Field-aligned coordinates

B.Dudson§, M.V.Umansky†, L.C.Wang‡, X.Q.Xu†, L.L.LoDestro†

§Department of Physics, University of York, UK
†Lawrence Livermore National Laboratory, USA
‡IFTS, China

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Contents

1 Introduction

This manual covers the field-aligned coordinate system used in many BOUT++ tokamak models, and useful derivations and expressions.

2 Orthogonal toroidal coordinates

Starting with an orthogonal toroidal coordinate system (ψ, θ, ζ), where ψ is the poloidal flux, θ the poloidal angle (from 0 to 2π), and ζ the toroidal angle (also 0 to 2π). The magnitudes of the unit vectors are

$$|\mathbf{e}_\psi| = \frac{1}{R\,|B_p|} \qquad |\mathbf{e}_\theta| = h_\theta \qquad |\mathbf{e}_\zeta| = R \tag{1}$$

where h_θ is the poloidal arc length per radian. The coordinate system is right handed, so $\hat{\mathbf{e}}_\psi \times \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\zeta$, $\hat{\mathbf{e}}_\psi \times \hat{\mathbf{e}}_\zeta = -\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\zeta = \hat{\mathbf{e}}_\psi$. The covariant metric coefficients are

$$g_{\psi\psi} = \frac{1}{(R\,|B_p|)^2} \qquad g_{\theta\theta} = h_\theta^2 \qquad g_{\zeta\zeta} = R^2 \tag{2}$$

and the magnitudes of the reciprocal vectors are therefore

$$|\nabla\psi| = R\,|B_p| \qquad |\nabla\theta| = \frac{1}{h_\theta} \qquad |\nabla\zeta| = \frac{1}{R} \tag{3}$$

Because the coordinate system is orthogonal, $g^{ii} = 1/g_{ii}$ and so the cross-products can be calculated as

$$\begin{aligned}
\nabla\psi \times \nabla\theta = & \quad \mathbf{e}^\psi \times \mathbf{e}^\theta = g^{\psi\psi}\mathbf{e}_\psi \times g^{\theta\theta}\mathbf{e}_\theta \\
= & \quad g^{\psi\psi}g^{\theta\theta}h_\psi h_\theta \hat{\mathbf{e}}_\psi \times \hat{\mathbf{e}}_\theta \\
= & \quad \tfrac{1}{h_\psi h_\theta}\hat{\mathbf{e}}_\zeta = \tfrac{R|B_p|}{h_\theta}\hat{\mathbf{e}}_\zeta
\end{aligned}$$

Similarly,

$$\nabla\psi \times \nabla\zeta = -\,|B_p|\,\hat{\mathbf{e}}_\theta \qquad \nabla\theta \times \nabla\zeta = \frac{1}{Rh_\theta}\hat{\mathbf{e}}_\psi = \frac{1}{h_\theta R^2\,|B_\theta|}\nabla\psi$$

3 Field-aligned coordinates

In order to efficiently simulate (predominantly) field-aligned structures, grid-points are placed in a field-aligned coordinate system. We define $\sigma_{B\theta} \equiv B_\theta / |B_\theta|$ i.e. the sign of the poloidal field. The new coordinates (x, y, z) are defined by:

$$x = \sigma_{B\theta}\left(\psi - \psi_0\right) \qquad y = \theta \qquad z = \sigma_{B\theta}\left(\zeta - \int_{\theta_0}^{\theta}\nu\left(\psi, \theta\right) d\theta\right) \tag{4}$$

Where ν is the local field-line pitch given by

$$\nu\left(\psi, \theta\right) = \frac{\mathbf{B}\cdot\nabla\zeta}{\mathbf{B}\cdot\nabla\theta} = \frac{B_\zeta h_\theta}{B_\theta R} = \frac{\left(F/R\right)h_\theta}{B_\theta R} = FJ/R^2 \tag{5}$$

The coordinate system is chosen so that x increases radially outwards, from plasma to the wall. The sign of the toroidal field B_ζ can then be either +ve or -ve.

The contravariant basis vectors are therefore

$$\nabla x = \sigma_{B\theta}\nabla\psi \qquad \nabla y = \nabla\theta \qquad \nabla z = \sigma_{B\theta}\left(\nabla\zeta - \left[\int_{\theta_0}^{\theta}\frac{\partial\nu\left(\psi, \theta\right)}{\partial\psi}d\theta\right]\nabla\psi - \nu\left(\psi, \theta\right)\nabla\theta\right)$$

The term in square brackets is the integrated local shear:

$$I = \int_{y_0}^{y}\frac{\partial\nu\left(x, y\right)}{\partial\psi}dy$$

3.1 Magnetic field

Magnetic field is given in Clebsch form by:

$$\mathbf{B} = \nabla z \times \nabla x = \frac{1}{J}\underline{e}_y$$

The contravariant components of this are then

$$B^y = \frac{B_\theta}{h_\theta} \qquad B^x = B^z = 0 \tag{6}$$

i.e. $\mathbf{B}$ can be written as

$$\mathbf{B} = \frac{B_\theta}{h_\theta}\mathbf{e}_y \tag{7}$$

and the covariant components calculated using g_{ij} as

$$B_x = \sigma_{B\theta}B_\zeta I R \qquad B_y = \frac{B^2 h_\theta}{B_\theta} \qquad B_z = \sigma_{B\theta}B_\zeta R \tag{8}$$

The unit vector in the direction of equilibrium $\mathbf{B}$ is therefore

$$\underline{b} = \frac{1}{JB}\underline{e}_y = \frac{1}{JB}\left[g_{xy}\nabla x + g_{yy}\nabla y + g_{yz}\nabla z\right]$$

3.2 Jacobian and metric tensors

The Jacobian of this coordinate system is

$$J^{-1} \equiv (\nabla x \times \nabla y) \cdot \nabla z = B_\theta / h_\theta$$

which can be either positive or negative, depending on the sign of B_θ. The contravariant metric tensor is given by:

$$g^{ij} \equiv \mathbf{e}^i \cdot \mathbf{e}^j \equiv \nabla u^i \cdot \nabla u^j = \left(\begin{array}{ccc} \left(RB_\theta\right)^2 & 0 & -I\left(RB_\theta\right)^2 \\ 0 & 1/h_\theta^2 & -\sigma_{B\theta}\nu/h_\theta^2 \\ -I\left(RB_\theta\right)^2 & -\sigma_{B\theta}\nu/h_\theta^2 & I^2\left(RB_\theta\right)^2 + B^2/\left(RB_\theta\right)^2 \end{array} \right)$$

and the covariant metric tensor:

$$g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j = \left(\begin{array}{ccc} I^2R^2 + 1/\left(RB_\theta\right)^2 & \sigma_{B\theta}B_\zeta h_\theta IR/B_\theta & IR^2 \\ \sigma_{B\theta}B_\zeta h_\theta IR/B_\theta & B^2h_\theta^2/B_\theta^2 & \sigma_{B\theta}B_\zeta h_\theta R/B_\theta \\ IR^2 & \sigma_{B\theta}B_\zeta h_\theta R/B_\theta & R^2 \end{array} \right)$$

3.3 Differential operators

The derivative of a scalar field F along the unperturbed magnetic field is given by

$$\partial^0_{||}F \equiv \mathbf{b}_0 \cdot \nabla F = \frac{1}{\sqrt{g_{yy}}}\frac{\partial F}{\partial y} = \frac{B_\theta}{Bh_\theta}\frac{\partial F}{\partial y}$$

whilst the parallel divergence is given by

$$\nabla^0_{||}F = B_0\partial^0_{||}\left(\frac{F}{B_0}\right)$$

Using equation (41), the laplacian operator is given by

$$\begin{aligned} \nabla^2 = \quad & \tfrac{\partial^2}{\partial x^2}\left|\nabla x\right|^2 + \tfrac{\partial^2}{\partial y^2}\left|\nabla y\right|^2 + \tfrac{\partial^2}{\partial z^2}\left|\nabla z\right|^2 \\ & -2\tfrac{\partial^2}{\partial x\partial z}I\left(RB_p\right)^2 - 2\tfrac{\partial^2}{\partial y\partial z}\tfrac{\nu}{h_\theta^2} \\ & +\tfrac{\partial}{\partial x}\nabla^2 x + \tfrac{\partial}{\partial y}\nabla^2 y + \tfrac{\partial}{\partial z}\nabla^2 z \end{aligned} \tag{9}$$

Using equation (40) for $\nabla^2 x = -\Gamma^x$ etc, the values are

$$\nabla^2 x = \frac{B_p}{h_\theta}\frac{\partial}{\partial x}\left(h_\theta R^2 B_p\right) \qquad \nabla^2 y = \frac{B_p}{h_\theta}\frac{\partial}{\partial y}\left(\frac{1}{B_p h_\theta}\right) \tag{10}$$

$$\nabla^2 z = -\frac{B_p}{h_\theta}\left[\frac{\partial}{\partial x}\left(IR^2B_ph_\theta\right) + \frac{\partial}{\partial y}\left(\frac{\nu}{B_ph_\theta}\right)\right]$$

Neglecting some parallel derivative terms, the perpendicular Laplacian can be written:

$$\nabla_\perp^2 = (RB_\theta)^2\left[\frac{\partial^2}{\partial x^2} - 2I\frac{\partial^2}{\partial z\partial x} + \left(I^2 + \frac{B^2}{(RB_\theta)^4}\right)\frac{\partial^2}{\partial z^2}\right] + \nabla^2 x\frac{\partial}{\partial x} + \nabla^2 z\frac{\partial}{\partial z}$$

The second derivative along the equilibrium field

$$\partial_{||}^2\phi = \partial_{||}^0\left(\partial_{||}^0\phi\right) = \frac{1}{\sqrt{g_{yy}}}\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{g_{yy}}}\right)\frac{\partial\phi}{\partial y} + \frac{1}{g_{yy}}\frac{\partial^2\phi}{\partial y^2}$$

A common expression (Poisson bracket in reduced MHD) is:

$$\mathbf{b}_0\cdot\nabla\phi\times\nabla A = \frac{1}{J\sqrt{g_{yy}}}\left[\left(g_{yy}\frac{\partial\phi}{\partial z} - g_{yz}\frac{\partial\phi}{\partial y}\right)\frac{\partial A}{\partial x} + \left(g_{yz}\frac{\partial\phi}{\partial x} - g_{xy}\frac{\partial\phi}{\partial z}\right)\frac{\partial A}{\partial y} + \left(g_{xy}\frac{\partial\phi}{\partial y} - g_{yy}\frac{\partial\phi}{\partial x}\right)\frac{\partial A}{\partial z}\right]$$

The perpendicular gradient operator:

$$\begin{aligned}\nabla_\perp &\equiv \nabla - \underline{b}\left(\underline{b}\cdot\nabla\right)\\ &= \nabla x\left(\frac{\partial}{\partial x} - \frac{g_{xy}}{(JB)^2}\frac{\partial}{\partial y}\right) + \nabla z\left(\frac{\partial}{\partial z} - \frac{g_{yz}}{(JB)^2}\frac{\partial}{\partial y}\right)\end{aligned}$$

3.4 $J \times B$ in field-aligned coordinates

Components of the magnetic field in field-aligned coordinates:

$$B^y = \frac{B_\theta}{h_\theta} \qquad B^x = B^z = 0$$

and

$$B_x = \sigma_{B\theta}B_\zeta IR \qquad B_y = \frac{B^2 h_\theta}{B_\theta} \qquad B_z = \sigma_{B\theta}B_\zeta R$$

Calculate current $\mathbf{J} = \frac{1}{\mu}\nabla\times\mathbf{B}$

$$\left(\nabla\times\mathbf{B}\right)^x = \frac{1}{J}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) = 0$$

since $B_\zeta R$ is a flux-surface quantity, and $\mathbf{B}$ is axisymmetric.

$$\begin{aligned}\left(\nabla\times\mathbf{B}\right)^y &= -\sigma_{B\theta}\frac{B_\theta}{h_\theta}\frac{\partial}{\partial x}\left(B_\zeta R\right)\\ \left(\nabla\times\mathbf{B}\right)^z &= \frac{B_\theta}{h_\theta}\left[\frac{\partial}{\partial x}\left(\frac{B^2 h_\theta}{B_\theta}\right) - \sigma_{B\theta}\frac{\partial}{\partial y}\left(B_\zeta IR\right)\right]\end{aligned}$$

The second term can be simplified, again using $B_\zeta R$ constant on flux-surfaces:

$$\frac{\partial}{\partial y}\left(B_\zeta I R\right) = \sigma_{B\theta} B_\zeta R \frac{\partial \nu}{\partial x} \qquad \nu = \frac{h_\theta B_\zeta}{R B_\theta}$$

From these, calculate covariant components:

$$\begin{aligned}
\left(\nabla\times\mathbf{B}\right)_x &= -B_\zeta I R\frac{\partial}{\partial x}\left(B_\zeta R\right) + \frac{IR^2B_\theta}{h_\theta}\left[\frac{\partial}{\partial x}\left(\frac{B^2h_\theta}{B_\theta}\right) - B_\zeta R\frac{\partial \nu}{\partial x}\right] \\
\left(\nabla\times\mathbf{B}\right)_y &= -\sigma_{B\theta}\frac{B^2h_\theta}{B_\theta}\frac{\partial}{\partial x}\left(B_\zeta R\right) + \sigma_{B\theta}B_\zeta R\left[\frac{\partial}{\partial x}\left(\frac{B^2h_\theta}{B_\theta}\right) - B_\zeta R\frac{\partial \nu}{\partial x}\right] \\
\left(\nabla\times\mathbf{B}\right)_z &= -B_\zeta R\frac{\partial}{\partial x}\left(B_\zeta R\right) + \frac{R^2B_\theta}{h_\theta}\left[\frac{\partial}{\partial x}\left(\frac{B^2h_\theta}{B_\theta}\right) - B_\zeta R\frac{\partial \nu}{\partial x}\right]
\end{aligned} \tag{11}$$

Calculate $\mathbf{J}\times\mathbf{B}$ using

$$\mathbf{e}^i = \frac{1}{J}\left(\mathbf{e}_j\times\mathbf{e}_k\right) \qquad \mathbf{e}_i = J\left(\mathbf{e}^j\times\mathbf{e}^k\right) \qquad i,j,k \;\texttt{cyc}\; 1,2,3 \tag{12}$$

gives

$$\begin{aligned}
\mu_0\left(\mathbf{J}\times\mathbf{B}\right)^x &= \frac{1}{J}\left[\left(\nabla\times\mathbf{B}\right)_y B_z - \left(\nabla\times\mathbf{B}\right)_z B_y\right] \\
&= -\frac{B_\theta^3R^2}{h_\theta}\left[\frac{\partial}{\partial x}\left(\frac{B^2h_\theta}{B_\theta}\right) - B_\zeta R\frac{\partial \nu}{\partial x}\right]
\end{aligned}$$

Covariant components of ∇P:

$$\left(\nabla P\right)_x = \frac{\partial P}{\partial x} \qquad \left(\nabla P\right)_y = \left(\nabla P\right)_z = 0$$

and contravariant:

$$\left(\nabla P\right)^x = \left(RB_\theta\right)^2\frac{\partial P}{\partial x} \qquad \left(\nabla P\right)^y = 0 \qquad \left(\nabla P\right)^z = -I\left(RB_\theta\right)^2\frac{\partial P}{\partial x}$$

Hence equating contravariant x components of $\mathbf{J}\times\mathbf{B} = \nabla P$,

$$\frac{\partial}{\partial x}\left(\frac{B^2h_\theta}{B_\theta}\right) - B_\zeta R\frac{\partial}{\partial x}\left(\frac{B_\zeta h_\theta}{RB_\theta}\right) + \frac{\mu_0 h_\theta}{B_\theta}\frac{\partial P}{\partial x} = 0 \tag{13}$$

Use this to calculate h_θ profiles (need to fix h_θ at one radial location).

Close to x-points, the above expression becomes singular, so a better way to write it is:

$$\frac{\partial}{\partial x}\left(B^2h_\theta\right) - h_\theta B_\theta\frac{\partial B_\theta}{\partial x} - B_\zeta R\frac{\partial}{\partial x}\left(\frac{B_\zeta h_\theta}{R}\right) + \mu_0 h_\theta\frac{\partial P}{\partial x} = 0$$

For solving force-balance by adjusting P and f profiles, the form used is

$$B_\zeta h_\theta \frac{\partial B_\zeta}{\partial x} + \frac{B_\zeta^2 h_\theta}{R}\frac{\partial R}{\partial x} + \mu_0 h_\theta \frac{\partial P}{\partial x} = -B_\theta \frac{\partial}{\partial x}\left(B_\theta h_\theta\right)$$

A quick way to calculate f is to rearrange this to:

$$\frac{\partial B_\zeta}{\partial x} = B_\zeta\left[-\frac{1}{R}\frac{\partial R}{\partial x}\right] + \frac{1}{B_\zeta}\left[-\mu_0\frac{\partial P}{\partial x} - \frac{\partial B_\theta}{\partial h_\theta}\frac{\partial}{\partial x}\left(B_\theta h_\theta\right)\right]$$

and then integrate this using LSODE.

3.5 Parallel current

$$J_{||} = \mathbf{b}\cdot\mathbf{J} \qquad b^y = \frac{B_\theta}{B h_\theta}$$

and from equation 11:

$$J_y = \frac{\sigma_{B\theta}}{\mu_0}\left\{-\frac{B^2 h_\theta}{B_\theta}\frac{\partial}{\partial x}\left(B_\zeta R\right) + B_\zeta R\left[\frac{\partial}{\partial x}\left(\frac{B^2 h_\theta}{B_\theta}\right) - \sigma_{B\theta} B_\zeta R \frac{\partial \nu}{\partial x}\right]\right\}$$

since $J_{||} = b^y J_y$,

$$\mu_0 J_{||} = \sigma_{B\theta}\frac{B_\theta B_\zeta R}{B h_\theta}\left[\frac{\partial}{\partial x}\left(\frac{B^2 h_\theta}{B_\theta}\right) - B_\zeta R\frac{\partial \nu}{\partial x}\right] - \sigma_{B\theta} B \frac{\partial}{\partial x}\left(B_\zeta R\right)$$

3.6 Curvature

For reduced MHD, need to calculate curvature term $\mathbf{b}\times\kappa$, where $\kappa = \left(\mathbf{b}\cdot\nabla\right)\mathbf{b} = -\mathbf{b}\times\left(\nabla\times\mathbf{b}\right)$. Re-arranging, this becomes:

$$\mathbf{b}\times\kappa = \nabla\times\mathbf{b} - \mathbf{b}\left(\mathbf{b}\cdot\left(\nabla\times\mathbf{b}\right)\right)$$

Components of $\nabla\times\mathbf{b}$ are:

$$\begin{aligned}
\left(\nabla\times\mathbf{b}\right)^x &= \sigma_{B\theta}\frac{B_\theta}{h_\theta}\frac{\partial}{\partial y}\left(\frac{B_\zeta R}{B}\right) \\
\left(\nabla\times\mathbf{b}\right)^y &= -\sigma_{B\theta}\frac{B_\theta}{h_\theta}\frac{\partial}{\partial x}\left(\frac{B_\zeta R}{B}\right) \\
\left(\nabla\times\mathbf{b}\right)^z &= \frac{B_\theta}{h_\theta}\frac{\partial}{\partial x}\left(\frac{B h_\theta}{B_\theta}\right) - \sigma_{B\theta}\frac{B_\theta B_\zeta R}{h_\theta B}\frac{\partial \nu}{\partial x} - \sigma_{B\theta} I \frac{B_\theta}{h_\theta}\frac{\partial}{\partial y}\left(\frac{B_\zeta R}{B}\right)
\end{aligned}$$

giving:

$$
\begin{aligned}
\kappa &= -\frac{B_\theta}{Bh_\theta}\left[\frac{\partial}{\partial x}\left(\frac{Bh_\theta}{B_\theta}\right) - \sigma_{B\theta}\frac{\partial}{\partial y}\left(\frac{B_\zeta IR}{B}\right)\right]\nabla x \\
&+ \sigma_{B\theta}\frac{B_\theta}{Bh_\theta}\frac{\partial}{\partial y}\left(\frac{B_\zeta R}{B}\right)\nabla z
\end{aligned} \tag{14}
$$

$$
\mathbf{b}\cdot(\nabla\times\mathbf{b}) = -\sigma_{B\theta}B\frac{\partial}{\partial x}\left(\frac{B_\zeta R}{B}\right) + \sigma_{B\theta}\frac{B_\zeta B_\theta R}{Bh_\theta}\frac{\partial}{\partial x}\left(\frac{Bh_\theta}{B_\theta}\right) - \frac{B_\theta B_\zeta^2 R^2}{h_\theta B^2}\frac{\partial \nu}{\partial x}
$$

therefore,

$$
\begin{aligned}
(\mathbf{b}\times\kappa)^x &= \sigma_{B\theta}\frac{B_\theta}{h_\theta}\frac{\partial}{\partial y}\left(\frac{B_\zeta R}{B}\right) = -\sigma_{B\theta}\frac{B_\theta B_\zeta R}{h_\theta B^2}\frac{\partial B}{\partial y} \\
(\mathbf{b}\times\kappa)^y &= \frac{B_\theta^2 B_\zeta^2 R^2}{B^3 h_\theta^2}\frac{\partial \nu}{\partial x} - \sigma_{B\theta}\frac{B_\theta^2 B_\zeta R}{B^2 h_\theta^2}\frac{\partial}{\partial x}\left(\frac{Bh_\theta}{B_\theta}\right) \\
(\mathbf{b}\times\kappa)^z &= \frac{B_\theta}{h_\theta}\frac{\partial}{\partial x}\left(\frac{Bh_\theta}{B_\theta}\right) - \sigma_{B\theta}\frac{B_\theta B_\zeta R}{h_\theta B}\frac{\partial \nu}{\partial x} - I(\mathbf{b}\times\kappa)^x
\end{aligned}
$$

Using equation 13:

$$
B\frac{\partial}{\partial x}\left(\frac{Bh_\theta}{B_\theta}\right) + \frac{Bh_\theta}{B_\theta}\frac{\partial B}{\partial x} - \sigma_{B\theta}B_\zeta R\frac{\partial}{\partial x}\left(\frac{B_\zeta h_\theta}{RB_\theta}\right) + \frac{\mu_0 h_\theta}{B_\theta}\frac{\partial P}{\partial x} = 0
$$

we can re-write the above components as:

$$
\begin{aligned}
(\mathbf{b}\times\kappa)^y &= \sigma_{B\theta}\frac{B_\theta B_\zeta R}{B^2 h_\theta}\left[\frac{\mu_0}{B}\frac{\partial P}{\partial x} + \frac{\partial B}{\partial x}\right] \\
(\mathbf{b}\times\kappa)^z &= -\frac{\mu_0}{B}\frac{\partial P}{\partial x} - \frac{\partial B}{\partial x} - I(\mathbf{b}\times\kappa)^x
\end{aligned}
$$

3.7 Curvature from $\nabla \times (\mathbf{b}/B)$

The vector $\mathbf{b}\times\kappa$ is an approximation of

$$
\frac{B}{2}\nabla\times\left(\frac{\mathbf{b}}{B}\right) \simeq \mathbf{b}\times\kappa
$$

so can just derive from the original expression. Using the contravariant components of $\mathbf{b}$, and the curl operator in curvilinear coordinates (see appendix):

$$\begin{aligned}\nabla\times\left(\frac{\mathbf{b}}{B}\right) &= \frac{B_\theta}{h_\theta}\left[\left(\frac{\partial}{\partial x}\left(\frac{h_\theta}{B_\theta}\right)-\frac{\partial}{\partial y}\left(\frac{\sigma_{B\theta}B_\zeta IR}{B^2}\right)\right)\mathbf{e}_z\right.\\ &+\frac{\partial}{\partial y}\left(\frac{\sigma_{B\theta}B_\zeta R}{B^2}\right)\mathbf{e}_x\\ &\left.+\frac{\partial}{\partial x}\left(\frac{\sigma_{B\theta}B_\zeta R}{B^2}\right)\mathbf{e}_y\right]\end{aligned}$$

This can be simplified using

$$\frac{\partial}{\partial y}\left(\frac{\sigma_{B\theta}B_\zeta IR}{B^2}\right) = I\sigma_{B\theta}B_\zeta R\frac{\partial}{\partial y}\left(\frac{1}{B^2}\right)+\frac{B_\zeta R}{B^2}\frac{\partial \nu}{\partial x}$$

to give

$$\begin{aligned}(\mathbf{b}\times\kappa)^x &= -\sigma_{B\theta}\frac{B_\theta B_\zeta R}{h_\theta B^2}\frac{\partial B}{\partial y}\\ (\mathbf{b}\times\kappa)^y &= -\sigma_{B\theta}\frac{BB_\theta}{2h_\theta}\frac{\partial}{\partial x}\left(\frac{B_\zeta R}{B^2}\right)\\ (\mathbf{b}\times\kappa)^z &= \frac{BB_\theta}{2h_\theta}\frac{\partial}{\partial x}\left(\frac{h_\theta}{B_\theta}\right)-\frac{B_\theta B_\zeta R}{2h_\theta B}\frac{\partial \nu}{\partial x}-I\left(\mathbf{b}\times\kappa\cdot\nabla\right)^x\end{aligned}$$

The first and second terms in $(\mathbf{b}\times\kappa\cdot\nabla)^z$ almost cancel, so by expanding out ν a better expression is

$$(\mathbf{b}\times\kappa)^z = \frac{B_\theta^3}{2h_\theta B}\frac{\partial}{\partial x}\left(\frac{h_\theta}{B_\theta}\right)-\frac{B_\zeta R}{2B}\frac{\partial}{\partial x}\left(\frac{h_\theta}{B_\theta}\right)$$

3.8 Curvature of a single line

The curvature vector can be calculated from the field-line toroidal coordinates (R, Z, ϕ) as follows. The line element is given by

$$d\mathbf{r} = dR\hat{\mathbf{R}} + dZ\hat{\mathbf{Z}} + Rd\phi\hat{\phi}$$

Hence the tangent vector is

$$\hat{\mathbf{T}} \equiv \frac{d\mathbf{r}}{ds} = \frac{dR}{ds}\hat{\mathbf{R}} + \frac{dZ}{ds}\hat{\mathbf{Z}} + R\frac{d\phi}{ds}\hat{\phi}$$

where s is the distance along the field-line. From this, the curvature vector is given by

$$\kappa \equiv \frac{d\mathbf{T}}{ds} \quad = \quad \frac{d^2R}{ds^2}\hat{\mathbf{R}} + \frac{dR}{ds}\frac{d\phi}{ds}\hat{\phi}$$

$$
\begin{aligned}
&+ \quad \frac{d^2Z}{ds^2}\hat{\mathbf{Z}} \\
&+ \quad \frac{dR}{ds}\frac{d\phi}{ds}\hat{\phi} + R\frac{d^2\phi}{ds^2}\hat{\phi} - R\left(\frac{d\phi}{ds}\right)^2\hat{\mathbf{R}}
\end{aligned}
$$

i.e.

$$
\kappa = \left[\frac{d^2R}{ds^2} - R\left(\frac{d\phi}{ds}\right)^2\right]\hat{\mathbf{R}} + \frac{d^2Z}{ds^2}\hat{\mathbf{Z}} + \left[2\frac{dR}{ds}\frac{d\phi}{ds} + R\frac{d^2\phi}{ds^2}\right]\hat{\phi} \tag{15}
$$

Want the components of $\mathbf{b}\times\kappa$, and since the vector $\mathbf{b}$ is just the tangent vector $\mathbf{T}$ above, this can be written using the cross-products

$$
\hat{\mathbf{R}}\times\hat{\mathbf{Z}} = -\hat{\phi} \qquad \hat{\phi}\times\hat{\mathbf{Z}} = \hat{\mathbf{R}} \qquad \hat{\mathbf{R}}\times\hat{\phi} = \hat{\mathbf{Z}}
$$

This vector must then be dotted with $\nabla\psi$, $\nabla\theta$, and $\nabla\phi$. This is done by writing these vectors in cylindrical coordinates:

$$
\begin{aligned}
\nabla\psi &= \frac{\partial\psi}{\partial R}\hat{\mathbf{R}} + \frac{\partial\psi}{\partial Z}\hat{\mathbf{Z}} \\
\nabla\theta &= \frac{1}{B_\theta h_\theta}\nabla\phi\times\nabla\psi = \frac{1}{RB_\theta h_\theta}\left(\frac{\partial\psi}{\partial Z}\hat{\mathbf{R}} - \frac{\partial\psi}{\partial R}\hat{\mathbf{Z}}\right)
\end{aligned}
$$

An alternative is to use

$$
\mathbf{b}\times\nabla\phi = \frac{\sigma_{B\theta}}{BR^2}\nabla\psi
$$

and that the tangent vector $\mathbf{T} = \mathbf{b}$. This gives

$$
\nabla\psi = \sigma_{B\theta}BR\left[\frac{dR}{ds}\mathbf{Z} - \frac{dZ}{ds}\mathbf{R}\right] \tag{16}
$$

and so because $d\phi/ds = B_\zeta/(RB)$

$$
\kappa\cdot\nabla\psi = \sigma_{B\theta}BR\left[\left(\frac{B_\zeta^2}{RB^2} - \frac{d^2R}{ds^2}\right)\frac{dZ}{ds} + \frac{d^2Z}{ds^2}\frac{dR}{ds}\right] \tag{17}
$$

Taking the cross-product of the tangent vector with the curvature in equation 15 above gives

$$
\begin{aligned}
\mathbf{b}\times\kappa &= \left[\frac{B_\zeta}{B}\frac{d^2Z}{ds^2} - \frac{dZ}{ds}\left(2\frac{dR}{ds}\frac{d\phi}{ds} + R\frac{d^2\phi}{ds^2}\right)\right]\mathbf{R} \\
&+ \left[\frac{dR}{ds}\left(2\frac{dR}{ds}\frac{d\phi}{ds} + R\frac{d^2\phi}{ds^2}\right) - \frac{B_\zeta}{B}\left(\frac{d^2R}{ds^2} - R\left(\frac{d\phi}{ds}\right)^2\right)\right]\mathbf{Z} \\
&+ \left[\frac{dZ}{ds}\left(\frac{d^2R}{ds^2} - R\left(\frac{d\phi}{ds}\right)^2\right) - \frac{dR}{ds}\frac{d^2Z}{ds^2}\right]\hat{\phi}
\end{aligned}
$$

The components in field-aligned coordinates can then be calculated:

$$
\begin{aligned}
(\mathbf{b}\times\kappa)^x &= \sigma_{B\theta}\left(\mathbf{b}\times\kappa\right)\cdot\nabla\psi \\
&= \frac{RB_\theta^2}{B}\left(2\frac{dR}{ds}\frac{d\phi}{ds} + R\frac{d^2\phi}{ds^2}\right) - RB_\zeta\left(\frac{dR}{ds}\frac{d^2R}{ds^2} + \frac{dZ}{ds}\frac{d^2Z}{ds^2}\right) + \frac{B_\zeta^3}{B^2}\frac{dR}{ds}
\end{aligned}
$$

3.9 Curvature in toroidal coordinates

In toroidal coordinates (ψ, θ, ϕ), the $\mathbf{b}$ vector is

$$
\begin{aligned}
\mathbf{b} &= \frac{B_\theta}{B}\hat{\mathbf{e}}_\theta + \frac{B_\zeta}{B}\hat{\mathbf{e}}_\phi \\
&= \frac{B_\theta h_\theta}{B}\nabla\theta + \frac{RB_\zeta}{B}\nabla\phi
\end{aligned}
$$

The curl of this vector is

$$
\begin{aligned}
(\nabla\times\mathbf{b})^\psi &= \frac{1}{\sqrt{g}}\left(\frac{\partial b_\phi}{\partial\theta} - \frac{\partial b_\theta}{\partial\phi}\right) \\
(\nabla\times\mathbf{b})^\theta &= \frac{1}{\sqrt{g}}\left(\frac{\partial b_\psi}{\partial\phi} - \frac{\partial b_\phi}{\partial\psi}\right) \\
(\nabla\times\mathbf{b})^\phi &= \frac{1}{\sqrt{g}}\left(\frac{\partial b_\theta}{\partial\psi} - \frac{\partial b_\psi}{\partial\theta}\right)
\end{aligned}
$$

where $1/\sqrt{g} = B_\theta/h_\theta$. Therefore, in terms of unit vectors:

$$
\nabla\times\mathbf{b} = \frac{1}{Rh_\theta}\frac{\partial}{\partial\theta}\left(\frac{RB_\zeta}{B}\right)\hat{\mathbf{e}}_\psi - B_\theta\frac{\partial}{\partial\psi}\left(\frac{RB_\zeta}{B}\right)\hat{\mathbf{e}}_\theta + \frac{B_\theta R}{h_\theta}\frac{\partial}{\partial\psi}\left(\frac{h_\theta B_\theta}{B}\right)\hat{\mathbf{e}}_\phi
$$

3.10 ψ derivative of B field

Needed to calculate magnetic shear, and one way to get the curvature. The simplest way is to use finite differencing, but there is another way using local derivatives (implemented using DCT).

$$
B_\theta = \frac{|\nabla\psi|}{R} = \frac{1}{R}\sqrt{\left(\frac{\partial\psi}{\partial R}\right)^2 + \left(\frac{\partial\psi}{\partial R}\right)^2}
$$

Using

$$
\nabla B_\theta = \frac{\partial B_\theta}{\partial\psi}\nabla\psi + \frac{\partial B_\theta}{\partial\theta}\nabla\theta + \frac{\partial B_\theta}{\partial\phi}\nabla\phi
$$

we get

$$\nabla B_\theta \cdot \nabla\psi = \frac{\partial B_\theta}{\partial\psi}\left|\nabla\psi\right|^2$$

and so

$$\frac{\partial B_\theta}{\partial\psi} = \nabla B_\theta \cdot \nabla\psi / \left(RB_\theta\right)^2$$

The derivatives of B_θ in R and Z are:

$$\begin{aligned}\frac{\partial B_\theta}{\partial R} &= -\frac{B_\theta}{R} + \frac{1}{B_\theta R^2}\left[\frac{\partial\psi}{\partial R}\frac{\partial^2\psi}{\partial R^2} + \frac{\partial\psi}{\partial Z}\frac{\partial^2\psi}{\partial R\partial Z}\right] \\ \frac{\partial B_\theta}{\partial Z} &= \frac{1}{B_\theta R^2}\left[\frac{\partial\psi}{\partial Z}\frac{\partial^2\psi}{\partial Z^2} + \frac{\partial\psi}{\partial R}\frac{\partial^2\psi}{\partial R\partial Z}\right]\end{aligned}$$

For the toroidal field, $B_\zeta = f/R$

$$\frac{\partial B_\zeta}{\partial\psi} = \frac{1}{R}\frac{\partial f}{\partial\psi} - \frac{f}{R^2}\frac{\partial R}{\partial\psi}$$

As above, $\frac{\partial R}{\partial\psi} = \nabla R \cdot \nabla\psi / \left(RB_\theta\right)^2$, and since $\nabla R \cdot \nabla R = 1$,

$$\frac{\partial R}{\partial\psi} = \frac{\partial\psi}{\partial R} / \left(RB_\theta\right)^2$$

similarly,

$$\frac{\partial Z}{\partial\psi} = \frac{\partial\psi}{\partial Z} / \left(RB_\theta\right)^2$$

and so the variation of toroidal field with ψ is

$$\frac{\partial B_\zeta}{\partial\psi} = \frac{1}{R}\frac{\partial f}{\partial\psi} - \frac{B_\zeta}{R^3 B_\theta^2}\frac{\partial\psi}{\partial R}$$

From the definition $B = \sqrt{B_\zeta^2 + B_\theta^2}$,

$$\frac{\partial B}{\partial\psi} = \frac{1}{B}\left(B_\zeta\frac{\partial B_\zeta}{\partial\psi} + B_\theta\frac{\partial B_\theta}{\partial\psi}\right)$$

3.11 Parallel derivative of B field

To get the parallel gradients of the B field components, start with

$$\frac{\partial}{\partial s}\left(B^2\right) = \frac{\partial}{\partial s}\left(B_\zeta^2\right) + \frac{\partial}{\partial s}\left(B_\theta^2\right)$$

Using the fact that RB_ζ is constant along s,

$$\frac{\partial}{\partial s}\left(R^2 B_\zeta^2\right) = R^2\frac{\partial}{\partial s}\left(B_\zeta^2\right) + B_\zeta^2\frac{\partial}{\partial s}\left(R^2\right) = 0$$

which gives

$$\frac{\partial}{\partial s}\left(B_\zeta^2\right) = -\frac{B_\zeta^2}{R^2}\frac{\partial}{\partial s}\left(R^2\right) \tag{18}$$

The poloidal field can be calculated from

$$\frac{\partial}{\partial s}\left(\nabla\psi\cdot\nabla\psi\right) = \frac{\partial}{\partial s}\left(R^2 B_\theta^2\right) = R^2\frac{\partial}{\partial s}\left(B_\theta^2\right) + B_\theta^2\frac{\partial}{\partial s}\left(R^2\right)$$

Using equation 16, $\nabla\psi\cdot\nabla\psi$ can also be written as

$$\nabla\psi\cdot\nabla\psi = B^2R^2\left[\left(\frac{\partial R}{\partial s}\right)^2 + \left(\frac{\partial Z}{\partial s}\right)^2\right]$$

and so (unsurprisingly)

$$\frac{B_\theta^2}{B^2} = \left[\left(\frac{\partial R}{\partial s}\right)^2 + \left(\frac{\partial Z}{\partial s}\right)^2\right]$$

Hence

$$\frac{\partial}{\partial s}\left(B_\theta^2\right) = B^2\frac{\partial}{\partial s}\left[\left(\frac{\partial R}{\partial s}\right)^2 + \left(\frac{\partial Z}{\partial s}\right)^2\right] + \frac{B_\theta^2}{B^2}\frac{\partial}{\partial s}\left(B^2\right)$$

Which gives

$$\frac{\partial}{\partial s}\left(B^2\right) = -\frac{B^2}{R^2}\frac{\partial}{\partial s}\left(R^2\right) + \frac{B^4}{B_\zeta^2}\frac{\partial}{\partial s}\left[\left(\frac{\partial R}{\partial s}\right)^2 + \left(\frac{\partial Z}{\partial s}\right)^2\right] \tag{19}$$

$$\frac{\partial}{\partial s}\left(B_\theta^2\right) = \left(1 + \frac{B_\theta^2}{B_\zeta^2}\right)B^2\frac{\partial}{\partial s}\left[\left(\frac{\partial R}{\partial s}\right)^2 + \left(\frac{\partial Z}{\partial s}\right)^2\right] - \frac{B_\theta^2}{R^2}\frac{\partial}{\partial s}\left(R^2\right) \tag{20}$$

3.12 Magnetic shear from $J \times B$

Re-arranging the radial force balance equation 13 gives

$$\frac{B_\theta^2 R}{B_\zeta}\frac{\partial\nu}{\partial\psi} + \nu\left(\frac{2RB}{B_\zeta}\frac{\partial B}{\partial\psi} + \frac{B^2}{B_\zeta}\frac{\partial R}{\partial\psi} - \frac{B^2R}{B_\zeta^2}\frac{\partial B_\zeta}{\partial\psi}\right) + \frac{\mu_0 h_\theta}{B_\theta}\frac{\partial P}{\partial\psi} = 0$$

3.13 Magnetic shear

The field-line pitch is given by

$$\nu = \frac{h_\theta B_\zeta}{B_\theta R}$$

and so

$$\frac{\partial \nu}{\partial \psi} = \frac{\nu}{h_\theta}\frac{\partial h_\theta}{\partial \psi} + \frac{\nu}{B_\zeta}\frac{\partial B_\zeta}{\partial \psi} - \frac{\nu}{B_\theta}\frac{\partial B_\theta}{\partial \psi} - \frac{\nu}{R}\frac{\partial R}{\partial \psi}$$

The last three terms are given in the previous section, but $\partial h_\theta / \partial \psi$ needs to be evaluated

3.14 ψ derivative of h_θ

From the expression for curvature 14, and using $\nabla x \cdot \nabla \psi = \sigma_{B\theta}\left(RB_\theta\right)^2$ and $\nabla z \cdot \nabla \psi = -\sigma_{B\theta} I \left(RB_\theta\right)^2$

$$\begin{aligned}
\kappa \cdot \nabla \psi &= -\sigma_{B\theta}\frac{B_\theta}{Bh_\theta}\left(RB_\theta\right)^2\left[\frac{\partial}{\partial x}\left(\frac{Bh_\theta}{B_\theta}\right) - \sigma_{B\theta}\frac{\partial}{\partial y}\left(\frac{B_\zeta I R}{B}\right)\right] \\
&\quad -I\left(RB_\theta\right)^2\frac{B_\theta}{Bh_\theta}\frac{\partial}{\partial y}\left(\frac{B_\zeta R}{B}\right)
\end{aligned}$$

The second and third terms partly cancel, and using $\frac{\partial I}{\partial y} = \sigma_{B\theta}\frac{\partial \nu}{\partial x}$

$$\begin{aligned}
\frac{\kappa \cdot \nabla \psi}{\left(RB_\theta\right)^2} &= -\sigma_{B\theta}\frac{B_\theta}{Bh_\theta}\frac{\partial}{\partial x}\left(\frac{Bh_\theta}{B_\theta}\right) + \sigma_{B\theta}\frac{B_\theta}{Bh_\theta}\frac{B_\zeta R}{B}\frac{\partial \nu}{\partial x} \\
&= -\sigma_{B\theta}\frac{B_\theta}{Bh_\theta}\left[\frac{\partial}{\partial x}\left(\frac{Bh_\theta}{B_\theta}\right) - \frac{B_\zeta R}{B}\frac{\partial}{\partial x}\left(\frac{B_\zeta h_\theta}{B_\theta R}\right)\right] \\
&= -\sigma_{B\theta}\frac{B_\theta}{Bh_\theta}\left[h_\theta\frac{\partial}{\partial x}\left(\frac{B}{B_\theta}\right) - h_\theta\frac{B_\zeta R}{B}\frac{\partial}{\partial x}\left(\frac{B_\zeta}{B_\theta R}\right) + \frac{B^2}{BB_\theta}\frac{\partial h_\theta}{\partial x} - \frac{B_\zeta^2}{BB_\theta}\frac{\partial h_\theta}{\partial x}\right] \\
&= -\sigma_{B\theta}\frac{B_\theta}{B^2 h_\theta}\frac{\partial h_\theta}{\partial x} - \sigma_{B\theta}\frac{B_\theta}{B^2}\left[B\frac{\partial}{\partial x}\left(\frac{B}{B_\theta}\right) - B_\zeta R\frac{\partial}{\partial x}\left(\frac{B_\zeta}{B_\theta R}\right)\right]
\end{aligned}$$

Writing

$$\begin{aligned}
B\frac{\partial}{\partial x}\left(\frac{B}{B_\theta}\right) &= \frac{\partial}{\partial x}\left(\frac{B^2}{B_\theta}\right) - \frac{B}{B_\theta}\frac{\partial B}{\partial x} \\
B_\zeta R\frac{\partial}{\partial x}\left(\frac{B_\zeta}{B_\theta R}\right) &= \frac{\partial}{\partial x}\left(\frac{B_\zeta^2}{B_\theta}\right) - \frac{B_\zeta}{B_\theta R}\frac{\partial}{\partial x}\left(B_\zeta R\right)
\end{aligned}$$

and using $B\frac{\partial B}{\partial x} = B_\zeta\frac{\partial B_\zeta}{\partial x} + B_\theta\frac{\partial B_\theta}{\partial x}$, this simplifies to give

$$\frac{\kappa\cdot\nabla\psi}{\left(RB_\theta\right)^2} = -\sigma_{B\theta}\frac{B_\theta^2}{B^2h_\theta}\frac{\partial h_\theta}{\partial x} - \sigma_{B\theta}\frac{B_\zeta^2}{B^2R}\frac{\partial R}{\partial x} \tag{21}$$

This can be transformed into an expression for $\frac{\partial h_\theta}{\partial x}$ involving only derivatives along field-lines. Writing $\nabla R = \frac{\partial R}{\partial\psi}\nabla\psi + \frac{\partial R}{\partial\theta}\nabla\theta$,

$$\nabla R\cdot\nabla\psi = \frac{\partial R}{\partial\psi}\left(RB_\theta\right)^2$$

Using 16,

$$\nabla\psi\cdot\nabla R = -\sigma_{B\theta}BR\frac{dZ}{ds}$$

and so

$$\frac{\partial R}{\partial x} = -\frac{BR}{\left(RB_\theta\right)^2}\frac{dZ}{ds}$$

Substituting this and equation 17 for $\kappa\cdot\nabla\psi$ into equation 21 the $\frac{\partial R}{\partial x}$ term cancels with part of the $\kappa\cdot\nabla\psi$ term, simplifying to

$$\frac{\partial h_\theta}{\partial x} = -h_\theta\frac{B^3R}{B_\theta^2\left(RB_\theta\right)^2}\left[\frac{d^2Z}{ds^2}\frac{dR}{ds} - \frac{d^2R}{ds^2}\frac{dZ}{ds}\right] \tag{22}$$

4 Shifted radial derivatives

The coordinate system given by equation 4 and used in the above sections has a problem: There is a special poloidal location θ_0 where the radial basis vector $\mathbf{e}_x$ is purely in the $\nabla\psi$ direction. Moving away from this location, the coordinate system becomes sheared in the toroidal direction.

Making the substitution

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial\psi} + I\frac{\partial}{\partial z} \tag{23}$$

we also get the mixed derivative

$$\begin{aligned}\frac{\partial}{\partial z\partial x} &= \frac{\partial}{\partial z}\frac{\partial}{\partial\psi} + \frac{\partial I}{\partial z}\frac{\partial}{\partial z} + I\frac{\partial^2}{\partial z^2} \\ &= \frac{\partial^2}{\partial z\partial\psi} + I\frac{\partial^2}{\partial z^2}\end{aligned} \tag{24}$$

and second-order x derivative

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial\psi^2} + \frac{\partial}{\partial\psi}\left(I\frac{\partial}{\partial z}\right) + I\frac{\partial}{\partial z}\left(\frac{\partial}{\partial\psi} + I\frac{\partial}{\partial z}\right) \\ &= \frac{\partial^2}{\partial\psi^2} + I^2\frac{\partial^2}{\partial z^2} + 2I\frac{\partial^2}{\partial z\partial\psi} + \frac{\partial I}{\partial\psi}\frac{\partial}{\partial z} \qquad (25)\end{aligned}$$

4.1 Perpendicular Laplacian

$$\nabla_\perp^2 = \left(RB_\theta\right)^2\left[\frac{\partial^2}{\partial x^2} - 2I\frac{\partial^2}{\partial z\partial x} + \left(I^2 + \frac{B^2}{\left(RB_\theta\right)^4}\right)\frac{\partial^2}{\partial z^2}\right]$$

transforms to

$$\nabla_\perp^2 = \left(RB_\theta\right)^2\left[\frac{\partial^2}{\partial\psi^2} + \frac{\partial I}{\partial\psi}\frac{\partial}{\partial z} + \frac{B^2}{\left(RB_\theta\right)^4}\frac{\partial^2}{\partial z^2}\right] \qquad (26)$$

The extra term involving I disappears, but only if **both** the x and z first derivatives are taken into account:

$$\nabla_\perp^2 = \left(RB_\theta\right)^2\left[\frac{\partial^2}{\partial x^2} - 2I\frac{\partial^2}{\partial z\partial x} + \left(I^2 + \frac{B^2}{\left(RB_\theta\right)^4}\right)\frac{\partial^2}{\partial z^2}\right] + \nabla^2 x\frac{\partial}{\partial x} + \nabla^2 z\frac{\partial}{\partial z}$$

with

$$\nabla^2 x = \frac{1}{J}\frac{\partial}{\partial x}\left[J\left(RB_\theta\right)^2\right]$$

$$\begin{aligned}\nabla^2 z &= \frac{1}{J}\left[-\frac{\partial}{\partial x}\left(JI\left(RB_\theta\right)^2\right) - \frac{\partial}{\partial y}\left(\frac{B_\zeta}{B_\theta^2 R}\right)\right] \\ &= \frac{1}{J}\left[-I\frac{\partial}{\partial x}\left(J\left(RB_\theta\right)^2\right) - \frac{\partial I}{\partial x}J\left(RB_\theta\right)^2 - \frac{\partial}{\partial y}\left(\frac{B_\zeta}{B_\theta^2 R}\right)\right] \qquad (27)\end{aligned}$$

where $J = h_\theta/B_\theta$ is the Jacobian. Transforming into ψ derivatives, the middle term of equation 27 cancels the I term in equation 26, but introduces another I term (first term in equation 27). This term cancels with the $\nabla^2 x$ term when $\frac{\partial}{\partial x}$ is expanded, so the full expression for $\nabla_\perp^2$ using ψ derivatives is:

$$\begin{aligned}\nabla_\perp^2 &= \left(RB_\theta\right)^2\left[\frac{\partial^2}{\partial\psi^2} + \frac{B^2}{\left(RB_\theta\right)^4}\frac{\partial^2}{\partial z^2}\right] \\ &+ \frac{1}{J}\frac{\partial}{\partial\psi}\left[J\left(RB_\theta\right)^2\right]\frac{\partial}{\partial\psi} - \frac{1}{J}\frac{\partial}{\partial y}\left(\frac{B_\zeta}{B_\theta^2 R}\right)\frac{\partial}{\partial z} \qquad (28)\end{aligned}$$

4.1.1 In orthogonal (ψ,θ,ζ) flux coordinates

For comparison, the perpendicular Laplacian can be derived in orthogonal “flux” coordinates

$$|\nabla\psi| = RB_\theta \qquad |\nabla\theta| = 1/h_\theta \qquad |\nabla\zeta| = 1/R$$

The Laplacian operator is given by

$$\begin{aligned}\nabla^2 A &= (RB_\theta)^2\frac{\partial^2 A}{\partial\psi^2} + \frac{1}{h_\theta^2}\frac{\partial^2 A}{\partial\theta^2} + \frac{1}{R^2}\frac{\partial^2 A}{\partial\zeta^2} \\ &+ \frac{1}{J}\frac{\partial}{\partial\psi}\left[J\left(RB_\theta\right)^2\right]\frac{\partial A}{\partial\psi} + \frac{1}{J}\frac{\partial}{\partial\theta}\left(J/h_\theta^2\right)\frac{\partial A}{\partial\theta}\end{aligned} \tag{29}$$

parallel derivative by

$$\partial_{||} \equiv \mathbf{b}\cdot\nabla = \frac{B_\theta}{Bh_\theta}\frac{\partial}{\partial\theta} + \frac{B_\zeta}{RB}\frac{\partial}{\partial\zeta} \tag{30}$$

and so

$$\begin{aligned}\partial^2_{||}A \equiv \partial_{||}\left(\partial_{||}A\right) &= \left(\frac{B_\theta}{Bh_\theta}\right)^2\frac{\partial^2 A}{\partial\theta^2} + \left(\frac{B_\zeta}{RB}\right)^2\frac{\partial^2 A}{\partial\zeta^2} \\ &+ 2\frac{B_\theta B_\zeta}{B^2 h_\theta R}\frac{\partial^2 A}{\partial\theta\partial\zeta} \\ &+ \frac{\partial}{\partial\theta}\left(\frac{B_\theta}{Bh_\theta}\right)\frac{\partial A}{\partial\theta} + \frac{\partial}{\partial\theta}\left(\frac{B_\zeta}{RB}\right)\frac{\partial A}{\partial\zeta}\end{aligned} \tag{31}$$

Hence in orthogonal flux coordinates, the perpendicular Laplacian is:

$$\nabla^2_\perp \equiv \nabla^2 - \partial^2_{||} = (RB_\theta)^2\left[\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^4B^2}\frac{\partial^2}{\partial\zeta^{2^2}}\right] + \frac{B_\zeta^2}{h_\theta^2 B^2}\frac{\partial^2}{\partial\theta^2} + \cdots \tag{32}$$

where the neglected terms are first-order derivatives. The coefficient for the second-order z derivative differs from equation 28, and equation 32 still contains a derivative in θ. This shows that the transformation made to get equation 28 doesn't result in the same answer as orthogonal flux coordinates: equation 28 is in field-aligned coordinates.

Note that in the limit of $B_\theta = B$, both equations 28 and 32 are the same, as they should be.

4.2 Operator $\mathbf{B}\times\nabla\phi\cdot\nabla A$

$$\mathbf{B}\times\nabla\phi\cdot\nabla A = \left(\frac{\partial\phi}{\partial x}\frac{\partial A}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial A}{\partial x}\right)\left(-B_\zeta\frac{RB_\theta}{h_\theta}\right)$$

$$
\begin{aligned}
&+ \left(\frac{\partial\phi}{\partial x}\frac{\partial A}{\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial A}{\partial x}\right)\left(-B^2\right) \\
&- \left(\frac{\partial\phi}{\partial y}\frac{\partial A}{\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial A}{\partial y}\right)\left(IB_\zeta\frac{RB_\theta}{h_\theta}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}\times\nabla\phi\cdot\nabla A &= \left(\frac{\partial\phi}{\partial\psi}\frac{\partial A}{\partial y} + I\frac{\partial\phi}{\partial z}\frac{\partial A}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial A}{\partial\psi} - I\frac{\partial\phi}{\partial y}\frac{\partial A}{\partial z}\right)\left(-B_\zeta\frac{RB_\theta}{h_\theta}\right) \\
&+ \left(\frac{\partial\phi}{\partial\psi}\frac{\partial A}{\partial z} + I\frac{\partial\phi}{\partial z}\frac{\partial A}{\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial A}{\partial\psi} - I\frac{\partial\phi}{\partial z}\frac{\partial A}{\partial z}\right)\left(-B^2\right) \\
&- \left(\frac{\partial\phi}{\partial y}\frac{\partial A}{\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial A}{\partial y}\right)\left(IB_\zeta\frac{RB_\theta}{h_\theta}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}\times\nabla\phi\cdot\nabla A &= \left(\frac{\partial\phi}{\partial\psi}\frac{\partial A}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial A}{\partial\psi}\right)\left(-B_\zeta\frac{RB_\theta}{h_\theta}\right) \\
&+ \left(\frac{\partial\phi}{\partial\psi}\frac{\partial A}{\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial A}{\partial\psi}\right)\left(-B^2\right) \qquad (33)
\end{aligned}
$$

References

[1] R D Hazeltine and J D Meiss. *Plasma Confinement*. Dover publications, 2003.

A Differential geometry

The following is notes from [**?**].

Sets of vectors $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ and $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ are **reciprocal** if

$$\mathbf{A}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{b} = \mathbf{C}\cdot\mathbf{c} = 1 \qquad (34)$$

$$\mathbf{A}\cdot\mathbf{b} = \mathbf{A}\cdot\mathbf{c} = \mathbf{B}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{c} = \mathbf{C}\cdot\mathbf{a} = \mathbf{C}\cdot\mathbf{b} = 0 \qquad (35)$$

$$(36)$$

which implies that $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ and $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ are each linearly independent. Equivalently,

$$\mathbf{a} = \frac{\mathbf{B}\times\mathbf{C}}{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})} \qquad \mathbf{b} = \frac{\mathbf{C}\times\mathbf{A}}{\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})} \qquad \mathbf{c} = \frac{\mathbf{A}\times\mathbf{B}}{\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})} \qquad (37)$$

Either of these sets can be used as a basis, and any vector $\mathbf{w}$ can be represented as $\mathbf{w} = (\mathbf{w}\cdot\mathbf{a})\,\mathbf{A} + (\mathbf{w}\cdot\mathbf{b})\,\mathbf{B} + (\mathbf{w}\cdot\mathbf{c})\,\mathbf{C}$ or as $\mathbf{w} = (\mathbf{w}\cdot\mathbf{A})\,\mathbf{a} + (\mathbf{w}\cdot\mathbf{B})\,\mathbf{b} + (\mathbf{w}\cdot\mathbf{C})\,\mathbf{c}$. In the cartesian coordinate system, the basis vectors are reciprocal to themselves so this distinction is not needed. For a general set of coordinates $\{u^1, u^2, u^3\}$, **tangent basis vectors** can be defined. If the cartesian coordinates of a point are given by $(x, y, z) = \mathbf{R}\,(u^1, u^2, u^3)$ then the tangent basis vectors are:

$$\mathbf{e}_i = \frac{\partial \mathbf{R}}{\partial u^i} \tag{38}$$

and in general these will vary from point to point. The **scale factor** or **metric coefficient** $h_i = |\mathbf{e}_i|$ is the distance moved for a unit change in u^i. The unit vector $\hat{\mathbf{e}}_i = \mathbf{e}_i / h_i$. Definitiion of **grad operator**:

$$\boxed{\nabla\Phi \text{ of a function } \Phi \text{ is defined so that } d\Phi = \nabla\Phi\cdot d\mathbf{R}}$$

From the chain rule, $d\mathbf{R} = \frac{\partial\mathbf{R}}{\partial u^i}du^i = \mathbf{e}_i du^i$ and substituting $\Phi = u^i$

$$du^i = \nabla u^i \cdot \mathbf{e}_j du^j$$

which can only be true if $\nabla u^i \cdot \mathbf{e}_j = \delta^i_j$ i.e. if

$$\boxed{\text{Sets of vectors } \mathbf{e}^i \equiv \nabla u^i \text{ and } \mathbf{e}_j \text{ are reciprocal}}$$

Since the sets of vectors $\{\mathbf{e}^i\}$ and $\{\mathbf{e}_i\}$ are reciprocal, any vector $\mathbf{D}$ can be written as $\mathbf{D} = D_i\mathbf{e}^i = D^i\mathbf{e}_i$ where $D_i = \mathbf{D}\cdot\mathbf{e}_i$ are the **covariant components** and $D^i = \mathbf{D}\cdot\mathbf{e}^i$ are the **contravariant components**. To convert between covariant and contravariant components, define the **metric coefficients** $g_{ij} = \mathbf{e_i}\cdot\mathbf{e_j}$ and $g^{ij} = \mathbf{e^i}\cdot\mathbf{e^j}$ so that $\mathbf{e}_i = g_{ij}\mathbf{e}^j$. g_{ij} and g^{ij} are symmetric and if the basis is orthogonal then $g_{ij} = g^{ij} = 0$ for $i \neq j$ i.e. the metric is diagonal.

$$\boxed{g_{ij} = h_i h_j \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \text{ and so } g_{ii} = h_i^2}$$

For a general set of coordinates, the grad operator can be expressed as

$$\nabla = \frac{\partial}{\partial u_i}\nabla u_i$$

and for a general set of (differentiable) coordinates $\{u^i\}$, the laplacian is given by

$$\nabla^2\phi = \frac{1}{J}\frac{\partial}{\partial u^i}\left(J g^{ij}\frac{\partial\phi}{\partial u^j}\right) \tag{39}$$

which can be expanded as

$$\nabla^2\phi = g^{ij}\frac{\partial^2\phi}{\partial u^i\partial u^j} + \underbrace{\frac{1}{J}\frac{\partial}{\partial u^i}\left(Jg^{ij}\right)}_{-\Gamma^j}\frac{\partial\phi}{\partial u^j} \tag{40}$$

where Γ^j are the **connection coefficients**. Setting $\phi = u^k$ in equation (39) gives $\nabla^2 u^k = -\Gamma^k$. Expanding (39) and setting $\{u^i\} = \{x, y, z\}$ gives

$$\begin{aligned}\nabla^2 f = \nabla\cdot\nabla f = \nabla\cdot\left(\frac{\partial}{\partial x}\nabla x + \frac{\partial}{\partial y}\nabla y + \frac{\partial}{\partial z}\nabla z\right)&\\ = \frac{\partial^2 f}{\partial x^2}\left|\nabla x\right|^2 + \frac{\partial^2 f}{\partial y^2}\left|\nabla y\right|^2 + \frac{\partial^2 f}{\partial z^2}\left|\nabla z\right|^2&\\ +2\frac{\partial^2 f}{\partial x\partial y}\left(\nabla x\cdot\nabla y\right) + 2\frac{\partial^2 f}{\partial x\partial z}\left(\nabla x\cdot\nabla z\right) + 2\frac{\partial^2 f}{\partial y\partial z}\left(\nabla y\cdot\nabla z\right)&\\ +\nabla^2 x\frac{\partial f}{\partial x} + \nabla^2 y\frac{\partial f}{\partial y} + \nabla^2 z\frac{\partial f}{\partial z}&\end{aligned} \tag{41}$$

Curl defined as:

$$\nabla\times\mathbf{A} = \frac{1}{\sqrt{g}}\sum_k\left(\frac{\partial A_j}{\partial u_i} - \frac{\partial A_i}{\partial u_j}\right)\mathbf{e}_k \qquad i,j,k\ \texttt{cyc}\ 1,2,3 \tag{42}$$

Cross-product relation between contravariant and covariant vectors:

$$\mathbf{e}^i = \frac{1}{J}\left(\mathbf{e}_j\times\mathbf{e}_k\right) \qquad \mathbf{e}_i = J\left(\mathbf{e}^j\times\mathbf{e}^k\right) \qquad i,j,k\ \texttt{cyc}\ 1,2,3 \tag{43}$$

B Divergence of ExB velocity

$$\underline{v}_{ExB} = \frac{\underline{b}\times\nabla\phi}{B}$$

Using

$$\nabla\cdot\left(\underline{F}\times\underline{G}\right) = \left(\nabla\times\underline{F}\right)\cdot\underline{G} - \underline{F}\cdot\left(\nabla\times\underline{G}\right)$$

the divergence of the $\underline{E}\times\underline{B}$ velocity can be written as

$$\nabla\cdot\left(\frac{1}{B}\underline{b}\times\nabla\phi\right) = \left[\nabla\times\left(\frac{1}{B}\underline{b}\right)\right]\cdot\nabla\phi - \frac{1}{B}\underline{b}\cdot\nabla\times\nabla\phi \tag{44}$$

The second term on the right is identically zero (curl of a gradient). The first term on the right can be expanded as

$$\left[\nabla\times\left(\frac{1}{B}\underline{b}\right)\right]\cdot\nabla\phi = \left[\nabla\left(\frac{1}{B}\right)\times\underline{b} + \frac{1}{B}\nabla\times\underline{b}\right]\cdot\nabla\phi$$

Using

$$\underline{b}\times\underline{\kappa} = \nabla\times\underline{b} - \underline{b}\left[\underline{b}\cdot\left(\nabla\times\underline{b}\right)\right]$$

this becomes:

$$\begin{aligned}\nabla\cdot\left(\frac{1}{B}\underline{b}\times\nabla\phi\right) = & - & \underline{b}\times\nabla\left(\frac{1}{B}\right)\cdot\nabla\phi \\ & + & \frac{1}{B}\underline{b}\times\underline{\kappa}\cdot\nabla\phi \\ & + & \left[\underline{b}\cdot\left(\nabla\times\underline{b}\right)\right]\underline{b}\cdot\nabla\phi\end{aligned}$$

Alternatively, equation 44 can be expanded as

$$\begin{aligned}\nabla\cdot\left(\frac{1}{B}\underline{b}\times\nabla\phi\right) &= -B\underline{b}\times\nabla\left(\frac{1}{B^2}\right)\cdot\nabla\phi + \frac{1}{B^2}\nabla\times\underline{B}\cdot\nabla\phi \\ &= -B\underline{b}\times\nabla\left(\frac{1}{B^2}\right)\cdot\nabla\phi + \frac{1}{B^2}\underline{J}\cdot\nabla\phi\end{aligned}$$