7

The binomial series

7.1 Pascal's triangle

A **binomial expression** is one which contains two terms connected by a plus or minus sign. Thus $(p+q)$, $(a+x)^2$, $(2x+y)^3$ are examples of binomial expressions. Expanding $(a+x)^n$ for integer values of n from 0 to 6 gives the results as shown at the bottom of the page.

From these results the following patterns emerge:

(i) 'a' decreases in power moving from left to right.
(ii) 'x' increases in power moving from left to right.
(iii) The coefficients of each term of the expansions are symmetrical about the middle coefficient when n is even and symmetrical about the two middle coefficients when n is odd.
(iv) The coefficients are shown separately in Table 7.1 and this arrangement is known as **Pascal's triangle**. A coefficient of a term may be obtained by adding the two adjacent coefficients immediately above in the previous row. This is shown by the triangles in Table 7.1, where, for example, $1+3=4$, $10+5=15$, and so on.
(v) Pascal's triangle method is used for expansions of the form $(a+x)^n$ for integer values of n less than about 8.

Table 7.1

$(a+x)^0$							1						
$(a+x)^1$						1		1					
$(a+x)^2$					1		2		1				
$(a+x)^3$				1		3		3		1			
$(a+x)^4$			1		4		6		4		1		
$(a+x)^5$		1		5		10		10		5		1	
$(a+x)^6$	1		6		15		20		15		6		1

Problem 1. Use the Pascal's triangle method to determine the expansion of $(a+x)^7$.

From Table 7.1, the row of Pascal's triangle corresponding to $(a+x)^6$ is as shown in (1) below. Adding adjacent coefficients gives the coefficients of $(a+x)^7$ as shown in (2) below.

	1		6		15		20		15		6		1		(1)
1		7		21		35		35		21		7		1	(2)

The first and last terms of the expansion of $(a+x)^7$ are a^7 and x^7 respectively. The powers of 'a' decrease and the powers of 'x' increase moving from left to right.

Hence

$$\begin{aligned}\boldsymbol{(a+x)^7} &\boldsymbol{= a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3} \\ &\boldsymbol{+ 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7}\end{aligned}$$

Problem 2. Determine, using Pascal's triangle method, the expansion of $(2p-3q)^5$.

Comparing $(2p-3q)^5$ with $(a+x)^5$ shows that $a=2p$ and $x=-3q$.

$$\begin{aligned}
(a+x)^0 &= & &1\\
(a+x)^1 &= a+x & &a+x\\
(a+x)^2 &= (a+x)(a+x) &= &a^2+2ax+x^2\\
(a+x)^3 &= (a+x)^2(a+x) &= &a^3+3a^2x+3ax^2+x^3\\
(a+x)^4 &= (a+x)^3(a+x) &= &a^4+4a^3x+6a^2x^2+4ax^3+x^4\\
(a+x)^5 &= (a+x)^4(a+x) &= &a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5\\
(a+x)^6 &= (a+x)^5(a+x) &= &a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6
\end{aligned}$$

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Using Pascal's triangle method:

$$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + \cdots$$

Hence

$$\begin{aligned}(2p-3q)^5 &= (2p)^5 + 5(2p)^4(-3q) \\ &\quad + 10(2p)^3(-3q)^2 \\ &\quad + 10(2p)^2(-3q)^3 \\ &\quad + 5(2p)(-3q)^4 + (-3q)^5\end{aligned}$$

i.e. $\mathbf{(2p-3q)^5 = 32p^5 - 240p^4q + 720p^3q^2 - 1080p^2q^3 + 810pq^4 - 243q^5}$

Now try the following exercise.

Exercise 32 Further problems on Pascal's triangle

1. Use Pascal's triangle to expand $(x-y)^7$

$$\left[\begin{array}{l} x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 \\ \quad + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7 \end{array}\right]$$

2. Expand $(2a+3b)^5$ using Pascal's triangle

$$\left[\begin{array}{l} 32a^5 + 240a^4b + 720a^3b^2 \\ \quad + 1080a^2b^3 + 810ab^4 + 243b^5 \end{array}\right]$$

7.2 The binomial series

The **binomial series** or **binomial theorem** is a formula for raising a binomial expression to any power without lengthy multiplication. The general binomial expansion of $(a+x)^n$ is given by:

$$\boldsymbol{(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots}$$

where 3! denotes $3\times2\times1$ and is termed 'factorial 3'. With the binomial theorem n may be a fraction, a decimal fraction or a positive or negative integer. When n is a positive integer, the series is finite, i.e., it comes to an end; when n is a negative integer, or a fraction, the series is infinite. In the general expansion of $(a+x)^n$ it is noted that the 4th term is: $\dfrac{n(n-1)(n-2)}{3!}a^{n-3}x^3$. The number 3 is very evident in this expression.

For any term in a binomial expansion, say the r'th term, $(r-1)$ is very evident. It may therefore be reasoned that **the r'th term of the expansion $(a+x)^n$ is:**

$$\boldsymbol{\frac{n(n-1)(n-2)\ldots \text{ to } (r-1) \text{ terms}}{(r-1)!}a^{n-(r-1)}x^{r-1}}$$

If $a=1$ in the binomial expansion of $(a+x)^n$ then:

$$\boldsymbol{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots}$$

which is valid for $-1 < x < 1$.
When x is small compared with 1 then:

$$(1+x)^n \approx 1 + nx$$

7.3 Worked problems on the binomial series

Problem 3. Use the binomial series to determine the expansion of $(2+x)^7$.

The binomial expansion is given by:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots$$

When $a=2$ and $n=7$:

$$\begin{aligned}(2+x)^7 &= 2^7 + 7(2)^6x + \frac{(7)(6)}{(2)(1)}(2)^5x^2 \\ &\quad + \frac{(7)(6)(5)}{(3)(2)(1)}(2)^4x^3 + \frac{(7)(6)(5)(4)}{(4)(3)(2)(1)}(2)^3x^4 \\ &\quad + \frac{(7)(6)(5)(4)(3)}{(5)(4)(3)(2)(1)}(2)^2x^5 \\ &\quad + \frac{(7)(6)(5)(4)(3)(2)}{(6)(5)(4)(3)(2)(1)}(2)x^6 \\ &\quad + \frac{(7)(6)(5)(4)(3)(2)(1)}{(7)(6)(5)(4)(3)(2)(1)}x^7\end{aligned}$$

i.e. $\mathbf{(2+x)^7 = 128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x^7}$

Problem 4. Expand $\left(c-\frac{1}{c}\right)^5$ using the binomial series.

$$\left(c-\frac{1}{c}\right)^5 = c^5 + 5c^4\left(-\frac{1}{c}\right) + \frac{(5)(4)}{(2)(1)}c^3\left(-\frac{1}{c}\right)^2 + \frac{(5)(4)(3)}{(3)(2)(1)}c^2\left(-\frac{1}{c}\right)^3 + \frac{(5)(4)(3)(2)}{(4)(3)(2)(1)}c\left(-\frac{1}{c}\right)^4 + \frac{(5)(4)(3)(2)(1)}{(5)(4)(3)(2)(1)}\left(-\frac{1}{c}\right)^5$$

i.e. $\boldsymbol{\left(c-\frac{1}{c}\right)^5 = c^5 - 5c^3 + 10c - \frac{10}{c} + \frac{5}{c^3} - \frac{1}{c^5}}$

Problem 5. Without fully expanding $(3+x)^7$, determine the fifth term.

The r'th term of the expansion $(a+x)^n$ is given by:

$$\frac{n(n-1)(n-2)\ldots \text{ to } (r-1) \text{ terms}}{(r-1)!}a^{n-(r-1)}x^{r-1}$$

Substituting $n=7$, $a=3$ and $r-1=5-1=4$ gives:

$$\frac{(7)(6)(5)(4)}{(4)(3)(2)(1)}(3)^{7-4}x^4$$

i.e. the fifth term of $(3+x)^7 = 35(3)^3x^4 = \mathbf{945x^4}$

Problem 6. Find the middle term of $\left(2p-\frac{1}{2q}\right)^{10}$

In the expansion of $(a+x)^{10}$ there are $10+1$, i.e. 11 terms. Hence the middle term is the sixth. Using the general expression for the r'th term where $a=2p$, $x=-\frac{1}{2q}$, $n=10$ and $r-1=5$ gives:

$$\frac{(10)(9)(8)(7)(6)}{(5)(4)(3)(2)(1)}(2p)^{10-5}\left(-\frac{1}{2q}\right)^5 = 252(32p^5)\left(-\frac{1}{32q^5}\right)$$

Hence the middle term of $\left(2p-\frac{1}{2q}\right)^{10}$ is $\boldsymbol{-252\frac{p^5}{q^5}}$

Problem 7. Evaluate $(1.002)^9$ using the binomial theorem correct to (a) 3 decimal places and (b) 7 significant figures.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(1.002)^9 = (1+0.002)^9$$

Substituting $x=0.002$ and $n=9$ in the general expansion for $(1+x)^n$ gives:

$$\begin{aligned}(1+0.002)^9 &= 1 + 9(0.002) + \frac{(9)(8)}{(2)(1)}(0.002)^2 + \frac{(9)(8)(7)}{(3)(2)(1)}(0.002)^3 + \cdots \\ &= 1 + 0.018 + 0.000144 + 0.000000672 + \cdots \\ &= 1.018144672\ldots\end{aligned}$$

Hence $(1.002)^9 = $ **1.018, correct to 3 decimal places**

$=$ **1.018145, correct to 7 significant figures**

Problem 8. Evaluate $(0.97)^6$ correct to 4 significant figures using the binomial expansion.

$(0.97)^6$ is written as $(1-0.03)^6$
Using the expansion of $(1+x)^n$ where $n=6$ and $x=-0.03$ gives:

$$\begin{aligned}(1-0.03)^6 &= 1 + 6(-0.03) + \frac{(6)(5)}{(2)(1)}(-0.03)^2 + \frac{(6)(5)(4)}{(3)(2)(1)}(-0.03)^3 + \frac{(6)(5)(4)(3)}{(4)(3)(2)(1)}(-0.03)^4 + \cdots \\ &= 1 - 0.18 + 0.0135 - 0.00054 + 0.00001215 - \cdots \\ &\approx 0.83297215\end{aligned}$$

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i.e. $\quad \mathbf{(0.97)^6 = 0.8330}$**, correct to 4 significant figures**

Problem 9. Determine the value of $(3.039)^4$, correct to 6 significant figures using the binomial theorem.

$(3.039)^4$ may be written in the form $(1+x)^n$ as:

$$(3.039)^4 = (3+0.039)^4$$
$$= \left[3\left(1+\frac{0.039}{3}\right)\right]^4$$
$$= 3^4(1+0.013)^4$$
$$(1+0.013)^4 = 1+4(0.013) + \frac{(4)(3)}{(2)(1)}(0.013)^2 + \frac{(4)(3)(2)}{(3)(2)(1)}(0.013)^3 + \cdots$$
$$= 1+0.052+0.001014 + 0.000008788 + \cdots$$
$$= 1.0530228$$
correct to 8 significant figures

Hence $(3.039)^4 = 3^4(1.0530228)$
$= \mathbf{85.2948}$**, correct to 6 significant figures**

Now try the following exercise.

Exercise 33 Further problems on the binomial series

1. Use the binomial theorem to expand $(a+2x)^4$.
$$\left[\begin{array}{l} a^4+8a^3x+24a^2x^2 \\ \quad +32ax^3+16x^4 \end{array}\right]$$
2. Use the binomial theorem to expand $(2-x)^6$.
$$\left[\begin{array}{l} 64-192x+240x^2-160x^3 \\ \quad +60x^4-12x^5+x^6 \end{array}\right]$$
3. Expand $(2x-3y)^4$
$$\left[\begin{array}{l} 16x^4-96x^3y+216x^2y^2 \\ \quad -216xy^3+81y^4 \end{array}\right]$$
4. Determine the expansion of $\left(2x+\dfrac{2}{x}\right)^5$.
$$\left[\begin{array}{l} 32x^5+160x^3+320x+\dfrac{320}{x} \\ \quad +\dfrac{160}{x^3}+\dfrac{32}{x^5} \end{array}\right]$$
5. Expand $(p+2q)^{11}$ as far as the fifth term.
$$\left[\begin{array}{l} p^{11}+22p^{10}q+220p^9q^2 \\ +1320p^8q^3+5280p^7q^4 \end{array}\right]$$
6. Determine the sixth term of $\left(3p+\dfrac{q}{3}\right)^{13}$.
[$34749\,p^8q^5$]
7. Determine the middle term of $(2a-5b)^8$.
[$700000\,a^4b^4$]
8. Use the binomial theorem to determine, correct to 4 decimal places:
(a) $(1.003)^8$ (b) $(1.042)^7$
[(a) 1.0243 (b) 1.3337]
9. Use the binomial theorem to determine, correct to 5 significant figures:
(a) $(0.98)^7$ (b) $(2.01)^9$
[(a) 0.86813 (b) 535.51]
10. Evaluate $(4.044)^6$ correct to 3 decimal places.
[4373.880]

7.4 Further worked problems on the binomial series

Problem 10.
(a) Expand $\dfrac{1}{(1+2x)^3}$ in ascending powers of x as far as the term in x^3, using the binomial series.
(b) State the limits of x for which the expansion is valid.

(a) Using the binomial expansion of $(1+x)^n$, where $n=-3$ and x is replaced by $2x$ gives:

$$\frac{1}{(1+2x)^3} = (1+2x)^{-3}$$

$$= 1 + (-3)(2x) + \frac{(-3)(-4)}{2!}(2x)^2 + \frac{(-3)(-4)(-5)}{3!}(2x)^3 + \cdots$$

$$= \mathbf{1 - 6x + 24x^2 - 80x^3 + \cdots}$$

(b) The expansion is valid provided $|2x| < 1$,

i.e. $\mathbf{|x| < \frac{1}{2}}$ or $\mathbf{-\frac{1}{2} < x < \frac{1}{2}}$

> Problem 11.
> (a) Expand $\frac{1}{(4-x)^2}$ in ascending powers of x as far as the term in x^3, using the binomial theorem.
> (b) What are the limits of x for which the expansion in (a) is true?

(a) $$\frac{1}{(4-x)^2} = \frac{1}{\left[4\left(1-\frac{x}{4}\right)\right]^2} = \frac{1}{4^2\left(1-\frac{x}{4}\right)^2} = \frac{1}{16}\left(1-\frac{x}{4}\right)^{-2}$$

Using the expansion of $(1+x)^n$

$$\frac{1}{(4-x)^2} = \frac{1}{16}\left(1-\frac{x}{4}\right)^{-2}$$

$$= \frac{1}{16}\left[1 + (-2)\left(-\frac{x}{4}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{x}{4}\right)^3 + \cdots\right]$$

$$= \mathbf{\frac{1}{16}\left(1 + \frac{x}{2} + \frac{3x^2}{16} + \frac{x^3}{16} + \cdots\right)}$$

(b) The expansion in (a) is true provided $\left|\frac{x}{4}\right| < 1$,

i.e. $\mathbf{|x| < 4}$ or $\mathbf{-4 < x < 4}$

> Problem 12. Use the binomial theorem to expand $\sqrt{4+x}$ in ascending powers of x to four terms. Give the limits of x for which the expansion is valid.

$$\sqrt{4+x} = \sqrt{\left[4\left(1+\frac{x}{4}\right)\right]} = \sqrt{4}\sqrt{\left(1+\frac{x}{4}\right)} = 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$

Using the expansion of $(1+x)^n$,

$$2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$

$$= 2\left[1 + \left(\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{(1/2)(-1/2)}{2!}\left(\frac{x}{4}\right)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x}{4}\right)^3 + \cdots\right]$$

$$= 2\left(1 + \frac{x}{8} - \frac{x^2}{128} + \frac{x^3}{1024} - \cdots\right)$$

$$= \mathbf{2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} - \cdots}$$

This is valid when $\left|\frac{x}{4}\right| < 1$,

i.e. $\mathbf{|x| < 4}$ **or** $\mathbf{-4 < x < 4}$

> Problem 13. Expand $\frac{1}{\sqrt{(1-2t)}}$ in ascending powers of t as far as the term in t^3.
> State the limits of t for which the expression is valid.

$$\frac{1}{\sqrt{(1-2t)}}$$

$$= (1-2t)^{-\frac{1}{2}}$$

$$= 1 + \left(-\frac{1}{2}\right)(-2t) + \frac{(-1/2)(-3/2)}{2!}(-2t)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}(-2t)^3 + \cdots,$$

using the expansion for $(1+x)^n$

$$= \mathbf{1 + t + \frac{3}{2}t^2 + \frac{5}{2}t^3 + \cdots}$$

The expression is valid when $|2t| < 1$,

i.e. $\mathbf{|t| < \frac{1}{2}}$ or $\mathbf{-\frac{1}{2} < t < \frac{1}{2}}$

Problem 14. Simplify $\dfrac{\sqrt[3]{(1-3x)}\,\sqrt{(1+x)}}{\left(1+\dfrac{x}{2}\right)^3}$ given that powers of x above the first may be neglected.

$$\frac{\sqrt[3]{(1-3x)}\sqrt{(1+x)}}{\left(1+\frac{x}{2}\right)^3}$$

$$= (1-3x)^{\frac{1}{3}}(1+x)^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{-3}$$

$$\approx \left[1+\left(\frac{1}{3}\right)(-3x)\right]\left[1+\left(\frac{1}{2}\right)(x)\right]\left[1+(-3)\left(\frac{x}{2}\right)\right]$$

when expanded by the binomial theorem as far as the x term only,

$$= (1-x)\left(1+\frac{x}{2}\right)\left(1-\frac{3x}{2}\right)$$

$$= \left(1-x+\frac{x}{2}-\frac{3x}{2}\right)$$ when powers of x higher than unity are neglected

$$= \mathbf{(1-2x)}$$

Problem 15. Express $\dfrac{\sqrt{(1+2x)}}{\sqrt[3]{(1-3x)}}$ as a power series as far as the term in x^2. State the range of values of x for which the series is convergent.

$$\frac{\sqrt{(1+2x)}}{\sqrt[3]{(1-3x)}} = (1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}}$$

$$(1+2x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)(2x) + \frac{(1/2)(-1/2)}{2!}(2x)^2+\cdots$$

$$= 1+x-\frac{x^2}{2}+\cdots$$ which is valid for $|2x|<1$, i.e. $|x|<\dfrac{1}{2}$

$$(1-3x)^{-\frac{1}{3}} = 1+(-1/3)(-3x) + \frac{(-1/3)(-4/3)}{2!}(-3x)^2+\cdots$$

$$= 1+x+2x^2+\cdots$$ which is valid for $|3x|<1$, i.e. $|x|<\dfrac{1}{3}$

Hence

$$\frac{\sqrt{(1+2x)}}{\sqrt[3]{(1-3x)}} = (1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}}$$

$$= \left(1+x-\frac{x^2}{2}+\cdots\right)(1+x+2x^2+\cdots)$$

$$= 1+x+2x^2+x+x^2-\frac{x^2}{2},$$

neglecting terms of higher power than 2,

$$= \mathbf{1+2x+\frac{5}{2}x^2}$$

The series is convergent if $\mathbf{-\dfrac{1}{3}<x<\dfrac{1}{3}}$

Now try the following exercise.

Exercise 34 Further problems on the binomial series

In problems 1 to 5 expand in ascending powers of x as far as the term in x^3, using the binomial theorem. State in each case the limits of x for which the series is valid.

1. $\dfrac{1}{(1-x)}$

   $[1+x+x^2+x^3+\cdots,\ |x|<1]$

2. $\dfrac{1}{(1+x)^2}$

   $[1-2x+3x^2-4x^3+\cdots,\ |x|<1]$

3. $\dfrac{1}{(2+x)^3}$

   $\left[\dfrac{1}{8}\left(1-\dfrac{3x}{2}+\dfrac{3x^2}{2}-\dfrac{5x^3}{4}+\cdots\right)\ |x|<2\right]$

4. $\sqrt{2+x}$

   $\left[\sqrt{2}\left(1+\dfrac{x}{4}-\dfrac{x^2}{32}+\dfrac{x^3}{128}-\cdots\right)\ |x|<2\right]$

5. $\dfrac{1}{\sqrt{1+3x}}$

$$\left[\begin{array}{l}\left(1-\dfrac{3}{2}x+\dfrac{27}{8}x^2-\dfrac{135}{16}x^3+\cdots\right)\\ |x|<\dfrac{1}{3}\end{array}\right]$$

6. Expand $(2+3x)^{-6}$ to three terms. For what values of x is the expansion valid?

$$\left[\begin{array}{l}\dfrac{1}{64}\left(1-9x+\dfrac{189}{4}x^2\right)\\ |x|<\dfrac{2}{3}\end{array}\right]$$

7. When x is very small show that:

(a) $\dfrac{1}{(1-x)^2\sqrt{(1-x)}}\approx 1+\dfrac{5}{2}x$

(b) $\dfrac{(1-2x)}{(1-3x)^4}\approx 1+10x$

(c) $\dfrac{\sqrt{1+5x}}{\sqrt[3]{1-2x}}\approx 1+\dfrac{19}{6}x$

8. If x is very small such that x^2 and higher powers may be neglected, determine the power series for $\dfrac{\sqrt{x+4}\sqrt[3]{8-x}}{\sqrt[5]{(1+x)^3}}$

$$\left[4-\frac{31}{15}x\right]$$

9. Express the following as power series in ascending powers of x as far as the term in x^2. State in each case the range of x for which the series is valid.

(a) $\sqrt{\left(\dfrac{1-x}{1+x}\right)}$ (b) $\dfrac{(1+x)\sqrt[3]{(1-3x)^2}}{\sqrt{(1+x^2)}}$

$$\left[\begin{array}{l}\text{(a) } 1-x+\dfrac{1}{2}x^2,\ |x|<1\\ \text{(b) } 1-x-\dfrac{7}{2}x^2,\ |x|<\dfrac{1}{3}\end{array}\right]$$

7.5 Practical problems involving the binomial theorem

Binomial expansions may be used for numerical approximations, for calculations with small variations and in probability theory (see Chapter 57).

Problem 16. The radius of a cylinder is reduced by 4% and its height is increased by 2%. Determine the approximate percentage change in (a) its volume and (b) its curved surface area, (neglecting the products of small quantities).

Volume of cylinder $=\pi r^2h$.
Let r and h be the original values of radius and height.
The new values are $0.96r$ or $(1-0.04)r$ and $1.02h$ or $(1+0.02)h$.

(a) New volume $=\pi[(1-0.04)r]^2[(1+0.02)h]$

$$=\pi r^2h(1-0.04)^2(1+0.02)$$

Now $(1-0.04)^2=1-2(0.04)+(0.04)^2$
$=(1-0.08)$,
neglecting powers of small terms.

Hence new volume

$\approx \pi r^2h(1-0.08)(1+0.02)$

$\approx \pi r^2h(1-0.08+0.02)$, neglecting products of small terms

$\approx \pi r^2h(1-0.06)$ or $0.94\pi r^2h$, i.e. 94% of the original volume

Hence the volume is reduced by approximately 6%.

(b) Curved surface area of cylinder $=2\pi rh$.

New surface area

$=2\pi[(1-0.04)r][(1+0.02)h]$

$=2\pi rh(1-0.04)(1+0.02)$

$\approx 2\pi rh(1-0.04+0.02)$, neglecting products of small terms

$\approx 2\pi rh(1-0.02)$ or $0.98(2\pi rh)$, i.e. 98% of the original surface area

Hence the curved surface area is reduced by approximately 2%.

Problem 17. The second moment of area of a rectangle through its centroid is given by $\dfrac{bl^3}{12}$. Determine the approximate change in the second moment of area if b is increased by 3.5% and l is reduced by 2.5%.

New values of b and l are $(1+0.035)b$ and $(1-0.025)l$ respectively.

New second moment of area

$$= \frac{1}{12}[(1+0.035)b][(1-0.025)l]^3$$
$$= \frac{bl^3}{12}(1+0.035)(1-0.025)^3$$
$$\approx \frac{bl^3}{12}(1+0.035)(1-0.075), \text{ neglecting powers of small terms}$$
$$\approx \frac{bl^3}{12}(1+0.035-0.075), \text{ neglecting products of small terms}$$
$$\approx \frac{bl^3}{12}(1-0.040) \text{ or } (0.96)\frac{bl^3}{12}, \text{ i.e. } 96\% \text{ of the original second moment of area}$$

Hence the second moment of area is reduced by approximately 4%.

Problem 18. The resonant frequency of a vibrating shaft is given by: $f = \frac{1}{2\pi}\sqrt{\frac{k}{I}}$, where k is the stiffness and I is the inertia of the shaft. Use the binomial theorem to determine the approximate percentage error in determining the frequency using the measured values of k and I when the measured value of k is 4% too large and the measured value of I is 2% too small.

Let f, k and I be the true values of frequency, stiffness and inertia respectively. Since the measured value of stiffness, k_1, is 4% too large, then

$$k_1 = \frac{104}{100}k = (1+0.04)k$$

The measured value of inertia, I_1, is 2% too small, hence

$$I_1 = \frac{98}{100}I = (1-0.02)I$$

The measured value of frequency,

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{k_1}{I_1}} = \frac{1}{2\pi}k_1^{\frac{1}{2}}I_1^{-\frac{1}{2}}$$
$$= \frac{1}{2\pi}[(1+0.04)k]^{\frac{1}{2}}[(1-0.02)I]^{-\frac{1}{2}}$$
$$= \frac{1}{2\pi}(1+0.04)^{\frac{1}{2}}k^{\frac{1}{2}}(1-0.02)^{-\frac{1}{2}}I^{-\frac{1}{2}}$$
$$= \frac{1}{2\pi}k^{\frac{1}{2}}I^{-\frac{1}{2}}(1+0.04)^{\frac{1}{2}}(1-0.02)^{-\frac{1}{2}}$$

i.e. $f_1 = f(1+0.04)^{\frac{1}{2}}(1-0.02)^{-\frac{1}{2}}$

$$\approx f\left[1+\left(\frac{1}{2}\right)(0.04)\right]\left[1+\left(-\frac{1}{2}\right)(-0.02)\right]$$
$$\approx f(1+0.02)(1+0.01)$$

Neglecting the products of small terms,

$$f_1 \approx (1+0.02+0.01)f \approx 1.03f$$

Thus the percentage error in f based on the measured values of k and I is approximately $[(1.03)(100)-100]$, i.e. **3% too large**.

Now try the following exercise.

Exercise 35 Further practical problems involving the binomial theorem

1. Pressure p and volume v are related by $pv^3 = c$, where c is a constant. Determine the approximate percentage change in c when p is increased by 3% and v decreased by 1.2%. [0.6% decrease]

2. Kinetic energy is given by $\frac{1}{2}mv^2$. Determine the approximate change in the kinetic energy when mass m is increased by 2.5% and the velocity v is reduced by 3%. [3.5% decrease]

3. An error of +1.5% was made when measuring the radius of a sphere. Ignoring the products of small quantities determine the approximate error in calculating (a) the volume, and (b) the surface area. [(a) 4.5% increase (b) 3.0% increase]

4. The power developed by an engine is given by $I = k$ PLAN, where k is a constant. Determine the approximate percentage change in the power when P and A are each increased by 2.5% and L and N are each decreased by 1.4%. [2.2% increase]

5. The radius of a cone is increased by 2.7% and its height reduced by 0.9%. Determine the approximate percentage change in its volume, neglecting the products of small terms. [4.5% increase]

6. The electric field strength H due to a magnet of length $2l$ and moment M at a point on its axis distance x from the centre is given by

$$H = \frac{M}{2l}\left\{\frac{1}{(x-l)^2} - \frac{1}{(x+l)^2}\right\}$$

Show that if l is very small compared with x, then $H \approx \frac{2M}{x^3}$.

7. The shear stress τ in a shaft of diameter D under a torque T is given by: $\tau = \frac{kT}{\pi D^3}$. Determine the approximate percentage error in calculating τ if T is measured 3% too small and D 1.5% too large. [7.5% decrease]

8. The energy W stored in a flywheel is given by: $W = kr^5N^2$, where k is a constant, r is the radius and N the number of revolutions. Determine the approximate percentage change in W when r is increased by 1.3% and N is decreased by 2%. [2.5% increase]

9. In a series electrical circuit containing inductance L and capacitance C the resonant frequency is given by: $f_r = \frac{1}{2\pi\sqrt{LC}}$. If the values of L and C used in the calculation are 2.6% too large and 0.8% too small respectively, determine the approximate percentage error in the frequency. [0.9% too small]

10. The viscosity η of a liquid is given by: $\eta = \frac{kr^4}{\nu l}$, where k is a constant. If there is an error in r of +2%, in ν of +4% and l of −3%, what is the resultant error in η? [+7%]

11. A magnetic pole, distance x from the plane of a coil of radius r, and on the axis of the coil, is subject to a force F when a current flows in the coil. The force is given by: $F = \frac{kx}{\sqrt{(r^2+x^2)^5}}$, where k is a constant. Use the binomial theorem to show that when x is small compared to r, then

$$F \approx \frac{kx}{r^5} - \frac{5kx^3}{2r^7}$$

12. The flow of water through a pipe is given by: $G = \sqrt{\frac{(3d)^5H}{L}}$. If d decreases by 2% and H by 1%, use the binomial theorem to estimate the decrease in G. [5.5%]