in $\boldsymbol{C}$ such that $h_{t(e)}f_e = g_e h_{s(e)}$ for all $e \in E$.*

a) Show that the D-diagrams in $\boldsymbol{C}$ form a category. We denote this category by $\boldsymbol{C}^D$.

b) Show that there exists a functor $\Gamma : \boldsymbol{C} \to \boldsymbol{C}^D$ mapping an object X to the constant D-diagram with $X_v = X$ for all $v \in V$ and $f_e = id_X$ for all $e \in E$, and mapping a morphism $h : X \to Y$ to the morphism $(h_v)_{v\in V}$ with all $h_v = h$.

c) A left limit of a D-diagram A in $\boldsymbol{C}$ is an object $\varprojlim A$ of $\boldsymbol{C}$ such that

$$\mathrm{Hom}_{\boldsymbol{C}}(-, \varinjlim A) \cong \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(-), A)$$

as functors on $\boldsymbol{C}$. Prove that $\varprojlim A$ is unique up to isomorphism if it exists, and that the notion of left limit generalizes that of a projective limit.

d) Show that $\boldsymbol{C}$ admits left limits of all D-diagrams in $\boldsymbol{C}$ is and only if the functor $\Gamma : \boldsymbol{C} \to \boldsymbol{C}^D$ has a right adjoint $\varprojlim : \boldsymbol{C}^D \to \boldsymbol{C}$, i.e.

$$\mathrm{Hom}_{\boldsymbol{C}}(-, \varprojlim -) \cong \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(-), -)$$

If this right adjoint exists, we say that $\boldsymbol{C}$ admits left limits over D.

e) A right limit of a D-diagram A in $\boldsymbol{C}$ is an object $\varinjlim A$ of $\boldsymbol{C}$ such that

$$\mathrm{Hom}_{\boldsymbol{C}}(\varinjlim A, -) \cong \mathrm{Hom}_{\boldsymbol{C}^D}(A, \Gamma(-))$$

Formulate and prove the analogues of the assertions in (c) and (d). If Γ has a left adjoint $\varinjlim : \boldsymbol{C}^D \to \boldsymbol{C}$ we say that $\boldsymbol{C}$ admits right limits over D.

Solution. a) Note that the statement of the problem already defines a set of objects of $\boldsymbol{C}^D$ and a set of morphisms $\mathrm{Hom}_{\boldsymbol{C}^D}((X_v), (f_e), ((Y_v), g_e))$. Given $(\phi_v) \in \mathrm{Hom}_{\boldsymbol{C}^D}((X_v), (f_e), ((Y_v), g_e))$ and $(\psi_v) \in \mathrm{Hom}_{\boldsymbol{C}^D}((Y_v), (g_e), ((Z_v), h_e))$ we have a composition $(\psi_v \circ \phi_v)$ defined by the composition of $\boldsymbol{C}$ at each $v \in V$. We just have to check that it is indeed an element of $\mathrm{Hom}_{\boldsymbol{C}^D}((X_v), (f_e), ((Z_v), h_e))$. Indeed, we have

$$(\psi \circ \phi)_{t(e)} f_e = \psi_{t(e)}\phi_{t(e)} f_e = \psi_{t(e)} g_e \phi_{s(e)} = h_e \psi_{s(e)}\phi_{s(e)} = h_e(\psi \circ \phi)_{s(e)}$$

It is clear that for every D-diagram $((X_v), (f_e))$, the set of morphisms $\mathrm{Hom}_{\boldsymbol{C}^D}((X_v), (f_e), ((X_v), f_e))$ has an identity map $\mathrm{id}_{((X_v),(f_e))}$ which is the morphism (h_v) defined by $(h_v = \mathrm{id}_{X_v})$. The composition of morphisms of $\boldsymbol{C}^D$ is associative because the composition of morphisms of $\boldsymbol{C}$ is. So $\boldsymbol{C}^D$ satisfies the definition of a category.

b) The statement already defines how the functor acts on objects and morphisms. We only have to check that the 2 properties of functors are satisfied: The identity must be mapped to the identity and the composition to the composition. But this is straightforward because $\Gamma(\mathrm{id}_X) = (\mathrm{id}_v)$ which is the identity over $\Gamma(X)$, and given morphisms $g : X \to Y$, $h : Y \to Z$, $\Gamma(h \circ g) = ((h \circ g)_v) = (h_v \circ g_v) = (h_v) \circ (g_v) = \Gamma(h) \circ \Gamma(g)$.

c) Suppose that we have $\varprojlim_1 A$ and $\varprojlim_2 A$ objects of $\boldsymbol{C}$ that satisfy this property. Let θ^1 be the isomorphism of functors $\mathrm{Hom}_{\boldsymbol{C}}(-, \varinjlim_1 A) \cong \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(-), A)$ and θ^2 be the isomorphism of functors $\mathrm{Hom}_{\boldsymbol{C}}(-, \varinjlim_2 A) \cong \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(-), A)$.

The isomorphism of functors means that for every elements $X, Y \in \boldsymbol{C}$ and every morphism $f : X \to Y$ we have isomorphisms θ^1_X, θ^1_Y that make the following diagram commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\boldsymbol{C}}(X, \varprojlim_1 A) \arrow[r, leftrightarrow, "\theta^1_X"] & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(X), A) \\
\mathrm{Hom}_{\boldsymbol{C}}(Y, \varprojlim_1 A) \arrow[r, leftrightarrow, "\theta^1_Y"] \arrow[u, "\mathrm{Hom}_{\boldsymbol{C}}(f)"] & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(Y), A) \arrow[u, "\mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(f))"']
\end{tikzcd}$$

And the same holds for θ^2. So in fact we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\boldsymbol{C}}(X, \varprojlim_1 A) \arrow[r, leftrightarrow, "\theta^1_X"] & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(X), A) \arrow[r, leftrightarrow, "(\theta^2_X)^{-1}"] & \mathrm{Hom}_{\boldsymbol{C}}(X, \varprojlim_2 A) \\
\mathrm{Hom}_{\boldsymbol{C}}(Y, \varprojlim_1 A) \arrow[r, leftrightarrow, "\theta^1_Y"] \arrow[u, "\mathrm{Hom}_{\boldsymbol{C}}(f)"] & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(Y), A) \arrow[r, leftrightarrow, "(\theta^2_Y)^{-1}"] \arrow[u, "\mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(f))"'] & \mathrm{Hom}_{\boldsymbol{C}}(Y, \varprojlim_2 A) \arrow[u, "\mathrm{Hom}_{\boldsymbol{C}}(f)"']
\end{tikzcd}$$

Now take $X = \varprojlim_1 A$ and $Y = \varprojlim_2 A$. Consider the morphism $f := ((\theta^2_{\varprojlim_1 A})^{-1} \circ \theta^1_{\varprojlim_2 A})(\mathrm{id}_{\varprojlim_1 A}) \in \mathrm{Hom}_{\boldsymbol{C}}(\varprojlim_1 A, \varprojlim_2 A)$. We will show that this is in fact an isomorphism. For this morphism f, we have the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\boldsymbol{C}}(\varprojlim_1 A, \varprojlim_1 A) \arrow[r, leftrightarrow, "\theta^1_{\varprojlim_1 A}"] & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(\varprojlim_1 A), A) \arrow[r, leftrightarrow, "\left(\theta^2_{\varprojlim_1 A}\right)^{-1}"] & \mathrm{Hom}_{\boldsymbol{C}}(\varprojlim_1 A, \varprojlim_2 A) \\
\mathrm{Hom}_{\boldsymbol{C}}(\varprojlim_2 A, \varprojlim_1 A) \arrow[r, leftrightarrow, "\theta^1_{\varprojlim_2 A}"] \arrow[u, "\mathrm{Hom}_{\boldsymbol{C}}(f)"] & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(Y), A) \arrow[r, leftrightarrow, "\left(\theta^2_{\varprojlim_2 A}\right)^{-1}"] \arrow[u, "\mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(f))"'] & \mathrm{Hom}_{\boldsymbol{C}}(\varprojlim_2 A, \varprojlim_2 A) \arrow[u, "\mathrm{Hom}_{\boldsymbol{C}}(f)"']
\end{tikzcd}$$

Now let's follow the two paths that can follow the morphism $\mathrm{id}_{\varprojlim_2 A} \in \mathrm{Hom}_{\boldsymbol{C}}(\varprojlim_2 A, \varprojlim_2 A)$ to $\mathrm{Hom}(\varprojlim_1 A, \varprojlim_1 A)$. Going first up an then left, we have the identity on $\varprojlim_1 A$ (because of the definition of f). If we go first left and then up, we have $((\theta^1_{\varprojlim_1 A})^{-1} \circ \theta^2_{\varprojlim_2 A})(\mathrm{id}_{\varprojlim_2 A}) \circ f$, the two paths must agree as the diagram commutes, so we have proven that $g \circ f = \mathrm{id}_{\varprojlim_1 A}$, for g being defined as $g := ((\theta^1_{\varprojlim_1 A})^{-1} \circ \theta^2_{\varprojlim_2 A})(\mathrm{id}_{\varprojlim_2 A}) \in \mathrm{Hom}_{\boldsymbol{C}}(\varprojlim_2 A, \varprojlim_1 A)$. The symmetric calculation exchanging the roles of f and g proves that $f \circ g$ is also the identity. On conclusion, f is an invertible morphism and so $\varprojlim_1 A \cong \varprojlim_2 A$ as we wanted.

Now we want to prove that the notion of left limit generalizes that of projective limit. To do that, we will use the characterization of projective limit of exercise 1.8: $\forall T$ and morphisms $g_j : T \to S_j$ such that $f_{ij} g_i = g_j$, $\exists! g : T \to \varprojlim S_i$ with $g_j = f_j g$. If we turn the partially ordered set into a directed graph by putting $V = I$ and $E = \{(i,j)$, such that $i, j \in I, i \geq j\}$, $s : E \to V, e = (i,j) \mapsto i$ and $t : E \to V, e = (i,j) \mapsto J$. Then we build a D-diagram A by $V \ni i \mapsto S_i$ and $E \ni (i,j) \mapsto f_{ij}$. In that language the above characterization of projective limits implies that $\forall T$ there is a bijective correspondence between $\mathrm{Hom}(\Gamma(T), A)$ and $\mathrm{Hom}(T, \varprojlim S_i)$, which means that $\varprojlim S_i$ is in fact the left limit of the D-diagram A.

d) It is clear that, if Γ has a right adjoint $\varprojlim -$, then for every D-diagram A, $\varprojlim A$ satisfies $\mathrm{Hom}_{\boldsymbol{C}}(-, \varprojlim A) \cong \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(-), A)$, so it is a left limit of A, and in consequence, every D-diagram admits left limit.

Reciprocally, we have to show that the assignation $A \mapsto \varprojlim A$ is functorial. For that we have to define, for each A, B D-diagrams, and every morphism of D diagrams $f : A \to B$ a map $\varprojlim f : \varprojlim A \to \varprojlim B$ that preserves identities and composition. Let's define $\varprojlim f := (\theta^B_{\varprojlim A})^{-1}(f \circ \theta^A_{\varprojlim A}(\mathrm{id}_{\varprojlim A}))$. (here θ^A denotes the isomorphism of functors of c for the D-diagram A). Note that if $A = B$ and $f = \mathrm{id}_A$ then $\varprojlim f = \mathrm{id}_{\varprojlim A}$. Now let A, B, C be 3 D-diagrams and $f : A \to B$, $g : B \to C$, $h = g \circ f : A \to C$. By definition of the functor $\varprojlim -$, we have that

$$\begin{aligned}
\theta^B_{\varprojlim A}(\varprojlim f) &= f \circ \theta^A_{\varprojlim A}(\mathrm{id}_{\varprojlim A}) \\
\theta^C_{\varprojlim B}(\varprojlim g) &= g \circ \theta^B_{\varprojlim B}(\mathrm{id}_{\varprojlim A}) \\
\theta^C_{\varprojlim A}(\varprojlim h) &= g \circ f \circ \theta^A_{\varprojlim A}(\mathrm{id}_{\varprojlim A})
\end{aligned}$$

We have the following commutative diagrams due to the isomorphisms between the functors Hom in (c).

$$\begin{tikzcd}
\mathrm{Hom}_{\boldsymbol{C}}(\varprojlim B, \varprojlim B) \arrow[r, leftrightarrow, "\theta^B_{\varprojlim B}"] \arrow[d, "- \circ \varprojlim f"] & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(\varprojlim B), B) \arrow[d, "- \circ \Gamma\left(\varprojlim f\right)"] \\
\mathrm{Hom}_{\boldsymbol{C}}(\varprojlim A, \varprojlim B) \arrow[r, leftrightarrow, "\theta^B_{\varprojlim A}"] & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(\varprojlim A), B)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\boldsymbol{C}}(\varprojlim B, \varprojlim C) \arrow[r, leftrightarrow, "\theta^C_{\varprojlim B}"] \arrow[d, "- \circ \varprojlim f"] & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(\varprojlim B), C) \arrow[d, "- \circ \Gamma\left(\varprojlim f\right)"] \\
\mathrm{Hom}_{\boldsymbol{C}}(\varprojlim A, \varprojlim C) \arrow[r, leftrightarrow, "\theta^C_{\varprojlim A}"] & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(\varprojlim A), C)
\end{tikzcd}$$

Now following the first diagram from top-left to bottom-right, starting with the application $\mathrm{id}_{\varprojlim B}$ we get the following identity:

$$\theta^B_{\varprojlim A}(\varprojlim f) = \theta^B_{\varprojlim_B}(\mathrm{id}_{\varprojlim B}) \circ \Gamma(\varprojlim f)$$

Now composing with g on both sides we get

$$g \circ \theta^B_{\varprojlim A}(\varprojlim f) = g \circ \theta^B_{\varprojlim_B}(\mathrm{id}_{\varprojlim B}) \circ \Gamma(\varprojlim f)$$

Now using the definitions, we have that $g \circ \theta^B_{\varprojlim A}(\varprojlim f) = \theta^C_{\varprojlim A}(\varprojlim h)$ and $\theta^C_{\varprojlim B}(\varprojlim g) = g \circ \theta^B_{\varprojlim B}(\mathrm{id}_{\varprojlim A})$. Substituting into our equation, we have

$$\theta^C_{\varprojlim A}(\varprojlim h) = \theta^C_{\varprojlim B}(\varprojlim g) \circ \Gamma(\varprojlim f)$$

Now let's follow the second diagram from the top-left to the bottom right starting with the morphism $\varprojlim g \in \mathrm{Hom}_{\boldsymbol{C}}(\varprojlim B, \varprojlim C)$. We obtain that $\theta^C_{\varprojlim_A}(\varprojlim g \circ \varprojlim f) = \theta^C_{\varprojlim B}(\varprojlim g) \circ$

$\Gamma(\varprojlim f)$. So substituting this into our previous equation we finally have $\theta^C_{\varprojlim A}(\varprojlim h) = \theta^C_{\varprojlim_A}(\varprojlim g \circ \varprojlim f)$ and so

$$\varprojlim h = \varprojlim g \circ \varprojlim f$$

And so we have defined a functor $\varprojlim -$ that is right adjoint to Γ.

e) A symmetric argument holds and allows us to prove that right limits are unique if they exist, and that $\boldsymbol{C}$ admits right limits over every D-diagram if and only if $\Gamma(-)$ has a left adjoint $\varinjlim -$.

Exercise 2.2. *Let $\boldsymbol{C}$ be a category.*

a) Prove that $\boldsymbol{C}$ admits left limits over the empty directed graph (with $V = E = \varnothing$) if and only if $\boldsymbol{C}$ has a terminal object.

b) Prove that $\boldsymbol{C}$ admits left limits over the directed graph () if and only if the fibred product of any two objects over a third one exists in $\boldsymbol{C}$.

Solution. a) There is only one possible D-diagram over the empty directed graph, which is the empty D-diagram $\varnothing$. Then, if $\boldsymbol{C}$ admits left limits over the empty directed graph, then for every $X \in \boldsymbol{C}$ there is only one possible morphism of D-diagrams $\varnothing : \Gamma(X) \to \varnothing$, which by the isomorphism of Hom functors implies that there is only one morphism $X \to \varprojlim \varnothing$. Then, the object $\varprojlim \varnothing$ satisfies the definition of terminal object in $\boldsymbol{C}$. Reciprocally, if $\boldsymbol{C}$ has a terminal object, 0, that means that for every $X \in \boldsymbol{C} \exists!$ element in $\mathrm{Hom}_{\boldsymbol{C}}(X, 0)$. Then, we have a bijective map $\mathrm{Hom}_{\boldsymbol{C}}(X, 0) \cong \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(X), \varnothing)$ that maps the only element in $\mathrm{Hom}_{\boldsymbol{C}}(X, 0)$ to the empty morphism (the only element in $\mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(X), \varnothing)$), so $0 = \varprojlim \varnothing$ and so the category admits left limits over the empty directed graph.

b) If $\boldsymbol{C}$ admits left limits over that directed graph, it means that for every D-diagram A, composed by $X, Y, S \in \boldsymbol{C}$ and $a : X \to S$, $b : Y \to S$ morphisms, $\exists$ an object $\varprojlim A$ such that $\mathrm{Hom}_{\boldsymbol{C}}(Z, \varprojlim A) \cong \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(Z), A)$, for every $Z \in \boldsymbol{C}$.

In particular, let $(h)_A \in \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(Z), A)$, which means that $(h)_A$ is has the following elements: Morphisms $h_X : Z \to X$, $h_Y : Z \to Y$ and $h_S : Z \to S$ satisfying $h_s = ah_X = bh_Y$. Let $h = \theta_Z((h)_A)$, where θ is the isomorphism of functors. As usual, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\boldsymbol{C}}(\varprojlim A, \varprojlim A) & \xleftrightarrow{\theta_{\varprojlim A}} & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(\varprojlim A), A) \\
\Big\downarrow{\scriptstyle -\circ h} & & \Big\downarrow{\scriptstyle -\circ\Gamma(h)} \\
\mathrm{Hom}_{\boldsymbol{C}}(Z, \varprojlim A) & \xleftrightarrow{\theta_{\varprojlim A}} & \mathrm{Hom}_{\boldsymbol{C}^D}(\Gamma(Z), A)
\end{array}$$

Taking the identity on the top-left and following the diagram in both directions, we have the following equality: $\theta_{\varprojlim A}(\mathrm{id}_{\varprojlim A} \circ \Gamma(h) = \theta_Z(h) = (h)_A$. That means that $\exists p_X, p_Y = \theta_{\varprojlim A}(\mathrm{id}_{\varprojlim A})_{X,Y}$ such that $h_{X,Y} = p_{X,Y} \circ h$, for h a uniquely determined morphism such. This is exactly the definition of fibred product of X and Y over S.

Reciprocally, if $\boldsymbol{C}$ admits a fibred product, given a D-diagram A (defined by X, Y, S, a, b with the same notation we have been using), an object Z and a morphism of D-diagrams $\Gamma(Z) \to A$, that is, applications $h_X : Z \to X$, $h_Y : Z \to Y$ satisfying $ah_X = bh_y : Z \to S$, consider the fibred product $X \times_S Y$. That object satisfies that there is a unique morphism $h : Z \to X \times_S Y$ such that the morphisms $h_{X,Y}$ factor through h. In particular, the existence and uniqueness of this h allows us to define for every Z a bijection $\mathrm{Hom}_{\boldsymbol{C}}(Z, X \times_S Y) \cong \mathrm{Hom}_{Cat^D}(\Gamma(Z), A)$, so $X \times_S Y$ is in fact the left limit of the D-diagram A. On conclusion, $\boldsymbol{C}$ admits left limits over D.