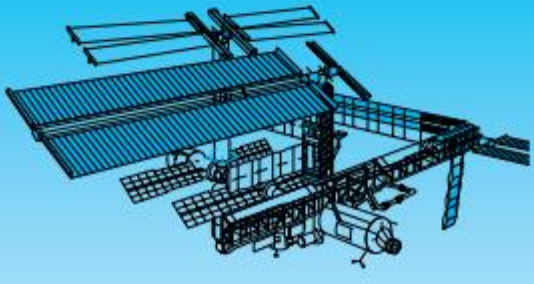
CHAPTER 10



Fourier Integrals and the Fourier Transform

Fourier series enable functions and solutions of linear systems defined over a finite interval to be represented as an infinite series of sines and cosines. This suffices for many physical problems, but often the interval involved is either semi-infinite or infinite, in which case a somewhat different representation becomes necessary. This happens, for example, when working with the partial differential equations that describe heat conduction and diffusion in a half-space for which Fourier series cannot be used.

The Fourier integral can be regarded as the limiting case of a Fourier series representation of a function $f(x)$ defined over an interval $-L < x < L$ as $L \to \infty$. The meaning of the integral representation when the function to be represented is discontinuous is considered, and the special cases of the sine and cosine integral representations are introduced.

Fourier sine and cosine transforms are considered, tables of their transform pairs are given, and the transform of derivatives is discussed. In anticipation of Chapter 18, an application of the Fourier transform is made to the problem of the one-dimensional time dependent heat equation.

10.1 The Fourier Integral

A Fourier series has been shown to represent an arbitrary function $f(x)$ over an interval $-L \leq x \leq L$, and because the series is periodic with period $2L$ the representation of $f(x)$ in this fundamental interval is repeated by periodicity for all x outside the interval. However, even if $f(x)$ is defined outside the fundamental interval, it does not necessarily follow that the function and its periodic extensions coincide outside the interval. This means that if a nonperiodic function is to be represented over an arbitrarily large interval, some generalization of a Fourier series is required.

Letting $L \to \infty$ in a Fourier series leads to the introduction of a different type of representation called a **Fourier integral representation**, where the function $f(x)$ is defined for all x and need not be periodic. This representation forms the basis of an integral transform called the **Fourier transform** that is similar to the Laplace transform. As with the Laplace transform, one of the the main uses of the Fourier transform is in the solution of differential equations.

The derivation of the Fourier integral representation given here is heuristic, because a rigorous one requires techniques that are not needed elsewhere in the book. We start from the definition of a Fourier series of $f(x)$ over an interval $-L \leq x \leq L$ given in (18) and (19) of Section 9.1 by writing

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \quad \text{for } n = 1, 2, \ldots. \end{aligned} \tag{2}$$

Substituting the Fourier coefficients (2) into Fourier series (1) allows it to be written in the integral form

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(u)du + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(u)\cos\frac{n\pi(u-x)}{L}du. \tag{3}$$

To proceed further, if the representation is to remain valid as $L \to \infty$ the first term must not become either infinite or indeterminate. This will certainly be true if $\lim_{L\to\infty}\int_{-L}^{L}|f(x)|dx$ is finite, because then the integral involved in the first term will be *absolutely convergent* and the first term in (3) will vanish in the limit as $L \to \infty$. From now on we will assume this condition to be satisfied. We can now write (3) as

$$f(x) = \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(u)\cos\frac{n\pi(u-x)}{L}du. \tag{4}$$

It is from this point onward that our derivation of the Fourier integral representation becomes heuristic, because the arguments used to convert (4) to an integral over the interval $(-\infty, \infty)$ are merely intuitive. A careful examination of the convergence of the double integral involved would be necessary to provide a rigorous justification.

Setting $\Delta_n\omega = \pi/L$, and defining the frequency $\omega_n = n\pi/L$, allows (4) to be rewritten as

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\Delta_n\omega\int_{-L}^{L} f(u)\cos[\omega_n(u-x)]du. \tag{5}$$

Examination of (5) suggests it is equivalent to the pre-limit sum approximation used in the definition of the definite (Riemann) integral of the function

$$F(u) = \frac{1}{\pi}\int_{-L}^{L} f(u)\cos\omega(u-x)du.$$

Using this last result in (5), and proceeding to the limit as $L \to \infty$, we obtain

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}d\omega\int_{-\infty}^{\infty} f(u)\cos\omega(u-x)du, \tag{6}$$

the Fourier integral representation

which is called the **Fourier integral representation** of $f(x)$.

By defining the functions $A(\omega)$ and $B(\omega)$ as

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\omega u du \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\omega u du, \tag{7}$$

the Fourier integral representation in (6) can be written in the simpler form

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega. \tag{8}$$

Dirichlet conditions

The convergence properties of Fourier series recorded in Theorem 9.1 can be shown to be transferred to the Fourier integral representation of $f(x)$ if, in addition to the integral of $f(x)$ being absolutely convergent over $(-\infty, \infty)$, it also satisfies certain other conditions. These conditions, called **Dirichlet conditions**, are as follows:

(i) In any finite interval $f(x)$ has only a finite number of maxima and minima

(ii) In any finite interval $f(x)$ has only a finite number of bounded jump discontinuities and no infinite jump discontinuities.

We now state the following theorem for the Fourier integral without proof.

PETER GUSTAV LEJEUNE DIRICHLET (1805–1859)
A German mathematician who studied under Gauss, was the son-in-law of Jacobi and succeeded Gauss as Professor of Mathematics at Göttingen. He did much to make some of the more abstruse contributions by Gauss better understood. His most important contributions to mathematics were his major contribution to the understanding of the convergence of Fourier series, and his work on number theory and the theory of potential.

THEOREM 10.1

the fundamental Fourier integral theorem

Fourier integral theorem Let $f(x)$ satisfy Dirichlet conditions, and suppose the (sufficiency) conditions that $f(x)$ be both integrable and absolutely integrable over the interval $-\infty < x < \infty$ are both satisfied, so each of the integrals $\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} |f(x)|dx$ exists. Then

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(u)\cos\omega(u-x)du$$

or, equivalently,

$$\frac{1}{2}[f(x+0) + f(x-0)] = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\omega u du \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\omega u du.$$

■

EXAMPLE 10.1

Find the Fourier integral representation of $f(x) = e^{-|x|}$.

Solution The function $e^{-|x|}$ satisfies the Dirichlet conditions, and $\int_{-\infty}^{\infty} |e^{-|x|}|dx = 2$, so the integral of $f(x) = e^{-|x|}$ over $(-\infty, \infty)$ is absolutely convergent. This confirms that $f(x) = e^{-|x|}$ has a Fourier integral representation.

The function $e^{-|x|}$ is even in x, so $e^{-|u|}\cos\omega u$ is also even, and

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|u|}\cos\omega u du = \frac{2}{\pi}\int_0^{\infty} e^{-u}\cos\omega u du = \frac{2}{\pi(1+\omega^2)}.$$

As the function $e^{-|u|}\sin\omega u$ is odd in u,

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|u|}\sin\omega u du = \frac{2}{\pi}\int_0^{\infty} e^{-u}\sin\omega u du = 0,$$

so from (8) the Fourier integral representation of $e^{-|x|}$ is seen to be

$$e^{-|x|} = \frac{2}{\pi}\int_0^{\infty}\frac{\cos\omega x}{1+\omega^2}d\omega.$$

EXAMPLE 10.2 Find the Fourier integral representation of

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$$

and use Theorem 10.1 to find the value of the resulting integral when (a) $x < 0$, (b) $x = 0$, and (c) $x > 0$.

Solution The function $f(x)$ satisfies the Dirichlet conditions and the integral $\int_{-\infty}^{\infty}|f(x)|dx = \int_0^{\infty} e^{-x}dx = 1$, so as the conditions of Theorem 10.1 are satisfied the function has a Fourier integral representation.

We have

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\omega u du = \frac{1}{\pi}\int_0^{\infty} e^{-u}\cos\omega u du = \frac{1}{\pi(1+\omega^2)}$$

and

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\omega u du = \frac{1}{\pi}\int_0^{\infty} e^{-u}\sin\omega u du = \frac{\omega}{\pi(1+\omega^2)}.$$

Substituting into (8) shows the Fourier integral representation to be

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega \quad \text{for } -\infty < x < \infty.$$

Applying the results of Theorem 10.1 to this integral, we find that

$$\pi f(x) = \int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega = \begin{cases} 0, & x < 0 \\ \pi/2, & x = 0 \\ \pi e^{-x}, & x > 0. \end{cases}$$

When $x = 0$, this last result is seen to reduce to the familiar definite integral

$$\int_0^{\infty}\frac{d\omega}{1+\omega^2} = \frac{\pi}{2}.$$

Special forms of the Fourier integral representation arise according to whether $f(x)$ is even or odd. When $f(x)$ is an even function, $f(u)\sin\omega u$ is an odd function of u, so $B(\omega) \equiv 0$ and

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(u)\cos\omega u du, \tag{9}$$

so that (8) simplifies to the **Fourier cosine integral representation** of $f(x)$

$$f(x) = \int_0^\infty A(\omega)\cos\omega x d\omega. \tag{10}$$

Similarly, when $f(x)$ is an odd function, $f(u)\cos\omega u$ is an odd function of u, so $A(\omega) \equiv 0$ and

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(u)\sin\omega u du, \tag{11}$$

causing (8) to simplify to the **Fourier sine integral representation** of $f(x)$ given by

$$f(x) = \int_0^\infty B(\omega)\sin\omega x d\omega. \tag{12}$$

different Fourier integral representations

Summary of Fourier integral representations

(a) An *arbitrary* function $f(x)$ satisfying the conditions of Theorem 10.1 has the **general Fourier integral representation**

$$\frac{1}{2}[f(x+0) + f(x-0)] = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega. \tag{13}$$

(b) An *even* function $f(x)$ satisfying the conditions of Theorem 10.1 has the **Fourier cosine integral representation**

$$\frac{1}{2}[f(x+0) + f(x-0)] = \int_0^\infty A(\omega)\cos\omega x d\omega. \tag{14}$$

(c) An *odd* function $f(x)$ satisfying the conditions of Theorem 10.1 has the **Fourier sine integral representation**

$$\frac{1}{2}[f(x+0) + f(x-0)] = \int_0^\infty B(\omega)\sin\omega x d\omega, \tag{15}$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\cos\omega u du \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\sin\omega u du. \tag{16}$$

Summary

The Fourier integral representation of a function $f(x)$ was introduced as the natural extension of a Fourier series representation as the interval of the representation extends to become the interval $-\infty < x < \infty$. A fundamental representation theorem was given and illustrated by example, and some useful special cases of the theorem were considered.

EXERCISES 10.1

Find the Fourier integral representation of the given functions.

1. The rectangular pulse function $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ (Fig. 10.1).

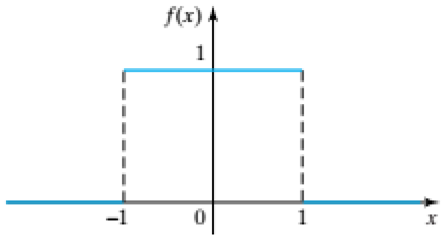


FIGURE 10.1 The rectangular pulse function.

2. The triangular function

$$f(x) = \begin{cases} 0, & |x| > a \\ b\left(1 + \dfrac{x}{a}\right), & -a \le x \le 0 \\ b\left(1 - \dfrac{x}{a}\right), & 0 \le x \le a \end{cases} \quad \text{(Fig. 10.2).}$$

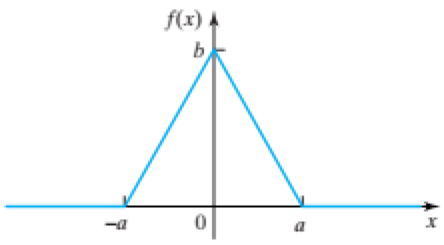


FIGURE 10.2 The triangular function.

3. $f(x) = \begin{cases} 0, & |x| > a \\ bx/a, & -a \le x \le a \end{cases}$ (Fig. 10.3).

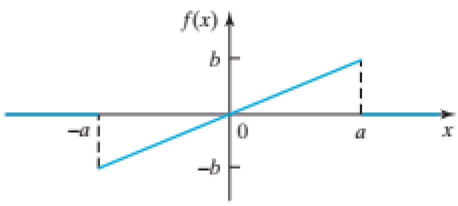


FIGURE 10.3 The truncated straight line function.

4. $f(x) = \begin{cases} 0, & x \le 0 \\ \sin x, & 0 \le x \le \pi \\ 0, & x \ge \pi \end{cases}$ (Fig. 10.4).

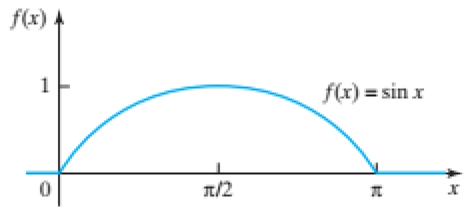


FIGURE 10.4 The asymmetric truncated sine function.

5. $f(x) = \begin{cases} (\pi/2)\cos x, & |x| < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$ (Fig. 10.5).

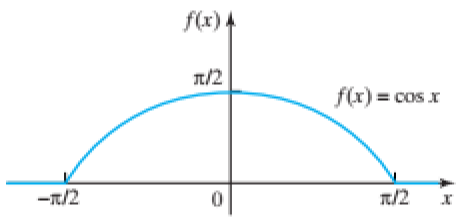


FIGURE 10.5 The truncated cosine function.

6. $f(x) = \begin{cases} (\pi/2)\sin x, & |x| < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$ (Fig. 10.6).

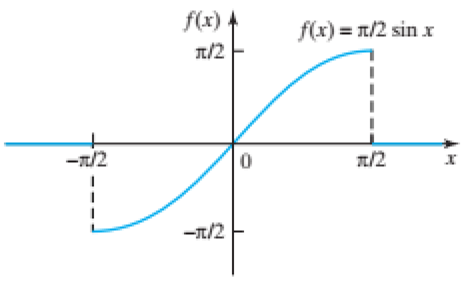


FIGURE 10.6 The truncated sine function.

7. $f(x) = \begin{cases} 0, & x < 0 \\ \cos x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ (Fig. 10.7).

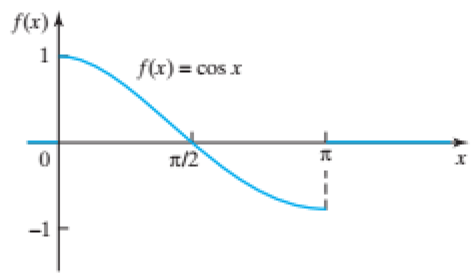


FIGURE 10.7 The asymmetric truncated cosine function.

8. The hump function $f(x) = 1/(1 + x^2)$ (Fig. 10.8). (Hint: Use the result of Example 10.16 with a change of notation.)

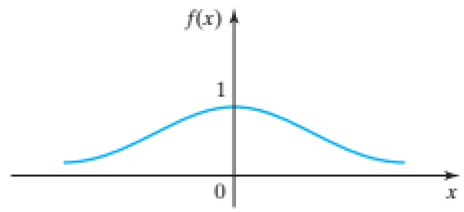


FIGURE 10.8 The hump function.

10.2 The Fourier Transform

The starting point for the development of the *Fourier transform* is the complex form of the Fourier integral representation of a function $f(x)$. To derive this representation in which $f(x)$ is defined over the interval $(-\infty, \infty)$, we substitute into (8) of Section 10.1 the expressions for $A(\omega)$ and $B(\omega)$ given in (7) to obtain

$$\begin{aligned}\frac{1}{2}[f(x+0) + f(x-0)] &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(u)[\cos\omega u \cos\omega x + \sin\omega u \sin\omega x]du\right]d\omega \\ &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(u)\cos\{\omega(u-x)\}du\right]d\omega \\ &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(u)\cos\{\omega(x-u)\}du\right]d\omega,\end{aligned}$$

where we have used the result $\cos\omega(u-x) = \cos\omega(x-u)$.

As the integrand in the last integral is an even function of ω, the interval of integration with respect to ω can be doubled and the result compensated by the introduction of a multiplicative factor 1/2 to give

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(u)\cos\omega(x-u)du\right]d\omega. \tag{17}$$

The function $\sin\omega(x-u)$ is an odd function of ω, so it follows directly that

$$0 = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(u)\sin\{\omega(x-u)\}du\right]d\omega. \tag{18}$$

the complex Fourier integral representation

Multiplying equation (18) by i, adding the result to equation (17), and using the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, we arrive at the **complex Fourier integral**

representation

$$\frac{1}{2}[f(x+0)+f(x-0)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(u)\exp\{i\omega(x-u)\}du\right]d\omega. \tag{19}$$

The brackets in (17) to (19) were retained to clarify the order in which the integrations are performed, but they are usually omitted in (19), which then becomes

$$\frac{1}{2}[f(x+0)+f(x-0)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)\exp\{i\omega(x-u)\}du\,d\omega. \tag{20}$$

Clearly, the left-hand side of (20) reduces to $f(x)$ wherever the function is continuous.

To arrive at the definitions of a Fourier transform and its inverse we write the factor $\exp\{i\omega(x-u)\}$ in (19) (equivalently (20)) as the product $\exp\{i\omega x\}\cdot\exp\{-i\omega u\}$. Then, as the inner integral only involves integration with respect to u, we rewrite (19) as

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\{i\omega x\}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\exp\{-i\omega u\}du\right]d\omega, \tag{21}$$

where the left-hand side is to be replaced by $(1/2)[f(x+0)+f(x-0)]$ whenever $f(x)$ is discontinuous.

If we now define the function $F(\omega)$ as

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\exp\{-i\omega u\}du,$$

then because u is a dummy variable it can be replaced by x and the result rewritten as

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\exp\{-i\omega x\}dx, \tag{22}$$

so that (19) becomes

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)\exp\{i\omega x\}d\omega. \tag{23}$$

Fourier transforms and transform pairs

The function $F(\omega)$ in (22) is called the **Fourier transform** of $f(x)$, or sometimes the **exponential Fourier transform**, and because integral (23) recovers $f(x)$ from $F(\omega)$ it is called the **inversion integral** for the Fourier transform. As with the Laplace transform, when working with the Fourier transform the function $f(x)$ and the associated Fourier transform $F(\omega)$ are called a **Fourier transform pair**. A short table of Fourier transform pairs is to be found at the end of this section.

Various other notations are used to indicate the Fourier transform of $f(x)$, the most common of which involves representing it by $\hat{f}(\omega)$, so in terms of the notation used here, $\hat{f}(\omega)=F(\omega)$.

Another notation that is often useful involves representing the Fourier transform of $f(x)$ by $\mathcal{F}\{f(x)\}$, so that $\mathcal{F}\{f(x)\} = F(\omega)$, and when this notation is used the inverse Fourier transform is written $\mathcal{F}^{-1}\{F(\omega)\} = f(x)$. In what follows a function to be transformed is denoted by a lowercase letter, and the corresponding uppercase letter is then used to denote its Fourier transform. So, for example, $\mathcal{F}\{g(x)\} = G(\omega)$ and $\mathcal{F}\{h(x)\} = H(\omega)$.

The choice of the normalizing factors $1/\sqrt{2\pi}$ in integrals (22) and (23) is optional, and it is chosen here to introduce as much symmetry as possible into the definitions of a Fourier transform and its inverse. All that is required of the normalizing factors is that their product be $1/(2\pi)$, so in many reference works the factor $1/\sqrt{2\pi}$ in (22) is replaced by 1, while the factor $1/\sqrt{2\pi}$ in (23) is replaced by $1/(2\pi)$. It is impossible to achieve complete symmetry in the definitions of a Fourier integral and its inverse because the exponential factor occurs with opposite signs in (22) and (23).

When Fourier transforms listed in reference works are used, another source of confusion can arise because sometimes the signs in the exponential factors occurring in integrals (22) and (23) are interchanged. When this happens a Fourier transform obtained using this sign convention can be converted to the one used here by reversing the sign of ω. However, each definition of the Fourier transform and the corresponding inversion integral conform to the general pattern

$$\mathcal{F}\{f(x)\} = \frac{k}{2\pi}\int_{-\infty}^{\infty} f(x)\exp\{\pm i\omega x\}dx \quad \text{and}$$
$$\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{k}\int_{-\infty}^{\infty} F(\omega)\exp\{\mp i\omega x\}d\omega, \tag{24}$$

where k is an arbitrary scale factor.

In view of the different conventions that are in use, when working with Fourier transforms and referring to reference works, it is essential that the normalizing factor k and the sign convention employed in the exponential factors be established before any use is made of the results.

When we considered the convergence of Fourier series, the Riemann–Lebesgue lemma was established the results of which were that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\,\cos nx dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\,\sin nx dx = 0. \tag{25}$$

A limiting argument similar to the one used in Section 10.1 when deriving the Fourier integral representation of $f(x)$ shows that, provided $f(x)$ has a Fourier transform,

$$\lim_{|\omega|\to\infty}\int_{-\infty}^{\infty} f(x)\,\cos\omega x dx = \lim_{|\omega|\to\infty}\int_{-\infty}^{\infty} f(x)\,\sin\omega x dx = 0. \tag{26}$$

As the Fourier transform $F(\omega)$ of $f(x)$ can be written

$$F(\omega) = \tfrac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} f(x)\,\cos\omega x dx - i\int_{-\infty}^{\infty} f(x)\,\sin\omega x dx\right], \tag{27}$$

an application of limits (26) in (27) establishes the important property of a Fourier transform that

$$\lim_{|\omega|\to\infty} F(\omega) = 0. \tag{28}$$

EXAMPLE 10.3 Find the Fourier transforms of

(a) $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a, \end{cases}$ (b) $g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise}, \end{cases}$ (c) $p(x) = \dfrac{1}{x^2 + a^2}$

by making use of the standard integral $\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2+a^2} dx = \frac{\pi}{a} e^{-|\omega|a}$ $(a > 0)$ and (d) $q(x) = \begin{cases} e^{iax}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$. In each case confirm that the Fourier transform vanishes as $\omega \to \pm\infty$.

Solution

(a)

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\omega x} dx = \frac{1}{\omega\sqrt{2\pi}} \left[\frac{e^{i\omega a} - e^{-i\omega a}}{i} \right]$$

$$= \frac{1}{\omega}\sqrt{\frac{2}{\pi}} \left[\frac{e^{i\omega a} - e^{-i\omega a}}{2i} \right] = \sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}.$$

As $\sin \omega a$ is bounded, it follows directly that $\lim_{|\omega|\to\infty} F(\omega) = 0$.

(b)

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^a e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{1 - e^{-i\omega a}}{i\omega} \right).$$

As the numerator of $G(\omega)$ is bounded, it follows that $\lim_{|\omega|\to\infty} G(\omega) = 0$. This example shows that although $f(x)$ may be real, its Fourier transform can be complex.

(c)

$$P(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{x^2 + a^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + a^2} dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin \omega x}{x^2 + a^2} dx.$$

The integrand of the second integral is odd, so the value of the integral is zero. Using the standard result

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + a^2} dx = \frac{\pi}{a} e^{-|\omega|a}$$

in the remaining integral on the right, we find that

$$P(\omega) = \sqrt{\frac{\pi}{2}} \frac{e^{-|\omega|a}}{a} \quad (a > 0).$$

In this case the factor $e^{-|\omega|a}$ ensures that $\lim_{|\omega|\to\infty} P(\omega) = 0$.

(d)

$$Q(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i(\omega - a)x} dx$$

$$= \frac{i}{\sqrt{2\pi}} \left(\frac{1 - e^{-i(\omega - a)}}{a - \omega} \right).$$

As the numerator of the Fourier transform is bounded, the denominator causes the transform to vanish as $|\omega| \to \infty$. This example shows that a complex function can also have a Fourier transform and, in general, that the transform will be complex.

the main operational properties of Fourier transforms

The fundamental properties contained in Theorems 10.2 to 10.8 that follow are called **operational properties** of the Fourier transform. Familiarity with these properties is essential, because they simplify calculations involving Fourier transforms and can lead to results that are difficult to obtain without their use.

THEOREM 10.2

Linearity of the Fourier transform Let the functions $f(x)$ and $g(x)$ have the respective Fourier transforms $F(\omega)$ and $G(\omega)$, and let a and b be arbitrary constants. Then

$$\mathcal{F}\{af(x)+bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}.$$

Proof As the Fourier integral involves the operation of integration, the linearity property of the transform follows directly from the linearity property of the definite integral. ■

Theorem 10.2 is important when the Fourier transform of a sum of functions is required, because it is this result that allows each term involved in the sum to be transformed separately before the results are added.

EXAMPLE 10.4

Find the Fourier transform of $3f(x) - 2g(x)$, where $f(x)$ and $g(x)$ are the functions in (a) and (b) of Example 10.3.

Solution Using the results of Example 10.3 and applying Theorem 10.2, we have

$$\begin{aligned}\mathcal{F}\{3f(x) - 2g(x)\} &= 3\mathcal{F}\{f(x)\} - 2\mathcal{F}\{g(x)\}\\ &= \sqrt{\frac{2}{\pi}}\left\{\frac{3\sin\omega a}{\omega} - \left(\frac{1-e^{-i\omega a}}{i\omega}\right)\right\}.\end{aligned}$$

■

THEOREM 10.3

Fourier transform of a derivative of $f(x)$ Let $f(x)$ be a continuous function of x with the property that $\lim_{|x|\to\infty} f(x) = 0$, and such that $f'(x)$ is absolutely integrable over $(-\infty, \infty)$. Then:

(a) $\mathcal{F}\{f'(x)\} = i\omega F(\omega)$.

(b) For all n such that the derivatives $f^{(r)}(x)$ with $r = 1, 2, \ldots, n$ satisfy Dirichlet conditions, are absolutely integrable over $(-\infty, \infty)$, and $\lim_{|x|\to\infty} f^{(n-1)}(x) = 0$,

$$\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n F(\omega),$$

where $f^{(n)}(x) = d^n f/dx^n$.

Proof

(a) Integration by parts coupled with the condition that $\lim_{|x|\to\infty} f(x) = 0$ gives

$$\begin{aligned}\mathcal{F}\{f'(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty} - (-i\omega)\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx\right]\\ &= i\omega\,\mathcal{F}\{f(x)\} = i\omega F(\omega),\end{aligned}$$

where the term $f(x)e^{-i\omega x}|_{-\infty}^{\infty}$ vanishes because of the condition $\lim_{|x|\to\infty} f(x) = 0$.

(b) The second part of the theorem follows by repeated application of result (a), and the conditions imposed on $f^{(n)}(x)$ are necessary to ensure that its Fourier transform exists. ■

EXAMPLE 10.5

Find the Fourier transform of $p'(x)$ from the Fourier transform of $p(x)$, where $p(x)$ is the function in Example 10.3(c).

Solution It was shown in Example 10.3(c) that $P(\omega) = \sqrt{\frac{\pi}{2}}\frac{e^{-|\omega|a}}{a}$, so it follows from Theorem 10.3 (a) that $\mathcal{F}\{p'(x)\} = i\omega P(\omega) = i\omega\sqrt{\frac{\pi}{2}}\frac{e^{-|\omega|a}}{a}$. ■

THEOREM 10.4

Fourier transform of $x^n f(x)$ Let $f(x)$ be a continuous and differentiable function with an n times differentiable Fourier transform $F(\omega)$. Then

(a) $$\mathcal{F}\{xf(x)\} = i\frac{d}{d\omega}[F(\omega)]$$

and

(b) $$\mathcal{F}\{x^n f(x)\} = i^n\frac{d^n}{d\omega^n}[F(\omega)],$$

for all n such that $\lim_{|\omega|\to\infty} F^{(n)}(\omega) = 0$.

Proof The proof of the theorem follows directly by the application of *Leibniz's rule* that governs differentiation under the integral sign. The rule may be stated as follows:

Leibniz' rule: Let $f(x,\omega)$ and $\partial f/\partial\omega$ be continuous functions of their variables with $-\infty < x < \infty$ and $-\infty < \omega < \infty$. Furthermore, let $\int_{-\infty}^{\infty}|f(x,\omega)|dx$ be finite and $|\partial f/\partial\omega| \le h(x)$ where $h(x)$ is piecewise continuous and such that $\int_{-\infty}^{\infty} h(x)dx$ is finite. Then

$$\frac{d}{d\omega}\int_{-\infty}^{\infty} f(x,\omega)dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial\omega}[f(x,\omega)]dx.$$

(a) Using Leibniz' rule to differentiate the Fourier transform of $f(x)$, we obtain

$$\frac{d}{d\omega}[F(\omega)] = \frac{1}{\sqrt{2\pi}}\frac{d}{d\omega}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xf(x)e^{-i\omega x}dx.$$

The required result follows from this after multiplication by i, because the expression on the right is then $\mathcal{F}\{xf(x)\}$.

(b) The proof for the case when $n > 1$ follows by repeated application of result (a). The conditions imposed on $x^n f(x)$ and $F(\omega)$ are necessary to ensure the existence of the Fourier transform. ■

THEOREM 10.5

Fourier transform of $x^m f^{(n)}(x)$ Let $f(x)$ be a continuous n times differentiable function. Furthermore, let $x^m f^{(r)}(x)$ for $r = 1, 2, \ldots, n$ satisfy Dirichlet conditions and be absolutely integrable over $(-\infty, \infty)$, and let $\omega^n F(\omega)$ possess an m times differentiable inverse Fourier transform. Then, provided $\lim_{|x|\to\infty} f^{(n-1)}(x) = 0$,

$$\mathcal{F}\{x^m f^{(n)}(x)\} = (i)^{m+n}\frac{d^m}{d\omega^m}[\omega^n F(\omega)].$$

Proof The result follows directly by combining Theorems 10.3 and 10.4, because

$$\mathcal{F}\{x^m f^{(n)}(x)\} = (i)^m \frac{d^m}{d\omega^m}\mathcal{F}\{f^{(n)}(x)\} = (i)^{m+n}\frac{d^m}{d\omega^m}[\omega^n F(\omega)].$$

The conditions imposed on $x^m f^{(n)}(x)$ and $\omega^n F(\omega)$ are necessary to ensure the existence of the Fourier transform. ■

The examples that follow illustrate how Theorems 10.3 to 10.5 may be used to find the Fourier transforms of more complicated functions.

EXAMPLE 10.6 Find the Fourier transform of $f(x) = \exp(-a^2x^2)(a > 0)$.

Solution The function $f(x)$ is continuous and differentiable for all x and

$$\int_{-\infty}^{\infty} |\exp(-a^2x^2)|dx = \int_{-\infty}^{\infty} \exp(-a^2x^2)dx = \frac{1}{a}\int_{-\infty}^{\infty} \exp(-u^2)du = \frac{\sqrt{\pi}}{a},$$

where we have made use of the standard integral $\int_{-\infty}^{\infty} \exp(-u^2)du = \sqrt{\pi}$. This shows that $f(x)$ is absolutely integrable over the interval $(-\infty, \infty)$, and so $f(x)$ has a Fourier transform. A straightforward calculation establishes that $f(x)$ satisfies the differential equation

$$f' + 2a^2xf = 0.$$

Taking the Fourier transform of this equation using Theorem 10.2 gives

$$\mathcal{F}\{f'(x)\} + 2a^2\mathcal{F}\{xf(x)\} = 0.$$

Applying Theorem 10.3 to the first term and Theorem 10.4 to the second term and cancelling a factor i reduces this to the variables separable equation for $F(\omega)$,

$$2a^2F' + \omega F = 0, \quad \text{where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-a^2x^2)e^{-i\omega x}dx.$$

When variables are separated, the equation becomes

$$\int \frac{F'}{F}d\omega = -\frac{1}{2a^2}\int \omega d\omega,$$

so

$$\ln F(\omega) = -\frac{\omega^2}{4a^2} + \ln A, \quad \text{or} \quad F(\omega) = A\exp\left[-\frac{\omega^2}{4a^2}\right],$$

where, for convenience, the arbitrary integration constant has been written in the form $\ln A$. To determine A we use the fact that $A = F(0)$, but

$$F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-a^2x^2)dx = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{a} = \frac{1}{a\sqrt{2}},$$

and so

$$\mathcal{F}\{\exp(-a^2x^2)\} = F(\omega) = \frac{1}{a\sqrt{2}}\exp\left\{-\frac{\omega^2}{4a^2}\right\} \quad (a > 0).$$ ■

EXAMPLE 10.7

finding the Fourier transform of a function defined by a differential equation

Find the Fourier transform of the Bessel function $J_0(x)$.

Solution The Bessel function $J_0(x)$ does not satisfy the absolute integrability condition found in Theorem 10.1. However, this is merely a sufficient condition that ensures the existence of the Fourier transform of a function $f(x)$, though not a necessary one. Functions exist that possess a Fourier transform even though this condition is violated, and $J_0(x)$ is such a function. The function $f(x) = J_0(x)$ is an even function that is defined for all x and satisfies Bessel's differential equation of order zero

$$xf'' + f' + xf = 0.$$

Taking the Fourier transform of the differential equation by using Theorem 10.2 and then applying Theorem 10.5 to the first term, Theorem 10.3 to the second term, and Theorem 10.4 to the last term, we find, after the cancellation of a factor i and the combination of terms, that

$$(1-\omega^2)F' - \omega F = 0, \quad \text{where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} J_0(x)e^{-i\omega x}dx.$$

This is a linear first order variables separable differential equation that can be written

$$\int \frac{F'}{F}d\omega = \int \frac{\omega}{1-\omega^2}d\omega,$$

so integration gives

$$\ln F(\omega) = -\frac{1}{2}\ln(1-\omega^2) + \ln A, \quad \text{or} \quad F(\omega) = \frac{A}{(1-\omega^2)^{1/2}}, \quad \text{with } 0 < \omega^2 < 1.$$

In this equation, the arbitrary integration constant has again been written in the form $\ln A$, and the restriction on ω^2 is necessary because the real logarithmic function is not defined for negative arguments.

To determine A we use the fact that $A = F(0)$, together with the standard result $\int_0^\infty J_0(x)dx = 1$ and the fact that $J_0(x)$ is an even function, to obtain

$$A = F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} J_0(x)dx = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} J_0(x)dx = \sqrt{\frac{2}{\pi}}.$$

Substituting A into $F(\omega)$ gives

$$\mathcal{F}\{J_0(x)\} = F(\omega) = \sqrt{\frac{2}{\pi}}\frac{1}{(1-\omega^2)^{1/2}}H(1-|\omega|),$$

where the Heaviside unit step function $H(1-|\omega|)$ is necessary because of the restriction imposed by the real logarithmic function that requires ω to be such that $0 < \omega^2 < 1$. ■

When working with Fourier integrals, as with the Laplace transform, it is useful to introduce the convolution operation to establish the relationship between the functions $f(x)$ and $g(x)$ and their respective Fourier transforms $F(\omega)$ and $G(\omega)$.

The **convolution** of functions $f(x)$ and $g(x)$ denoted by $f * g$ is a function of x, and if the dependence on a variable x in the convolution is to be emphasized, it is then denoted by $(f * g)(x)$. The convolution of $f(x)$ and $g(x)$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} f(x-t)g(t)dt. \tag{29}$$

A slightly different definition of the convolution operation for the Fourier transform is also to be found in the literature, where it is defined as

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

When this definition is employed, the form taken by the next theorem (the convolution theorem for Fourier transforms) will require modification. This is because its form will depend on the factor $1/\sqrt{2\pi}$ and the way the constant 2π enters in the definition of the Fourier transform that is used.

THEOREM 10.6

relating the convolution of $f(x)$ and $g(x)$ and the product of their transforms

The convolution theorem for Fourier transforms Let the functions $f(x)$ and $g(x)$ be piecewise continuous, bounded, and absolutely integrable over $(-\infty, \infty)$ with the respective Fourier transforms $F(\omega)$ and $G(\omega)$. Then

(a) $\mathcal{F}\{(f * g)(x)\} = 2\pi\, \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}$, or $\mathcal{F}\{f * g\} = 2\pi\, F(\omega)G(\omega)$

and, conversely,

(b) $$(f * g)(x) = \sqrt{2\pi} \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x} d\omega.$$

Proof **(a)** By definition,

$$\mathcal{F}\{(f * g)(x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)e^{-i\omega x} dt \right] dx$$
$$= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)g(x-t)e^{-i\omega x} dx \right] dt, \right.$$

where the second result follows from the first by a change in the order of integration. If we set $v = x - t$, this becomes

$$\mathcal{F}\{(f * g)(x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[f(t)g(v)e^{-i\omega(t+v)} dt \right] dv$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \int_{-\infty}^{\infty} g(v)e^{-i\omega v} dv.$$

However, t and v are dummy variables and so may be replaced by x, causing the preceding result to become

$$\mathcal{F}\{(f * g)(x)\} = \mathcal{F}\{f(x)\}2\pi\mathcal{F}\{g(x)\},$$

showing that

$$\mathcal{F}\{(f * g)(x)\} = 2\pi\, \mathcal{F}\{f(x)\}\,\mathcal{F}\{g(x)\}, \quad \text{or} \quad \mathcal{F}\{(f * g)(x)\} = 2\pi\, F(\omega)G(\omega).$$

Result (b) follows directly from the last result by taking the inverse Fourier transform that causes a factor $\sqrt{2\pi}$ to cancel. ■

EXAMPLE 10.8

It was shown in Example 10.3(a) that the function $f(x)=\begin{cases}1, |x|<a\\ 0, |x|>a\end{cases}$ has the Fourier transform $F(\omega)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega a}{\omega}\right)$, so by the convolution theorem it follows that

$$\mathcal{F}\{(f*f)(x)\}=\sqrt{2\pi}\left[\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega a}{\omega}\right)\right]^2=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\omega a}{\omega^2}\right).$$

Confirm this result by calculating $(f*f)(x)$ and finding its Fourier transform.

Solution In terms of the Heaviside unit step function we can write $f(t)=H(a-|t|)$ and $f(x-t)=H(a-|x-t|)$, after which consideration of the product $f(t)f(x-t)$ shows that

$$f(t)f(x-t)=\begin{cases}1, & -a<t<x+a, (-2a<x<0)\\ 0, & \text{otherwise}\end{cases}$$

and

$$f(t)f(x-t)=\begin{cases}1, & x-a<t<a, (0<x<2a)\\ 0, & \text{otherwise.}\end{cases}$$

The required convolution is then given by

$$(f*f)(x)=\begin{cases}\int_{-a}^{x+a}dt=2a+x, (-2a<x<0)\\ \int_{x-a}^{a}dt=2a-x, (0<x<2a)\end{cases} \quad \text{and} \quad (f*f)(x)=0 \text{ otherwise.}$$

Taking the Fourier transform of $(f^*f)(x)$, we have

$$\begin{aligned}\mathcal{F}\{(f*f)(x)\} &= \frac{1}{\sqrt{2\pi}}\left\{\int_{-2a}^{0}(2a+x)e^{-i\omega x}dx+\int_{0}^{2a}(2a-x)e^{-i\omega x}dx\right\}\\ &= \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos 2\omega a}{\omega^2}\right),\end{aligned}$$

but $1-\cos 2\omega a=2\sin^2\omega a$, so

$$\mathcal{F}\{(f*f)(x)\}=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\omega a}{\omega^2}\right),$$

as required. ■

THEOREM 10.7

the Parseval relation extended to Fourier transforms

The Parseval relation for the Fourier transform If $f(x)$ has the Fourier transform $F(\omega)$, then

$$\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|F(\omega)|^2d\omega.$$

Proof Setting $x=0$ in result (b) of the convolution theorem gives

$$\int_{-\infty}^{\infty}f(t)g(-t)dt=\int_{-\infty}^{\infty}F(\omega)G(\omega)d\omega.$$

As the Fourier transform is defined for both real and complex functions, it follows from the definition of the transform that $\mathcal{F}\{\bar{f}(-x)\}=\overline{F}(\omega)$, where the bar indicates complex conjugation. If we set $g(t) = \bar{f}(-t)$, the preceding result becomes

$$\int_{-\infty}^{\infty} f(t)\bar{f}(t)dt = \int_{-\infty}^{\infty} F(\omega)\overline{F}(\omega)d(\omega),$$

or

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega,$$

and the result is proved. ■

EXAMPLE 10.9

Using the result of Example 10.3(a) and the Parseval relation, show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega a}{\omega^2} d\omega = \pi a.$$

Solution Substituting $f(x) = \left\{\begin{smallmatrix} 1, & |x| < a \\ 0, & |x| > a \end{smallmatrix}\right.$ and the corresponding Fourier transform $F(\omega) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin \omega a}{\omega}\right)$ found in Example 10.3(a) into the Parseval relation gives

$$\int_{-a}^{a} 1^2 dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 \omega a}{\omega^2}\right)d\omega, \text{ and so } 2a = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 \omega a}{\omega^2}\right)d\omega \ (a > 0),$$

from which the required result follows. ■

The final theorem describes the effect on the Fourier transform of $f(x)$ caused by scaling x by a factor a, shifting x by a and shifting ω by λ.

THEOREM 10.8

some useful properties of Fourier transforms

Fourier transforms involving scaling x by a, shifting x by a, and shifting ω by λ If $f(x)$ has a Fourier transform $F(\omega)$, then

(i) $\mathcal{F}\{f(ax)\} = \dfrac{1}{a}F(\omega/a) \quad (a > 0)$

(ii) $\mathcal{F}\{f(x-a)\} = e^{-i\omega a}F(\omega)$

(iii) $\mathcal{F}\{e^{i\lambda x}f(x)\} = F(\omega - \lambda)$

Proof As the results of the theorem follow immediately from the definition of the Fourier transform, only result (i) will be proved, and the derivation of results (ii) and (iii) left as exercises. Starting from the definition of $\mathcal{F}\{f(ax)\}$ and making the variable change $u = ax$ we have

$$\begin{aligned}\mathcal{F}\{f(ax)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{-i\omega x}dx = \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\omega u/a}du \\ &= \frac{1}{a}F(\omega/a)(a > 0).\end{aligned}$$

■

EXAMPLE 10.10

Using the function $f(x)$ and its Fourier transform $F(\omega)$ from Example 10.9, find (a) $\mathcal{F}\{f(2x)\}$, (b) $\mathcal{F}\{f(x-\pi)\}$, and (c) $\mathcal{F}\{e^{ix}f(x)\}$.

Solution Using the results of Theorem 10.8 we have:

(a) $\mathcal{F}\{f(2x)\} = \dfrac{1}{2}\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin(\omega a/2)}{(\omega/2)}\right) = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin(\omega a/2)}{\omega}\right)$

(b) $\mathcal{F}\{f(x-\pi)\} = e^{-i\pi\omega}\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin \omega a}{w}\right)$

(c) $\mathcal{F}\{e^{ix} f(x)\} = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin(\omega - 1)a}{\omega - 1}\right)$ ■

the Dirac delta function and the Fourier transform

The **Dirac delta function** $\delta(x)$ was introduced in connection with the Laplace transform, where it was recognized that it is not a function in the usual sense, but an *operation* that only has meaning when it appears in the integrand of a definite integral. Because of its many uses in connection with physical problems described by differential equations, we now extend its definition in a way that is suitable for use with Fourier transforms. This is accomplished by defining $\delta(x-a)$ in a symmetrical manner about $x = a$ in terms of the integrals

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)dx = \int_{-\infty}^{\infty} \delta(a-x) f(x)dx = f(a), \tag{30}$$

where a is any real number.

This definition allows the Fourier transform of $\delta(x-a)$ to be represented as

$$\mathcal{F}\{\delta(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-a)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}e^{-i\omega a}. \tag{31}$$

EXAMPLE 10.11 Find the Fourier transform of $f(x) = \delta(x-a)\exp[-b^2x^2]$ $(b > 0)$.

Solution By definition

$$\begin{aligned}\mathcal{F}\{\delta(x-a)\exp[-b^2x^2]\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-a)\exp[-b^2x^2]e^{-i\omega x}dx \\ &= \frac{1}{\sqrt{2\pi}}\exp[-(a^2b^2 + i\omega a)].\end{aligned}$$ ■

Fourier Transforms of Partial Derivatives with Respect to *x* of a Function *f*(*x*, *t*) of Two Independent Variables

transforming partial derivatives

The Fourier transform with respect to x of a function $f(x,t)$ of two independent variables x and t, denoted by $F(\omega, t)$, is defined as

$$_x\mathcal{F}\{f(x,t)\} = F(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t)e^{-i\omega x}dx, \tag{32}$$

where the prefix suffix x shows the variable that is being transformed.

In (32) the variable t is not involved in the integration with respect to x, so it follows that the integral by which $f(x, t)$ is recovered from $F(\omega, t)$ and the transform of partial derivatives of $f(x, t)$ with respect to x obey the same rules as those for the function of a single variable $f(x)$. Thus, the inversion integral is given by

$$f(x, t) = {}_x\mathcal{F}^{-1}\{F(\omega, t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega, t) e^{i\omega x} d\omega, \tag{33}$$

and the Fourier transforms of the partial derivatives of $f(x, t)$ with respect to x are given by

$$ {}_x\mathcal{F}\left\{\frac{\partial^n}{\partial x^n}[f(x, t)]\right\} = (i\omega)^n F(\omega, t) \tag{34}$$

$$ {}_x\mathcal{F}\{x^n f(x, t)\} = i^n \frac{\partial^n}{\partial \omega^n}[F(\omega, t)] \tag{35}$$

$$ {}_x\mathcal{F}\left\{x^m \frac{\partial^n}{\partial x^n}[f(x, t)]\right\} = i^{m+n} \frac{\partial^m}{\partial \omega^m}[\omega^n F(\omega, t)]. \tag{36}$$

These results are necessary when using the Fourier transform to solve partial differential equations involving a function $f(x, t)$ of two independent variables x and t where $-\infty < x < \infty$. Once the partial differential equation has been transformed, it becomes an ordinary differential equation for $F(\omega, t)$, with t as the independent variable and ω as a parameter. When $F(\omega, t)$ has been found by solving the differential equation, the solution $f(x, t)$ of the partial differential equation is recovered from $F(\omega, t)$ by means of the inversion integral (33).

an application to the heat equation

To illustrate the application of the Fourier transform to a partial differential equation we take as an example the **one-dimensional heat equation**, the derivation of which can be found in Section 18.5. This same partial differential equation was used when developing applications of the Laplace transform in Chapter 7. The heat equation that determines the one-dimensional temperature distribution $T(x, t)$ on a plane $x =$ constant at time t in an infinite block of metal with heat conduction properties characterized by the constant κ is given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa} \frac{\partial T}{\partial t}.$$

The problem we now consider is finding the temperature distribution throughout the metal at a time t when at $t = 0$ the one-dimensional temperature distribution throughout the block is given by

$$T(x, 0) = f(x),$$

where $f(x)$ is a prescribed function. Our objective will be to find the temperature $T(x, t)$ on a plane $x =$ constant at a time $t > 0$ caused by the redistribution of heat as time increases.

The Laplace transform cannot be used because when applied to the spatial variable x it is only valid for $x \geq 0$, so instead we must make use of the Fourier transform with respect to x because this applies for $-\infty \leq x \leq \infty$. Taking the Fourier transform of the heat equation with respect to x gives

$$ {}_x\mathcal{F}\left\{\frac{\partial^2 T}{\partial x^2}\right\} = {}_x\mathcal{F}\frac{1}{\kappa}\left\{\frac{\partial T}{\partial t}\right\},$$

so if we apply (34) with $n = 2$, while regarding ω as a parameter, this becomes

$$-\omega^2\kappa F(\omega, t) = \frac{d}{dt}[F(\omega, t)], \quad \text{where} \quad F(\omega, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} T(x, t)e^{-i\omega x}dx.$$

The transform $F(\omega, t)$ satisfies the ordinary differential equation

$$F' + \omega^2\kappa F = 0,$$

with the solution

$$F(\omega, t) = A(\omega)\exp\{-\omega^2\kappa t\},$$

where $A(\omega)$ is to be determined (remember that ω is a constant with respect to t).

As

$$F(\omega, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} T(x, t)e^{-i\omega x}dx,$$

it follows from the initial condition that

$$F(\omega, 0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx,$$

but $F(\omega, 0) = A(\omega)$, so

$$F(\omega, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\exp\{-i\omega x' - \omega^2\kappa t\}dx',$$

where to avoid confusion in the next step of the calculation the dummy variable x has been replaced by x'.

Applying the inversion integral to this result gives

$$\begin{aligned} T(x, t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\{i\omega x\}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\exp\{-i\omega x' - \omega^2\kappa t\}dx'\right]d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\left[\int_{-\infty}^{\infty} \exp\{i\omega(x - x') - \omega^2\kappa t\}d\omega\right]dx'. \end{aligned}$$

We show separately that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x - x') - \omega^2\kappa t\}d\omega = \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x - x')^2}{4\kappa t}\right\},$$

so the required solution is seen to be given by

$$T(x, t) = \sqrt{\frac{1}{4\pi\kappa t}}\int_{-\infty}^{\infty} f(x')\exp\left\{-\frac{(x - x')^2}{4\kappa t}\right\}dx'.$$

OPTIONAL To show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x - x') - \omega^2\kappa t\}d\omega = \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x - x')^2}{4\kappa t}\right\}$$

we need to use a complex analysis method from Chapter 15. However, before we can use this technique, the integrand of the integral on the left must be rewritten. We multiply the exponential function by $e^P e^{-P}$ (that is, by 1), where P is to be determined later, and as a result obtain

$$\exp\{i\omega(x - x') - \omega^2\kappa t\} = e^P\exp\{-P + i\omega(x - x') - \omega^2\kappa t\}.$$

We now choose P so that the exponent in the exponential can be expressed in the form $-(\alpha - i\beta\omega)^2$. When this is done it turns out that

$$\alpha = -\frac{i(x-x')}{2\sqrt{\kappa t}}, \quad \beta = i\sqrt{\kappa t}, \quad \text{and} \quad P = -\frac{(x-x')^2}{4\kappa t},$$

so

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\int_{-\infty}^{\infty}\exp\left\{-\left(-\frac{i(x-x')}{2\sqrt{\kappa t}} + \omega\sqrt{\kappa t}\right)^2\right\}d\omega$$

Making the change of variable

$$u = -\frac{i(x-x')}{2\sqrt{\kappa t}} + \omega\sqrt{\kappa t},$$

we find that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\frac{1}{\sqrt{\kappa t}}\int_{ic-\infty}^{ic+\infty}\exp\{-u^2\}du,$$

where $c = (x-x')^2/\sqrt{(4\kappa t)}$. If we integrate $\exp\{-u^2\}$ around the rectangle with corners located at $-R$, R, $R+ic$, and $-R+ic$ in the complex plane, and proceed to the limit as $R \longrightarrow \infty$, it follows that the integrals from $-R$ to $-R+ic$ and from R to $R+ic$ vanish, so as $\exp\{-u^2\}$ has no poles inside the rectangle, we have

$$\int_{ic-\infty}^{ic+\infty}\exp\{-u^2\}du = \int_{-\infty}^{\infty}\exp\{-u^2\}du.$$

The integral on the right is related to the error function $\mathrm{erf}(v)$ because

$$\int_0^v \exp\{-u^2\}du = \frac{\sqrt{\pi}}{2}\mathrm{erf}(v),$$

where $\mathrm{erf}(-v) = -\mathrm{erf}(v)$ and $\mathrm{erf}\,(\infty) = 1$.
Thus,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\frac{1}{\sqrt{\kappa t}}\frac{\sqrt{\pi}}{2}[\mathrm{erf}(\infty) - \mathrm{erf}(-\infty)]$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\frac{1}{\sqrt{\kappa t}}\frac{\sqrt{\pi}}{2}2$$
$$= \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\},$$

so we have shown that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega = \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}. \qquad (37)$$

Fourier integrals are discussed in references [4.3] and [4.4]. Tables of Fourier transform pairs are given in references [4.2] and [3.11].

Summary

The Fourier transform was introduced and its most important operational properties were established. The transforms of derivatives and partial derivatives were considered, and applications were made to functions defined by an ordinary differential equation and also to the unsteady one-dimensional heat equation. Partial differential equations such as the heat equation, and the use of integral transforms in their solution, will be considered in more detail in Chapter 18.

TABLE 10.1 Fourier Transform Pairs

$f(x)$	$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $f^{(n)}(x)$	$(i\omega)^n F(\omega)$
3. $x^n f(x)$	$(i)^n \frac{d^n}{d\omega^n}[F(\omega)]$
4. $x^m f^{(n)}(x)$	$(i)^{m+n} \frac{d^m}{d\omega^m}[\omega^n F(\omega)]$
5. $f(ax)(a > 0)$	$\frac{1}{a} F(\omega/a)$
6. $f(x - a)$	$e^{-i\omega a} F(\omega)$
7. $e^{i\lambda x} f(x)$	$F(\omega - \lambda)$
8. $(f * g)(x)$	$\sqrt{2\pi} F(\omega) G(\omega)$ (convolution theorem)
9. $\int_{-\infty}^{\infty} \lvert f(x) \rvert^2 dx$	$\int_{-\infty}^{\infty} \lvert F(\omega) \rvert^2 d\omega$ (Parseval relation)
10. $\begin{cases} 1, & \lvert x \rvert < a \\ 0, & \lvert x \rvert > a \end{cases} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \left(\frac{\sin a\omega}{\omega} \right)$
11. $\frac{\sin ax}{x} \quad (a > 0)$	$\begin{cases} \sqrt{\frac{\pi}{2}}, & \lvert \omega \rvert < a \\ 0, \lvert \omega \rvert > a \end{cases}$
12. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases} \quad (0 < a < b)$	$\frac{1}{\sqrt{2\pi}} \left(\frac{e^{-ia\omega} - e^{-ib\omega}}{i\omega} \right)$
13. $\begin{cases} a - \lvert x \rvert, & \lvert x \rvert < a \\ 0, & \lvert x \rvert > a \end{cases}$	$\sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos \omega a}{\omega^2} \right)$
14. $\frac{1}{a^2 + x^2} (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert \omega \rvert}}{a}$
15. $\begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \end{cases} (a > 0)$	$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{a + i\omega} \right)$
16. $\begin{cases} e^{ax}, & b < x < c \\ 0, & \text{otherwise} \end{cases} (a > 0)$	$\frac{1}{\sqrt{2\pi}} \left[\frac{e^{(a-i\omega)c} - e^{(a-i\omega)b}}{a - i\omega} \right]$

(*continued*)

TABLE 10.1 *(continued)*

$f(x)$	$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$
17. $e^{-a\|x\|}$ $(a > 0)$	$\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\omega^2}\right)$
18. $xe^{-a\|x\|}$ $(a > 0)$	$-\sqrt{\frac{2}{\pi}}\frac{2ia\omega}{(a^2+\omega^2)^2}$
19. $\begin{cases} e^{iax}, & \|x\| < b \\ 0, & \|x\| > b \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin b(\omega - a)}{\omega - a}\right)$
20. $\exp(-a^2x^2)$ $(a > 0)$	$\frac{1}{a\sqrt{2}}\exp\left\{-\frac{\omega^2}{4a^2}\right\}$
21. $\begin{cases} e^{-x}x^a, & x > 0 \\ 0, & x \le 0 \end{cases}$	$\frac{\Gamma(a)}{\sqrt{2\pi}(1+i\omega)^a}$
22. $J_0(ax)$ $(a > 0)$	$\sqrt{\frac{2}{\pi}}\frac{H(a-\|\omega\|)}{(a^2-\omega^2)^{1/2}}$
23. $\delta(x-a)$ $(a$ real$)$	$\frac{1}{\sqrt{2\pi}}e^{-ia\omega}$

EXERCISES 10.2

In Exercises 1 through 10 establish the Fourier transform of the stated entry in Table 10.1.

1. Entry 11.

2. Entry 12.

3. Entry 13.

4. Entry 15.

5. Entry 16.

6. Entry 17.

7. Entry 18.

8. Entry 19.

9. Entry 21.

10. Entry 22, by using the fact that $f(x) = J_0(ax)$ satisfies the Bessel's differential equation of order zero

$$xf'' + f' + a^2xf = 0 \quad (a > 0),$$

together with the standard result $\int_0^\infty J_0(ax)dx = 1/a$.

11. Use integration by parts to show that if $f(x)$ has a finite jump discontinuity at $x = a$, then $\mathcal{F}\{f'(x)\} = i\omega F(\omega) - \frac{1}{\sqrt{2\pi}}[f(a+) - f(a-)]e^{-iwa}$.

12. (a) Use the result of Exercise 11 to find the Fourier transform of $f'(x)$ given that

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Calculate $f'(x)$ and use entry 12 of Table 10.1 to find $\mathcal{F}\{f'(x)\}$ directly. Hence, show that the result obtained by this direct method is in agreement with the Fourier transform found in (a). So $f'(x) = -\delta(x-1) + \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$.

10.3 Fourier Cosine and Sine Transforms

The Fourier *cosine and sine transforms* arise as special cases of the Fourier transform, according to whether $f(x)$ is even or odd. Let us start by considering the Fourier cosine transform of $f(x)$ that can be defined when $f(x)$ is an even function that is absolutely integrable over $(-\infty, \infty)$, and so possesses a Fourier transform. Result (22) of Section 10.2 can be written

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\{\cos\omega x - i\sin\omega x\}dx, \tag{38}$$

but if $f(x)$ is an even function, the product $f(x)\cos\omega x$ is also even, so its integral over $(-\infty,\infty)$ does not vanish, though the product $f(x)\sin\ \omega x$ is an odd function, so its integral over $(-\infty,\infty)$ vanishes, causing (38) to simplify to

$$F_C(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos\omega x dx.$$

If we use the result $f(-x) = f(x)$ to change the interval of integration to $[0,\infty)$ this last result becomes

$$F_C(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos\omega x dx, \tag{39}$$

Fourier sine and cosine transforms

where the integral on the right is called the **Fourier cosine transform** of $f(x)$, and to distinguish it from the ordinary Fourier transform we write $\mathcal{F}_C\{f(x)\} = F_C(\omega)$. The **Fourier cosine inversion integral** corresponding to equation (23) of Section 10.2 becomes $f(x) = \mathcal{F}_C^{-1}\{F_C(\omega)\}$, where

$$f(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_C(\omega)\cos\omega x d\omega. \tag{40}$$

inversion integrals

A similar argument applied to (16) of Section 10.2 when $f(x)$ is an odd function leads to the result

$$F_S(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin\omega x dx, \tag{41}$$

where the integral on the right is called the **Fourier sine transform** of $f(x)$ and we write $\mathcal{F}_S\{f(x)\} = F_S(\omega)$. The corresponding **Fourier cosine inversion integral** becomes $f(x) = \mathcal{F}_S^{-1}\{F_S(\omega)\}$, where

$$f(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_S(\omega)\sin\omega x d\omega. \tag{42}$$

The Fourier cosine transform of $f(x)$ in (39) only involves $f(x)$ for $x \geq 0$, though it was derived from the Fourier transform on the assumption that $f(x)$ was an even function defined for all x. Consequently, taking the Fourier cosine transform of an arbitrary function $f(x)$ defined for $x \geq 0$ is equivalent to transforming an *even* function $f_e(x)$ obtained from $f(x)$ by setting $f_e(x) = f(x)$ for $x \geq 0$ and defining $f_e(x)$ for $x < 0$ by $f_e(-x) = f(x)$. Similarly, the Fourier sine transform of $f(x)$ in (41) only involves $f(x)$ for $x \geq 0$, though it was derived on the assumption that $f(x)$ was an odd function. So, taking the Fourier sine transform of an arbitrary function $f(x)$ defined for $x \geq 0$ is equivalent to transforming *odd* function $f_o(x)$ obtained from $f(x)$ by setting $f_o(x) = f(x)$ for $x \geq 0$ and defining $f_e(x)$for $x < 0$ by $f_e(-x) = -f(x)$.

Because (40) and (41) have been derived from (22) of Section 10.2, it follows that whenever $f(x)$ is discontinuous, the expression on the left must be replaced by $(1/2)[f(x+0) + f(x-0)]$, because the Fourier cosine and sine transforms have the same convergence properties as the Fourier transform.

EXAMPLE 10.12

Find $\mathcal{F}_C\{e^{-ax}\}$ and $\mathcal{F}_S\{e^{-ax}\}$ when $a > 0$, and use the results with the Fourier cosine and sine inversion integrals and an interchange of variables to show that

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-a\omega}}{a} \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}e^{-a\omega}.$$

Solution By definition

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos\omega x dx$$
$$= \text{Re}\sqrt{\frac{2}{\pi}}\left\{\int_0^\infty e^{-ax}e^{i\omega x}dx\right\} = \sqrt{\frac{2}{\pi}}\,\text{Re}\left\{\frac{1}{a-i\omega}\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2+a^2}\right).$$

Similarly,

$$\mathcal{F}_S\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin\omega x dx$$
$$= \text{Im}\sqrt{\frac{2}{\pi}}\left\{\int_0^\infty e^{-ax}e^{i\omega x}dx\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right).$$

Using these results in the Fourier cosine and sine inversion integrals gives

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}d\omega = \frac{2}{\pi}\int_0^\infty \frac{\omega\sin\omega x}{\omega^2+a^2}d\omega, \text{ for } a > 0,$$

so after x and ω are interchanged, these results become

$$e^{-a\omega} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{x^2+a^2}dx = \frac{2}{\pi}\int_0^\infty \frac{x\cos\omega x}{x^2+a^2}dx.$$

However,

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos\omega x}{x^2+a^2}dx \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin\omega x}{x^2+a^2}dx,$$

so combining results gives

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-a\omega}}{a} \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}e^{-a\omega}.$$ ■

THEOREM 10.9

Linearity of the Fourier cosine and sine transforms Let the functions $f(x)$ and $g(x)$ have Fourier cosine and sine transforms, and let a and b be arbitrary constants. Then

$$\mathcal{F}_C\{af(x)+bg(x)\} = a\,\mathcal{F}_C\{f(x)\} + b\,\mathcal{F}_C\{g(x)\} = aF_C(\omega) + bG_C(\omega)$$

and

$$\mathcal{F}_S\{af(x)+bg(x)\} = a\,\mathcal{F}_S\{f(x)\} + b\,\mathcal{F}_S\{g(x)\} = aF_S(\omega) + bG_S(\omega).$$

Proof The linearity properties of the Fourier cosine and sine transforms follow directly from the linearity property of the Fourier transform from which they are derived. ■

linearity of sine and cosine transforms and the transformation of derivatives

The expressions for the Fourier cosine and sine transforms of derivatives of a function $f(x)$ are slightly more complicated than those for the Fourier transform because they involve the initial values of the function and its derivatives.

THEOREM 10.10

Fourier cosine and sine transforms of derivatives Let $f(x)$ be continuous and absolutely integrable over $[0, \infty)$ and such that $\lim_{x\to\infty} f(x) = 0$. Then if $f'(x)$ and $f''(x)$ are piecewise continuous on each finite subinterval of $[0, \infty)$,

(i) $$\mathcal{F}_C\{f'(x)\} = \omega\mathcal{F}_S\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0)$$

(ii) $$\mathcal{F}_S\{f'(x)\} = -\omega\mathcal{F}_C\{f(x)\}$$

(iii) $$\mathcal{F}_C\{f''(x)\} = -\omega^2\mathcal{F}_C\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0)$$

(iv) $$\mathcal{F}_S\{f''(x)\} = -\omega^2\mathcal{F}_S\{f(x)\} + \sqrt{\frac{2}{\pi}}\omega f(0).$$

Proof The proof of each result is similar, so only result (i) will be derived in detail and outlines given for the proofs of the remaining results. To obtain (i) we integrate by parts and make use of the definition of $\mathcal{F}_C\{f(x)\}$ as follows:

$$\begin{aligned}\mathcal{F}_C\{f'(x)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos\omega x dx\\ &= \sqrt{\frac{2}{\pi}}\left[f(x)\cos\omega x\Big|_0^\infty + \omega\int_0^\infty f(x)\sin\omega x dx\right]\\ &= -\sqrt{\frac{2}{\pi}}f(0) + \omega\mathcal{F}_S\{f(x)\}.\end{aligned}$$

Result (iii) follows from (i) by replacing f by f'. Result (ii) follows in similar fashion, and (iv) follows from (ii) by replacing f by f'. ■

When Theorem 10.10 is used in the solution of second order differential equations, the initial conditions involved will help decide whether to use the cosine or sine transform. Thus, for example, if $f(0)$ is given but $f'(0)$ is unknown, the Fourier sine transform should be used to transform $f''(x)$ because result (iv) does not involve $f'(0)$. Conversely, if $f(0)$ is unknown but $f'(0)$ is given, then the Fourier cosine transform should be used to transform $f''(x)$, because result (iii) does not involve $f(0)$.

The Fourier cosine and sine transforms have Parseval relations that are analogous to the Parseval relation for the Fourier transform given in Theorem 10.7. To arrive at the first of these results we consider two functions $f(x)$ and $g(x)$ with the respective Fourier cosine transforms $F_C(\omega)$ and $G_C(\omega)$ and, using the definition of $G_C(\omega)$, write

$$\int_0^\infty F_C(\omega)G_C(\omega)\cos\omega x d\omega = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C(\omega)\cos\omega x d\omega\int_0^\infty g(x)\cos\omega x dx.$$

Changing the order of integration in the expression on the right gives

$$\sqrt{\frac{2}{\pi}}\int_0^\infty F_C(\omega)\cos\omega x d\omega\int_0^\infty g(v)\cos\omega v dv$$
$$=\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)dx\int_0^\infty F_C(\omega)\cos\omega x\cos\omega v d\omega$$
$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{2}[\cos\omega(x+v)+\cos\omega|x-v|]F_C(\omega)d\omega$$
$$=\frac{1}{2}\int_0^\infty g(v)[f(x+v)+f(|x-v|)]dv,$$

where use has first been made of the identity $\cos u\cos v=\frac{1}{2}[\cos(u+v)+\cos(u-v)]$ and then of the Fourier cosine inversion integral.

We have established the result

$$\int_0^\infty F_C(\omega)G_C(\omega)\cos\omega x d\omega=\frac{1}{2}\int_0^\infty g(v)[f(x+v)+f(|x-v|)]dv.$$

Setting $x=0$ in this last result shows that

$$\int_0^\infty F_C(\omega)G_C(\omega)d\omega=\int_0^\infty f(v)g(v)dv. \tag{43}$$

The **Parseval relation** for the **Fourier cosine transform** follows from this result by identifying $g(v)$ with $\bar{f}(v)$, for then (43) becomes

$$\int_0^\infty |F_C(\omega)|^2 d\omega=\int_0^\infty |f(x)|^2 dx, \tag{44}$$

where in the last integral the dummy variable v has been replaced by x.

A similar argument involving the Fourier sine transform establishes the corresponding results

$$\int_0^\infty F_S(\omega)G_S(\omega)d\omega=\int_0^\infty f(v)g(v)dv \tag{45}$$

and the **Parseval relation** for the **Fourier sine transform**

$$\int_0^\infty |F_S(\omega)|^2 d\omega=\int_0^\infty |f(x)|^2 dx. \tag{46}$$

We have arrived at the following theorem.

THEOREM 10.11

the Parseval relation extended to Fourier sine and cosine transforms

The Parseval relation for the Fourier cosine and sine transforms Let $f(x)$ have the respective Fourier cosine and sine transforms $F_C(\omega)$ and $F_S(\omega)$. Then the Parseval relation for the Fourier cosine transform is

$$\int_0^\infty |F_C(\omega)|^2 d\omega=\int_0^\infty |f(x)|^2 dx,$$

and the Parseval relation for the Fourier sine transform is

$$\int_0^\infty |F_S(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx.$$ ■

Results (44) and (46) often provide a simple way of evaluating improper integrals, as shown by the following example.

EXAMPLE 10.13

Apply result (43) to $f(x) = xe^{-ax}$ and $g(x) = xe^{-bx}$, where $a > 0$, $b > 0$, given that

$$\mathcal{F}_C\{f(x)\} = \sqrt{\frac{2}{\pi}}\frac{(a^2-\omega^2)}{(a^2+\omega^2)^2} \quad \text{and} \quad \mathcal{F}_C\{g(x)\} = \sqrt{\frac{2}{\pi}}\frac{(b^2-\omega^2)}{(b^2+\omega^2)^2}.$$

Solution Substituting into (43) gives

$$\frac{2}{\pi}\int_0^\infty \frac{(a^2-\omega^2)(b^2-\omega^2)}{(a^2+\omega^2)^2(b^2+\omega^2)^2}d\omega = \int_0^\infty x^2 e^{-(a+b)x}dx,$$

and after integrating the expression on the right and multiplying by $\pi/2$ we find that

$$\int_0^\infty \frac{(a^2-\omega^2)(b^2-\omega^2)}{(a^2+\omega^2)^2(b^2+\omega^2)^2}d\omega = \frac{\pi}{(a+b)^3}.$$

This integral can be evaluated by other techniques, but the preceding method is one of the simplest. ■

The final theorem in this section is the analogue of Theorem 10.8, and it is useful when transforming known Fourier cosine and sine transforms.

THEOREM 10.12

shifting and scaling Fourier sine and cosine transforms

Shifting ω and scaling x in Fourier cosine and sine transforms Let $f(x)$ have the respective Fourier cosine and sine transforms $F_C(\omega)$ and $F_S(\omega)$. Then

(i) $\mathcal{F}_C\{\cos(ax)f(x)\} = \frac{1}{2}\{F_C(\omega+a) + F_C(\omega-a)\}$

(ii) $\mathcal{F}_C\{\sin(ax)f(x)\} = \frac{1}{2}\{F_S(a+\omega) + F_S(a-\omega)\}$

(iii) $\mathcal{F}_S\{\cos(ax)f(x)\} = \frac{1}{2}\{F_S(\omega+a) + F_S(\omega-a)\}$

(iv) $\mathcal{F}_S\{\sin(ax)f(x)\} = \frac{1}{2}\{F_C(\omega-a) - F_C(\omega+a)\}$

(v) $\mathcal{F}_C\{f(ax)\} = \dfrac{1}{a}F_C(\omega/a) \quad (a > 0)$

(vi) $\mathcal{F}_S\{f(ax)\} = \dfrac{1}{a}F_S(\omega/a) \quad (a > 0).$

Proof (i) $\mathcal{F}_C\{\cos(ax)f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos(\omega x)\cos(ax)f(x)dx$, but

$$\cos(ax)\cos(\omega x) = \frac{1}{2}[\cos\{(a+\omega)x\} + \cos\{(a-\omega)x\}].$$

so

$$\begin{aligned}\mathcal{F}_C\{\cos(ax)f(x)\} &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \cos\{(a+\omega)x\}f(x)dx \\ &\quad+\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \cos\{(a-\omega)x\}f(x)dx \\ &= \frac{1}{2}\{F_C(\omega+a)+F_C(\omega-a)\}.\end{aligned}$$

Results (ii) to (iv) follow in similar fashion, whereas results (v) and (vi) follow from the definitions of the Fourier cosine and sine transforms after making the change of variable $u = ax$. ■

EXAMPLE 10.14

Given $f(x) = e^{-ax}$ with $a > 0$, use the results of Theorem 10.12 to find (a) $\mathcal{F}_C\{\cos bx f(x)\}$ and (b) $\mathcal{F}_S\{f(bx)\}$, when $b > 0$.

Solution

(a) Using Theorem 10.12 (i) with

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2+a^2}\right),$$

gives

$$\begin{aligned}\mathcal{F}_C\{\cos bxe^{-ax}\} &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{a}{(\omega+b)^2+a^2}\right)+\frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{a}{(\omega-b)^2+a^2}\right) \\ &= \sqrt{\frac{2}{\pi}}\frac{a(\omega^2+a^2+b^2)}{[(\omega+b)^2+a^2][(\omega-b)^2+a^2]}.\end{aligned}$$

(b) Using Theorem 10.12 (vi) with

$$\mathcal{F}_S\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right)$$

gives

$$\mathcal{F}_S\{f(bx)\} = \mathcal{F}_S\{e^{-abx}\} = \frac{1}{b}\sqrt{\frac{2}{\pi}}\left(\frac{\omega/b}{(\omega/b)^2+a^2}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2b^2}\right).$$

This result is to be expected, as it follows directly from the original result when a is replaced by ab. ■

When Fourier cosine and sine transforms are used in the solution of partial differential equations, the function to be transformed is a function of more than one variable. So, for example, the operation of taking the Fourier cosine transform of $f(x, y)$ with respect to x, denoted by $F_C(\omega, y)$, is given by

$${}_x\mathcal{F}_C\{f(x, y)\} = F_C(\omega, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x, y)\cos\omega x dx. \tag{47}$$

Similarly, the operation of taking the Fourier sine transform of $f(x, y)$ with respect to y, denoted by $F_S(x, \omega)$, is given by

$$ {}_y\mathcal{F}_S\{f(x,\ y)\} = F_S(x,\ \omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x,\ y)\sin \omega y dy. \tag{48}$$

As a variable that has not been transformed only appears as a parameter in the transform, it follows immediately that the rules for transforming partial derivatives follow directly from the rules for transforming derivatives of functions of a single independent variable. As a result, when interpreted in terms of a function $f(x, y)$, the entries in Theorem 10.10 take the following form.

transform of partial derivatives by Fourier sine and cosine transforms

Fourier cosine and sine transforms of partial derivatives of a function f(x, y)

$$ {}_x\mathcal{F}_C\{f'(x,\ t)\} = \omega F_S(\omega,\ t) - \sqrt{\frac{2}{\pi}} f(0,\ t) \tag{49}$$

$$ {}_x\mathcal{F}_S\{f'(x,\ t)\} = -\omega F_C(\omega,\ t) \tag{50}$$

$$ {}_x\mathcal{F}_C\{f''(x,\ t)\} = -\omega^2 F_S(\omega,\ t) - \sqrt{\frac{2}{\pi}} f'(0,\ t) \tag{51}$$

$$ {}_x\mathcal{F}_S\{f''(x,\ t)\} = -\omega^2 F_S(\omega,\ t) + \sqrt{\frac{2}{\pi}} \omega f(0,\ t) \tag{52}$$

It also follows that when transforming with respect to x partial derivatives of $f(x,\ y)$ with respect to y, the function f is transformed and the partial derivative of $f(x,\ y)$ with respect to y becomes an ordinary derivative with respect to y of the transformed function. So, for example,

$$ {}_x\mathcal{F}_C\left\{\frac{\partial^n f(x,\ y)}{\partial y^n}\right\} = \frac{d^n F_C(\omega,\ y)}{dy^n}, $$

with corresponding results for mixed derivatives.

To provide a motivation for these results we again anticipate the discussion of partial differential equations that is to follow in Chapter 18. Our objective now will be to solve the same **initial boundary value problem** for the one-dimensional **heat equation** that was solved previously by means of the Laplace transform. The one-dimensional heat equation governing the temperature $T(x,\ t)$ in a semi-infinite slab of metal at a distance x from its plane face at time t is

another application to the heat equation

$$ \frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}, \tag{53}$$

and as before we will seek a solution subject to the initial condition

$$ T(x,\ 0) = 0 \tag{54}$$

and the boundary condition

$$ T(0,\ t) = T_0, \quad t \geq 0. \tag{55}$$

The initial condition (54) says that at time $t = 0$ all the metal in the slab is at temperature $T = 0$, whereas the boundary condition (55) says that for $t > 0$ the plane face of the slab of metal is suddenly maintained at the constant temperature $T = T_0$.

As an initial temperature is known, but $\partial T/\partial x$ is unknown, consideration of results (49) to (52) suggests that we use the Fourier sine transform because it is valid for $x \geq 0$ and it only requires knowledge of $T(0,\ t) = T_0$. Accordingly, taking the Fourier sine transform of (53) with $\mathcal{F}_S\{T(x,\ t)\} = T_S(\omega,\ t)$, we have

$$\mathcal{F}_S\left\{\frac{\partial^2 T}{\partial x^2}\right\} = \frac{1}{\kappa} \quad \mathcal{F}_S\left\{\frac{\partial T}{\partial t}\right\},$$

so using (52) and regarding ω as a parameter (it is independent of t), we obtain

$$\kappa\left(-\omega^2 T_S(\omega,\ t) + \omega T_0\sqrt{\frac{2}{\pi}}\right) = \frac{d}{dt}[T_S(\omega,\ t)].$$

Thus, $T_S(\omega, t)$ satisfies the linear differential equation

$$T_S' + \omega^2\kappa T_S = \omega\kappa T_0\sqrt{\frac{2}{\pi}}$$

with the solution

$$T_S(\omega,\ t) = \frac{T_0}{\omega}\sqrt{\frac{2}{\pi}} + A(\omega)\exp\{-\omega^2\kappa t\},$$

where the arbitrary function $A(\omega)$ enters as the integration "constant" when $T_S(\omega,\ t)$ is integrated with respect to t, during which ω behaves as a constant.

Applying the inverse Fourier sine transform to this last result gives

$$T(x,\ t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{\frac{T_0}{\omega}\sqrt{\frac{2}{\pi}} + A(\omega)\exp\{-\omega^2\kappa t\}\right\}\sin\omega x d\omega.$$

To determine $A(\omega)$ we now apply the initial condition $T(x,\ 0) = 0$ to the preceding result, which then becomes

$$0 = \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{\frac{T_0}{\omega}\sqrt{\frac{2}{\pi}} + A(\omega)\right\}\sin\omega x d\omega.$$

This must be true for all ω, but this is only possible if $A(\omega) = -\frac{T_0}{\omega}\sqrt{\frac{2}{\pi}}$, and so

$$T(x, t) = T_0\sqrt{\frac{2}{\pi}}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{1-\exp(-\kappa t\omega^2)}{\omega}\right)\sin\omega x d\omega\right\}.$$

The bracketed term is the inverse Fourier sine transform of $\{[1 - \exp(-\kappa\omega^2)]/\omega\}$, so if we use entry 17 in Table 10.3, the solution becomes

$$T(x, t) = T_0\,\mathrm{erfc}\left\{\frac{x}{2\sqrt{\kappa t}}\right\}.$$

This is the result that was obtained in Section 7.3 (e) (ii) by means of the Laplace transform. The result agrees with physical intuition because for any fixed x we have $\lim_{t\to\infty}\,\mathrm{erfc}\{\frac{x}{2\sqrt{\kappa t}}\} = 1$, showing that as $t \to \infty$, so $T(x, t) \to T_0$ the constant temperature of the plane face of the metal.

Summary

The Fourier sine and cosine transforms were introduced, their inversion integrals were stated, and the main operational properties of the transforms were established. The sine and cosine transforms of ordinary and partial derivatives were derived and applications were made to the unsteady one-dimensional heat equation.

TABLE 10.2 Fourier Cosine Transform Pairs

$f(x)$	$F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $\cos(ax) f(x)$	$\frac{1}{2}\{F_C(\omega + a) + F_C(\omega - a)\}$
3. $\sin(ax) f(x)$	$\frac{1}{2}\{F_S(a + \omega) + F_S(a - \omega)\}$
4. $f(ax)$	$\frac{1}{a} F_C\left(\frac{\omega}{a}\right)(a > 0)$
5. $f'(x)$	$\omega F_S(\omega) - \sqrt{\frac{2}{\pi}} f(0)$
6. $f''(x)$	$-\omega^2 F_C(\omega) - \sqrt{\frac{2}{\pi}} f'(0)$
7. $\int_0^\infty \lvert f(x)\rvert^2 dx$	$\int_0^\infty \lvert F(\omega)\rvert^2 d\omega$ (Parseval relation)
8. $\int_0^\infty f(x)g(x)dx$	$\int_0^\infty F_C(\omega)G_C(\omega)d\omega$
9. $\begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin a\omega}{\omega}\right)$
10. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin b\omega - \sin a\omega}{\omega}\right)$
11. $x^{\alpha-1}(0 < \alpha < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{\omega^\alpha} \cos \frac{\alpha\pi}{2}$
12. $\begin{cases} x, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\cos b\omega + b\omega \sin b\omega - \cos a\omega - a\omega \sin a\omega}{\omega^2}\right)$
13. $e^{-ax}(a > 0)$	$\sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2 + a^2}\right)$
14. $xe^{-ax}(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{(a^2 - \omega^2)}{(a^2 + \omega^2)^2}$
15. $\exp\{-ax^2\}\ (a > 0)$	$\frac{1}{\sqrt{2a}} \exp\left\{-\frac{\omega^2}{4a}\right\}$
16. $\frac{1}{x^2 + a^2}(a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\omega}}{a}$
17. $J_0(ax)(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{H(a - \omega)}{(a^2 - \omega^2)^{1/2}}$
18. $\frac{\sin ax}{x}(a > 0)$	$\sqrt{\frac{2}{\pi}} H(a - \omega)$

TABLE 10.3 Fourier Sine Transform Pairs

$f(x)$	$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $\cos(ax) f(x)$	$\frac{1}{2}\{F_S(\omega + a) + F_S(\omega - a)\}$
3. $\sin(ax) f(x)$	$\frac{1}{2}\{F_C(\omega - a) - F_C(\omega + a)\}$
4. $f(ax)$	$\frac{1}{a} F_S\left(\frac{\omega}{a}\right) \quad (a > 0)$
5. $f'(x)$	$-\omega F_C(\omega)$
6. $f''(x)$	$-\omega^2 F_S(\omega) + \sqrt{\frac{2}{\pi}} \omega f(0)$
7. $\int_0^\infty \lvert f(x)\rvert^2 dx$	$\int_0^\infty \lvert F(\omega)\rvert^2 d\omega$ (Parseval relation)
8. $\int_0^\infty f(x)g(x)dx$	$\int_0^\infty F_S(\omega)G_S(\omega)d\omega$
9. $\begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos a\omega}{\omega} \right)$
10. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \left(\frac{\cos a\omega - \cos b\omega}{\omega} \right)$
11. $x^{\alpha-1}$ $(0 < \alpha < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{\omega^\alpha} \sin \frac{\alpha\pi}{2}$
12. e^{-ax} $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{\omega}{(\omega^2 + a^2)}$
13. xe^{-ax} $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{2a\omega}{(\omega^2 + a^2)^2}$
14. $x \exp\{-ax^2\}$ $(a > 0)$	$\frac{\omega}{(2a)^{3/2}} \exp\left\{-\frac{\omega^2}{4a}\right\}$
15. $\frac{x}{x^2 + a^2}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} e^{-a\omega}$
16. $\frac{\cos ax}{x}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} H(\omega - a)$
17. $\operatorname{erfc}\left\{\frac{x}{2a}\right\}$ $(a > 0)$	$\sqrt{\frac{2}{\pi}} \left\{ \frac{1 - \exp(-a^2\omega^2)}{\omega} \right\}$

EXERCISES 10.3

In Exercises 1 through 10 establish the Fourier cosine transform of the stated entry in Table 10.2.

1. Entry 9.
2. Entry 10.
3. Entry 11.
4. Entry 12.
5. Entry 13.
6. Entry 14.
7. Entry 15.
8. Entry 16.
9. Entry 17.
10. Entry 18.

In Exercises 11 through 15 find the Fourier cosine transform of the stated function.

11. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

12. $f(x) = \begin{cases} \cos x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

13. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

14. $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

15. $f(x) = \begin{cases} 1 - x^2, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$

In Exercises 16 through 23 establish the Fourier sine transform of the stated entry in Table 10.3.

16. Entry 9.

17. Entry 10.

18. Entry 11.

19. Entry 12.

20. Entry 13.

21. Entry 14.

22. Entry 15.

23. Entry 16.

In Exercises 24 through 28 find the Fourier sine transform of the stated function.

24. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

25. $f(x) = \begin{cases} \cos x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

26. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

27. $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

28. $f(x) = \begin{cases} 1 - x^2, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$