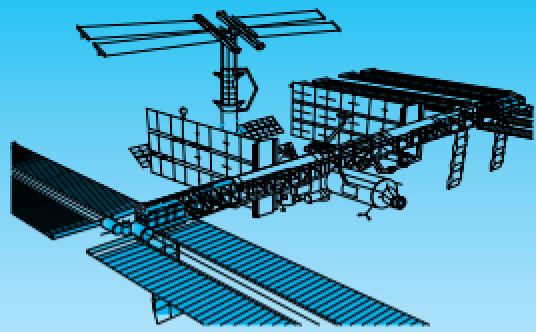
CHAPTER 11



Vector Differential Calculus

Many physical quantities that occur in engineering and science require more than a single number to characterize them. When describing quantities such as force and velocity it is necessary to specify both a magnitude and a direction, and these are examples of vector quantities, whereas the air temperature, which can be specified by giving a single number, is an example of a scalar quantity. Physical problems are often best described in terms of vectors, so the objective of this chapter is to develop the most important aspects of vector differential calculus.

Scalar and vector fields are defined in Section 11.1, and these concepts are then related to the limit, continuity, and differentiability of a vector function of a single real variable. The rules for the differentiation of vector functions of a single real variable are established and used to develop the basic geometry of space curves. The definition of the derivative at a point on a space curve is used when defining the unit tangent vector **T** to such a curve, its curvature κ, its principal normal **N**, and its binormal **B**.

The integration of scalar and vector functions of a single real variable is developed in Section 11.2, after which the line integral of a vector function of position $\mathbf{F}(x, y, z)$ is defined, and by way of example it is then used to define the circulation in a fluid flow and the flux of a vector function of position.

A directional derivative of a scalar function $w = f(x, y, z)$ is defined in Section 11.3 where its most important properties are established. The directional derivative is used when developing the concept of the gradient of f, written either grad f or ∇f, after which rules for its use are developed.

The important property of path invariance of integrals in conservative fields is proved in Section 11.4. The potential function is introduced, a test for a conservative field is given, and the determination of the related potential function is discussed, all of which concepts have important applications throughout engineering and science.

The two other vector operators divergence and curl, written div **F** and curl **F**, respectively, are defined and their physical meaning is explained in Section 11.5. The properties of the divergence operator are established, and then used to prove the properties of the most important combinations of the gradient, divergence, and curl operators.

Applications involving vector operators are often simplified if an appropriate system of coordinates is adopted. The purpose of Section 11.6 is to establish the forms taken by the gradient, divergence, and curl operators in a general system of orthogonal curvilinear coordinates, with special emphasis on cylindrical and spherical polar coordinates.

11.1 Scalar and Vector Fields, Limits, Continuity, and Differentiability

A scalar function $F(x, y, z)$ defined over some region of space D is a function that assigns to each point P_0 in D with coordinates (x_0, y_0, z_0) the *number* $F(P_0) = F(x_0, y_0, z_0)$. The set of all numbers $F(P)$ for all points P in D are said to form a **scalar field** over D. If P has position vector $\mathbf{r}$, we can write the scalar field $F(x, y, z)$ in the form $F(P) = F(\mathbf{r})$ to emphasize the fact that a *scalar* value $F(\mathbf{r})$ is associated with the position vector $\mathbf{r}$ in D. In physical problems P is usually a point in space, and in addition to depending on P, the function F often also depends on the time t, so then $F(P, t) = F(x, y, z, t)$ and in this case we can write $F(P, t) = F(\mathbf{r}, t)$. A typical example of a time dependent scalar field is provided by the temperature distribution throughout a block of metal heated in such a way that the temperatures on its sides vary with time.

scalar and vector fields

More general than a scalar field $F(x, y, z)$ is a **vector field** defined by a vector function $\mathbf{F}(x, y, z)$ over some region of space D that assigns to each point P_0 in D with coordinates (x_0, y_0, z_0) the vector $\mathbf{F}(P_0) = \mathbf{F}(x_0, y_0, z_0)$ with its tail at P_0. Functions of this type are called either **vector functions** or **vector-valued functions**, and if P has position vector $\mathbf{r}$ we can write $\mathbf{F}(P) = \mathbf{F}(\mathbf{r})$ to emphasize the fact that in this case a *vector* $\mathbf{F}(P)$ is associated with each position vector $\mathbf{r}$ in D. Like scalar fields, vector fields over D often depend on both position and the time t, so then $\mathbf{F} = \mathbf{F}(x, y, z, t)$, and in this case we can write $\mathbf{F}(P, t) = \mathbf{F}(\mathbf{r}, t)$. An example of a time dependent vector field is provided by the fluid velocity vector in the unsteady flow of water around a bridge support column, because there the velocity depends on both the position vector $\mathbf{r}$ in the water and the time t at which the velocity is observed. In general, in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, a time-dependent vector-valued function can be defined by setting

$$\mathbf{F}(\mathbf{r}, t) = f_1(\mathbf{r}, t)\mathbf{i} + f_2(\mathbf{r}, t)\mathbf{j} + f_3(\mathbf{r}, t)\mathbf{k}, \tag{1}$$

where the scalars $f_1(\mathbf{r}, t)$, $f_2(\mathbf{r}, t)$, and $f_3(\mathbf{r}, t)$ are the components of $\mathbf{F}(\mathbf{r}, t)$ that depend on both position and time and, at a point $\mathbf{r}_0$, translating the vector $\mathbf{F}(\mathbf{r}_0, t)$ until its tail is located at $\mathbf{r}_0$.

EXAMPLE 11.1

(a) The scalar function of position $F(x, y, z) = xyz^2$ for (x, y, z) inside the unit sphere $x^2 + y^2 + z^2 = 1$ defines a scalar field throughout the unit sphere.

(b) The vector-valued function $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (xyz - 2)\mathbf{k}$, for (x, y, z) inside the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, defines a vector field throughout the ellipsoid. ■

In order to perform calculus on vectors it is necessary to introduce the idea of a vector as a function. The simplest example of this kind is a vector $\mathbf{F}(t)$ of a single real variable t, which in terms of cartesian coordinates can be written

$$\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}, \tag{2}$$

where the components $f_1(t)$, $f_2(t)$, and $f_3(t)$ of $\mathbf{F}(t)$ are functions of t defined over some interval $a \le t \le b$. Vectors of this type are called **vector functions of a single real variable**.

v

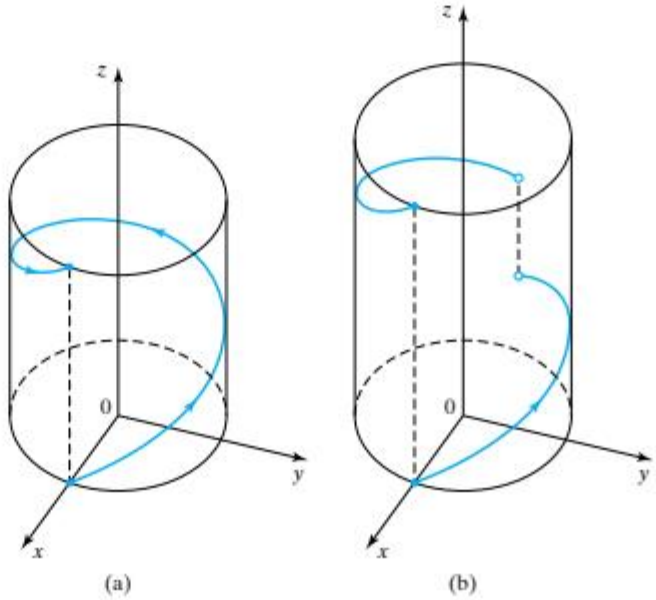


FIGURE 11.1 (a) A single turn of a helix. (b) A single turn of a broken helix.

If $\mathbf{F}(t)$ is regarded as a position vector $\mathbf{r}(t)$ in space, (2) can be interpreted as a curve in space traced out by the tip of the vector $\mathbf{r}(t)$ as t increases from a to b. Notice that a *sense* (of direction) along the curve is determined by the direction in which $\mathbf{r}(t)$ moves along the curve as t increases. When the components of $\mathbf{r}(t)$ are all continuous functions the curve, or path, traced out by the tip of $\mathbf{r}(t)$ will be an unbroken curve in space and $\mathbf{r}'(t) \neq \mathbf{0}$, though the curve will only be *smooth* if in addition to the components of $\mathbf{r}(t)$ being continuous they are also continuously differentiable for $a \leq t \leq b$, but more will be said about this later. If t is allowed to *decrease* from b to a, then the sense along the curve is *reversed*, and this fact will be important later when line integrals are considered.

EXAMPLE 11.2

(a) When interpreted as a position vector, the vector function of a single real variable $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ for $0 \leq t \leq 2\pi$ describes a single turn of the space curve called a **helix** that is shown in Fig. 11.1(a). The fact that each component of $\mathbf{r}(t)$ is both continuous and continuously differentiable and $|d\mathbf{r}/dt| \neq 0$ ensures that the helix is a smooth curve. The form of the helix can be visualized by recognizing that, as t increases, so the projection of $\mathbf{r}(t)$ onto the (x, y)-plane given by the vector $\mathbf{r}_{(x,y)}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ moves once in a counterclockwise direction around a unit circle centered on the origin, while the $\mathbf{k}$ component increases linearly with t.

(b) The vector function of a single real variable $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \theta(t + H(t - \pi))\mathbf{k}$ for $0 \leq t \leq 2\pi$, where $H(t)$ is the Heaviside unit step function, has a discontinuous $\mathbf{k}$ component, and so describes the broken helix shown in Fig. 11.1(b), where the jump in the $\mathbf{k}$ component of $\mathbf{r}(t)$ occurs at $t = \pi$. ■

It is important to recognize that because vector quantities are independent of a coordinate system, vector-valued functions and vector fields do not depend for their existence on any particular coordinate system. The choice of coordinate system used to describe vector functions is usually taken to be the one that is most appropriate for the geometry of the situation involved. So, for example, when a vector of interest depends only on distance along a straight axis and on the position on a circle centered on the axis and lying in a plane normal to the axis, it is natural to describe it in terms of the cylindrical polar coordinates (r, θ, z).

To make further progress it is necessary to generalize the related concepts of the limit and continuity of a real function of a single real variable to vector functions of a single real variable.

limits and continuity of vector functions

Limits and continuity of vector functions of a single real variable

A vector function of a single real variable $\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ is said to have $\mathbf{L}$ as its **limit** at t_0, written $\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{L}$, where $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, if

$$\lim_{t\to t_0} f_1(t) = L_1, \quad \lim_{t\to t_0} f_2(t) = L_2, \quad \text{and} \quad \lim_{t\to t_0} f_3(t) = L_3.$$

If, in addition, the vector function is defined at t_0 and $\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{F}(t_0)$, then $\mathbf{F}(t)$ is said to be **continuous** at t_0. A vector function $\mathbf{F}(t)$ that is continuous for each t in the interval $a \le t \le b$ is said to be continuous over the interval. A vector function of a single real variable that is not continuous at a point t_0 is said to be **discontinuous** at t_0.

It can be seen from the preceding definitions that the limit and continuity properties of a parametrically defined vector function can be determined by examination of the behavior of its components. So, for example, the parametrically defined vector function describing the helix in Example 11.1(a) is seen to be continuous, whereas the broken helix in Example 11.1(b) is seen to be discontinuous at one point because of the behavior of its $\mathbf{k}$ component when $t = \pi$.

The notion of a limit of a vector function of a single real variable leads naturally to the definition of the differentiability of such a function. Returning to (2) we see that if t is increased to $t + \Delta t$, the change $\Delta\mathbf{F}$ produced in $\mathbf{F}$ is

$$\begin{aligned}\Delta\mathbf{F} &= \mathbf{F}(t + \Delta t) - \mathbf{F}(t)\\ &= \{f_1(t + \Delta t)\mathbf{i} + f_2(t + \Delta t)\mathbf{j} + f_3(t + \Delta t)\mathbf{k}\} - \{f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}\},\end{aligned}$$

so

$$\frac{\Delta\mathbf{F}}{\Delta t} = \left(\frac{f_1(t + \Delta t) - f_1(t)}{\Delta t}\right)\mathbf{i} + \left(\frac{f_2(t + \Delta t) - f_2(t)}{\Delta t}\right)\mathbf{j} + \left(\frac{f_3(t + \Delta t) - f_3(t)}{\Delta t}\right)\mathbf{k}.$$

If the functions $f_1(t)$, $f_2(t)$, and $f_3(t)$ are differentiable, by letting $\Delta t \to 0$ it follows at once that the derivative of $\mathbf{F}(t)$, denoted by $d\mathbf{F}/dt$, can be expressed in terms of the derivatives of the components of $\mathbf{F}(t)$ as

$$\frac{d\mathbf{F}}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{df_2}{dt}\mathbf{j} + \frac{df_3}{dt}\mathbf{k}. \tag{3}$$

We have arrived at the following definitions of the differentiability of $\mathbf{F}(t)$ and the derivative $d\mathbf{F}/dt$.

Differentiability and the derivative of a vector function of a single real variable

The vector function of a single real variable $\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ defined over the interval $a \leq t \leq b$ is said to be **differentiable** at a point t_0 in the interval if its components are differentiable at t_0. It is said to be **differentiable over the interval** if it is differentiable at each point of the interval, and when $\mathbf{F}(t)$ is differentiable its **derivative** with respect to t is

$$\frac{d\mathbf{F}}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{df_2}{dt}\mathbf{j} + \frac{df_3}{dt}\mathbf{k}.$$

If $\mathbf{F}(t)$ is continuous over $a \leq t \leq b$, but $d\mathbf{F}/dt$ is discontinuous at a point t_0 in the interval, the derivative $d\mathbf{F}/dt$ will only be defined in the one-sided sense to the left and right of t_0 at the points $t = t_0 - 0$ and $t = t_0 + 0$.

When $d\mathbf{F}/dt$ is differentiable, the second order derivative $d^2\mathbf{F}/dt^2$ is defined as

$$\frac{d^2\mathbf{F}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{F}}{dt}\right)$$

and, in general, provided the derivatives exist,

$$\frac{d^n\mathbf{F}}{dt^n} = \frac{d}{dt}\left(\frac{d^{n-1}\mathbf{F}}{dt^{n-1}}\right), \quad \text{for } n \geq 2.$$

If $\mathbf{F}(t)$ is taken to be a differentiable position vector $\mathbf{r}(t)$, it follows from the definition of a derivative that $d\mathbf{r}/dt$ is a vector that is tangent to the point $\mathbf{r}(t)$ on the curve Γ traced out by the tip of the vector as t increases from $t = a$ to $t = b$. This situation, illustrated in Fig. 11.2, shows the relationship between $\mathbf{r}(t + \Delta t)$, $\mathbf{r}(t)$, and $\Delta\mathbf{r}$ before proceeding to the limit as $\Delta t \to 0$. It can be seen from this that as $\Delta t \to 0$, so $\Delta\mathbf{r}$ tends to coincidence with the tangent line T to the curve Γ at the point $\mathbf{r}(t)$. Furthermore, if $\mathbf{r}(t)$ is a position vector in space and t is the time, $d\mathbf{r}/dt$ is the **velocity** of the point with position vector $\mathbf{r}(t)$ and $d^2\mathbf{r}/dt^2$ is its **acceleration**.

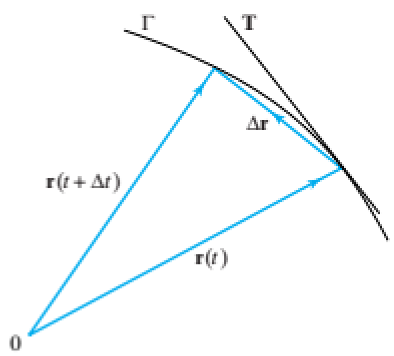


FIGURE 11.2 As $\Delta t \to 0$, so the vector $\Delta\mathbf{r}$ tends to coincidence with the tangent line T to the space curve Γ at $\mathbf{r}(t)$.

The differentiability properties of vector functions of a single real variable have been seen to be determined by the differentiability properties of the components. Consequently, as $\mathbf{F}(t)$ is a linear combination of its components in the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ directions, it follows that the rules for the differentiation of vector functions of a single real variable follow directly by applying the rules for the differentiation of a real function of a single real variable to each component in turn. The theorem that follows summarizes the basic rules for differentiation, and because vectors are independent of a coordinate system the results can be formulated without reference to a coordinate system.

THEOREM 11.1

differentiation of vector functions

Differentiation of vector functions of a single real variable Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be differentiable functions of t over some interval $a \le t \le b$, with $\mathbf{C}$ an arbitrary constant vector and c an arbitrary constant scalar. Then rules for differentiation of vector functions of a single real variable over the interval $a \le t \le b$ are:

(i) $\dfrac{d\mathbf{C}}{dt} = \mathbf{0}$ (differentiation of a constant vector)

(ii) $\dfrac{d}{dt}(c\mathbf{u}) = c\dfrac{d\mathbf{u}}{dt}$ (differentiation of a vector scaled by c)

(iii) $\dfrac{d}{dt}(\mathbf{u} \pm \mathbf{v}) = \dfrac{d\mathbf{u}}{dt} \pm \dfrac{d\mathbf{v}}{dt}$ (differentiation of a sum or difference)

(iv) $\dfrac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \dfrac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \mathbf{u} \cdot \dfrac{d\mathbf{v}}{dt}$ (differentiation of a dot product)

(v) $\dfrac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \dfrac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \dfrac{d\mathbf{v}}{dt}$ (differentiation of a cross product)

(vi) If $\mathbf{u}(t)$ is a differentiable function of t and $t = t(s)$ is a differentiable function of s, then

$$\frac{d\mathbf{u}}{ds} = \frac{d\mathbf{u}}{dt}\frac{dt}{ds}$$

or, explicitly, if $\mathbf{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$, then

$$\frac{d\mathbf{u}}{ds} = \frac{du_1}{dt}\frac{dt}{ds}\mathbf{i} + \frac{du_2}{dt}\frac{dt}{ds}\mathbf{j} + \frac{du_3}{dt}\frac{dt}{ds}\mathbf{k}$$

(the chain rule for differentiation of $\mathbf{u}(t)$).

Proof The proof of each result is straightforward and similar, so only the proof of result (iv) will be given, and for convenience the vectors $\mathbf{u}$ and $\mathbf{v}$ will be expressed in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The proofs of the remaining results will be left as exercises.

Letting $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, we have

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

We now differentiate the scalar function $\mathbf{u} \cdot \mathbf{v}$ with respect to t, using the result

$$\frac{d(u_iv_i)}{dt} = \frac{du_i}{dt}v_i + u_i\frac{dv_i}{dt}, \quad \text{for } i = 1, 2, 3,$$

which when $i = 1$ can be written

$$\frac{d(u_1v_1\mathbf{i})}{dt} = \left(\frac{du_1}{dt}\mathbf{i}\right)\cdot(v_1\mathbf{i}) + (u_1\mathbf{i})\cdot\left(\frac{dv_i}{dt}\mathbf{i}\right),$$

with corresponding results for $d(u_2v_2)/dt$ and $d(u_3v_3)/dt$. Summing the results for $d(u_iv_i)dt$ corresponding to $i = 1, 2, 3$, we arrive at result (iv), and the proof is complete. ■

EXAMPLE 11.3 Given that $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, find the first three derivatives of $\mathbf{r}$ with respect to t.

Solution $\dfrac{d\mathbf{r}}{dt} = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$, $\dfrac{d^2\mathbf{r}}{dt^2} = -\cos t\mathbf{i} - \sin t\mathbf{j}$, and $\dfrac{d^3\mathbf{r}}{dt^3} = \sin t\mathbf{i} - \cos t\mathbf{j}$. ■

EXAMPLE 11.4 Given that $\mathbf{u} = t\mathbf{i} - 2t\mathbf{j} + t^2\mathbf{k}$, $\mathbf{v} = t\mathbf{j} + 3t\mathbf{k}$ and $\mathbf{w} = t\mathbf{i} - t^2\mathbf{k}$, find

$$\frac{d}{dt}[(\mathbf{u}\cdot\mathbf{v})\mathbf{w}].$$

Solution The scalar $\mathbf{u}\cdot\mathbf{v} = -2t^2 + 3t^3$, so $(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = (3t^4 - 2t^3)\mathbf{i} - (3t^5 - 2t^4)\mathbf{k}$, and so

$$\frac{d}{dt}[(\mathbf{u}\cdot\mathbf{v})\mathbf{w}] = (12t^3 - 6t^2)\mathbf{i} - (15t^4 - 8t^3)\mathbf{k}.$$ ■

vector differential

The concept of a **vector differential** is often useful, and by analogy with the real variable calculus, if $\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$, the vector differential $d\mathbf{F}$ is defined as

$$d\mathbf{F} = \left(\frac{df_1}{dt}\mathbf{i} + \frac{df_2}{dt}\mathbf{j} + \frac{df_3}{dt}\mathbf{k}\right)dt. \tag{4}$$

A simple and useful application of the vector differential is to the element of arc length along a space curve Γ defined by the position vector $\mathbf{r}(t) = x_1(t)\mathbf{i} + x_2(t)\mathbf{j} + x_3(t)\mathbf{k}$ for $t \geq t_0$. If s is the arc length measured along Γ from some fixed point, then by applying Pythagoras' theorem to the differential elements

$$dx_1 = \frac{dx_1}{dt}dt, \quad dx_2 = \frac{dx_2}{dt}dt, \quad \text{and} \quad dx_3 = \frac{dx_3}{dt}dt,$$

it is seen from Fig. 11.3 that the differential element of arc length ds along Γ is given by

$$ds = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2} dt, \tag{5}$$

and so

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2}. \tag{6}$$

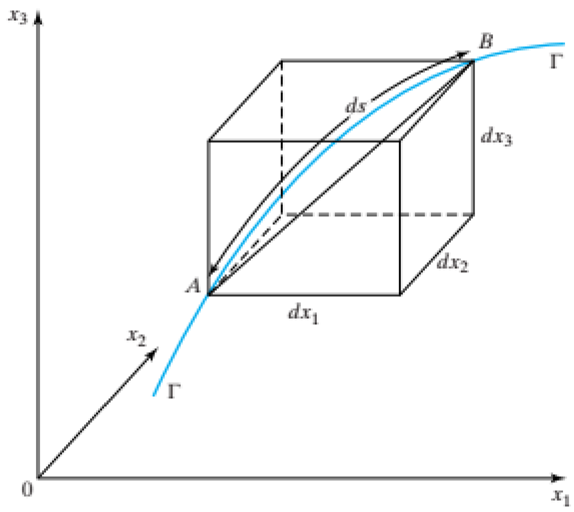


FIGURE 11.3 The geometrical relationship between the differentials ds, dx_1, dx_2, and dx_3.

This result shows that when t is the time and $\mathbf{r}(t)$ is a position vector in space, $\frac{ds}{dt} = |\frac{d\mathbf{r}}{dt}|$ is the *speed* with which the tip of position vector $\mathbf{r}(t)$ traces out a space curve Γ.

tangent vector

Examination of Fig. 11.2 and consideration of the definition of $d\mathbf{r}/dt$ shows that the *unit tangent vector* $\mathbf{T}$ along Γ as a function of t is given by

$$\mathbf{T} = \frac{d\mathbf{r}}{dt} \Big/ \left|\frac{d\mathbf{r}}{dt}\right|, \tag{7}$$

and as $ds/dt = |d\mathbf{r}/dt|$, this can be rewritten in the form

$$\frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\mathbf{T}. \tag{8}$$

EXAMPLE 11.5

If $\mathbf{r}(t)$ is a position vector and t is the time, find the velocity, speed, and acceleration of a particle with position vector $\mathbf{r}(t) = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$, where a and ω are constants, and interpret the results.

Solution We have $|\mathbf{r}(t)| = (a^2\cos^2\omega t + a^2\sin^2\omega t)^{1/2} = a$, so as the motion is two-dimensional in the plane containing $\mathbf{i}$ and $\mathbf{j}$, it takes place in a circle of radius a with its center at the origin of the coordinate system. Differentiation of $\mathbf{r}(t)$ gives

$$\frac{d\mathbf{r}}{dt} = -\omega a\sin\omega t\mathbf{i} + \omega a\cos\omega t\mathbf{j} \quad \text{and} \quad \frac{d^2\mathbf{r}}{dt^2} = -\omega^2 a\cos\omega t\mathbf{i} - \omega^2 a\sin\omega t\mathbf{j}.$$

The speed $ds/dt = |d\mathbf{r}/dt| = \omega a$ is constant, and the velocity $d\mathbf{r}/dt$ is seen to be tangential to the circular path, because $\mathbf{r}\cdot(d\mathbf{r}/dt) = 0$. The acceleration $d^2\mathbf{r}/dt^2$ is proportional to $\mathbf{r}$, but oppositely directed, so it is always directed toward the origin. Figure 11.4 illustrates the relationship between the velocity and acceleration as the particle moves around the circle at a constant speed ωa.

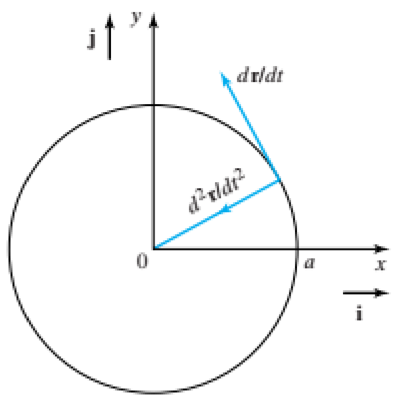


FIGURE 11.4 Uniform motion around the circle $\mathbf{r} = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$.

In dealing with the geometry of a space curve Γ, it is often convenient to specify the position vector $\mathbf{r}$ of a point on the curve in terms of the arc length s measured along the curve from some fixed point, so that then $\mathbf{r} = \mathbf{r}(s)$. When $\mathbf{r}$ is expressed in this manner the equation $\mathbf{r} = \mathbf{r}(s)$ is called the **intrinsic equation** of Γ. In addition to the unit tangent $\mathbf{T}$ at any point $\mathbf{r} = \mathbf{r}(s)$ of Γ, two other important unit vectors $\mathbf{N}$ and $\mathbf{B}$ can also be defined at that point.

intrinsic vector equation

To arrive at definitions of vectors $\mathbf{N}$ and $\mathbf{B}$, we start from the fact that as $\mathbf{T}$ is a unit vector $\mathbf{T}\cdot\mathbf{T} = 1$, so differentiating with respect to t and using Theorem 11.1(iv) we have

$$\frac{d\mathbf{T}}{ds}\cdot\mathbf{T} + \mathbf{T}\cdot\frac{d\mathbf{T}}{ds} = 0.$$

However, as the scalar product is commutative, this last result is seen to be equivalent to

$$\mathbf{T}\cdot\frac{d\mathbf{T}}{ds} = 0,$$

showing that $\mathbf{T}$ and $d\mathbf{T}/ds$ are orthogonal. The unit vector $\mathbf{N}$ in the direction of $d\mathbf{T}/ds$ at a point $\mathbf{r} = \mathbf{r}(s)$ on Γ is called the **principal normal** to Γ at $\mathbf{r}(s)$, and so

$$\mathbf{N} = \frac{d\mathbf{T}}{ds}\Big/\left|\frac{d\mathbf{T}}{ds}\right| \quad \text{for} \quad \left|\frac{d\mathbf{T}}{ds}\right| \neq 0. \tag{9}$$

When the connection between $d\mathbf{T}/ds$ and $\mathbf{N}$ at a point $\mathbf{r} = \mathbf{r}(s)$ on Γ is written in the form

$$\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{N}, \tag{10}$$

curvature, normal and binormal

the nonnegative number $\kappa(s)$ is called the **curvature** of the curve Γ at $\mathbf{r} = \mathbf{r}(s)$, and $\rho(s) = 1/\kappa(s)$ is called the **radius of curvature** of the curve Γ at $\mathbf{r} = \mathbf{r}(s)$. As $\mathbf{N}$ is a unit vector, taking the modulus of (10) gives

$$\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|. \tag{11}$$

In the case of a smooth plane curve Γ, the circle of curvature at a point P on Γ is tangent to Γ at P with radius $\rho = 1/\kappa$, and such that its center lies on the concave side of Γ.

If the curvature is required in terms of the parameter t, the relationship between $\kappa(s)$ and $\kappa(t)$ follows from the chain rule

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt},$$

showing that

$$\left|\frac{d\mathbf{T}}{dt}\right| = \kappa(t)\left|\frac{ds}{dt}\right|. \tag{12}$$

As $dt/ds = 1/(ds/dt) = 1/|d\mathbf{r}/dt|$, this last result can be written in the convenient form

$$\kappa(t) = \left|\frac{d\mathbf{T}}{dt}\right| \bigg/ \left|\frac{d\mathbf{r}}{dt}\right|. \tag{13}$$

Finally, the vector $\mathbf{B}$, defined as

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}, \tag{14}$$

is called the **unit binormal** to the curve Γ at $\mathbf{r} = \mathbf{r}(s)$. The three unit vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at a point $\mathbf{r} = \mathbf{r}(s)$ on the space curve Γ form a *triad* of mutually orthogonal unit vectors whose orientation depends on the location of the point on Γ. When studying the geometry of space curves it proves to be more convenient to use the unit vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$, whose orientation depends on the point on the curve under consideration, than a fixed reference system of unit vectors such as $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. ■

EXAMPLE 11.6 Show that the straight line $\mathbf{r}(t) = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k} + \mathbf{C}$, with a, b, and c scalar constants and $\mathbf{C}$ a constant vector, has an infinite radius of curvature at every point.

Solution Differentiation shows that $|d\mathbf{r}/dt| = (a^2 + b^2 + c^2)^{1/2} \neq 0$, and the tangent vector $\mathbf{T} = d\mathbf{r}/dt/|d\mathbf{r}/dt| = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})/(a^2 + b^2 + c^2)^{1/2}$, so $d\mathbf{T}/dt \equiv 0$, and $\mathbf{N}$ has to be chosen arbitrarily except for $\mathbf{T} \cdot \mathbf{N} = 0$. Consequently, from (13) $\kappa(t) \equiv 0$, and so the radius of curvature $\rho(t) = 1/\kappa(t) = \infty$ for all t. ■

EXAMPLE 11.7 Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and $\kappa(t)$ for the helix $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$.

Solution From $ds/dt = |d\mathbf{r}/dt|$ we have

$$ds/dt = [(-a\sin t)^2 + (a\cos t)^2 + b^2]^{1/2} = (a^2 + b^2)^{1/2},$$

and so

$$\mathbf{T} = \frac{d\mathbf{r}}{dt} \bigg/ \frac{ds}{dt} = \frac{1}{(a^2+b^2)^{1/2}}(-a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}).$$

By definition,

$$\mathbf{N} = \frac{d\mathbf{T}}{ds} \bigg/ \left|\frac{d\mathbf{T}}{ds}\right| = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} \bigg/ \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| = \frac{d\mathbf{T}}{dt} \bigg/ \left|\frac{d\mathbf{T}}{dt}\right| = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

and

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{(a^2+b^2)^{1/2}}(b\sin t\mathbf{i} - b\cos t\mathbf{j} + a\mathbf{k}).$$

A simple calculation shows that $|d\mathbf{T}/dt| = a/(a^2+b^2)^{1/2}$, $|d\mathbf{r}/dt| = (a^2+b^2)^{1/2}$, so it follows from (13) that the curvature $\kappa(t) = a/(a^2+b^2)$ for all t. This is to be expected, because the uniform shape of the helix implies that the curvature, and hence the radius of curvature, are constant along the helix. ■

Summary

Scalar and vector fields have been introduced, vector functions of a single real variable have been defined, and their differentiability properties have been derived. Applications to dynamics and the geometry of space curves have been made.

EXERCISES 11.1

In Exercises 1 through 6 find the first and second derivatives of the function and their values at the given value of t.

1. $\mathbf{r} = t\sin t\mathbf{i} + t\cos t\mathbf{j} + t^2\mathbf{k}$, $\quad t = \pi/2$.
2. $\mathbf{r} = (1+t^2)\mathbf{i} + e^{-2t}\mathbf{j} + \sqrt{t}\mathbf{k}$, $\quad t = 1$.
3. $\mathbf{r} = (2-\cos^2 t)\mathbf{i} + \sin^2 t\mathbf{j} + (\pi - t)\mathbf{k}$, $\quad t = \pi/4$.
4. $\mathbf{r} = \ln(1+t)\mathbf{i} + \ln(1+r^2)\mathbf{j} + e^{3t}\mathbf{k}$, $\quad t = 0$.
5. $\mathbf{r} = (t - \sin t)\mathbf{i} + (1-\cos t)\mathbf{j}$, $\quad t = \pi/2 \quad$ (a **cycloid**).

   Notice that $\mathbf{r}$ is arbitrarily many times differentiable, yet the cycloid has cusps for $t = n\pi$.
6. $\mathbf{r} = 4\cos t\mathbf{i} + 3\sin t\mathbf{j} + 2t\mathbf{k}$, $\quad t = \pi/4 \quad$ (an **elliptical "helix"**).
7. Prove result (iii) in Theorem 11.1 by expressing the vectors in terms of their cartesian components.
8. Prove result (v) in Theorem 11.1 by expressing the vectors in terms of their cartesian components.
9. Given that $\mathbf{r} = t\mathbf{i} + 3t^2\mathbf{j} - (t-1)\mathbf{k}$ and $t = \ln(1+s^2)$, use result (vi) in Theorem 11.1 to find $d\mathbf{r}/ds$.
10. Given that $\mathbf{r} = \sin t\mathbf{i} + \cos t\mathbf{j} + \tan t\mathbf{k}$ and $t = 2 + s^2$, use result (vi) in Theorem 11.1 to find $d\mathbf{r}/ds$.
11. A particle has a position vector at time t given by

$$\mathbf{r} = t^2\mathbf{i} + 4\cos 2t\mathbf{j} + 3\sin 2t\mathbf{k}.$$

    Find the component of its velocity in the direction $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ at time t.
12. A particle has a position vector at time t given by

$$\mathbf{r} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + (t^2 - 2)\mathbf{k}.$$

    Find the component of its velocity in the direction $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ at time t.
13. If $\phi(t)$ is a differentiable function of t and $\mathbf{u}(t)$ is a differentiable parametrically defined function of t, prove that

$$\frac{d}{dt}(\phi\mathbf{u}) = \phi\frac{d\mathbf{u}}{dt} + \frac{d\phi}{dt}\mathbf{u}.$$

14. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are differentiable parametrically defined functions of t, prove that

$$\frac{d}{dt}(\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})) = \mathbf{u}\cdot\left(\mathbf{v}\times\frac{d\mathbf{w}}{dt}\right) + \mathbf{u}\cdot\left(\frac{d\mathbf{v}}{dt}\times\mathbf{w}\right) + \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}),$$

    where the order in the products must be preserved.
15. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are differentiable parametrically defined functions of t, prove that

$$\frac{d}{dt}(\mathbf{u}\times(\mathbf{v}\times\mathbf{w})) = \mathbf{u}\times\left(\mathbf{v}\times\frac{d\mathbf{w}}{dt}\right) + \mathbf{u}\times\left(\frac{d\mathbf{v}}{dt}\times\mathbf{w}\right) + \frac{d\mathbf{u}}{dt}\times(\mathbf{v}\times\mathbf{w}),$$

    where the order in the products must be preserved.

16. If $\mathbf{u}$ is a differentiable parametrically defined function of t, prove that

$$\frac{d\mathbf{u}}{dt}\times\frac{d}{dt}\left(\frac{d\mathbf{u}}{dt}\times\frac{d^2\mathbf{u}}{dt^2}\right)=\frac{d\mathbf{u}}{dt}\left(\frac{d\mathbf{u}}{dt}\cdot\frac{d^3\mathbf{u}}{dt^3}\right)-\frac{d^3\mathbf{u}}{dt^3}\left(\frac{d\mathbf{u}}{dt}\right)^2.$$

17. If $\mathbf{u}$ is a differentiable parametrically defined function of t, prove that

$$\frac{d\mathbf{u}}{dt}\times\frac{d}{dt}\left(\mathbf{u}\times\frac{d\mathbf{u}}{dt}\right)=\mathbf{u}\left(\frac{d\mathbf{u}}{dt}\cdot\frac{d^2\mathbf{u}}{dt^2}\right)-\frac{d^2\mathbf{u}}{dt^2}\left(\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}\right).$$

18. Given that $\phi(t)=t^2\cos t$ and $\mathbf{u}=\sin t\mathbf{i}+2\cos t\mathbf{j}+(1+t^2)^{1/2}\mathbf{k}$, use the result of Exercise 13 to find $\frac{d}{dt}(\phi\mathbf{u})$, and confirm the result by direct differentiation of $\phi\mathbf{u}$ with respect to t.

19. Given that $\mathbf{u}=2t\mathbf{i}-t^2\mathbf{j}+\mathbf{k}$, $\mathbf{v}=2\mathbf{i}+3t\mathbf{j}+t\mathbf{k}$, and $\mathbf{w}=t\mathbf{i}+2t\mathbf{j}-t\mathbf{k}$, use the result of Exercise 14 to find $\frac{d}{dt}(\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}))$. Confirm the result by finding $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ and differentiating the result with respect to t.

20. Given that $\mathbf{u}=t\mathbf{i}-t\mathbf{j}+t^2\mathbf{k}$, $\mathbf{v}=-t\mathbf{i}+2t\mathbf{j}-t^2\mathbf{k}$, and $\mathbf{w}=2t\mathbf{i}-2t\mathbf{j}+t\mathbf{k}$, use the result of Exercise 15 to find $\frac{d}{dt}(\mathbf{u}\times(\mathbf{v}\times\mathbf{w}))$. Confirm the result by finding $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ and differentiating the result with respect to t.

21. Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ as functions of t for the helix $\mathbf{r}(t)=a\cos\omega t\mathbf{i}+a\sin\omega t\mathbf{j}+bt\mathbf{k}$.

22. By differentiating $\mathbf{B}=\mathbf{T}\times\mathbf{N}$ with respect to s, show that

$$\frac{d\mathbf{B}}{ds}=\mathbf{T}\times\frac{d\mathbf{N}}{ds},$$

and then by forming the product $\mathbf{N}\times d\mathbf{B}/ds$, show that

$$\mathbf{N}\times\frac{d\mathbf{B}}{ds}=\mathbf{0}.$$

Introduce a constant of proportionality called the **torsion** of the curve Γ at P, which by convention is denoted by $-\tau$, and deduce from this last result that

$$\frac{d\mathbf{B}}{ds}=-\tau\mathbf{N}.$$

Finally, by differentiating $\mathbf{N}=\mathbf{B}\times\mathbf{T}$ with respect to s show that

$$\frac{d\mathbf{N}}{ds}=\tau\mathbf{B}-\kappa\mathbf{T}.$$

The three equations relating the derivatives of $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ with respect to s to $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, and τ found earlier, namely,

$$\frac{d\mathbf{T}}{ds}=\kappa\mathbf{N},\quad \frac{d\mathbf{N}}{ds}=\tau\mathbf{B}-\kappa\mathbf{T},\quad\text{and}\quad \frac{d\mathbf{B}}{ds}=-\tau\mathbf{N},$$

are called the **Frenet–Serret equations**, and they are fundamental to the study of the differential geometry of space curves.

11.2 Integration of Scalar and Vector Functions of a Single Real Variable

As with real functions of a single real variable, a differentiable vector function of a single real variable $\mathbf{F}(t)$ will be called an **antiderivative** of the vector function $\mathbf{f}(t)$ on some interval $a<t<b$ if at each point of the interval $d\mathbf{F}(t)/dt=\mathbf{f}(t)$. Because differentiation of a vector constant yields the null vector $\mathbf{0}$, an antiderivative of $\mathbf{f}$ is only determined up to an arbitrary additive vector constant $\mathbf{C}$. An **indefinite integral** of $\mathbf{f}$ is any antiderivative of $\mathbf{f}$ to which has been added an arbitrary vector constant.

indefinite and definite integrals of vector functions of a single real variable

Indefinite and definite integrals of a vector function of a single real variable

If $\mathbf{F}(t)$ is any antiderivative of $\mathbf{f}(t)$, then an **indefinite integral** of the function $\mathbf{f}$ with respect to t, written $\int\mathbf{f}(t)dt$, is

$$\int\mathbf{f}(t)dt=\mathbf{F}(t)+\mathbf{C},$$

where $\mathbf{C}$ is an arbitrary vector constant.

If $\mathbf{f}(t)=f_1(t)\mathbf{i}+f_2(t)\mathbf{j}+f_3(t)\mathbf{k}$, the indefinite integral of $\mathbf{f}(t)$ is determined by integrating each component of $\mathbf{f}(t)$ with respect to t and combining the results to give

$$\int f_1(t)dt\mathbf{i} + \int f_2(t)dt\mathbf{j} + \int f_3(t)dt\mathbf{k} = \mathbf{F}(t) + \mathbf{C}.$$

The **definite integral** of $\mathbf{f}(t)$ over the interval $a \le t \le b$ is defined as

$$\int_a^b \mathbf{f}(t)dt = \int_a^b f_1(t)dt\mathbf{i} + \int_a^b f_2(t)dt\mathbf{j} + \int_a^b f_3(t)dt\mathbf{k}.$$

EXAMPLE 11.8

Given that $\mathbf{f}(t) = \sin t\mathbf{i} + (1 - t^2)\mathbf{j} + e^{-t}\mathbf{k}$, find

$$\text{(a)} \int \mathbf{f}(t)dt \quad \text{and} \quad \text{(b)} \int_0^2 \mathbf{f}(t)dt.$$

Solution

(a)

$$\int \mathbf{f}(t)dt = \int \sin t\, dt\mathbf{i} + \int (1 - t^2)dt\mathbf{j} + \int e^{-t}\, dt\mathbf{k}$$
$$= -\cos t\mathbf{i} + \left(t - \frac{1}{3}t^3\right)\mathbf{j} - e^{-t}\mathbf{k} + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k},$$

where c_1, c_2, and c_3 are arbitrary real constants, so

$$\int \mathbf{f}(t)dt = -\cos t\mathbf{i} + \left(t - \frac{1}{3}t^3\right)\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary vector constant.

(b)

$$\int_0^2 \mathbf{f}(t)dt = \int_0^2 \sin t dt\mathbf{i} + \int_0^2 (1 - t^2)dt\mathbf{j} + \int_0^2 e^{-t}\, dt\mathbf{k}$$
$$= (1 - \cos 2)\mathbf{i} - \frac{2}{3}\mathbf{j} + (1 - e^{-2})\mathbf{k}.$$

It is sometimes necessary to find the length of arc between two points on a curve defined by a vector function of a single real variable. This can be accomplished by making use of result (6), which showed that the rate of change of distance s with respect to t along the curve Γ defined by

$$\mathbf{r}(t) = x_1(t)\mathbf{i} + x_2(t)\mathbf{j} + x_3(t)\mathbf{k}$$

is given by

$$\frac{ds}{dt} = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2}.$$

arc length along a space curve

Consequently, if the length of arc $s = s(t_2) - s(t_1)$ between the points corresponding to $t = t_1$ and $t = t_2$ is required, where $t_2 > t_1$, integration of this result gives

$$\int_{t_1}^{t_2} \frac{ds}{dt}dt = \int_{t_1}^{t_2} \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2} dt,$$

so the required arc length is given by the definite integral

$$s = s(t_2) - s(t_1) = \int_{t_1}^{t_2} \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2 \right]^{1/2} dt. \qquad (15)$$

EXAMPLE 11.9

Find the length of arc along the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \alpha t\mathbf{k}$ between the points corresponding to $t = 0$ and $t = 2\pi$, where α is a scalar constant.

Solution Making the identifications $x_1(t) = \cos t$, $x_2(t) = \sin t$, $x_3(t) = \alpha t$, $t_1 = 0$, and $t_2 = 2\pi$, and substituting into (15) gives

$$\begin{aligned} s &= \int_0^{2\pi} [(-\sin t)^2 + (\cos t)^2 + \alpha^2]^{1/2}\, dt \\ &= \sqrt{1+\alpha^2} \int_0^{2\pi} dt = 2\pi\sqrt{1+\alpha^2}. \end{aligned}$$

When $\alpha = 0$ the helix reduces to a circle of unit radius, and as expected s then becomes the circumference 2π of a unit circle. ■

Let the vector $\mathbf{F}(x, y, z)$ be defined along a piecewise smooth space curve Γ along which the arc length is s, and let Γ extend from the point $\mathbf{r}_1$ at which $s = s_1$ to the point $\mathbf{r}_2$ at which $s = s_2$. Then, if $\mathbf{T}(s)$ is the unit tangent vector to Γ at arc length s, an expression of the form

$$I = \int_{s_1}^{s_2} \mathbf{F} \cdot \mathbf{T}\, ds$$

scalar line integrals

is called a **line integral** of **F**, or more precisely, the **scalar line integral** of **F** along the space curve Γ. It follows from (8) that $\mathbf{T}ds = d\mathbf{r}$, so the line integral of **F** along Γ can be written in the simpler form

$$I = \int_{s_1}^{s_2} \mathbf{F} \cdot d\mathbf{r}. \qquad (16)$$

Integrals of this type have many applications, two of the most important of which are described in what follows. The first application is to mechanics, where when a constant force **F** moves its point of application a distance d along a straight line L, the **work** that is done by the force is $W = f_L d$, where f_L is the component of **F** along the line L. To find the work done by a variable force $\mathbf{F}(t)$ as it moves its point of application along a parametrically defined curve Γ, it is necessary to generalize this simple result by appealing to the notion of a line integral along the space curve Γ.

If the vector differential along Γ is denoted by $d\mathbf{r}$, its length $|d\mathbf{r}| = dr$, so the unit vector **T** in the direction $d\mathbf{r}$ will be $\mathbf{T} = d\mathbf{r}/dr$. Consequently, the component of force **F** in the direction of $d\mathbf{r}$ is given by $\mathbf{F} \cdot \mathbf{T} = (\mathbf{F} \cdot d\mathbf{r})/dr$, so the element of work dW performed by the force in moving its point of application along $d\mathbf{r}$ will be

$$dW = \mathbf{F} \cdot \left(\frac{d\mathbf{r}}{dr}\right) dr = \mathbf{F} \cdot d\mathbf{r}.$$

Integration of this result shows the work performed by the force in moving its point of application along Γ from $\mathbf{r} = \mathbf{r}_1$ to $\mathbf{r} = \mathbf{r}_2$, corresponding to $s = s_1$ and $s = s_2$, respectively, is given by the line integral

$$W = \int_{s_1}^{s_2} \mathbf{F} \cdot d\mathbf{r}. \tag{17}$$

When $\mathbf{r} = \mathbf{r}(t)$ is known as a function of t, but t is not the arc length s along Γ, and integration is between $\mathbf{r} = \mathbf{r}(t_1)$ and $\mathbf{r} = \mathbf{r}(t_2)$, $d\mathbf{r} = (d\mathbf{r}/dt)dt$ and (17) becomes

$$W = \int_{t(s_1)}^{t(s_2)} \mathbf{F}(\mathbf{r}(t)) \cdot (d\mathbf{r}/dt)dt. \tag{18}$$

Integrals of this type arise when particles move in a gravitational field or a charged particle moves in an electric field. The sign of W depends on the direction of integration, so reversing its direction changes the sign of W. Work is done by the vector field $\mathbf{F}$ when W is positive, and work is recovered from the field when W is negative.

For the second example we consider the case of fluid mechanics and identify $\mathbf{F}$ with the fluid velocity vector $\mathbf{q}$. In this case a line integral of the form (16) is called the **flow** of the fluid along Γ, because $d\mathbf{r} = (d\mathbf{r}/ds)ds = \mathbf{T}ds$, where $\mathbf{T}$ is the unit tangent along Γ, so that $\mathbf{q} \cdot \mathbf{T}$ is the component of the flow along Γ. The **circulation** k of fluid is defined as the flow around a *closed* curve Γ, so it is given by

circulation and irrotational flow

$$k = \oint_\Gamma \mathbf{q} \cdot d\mathbf{r} = \oint_\Gamma \mathbf{q} \cdot \mathbf{T}\, ds, \tag{19}$$

where the symbol $\oint_\Gamma$ is used to indicate that the line integral of $\mathbf{q} \cdot d\mathbf{r}$ is taken *once* around the closed curve Γ.

In fluid mechanics the circulation k describes an important characteristic of the fluid motion, and it can be seen from (19) that reversing the direction of integration around Γ reverses the sign of $\mathbf{T}$, and so leads to a reversal of the sign of the circulation. The fundamental class of fluid flow in which there is zero circulation around every simple closed curve Γ, so that $k \equiv 0$, is called **irrotational** flow.

In general, the line integral (16) depends not only on $\mathbf{F}$ and the end points of integration, but also on the path Γ along which the integral is evaluated. The method of evaluating line integrals, and the fact that they usually depend on the path, is illustrated in the next example.

EXAMPLE 11.10

Find the line integral of $\mathbf{F} = -yz^2\mathbf{i} + xz^2\mathbf{j} + yz\mathbf{k}$ (a) along the helix Γ given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$ from $t = 0$ to $t = 2\pi$, and (b) along the straight line path γ joining the points $\mathbf{r}(0)$ to $\mathbf{r}(2\pi)$.

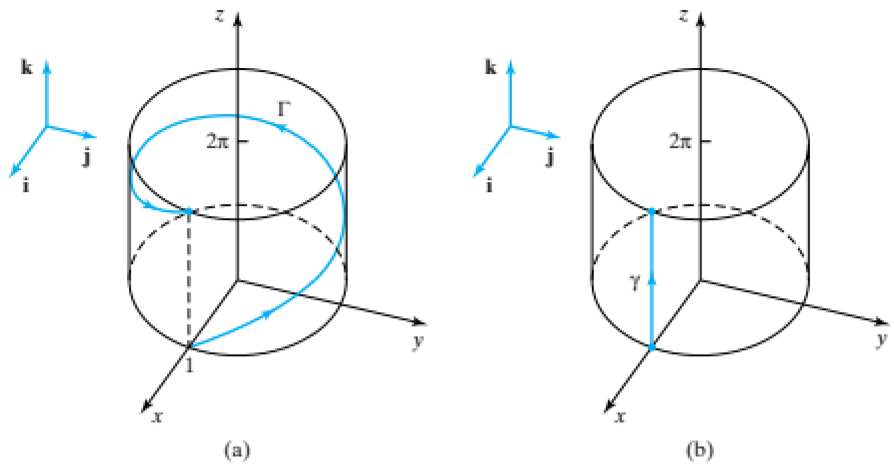


FIGURE 11.5 (a) The helix Γ. (b) The straight line path γ.

Solution

(a) The helix Γ is shown in Fig. 11.5(a).
Differentiation of $\mathbf{r}(t)$ gives

$$\frac{d\mathbf{r}}{dt} = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k},$$

but on the helix $x = \cos t$, $y = \sin t$, and $z = t$, so in the line integral along Γ the general vector-valued function $\mathbf{F}$ becomes the vector function of the single real variable t given by $\mathbf{F}(t) = -t^2 \sin t\mathbf{i} + t^2 \cos t\mathbf{j} + t \sin t\mathbf{k}$. As a result,

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= (-t^2 \sin t\mathbf{i} + t^2 \cos t\mathbf{j} + t \sin t\mathbf{k}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k})dt \\ &= (t^2 + t \sin t)dt,\end{aligned}$$

and so the required line integral is

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (t^2 + t \sin t)dt = \frac{8}{3}\pi^3 - 2\pi.$$

(b) The straight line path γ shown in Fig. 11.5(b) joins the points $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}(2\pi) = \mathbf{i} + 2\pi\mathbf{k}$, so in terms of the parameter t its vector equation can be written $\mathbf{r}(t) = \mathbf{i} + t\mathbf{k}$ with $0 \le t \le 2\pi$. This shows that on the path γ we have $x = 1$, $y = 0$, and $z = t$, and $d\mathbf{r} = dt\mathbf{k}$.

Consequently, on γ the vector-valued function $\mathbf{F}$ becomes $\mathbf{F} = t^2\mathbf{j}$, and so

$$\mathbf{F} \cdot d\mathbf{r} = t^2\mathbf{j} \cdot (dt\mathbf{k}) = 0,$$

showing that

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = 0.$$

In the next section, after the introduction of the *gradient* of a function, we will find a condition to be satisfied by $\mathbf{F}$ in order that the line integral in (16) is independent of the path Γ, and so depends only on $\mathbf{F}$ and the end points of the integration.

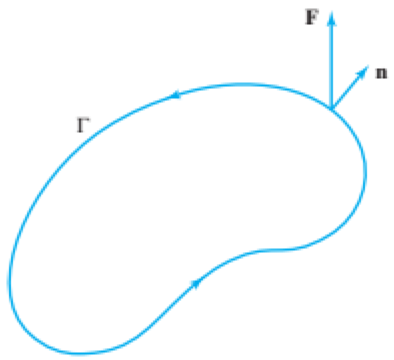


FIGURE 11.6 $\Phi_\Gamma = \oint_\Gamma \mathbf{F} \cdot \mathbf{n}\, ds$ is the flux of $\mathbf{F}$ across Γ.

As a final example of an application of line integrals we determine the *flux* of a vector $\mathbf{F}(x, y)$ across a closed two-dimensional smooth curve Γ in the (x, y)-plane. If $\mathbf{n}$ is a unit vector normal to Γ that is directed *outward* from Γ, as shown in Fig. 11.6, the **flux** Φ_Γ across the curve Γ is defined as the line integral

$$\Phi_\Gamma = \int_\Gamma \mathbf{F} \cdot \mathbf{n}\, ds,$$

the flux of a vector across a plane curve

where s is the arc length around Γ and integration is in the *counterclockwise* sense around Γ. As $\mathbf{F} \cdot \mathbf{n}$ is the component of $\mathbf{F}$ in the direction of the outward drawn normal to Γ, the flux Φ_Γ is seen to measure the total amount of the normal component of $\mathbf{F}$ that crosses the curve Γ.

For a physical illustration of the meaning of flux, let us consider a long block of metal with its axis in the z-direction in which there is a steady-state temperature distribution that is only a function of x and y. This means that the temperature distribution is the same in every plane $z =$ constant. Let us now consider a cylindrical region in the block of unit height and cross-section Γ with its axis in the z-direction. Then if $\mathbf{F}$ is identified with a heat flow vector $\mathbf{h}(x, y)$, the flux Φ_Γ is the amount of the heat that crosses the curved walls of this cylinder in Fig. 11.7 in a unit time. If $\Phi_\Gamma > 0$ there is a net *outflow* of heat from the region bounded by Γ, and if $\Phi_\Gamma < 0$ there is a net *inflow* of heat into the region. When $\Phi_\Gamma = 0$ the amount of heat in the region remains constant.

In two space dimensions it is important to recognize the difference between the circulation and flux of $\mathbf{F}$ in relation to the curve Γ. Whereas the determination of the *circulation* of $\mathbf{F}$ involves the line integral of the component of $\mathbf{F}$ *along* the tangent to curve Γ with respect to the arc length s, the *flux* of $\mathbf{F}$ involves the line integral of the component of $\mathbf{F}$ normal to (*across*) the curve Γ with respect to the arc length.

To determine the flux we proceed as follows. Let $\mathbf{F}(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}$ and Γ have the equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then, as integration around Γ is in the counterclockwise sense, we see from Fig. 11.6 that if $\mathbf{T}$ is the unit tangent to Γ, then $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. As $\mathbf{T} = (dx/ds)\mathbf{i} + (dy/ds)\mathbf{j}$, it follows that

$$\begin{aligned}\mathbf{n} &= \mathbf{T} \times \mathbf{k} = [(dx/ds)\mathbf{i} + (dy/ds)\mathbf{j}] \times \mathbf{k}\\ &= (dy/ds)\mathbf{i} - (dx/ds)\mathbf{j},\end{aligned}$$

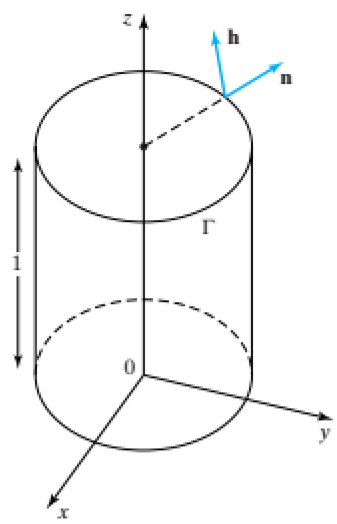


FIGURE 11.7 A cylinder of unit height and cross-section Γ with its axis in the z-direction.

and so

$$\Phi_\Gamma = \int_\Gamma \mathbf{F}\cdot\mathbf{n}\,ds = \int_\Gamma (f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j})\cdot((dy/ds)\mathbf{i} - (dx/ds)\mathbf{j})ds$$
$$= \int_\Gamma f_1(x,y)dy - f_2(x,y)dx.$$

EXAMPLE 11.11 Find the flux of $\mathbf{F} = (2x+y)\mathbf{i} + (y-x)\mathbf{j}$ across the ellipse with the equation $x^2/a^2 + y^2/b^2 = 1$.

Solution By setting $x = a\cos t$ and $y = b\sin t$ and restricting t to the interval $0 \le t \le 2\pi$, the ellipse is traversed once in the counterclockwise sense as required. As $dx = -a\sin t\,dt$ and $dy = b\cos t\,dt$, substitution into the expression for Φ_Γ gives

$$\Phi_\Gamma = \int_0^{2\pi} [(2a\cos t + b\sin t)b\cos t - (b\sin t - a\cos t)(-a\sin t)]\,dt = 3ab\pi. \quad ■$$

Finally we define a different integral called a *vector line integral* of $\mathbf{F}$. To do this we let a curve Γ have the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{for } a \le t \le b$$

and introduce a general vector function $\mathbf{F} = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$ defined along the curve Γ. Then the **vector line integral** of $\mathbf{F}$ along Γ from $t = a$ to $t = b$ is defined as

$$\int_a^b \mathbf{F}\,dt = \mathbf{i}\int_a^b F_1(t)dt + \mathbf{j}\int_a^b F_2(t)dt + \mathbf{k}\int_a^b F_3(t)dt, \tag{20}$$

where $F_i(t) = F_i(x(t), y(t), z(t))$, for $i = 1, 2, 3$.

EXAMPLE 11.12

Find the vector line integral of the vector function $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + z\mathbf{k}$ along the curve $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + t\mathbf{k}$ over the interval $0 \le t \le \pi$.

Solution

$$\int_0^\pi \mathbf{F}\,dt = \mathbf{i}\int_0^\pi at\cos t\,dt + \mathbf{j}\int_0^\pi at\sin t\,dt + \mathbf{k}\int_0^\pi t\,dt = -2a\mathbf{i} + \pi a\mathbf{j} + \frac{1}{2}\pi^2\mathbf{k}.$$

Summary

Indefinite and definite integrals of vector functions of a single real variable have been defined and illustrated by example. The scalar line integral of a vector $\mathbf{F}(x, y, z)$ has been defined and its application illustrated by considering the work done by a force as it moves along a space curve between two fixed points. The line integral has also been applied to fluid flow and used to define the circulation of the fluid, and the related concept of an irrotational flow for which the circulation around any closed curve in the fluid is zero. Finally, the flux of a vector across a plane curve has been defined.

EXERCISES 11.2

In Exercises 1 through 4 find the required indefinite and definite integrals.

1. (a) $\int (t\sin t\mathbf{i} + 3t^2\mathbf{j} - 3t\mathbf{k})dt.$

   (b) $\int_0^2 (\ln(1+3t)\mathbf{i} + (t^3 - 2t)\mathbf{j} + te^t\mathbf{k})dt.$

2. (a) $\int (\cosh^2 t\mathbf{i} + 2\sin^2 2t\mathbf{j} + \mathbf{k})dt.$

   (b) $\int_0^2 ((1+t^2)^{-1}\mathbf{i} - t\sin t\mathbf{j} - (1-3t^2)\mathbf{k})dt.$

3. (a) $\int (\cos^2 3t\mathbf{i} + \sin^2 t\mathbf{j} + t\mathbf{k})dt.$

   (b) $\int_0^\pi ((1+3t^2)\mathbf{i} + \cos 4t\mathbf{j} + \sin 3t\mathbf{k})dt.$

4. (a) $\int (t(1+t)^{-1}\mathbf{i} + \sec^2 3t\mathbf{j} + (t^2-4)\mathbf{k})dt.$

   (b) $\int_0^4 (t(1+3t^2)^{-1}\mathbf{i} + (1+t^2)^{1/2}\mathbf{j} + t^2e^{-t}\mathbf{k})dt.$

5. Find the arc length along the circular helix $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + \alpha t\mathbf{k}$ between the points corresponding to $t = \pi$ and $t = 3\pi/2$.

6. Find the arc length along the curve $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$ between the points corresponding to $t = 0$ and $t = 2\pi$.

7. Given the vector valued function $\mathbf{F} = -z\mathbf{i} + x\mathbf{j} - y\mathbf{k}$, find the scalar line integral of $\mathbf{F}$ along the space curve $\mathbf{r}(t) = \sin t\mathbf{i} - \cos t\mathbf{j} + e^t\mathbf{k}$ between the points on the curve corresponding to $t = 0$ and $t = \pi/2$.

8. Given the vector valued function $\mathbf{F} = 2y\mathbf{i} + x^2\mathbf{j} - 3z\mathbf{k}$, find the line integral of $\mathbf{F}$ along the space curve $\mathbf{r}(t) = t\mathbf{i} + (1+2t^3)\mathbf{j} + t^2\mathbf{k}$ between the points on the curve corresponding to $t = 1$ and $t = 3$.

9. Let $\mathbf{F}$ be the vector-valued function $\mathbf{F} = -x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that the line integrals of $\mathbf{F}$ along the helix $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k}$ between the points on the helix corresponding to $t = 0$ and $t = 2\pi$ and along the straight line path joining the points $\mathbf{r}(0)$ to $\mathbf{r}(2\pi)$ are the same.

10. Let $\mathbf{F}$ be the vector-valued function $\mathbf{F} = 2xy^2z\mathbf{i} + 2x^2yz\mathbf{j} + x^2y^2\mathbf{k}$. Find the line integral of $\mathbf{F}$ along the straight line Γ with the equation $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$ between the points corresponding to $t = 0$ and $t = 1$. Let γ be the path formed by the straight line segments joining the points $PQRS$, in this order, where P is the point $\mathbf{r} = \mathbf{0}$, Q is the point $\mathbf{r} = \mathbf{i}$, R is the point $\mathbf{r} = \mathbf{i} + 2\mathbf{j}$, and S is the point $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find the line integral of $\mathbf{F}$ along γ from P to S, and hence show that it has the same value as the integral along Γ.

11. The velocity vector in a two-dimensional fluid flow is $\mathbf{v} = y\mathbf{i} + x^2y\mathbf{j}$. Find the circulation (a) around the ellipse $x^2 + \frac{1}{4}y^2 = 1$ and (b) around the unit circle $x^2 + y^2 = 1$, and hence show the flow is *not* irrotational.

12. The velocity vector in a two-dimensional fluid flow is $\mathbf{v} = (2x + 3y^2)\mathbf{i} + 6x\mathbf{j}$. Show that there is zero circulation around all the circles $(x-a)^2 + (y-b)^2 = c^2$, where a, b, and $c > 0$ are arbitrary real numbers. Is it correct to say this proves that the flow is irrotational? Give reasons justifying your answer.

13. Find the flux of $\mathbf{F} = (3x + 2y)\mathbf{i} + (2x - y)\mathbf{j}$ across the circle $x^2 + y^2 = 4$.

11.3 Directional Derivatives and the Gradient Operator

Consider a scalar function $w = f(x, y, z)$ with continuous first order partial derivatives with respect to x, y, and z that is defined in some region D of space, and let a space curve Γ in D have the parametric equations $x = x(t)$, $y = y(t)$, and $z = z(t)$. Then from the chain rule

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}, \tag{21}$$

and it is seen from this that dw/dt can be interpreted as the scalar product of the two vectors

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \quad \text{and} \quad \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

The first vector, denoted by

$$\text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \tag{22}$$

the gradient of a scalar function of position

is called the **gradient** of the scalar function f expressed in terms of cartesian coordinates, whereas from Section 11.1 the second vector

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{23}$$

is seen to be a vector that is tangent to the space curve Γ. Consequently, dw/dt is the scalar product of grad f and $d\mathbf{r}/dt$ at the point $x = x(t)$, $y = y(t)$, and $z = z(t)$ for any given value of t.

Another notation for grad f that is also used is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \tag{24}$$

where the symbol ∇f is either read "del f" or "grad f." In this notation, the **vector operator**

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{25}$$

is the **gradient operator** expressed in terms of cartesian coordinates, and if ϕ is a suitably differentiable scalar function of x, y, and z, it is to be understood that

$$\nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}. \tag{26}$$

Let us now introduce the unit vector $\mathbf{v}$ defined as

$$\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}, \tag{27}$$

where l, m, and n are the direction cosines of the tangent to the space curve Γ in (23), so that

$$l = \frac{dx}{dt} \bigg/ \left|\frac{d\mathbf{r}}{dt}\right|, \quad m = \frac{dy}{dt} \bigg/ \left|\frac{d\mathbf{r}}{dt}\right|, \quad n = \frac{dz}{dt} \bigg/ \left|\frac{d\mathbf{r}}{dt}\right|, \tag{28}$$

with

$$\left|\frac{d\mathbf{r}}{dt}\right| = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2}. \tag{29}$$

Then as the scalar product of a vector **F** and the unit vector **v** is the *projection* of **F** in the direction **v**, it follows at once that

$$D_v f = \mathbf{v} \cdot \operatorname{grad} f = l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z} \tag{30}$$

the directional derivative and its properties

is the **directional derivative** of f in the direction **v**. This last result has meaning irrespective of whether **v** is tangent to a space curve, so from now on **v** can be taken to be an arbitrary unit vector in space.

The directional derivative $D_v f$ can be interpreted in terms of the ordinary operation of differentiation by considering Fig. 11.8. In the diagram, a straight line T in space in the direction of a given vector **v** passes through a fixed point P, and Q is a general point on line T at a distance s from P. The directional derivative $D_v f$ is then given by

$$D_v f = \frac{df}{dv} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}. \tag{31}$$

In the two-dimensional case in the (x, y)-plane, the directional derivative defined in (30) simplifies to

$$D_v f = \mathbf{v} \cdot \operatorname{grad} f = l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y}, \tag{32}$$

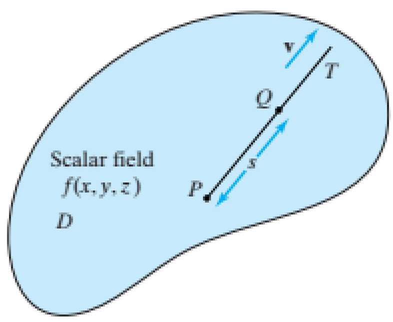


FIGURE 11.8 The directional derivative $D_v f$.

where now the unit vector $\mathbf{v} = l\mathbf{i} + m\mathbf{j}$, with $l^2 + m^2 = 1$, and the grad f in (22) simplifies to

$$\text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}, \tag{33}$$

where again the unit vector $\mathbf{v} = l\mathbf{i} + m\mathbf{j}$, with $l^2 + m^2 = 1$.

EXAMPLE 11.13 Find the directional derivative of $f = x^2 + 3y^2 + 2z^2$ in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, and determine its value at the point $(1, -3, 2)$.

Solution grad $f = 2x\mathbf{i} + 6y\mathbf{j} + 4z\mathbf{k}$ and the unit vector in the required direction is $v = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$, and so the required directional derivative is

$$D_v f = \left(\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) \cdot (2x\mathbf{i} + 6y\mathbf{j} + 4z\mathbf{k}),$$

and so

$$D_v f = \frac{4}{3}x - 2y - \frac{8}{3}z.$$

This shows that the directional derivative $D_v f$ at the point $(1, -3, 2)$ is

$$D_v f(1, -3, 2) = \frac{4}{3} + 6 - \frac{16}{3} = 2.$$

Inspection of definition (30) shows immediately that $D_v f$, which is the rate of change of f in the direction $\mathbf{v}$, must take its greatest value when $\mathbf{v}$ is in the direction of grad f, its smallest value when $\mathbf{v}$ and grad f are oppositely directed, and the value zero when $\mathbf{v}$ and grad f are orthogonal. These simple properties of a directional derivative are sufficiently important for them to be recorded separately in the following form.

Properties of directional derivatives

1. The most rapid increase of a differentiable function $f(x, y, z)$ at a point P in space occurs in the direction of the vector $\mathbf{v}_P = \text{grad } f(P)$. The directional derivative at P is then given by

$$D_v f(P) = |\text{grad } f(P)| = \left((\partial f/\partial x)_P^2 + (\partial f/\partial y)_P^2 + (\partial f/\partial z)_P^2\right)^{1/2}.$$

2. The most rapid decrease of a differentiable function $f(x, y, z)$ at a point P in space occurs when the vector $\mathbf{v}_P$ just defined in 1 and grad f are oppositely directed, so that $\mathbf{v}_P = -\text{grad } f(P)$. The directional derivative at P is then the *negative* of the result in 1 and so is given by

$$\begin{aligned} D_v f(P) &= -|\text{grad } f(P)| \\ &= -\left((\partial f/\partial x)_P^2 + (\partial f/\partial y)_P^2 + (\partial f/\partial z)_P^2\right)^{1/2}. \end{aligned}$$

3. There is a zero local rate of change of a differentiable function $f(x, y, z)$ at a point P in space in the direction of any vector $\mathbf{v}_P$ that is orthogonal to grad f at P, so that $\mathbf{v}_P \cdot \text{grad } f(P) = 0$.

When a scalar function f defined over a region D of space is suitably differentiable, the vector-valued function grad f defines a *vector field* over D in terms of the *scalar field* defined by f. The next theorem establishes the result of performing the gradient operation on combinations of scalar functions.

THEOREM 11.2

properties of the gradient operator

Rules for the gradient operator Let the gradients of f and g be defined over a region D. Then the gradient operator has the following properties.

(i) Gradient of a constant multiple of f:

$$\text{grad}\,(cf) = c\,\text{grad}\,f; \qquad (c \text{ a scalar constant})$$

(ii) Gradient of a sum or difference of functions:

$$\text{grad}\,(f \pm g) = \text{grad}\,f \pm \text{grad}\,g;$$

(iii) Gradient of a product of functions:

$$\text{grad}\,(fg) = f\,\text{grad}\,g + g\,\text{grad}\,f;$$

(iv) Gradient of a quotient of functions:

$$\text{grad}\left(\frac{f}{g}\right) = (g\,\text{grad}\,f - f\,\text{grad}\,g)/g^2 \qquad (g \neq 0).$$

Proof These results all follow by applying the usual rules for partial differentiation to each component of the gradient function on the left, and then recombining the results to obtain the expression on the right. To illustrate the form of argument involved, we prove result (iii) concerning the gradient of a product of functions. By definition,

$$\begin{aligned}\text{grad}(fg) &= \frac{\partial(fg)}{\partial x}\mathbf{i} + \frac{\partial(fg)}{\partial y}\mathbf{j} + \frac{\partial(fg)}{\partial z}\mathbf{k} \\ &= \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\mathbf{j} + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\mathbf{k} \\ &= f\,\text{grad}\,g + g\,\text{grad}\,f. \qquad \blacksquare\end{aligned}$$

A simple application of the gradient of a function involves the determination of the tangent plane to the surface S defined by the function $f(x, y, z) = \text{constant}$ at a point $P_0(x_0, y_0, z_0)$ on the surface S.

Define the function $w = f(x, y, z) - c$, where $c = \text{constant}$, so that the surface S then has the equation $w = 0$. Let any space curve Γ in the surface S have the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{and} \quad z = z(t).$$

Then differentiation of $w = f(x, y, z) - c$ with respect to t gives

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt},$$

but on S the function $w \equiv 0$, so this reduces to

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0.$$

This result shows that any curve Γ in S must be orthogonal to grad f, and so at every point P of the surface S the vector grad f is normal to the surface. The vector equation of a plane with normal $\mathbf{n}$ containing the point P_0 with position vector $\mathbf{r}_0$ is

$$(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{n}=0,$$

where $\mathbf{r}$ is the position vector of an arbitrary point on the plane. If we set $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $\mathbf{r}_0=x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}$, and identify $\mathbf{n}$ with grad f at P_0, where

$$\text{grad } f(P_0)=\left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i}+\left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j}+\left(\frac{\partial f}{\partial z}\right)_{P_0}\mathbf{k},$$

the required tangent plane to the surface at $P_0(x_0, y_0, z_0)$ is seen to be given by

$$(x-x_0)\left(\frac{\partial f}{\partial x}\right)_{P_0}+(y-y_0)\left(\frac{\partial f}{\partial y}\right)_{P_0}+(z-z_0)\left(\frac{\partial f}{\partial z}\right)_{P_0}=0 \tag{34}$$

EXAMPLE 11.14 Find the tangent plane at the point $(2, -1, 3)$ on the sphere

$$(x-1)^2+(y+2)^2+(z-4)^2=3.$$

Solution It is first necessary to check that the point $(2, -1, 3)$ does actually lie on the sphere, and this is confirmed by showing that $x=2$, $y=-1$, and $z=3$ satisfies the equation of the sphere. Writing $f=(x-1)^2+(y+2)^2+(z-4)^2$, we find that $\partial f/\partial x=2x$, $\partial f/\partial y=2y$, and $\partial f/\partial z=2z$, so that $(\partial f/\partial x)_{(2,-1,3)}=4$, $(\partial f/\partial y)_{(2,-1,3)}=-2$, and $(\partial f/\partial z)_{(2,-1,3)}=6$. Substitution into (34) shows that the equation of the tangent plane to the sphere at the point $(2, -1, 3)$ is

$$4(x-2)-2(y+1)+6(z-3)=0,$$

and after simplification this reduces to

$$4x-2y+6z=28.$$

In applications, the geometry of a problem often makes it necessary to express the gradient operator in terms of different coordinate systems. The coordinate systems that occur most frequently as a result of formulating problems involving either a cylindrical or a spherical geometry are the cylindrical polar coordinate system (r, θ, z) illustrated in Fig. 11.9a and the spherical polar coordinate system (r, θ, ϕ) illustrated in Fig. 11.9b, and shown in a different form in Fig. 1.15.

Consideration of the geometry of Figs. 11.9a,b establishes that the connection between these coordinate systems and the cartesian coordinates (x, y, z) is given by:

Cylindrical polar coordinates (r, θ, z)

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z \tag{35}$$

Spherical polar coordinates (r, θ, ϕ)

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta. \tag{36}$$

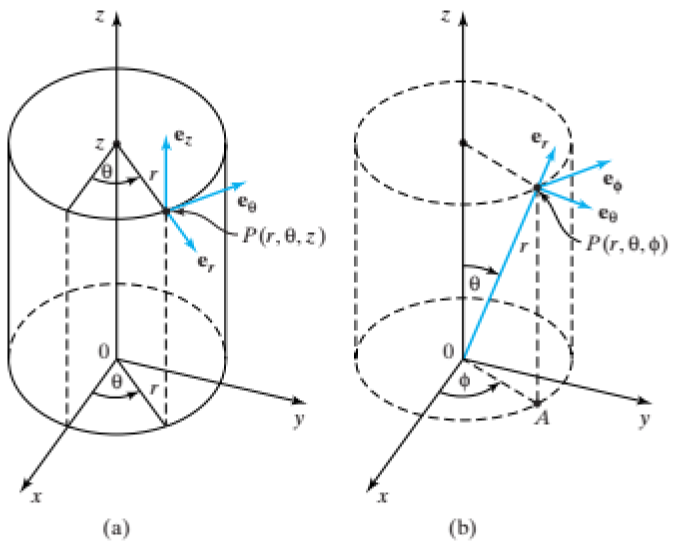


FIGURE 11.9 (a) Cylindrical polar coordinates. (b) Spherical polar coordinates.

The forms taken by grad f in cylindrical and spherical polar coordinates are given next for reference, though the derivation of these results together with related results in terms of general orthogonal curvilinear coordinates will be postponed until Section 11.6.

gradient operator in cylindrical polar coordinates

grad f in cylindrical polar coordinates (r, θ, z)

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z, \tag{37}$$

where $\mathbf{e}_r$ is a unit vector parellel to the (x, y)-plane along the radial line r, $\mathbf{e}_\theta$ is a unit vector in the (x, y)-plane normal to $\mathbf{e}_r$ in the direction of increasing θ, and $\mathbf{e}_z$ is a unit vector in the positive z-direction as shown in Fig. 11.9a, so that $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_z$.

grad f in spherical polar coordinates (r, θ, ϕ)

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi, \tag{38}$$

where $\mathbf{e}_r$ is a unit vector along the radial line r, $\mathbf{e}_\theta$ is a unit vector in the direction of increasing θ, and $\mathbf{e}_\phi$ is a unit vector in the direction of increasing ϕ that is normal to the plane containing $\mathbf{e}_r$ and $\mathbf{e}_\theta$, as shown in Fig. 11.9b, so that $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\phi$.

Notations for cylindrical and spherical polar coordinates are not uniform, so when consulting other works it is advisable to check the notation and conventions that are in use. This is particularly important in the case of spherical polar coordinates, where the r used here is sometimes replaced by ρ, with r then used to denote the distance OA in Fig. 11.9b; in addition, the symbols θ and ϕ are often interchanged.

Summary The gradient of a scalar function of position is a vector, and it has been defined and used to define the concept of a directional derivative. The properties of directional derivatives have been established and the gradient operator has been used to determine the tangent plane to a sphere at a given point on its surface. For future use, the gradient operator has been expressed in terms of both cylindrical and spherical polar coordinates.

EXERCISES 11.3

In Exercises 1 through 8 find the derivative of the scalar function f in the direction of the vector v and find its value at the point P.

1. $f = x\sin y + y\cos x$, with $v = \mathbf{i} + 2\mathbf{j}$ and P the point $(\pi/4, 0)$.
2. $f = x\sinh(x + 2y)$, with $v = 3\mathbf{i} - \mathbf{j}$ and P the point $(1, -2)$.
3. $f = xe^{xy} + 2x - y$, with $v = \mathbf{i} + 4\mathbf{j}$ and P the point $(-2, 1)$.
4. $f = \ln(x + 2y^2)$, with $v = -\mathbf{i} + 2\mathbf{j}$ and P the point $(1, 3)$.
5. $f = \sin(xy) + e^{3xz}$, with $v = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and P the point $(1, \pi/4, 1)$.
6. $f = (x^2y + z)^{1/2}$, with $v = \mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ and P the point $(2, -3, 1)$.
7. $f = \sinh(xy^2z + 3y)$, with $v = 2\mathbf{i} + \mathbf{k}$ and P the point $(1, -2, 2)$.
8. $f = (xz^2 + 3y)^{-1}$, with $v = -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and P the point $(1, -1, 1)$.
9. Prove result (iv) in Theorem 11.2.
10. Use result (iv) in Theorem 11.2 to find grad (f/g) given that $f = ye^{xy} + z$ and $g = xyz^2 + 1$, and confirm the result by direct calculation.

In Exercises 11 through 14 find grad f and evaluate it at the point P.

11. $f = x^2 + 3xyz - yz^2$, with P the point $(1, 3, -1)$.
12. $f = (x^2 + 2y^2 + 4z^2)^{-1}$, with P the point $(1, 2, 1)$.
13. $f = \exp(xy + 2yz - 3xz)$, with P the point $(1, 0, 2)$.
14. $f = (x^2 + yz + 3z^2)^{1/2}$, with P the point $(1, -1, 2)$.
15. Derive the cartesian form of the equation of the straight line that is normal to the curve $f(x, y) = \text{constant}$ at a point (x_0, y_0) on the curve.
16. Derive the cartesian form of the equation of the tangent line to the curve $f(x, y) = \text{constant}$ at a point (x_0, y_0) on the curve.
17. Find the equation of the tangent plane to the surface $x^3 + 3xy + z^2 = 11$ at the point on the surface $(1, 2, 2)$.
18. Find the equation of the tangent plane to the surface $\sin(xy) + 2\cos(yz) + 3x = 4$ at the point on the surface $(1, \pi/2, 1)$.
19. Derive the vector equation of the straight line that is normal to the surface $f(x, y, z) = \text{constant}$ at a point with position vector $\mathbf{r}_0$ on the surface.
20. If two surfaces $f(x, y, z) = \text{constant}$ and $g(x, y, z) = \text{Constant}$ intersect at a point with position vector $\mathbf{r}_0$, find a vector that is tangent to their curve of intersection of the two surfaces at $\mathbf{r}_0$.
21. Find grad f, given that $f(r, \theta, z) = r^2\sin\theta + rz^2 + 1$.
22. Find grad f, given that $f(r, \phi, \theta) = r\sin\theta\cos\phi + \sin^2\phi$.
23. If $\mathbf{F} = \text{grad}\, f$, prove that
$$\text{grad}(f^n) = nf^{n-1}\mathbf{F}.$$
Use the result to show that when $f = r$ is the distance of a point $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ from the origin, then
$$\text{grad}\, r = \hat{\mathbf{r}} \quad \text{and} \quad \text{grad}\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3},$$
where $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$, so $\hat{\mathbf{r}} = \mathbf{r}/r$.

11.4 Conservative Fields and Potential Functions

conservative fields and path invariance

Let us reconsider the line integral $\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}$ along a path Γ joining the two points $\mathbf{r}_1$ and $\mathbf{r}_2$ in a region D of space. If the value of this line integral is *independent* of the choice of path Γ in D, the vector field $\mathbf{F}$ is called a **conservative field**. The name *conservative* comes from mechanics, where it refers to the study of dynamics in which dissipative effects such as friction can be ignored, so that the sum of the kinetic and potential energy in a system remains constant (is *conserved*), though conservative fields of different types play key roles throughout physics and engineering.

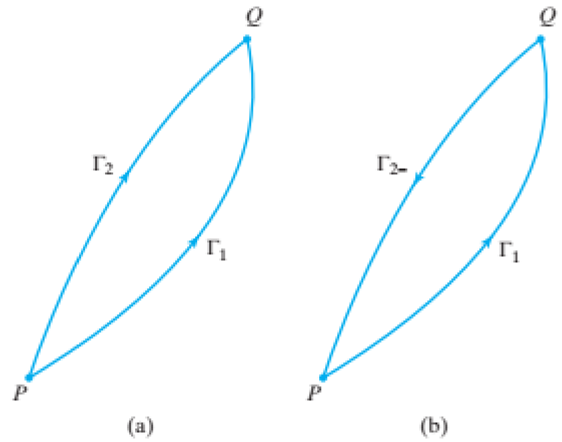


FIGURE 11.10 (a) The two paths Γ_1 and Γ_2. (b) The loop containing P and Q.

The next theorem shows that the definition of a conservative field in terms of the independence of the line integral of the path from $\mathbf{r}_1$ to $\mathbf{r}_2$ is equivalent to the vanishing of the line integral of a conservative field around any closed loop in D.

THEOREM 11.3

Path invariance and integrals around loops If $\mathbf{F}$ is a conservative field in a region D, then $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ around every closed loop Γ in D and, conversely, if $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ around every closed loop Γ in region D, then $\mathbf{F}$ is a conservative field in D.

Proof The proof of this result is straightforward, and it involves two steps. One is to show that if $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ around every closed loop Γ in D, then the field is conservative, and the other involves showing that the converse result is true.

STEP 1 Let the points P and Q shown in Fig. 11.10(a) be any two points in a region D throughout which $\mathbf{F}$ is a conservative field, and let Γ_1 and Γ_2 be any two paths in D connecting P to Q.

As $\mathbf{F}$ is a conservative field, by definition

$$\int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{r} = \int_{\Gamma_2} \mathbf{F}\cdot d\mathbf{r} \quad \text{and so} \quad \int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\Gamma_2} \mathbf{F}\cdot d\mathbf{r} = 0.$$

If we reverse the direction of integration in the second integral, thereby changing its sign, and indicate the path from Q to P by Γ_{2-}, this last result becomes

$$\int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_{2-}} \mathbf{F}\cdot d\mathbf{r} = 0.$$

However, the reversal of direction of integration on path Γ_2 makes the successive paths Γ_1 and Γ_{2-} into the loop in D shown in Fig. 11.10(b). So as P and Q were any two points in D, and Γ_1 and Γ_2 were any two paths in D joining P and Q; this proves the first part of the theorem.

STEP 2 We must now prove the converse result, that if $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ around every closed loop Γ in region D, then the field $\mathbf{F}$ is conservative in D. The proof involves reversing the argument used in Step 1. Let the arbitrary paths Γ_1 and Γ_{2-} in Fig. 11.10(b) form any loop in D, and let P and Q be any two points on the loop.

Then

$$\int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\Gamma_{2-}} \mathbf{F} \cdot d\mathbf{r} = 0,$$

but if we reverse the direction of integration along Γ_{2-}, and compensate by reversing the sign of the integral, this becomes

$$\int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r}.$$

As P and Q were arbitrary points, and Γ_1 and Γ_2 are any two paths joining these points, we have succeeded in showing that the integral is path independent, so the theorem is proved. ■

Let f be a differentiable scalar function defined over a region D and let $\mathbf{F} = \text{grad } f$ be a vector field defined in terms of f. Then f is called the **potential function** for the vector field $\mathbf{F}$. The connection between potential functions and conservative fields will become clear later.

Let us now show that if a vector field $\mathbf{F}$ has a potential function f, then the function f is unique to within an arbitrary additive constant. The proof is simple. Suppose the scalar fields f and g have the same gradient in some region D, so we can write

$$\text{grad}(f - g) \equiv 0.$$

Then if $\mathbf{v} \neq \mathbf{0}$ is an arbitrary vector in D, it follows from the preceding result that $\mathbf{v} \cdot \text{grad } (f - g) = 0$. This shows that the directional derivative of $f - g$ is equal to zero in every direction at each point of D, and this in turn implies that $f - g =$ constant, so the result is proved.

We now establish the fundamental connection between $\mathbf{F} = \text{grad } f$ and the line integral of $\mathbf{F}$ along any path Γ joining two points in a region D of space. In order to achieve this it is necessary to place some restrictions on the scalar potential function $f(x, y, z)$, the path Γ, and the region D. The function f will be assumed to have continuous first order partial derivatives in D, the path Γ in D must be continuous and piecewise smooth and comprise finitely many segments, and the region D must be open and simply connected.

The terms *open* and *simply connected* need explanation. In straightforward terms, a **simply connected** region in space can be regarded as any region that can be continuously deformed into a sphere inside of which no voids, curves, or points are missing, so it has the property that every loop in the region can be shrunk to a point that belongs to the region, without any part of the loop ever leaving the region. To understand this, consider the case of a region in space from which the points on a line are missing, and let the the loop encircle the line. Then there is no way the loop can be shrunk to a point without leaving the region, so the region is *not* simply connected (it is **multiply connected**). A region in space will be **open** if only the points on the surface of the region (its **boundary points**) are missing. A region in space is **connected** if every point in the region can be joined to every other point in the region by a piecewise continuous line that lies entirely within the region.

simply and multiply connected regions

For example, the points between two concentric spheres, the points on the surface of each of which are missing, form an *open* region that is *connected*. The region is open because its boundary points are not included in the region, and it is connected because any two points in the region can always be joined by a space curve that lies inside the region.

As another example, consider the points inside two adjacent nonintersecting spheres, each of which is connected within itself. Then the region formed by the points inside the two spheres is *not* connected, because every path joining a point in one sphere to a point in the other sphere contains points that belong to neither sphere.

THEOREM 11.4

Condition for the path independence of a line integral Let $\mathbf{F}$ be a vector field defined in an open connected region D of space, and let Γ be any path in D connecting two arbitrary points P at $\mathbf{r}_1$ and Q at $\mathbf{r}_2$ in D. Then:

(i) If the line integral $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r}$ is independent of the path Γ joining $\mathbf{r}_1$ to $\mathbf{r}_2$, a scalar field f exists such that $\mathbf{F} = \text{grad } f$.

(ii) If $\mathbf{F} = \text{grad } f$ with $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int_P^Q (F_1\,dx + F_2\,dy + F_3\,dz) = f(Q) - f(P).$$

a condition that ensures path invariance

Proof Although not difficult, the proof of result (i) is a little harder than that of result (ii). To prove (i) it is necessary to show that if P and Q are any two points in an open connected region D, and the integral $f = \int_P^Q \mathbf{F} \cdot d\mathbf{r}$ is independent of the path Γ joining P to Q, then $\mathbf{F} = \text{grad } f$.

Let P be an arbitrary point in D with coordinates (x_0, y_0, z_0), and Q be a point with coordinates (x, y_0, z_0), so that P and Q only differ in their x coordinates. By hypothesis f is independent of the path Γ from P to Q, so we can take it to be a straight line on which the general point can be written $\mathbf{r} = t\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ for $x_0 \le t \le x_1$. Let $P(x)$ be any point on Γ corresponding to $\mathbf{r} = x\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, so $d\mathbf{r}/dt = \mathbf{i}$, and denote by $f(x)$ the integral

$$f(x) = \int_{x_0}^{x} \mathbf{F} \cdot d\mathbf{r}.$$

Then, setting $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, on path Γ we can write

$$f(x) = \int_{x_0}^{x} \mathbf{F} \cdot \left(\frac{d\mathbf{r}}{dt}\right) dt = \int_{x_0}^{x} F_1(t, y_0, z_0)dt,$$

and so

$$\begin{aligned} f(x+h) - f(x) &= \int_{x_0}^{x+h} F_1(t, y_0, z_0)dt - \int_{x_0}^{x} F_1(t, y_0, z_0)dt \\ &= \int_{x}^{x+h} \mathbf{F}_1(t, y_0, z_0)dt. \end{aligned}$$

Applying the mean value theorem for integrals (see Theorem 1.4) to the integral on the right shows that

$$f(x+h) - f(x) = hF_1(\xi, y_0, z_0),$$

where the unknown number ξ is such that $x < \xi < x + h$. The preceding expression can be rewritten in the form

$$\frac{f(x+h) - f(x)}{h} = F_1(\xi, y_0, z_0),$$

and by proceeding to the limit as $h \to 0$, when $\xi \to x$, the expression on the left reduces to $\partial f/\partial x$, because f is a function of x, y, and z, but $y = y_0$ and $z = z_0$ remain constant during the limiting process. As P was an arbitrary point in D, it follows that y_0 and z_0 are arbitrary, so we have shown that $\partial f/\partial x = F_1$. Similar arguments in which first Q is taken to be the point (x_0, y, z_0), and then to be the point (x_0, y_0, z), show that $\partial f/\partial y = F_2$ and $\partial f/\partial z = F_3$. Combining these results gives $\mathbf{F} = \text{grad } f$, and the proof of (i) is complete.

To prove (ii), let the smooth path Γ joining any two points P and Q in D have the equation $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ for $a \le t \le b$. Then along Γ

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\\ &= \text{grad } f \cdot \left(\frac{d\mathbf{r}}{dt}\right) = \mathbf{F} \cdot \left(\frac{d\mathbf{r}}{dt}\right),\end{aligned}$$

and so

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \left(\frac{d\mathbf{r}}{dt}\right) dt = \int_a^b \left(\frac{df}{dt}\right) dt = f(Q) - f(P),$$

and the result is proved. ■

To make effective use of Theorem 11.4 (ii) it is necessary to know when $\mathbf{F}$ is the gradient of a scalar function f. Theorem 11.5, which follows, provides both a test for a conservative field and a way of finding its associated potential function f.

THEOREM 11.5

a test for a conservative field

Testing for a conservative field and finding the potential function The vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ with components that are continuous and differentiable is a conservative field, and so is derivable from a scalar potential f, if

(i) $$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$

When $\mathbf{F}$ is a conservative field the scalar potential function f is found by integrating the equations

(ii) $$\frac{\partial f}{\partial x} = F_1, \quad \frac{\partial f}{\partial y} = F_2, \quad \frac{\partial f}{\partial z} = F_3.$$

Proof If $\mathbf{F}$ is a conservative field, then a scalar potential f exists such that $\mathbf{F} = \text{grad } f$, and so

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Equating corresponding components gives

$$\frac{\partial f}{\partial x} = F_1, \quad \frac{\partial f}{\partial y} = F_2, \quad \frac{\partial f}{\partial z} = F_3.$$

As, by hypothesis, the components of $\mathbf{F}$ are differentiable, the equality of mixed derivatives requires that

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial F_2}{\partial x},$$

so we have established the first result in (i). The other two results are obtained in similar fashion by equating the other two mixed derivatives, so the first part of the theorem is proved. When **F** is a conservative field the scalar potential f follows by integrating the equations in (ii), and the proof of the theorem is complete. ■

EXAMPLE 11.15 Show that $\mathbf{F} = y^2 z\mathbf{i} + 2xyz\mathbf{j} + (2z + xy^2)\mathbf{k}$ is a conservative field in any open connected region of space, and find the associated scalar potential f. Use the result to evaluate the line integral $I = \int_P^Q \mathbf{F} \cdot d\mathbf{r}$, where P is the point (2, 1, 1) and Q is the point (3, 2, 2).

Solution In the notation of Theorem 11.5 the components of **F** are $F_1 = y^2 z$, $F_2 = 2xyz$, and $F_3 = 2z + xy^2$, and a routine calculation confirms that

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z},$$

in any region of space, so the **F** is a conservative field.

To find the scalar potential f we must integrate

$$\frac{\partial f}{\partial x} = y^2 z, \quad \frac{\partial f}{\partial y} = 2xyz, \quad \frac{\partial f}{\partial z} = 2z + xy^2.$$

Integrating the first equation with respect to x, while regarding y and z as constants, gives

$$f = xy^2 z + r(y, z),$$

where $r(y, z)$ is an arbitrary function of y and z. Combining this result with the expression for $\partial f/\partial y$ given earlier, we find that

$$\frac{\partial f}{\partial y} = 2xyz + \frac{\partial r}{\partial y} = 2xyz \quad \text{and so} \quad \frac{\partial r}{\partial y} = 0,$$

from which it follows that $r = s(z)$, with $s(z)$ an arbitrary function of z. Finally, using this result with the expression for $\partial f/\partial z$ given earlier we find that

$$\frac{\partial f}{\partial z} = xy^2 + \frac{ds}{dz} = 2z + xy^2 \quad \text{and so} \quad \frac{ds}{dz} = 2z,$$

from which it follows that $s(z) = z^2 + c$, where c is an arbitrary constant.

Combining results shows that the most general scalar potential function f associated with **F** is

$$f = xy^2 z + z^2 + c.$$

As **F** is a conservative field, the line integral between any two points in an open connected region D can be evaluated using result (ii) of Theorem 11.4. However, the arbitrary constant c in f can be omitted when evaluating a line integral using the result

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = \int_P^Q df = f(Q) - f(P),$$

because c occurs in both $f(Q)$ and $f(P)$, and so cancels. As a result, setting $f = xy^2 z + z^2$ and using the notation $(xy^2 z + z^2)_{(p,q,r)}$ to denote $xy^2 z + z^2$ evaluated with $x = p$, $y = q$, and $z = r$, we find that

$$I = \int_P^Q \mathbf{F} \cdot d\mathbf{r} = (xy^2z + z^2)_{(3,2,2)} - (xy^2z + z^2)_{(2,1,1)}$$
$$= 28 - 3 = 25.$$

The example that follows shows the necessity of the condition in Theorem 11.4 that the region D is *simply connected*, because if this is not the case, a line integral between two arbitrary points P and Q in D will *not* be independent of the path joining them.

EXAMPLE 11.16 Show that the two-dimensional vector field $\mathbf{F} = (\frac{-y}{x^2+y^2})\mathbf{i} + (\frac{x}{x^2+y^2})\mathbf{j}$ satisfies the conditions of Theorem 11.5 (i) in any region of space that does not contain the origin. Evaluate the integral $I = \int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ when (a) Γ is the circle $x^2 + y^2 = 2$ and and (b) Γ is the square with corners P at $(1, -1)$, Q at $(3, -1)$, R at $(3, 1)$, and S at $(1, 1)$, and comment on the results.

Solution The vector $\mathbf{F}$ is indeterminate at the origin, but is defined elsewhere in the plane, where it satisfies the condition

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right).$$

This shows that $\mathbf{F}$ satisfies the two-dimensional form of Theorem 11.5 (i) in any region of the plane that does not include the origin. When the origin is excluded from the plane, vector $\mathbf{F}$ is seen to be defined in a *nonsimply connected* region.

The circle $x^2 + y^2 = 2$ and the square with its corners at $PQRS$ are shown in Fig. 11.11, from which it can be seen that the points P and S are common, so both the circle and the square represent loops in the plane containing the points P and S. The circle encloses the origin, so the points in its interior are not simply connected, while the square excludes the origin, so the points in its interior are simply connected.

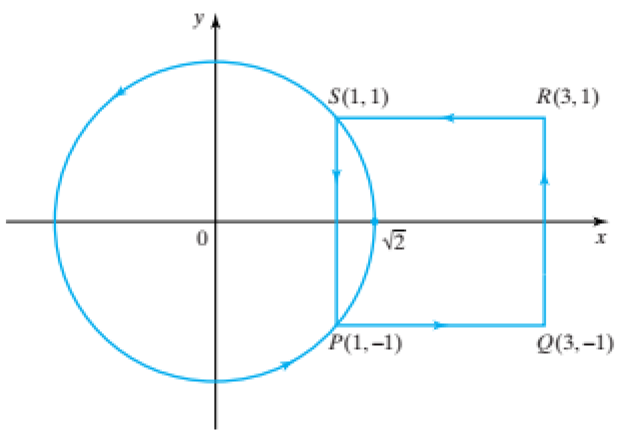


FIGURE 11.11 Two loops, each containing points P and S, in a nonsimply connected region.

Setting $x = \sqrt{2}\cos t$, $y = \sqrt{2}\sin t$ for $0 \le t \le 2\pi$ and evaluating the line integral I in case (a) gives

$$I = \int_{\Gamma} \left(\frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \right) = 2\pi.$$

In case (b) we have

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = \int_1^3 \frac{dx}{x^2 + 1}, \quad \int_Q^R \mathbf{F} \cdot d\mathbf{r} = 3 \int_{-1}^1 \frac{dy}{y^2 + 9}, \quad \int_R^S \mathbf{F} \cdot d\mathbf{r} = -\int_3^1 \frac{dx}{x^2 + 1}$$

and

$$\int_S^P \mathbf{F} \cdot d\mathbf{r} = \int_1^{-1} \frac{dy}{y^2 + 1}.$$

Evaluating these integrals and adding the results shows, as expected, that in case (b) the integral $I = 0$.

These results could be used to illustrate that when a region is not simply connected, the line integral between two points (in this case P and S) of a vector $\mathbf{F}$ that satisfies the conditions of Theorem 11.5 (i) will, in general, depend on the path joining the points. ■

FURTHER RESULTS

For the sake of completeness the definitions of the terms *open*, *connected*, and *simply connected* are given below in rather more detail, and they are then illustrated diagramatically by considering regions in the plane.

Definitions of open, connected, and simply connected regions

(i) A region D in space is said to be an **open** region if every point P in D can be enclosed in a sphere centered on P whose radius can always be chosen small enough that all points inside the sphere belong to D.

(ii) A region D in space is said to be **connected** if every pair of points in D can be joined by a piecewise smooth path with finitely many segments that lies entirely inside D.

(iii) A region D in space is said to be **simply connected** if every closed non-self-intersecting loop in D can be shrunk to a point in D in such a way that during the process every point on the loop remains in D.

Figure 11.12 illustrates these definitions in the case of two-dimensional regions, where a dashed boundary is used to indicate that the points on the boundary are omitted from the region. In (a), the region D is *open*, because however close P is taken to the dashed line, a circle (the two-dimensional equivalent of the sphere referred to in (i)) can always be drawn around P in such a way that all points in the circle lie in D. In (b) the region D represented by the interior of the two circles is *not* connected, because any line joining a point in one circle to a point in the other contains points that do not belong to either circle. In (c) the region D is *connected*, because any two points can always be joined by a line that lies entirely inside D.

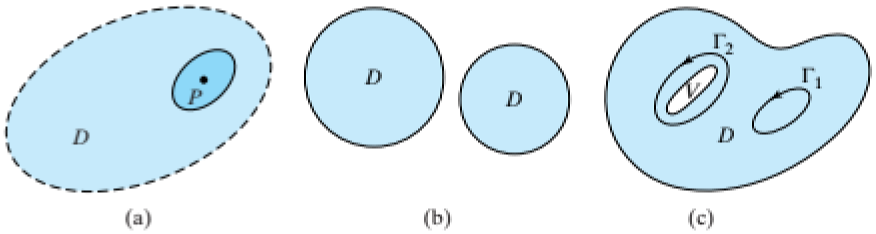


FIGURE 11.12 Regions in the plane illustrating connectivity.

However, in this case the region D is *not* simply connected, because although loop Γ_1 can be contracted to a point in such a way that every point on Γ_1 remains in D, this is not possible in the case of loop Γ_2, which encloses a void V. This last example can be visualized by considering the boundary of the void as a barrier and the loop as an elastic band. In the case of Γ_1 the elastic band can shrink to a point without hindrance, but in the case of Γ_2 this is prevented by the barrier surrounding the void.

Summary

A conservative field is one in which zero work is done when moving around a closed loop in the field and returning to the starting point. Expressed differently, a conservative field is one in which the work done when moving between two separate points is independent of the path followed between the two points. This property of conservative fields has led to this independence of a line integral on the path between two points being called the property of path invariance. The consequences of this definition have been explored and a condition has been found that ensures path invariance. A test for a conservative field has also been given.

EXERCISES 11.4

In Exercises 1 through 6 determine whether $\mathbf{F}$ is a conservative field, and if so, where.

1. $\mathbf{F} = (3x^2y^2 + yz^2)\mathbf{i} + (2x^3y + xz^2)\mathbf{j} + 2xyz\mathbf{k}$.
2. $\mathbf{F} = y\cos(xy + z^2)\mathbf{i} + x\cos(xy + z^2)\mathbf{j} + 2z\cos(xy + z^2)\mathbf{k}$.
3. $\mathbf{F} = e^x y^2\mathbf{i} + ye^x\mathbf{j} + 3xz\mathbf{k}$.
4. $\mathbf{F} = \dfrac{x}{(x^2+y^2+z^2)^{1/2}}\mathbf{i} - \dfrac{y}{(x^2+y^2+z^2)^{1/2}}\mathbf{j} + \dfrac{2z}{(x^2+y^2+z^2)^{1/2}}\mathbf{k}$.
5. $\mathbf{F} = \dfrac{-2xz}{(x^2+y^2+2z^2)^2}\mathbf{i} + \dfrac{-2yz}{(x^2+y^2+2z^2)^2}\mathbf{j} + \dfrac{x^2+y^2-z^2}{(x^2+y^2+2z^2)^2}\mathbf{k}$.
6. $\mathbf{F} = \dfrac{z}{x^2+y^2+z^2}\mathbf{i} - \dfrac{y}{x^2+y^2+z^2}\mathbf{j} + \dfrac{x}{x^2+y^2+z^2}\mathbf{k}$.

In Exercises 7 to 12 show $\mathbf{F}$ is a conservative field, and by finding the scalar potential f evaluate the integral $I = \int_P^Q \mathbf{F}\cdot d\mathbf{r}$ between the given points P and Q.

7. $\mathbf{F} = (z^3 + 6xy^2)\mathbf{i} + 6x^2y\mathbf{j} + 3xz^2\mathbf{k}$ with P at $(1, 0, 1)$ and Q at $(2, 1, 0)$.
8. $\mathbf{F} = 2xz^2\cosh(x^2 + 2y^2)\mathbf{i} + 4yz^2\cosh(x^2 + 2y^2)\mathbf{j} + 2z\sinh(x^2 + 2y^2)\mathbf{k}$, with P at $(1, 1, 1)$ and Q at $(0, 2, 1)$.
9. $\mathbf{F} = e^{xyz}(1 + xyz)\mathbf{i} + x^2ze^{xyz}\mathbf{j} + x^2ye^{xyz}\mathbf{k}$, with P at $(0, 0, 0)$ and Q at $(1, 1, 2)$.
10. $\mathbf{F} = \dfrac{yz(1-x^2)}{(1+x^2)^2}\mathbf{i} + \dfrac{xz}{1+x^2}\mathbf{j} + \dfrac{xy}{1+x^2}\mathbf{k}$, with P at $(1, 1, 1)$ and Q at $(2, 2, 0)$.
11. $\mathbf{F} = 2x(1 + yz^2)\mathbf{i} + x^2z^2\mathbf{j} + 2x^2yz\mathbf{k}$, with P at $(3, 1, -1)$ and Q at $(1, 0, 2)$.
12. $\mathbf{F} = 2x(y^2 + z^2)\mathbf{i} + 2y(1 + x^2)\mathbf{j} + 2z(1 + x^2)\mathbf{k}$, with P at $(0, 1, 2)$ and Q at $(2, 0, 1)$.
13. Verify the results of Example 11.15 by performing the indicated integrations along a straight line from P to Q.

11.5 Divergence and Curl of a Vector

It is necessary to introduce two new operations involving vectors. The first operation is called the *divergence* of a vector, and it associates a *scalar function* with a differentiable vector field $\mathbf{F}$. The second operation is called the *curl* of a vector, and it associates a *vector function* with the vector $\mathbf{F}$. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a differentiable vector field, the **divergence** of $\mathbf{F}$, written div $\mathbf{F}$, is the scalar function defined in terms of cartesian coordinates as

divergence of a vector

$$\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{39}$$

The divergence of the vector $\mathbf{F}$ can also be expressed in terms of the operator "del" defined in (25) as

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

by writing

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}), \tag{40}$$

where the mutual orthogonality of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ coupled with the fact that they are constant vectors causes the expression on the right of (40) to be reduced to the expression on the right of (39), with the operation $\nabla\cdot\mathbf{F}$ being read "del dot $\mathbf{F}$." The form taken by div $\mathbf{F}$ in more general coordinate systems is derived in Section 11.6.

At this stage, for simplicity, the definition of div $\mathbf{F}$ is expressed in terms of cartesian coordinates, though it will be shown later that div $\mathbf{F}$ is, in fact, independent of any coordinate system. In the next chapter it will be shown that div $\mathbf{F}$ can be interpreted as the flux of the normal component of the vector $\mathbf{F}$ that crosses the surface of a unit volume in a unit time. This means that when div $\mathbf{F}$ is positive, there is a net *flow* of $\mathbf{F}$ *out* of the volume, and when div $\mathbf{F}$ is negative, there is a net *flow* of $\mathbf{F}$ *into* the volume.

In anticipation of the next chapter, we give a heuristic derivation of div $\mathbf{F}$ in terms of cartesian coordinates that shows how div $\mathbf{F}$ can be defined differently, and at the same time illustrates its physical significance. Consider the small cube of side a shown in Fig. 11.13 with faces normal to the coordinate axes, and take the positive direction of the normal to each face of the cube to be the one directed *out* of the cube. The normal component of $\mathbf{F}$ *entering* face A is $F_2(x, y_0, z)$, and the normal component of $\mathbf{F}$ *leaving* face B is $F_2(x, y_0 + a, z)$, where from Taylor's theorem for functions of several variables, to first order in a we have $F_2(x, y_0 + a, z) = F_2(x, y_0, z) + a\partial F_2(x, y_0, z)/\partial y$.

Consequently, if we average $F_2(x, y_0, z)$ over face A and denote the result by $\tilde{F}_2$, the integral of $F_2(x, y_0, z)$ over face A is approximately equal to $a^2\tilde{F}_2$, while the integral over face B is approximately equal to $a^2[\tilde{F}_2 + a\partial\tilde{F}_2/\partial y]$, so the change of the flux of $\mathbf{F}$ from face A to face B is approximately $a^3\partial\tilde{F}_2/\partial y$. Similar results apply to the other pairs of faces, so denoting the surface of the cube by S, and letting F_n

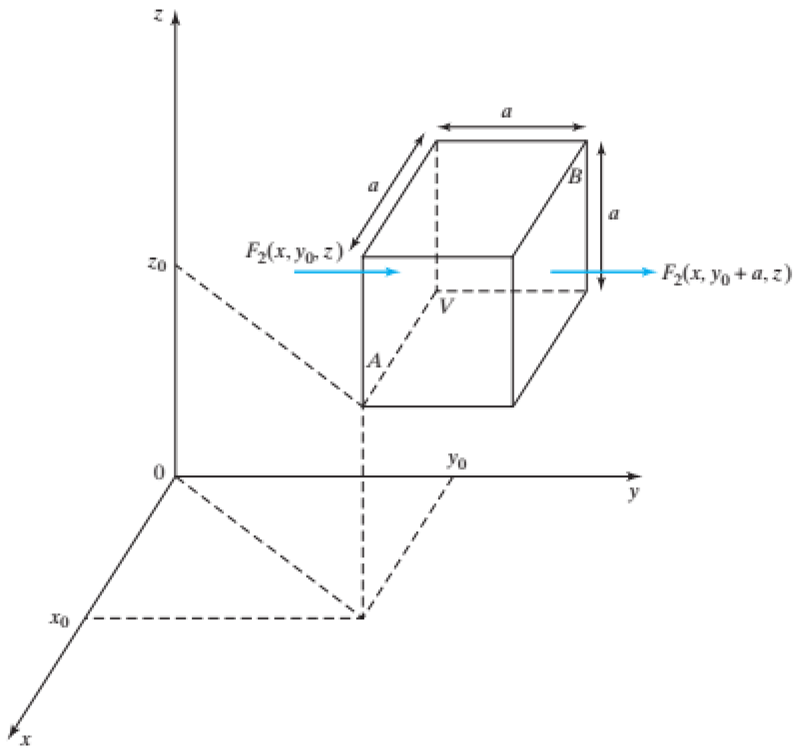


FIGURE 11.13 A representative cubic element.

denote the component of **F** normal to S, positive when outward, with dS a surface element of area of a face, we have

$$\lim_{a\to 0}\frac{1}{a^3}\iint_S F_n\,dS = \lim_{a\to 0}\frac{1}{a^3}\left(a^3\frac{\partial \tilde{F}_1}{\partial x} + a^3\frac{\partial \tilde{F}_2}{\partial y} + a^3\frac{\partial \tilde{F}_3}{\partial z}\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

The expression on the right is div **F**, so this result shows that the divergence of a vector field **F** in cartesian coordinates is the limit of the flux of the normal component of **F** through the surface S bounding a volume as the volume tends to zero. A different form of argument used in the next chapter will show that for *any* volume V with surface S and element of surface area dS, independently of any coordinate system

a different interpretation of div F

$$\operatorname{div} \mathbf{F} = \lim_{V\to 0}\frac{1}{V}\iint_S F_n\,dS.$$

It is helpful to interpret this result in terms of the flow of a liquid. If we identify **q** with the liquid velocity vector, V with the volume occupied by the liquid, and S with the surface enclosing V, the product $q_n\,dS$, with q_n the component of **q** normal to dS, is seen to be the volume of liquid crossing the surface element dS in a unit time. Consequently, $\iint_S F_n\,dS$ is the total volume of liquid leaving through the surface S in a unit time. As a liquid can be considered to be incompressible, provided the volume contains neither a *source* of liquid (a point in V through which liquid enters) nor a *sink* (a point in V through which liquid is extracted), it follows that $\iint_S F_n\,dS$ will be zero for an incompressible fluid.

Thus, in an incompressible liquid free from sources and sinks, div $\mathbf{q} = 0$. If sources and sinks occur in the liquid, their strengths can be found by enclosing each in a small volume and then letting it become arbitrarily small, in which case a *positive* value of div $\mathbf{q}$ will correspond to a source and a *negative* value to a sink.

If, instead of a liquid, the flow of a gas is involved, the compressibility of a gas causes its density to vary from point to point, so then, in general, the value of div $\mathbf{q}$ will depend on position and, if the flow is unsteady, also on the time.

EXAMPLE 11.17

Find div $\mathbf{F}$ when $\mathbf{F} = xy^2\mathbf{i} + 3yz\mathbf{j} - 4xz\mathbf{k}$.

Solution From (39) div $\mathbf{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(3yz) + \frac{\partial}{\partial z}(-4xz) = y^2 + 3z - 4x$. ■

We have seen that provided f is suitably differentiable, grad f is a vector, so when f is twice differentiable it is appropriate to examine the operation div (grad f). This is usually written div grad f, because no ambiguity arises when the brackets are omitted. By definition

$$\begin{aligned}\text{div grad } f &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right)\\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f, \qquad (41)\end{aligned}$$

and so div grad $f = \Delta f$ is simply the **Laplacian** of f.

THEOREM 11.6

fundamental properties of the divergence operator

Properties of the divergence operator Let the vector fields $\mathbf{F}$ and $\mathbf{G}$ and the scalar fields ϕ and ψ be a suitably differentiable, and let a and b be constants. Then the divergence operator has the following properties:

(i) $\text{div}(a\mathbf{F}) = a\,\text{div}\,\mathbf{F}$

(ii) $\text{div}(a\mathbf{F} + b\mathbf{G}) = a\,\text{div}\,\mathbf{F} + b\,\text{div}\,\mathbf{G}$

(iii) $\text{div}(\phi\mathbf{F}) = \phi\,\text{div}\,\mathbf{F} + \mathbf{F}\cdot\nabla\phi$

(iv) $\text{div}(\text{grad}\,\phi) = \Delta\phi$

(v) $\text{div}(\phi\nabla\psi) = \phi\Delta\psi + \text{grad}\,\phi\cdot\text{grad}\,\psi = \phi\Delta\psi + \nabla\phi\cdot\nabla\psi$

(vi) $\text{div}(\phi\nabla\psi) - \text{div}(\psi\nabla\phi) = \phi\Delta\psi - \psi\Delta\phi$

Proof The derivation of these results follows directly from the definition of the divergence of a vector in (39). So, as (iv) has already been established, we will only prove (iii) and leave the other results as exercises.

If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, it follows that $\phi\mathbf{F} = \phi F_1\mathbf{i} + \phi F_2\mathbf{j} + \phi F_3\mathbf{k}$, and so

$$\begin{aligned}\text{div}(\phi\mathbf{F}) &= \frac{\partial}{\partial x}(\phi F_1) + \frac{\partial}{\partial y}(\phi F_2) + \frac{\partial}{\partial z}(\phi F_3)\\ &= \phi\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + F_1\frac{\partial\phi}{\partial x} + F_2\frac{\partial\phi}{\partial y} + F_3\frac{\partial\phi}{\partial z}\\ &= \phi\,\text{div}\,\mathbf{F} + \mathbf{F}\cdot\nabla\phi.\end{aligned}$$

■

the definition of curl F

When expressed in terms of cartesian coordinates, the **curl** of the vector $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is defined as

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \tag{42}$$

This form of the definition of curl **F** is more easily remembered when expressed symbolically as the determinant

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}, \tag{43}$$

or in terms of the operator "del" as

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}), \tag{44}$$

where it is to be understood that the differentiations are to be performed before finding the cross products, and the operation $\nabla \times \mathbf{F}$ is read as "del cross **F**."

EXAMPLE 11.18

Find curl **F** given that $\mathbf{F} = xy\mathbf{i} + z\mathbf{j} + yz\mathbf{k}$.

Solution Using (43) we have

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & z & yz \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(z)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(xy)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(xy)\right)\mathbf{k} \\ &= (z-1)\mathbf{i} - x\mathbf{k}.\end{aligned}$$

■

EXAMPLE 11.19

Show that if ϕ is any scalar function with continuous first and second order derivatives, then $\operatorname{curl}(\operatorname{grad}\phi) \equiv \mathbf{0}$.

Solution By definition $\operatorname{grad}\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k}$, so from (44)

$$\operatorname{curl}(\operatorname{grad}\phi) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (\phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k}).$$

After we use the properties of the vector product with the mutually orthogonal unit vectors **i**, **j**, and **k**, this reduces to

$$\operatorname{curl}(\operatorname{grad}\phi) = \frac{\partial}{\partial x}(\phi_y)\mathbf{k} - \frac{\partial}{\partial x}(\phi_z)\mathbf{j} - \frac{\partial}{\partial y}(\phi_x)\mathbf{k} + \frac{\partial}{\partial y}(\phi_z)\mathbf{i} + \frac{\partial}{\partial z}(\phi_x)\mathbf{j} - \frac{\partial}{\partial z}(\phi_y)\mathbf{i}.$$

By hypothesis ϕ has continuous partial derivatives up to and including order 2, so there is equality of mixed derivatives. As a result $\phi_{xy} = \phi_{yx}$, showing that the **k** component of $\operatorname{curl}(\operatorname{grad}\phi)$ vanishes. The **j** and **i** components of $\operatorname{curl}(\operatorname{grad}\phi)$ vanish for the same reason so that $\operatorname{curl}(\operatorname{grad}\phi) \equiv \mathbf{0}$. ■

The operators grad, div, and curl can be combined in various ways that lead to identities, the results of which are listed in the next theorem. These identities are useful when manipulating vector operations. In some of the entries the notation $(\mathbf{F}\cdot\nabla)\mathbf{G}$ is used, and if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and $\mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$ this is to be interpreted as the vector

$$\begin{aligned}(\mathbf{F}\cdot\nabla)\mathbf{G} &= \left[(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\right](G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k})\\ &= \left(F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}\right)(G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}).\end{aligned}$$

THEOREM 11.7

combining grad, div, and curl

Properties of combinations of grad, div, and curl Let **F** and **G** be vector functions and let ϕ be a scalar function, all of which are suitably differentiable. Then the following identities hold.

(i) $\operatorname{curl}(\operatorname{grad}\phi) = \mathbf{0}$

(ii) $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$

(iii) $\operatorname{curl}(\phi\mathbf{F}) = \phi\operatorname{curl}\mathbf{F} - \mathbf{F}\times\operatorname{grad}\phi$

(iv) $\operatorname{grad}(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\times\operatorname{curl}\mathbf{G} + \mathbf{G}\times\operatorname{curl}\mathbf{F} + (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F}$

(v) $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$

(vi) $\operatorname{curl}(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$

(vii) $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \Delta\mathbf{F}$

Proof Result (i) has already been established. As the other results follow in similar fashion from the definitions of the gradient, divergence, and curl operators, the remaining proofs are left as exercises. ∎

The expression for curl **F** in more general coordinate systems is derived in Section 11.6, but a different definition of curl **F** together with a physical interpretation will be postponed until after the discusion of Stokes' theorem in the next chapter.

Theorem 11.7 provides a test for conservative vector fields **F**. Although the test is equivalent to the test in Theorem 11.5 (i), it is in a more easily remembered form. By definition, a vector field **F** is a *conservative field* if $\mathbf{F} = \operatorname{grad} f$, but from (i) of Theorem 11.7, if $\mathbf{F} = \operatorname{grad} f$ then $\operatorname{curl}\mathbf{F} = \mathbf{0}$, and it is this last result that provides the test. However, if after establishing that **F** is a conservative field its associated potential function f is required, it must be found by integrating the equations in Theorem 11.5 (ii), as illustrated in Example 11.14.

using curl F to test for a conservative field

Curl test for a conservative vector field

A vector field **F** is conservative, that is, it is $\mathbf{F} = \operatorname{grad} f$ where f is the associated scalar potential, if $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

EXAMPLE 11.20

For what values of a and b is the vector field $\mathbf{F} = (x+z)\mathbf{i} + a(y+z)\mathbf{j} + b(x+y)\mathbf{k}$ a conservative field?

Solution

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+z & a(y+z) & b(x+y) \end{vmatrix} = (b-a)\mathbf{i} + (1-b)\mathbf{j},$$

so curl $\mathbf{F} = \mathbf{0}$ if $b - a = 0$ and $1 - b = 0$. Consequently, $\mathbf{F}$ will be a conservative field if $a = b = 1$. ■

EXAMPLE 11.21 Find curl(curl $\mathbf{F}$) given that $\mathbf{F} = x^2y^2\mathbf{i} + y^2z^2\mathbf{j} + x^2z^2\mathbf{k}$.

Solution To calculate curl(curl $\mathbf{F}$), we will use result (vii) of Theorem 11.7. We have

$$\text{div}\,\mathbf{F} = 2xy^2 + 2yz^2 + 2zx^2,$$

so

$$\text{grad}(\text{div}\,\mathbf{F}) = (2y^2 + 4xz)\mathbf{i} + (2z^2 + 4xy)\mathbf{j} + (2x^2 + 4yz)\mathbf{k}.$$

Next,

$$\begin{aligned} \Delta\mathbf{F} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2y^2\mathbf{i} + y^2z^2\mathbf{j} + x^2z^2\mathbf{k}) \\ &= 2(x^2 + y^2)\mathbf{i} + 2(y^2 + z^2)\mathbf{j} + 2(x^2 + z^2)\mathbf{k}, \end{aligned}$$

so combining results gives

$$\text{curl}(\text{curl}\,\mathbf{F}) = (4xz - 2x^2)\mathbf{i} + (4xy - 2y^2)\mathbf{j} + (4yz - 2z^2)\mathbf{k}.$$ ■

Vector fields, line integrals, the theory, application, and evaluation of multiple integrals, and the vector operators grad, div, and curl are all defined and their properties developed in standard calculus and analytic geometry texts such as those in references [1.1], [1.2], [1.5], [1.6], and [1.7]. Reference [5.6] gives a concise summary of these results together with numerous examples. More advanced and detailed accounts, where the emphasis is placed on a vector treatment, are to be found in references [5.1], [5.2], and [1.4].

Summary

The previous section introduced the gradient operator, where it was shown that it acts on a scalar function of position to produce a vector. The present section introduced two more vector operators called the divergence and curl operators. The divergence operator was seen to act on a vector to produce a scalar, while the curl operator acted on a vector to produce another vector. The general operational properties of the divergence and curl operators were developed together with the results of combining all three vector operators.

EXERCISES 11.5

In Exercises 1 through 4, find div $\mathbf{F}$ for the given vector function $\mathbf{F}$.

1. $\mathbf{F} = x^2y\mathbf{i} + y^2z^2\mathbf{j} + xz^3\mathbf{k}$.
2. $\mathbf{F} = (1 - x^2)\mathbf{i} + \sin yz\mathbf{j} + e^{xyz}\mathbf{k}$.
3. $\mathbf{F} = 3x^2\mathbf{i} + 2x^2y^2\mathbf{j} + x\mathbf{k}$.
4. $\mathbf{F} = \cos x\mathbf{i} + \sin y\mathbf{j} + z^2\mathbf{k}$.
5. Prove that $\text{div}(\phi\mathbf{F}) = \phi\,\text{div}\,\mathbf{F} + \mathbf{F}\cdot\nabla\phi$ (Theorem 11.6 (iii)).
6. Prove that $\text{div}(\phi\nabla\psi) = \phi\Delta\psi + \nabla\phi\cdot\nabla\psi$ (Theorem 11.6 (v)).

In Exercises 7 through 10 find curl **F** for the given vector function **F**.

7. $\mathbf{F} = xyz^2\mathbf{i} + x^2yz\mathbf{j} + xy^2\mathbf{k}$.
8. $\mathbf{F} = \sinh xy\mathbf{i} + \cosh yz\mathbf{j} + xyz\mathbf{k}$.
9. $\mathbf{F} = \arctan \frac{x}{y}\mathbf{i} + \ln(x^2 + 2y^2)^{1/2}\mathbf{j} + y\mathbf{k}$.
10. $\mathbf{F} = (x^2 + y^2 + z^2)^{1/2}\mathbf{i} + (x^2 + y^2 + z^2)^{1/2}\mathbf{j} + x\mathbf{k}$.
11. Prove that $\operatorname{div}(\operatorname{curl}\mathbf{F}) \equiv 0$ (Theorem 11.7 (ii)).
12. Prove that $\operatorname{curl}(\phi\mathbf{F}) \equiv \phi \operatorname{curl}\mathbf{F} - \mathbf{F} \times \operatorname{grad}\phi$ (Theorem 11.7 (iii)).
13. Prove that $\operatorname{grad}(\mathbf{F} \cdot \mathbf{G}) \equiv \mathbf{F} \times \operatorname{curl}\mathbf{G} + \mathbf{G} \times \operatorname{curl}\mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$ (Theorem 11.7 (iv)).
14. Prove that $\operatorname{div}(\mathbf{F} \times \mathbf{G}) \equiv \mathbf{G} \cdot \operatorname{curl}\mathbf{F} - \mathbf{F} \cdot \operatorname{curl}\mathbf{G}$ (Theorem 11.7 (v)).
15. Prove that $\operatorname{curl}(\mathbf{F} \times \mathbf{G}) \equiv \mathbf{F} \operatorname{div}\mathbf{G} - \mathbf{G} \operatorname{div}\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$ (Theorem 11.7 (vi)).
16. Prove that $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \Delta\mathbf{F}$ (Theorem 11.7 (vii)).
17. Find curl(curl **F**) given that $\mathbf{F} = 3xyz\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}$.

In Exercises 17 and 20 use the curl test to see if or where the vector field **F** is conservative.

18. $\mathbf{F} = yz\cosh(xyz + y^2)\mathbf{i} + (xz + 2y)\cosh(xyz + y^2)\mathbf{j} + 2xy\cosh(xyz + y^2)\mathbf{k}$.
19. $\mathbf{F} = 2xy^2\mathbf{i} + (2x^2y + 6yz^3)\mathbf{j} + 9y^2z^2\mathbf{k}$.
20. $\mathbf{F} = \dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.
21. $\mathbf{F} = \dfrac{1}{(1 + x^2 + 2y^2z)}(2x\mathbf{i} + 4yz\mathbf{j} + 2y^2\mathbf{k})$.

11.6 Orthogonal Curvilinear Coordinates

The geometrical configuration of a physical problem often suggests the most appropriate coordinate system that should be used when seeking its solution. For example, heat conduction in a cylindrical rod suggests the use of cylindrical polar coordinates with the z-axis aligned with the axis of the rod, whereas the distribution of an electric field inside a spherical cavity suggests the use of spherical polar coordinates. When problems of this nature are expressed in terms of vectors, and the operators grad, div, and curl are involved, it becomes necessary to find the form taken by these operators in different systems of curvilinear coordinates. The reader who wishes to omit the derivation of the main results of this section should proceed directly to Theorem 11.8 after studying the definition of an orthogonal system of curvilinear coordinates and the meaning of the scale factors h_1, h_2, and h_3.

In what follows, in order to unify notation, it is convenient to denote the usual cartesian coordinates x, y, and z by x_1, x_2, and x_3 and a general system of curvilinear coordinates by q_1, q_2, and q_3, where the two systems are related by the equations

$$x_1 = x_1(q_1, q_2, q_3), \quad x_2 = x_2(q_1, q_2, q_3), \quad x_3 = x_3(q_1, q_2, q_3). \tag{45}$$

For the curvilinear coordinates q_1, q_2, and q_3 to be equivalent to the cartesian coordinate system x_1, x_2, and x_3 it is necessary that equations (45) can be solved uniquely in the form

$$q_1 = q_1(x_1, x_2, x_3), \quad q_2 = q_2(x_1, x_2, x_3), \quad q_3 = q_3(x_1, x_2, x_3), \tag{46}$$

so that one point in cartesian coordinates corresponds to only one point in curvilinear coordinates, and conversely. As derivatives of functions occur in grad, div, and curl, it is necessary that the coordinate functions x_1, x_2, and x_3, as functions of q_1, q_2, and q_3 in (45), are all suitably differentiable with respect to their arguments.

Taking the total differentials of the coordinate transformations in (45), we have

$$dx_1 = \frac{\partial x_1}{\partial q_1}dq_1 + \frac{\partial x_1}{\partial q_2}dq_2 + \frac{\partial x_1}{\partial q_3}dq_3, \quad dx_2 = \frac{\partial x_2}{\partial q_1}dq_1 + \frac{\partial x_2}{\partial q_2}dq_2 + \frac{\partial x_2}{\partial q_3}dq_3$$

$$dx_3 = \frac{\partial x_3}{\partial q_1}dq_1 + \frac{\partial x_3}{\partial q_2}dq_2 + \frac{\partial x_3}{\partial q_3}dq_3. \tag{47}$$

These results can be written in the matrix form

$$d\mathbf{x} = \mathbf{J}\,d\mathbf{q}, \tag{48}$$

where

$$d\mathbf{x} = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}, \quad d\mathbf{q} = \begin{bmatrix} dq_1 \\ dq_2 \\ dq_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial x_1}{\partial q_1} & \dfrac{\partial x_1}{\partial q_2} & \dfrac{\partial x_1}{\partial q_3} \\ \dfrac{\partial x_2}{\partial q_1} & \dfrac{\partial x_2}{\partial q_2} & \dfrac{\partial x_2}{\partial q_3} \\ \dfrac{\partial x_3}{\partial q_1} & \dfrac{\partial x_3}{\partial q_2} & \dfrac{\partial x_3}{\partial q_3} \end{bmatrix}. \tag{49}$$

The matrix vector linear differential elements $d\mathbf{x}$ and $d\mathbf{q}$ will be uniquely related by (48) provided matrix $\mathbf{J}$ is nonsingular, so the coordinate transformations (45) must be such that $J = \det \mathbf{J} \neq 0$, where

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial q_1} & \dfrac{\partial x_2}{\partial q_1} & \dfrac{\partial x_3}{\partial q_1} \\ \dfrac{\partial x_1}{\partial q_2} & \dfrac{\partial x_2}{\partial q_2} & \dfrac{\partial x_3}{\partial q_2} \\ \dfrac{\partial x_1}{\partial q_3} & \dfrac{\partial x_2}{\partial q_3} & \dfrac{\partial x_3}{\partial q_3} \end{vmatrix}. \tag{50}$$

the Jacobian of a transformation

The determinant J is called the **Jacobian** of the transformation, and it will be shown later that the absolute value of the Jacobian occurs as a scale factor in the **volume element** in orthogonal curvilinear coordinates. Thus, the vanishing of the Jacobian signifying nonuniqueness in the transformations (45) and (46) also corresponds to the failure of the curvilinear coordinate system to define a volume element.

Carl Gustav Jacobi (1804–1851)
A German mathematician who studied at the University of Berlin and obtained his doctorate in 1825. In 1827 he was appointed Extraordinary Professor of Mathematics at Königsberg and, after two years, he was promoted to Ordinary Professor of Mathematics. In 1842 he moved to Berlin where he remained until his death. His most important work was in connection with elliptic functions, but he also made important contributions to number theory, ordinary and partial differential equations, and the calculus of variations. He was an outstanding teacher of mathematics.

Keeping q_1 and $q_1 + dq_1$ constant defines two curvilinear surfaces in space, and four further curvilinear surfaces are defined by keeping q_2 and $q_2 + dq_2$ constant, and q_3 and $q_3 + dq_3$ constant. Taken together, the region between these six curvilinear surfaces defines the volume element dV in space shown in Fig. 11.14.

general and orthogonal curvilinear coordinates

Allowing q_1 to vary while holding q_2 and q_3 constant in (45) will generate a *curvilinear coordinate line* in space along which only q_1 changes. Similarly, allowing q_2 to vary while holding q_1 and q_3 constant, and then q_3 to vary while holding q_1 and q_2 constant, will generate curvilinear coordinate lines in space along which, respectively, only q_2 and q_3 vary. If a general point A in space shown in Fig. 11.14 is considered, there will be three curvilinear coordinate lines passing through the point. A curvilinear coordinate system will be said to be an **orthogonal** system if at every point in space the three tangents to the coordinate lines at their point of intersection

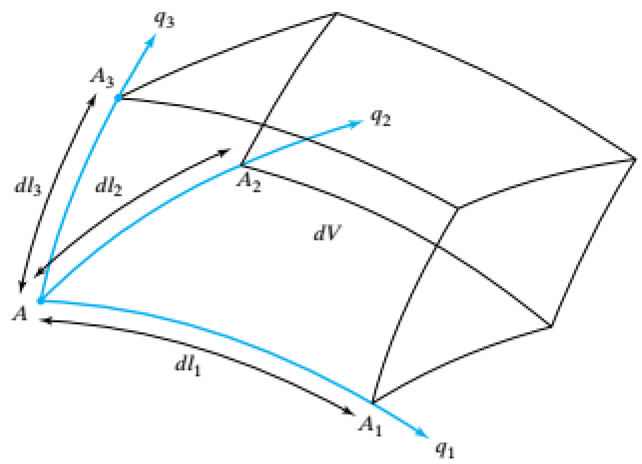


FIGURE 11.14 The curvilinear volume element dV.

are mutually orthogonal (perpendicular). Such coordinate systems are also considered to be *orthogonal* if the orthogonality condition fails at a single point or along a line. In what follows, only orthogonal coordinate systems will be considered.

the volume element

With the linear differential length elements $AA_1 = dl_1$, $AA_2 = dl_2$, and $AA_3 = dl_3$, the orthogonality of the curvilinear coordinate system implies that in terms of curvilinear coordinates the linear volume element dV in Fig. 11.14 is given by

$$dV = dl_1\, dl_2\, dl_3. \tag{51}$$

Now, in Fig. 11.14, let A be the point (x_1, x_2, x_3) and A_1 be the point $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$, where dx_1, dx_2, and dx_3 are the linear differential elements in cartesian coordinates. To find the linear differential length element dl_1 from A to A_1, we apply the Pythagoras theorem to the mutually orthogonal linear differential length elements dx_1, dx_2, and dx_3, when we obtain

$$dl_1^2 = dx_1^2 + dx_2^2 + dx_3^2, \tag{52}$$

However along AA_1 only q_1 varies, so as

$$dx_1 = \frac{\partial x_1}{\partial q_1} dq_1, \quad dx_2 = \frac{\partial x_2}{\partial q_1} dq_1, \quad dx_3 = \frac{\partial x_3}{\partial q_1} dq_1, \tag{53}$$

the square of the linear differential length element in (52) becomes

$$dl_1^2 = \left[\left(\frac{\partial x_1}{\partial q_1} \right)^2 + \left(\frac{\partial x_2}{\partial q_1} \right)^2 + \left(\frac{\partial x_3}{\partial q_1} \right)^2 \right] dq_1^2. \tag{54}$$

Similar arguments show that if dl_2 and dl_3 are the linear differential length elements along AA_2 and AA_3, then

$$dl_2^2 = \left[\left(\frac{\partial x_1}{\partial q_2} \right)^2 + \left(\frac{\partial x_2}{\partial q_2} \right)^2 + \left(\frac{\partial x_3}{\partial q_2} \right)^2 \right] dq_2^2, \tag{55}$$

and

$$dl_3^2 = \left[\left(\frac{\partial x_1}{\partial q_3}\right)^2 + \left(\frac{\partial x_2}{\partial q_3}\right)^2 + \left(\frac{\partial x_3}{\partial q_3}\right)^2\right] dq_3^2. \tag{56}$$

the scale factors h_1, h_2, h_3

We now adopt the standard notation and define the **scale factors** h_1, h_2, and h_3, with respect to the coordinates q_1, q_2, and q_3 in transformations (45), by

$$h_1 = \left[\left(\frac{\partial x_1}{\partial q_1}\right)^2 + \left(\frac{\partial x_2}{\partial q_1}\right)^2 + \left(\frac{\partial x_3}{\partial q_1}\right)^2\right]^{1/2} \tag{57}$$

$$h_2 = \left[\left(\frac{\partial x_1}{\partial q_2}\right)^2 + \left(\frac{\partial x_2}{\partial q_2}\right)^2 + \left(\frac{\partial x_3}{\partial q_2}\right)^2\right]^{1/2} \tag{58}$$

$$h_3 = \left[\left(\frac{\partial x_1}{\partial q_3}\right)^2 + \left(\frac{\partial x_2}{\partial q_3}\right)^2 + \left(\frac{\partial x_3}{\partial q_3}\right)^2\right]^{1/2}. \tag{59}$$

In terms of h_1, h_2, and h_3 the linear differential line elements dl_1, dl_2, and dl_3 in rectangular curvilinear coordinates defined in (54) to (56) become

$$dl_1 = h_1 dq_1, \quad dl_2 = h_2 dq_2, \quad dl_3 = h_3 dl_3. \tag{60}$$

If the general linear differential length element from A to B in Fig. 11.14 is denoted by ds, then as the coordinate system is orthogonal,

$$ds^2 = dl_1^2 + dl_2^2 + dl_3^2, \tag{61}$$

so it follows from (60) that

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2. \tag{62}$$

In terms of the scale factors the linear differential volume element dV in (51) becomes

$$dV = h_1 h_2 h_3 \, dq_1 \, dq_2 \, dq_3. \tag{63}$$

It can be seen from this last result that the coordinate transformations (45) will fail to define a volume element in curvilinear coordinates if a scale factor vanishes. From the definitions of the scale factors, this can only happen if all of the partial derivatives in a scale factor vanish, but when this occurs the Jacobian determinant J will have a zero row, and so will also vanish. This is to be expected, because it is known from calculus that when the Jacobian vanishes, the transformation between the coordinate systems ceases to be one to one.

To understand the geometrical interpretation of the Jacobian, we make use of the elementary result from vector analysis that the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ can be interpreted as the volume of the parallelepiped with sides given by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ that meet at a point. The value of this scalar triple product is equal to the determinant with the elements of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ as its first, second, and third rows, respectively. Considering dx_1, dx_2, and dx_3 in (47) as vectors in the curvilinear coordinate system, we see that the linear differential volume element $dV = dx_1 dx_2 dx_3$ can be written

$$\pm dV = \begin{vmatrix} \frac{\partial x_1}{\partial q_1}dq_1 & \frac{\partial x_2}{\partial q_1}dq_1 & \frac{\partial x_3}{\partial q_1}dq_1 \\ \frac{\partial x_1}{\partial q_2}dq_2 & \frac{\partial x_2}{\partial q_2}dq_2 & \frac{\partial x_3}{\partial q_2}dq_2 \\ \frac{\partial x_1}{\partial q_3}dq_3 & \frac{\partial x_2}{\partial q_3}dq_3 & \frac{\partial x_3}{\partial q_3}dq_3 \end{vmatrix} = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \frac{\partial x_3}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} dq_1\, dq_2\, dq_3. \tag{64}$$

the Jacobian and the volume element

As a volume element is essentially nonnegative, this can be expressed in terms of the Jacobian J of the transformation as

$$dV = \pm J\, dq_1\, dq_2\, dq_3, \tag{65}$$

where the sign in (65) is chosen to make the expression on the right positive. A comparison of (63) and (65) then shows that the absolute value of the Jacobian J is equal to the product of the scale factors forming the scale factor for the linear volume element dV, and so

$$h_1 h_2 h_3 = \pm J, \tag{66}$$

where the sign is chosen to make the expression on the right positive.

EXAMPLE 11.22

Find the scale factors, the linear differential length elements along the curvilinear coordinate lines, the square of the general linear differential length element ds, the linear differential volume element dV, and the Jacobian for (a) cylindrical polar coordinates and (b) spherical polar coordinates.

Solution

(a) In cylindrical polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, so to relate this system to the general one just considered, we must make the identifications $x_1 = x$, $x_2 = y$, $x_3 = z$, $q_1 = r$, $q_2 = \theta$, and $q_3 = z$. When this is done, substitution into (57) to (59) shows that

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1,$$

so from (60) the linear differential length elements along the curvilinear coordinate lines are

$$dl_1 = dr, \quad dl_2 = r\, d\theta, \quad dl_3 = dz.$$

It then follows from (62) that the square of the general linear differential length element ds is

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2,$$

and from (63) that the linear differential volume element in terms of cylindrical polar coordinates is

$$dV = r\, dr\, d\theta\, dz.$$

The Jacobian of the transformation

$$J = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r,$$

in agreement with (66).

The transformation ceases to be one to one when $r = 0$, because then $h_2 = 0$, though this is to be expected because $r = 0$ is the z-axis along which θ is indeterminate.

(b) In spherical polar coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, so to relate this system to the general one just considered we must make the identifications $x_1 = x$, $x_2 = y$, $x_3 = z$, $q_1 = r$, $q_2 = \phi$, and $q_3 = \theta$. When this is done, substitution into (57) to (59) shows that

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta,$$

so from (60) the linear differential length elements along the curvilinear coordinate lines are

$$dl_1 = dr, \quad dl_2 = r\,d\phi, \quad dl_3 = r\sin\theta\,d\theta$$

As in (a), it follows from (62) that the square of the general linear differential length element ds is

$$ds^2 = dr^2 + r^2\sin^2\theta\,d\theta^2 + r^2\,d\phi^2$$

and from (63) that the linear differential volume element in terms of spherical polar coordinates is

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi.$$

The Jacobian of the transformation

$$J = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \end{vmatrix} = -r^2\sin\theta,$$

and in agreement with (66) we see that $h_1h_2h_3 = |J| = r^2\sin\phi$.

The Jacobian vanishes when $r = 0$, causing h_2 and h_3 to vanish, but this corresponds to the origin where θ and ϕ are indeterminate. The Jacobian also vanishes when $\phi = 0$ and $\phi = \pi$, corresponding to points on the z-axis where θ is indeterminate. ■

To derive the form of the gradient, divergence, curl, and Laplacian operators in rectangular curvilinear coordinates, it is necessary to introduce the triad of unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ at a general point $(q_1^{(0)}, q_2^{(0)}, q_3^{(0)})$. Here, $\mathbf{e}_1$ is tangent to the q_1 coordinate line, $\mathbf{e}_2$ is tangent to the q_2 coordinate line, and $\mathbf{e}_3$ is tangent to the q_3 coordinate line at the point $(q_1^{(0)}, q_2^{(0)}, q_3^{(0)})$. If we denote a general vector in curvilinear coordinates by $\mathbf{q}(q_1, q_2, q_3)$, the vector forms of the three coordinate lines become

$$\mathbf{q} = \mathbf{q}\big(q_1, q_2^{(0)}, q_3^{(0)}\big), \quad \mathbf{q} = \mathbf{q}\big(q_1^{(0)}, q_2, q_3^{(0)}\big), \quad \text{and} \quad \mathbf{q} = \mathbf{q}\big(q_1^{(0)}, q_2^{(0)}, q_3\big). \tag{67}$$

As a result, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are, respectively, parallel to the derivatives $\partial\mathbf{q}/\partial q_1$, $\partial\mathbf{q}/\partial q_2$, and $\partial\mathbf{q}/\partial q_3$ at the point $(q_1^{(0)}, q_2^{(0)}, q_3^{(0)})$. The scale factors along these coordinate lines are h_1, h_2, and h_3, it follows that the unit vectors at $(q_1^{(0)}, q_2^{(0)}, q_3^{(0)})$ are

$$\mathbf{e}_1 = \frac{\partial\mathbf{q}}{\partial q_1}\bigg/\left|\frac{\partial\mathbf{q}}{\partial q_1}\right|, \quad \mathbf{e}_2 = \frac{\partial\mathbf{q}}{\partial q_2}\bigg/\left|\frac{\partial\mathbf{q}}{\partial q_2}\right|, \quad \text{and} \quad \mathbf{e}_3 = \frac{\partial\mathbf{q}}{\partial q_3}\bigg/\left|\frac{\partial\mathbf{q}}{\partial q_3}\right|,$$

where, of course, the scale factors h_1, h_2, and h_3 are given by

$$h_1 = \left|\frac{\partial\mathbf{q}}{\partial q_1}\right|, \quad h_2 = \left|\frac{\partial\mathbf{q}}{\partial q_2}\right|, \quad \text{and} \quad h_3 = \left|\frac{\partial\mathbf{q}}{\partial q_3}\right|,$$

so that

$$\mathbf{e}_1 = \frac{1}{h_1}\frac{\partial\mathbf{q}}{\partial q_1}, \quad \mathbf{e}_2 = \frac{1}{h_2}\frac{\partial\mathbf{q}}{\partial q_2}, \quad \mathbf{e}_3 = \frac{1}{h_3}\frac{\partial\mathbf{q}}{\partial q_3}. \tag{68}$$

It is important to recognize that unlike the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, which are parallel to the fixed x-, y-, and z-axes so their derivatives are zero, the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in curvilinear coordinates are functions of position, so when finding the form of vector operators, we must take into account the derivatives of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

THEOREM 11.8

grad, div, and curl in general rectangular curvilinear coordinates

Gradient, divergence, curl, and Laplacian in general rectangular curvilinear coordinates Let the scalar function $f(q_1, q_2, q_3)$, and the vector function

$$\mathbf{F} = F_1(q_1, q_2, q_3)\mathbf{e}_1 + F_2(q_1, q_2, q_3)\mathbf{e}_2 + F_3(q_1, q_2, q_3)\mathbf{e}_3$$

be suitably differentiable functions of the rectangular curvilinear coordinates q_1, q_2, and q_3, where $\mathbf{e}_1$ is the unit vector in the direction of increasing q_1, $\mathbf{e}_2$ is the unit vector in the direction of increasing q_2, and $\mathbf{e}_3$ is the unit vector in the direction of increasing q_3 at the point (q_1, q_2, q_3). Then:

(i) $$\operatorname{grad} f = \mathbf{e}_1\frac{1}{h_1}\frac{\partial f}{\partial q_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial f}{\partial q_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial f}{\partial q_3}$$

(ii) $$\operatorname{div}\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3F_1) + \frac{\partial}{\partial q_2}(h_1h_3F_2) + \frac{\partial}{\partial q_3}(h_1h_2F_3)\right]$$

(iii) $$\operatorname{curl}\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$

(iv) $$\Delta \equiv \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial q_3}\right)\right]$$

(the Laplacian operator)

Proof

(i) To find grad $f = \frac{\partial f}{\partial x_1}\mathbf{i} + \frac{\partial f}{\partial x_2}\mathbf{j} + \frac{\partial f}{\partial x_3}\mathbf{k}$ in terms of curvilinear coordinates it is necessary to find the components of this vector in the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ directions, and then to use them as the components of a vector expressed in terms of curvilinear coordinates. As only q_1 varies in the direction of $\mathbf{e}_1$, it follows from the first equations in (46) and (68) that

$$\mathbf{e}_1 = \frac{1}{h_1}\left(\frac{\partial x_1}{\partial q_1}\mathbf{i} + \frac{\partial x_2}{\partial q_1}\mathbf{j} + \frac{\partial x_3}{\partial q_1}\mathbf{k}\right).$$

Thus, the component of grad f in the direction of the unit vector $\mathbf{e}_1$ is

$$\mathbf{e}_1 \cdot \operatorname{grad} f = \frac{1}{h_1}\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial q_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial q_1} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial q_1}\right) = \frac{1}{h_1}\frac{\partial f}{\partial q_1},$$

where the last result follows directly from the chain rule.

Corresponding results apply for the components of grad f in the directions of the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$, so if we use these results as the components of grad f in curvilinear coordinates, it follows that

$$\operatorname{grad} f = \mathbf{e}_1\frac{1}{h_1}\frac{\partial f}{\partial q_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial f}{\partial q_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial f}{\partial q_3},$$

and result (i) is established.

In what follows, for conciseness when establishing results (ii) to (iv), the operator notations $\nabla \cdot (\cdot)$ and $\nabla \times (\cdot)$ will be used to signify the divergence and curl operators.

(ii) As $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are orthogonal unit vectors $\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3$. By identifying f in (i) with q_1 we see that $\mathbf{e}_1 = h_1\nabla q_1$ and, similarly, by identifying f with q_2 and q_3 it follows that $\mathbf{e}_2 = h_2\nabla q_2$ and $\mathbf{e}_3 = h_3\nabla q_3$, and so $\mathbf{e}_1 = h_2h_3\nabla q_2 \times \nabla q_3$.

To find div $\mathbf{F}$ it is necessary to compute $\nabla \cdot (F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3)$ taking into account the dependence of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ on position. Because of the linearity of the divergence operator, this can be accomplished by taking the divergence of each term in $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ and then summing the results. The divergence of the first term is given by $\nabla \cdot (F_1\mathbf{e}_1) = \nabla \cdot (F_1h_2h_3\nabla q_2 \times \nabla q_3)$, so using result (iii) of Theorem 11.6, this becomes

$$\nabla \cdot (F_1\mathbf{e}_1) = F_1h_1h_2\nabla \cdot (\nabla q_2 \times \nabla q_3) + (\nabla q_2 \times \nabla q_3) \cdot \nabla(F_1h_1h_2).$$

However, applying result (v) of Theorem 11.7 to the term $\nabla \cdot (\nabla q_2 \times \nabla q_3)$ and using the fact that curl(grad q_2) = curl(grad q_3) = 0 simplifies this result to

$$\nabla \cdot (F_1\mathbf{e}_1) = (\nabla q_2 \times \nabla q_3) \cdot \nabla(F_1h_1h_2),$$

but $\mathbf{e}_1 = h_2h_3\nabla q_2 \times \nabla q_3$, and so

$$\nabla \cdot (F_1\mathbf{e}_1) = \frac{1}{h_2h_3}\mathbf{e}_1 \cdot \nabla(F_1h_2h_3).$$

In the proof of (i) we saw that

$$\mathbf{e}_1 \cdot \operatorname{grad} f = \frac{1}{h_1}\frac{\partial f}{\partial q_1},$$

so identifying f with $F_1h_2h_3$ we find that

$$\nabla \cdot (F_1\mathbf{e}_1) = \frac{1}{h_1h_2h_3}\frac{\partial(F_1h_2h_3)}{\partial q_1}.$$

Corresponding results apply to $\nabla\cdot(F_2\mathbf{e}_2)$ and $\nabla\cdot(F_3\mathbf{e}_3)$, so summing the results we arrive at result (iii).

(iii) To find curl $\mathbf{F}$ it is necessary to compute $\nabla\times(F_1\mathbf{e}_1+F_2\mathbf{e}_2+F_3\mathbf{e}_3)$, so as curl is a linear operator, we may compute the curl of each term in $\mathbf{F}=F_1\mathbf{e}_1+F_2\mathbf{e}_2+F_3\mathbf{e}_3$ and then sum the results. Considering the term $\nabla\times(F_1\mathbf{e}_1)$ and writing $\mathbf{e}_1=h_1\nabla q_1$, we find that $\nabla\times(F_1\mathbf{e}_1)=\nabla\times(F_1h_1\nabla q_1)$. Applying result (iii) of Theorem 11.7 to this last result, we find that

$$\nabla\times(F_1\mathbf{e}_1)=F_1h_1\nabla\times(\nabla q_1)-(\nabla q_1)\times(\nabla F_1h_1),$$

but $\nabla\times(\nabla q_1)=0$, and so

$$\nabla\times(F_1\mathbf{e}_1)=-(\nabla q_1)\times(\nabla F_1h_1).$$

Now $\nabla q_1=\mathbf{e}_1/h_1$, so if we reverse the sign in the preceding result and compensate by interchanging the order of the factors, the result becomes

$$\nabla\times(F_1\mathbf{e}_1)=\left[\mathbf{e}_1\frac{1}{h_1}\frac{\partial(F_1h_1)}{\partial q_1}+\mathbf{e}_2\frac{1}{h_2}\frac{\partial(F_1h_1)}{\partial q_2}+\mathbf{e}_3\frac{1}{h_3}\frac{\partial(F_1h_1)}{\partial q_3}\right]\times\frac{\mathbf{e}_1}{h_1},$$

and so using the orthogonality of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, which implies $\mathbf{e}_1\times\mathbf{e}_1=\mathbf{0}$, $\mathbf{e}_2\times\mathbf{e}_1=-\mathbf{e}_3$, and $\mathbf{e}_3\times\mathbf{e}_1=\mathbf{e}_2$, this becomes

$$\nabla\times(F_1\mathbf{e}_1)=\mathbf{e}_2\frac{1}{h_1h_3}\frac{\partial}{\partial q_3}(h_1F_1)-\mathbf{e}_3\frac{1}{h_1h_2}\frac{\partial}{\partial q_2}(h_1F_1).$$

Corresponding results exist for $\nabla\times(F_2\mathbf{e}_2)$ and $\nabla\times(F_3\mathbf{e}_3)$, so combining them we find that

$$\begin{aligned}\nabla\times\mathbf{F}&=\mathbf{e}_2\frac{1}{h_1h_3}\frac{\partial}{\partial q_3}(h_1F_1)-\mathbf{e}_3\frac{1}{h_1h_2}\frac{\partial}{\partial q_2}(h_1F_1)+\mathbf{e}_3\frac{1}{h_1h_2}\frac{\partial}{\partial q_1}(h_2F_2)\\&\quad-\mathbf{e}_1\frac{1}{h_2h_3}\frac{\partial}{\partial q_3}(h_2F_2)+\mathbf{e}_1\frac{1}{h_2h_3}\frac{\partial}{\partial q_1}(h_3F_3)-\mathbf{e}_2\frac{1}{h_1h_3}\frac{\partial}{\partial q_2}(h_3F_3).\end{aligned}$$

This last result is seen to be the expansion of the determinant in (iii), so the proof is complete.

(iv) The Laplacian operator

$$\begin{aligned}\Delta&=\nabla\cdot\left[\mathbf{e}_1\frac{1}{h_1}\frac{\partial}{\partial q_1}+\mathbf{e}_2\frac{1}{h_2}\frac{\partial}{\partial q_2}+\mathbf{e}_3\frac{1}{h_3}\frac{\partial}{\partial q_3}\right]\\&=\operatorname{div}\left[\mathbf{e}_1\frac{1}{h_1}\frac{\partial}{\partial q_1}+\mathbf{e}_2\frac{1}{h_2}\frac{\partial}{\partial q_2}+\mathbf{e}_3\frac{1}{h_3}\frac{\partial}{\partial q_3}\right].\end{aligned}$$

Using result (ii) of the theorem with the operator $\frac{1}{h_1}\frac{\partial}{\partial q_1}$ in place of F_1, the operator $\frac{1}{h_2}\frac{\partial}{\partial q_2}$ in place of F_2 and the operator $\frac{1}{h_3}\frac{\partial}{\partial q_3}$ in place of F_3, we arrive at result (iv).

■

EXAMPLE 11.23

grad, div, curl, and the Laplacian in cylindrical and spherical polar coordinates

Find the forms taken by grad, div, curl, the Laplacian, and the Laplacian operator in (a) cylindrical polar coordinates and (b) spherical polar coordinates.

Solution **(a)** Using the notation of Example 11.22 and the scale factors $h_1=1$, $h_2=r$, and $h_3=1$ found in that example, routine calculations show that in

cylindrical polar coordinates, when $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z$,

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

$$\operatorname{div} \mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

$$\operatorname{curl} \mathbf{F} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$$

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad \text{(Laplacian operator).}$$

(b) Again using the notation of Example 11.21 and the scale factors $h_1 = 1$, $h_2 = r\sin\phi$, $h_3 = r$ found in that example, routine calculations show that in **spherical polar coordinates**, when $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$,

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$$

$$\operatorname{div} \mathbf{F} = \frac{1}{r^2}\frac{\partial (r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$\operatorname{curl} \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix};$$

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

(Laplacian operator). ■

Descriptions of general orthogonal curvilinear coordinates and the form taken by vector operators in different coordinate systems are to be found in references [1.3] and [5.2], whereas applications to continuum mechanics are to be found in reference [5.4] and to hydrodynamics in reference [6.5]. Further information can also be found in Chapters 23 and 24 of reference [G.3].

Summary

After introducing the concept of general orthogonal curvilinear coordinates, this section then derived expressions for grad, div, curl, and the Laplacian operators in terms of these coordinates. Because of the importance of cylindrical and spherical polar coordinates in applications, these operators were then expressed in terms of cylindrical and spherical polar coordinates.

EXERCISES 11.6

1. Write out the results of Theorem 11.6 using the operator notation $\nabla(.)$, $\nabla \cdot (.)$, $\nabla \times (.)$ in place of grad, div, and curl.
2. Write out the results of Theorem 11.7 using the operator notation $\nabla(.)$, $\nabla \cdot (.)$, $\nabla \times (.)$ in place of grad, div, and curl.
3. Complete the calculations leading to the results of Example 11.22(a) for cylindrical polar coordinates.
4. Complete the calculations leading to the results of Example 11.22(b) for spherical polar coordinates.
5. Show the curvilinear coordinate system defined in the region $q_3 \geq 0$ by the equations $x_1 = q_1 - q_2$, $x_2 = q_1 + q_2$, and $x_3 = \sinh q_3$ is orthogonal. Find the scale factors h_1, h_2, h_3, grad f, and div $\mathbf{F}$.
6. Show that the **parabolic cylindrical coordinates** (u, v, z) defined by the equations $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$, $z = z$ are orthogonal. Find the scale factors h_1, h_2, h_3, and $\nabla^2 f$.
7. Show that the **elliptic cylindrical coordinates** (ξ, η, z) defined by the equations $x = \cosh \xi \cos \eta$, $y = \sinh \xi \sin \eta$, $z = z$ for $0 \leq \xi < \infty$, $-\pi < \eta \leq \pi$, $-\infty < z < \infty$ are orthogonal. Find the scale factors h_1, h_2, h_3 and state the shapes of the surfaces $\xi =$ constant and $\eta =$ constant and find grad f.