# The Fourier Transform and the Wave Equation

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## 1 What is the Wave Equation?

Our goal in this expository paper is to study the solutions to the *d-dimensional wave equation*

$$\frac{\partial^2 u}{\partial x_1^2} + ... + \frac{\partial^2 u}{\partial x_d^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

This equation is a second order partial differential equation whose unknown $u = u(x,t)$ is a real-valued function with domain in $\mathbb{R}^d \times \mathbb{R}$. Here $c$ is a fixed positive constant. It is no harm to assume $c = 1$ since we can rescale the variable $t$ if necessary.

The wave equation (1) describes the transmission of waves within some medium. For example, when $d = 1$, it describes the motion of a vibrating string. We refer to Chapter 1 of the book [2] by Stein and Shakarchi for a derivation of the wave equation in this case.

In this expository paper, we focus on the Cauchy problem for the wave equation. That is, we want to find a solution to the equation (1) subject to the initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

where $f$ and $g$ are known functions on $\mathbb{R}^d$.

## 2 What is the Fourier Transform?

In order to solve the Cauchy problem, we introduce a useful tool called the Fourier transform. Given a complex-valued function $f$ with domain $\mathbb{R}^d$, we define its *Fourier transform* (at least formally) by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\, dx \tag{2}$$

for $\xi \in \mathbb{R}^d$.

It is important to note that the above definition does not make sense for all functions $f$. To resolve this issue, we introduce the *Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ which consists of all infinitely differentiable functions $f$ such that $f$ and all of its derivatives are rapidly decreasing. Here recall that a function $g$ is *rapidly decreasing* if for all positive integers $k$ we have

$$\sup_{x\in\mathbb{R}^d} |x|^k |g(x)| < \infty.$$

If $f \in \mathcal{S}(\mathbb{R}^d)$, then we can not only define $\hat{f}$ but also prove several important properties of $\hat{f}$. Here we only list those which will be used in this paper. For a complete list of properties and their proofs - we refer to Section 6.2 of [2].

(i) $\hat{f}$ is also a Schwartz function.

(ii) Fourier transform of derivatives.

$$((\frac{\partial}{\partial x})^\alpha f)^\wedge(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi). \tag{3}$$

(iii) Fourier inversion formula.

$$f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi i x\cdot\xi}\ d\xi. \tag{4}$$

For simplicity, from now on we assume that $f, g$ are Schwartz functions.

# 3 Fourier Transform of the Cauchy problem for the Wave Equation

In this section we will solve the Cauchy problem for the wave equation

$$\Delta u = \frac{\partial^2 u}{\partial t^2} \tag{5}$$

with the initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \tag{6}$$

where $f, g \in \mathcal{S}(\mathbb{R}^d)$. Here $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$ is the Laplacian.

Now to solve the Cauchy problem, we take the Fourier transform of the wave equation and its initial conditions with respect to the spatial variables $x_1, ..., x_d$. This will convert the wave equation into a solvable ODE. In fact, by (3) we get that (5) becomes

$$-4\pi^2|\xi|^2\hat{u}(\xi,t) = \frac{\partial^2\hat{u}}{\partial t^2}(\xi,t).$$

This is an ODE whose solution is given by

$$\hat{u}(\xi,t) = A(\xi)\cos(2\pi|\xi|t) + B(\xi)\sin(2\pi|\xi|t).$$

Now if we take the Fourier transform with respect to $x$ of the initial conditions of the Cauchy problem, we get that

$$\hat{u}(\xi,0) = \hat{f}(\xi)$$

and

$$\frac{\partial \hat{u}}{\partial t}(\xi,0) = \hat{g}(\xi).$$

If we solve for $A(\xi)$ and $B(\xi)$, we get that

$$A(\xi) = \hat{f}(\xi)$$

and

$$2\pi|\xi|B(\xi) = \hat{g}(\xi).$$

Therefore,

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}. \tag{7}$$

Now we use the Fourier inversion formula (4). In conclusion, a solution to the Cauchy problem (5) and (6) is formally defined as:

$$u(x,t) = \int_{\mathbb{R}^d} [\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}]e^{2\pi i x\cdot\xi} d\xi. \tag{8}$$

We can prove that this formula does indeed give a solution to the Cauchy problem. We refer to Theorem 6.3.1 in Stein-Shakarchi [2] for the proof.

One important thing to notice is that (8) is the *unique* solution to the Cauchy problem (5) and (6). For simplicity, we omit the proof of the uniqueness, and we refer our readers to Section 2.4 of [1].

# 4 Solution to the Cauchy problem for the Wave Equation in 3-D

We now focus on the case $d = 3$. In order to simplify (8) in Section 3, we first define the spherical mean of a function $f$. The *spherical mean* of a function in $\mathcal{S}(\mathbb{R}^3)$ over the sphere of radius $r = t$ centered at $x$, is defined by:

$$M_t(f)(x) = \frac{1}{4\pi}\int_{\mathbb{S}^2} f(x - t\sigma)\ d\sigma(\gamma)$$

where $d\sigma(\gamma)$ is the element of surface area for the unit sphere in $R^3$. We can interpret this spherical mean $M_t(f)$ as the average value of $f$ over the sphere centered at $x$ with radius $t$.

Why do we introduce $M_t(f)$? In fact we have the following formula:

$$\widehat{M_t(f)}(\xi) = \hat{f}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|t}.$$

We refer to [2] for the proof. Compare this formula with (7) above. We see that

$$\hat{u}(\xi, t) = \frac{\partial}{\partial t}(t\widehat{M_t(f)}(\xi)) + t\widehat{M_t(g)}(\xi).$$

By the Fourier inversion formula (4), we conclude that the solution of the Cauchy problem for the wave equation in 3-D

$$\Delta u = \frac{\partial^2 u}{\partial t^2} \qquad \text{subject to } u(x, 0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x, 0) = g(x)$$

is given by

$$u(x, t) = \frac{\partial}{\partial t}(tM_t(f)(x)) + tM_t(g)(x). \tag{9}$$

We refer to Theorem 6.3.6 in [2].

We can write (9) in a more explicit way. In fact,

$$u(x, t) = \frac{1}{|S(x, t)|}\int_{S(x,t)} [tg(y) + f(y) + \nabla f(y) \cdot (y - x)]\, d\sigma(y). \tag{10}$$

Here $S(x, t)$ denotes the sphere of center $x$ and radius $t$, and $|S(x, t)|$ denotes its area. This formula is called the *Kirchoff's formula*. For simplicity we skip its proof here, and we refer our readers to Section 2.4 of [1].

# 5 Solution to the Cauchy Problem for the Wave Equation in 2-D

Now, interestingly enough, the solution to the wave equation in 3-D leads to the solution of the wave equation in 2-D. This technique of using the solution of the wave equation in a higher odd dimension to find the solution to the wave equation in an even dimension is called the *method of descent*. This method works as follows.

Suppose $u = u(x, y, t)$ is a solution to the wave equation in 2-D such that $u(x, y, 0) = f(x, y)$ and $\frac{\partial}{\partial t}u(x, y, 0) = g(x, y)$. Define $\widetilde{u}(x, y, z, t) = u(x, y, t)$ and define $\widetilde{f}$ and $\widetilde{g}$ in a similar way. Then, we notice that $\widetilde{u}$ is a solution the wave equation in 3-D such that $\widetilde{f}$ and $\widetilde{g}$ are its initial data. While $\widetilde{f}$ and $\widetilde{g}$ are not necessarily Schwartz functions, we formally apply (10) to express $\widetilde{u}$ in terms of $\widetilde{f}$ and $\widetilde{g}$. This gives us an expression for $u$ in terms of $f, g$. Finally we prove that such an expression does give a solution to the wave equation in 2-D.

Using the method of descent, we get the following result which is Theorem 6.3.7 in [2]. If we define the corresponding means by:

$$\widetilde{M_t}(F)(x) = \frac{1}{2\pi}\int_{|y|\leq 1} F(x - ty)(1 - |y|^2)^{-1/2}dy,$$

then we can get a solution to the Cauchy problem for the wave equation in 2-D (with $f, g \in \mathcal{S}(\mathbb{R}^2)$) as follows:

$$u(x,t) = \frac{\partial}{\partial t}(t\widetilde{M}_t(f)(x)) + t\widetilde{M}_t(g)(x). \tag{11}$$

As in the 3-D case, we can write (11) in a more explicit way. In fact,

$$u(x,t) = \frac{1}{2|B(x,t)|}\int_{B(x,t)} \frac{tf(y) + t^2 g(y) + t\nabla f(y)\cdot(y-x)}{(t^2 - |x-y|^2)^{1/2}}\ dy. \tag{12}$$

Here $B(x,t)$ denotes the disk of center $x$ and radius $t$, and $|B(x,t)|$ denotes its area. This formula is called the *Poisson's formula.* We refer our readers to Section 2.4 of [1].

Here in the 2-D case, the solution $u(x,t)$ depends on $f$ and $g$ in whole disk whereas in the 3-D case the solution depends just on the values of the initial data near the boundary of that disk.

# 6 Huygens Principle and Finite Speed of Propagation

In this section, we discuss some properties of the solutions to the wave equation. First, we can observe the phenomenon known as Huygens principle through the solutions to the 1-D and 3-D wave equation. The 1-D and 3-D solutions are respectively:

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(y)\ dy$$

and

$$u(x,t) = \frac{1}{|S(x,t)|}\int_{S(x,t)} [tg(y) + f(y) + \Delta f(y)\cdot(y-x)]\ d\sigma(y).$$

In the 1-D case, the solution at $(x,t)$ is dependent only on the values of $f$ and $g$ in the interval between $[x-t, x+t]$. This is observed as follows[1]:

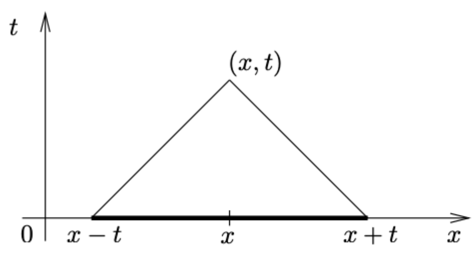


**Figure 1.** Huygens principle, $d = 1$

[1]All figures in this section are from [2].

In the 3-D case, the solution at $(x, t)$ is dependent on the values of $f$ and $g$ in a neighborhood of the sphere with center $x$ and radius $r = t$. This situation produces a cone called the *backward light cone* originating at $(x, t)$

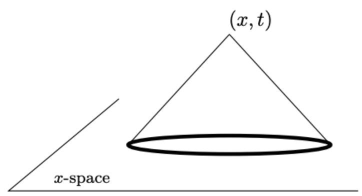


**Figure 2.** Backward light cone originating at $(x, t)$

In addition, if we narrow our view of the data to a point $x_0$ in the plan $t = 0$, we can observe the *forward light cone* shown as follows

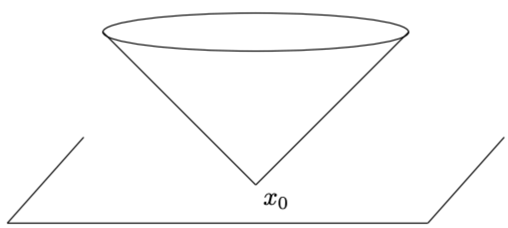


**Figure 3.** The forward light cone originating at $x_0$

The cones that depict the $u(x, t)$ solutions to the 1-D and 3-D wave equation allow us to observe the *Huygens principle*, which is a phenomenon that allows us to visualize the bending of waves when it enters a medium where it's speed is either increased or reduced in the faster or slower medium. It is also worth noting that in 2-D we still have Huygens principle - we just replace a sphere with an open disk in 2-D and we get a similar result.

Another important aspect that is connected to the results above is the *finite speed of propagation.* If we have an initial entrance at $x = x_0$, after a finite time $t$, the effects of Huygens principle will have propagated only inside the ball with center $x_0$ and radius $r = t$.

For a more detailed discussion of these two properties, we refer our readers to [1] and [2].

# References


[1] Lawrence C. Evans. *Partial differential equations.* Graduate studies in mathematics: v. 19. American Mathematical Society, 2010.

[2] Elias M. Stein and Rami Shakarchi. *Fourier analysis : an introduction.* Princeton lectures in analysis: 1. Princeton University Press, 2003.