2

Inequalities

2.1 Introduction to inequalities

An **inequality** is any expression involving one of the symbols $<$, $>$, $\leq$ or $\geq$

$p < q$ means p is less than q
$p > q$ means p is greater than q
$p \leq q$ means p is less than or equal to q
$p \geq q$ means p is greater than or equal to q

Some simple rules

(i) When a quantity is **added or subtracted** to both sides of an inequality, the inequality still remains.

For example, if $p < 3$

then $p + 2 < 3 + 2$ (adding 2 to both sides)

and $p - 2 < 3 - 2$ (subtracting 2 from both sides)

(ii) When **multiplying or dividing** both sides of an inequality by a **positive** quantity, say 5, the inequality **remains the same**. For example,

if $p > 4$ then $5p > 20$ and $\frac{p}{5} > \frac{4}{5}$

(iii) When **multiplying or dividing** both sides of an inequality by a **negative** quantity, say -3, **the inequality is reversed**. For example,

if $p > 1$ then $-3p < -3$ and $\frac{p}{-3} < \frac{1}{-3}$

(Note $>$ has changed to $<$ in each example.)

To **solve an inequality** means finding all the values of the variable for which the inequality is true.
Knowledge of simple equations and quadratic equations are needed in this chapter.

2.2 Simple inequalities

The solution of some simple inequalities, using only the rules given in section 2.1, is demonstrated in the following worked problems.

Problem 1. Solve the following inequalities:
(a) $3 + x > 7$ (b) $3t < 6$
(c) $z - 2 \geq 5$ (d) $\frac{p}{3} \leq 2$

(a) Subtracting 3 from both sides of the inequality: $3 + x > 7$ gives:

$$3 + x - 3 > 7 - 3, \text{ i.e. } \boldsymbol{x > 4}$$

Hence, all values of x greater than 4 satisfy the inequality.

(b) Dividing both sides of the inequality: $3t < 6$ by 3 gives:

$$\frac{3t}{3} < \frac{6}{3}, \text{ i.e. } \boldsymbol{t < 2}$$

Hence, all values of t less than 2 satisfy the inequality.

(c) Adding 2 to both sides of the inequality $z - 2 \geq 5$ gives:

$$z - 2 + 2 \geq 5 + 2, \text{ i.e. } \boldsymbol{z \geq 7}$$

Hence, all values of z greater than or equal to 7 satisfy the inequality.

(d) Multiplying both sides of the inequality $\frac{p}{3} \leq 2$ by 3 gives:

$$(3)\frac{p}{3} \leq (3)2, \text{ i.e. } \boldsymbol{p \leq 6}$$

Hence, all values of p less than or equal to 6 satisfy the inequality.

Problem 2. Solve the inequality: $4x + 1 > x + 5$

Subtracting 1 from both sides of the inequality: $4x + 1 > x + 5$ gives:

$$4x > x + 4$$

Subtracting x from both sides of the inequality: $4x > x + 4$ gives:

$$3x > 4$$

Dividing both sides of the inequality: $3x > 4$ by 3 gives:

$$x > \frac{4}{3}$$

Hence all values of x greater than $\frac{4}{3}$ satisfy the inequality:

$$4x + 1 > x + 5$$

Problem 3. Solve the inequality: $3 - 4t \leq 8 + t$

Subtracting 3 from both sides of the inequality: $3 - 4t \leq 8 + t$ gives:

$$-4t \leq 5 + t$$

Subtracting t from both sides of the inequality: $-4t \leq 5 + t$ gives:

$$-5t \leq 5$$

Dividing both sides of the inequality $-5t \leq 5$ by -5 gives:

$\boldsymbol{t \geq -1}$ (remembering to reverse the inequality)

Hence, all values of t greater than or equal to -1 satisfy the inequality.

Now try the following exercise.

Exercise 8 Further problems on simple inequalities

Solve the following inequalities:

1. (a) $3t > 6$ (b) $2x < 10$
   [(a) $t > 2$ (b) $x < 5$]

2. (a) $\frac{x}{2} > 1.5$ (b) $x + 2 \geq 5$
   [(a) $x > 3$ (b) $x \geq 3$]

3. (a) $4t - 1 \leq 3$ (b) $5 - x \geq -1$
   [(a) $t \leq 1$ (b) $x \leq 6$]

4. (a) $\frac{7 - 2k}{4} \leq 1$ (b) $3z + 2 > z + 3$
   $\left[\text{(a) } k \geq \frac{3}{2} \quad \text{(b) } z > \frac{1}{2}\right]$

5. (a) $5 - 2y \leq 9 + y$ (b) $1 - 6x \leq 5 + 2x$
   $\left[\text{(a) } y \geq -\frac{4}{3} \quad \text{(b) } x \geq -\frac{1}{2}\right]$

2.3 Inequalities involving a modulus

The **modulus** of a number is the size of the number, regardless of sign. Vertical lines enclosing the number denote a modulus.
For example, $|4| = 4$ and $|-4| = 4$ (the modulus of a number is never negative),
The inequality: $|t| < 1$ means that all numbers whose actual size, regardless of sign, is less than 1, i.e. any value between -1 and $+1$.
Thus $\boldsymbol{|t| < 1}$ **means** $\boldsymbol{-1 < t < 1}$.
Similarly, $|x| > 3$ means all numbers whose actual size, regardless of sign, is greater than 3, i.e. any value greater than 3 and any value less than -3.
Thus $\boldsymbol{|x| > 3}$ **means** $\boldsymbol{x > 3}$ **and** $\boldsymbol{x < -3}$.
Inequalities involving a modulus are demonstrated in the following worked problems.

Problem 4. Solve the following inequality: $|3x + 1| < 4$

Since $|3x + 1| < 4$ then $-4 < 3x + 1 < 4$

Now $-4 < 3x + 1$ becomes $-5 < 3x$,

i.e. $\boldsymbol{-\frac{5}{3} < x}$ and $3x + 1 < 4$ becomes $3x < 3$,

i.e. $\boldsymbol{x < 1}$

Hence, these two results together become $\boldsymbol{-\frac{5}{3} < x < 1}$ and mean that the inequality $|3x + 1| < 4$ is satisfied for any value of x greater than $-\frac{5}{3}$ but less than 1.

Problem 5. Solve the inequality: $|1 + 2t| \leq 5$

Since $|1 + 2t| \leq 5$ then $-5 \leq 1 + 2t \leq 5$

Now $-5 \leq 1 + 2t$ becomes $-6 \leq 2t$, i.e. $\boldsymbol{-3 \leq t}$

and $1 + 2t \leq 5$ becomes $2t \leq 4$ i.e. $\boldsymbol{t \leq 2}$

Hence, these two results together become: $\boldsymbol{-3 \leq t \leq 2}$

Problem 6. Solve the inequality: $|3z - 4| > 2$

$|3z-4|>2$ means $3z-4>2$ and $3z-4<-2$,
i.e. $3z>6$ and $3z<2$,
i.e. the inequality: $|3z-4|>2$ is satisfied when

$$\mathbf{z > 2 \text{ and } z < \frac{2}{3}}$$

Now try the following exercise.

Exercise 9 Further problems on inequalities involving a modulus

Solve the following inequalities:

1. $|t+1|<4$ $[-5<t<3]$
2. $|y+3|\le 2$ $[-5\le y\le -1]$
3. $|2x-1|<4$ $\left[-\frac{3}{2}<x<\frac{5}{2}\right]$
4. $|3t-5|>4$ $\left[t>3 \text{ and } t<\frac{1}{3}\right]$
5. $|1-k|\ge 3$ $[k\ge 4 \text{ and } k\le -2]$

2.4 Inequalities involving quotients

If $\frac{p}{q}>0$ then $\frac{p}{q}$ must be a **positive** value.

For $\frac{p}{q}$ to be positive, **either** p is positive **and** q is positive **or** p is negative **and** q is negative.

i.e. $\frac{+}{+}=+$ and $\frac{-}{-}=+$

If $\frac{p}{q}<0$ then $\frac{p}{q}$ must be a **negative** value.

For $\frac{p}{q}$ to be negative, **either** p is positive **and** q is negative **or** p is negative **and** q is positive.

i.e. $\frac{+}{-}=-$ and $\frac{-}{+}=-$

This reasoning is used when solving inequalities involving quotients, as demonstrated in the following worked problems.

Problem 7. Solve the inequality: $\dfrac{t+1}{3t-6}>0$

Since $\frac{t+1}{3t-6}>0$ then $\frac{t+1}{3t-6}$ must be **positive**.

For $\frac{t+1}{3t-6}$ to be positive,

either (i) $t+1>0$ **and** $3t-6>0$
or (ii) $t+1<0$ **and** $3t-6<0$

(i) If $t+1>0$ then $t>-1$ and if $3t-6>0$ then $3t>6$ and $t>2$
Both of the inequalities $t>-1$ **and** $t>2$ are only true when $t>2$,
i.e. the fraction $\frac{t+1}{3t-6}$ is positive when $\mathbf{t>2}$

(ii) If $t+1<0$ then $t<-1$ and if $3t-6<0$ then $3t<6$ and $t<2$
Both of the inequalities $t<-1$ **and** $t<2$ are only true when $t<-1$,
i.e. the fraction $\frac{t+1}{3t-6}$ is positive when $\boldsymbol{t<-1}$

Summarizing, $\frac{t+1}{3t-6}>0$ when $\boldsymbol{t>2}$ **or** $\boldsymbol{t<-1}$

Problem 8. Solve the inequality: $\dfrac{2x+3}{x+2}\le 1$

Since $\frac{2x+3}{x+2}\le 1$ then $\frac{2x+3}{x+2}-1\le 0$

i.e. $\frac{2x+3}{x+2}-\frac{x+2}{x+2}\le 0$,

i.e. $\frac{2x+3-(x+2)}{x+2}\le 0$ or $\frac{x+1}{x+2}\le 0$

For $\frac{x+1}{x+2}$ to be negative or zero,

either (i) $x+1\le 0$ **and** $x+2>0$
or (ii) $x+1\ge 0$ **and** $x+2<0$

(i) If $x+1\le 0$ then $x\le -1$ and if $x+2>0$ then $x>-2$
(Note that $>$ is used for the denominator, not $\ge$; a zero denominator gives a value for the fraction which is impossible to evaluate.)
Hence, the inequality $\frac{x+1}{x+2}\le 0$ is true when x is greater than -2 and less than or equal to -1, which may be written as $\boldsymbol{-2<x\le -1}$

(ii) If $x+1\ge 0$ then $x\ge -1$ and if $x+2<0$ then $x<-2$

It is not possible to satisfy both $x \geq -1$ and $x < -2$ thus no values of x satisfies (ii).

Summarizing, $\dfrac{2x+3}{x+2} \leq 1$ when $\mathbf{-2 < x \leq -1}$

Now try the following exercise.

Exercise 10 Further problems on inequalities involving quotients

Solve the following inequalities:

1. $\dfrac{x+4}{6-2x} \geq 0$ $\quad[-4 \leq x < 3]$
2. $\dfrac{2t+4}{t-5} > 1$ $\quad[t > 5 \text{ or } t < -9]$
3. $\dfrac{3z-4}{z+5} \leq 2$ $\quad[-5 < z \leq 14]$
4. $\dfrac{2-x}{x+3} \geq 4$ $\quad[-3 < x \leq -2]$

2.5 Inequalities involving square functions

The following two general rules apply when inequalities involve square functions:

(i) **if $x^2 > k$ then $x > \sqrt{k}$ or $x < -\sqrt{k}$** (1)

(ii) **if $x^2 < k$ then $-\sqrt{k} < x < \sqrt{k}$** (2)

These rules are demonstrated in the following worked problems.

Problem 9. Solve the inequality: $t^2 > 9$

Since $t^2 > 9$ then $t^2 - 9 > 0$, i.e. $(t+3)(t-3) > 0$ by factorizing
For $(t+3)(t-3)$ to be positive,

either (i) $(t+3) > 0$ **and** $(t-3) > 0$
or (ii) $(t+3) < 0$ **and** $(t-3) < 0$

(i) If $(t+3)>0$ then $t > -3$ and if $(t-3) > 0$ then $t > 3$
Both of these are true only when $\boldsymbol{t > 3}$

(ii) If $(t+3) < 0$ then $t < -3$ and if $(t-3) < 0$ then $t < 3$
Both of these are true only when $\boldsymbol{t < -3}$

Summarizing, $t^2 > 9$ when $\boldsymbol{t > 3}$ **or** $\boldsymbol{t < -3}$

This demonstrates the general rule:

if $x^2 > k$ then $x > \sqrt{k}$ or $x < -\sqrt{k}$ (1)

Problem 10. Solve the inequality: $x^2 > 4$

From the general rule stated above in equation (1):
if $x^2 > 4$ then $x > \sqrt{4}$ or $x < -\sqrt{4}$
i.e. the inequality: $x^2 > 4$ is satisfied when $\boldsymbol{x > 2}$ **or** $\boldsymbol{x < -2}$

Problem 11. Solve the inequality: $(2z+1)^2 > 9$

From equation (1), if $(2z+1)^2 > 9$ then

$$
\begin{array}{lrcl}
 & 2z+1 > \sqrt{9} & \text{or} & 2z+1 < -\sqrt{9} \\
\text{i.e.} & 2z+1 > 3 & \text{or} & 2z+1 < -3 \\
\text{i.e.} & 2z > 2 & \text{or} & 2z < -4, \\
\text{i.e.} & \boldsymbol{z > 1} & \text{or} & \boldsymbol{z < -2}
\end{array}
$$

Problem 12. Solve the inequality: $t^2 < 9$

Since $t^2 < 9$ then $t^2 - 9 < 0$, i.e. $(t+3)(t-3) < 0$ by factorizing. For $(t+3)(t-3)$ to be negative,

either (i) $(t+3) > 0$ **and** $(t-3) < 0$
or (ii) $(t+3) < 0$ **and** $(t-3) > 0$

(i) If $(t+3) > 0$ then $t > -3$ and if $(t-3) < 0$ then $t < 3$
Hence (i) is satisfied when $t > -3$ and $t < 3$ which may be written as: $\boldsymbol{-3 < t < 3}$

(ii) If $(t+3) < 0$ then $t < -3$ and if $(t-3) > 0$ then $t > 3$
It is not possible to satisfy both $t < -3$ and $t > 3$, thus no values of t satisfies (ii).

Summarizing, $t^2 < 9$ when $\boldsymbol{-3 < t < 3}$ which means that all values of t between -3 and $+3$ will satisfy the inequality.

This demonstrates the general rule:

if $x^2 < k$ then $-\sqrt{k} < x < \sqrt{k}$ (2)

Problem 13. Solve the inequality: $x^2 < 4$

From the general rule stated above in equation (2): if $x^2 < 4$ then $-\sqrt{4} < x < \sqrt{4}$
i.e. the inequality: $x^2 < 4$ is satisfied when:

$$\mathbf{-2 < x < 2}$$

Problem 14. Solve the inequality: $(y-3)^2 \le 16$

From equation (2), $-\sqrt{16} \le (y-3) \le \sqrt{16}$

i.e. $-4 \le (y-3) \le 4$

from which, $3-4 \le y \le 4+3$,

i.e. $\mathbf{-1 \le y \le 7}$

Now try the following exercise.

Exercise 11 Further problems on inequalities involving square functions

Solve the following inequalities:

1. $z^2 > 16$ $[z > 4 \text{ or } z < -4]$
2. $z^2 < 16$ $[-4 < z < 4]$
3. $2x^2 \ge 6$ $[x \ge \sqrt{3} \text{ or } x \le -\sqrt{3}]$
4. $3k^2 - 2 \le 10$ $[-2 \le k \le 2]$
5. $(t-1)^2 \le 36$ $[-5 \le t \le 7]$
6. $(t-1)^2 \ge 36$ $[t \ge 7 \text{ or } t \le -5]$
7. $7 - 3y^2 \le -5$ $[y \ge 2 \text{ or } y \le -2]$
8. $(4k+5)^2 > 9$ $\left[k > -\frac{1}{2} \text{ or } k < -2\right]$

2.6 Quadratic inequalities

Inequalities involving quadratic expressions are solved using either **factorization** or **'completing the square'**. For example,

$x^2 - 2x - 3$ is factorized as $(x+1)(x-3)$

and $6x^2 + 7x - 5$ is factorized as $(2x-1)(3x+5)$

If a quadratic expression does not factorize, then the technique of 'completing the square' is used. In general, the procedure for $x^2 + bx + c$ is:

$$x^2 + bx + c \equiv \left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$$

For example, $x^2 + 4x - 7$ does not factorize; completing the square gives:

$$x^2 + 4x - 7 \equiv (x+2)^2 - 7 - 2^2 \equiv (x+2)^2 - 11$$

Similarly,

$$x^2 - 6x - 5 \equiv (x-3)^2 - 5 - 3^2 \equiv (x-3)^2 - 14$$

Solving quadratic inequalities is demonstrated in the following worked problems.

Problem 15. Solve the inequality: $x^2 + 2x - 3 > 0$

Since $x^2 + 2x - 3 > 0$ then $(x-1)(x+3) > 0$ by factorizing. For the product $(x-1)(x+3)$ to be positive,

either (i) $(x-1) > 0$ **and** $(x+3) > 0$
or (ii) $(x-1) < 0$ **and** $(x+3) < 0$

(i) Since $(x-1) > 0$ then $x > 1$ and since $(x+3) > 0$ then $x > -3$
Both of these inequalities are satisfied only when $\boldsymbol{x > 1}$

(ii) Since $(x-1) < 0$ then $x < 1$ and since $(x+3) < 0$ then $x < -3$
Both of these inequalities are satisfied only when $\boldsymbol{x < -3}$

Summarizing, $x^2 + 2x - 3 > 0$ is satisfied when either $\boldsymbol{x > 1}$ **or** $\boldsymbol{x < -3}$

Problem 16. Solve the inequality: $t^2 - 2t - 8 < 0$

Since $t^2 - 2t - 8 < 0$ then $(t-4)(t+2) < 0$ by factorizing.
For the product $(t-4)(t+2)$ to be negative,

either (i) $(t-4) > 0$ **and** $(t+2) < 0$
or (ii) $(t-4) < 0$ **and** $(t+2) > 0$

(i) Since $(t-4) > 0$ then $t > 4$ and since $(t+2) < 0$ then $t < -2$
It is not possible to satisfy both $t > 4$ and $t < -2$, thus no values of t satisfies the inequality (i)

(ii) Since $(t-4) < 0$ then $t < 4$ and since $(t+2) > 0$ then $t > -2$
Hence, (ii) is satisfied when $-2 < t < 4$

Summarizing, $t^2 - 2t - 8 < 0$ is satisfied when $\mathbf{-2 < t < 4}$

Problem 17. Solve the inequality: $x^2 + 6x + 3 < 0$

$x^2 + 6x + 3$ does not factorize; completing the square gives:

$$x^2 + 6x + 3 \equiv (x + 3)^2 + 3 - 3^2$$
$$\equiv (x + 3)^2 - 6$$

The inequality thus becomes: $(x + 3)^2 - 6 < 0$ or $(x + 3)^2 < 6$

From equation (2), $-\sqrt{6} < (x + 3) < \sqrt{6}$

from which, $(-\sqrt{6} - 3) < x < (\sqrt{6} - 3)$

Hence, $x^2 + 6x + 3 < 0$ is satisfied when $\mathbf{-5.45 < x < -0.55}$ correct to 2 decimal places.

Problem 18. Solve the inequality: $y^2 - 8y - 10 \geq 0$

$y^2 - 8y - 10$ does not factorize; completing the square gives:

$$y^2 - 8y - 10 \equiv (y - 4)^2 - 10 - 4^2$$
$$\equiv (y - 4)^2 - 26$$

The inequality thus becomes: $(y - 4)^2 - 26 \geq 0$ or $(y - 4)^2 \geq 26$

From equation (1), $(y - 4) \geq \sqrt{26}$ or $(y - 4) \leq -\sqrt{26}$

from which, $\mathbf{y \geq 4 + \sqrt{26}}$ **or** $\mathbf{y \leq 4 - \sqrt{26}}$

Hence, $y^2 - 8y - 10 \geq 0$ is satisfied when $\mathbf{y \geq 9.10}$ **or** $\mathbf{y \leq -1.10}$ correct to 2 decimal places.

Now try the following exercise.

Exercise 12 Further problems on quadratic inequalities

Solve the following inequalities:

1. $x^2 - x - 6 > 0$ $[x > 3 \text{ or } x < -2]$
2. $t^2 + 2t - 8 \leq 0$ $[-4 \leq t \leq 2]$
3. $2x^2 + 3x - 2 < 0$ $\left[-2 < x < \frac{1}{2}\right]$
4. $y^2 - y - 20 \geq 0$ $[y \geq 5 \text{ or } y \leq -4]$
5. $z^2 + 4z + 4 \leq 4$ $[-4 \leq z \leq 0]$
6. $x^2 + 6x + 6 \leq 0$ $[(-\sqrt{3} - 3) \leq x \leq (\sqrt{3} - 3)]$
7. $t^2 - 4t - 7 \geq 0$ $[t \geq (\sqrt{11} + 2) \text{ or } t \leq (2 - \sqrt{11})]$
8. $k^2 + k - 3 \geq 0$ $\left[k \geq \left(\sqrt{\frac{13}{4}} - \frac{1}{2}\right) \text{ or } k \leq \left(-\sqrt{\frac{13}{4}} - \frac{1}{2}\right)\right]$