3

Partial fractions

3.1 Introduction to partial fractions

By algebraic addition,

$$\frac{1}{x-2}+\frac{3}{x+1}=\frac{(x+1)+3(x-2)}{(x-2)(x+1)}$$
$$=\frac{4x-5}{x^2-x-2}$$

The reverse process of moving from $\frac{4x-5}{x^2-x-2}$ to $\frac{1}{x-2}+\frac{3}{x+1}$ is called resolving into **partial fractions**.

In order to resolve an algebraic expression into partial fractions:

(i) the denominator must factorize (in the above example, x^2-x-2 factorizes as $(x-2)(x+1)$), and

(ii) the numerator must be at least one degree less than the denominator (in the above example $(4x-5)$ is of degree 1 since the highest powered x term is x^1 and (x^2-x-2) is of degree 2).

When the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of less degree than the denominator (see Problems 3 and 4).

There are basically three types of partial fraction and the form of partial fraction used is summarized in Table 3.1, where $f(x)$ is assumed to be of less degree than the relevant denominator and A, B and C are constants to be determined.

(In the latter type in Table 3.1, ax^2+bx+c is a quadratic expression which does not factorize without containing surds or imaginary terms.)

Resolving an algebraic expression into partial fractions is used as a preliminary to integrating certain functions (see Chapter 41) and in determining inverse Laplace transforms (see Chapter 66).

3.2 Worked problems on partial fractions with linear factors

Problem 1. Resolve $\frac{11-3x}{x^2+2x-3}$ into partial fractions.

The denominator factorizes as $(x-1)\,(x+3)$ and the numerator is of less degree than the denominator. Thus $\frac{11-3x}{x^2+2x-3}$ may be resolved into partial fractions.

Let $\frac{11-3x}{x^2+2x-3}\equiv\frac{11-3x}{(x-1)(x+3)}$

$$\equiv\frac{A}{(x-1)}+\frac{B}{(x+3)}$$

Table 3.1

Type	Denominator containing	Expression	Form of partial fraction
1	Linear factors (see Problems 1 to 4)	$\frac{f(x)}{(x+a)(x-b)(x+c)}$	$\frac{A}{(x+a)}+\frac{B}{(x-b)}+\frac{C}{(x+c)}$
2	Repeated linear factors (see Problems 5 to 7)	$\frac{f(x)}{(x+a)^3}$	$\frac{A}{(x+a)}+\frac{B}{(x+a)^2}+\frac{C}{(x+a)^3}$
3	Quadratic factors (see Problems 8 and 9)	$\frac{f(x)}{(ax^2+bx+c)(x+d)}$	$\frac{Ax+B}{(ax^2+bx+c)}+\frac{C}{(x+d)}$

where A and B are constants to be determined,

i.e. $\dfrac{11-3x}{(x-1)(x+3)} \equiv \dfrac{A(x+3)+B(x-1)}{(x-1)(x+3)}$,

by algebraic addition.

Since the denominators are the same on each side of the identity then the numerators are equal to each other.

Thus, $11-3x \equiv A(x+3)+B(x-1)$

To determine constants A and B, values of x are chosen to make the term in A or B equal to zero.

When $x=1$, then

$$11-3(1) \equiv A(1+3)+B(0)$$

i.e. $8=4A$

i.e. $\boldsymbol{A=2}$

When $x=-3$, then

$$11-3(-3) \equiv A(0)+B(-3-1)$$

i.e. $20=-4B$

i.e. $\boldsymbol{B=-5}$

Thus $\dfrac{\mathbf{11-3x}}{\mathbf{x^2+2x-3}} \equiv \dfrac{2}{(x-1)}+\dfrac{-5}{(x+3)}$

$$\equiv \frac{\mathbf{2}}{\mathbf{(x-1)}}-\frac{\mathbf{5}}{\mathbf{(x+3)}}$$

$$\left[\text{Check: } \frac{2}{(x-1)}-\frac{5}{(x+3)} = \frac{2(x+3)-5(x-1)}{(x-1)(x+3)} = \frac{11-3x}{x^2+2x-3}\right]$$

Problem 2. Convert $\dfrac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$ into the sum of three partial fractions.

Let $\dfrac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$

$$\equiv \frac{A}{(x+1)}+\frac{B}{(x-2)}+\frac{C}{(x+3)}$$

$$\equiv \frac{\begin{pmatrix}A(x-2)(x+3)+B(x+1)(x+3)\\ +C(x+1)(x-2)\end{pmatrix}}{(x+1)(x-2)(x+3)}$$

by algebraic addition.

Equating the numerators gives:

$$2x^2-9x-35 \equiv A(x-2)(x+3) + B(x+1)(x+3)+C(x+1)(x-2)$$

Let $x=-1$. Then

$$2(-1)^2-9(-1)-35 \equiv A(-3)(2) + B(0)(2)+C(0)(-3)$$

i.e. $-24=-6A$

i.e. $\boldsymbol{A} = \dfrac{-24}{-6} = \mathbf{4}$

Let $x=2$. Then

$$2(2)^2-9(2)-35 \equiv A(0)(5)+B(3)(5)+C(3)(0)$$

i.e. $-45=15B$

i.e. $\boldsymbol{B} = \dfrac{-45}{15} = \mathbf{-3}$

Let $x=-3$. Then

$$2(-3)^2-9(-3)-35 \equiv A(-5)(0)+B(-2)(0) + C(-2)(-5)$$

i.e. $10=10C$

i.e. $\boldsymbol{C=1}$

Thus $\dfrac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$

$$\equiv \frac{\mathbf{4}}{\mathbf{(x+1)}}-\frac{\mathbf{3}}{\mathbf{(x-2)}}+\frac{\mathbf{1}}{\mathbf{(x+3)}}$$

Problem 3. Resolve $\dfrac{x^2+1}{x^2-3x+2}$ into partial fractions.

The denominator is of the same degree as the numerator. Thus dividing out gives:

$$\begin{array}{r@{\,}l} & \ 1 \\ x^2-3x+2 & \overline{)\ x^2 \qquad +1} \\ & \ \underline{x^2-3x+2} \\ & \qquad \underline{3x-1} \end{array}$$

For more on polynomial division, see Section 1.4, page 6.

Hence $\dfrac{x^2+1}{x^2-3x+2} \equiv 1+\dfrac{3x-1}{x^2-3x+2}$

$$\equiv 1+\frac{3x-1}{(x-1)(x-2)}$$

Let $\dfrac{3x-1}{(x-1)(x-2)} \equiv \dfrac{A}{(x-1)}+\dfrac{B}{(x-2)}$

$$\equiv \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

Equating numerators gives:

$$3x-1 \equiv A(x-2)+B(x-1)$$

Let $x=1$. Then $2 = -A$

i.e. $\boldsymbol{A = -2}$

Let $x=2$. Then $\boldsymbol{5 = B}$

Hence $\dfrac{3x-1}{(x-1)(x-2)} \equiv \dfrac{-2}{(x-1)}+\dfrac{5}{(x-2)}$

Thus $\boldsymbol{\dfrac{x^2+1}{x^2-3x+2} \equiv 1-\dfrac{2}{(x-1)}+\dfrac{5}{(x-2)}}$

> Problem 4. Express $\dfrac{x^3-2x^2-4x-4}{x^2+x-2}$ in partial fractions.

The numerator is of higher degree than the denominator. Thus dividing out gives:

$$\begin{array}{r|l} & x \;\; -3 \\ \hline x^2+x-2 & x^3-2x^2-4x-4 \\ & \underline{x^3+\;x^2-2x} \\ & -3x^2-2x-4 \\ & \underline{-3x^2-3x+6} \\ & x-10 \end{array}$$

Thus $\dfrac{x^3-2x^2-4x-4}{x^2+x-2} \equiv x-3+\dfrac{x-10}{x^2+x-2}$

$$\equiv x-3+\frac{x-10}{(x+2)(x-1)}$$

Let $\dfrac{x-10}{(x+2)(x-1)} \equiv \dfrac{A}{(x+2)}+\dfrac{B}{(x-1)}$

$$\equiv \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

Equating the numerators gives:

$$x-10 \equiv A(x-1)+B(x+2)$$

Let $x=-2$. Then $-12=-3A$

i.e. $\boldsymbol{A=4}$

Let $x=1$. Then $-9=3B$

i.e. $\boldsymbol{B=-3}$

Hence $\dfrac{x-10}{(x+2)(x-1)} \equiv \dfrac{4}{(x+2)}-\dfrac{3}{(x-1)}$

Thus $\boldsymbol{\dfrac{x^3-2x^2-4x-4}{x^2+x-2}}$

$$\boldsymbol{\equiv x-3+\frac{4}{(x+2)}-\frac{3}{(x-1)}}$$

Now try the following exercise.

Exercise 13 Further problems on partial fractions with linear factors

Resolve the following into partial fractions.

1. $\dfrac{12}{x^2-9}$ $\quad\left[\dfrac{2}{(x-3)}-\dfrac{2}{(x+3)}\right]$

2. $\dfrac{4(x-4)}{x^2-2x-3}$ $\quad\left[\dfrac{5}{(x+1)}-\dfrac{1}{(x-3)}\right]$

3. $\dfrac{x^2-3x+6}{x(x-2)(x-1)}$ $\quad\left[\dfrac{3}{x}+\dfrac{2}{(x-2)}-\dfrac{4}{(x-1)}\right]$

4. $\dfrac{3(2x^2-8x-1)}{(x+4)(x+1)(2x-1)}$ $\quad\left[\dfrac{7}{(x+4)}-\dfrac{3}{(x+1)}-\dfrac{2}{(2x-1)}\right]$

5. $\dfrac{x^2+9x+8}{x^2+x-6}$ $\quad\left[1+\dfrac{2}{(x+3)}+\dfrac{6}{(x-2)}\right]$

6. $\dfrac{x^2-x-14}{x^2-2x-3}$ $\quad\left[1-\dfrac{2}{(x-3)}+\dfrac{3}{(x+1)}\right]$

7. $\dfrac{3x^3-2x^2-16x+20}{(x-2)(x+2)}$ $\quad\left[3x-2+\dfrac{1}{(x-2)}-\dfrac{5}{(x+2)}\right]$

3.3 Worked problems on partial fractions with repeated linear factors

Problem 5. Resolve $\dfrac{2x+3}{(x-2)^2}$ into partial fractions.

The denominator contains a repeated linear factor, $(x-2)^2$.

$$\text{Let } \frac{2x+3}{(x-2)^2} \equiv \frac{A}{(x-2)} + \frac{B}{(x-2)^2} \equiv \frac{A(x-2)+B}{(x-2)^2}$$

Equating the numerators gives:

$$2x+3 \equiv A(x-2)+B$$

Let $x = 2$. Then $7 = A(0) + B$

i.e. $\boldsymbol{B = 7}$

$$2x+3 \equiv A(x-2)+B \equiv Ax - 2A + B$$

Since an identity is true for all values of the unknown, the coefficients of similar terms may be equated.

Hence, equating the coefficients of x gives: $\boldsymbol{2 = A}$.

[Also, as a check, equating the constant terms gives:

$$3 = -2A + B$$

When $A = 2$ and $B = 7$,

$$\text{R.H.S.} = -2(2) + 7 = 3 = \text{L.H.S.}]$$

$$\text{Hence } \boldsymbol{\frac{2x+3}{(x-2)^2} \equiv \frac{2}{(x-2)} + \frac{7}{(x-2)^2}}$$

Problem 6. Express $\dfrac{5x^2-2x-19}{(x+3)(x-1)^2}$ as the sum of three partial fractions.

The denominator is a combination of a linear factor and a repeated linear factor.

$$\text{Let } \frac{5x^2-2x-19}{(x+3)(x-1)^2} \equiv \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$\equiv \frac{A(x-1)^2 + B(x+3)(x-1) + C(x+3)}{(x+3)(x-1)^2}$$

by algebraic addition.
Equating the numerators gives:

$$5x^2 - 2x - 19 \equiv A(x-1)^2 + B(x+3)(x-1) + C(x+3) \qquad (1)$$

Let $x = -3$. Then

$$5(-3)^2 - 2(-3) - 19 \equiv A(-4)^2 + B(0)(-4) + C(0)$$

i.e. $32 = 16A$

i.e. $\boldsymbol{A = 2}$

Let $x = 1$. Then

$$5(1)^2 - 2(1) - 19 \equiv A(0)^2 + B(4)(0) + C(4)$$

i.e. $-16 = 4C$

i.e. $\boldsymbol{C = -4}$

Without expanding the RHS of equation (1) it can be seen that equating the coefficients of x^2 gives: $5 = A + B$, and since $A = 2$, $\boldsymbol{B = 3}$.

[Check: Identity (1) may be expressed as:

$$5x^2 - 2x - 19 \equiv A(x^2 - 2x + 1) + B(x^2 + 2x - 3) + C(x+3)$$

i.e. $5x^2 - 2x - 19 \equiv Ax^2 - 2Ax + A + Bx^2 + 2Bx - 3B + Cx + 3C$

Equating the x term coefficients gives:

$$-2 \equiv -2A + 2B + C$$

When $A = 2$, $B = 3$ and $C = -4$ then

$$-2A + 2B + C = -2(2) + 2(3) - 4 = -2 = \text{LHS}$$

Equating the constant term gives:

$$-19 \equiv A - 3B + 3C$$

$$\text{RHS} = 2 - 3(3) + 3(-4) = 2 - 9 - 12 = -19 = \text{LHS}]$$

Hence $\dfrac{\mathbf{5x^2 - 2x - 19}}{\mathbf{(x+3)(x-1)^2}}$

$$\equiv \frac{\mathbf{2}}{\mathbf{(x+3)}} + \frac{\mathbf{3}}{\mathbf{(x-1)}} - \frac{\mathbf{4}}{\mathbf{(x-1)^2}}$$

Problem 7. Resolve $\dfrac{3x^2 + 16x + 15}{(x+3)^3}$ into partial fractions.

Let $\dfrac{3x^2 + 16x + 15}{(x+3)^3}$

$$\equiv \frac{A}{(x+3)} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3}$$

$$\equiv \frac{A(x+3)^2 + B(x+3) + C}{(x+3)^3}$$

Equating the numerators gives:

$$3x^2 + 16x + 15 \equiv A(x+3)^2 + B(x+3) + C \quad (1)$$

Let $x = -3$. Then

$$3(-3)^2 + 16(-3) + 15 \equiv A(0)^2 + B(0) + C$$

i.e. $\mathbf{-6 = C}$

Identity (1) may be expanded as:

$$3x^2 + 16x + 15 \equiv A(x^2 + 6x + 9) + B(x+3) + C$$

i.e. $3x^2 + 16x + 15 \equiv Ax^2 + 6Ax + 9A + Bx + 3B + C$

Equating the coefficients of x^2 terms gives: $\mathbf{3 = A}$

Equating the coefficients of x terms gives:

$$16 = 6A + B$$

Since $A = 3$, $\mathbf{B = -2}$

[Check: equating the constant terms gives:

$$15 = 9A + 3B + C$$

When $A = 3$, $B = -2$ and $C = -6$,

$$9A + 3B + C = 9(3) + 3(-2) + (-6) = 27 - 6 - 6 = 15 = \text{LHS}]$$

Thus $\dfrac{\mathbf{3x^2 + 16x + 15}}{\mathbf{(x+3)^3}}$

$$\equiv \frac{\mathbf{3}}{\mathbf{(x+3)}} - \frac{\mathbf{2}}{\mathbf{(x+3)^2}} - \frac{\mathbf{6}}{\mathbf{(x+3)^3}}$$

Now try the following exercise.

Exercise 14 Further problems on partial fractions with repeated linear factors

1. $\dfrac{4x-3}{(x+1)^2}$ $\quad \left[\dfrac{4}{(x+1)} - \dfrac{7}{(x+1)^2}\right]$
2. $\dfrac{x^2 + 7x + 3}{x^2(x+3)}$ $\quad \left[\dfrac{1}{x^2} + \dfrac{2}{x} - \dfrac{1}{(x+3)}\right]$
3. $\dfrac{5x^2 - 30x + 44}{(x-2)^3}$ $\quad \left[\dfrac{5}{(x-2)} - \dfrac{10}{(x-2)^2} + \dfrac{4}{(x-2)^3}\right]$
4. $\dfrac{18 + 21x - x^2}{(x-5)(x+2)^2}$ $\quad \left[\dfrac{2}{(x-5)} - \dfrac{3}{(x+2)} + \dfrac{4}{(x+2)^2}\right]$

3.4 Worked problems on partial fractions with quadratic factors

Problem 8. Express $\dfrac{7x^2 + 5x + 13}{(x^2+2)(x+1)}$ in partial fractions.

The denominator is a combination of a quadratic factor, $(x^2 + 2)$, which does not factorize without introducing imaginary surd terms, and a linear factor, $(x + 1)$. Let,

$$\frac{7x^2 + 5x + 13}{(x^2+2)(x+1)} \equiv \frac{Ax + B}{(x^2+2)} + \frac{C}{(x+1)} \equiv \frac{(Ax+B)(x+1) + C(x^2+2)}{(x^2+2)(x+1)}$$

Equating numerators gives:

$$7x^2 + 5x + 13 \equiv (Ax+B)(x+1) + C(x^2+2) \quad (1)$$

Let $x = -1$. Then

$$7(-1)^2 + 5(-1) + 13 \equiv (Ax+B)(0) + C(1+2)$$

i.e. $15 = 3C$

i.e. $\mathbf{C = 5}$

Identity (1) may be expanded as:

$7x^2 + 5x + 13 \equiv Ax^2 + Ax + Bx + B + Cx^2 + 2C$

Equating the coefficients of x^2 terms gives:

$$7 = A + C, \text{and since } C = 5, \boldsymbol{A = 2}$$

Equating the coefficients of x terms gives:

$$5 = A + B, \text{and since } A = 2, \boldsymbol{B = 3}$$

[Check: equating the constant terms gives:

$$13 = B + 2C$$

When $B = 3$ and $C = 5$,

$$B + 2C = 3 + 10 = 13 = \text{LHS}]$$

Hence $\boldsymbol{\dfrac{7x^2 + 5x + 13}{(x^2 + 2)(x + 1)} \equiv \dfrac{2x + 3}{(x^2 + 2)} + \dfrac{5}{(x + 1)}}$

Problem 9. Resolve $\dfrac{3 + 6x + 4x^2 - 2x^3}{x^2(x^2 + 3)}$ into partial fractions.

Terms such as x^2 may be treated as $(x + 0)^2$, i.e. they are repeated linear factors.

Let $\dfrac{3 + 6x + 4x^2 - 2x^3}{x^2(x^2 + 3)} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx + D}{(x^2 + 3)}$

$$\equiv \frac{Ax(x^2 + 3) + B(x^2 + 3) + (Cx + D)x^2}{x^2(x^2 + 3)}$$

Equating the numerators gives:

$$\begin{aligned} 3 + 6x + 4x^2 - 2x^3 &\equiv Ax(x^2 + 3) + B(x^2 + 3) \\ &\quad + (Cx + D)x^2 \\ &\equiv Ax^3 + 3Ax + Bx^2 + 3B \\ &\quad + Cx^3 + Dx^2 \end{aligned}$$

Let $x = 0$. Then $3 = 3B$

i.e. $\boldsymbol{B = 1}$

Equating the coefficients of x^3 terms gives:

$$-2 = A + C \qquad (1)$$

Equating the coefficients of x^2 terms gives:

$$4 = B + D$$

Since $B = 1$, $\boldsymbol{D = 3}$

Equating the coefficients of x terms gives:

$$6 = 3A$$

i.e. $\boldsymbol{A = 2}$

From equation (1), since $A = 2$, $\boldsymbol{C = -4}$

Hence $\boldsymbol{\dfrac{3 + 6x + 4x^2 - 2x^3}{x^2(x^2 + 3)}} \equiv \dfrac{2}{x} + \dfrac{1}{x^2} + \dfrac{-4x + 3}{x^2 + 3}$

$$\boldsymbol{\equiv \frac{2}{x} + \frac{1}{x^2} + \frac{3 - 4x}{x^2 + 3}}$$

Now try the following exercise.

Exercise 15 Further problems on partial fractions with quadratic factors

1. $\dfrac{x^2 - x - 13}{(x^2 + 7)(x - 2)}$ $\left[\dfrac{2x + 3}{(x^2 + 7)} - \dfrac{1}{(x - 2)}\right]$

2. $\dfrac{6x - 5}{(x - 4)(x^2 + 3)}$ $\left[\dfrac{1}{(x - 4)} + \dfrac{2 - x}{(x^2 + 3)}\right]$

3. $\dfrac{15 + 5x + 5x^2 - 4x^3}{x^2(x^2 + 5)}$ $\left[\dfrac{1}{x} + \dfrac{3}{x^2} + \dfrac{2 - 5x}{(x^2 + 5)}\right]$

4. $\dfrac{x^3 + 4x^2 + 20x - 7}{(x - 1)^2(x^2 + 8)}$

   $\left[\dfrac{3}{(x - 1)} + \dfrac{2}{(x - 1)^2} + \dfrac{1 - 2x}{(x^2 + 8)}\right]$

5. When solving the differential equation $\dfrac{d^2\theta}{dt^2} - 6\dfrac{d\theta}{dt} - 10\theta = 20 - e^{2t}$ by Laplace transforms, for given boundary conditions, the following expression for $\mathcal{L}\{\theta\}$ results:

   $$\mathcal{L}\{\theta\} = \frac{4s^3 - \frac{39}{2}s^2 + 42s - 40}{s(s - 2)(s^2 - 6s + 10)}$$

   Show that the expression can be resolved into partial fractions to give:

   $$\mathcal{L}\{\theta\} = \frac{2}{s} - \frac{1}{2(s - 2)} + \frac{5s - 3}{2(s^2 - 6s + 10)}$$