8 Maclaurin's series

8.1 Introduction

Some mathematical functions may be represented as power series, containing terms in ascending powers of the variable. For example,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

(as introduced in Chapter 5)

Using a series, called **Maclaurin's series**, mixed functions containing, say, algebraic, trigonometric and exponential functions, may be expressed solely as algebraic functions, and differentiation and integration can often be more readily performed.

8.2 Derivation of Maclaurin's theorem

Let the power series for $f(x)$ be

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \quad (1)$$

where $a_0, a_1, a_2, \ldots$ are constants.

When $x = 0$, $\boldsymbol{f(0) = a_0}$.

Differentiating equation (1) with respect to x gives:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \quad (2)$$

When $x = 0$, $\boldsymbol{f'(0) = a_1}$.

Differentiating equation (2) with respect to x gives:

$$f''(x) = 2a_2 + (3)(2)a_3x + (4)(3)a_4x^2 + (5)(4)a_5x^3 + \cdots \quad (3)$$

When $x = 0$, $f''(0) = 2a_2 = 2!a_2$, i.e. $\boldsymbol{a_2 = \frac{f''(0)}{2!}}$

Differentiating equation (3) with respect to x gives:

$$f'''(x) = (3)(2)a_3 + (4)(3)(2)a_4x + (5)(4)(3)a_5x^2 + \cdots \quad (4)$$

When $x = 0$, $f'''(0) = (3)(2)a_3 = 3!a_3$, i.e. $\boldsymbol{a_3 = \frac{f'''(0)}{3!}}$

Continuing the same procedure gives $\boldsymbol{a_4 = \frac{f^{iv}(0)}{4!}}$, $\boldsymbol{a_5 = \frac{f^{v}(0)}{5!}}$, and so on.

Substituting for $a_0, a_1, a_2, \ldots$ in equation (1) gives:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

i.e.

$$\boxed{\boldsymbol{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots}} \quad (5)$$

Equation (5) is a mathematical statement called **Maclaurin's theorem** or **Maclaurin's series**.

8.3 Conditions of Maclaurin's series

Maclaurin's series may be used to represent any function, say $f(x)$, as a power series provided that at $x = 0$ the following three conditions are met:

(a) $\boldsymbol{f(0) \neq \infty}$

For example, for the function $f(x) = \cos x$, $f(0) = \cos 0 = 1$, thus $\cos x$ meets the condition. However, if $f(x) = \ln x$, $f(0) = \ln 0 = -\infty$, thus $\ln x$ does not meet this condition.

(b) $\boldsymbol{f'(0), f''(0), f'''(0), \ldots \neq \infty}$

For example, for the function $f(x) = \cos x$, $f'(0) = -\sin 0 = 0, f''(0) = -\cos 0 = -1$, and so on; thus $\cos x$ meets this condition. However, if $f(x) = \ln x$, $f'(0) = \frac{1}{0} = \infty$, thus $\ln x$ does not meet this condition.

(c) **The resultant Maclaurin's series must be convergent**

In general, this means that the values of the terms, or groups of terms, must get progressively smaller and the sum of the terms must reach a limiting value.

For example, the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is convergent since the value of the terms is getting smaller and the sum of the terms is approaching a limiting value of 2.

8.4 Worked problems on Maclaurin's series

Problem 1. Determine the first four terms of the power series for $\cos x$.

The values of $f(0)$, $f'(0)$, $f''(0)$, ... in the Maclaurin's series are obtained as follows:

$$
\begin{aligned}
f(x) &= \cos x & f(0) &= \cos 0 = 1 \\
f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\
f''(x) &= -\cos x & f''(0) &= -\cos 0 = -1 \\
f'''(x) &= \sin x & f'''(0) &= \sin 0 = 0 \\
f^{\text{iv}}(x) &= \cos x & f^{\text{iv}}(0) &= \cos 0 = 1 \\
f^{\text{v}}(x) &= -\sin x & f^{\text{v}}(0) &= -\sin 0 = 0 \\
f^{\text{vi}}(x) &= -\cos x & f^{\text{vi}}(0) &= -\cos 0 = -1
\end{aligned}
$$

Substituting these values into equation (5) gives:

$$
f(x) = \cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) + \cdots
$$

i.e. $\mathbf{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$

Problem 2. Determine the power series for $\cos 2\theta$.

Replacing x with 2θ in the series obtained in Problem 1 gives:

$$
\begin{aligned}
\cos 2\theta &= 1 - \frac{(2\theta)^2}{2!} + \frac{(2\theta)^4}{4!} - \frac{(2\theta)^6}{6!} + \cdots \\
&= 1 - \frac{4\theta^2}{2} + \frac{16\theta^4}{24} - \frac{64\theta^6}{720} + \cdots
\end{aligned}
$$

i.e. $\mathbf{\cos 2\theta = 1 - 2\theta^2 + \frac{2}{3}\theta^4 - \frac{4}{45}\theta^6 + \cdots}$

Problem 3. Determine the power series for $\tan x$ as far as the term in x^3.

$$
\begin{aligned}
f(x) &= \tan x \\
f(0) &= \tan 0 = 0 \\
f'(x) &= \sec^2 x \\
f'(0) &= \sec^2 0 = \frac{1}{\cos^2 0} = 1 \\
f''(x) &= (2\sec x)(\sec x \tan x) \\
&= 2\sec^2 x \tan x \\
f''(0) &= 2\sec^2 0 \tan 0 = 0 \\
f'''(x) &= (2\sec^2 x)(\sec^2 x) \\
&\quad + (\tan x)(4\sec x \sec x \tan x), \\
&\qquad \text{by the product rule,} \\
&= 2\sec^4 x + 4\sec^2 x \tan^2 x \\
f'''(0) &= 2\sec^4 0 + 4\sec^2 0 \tan^2 0 = 2
\end{aligned}
$$

Substituting these values into equation (5) gives:

$$
f(x) = \tan x = 0 + (x)(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2)
$$

i.e. $\mathbf{\tan x = x + \frac{1}{3}x^3}$

Problem 4. Expand $\ln(1 + x)$ to five terms.

$$
\begin{aligned}
f(x) &= \ln(1 + x) & f(0) &= \ln(1 + 0) = 0 \\
f'(x) &= \frac{1}{(1 + x)} & f'(0) &= \frac{1}{1 + 0} = 1 \\
f''(x) &= \frac{-1}{(1 + x)^2} & f''(0) &= \frac{-1}{(1 + 0)^2} = -1 \\
f'''(x) &= \frac{2}{(1 + x)^3} & f'''(0) &= \frac{2}{(1 + 0)^3} = 2
\end{aligned}
$$

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$$f^{iv}(x) = \frac{-6}{(1+x)^4} \quad f^{iv}(0) = \frac{-6}{(1+0)^4} = -6$$
$$f^{v}(x) = \frac{24}{(1+x)^5} \quad f^{v}(0) = \frac{24}{(1+0)^5} = 24$$

Substituting these values into equation (5) gives:

$$f(x) = \ln(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(24)$$

i.e. $\mathbf{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots}$

Problem 5. Expand $\ln(1-x)$ to five terms.

Replacing x by $-x$ in the series for $\ln(1+x)$ in Problem 4 gives:

$$\ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \cdots$$

i.e. $\mathbf{\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots}$

Problem 6. Determine the power series for $\ln\left(\frac{1+x}{1-x}\right)$.

$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$ by the laws of logarithms, and from Problems 4 and 5,

$$\ln\left(\frac{1+x}{1-x}\right) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\right)$$
$$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots$$

i.e. $\mathbf{\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)}$

Problem 7. Use Maclaurin's series to find the expansion of $(2+x)^4$.

$$f(x) = (2+x)^4 \quad f(0) = 2^4 = 16$$
$$f'(x) = 4(2+x)^3 \quad f'(0) = 4(2)^3 = 32$$
$$f''(x) = 12(2+x)^2 \quad f''(0) = 12(2)^2 = 48$$
$$f'''(x) = 24(2+x)^1 \quad f'''(0) = 24(2) = 48$$
$$f^{iv}(x) = 24 \quad f^{iv}(0) = 24$$

Substituting in equation (5) gives:

$\mathbf{(2+x)^4}$

$$= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0)$$
$$= 16 + (x)(32) + \frac{x^2}{2!}(48) + \frac{x^3}{3!}(48) + \frac{x^4}{4!}(24)$$
$$\mathbf{= 16 + 32x + 24x^2 + 8x^3 + x^4}$$

(This expression could have been obtained by applying the binomial theorem.)

Problem 8. Expand $e^{\frac{x}{2}}$ as far as the term in x^4.

$$f(x) = e^{\frac{x}{2}} \quad f(0) = e^0 = 1$$
$$f'(x) = \frac{1}{2}e^{\frac{x}{2}} \quad f'(0) = \frac{1}{2}e^0 = \frac{1}{2}$$
$$f''(x) = \frac{1}{4}e^{\frac{x}{2}} \quad f''(0) = \frac{1}{4}e^0 = \frac{1}{4}$$
$$f'''(x) = \frac{1}{8}e^{\frac{x}{2}} \quad f'''(0) = \frac{1}{8}e^0 = \frac{1}{8}$$
$$f^{iv}(x) = \frac{1}{16}e^{\frac{x}{2}} \quad f^{iv}(0) = \frac{1}{16}e^0 = \frac{1}{16}$$

Substituting in equation (5) gives:

$$e^{\frac{x}{2}} = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$
$$= 1 + (x)\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(\frac{1}{4}\right) + \frac{x^3}{3!}\left(\frac{1}{8}\right) + \frac{x^4}{4!}\left(\frac{1}{16}\right) + \cdots$$

i.e. $\mathbf{e^{\frac{x}{2}} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots}$

Problem 9. Develop a series for $\sinh x$ using Maclaurin's series.

$$f(x) = \sinh x \qquad f(0) = \sinh 0 = \frac{e^0 - e^{-0}}{2} = 0$$

$$f'(x) = \cosh x \qquad f'(0) = \cosh 0 = \frac{e^0 + e^{-0}}{2} = 1$$

$$f''(x) = \sinh x \qquad f''(0) = \sinh 0 = 0$$

$$f'''(x) = \cosh x \qquad f'''(0) = \cosh 0 = 1$$

$$f^{iv}(x) = \sinh x \qquad f^{iv}(0) = \sinh 0 = 0$$

$$f^{v}(x) = \cosh x \qquad f^{v}(0) = \cosh 0 = 1$$

Substituting in equation (5) gives:

$$\begin{aligned}\sinh x &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) \\ &\quad + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \cdots \\ &= 0 + (x)(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(0) \\ &\quad + \frac{x^5}{5!}(1) + \cdots\end{aligned}$$

i.e. $\mathbf{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}$

(as obtained in Section 5.5)

Problem 10. Produce a power series for $\cos^2 2x$ as far as the term in x^6.

From double angle formulae, $\cos 2A = 2\cos^2 A - 1$ (see Chapter 18).

from which, $\quad \cos^2 A = \frac{1}{2}(1 + \cos 2A)$

and $\quad \cos^2 2x = \frac{1}{2}(1 + \cos 4x)$

From Problem 1,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

hence $\quad \cos 4x = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!} + \cdots$

$$= 1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \cdots$$

Thus $\quad \cos^2 2x = \frac{1}{2}(1 + \cos 4x)$

$$= \frac{1}{2}\left(1 + 1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \cdots\right)$$

i.e. $\mathbf{\cos^2 2x = 1 - 4x^2 + \frac{16}{3}x^4 - \frac{128}{45}x^6 + \cdots}$

Now try the following exercise.

Exercise 36 Further problems on Maclaurin's series

1. Determine the first four terms of the power series for $\sin 2x$ using Maclaurin's series.
$\left[\sin 2x = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots\right]$

2. Use Maclaurin's series to produce a power series for $\cosh 3x$ as far as the term in x^6.
$\left[1 + \frac{9}{2}x^2 + \frac{27}{8}x^4 + \frac{81}{80}x^6\right]$

3. Use Maclaurin's theorem to determine the first three terms of the power series for $\ln(1 + e^x)$. $\left[\ln 2 + \frac{x}{2} + \frac{x^2}{8}\right]$

4. Determine the power series for $\cos 4t$ as far as the term in t^6.
$\left[1 - 8t^2 + \frac{32}{3}t^4 - \frac{256}{45}t^6\right]$

5. Expand $e^{\frac{3}{2}x}$ in a power series as far as the term in x^3. $\left[1 + \frac{3}{2}x + \frac{9}{8}x^2 + \frac{9}{16}x^3\right]$

6. Develop, as far as the term in x^4, the power series for $\sec 2x$. $\left[1 + 2x^2 + \frac{10}{3}x^4\right]$

7. Expand $e^{2\theta}\cos 3\theta$ as far as the term in θ^2 using Maclaurin's series. $\left[1 + 2\theta - \frac{5}{2}\theta^2\right]$

8. Determine the first three terms of the series for $\sin^2 x$ by applying Maclaurin's theorem.
$\left[x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 \cdots\right]$

9. Use Maclaurin's series to determine the expansion of $(3+2t)^4$.
$[81+216t+216t^2+96t^3+16t^4]$

8.5 Numerical integration using Maclaurin's series

The value of many integrals cannot be determined using the various analytical methods. In Chapter 45, the trapezoidal, mid-ordinate and Simpson's rules are used to numerically evaluate such integrals. Another method of finding the approximate value of a definite integral is to express the function as a power series using Maclaurin's series, and then integrating each algebraic term in turn. This is demonstrated in the following worked problems.

Problem 11. Evaluate $\int_{0.1}^{0.4} 2\,e^{\sin\theta}\,d\theta$, correct to 3 significant figures.

A power series for $e^{\sin\theta}$ is firstly obtained using Maclaurin's series.

$$f(\theta)=e^{\sin\theta} \qquad f(0)=e^{\sin 0}=e^0=1$$

$$f'(\theta)=\cos\theta\,e^{\sin\theta} \qquad f'(0)=\cos 0\,e^{\sin 0}=(1)e^0=1$$

$$f''(\theta)=(\cos\theta)(\cos\theta\,e^{\sin\theta})+(e^{\sin\theta})(-\sin\theta),$$
by the product rule,
$$=e^{\sin\theta}(\cos^2\theta-\sin\theta);$$

$$f''(0)=e^0(\cos^2 0-\sin 0)=1$$

$$f'''(\theta)=(e^{\sin\theta})[(2\cos\theta(-\sin\theta)-\cos\theta)]+(\cos^2\theta-\sin\theta)(\cos\theta\,e^{\sin\theta})$$
$$=e^{\sin\theta}\cos\theta[-2\sin\theta-1+\cos^2\theta-\sin\theta]$$

$$f'''(0)=e^0\cos 0[(0-1+1-0)]=0$$

Hence from equation (5):

$$e^{\sin\theta}=f(0)+\theta f'(0)+\frac{\theta^2}{2!}f''(0)+\frac{\theta^3}{3!}f'''(0)+\cdots$$
$$=1+\theta+\frac{\theta^2}{2}+0$$

Thus
$$\int_{0.1}^{0.4}2\,e^{\sin\theta}\,d\theta=\int_{0.1}^{0.4}2\left(1+\theta+\frac{\theta^2}{2}\right)d\theta$$
$$=\int_{0.1}^{0.4}(2+2\theta+\theta^2)\,d\theta$$
$$=\left[2\theta+\frac{2\theta^2}{2}+\frac{\theta^3}{3}\right]_{0.1}^{0.4}$$
$$=\left(0.8+(0.4)^2+\frac{(0.4)^3}{3}\right)-\left(0.2+(0.1)^2+\frac{(0.1)^3}{3}\right)$$
$$=0.98133-0.21033$$
$$=\mathbf{0.771}, \text{ correct to 3 significant figures}$$

Problem 12. Evaluate $\int_0^1 \frac{\sin\theta}{\theta}\,d\theta$ using Maclaurin's series, correct to 3 significant figures.

Let
$$\begin{aligned}
f(\theta)&=\sin\theta & f(0)&=0\\
f'(\theta)&=\cos\theta & f'(0)&=1\\
f''(\theta)&=-\sin\theta & f''(0)&=0\\
f'''(\theta)&=-\cos\theta & f'''(0)&=-1\\
f^{iv}(\theta)&=\sin\theta & f^{iv}(0)&=0\\
f^{v}(\theta)&=\cos\theta & f^{v}(0)&=1
\end{aligned}$$

Hence from equation (5):

$$\sin\theta=f(0)+\theta f'(0)+\frac{\theta^2}{2!}f''(0)+\frac{\theta^3}{3!}f'''(0)+\frac{\theta^4}{4!}f^{iv}(0)+\frac{\theta^5}{5!}f^{v}(0)+\cdots$$
$$=0+\theta(1)+\frac{\theta^2}{2!}(0)+\frac{\theta^3}{3!}(-1)+\frac{\theta^4}{4!}(0)+\frac{\theta^5}{5!}(1)+\cdots$$

i.e. $\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots$

Hence

$$\int_0^1\frac{\sin\theta}{\theta}\,d\theta=\int_0^1\frac{\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\right)}{\theta}\,d\theta$$
$$=\int_0^1\left(1-\frac{\theta^2}{6}+\frac{\theta^4}{120}-\frac{\theta^6}{5040}+\cdots\right)d\theta$$

$$= \left[\theta - \frac{\theta^3}{18} + \frac{\theta^5}{600} - \frac{\theta^7}{7(5040)} + \cdots\right]_0^1$$

$$= 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{7(5040)} + \cdots$$

$= \mathbf{0.946}$, correct to 3 significant figures

Problem 13. Evaluate $\int_0^{0.4} x\ln(1+x)\,dx$ using Maclaurin's theorem, correct to 3 decimal places.

From Problem 4,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

Hence $\int_0^{0.4} x\ln(1+x)dx$

$$= \int_0^{0.4} x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)dx$$

$$= \int_0^{0.4} \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \cdots\right)dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{8} + \frac{x^5}{15} - \frac{x^6}{24} + \frac{x^7}{35} - \cdots\right]_0^{0.4}$$

$$= \left(\frac{(0.4)^3}{3} - \frac{(0.4)^4}{8} + \frac{(0.4)^5}{15} - \frac{(0.4)^6}{24} + \frac{(0.4)^7}{35} - \cdots\right) - (0)$$

$= 0.02133 - 0.0032 + 0.0006827 - \cdots$

$= \mathbf{0.019}$, correct to 3 decimal places

Now try the following exercise.

Exercise 37 Further problems on numerical integration using Maclaurin's series

1. Evaluate $\int_{0.2}^{0.6} 3e^{\sin\theta}\,d\theta$, correct to 3 decimal places, using Maclaurin's series. [1.784]
2. Use Maclaurin's theorem to expand $\cos 2\theta$ and hence evaluate, correct to 2 decimal places, $\int_0^1 \frac{\cos 2\theta}{\theta^{\frac{1}{3}}}\,d\theta$. [0.88]
3. Determine the value of $\int_0^1 \sqrt{\theta}\cos\theta\,d\theta$, correct to 2 significant figures, using Maclaurin's series. [0.53]
4. Use Maclaurin's theorem to expand $\sqrt{x}\ln(x+1)$ as a power series. Hence evaluate, correct to 3 decimal places, $\int_0^{0.5} \sqrt{x}\ln(x+1)\,dx$. [0.061]

8.6 Limiting values

It is sometimes necessary to find limits of the form $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\}$, where $f(a)=0$ and $g(a)=0$.
For example,

$$\lim_{x\to 1}\left\{\frac{x^2+3x-4}{x^2-7x+6}\right\} = \frac{1+3-4}{1-7+6} = \frac{0}{0}$$

and $\frac{0}{0}$ is generally referred to as indeterminate.

For certain limits a knowledge of series can sometimes help.

For example,

$$\lim_{x\to 0}\left\{\frac{\tan x - x}{x^3}\right\}$$

$$\equiv \lim_{x\to 0}\left\{\frac{x + \frac{1}{3}x^3 + \cdots - x}{x^3}\right\} \text{ from Problem 3}$$

$$= \lim_{x\to 0}\left\{\frac{\frac{1}{3}x^3 + \cdots}{x^3}\right\} = \lim_{x\to 0}\left\{\frac{1}{3}\right\} = \mathbf{\frac{1}{3}}$$

Similarly,

$$\lim_{x\to 0}\left\{\frac{\sinh x}{x}\right\}$$

$$\equiv \lim_{x\to 0}\left\{\frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} +}{x}\right\} \text{ from Problem 9}$$

$$= \lim_{x\to 0}\left\{1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right\} = \mathbf{1}$$

However, a knowledge of series does not help with examples such as $\lim_{x\to 1}\left\{\frac{x^2+3x-4}{x^2-7x+6}\right\}$

L'Hopital's rule will enable us to determine such limits when the differential coefficients of the numerator and denominator can be found.
L'Hopital's rule states:

$$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{x\to a}\left\{\frac{f'(x)}{g'(x)}\right\}$$

provided $g'(a) \neq 0$

It can happen that $\lim_{x\to a}\left\{\frac{f'(x)}{g'(x)}\right\}$ is still $\frac{0}{0}$; if so, the numerator and denominator are differentiated again (and again) until a non-zero value is obtained for the denominator.

The following worked problems demonstrate how L'Hopital's rule is used. Refer to Chapter 27 for methods of differentiation.

Problem 14. Determine $\lim_{x\to 1}\left\{\frac{x^2+3x-4}{x^2-7x+6}\right\}$

The first step is to substitute $x = 1$ into both numerator and denominator. In this case we obtain $\frac{0}{0}$. It is only when we obtain such a result that we then use L'Hopital's rule. Hence applying L'Hopital's rule,

$$\lim_{x\to 1}\left\{\frac{x^2+3x-4}{x^2-7x+6}\right\} = \lim_{x\to 1}\left\{\frac{2x+3}{2x-7}\right\}$$

i.e. both numerator and denominator have been differentiated

$$= \frac{5}{-5} = \mathbf{-1}$$

Problem 15. Determine $\lim_{x\to 0}\left\{\frac{\sin x - x}{x^2}\right\}$

Substituting $x = 0$ gives

$$\lim_{x\to 0}\left\{\frac{\sin x - x}{x^2}\right\} = \frac{\sin 0 - 0}{0} = \frac{0}{0}$$

Applying L'Hopital's rule gives

$$\lim_{x\to 0}\left\{\frac{\sin x - x}{x^2}\right\} = \lim_{x\to 0}\left\{\frac{\cos x - 1}{2x}\right\}$$

Substituting $x = 0$ gives

$$\frac{\cos 0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0} \quad \text{again}$$

Applying L'Hopital's rule again gives

$$\lim_{x\to 0}\left\{\frac{\cos x - 1}{2x}\right\} = \lim_{x\to 0}\left\{\frac{-\sin x}{2}\right\} = \mathbf{0}$$

Problem 16. Determine $\lim_{x\to 0}\left\{\frac{x-\sin x}{x-\tan x}\right\}$

Substituting $x = 0$ gives

$$\lim_{x\to 0}\left\{\frac{x-\sin x}{x-\tan x}\right\} = \frac{0-\sin 0}{0-\tan 0} = \frac{0}{0}$$

Applying L'Hopital's rule gives

$$\lim_{x\to 0}\left\{\frac{x-\sin x}{x-\tan x}\right\} = \lim_{x\to 0}\left\{\frac{1-\cos x}{1-\sec^2 x}\right\}$$

Substituting $x = 0$ gives

$$\lim_{x\to 0}\left\{\frac{1-\cos x}{1-\sec^2 x}\right\} = \frac{1-\cos 0}{1-\sec^2 0} = \frac{1-1}{1-1} = \frac{0}{0} \text{ again}$$

Applying L'Hopital's rule gives

$$\lim_{x\to 0}\left\{\frac{1-\cos x}{1-\sec^2 x}\right\} = \lim_{x\to 0}\left\{\frac{\sin x}{(-2\sec x)(\sec x \tan x)}\right\}$$
$$= \lim_{x\to 0}\left\{\frac{\sin x}{-2\sec^2 x \tan x}\right\}$$

Substituting $x = 0$ gives

$$\frac{\sin 0}{-2\sec^2 0 \tan 0} = \frac{0}{0} \quad \text{again}$$

Applying L'Hopital's rule gives

$$\lim_{x\to 0}\left\{\frac{\sin x}{-2\sec^2 x \tan x}\right\}$$
$$= \lim_{x\to 0}\left\{\frac{\cos x}{(-2\sec^2 x)(\sec^2 x) + (\tan x)(-4\sec^2 x \tan x)}\right\}$$

using the product rule

Substituting $x = 0$ gives

$$\frac{\cos 0}{-2\sec^4 0 - 4\sec^2 0 \tan^2 0} = \frac{1}{-2-0}$$
$$= -\frac{1}{2}$$

Hence $\lim_{x \to 0} \left\{ \frac{x - \sin x}{x - \tan x} \right\} = -\frac{1}{2}$

Now try the following exercise.

Exercise 38 Further problems on limiting values

Determine the following limiting values

1. $\lim_{x \to 1} \left\{ \frac{x^3 - 2x + 1}{2x^3 + 3x - 5} \right\}$ $\left[\frac{1}{9}\right]$

2. $\lim_{x \to 0} \left\{ \frac{\sin x}{x} \right\}$ [1]

3. $\lim_{x \to 0} \left\{ \frac{\ln(1 + x)}{x} \right\}$ [1]

4. $\lim_{x \to 0} \left\{ \frac{x^2 - \sin 3x}{3x + x^2} \right\}$ [−1]

5. $\lim_{\theta \to 0} \left\{ \frac{\sin\theta - \theta\cos\theta}{\theta^3} \right\}$ $\left[\frac{1}{3}\right]$

6. $\lim_{t \to 1} \left\{ \frac{\ln t}{t^2 - 1} \right\}$ $\left[\frac{1}{2}\right]$

7. $\lim_{x \to 0} \left\{ \frac{\sinh x - \sin x}{x^3} \right\}$ $\left[\frac{1}{3}\right]$

8. $\lim_{\theta \to \frac{\pi}{2}} \left\{ \frac{\sin\theta - 1}{\ln\sin\theta} \right\}$ [1]

9. $\lim_{t \to 0} \left\{ \frac{\sec t - 1}{t\sin t} \right\}$ $\left[\frac{1}{2}\right]$