5 Hyperbolic functions

5.1 Introduction to hyperbolic functions

Functions which are associated with the geometry of the conic section called a hyperbola are called **hyperbolic functions**. Applications of hyperbolic functions include transmission line theory and catenary problems. By definition:

(i) Hyperbolic sine of x,

$$\mathbf{sinh}\,\boldsymbol{x} = \frac{\mathbf{e}^{x} - \mathbf{e}^{-x}}{\mathbf{2}} \qquad (1)$$

'$\sinh x$' is often abbreviated to '$\mathrm{sh}\,x$' and is pronounced as 'shine x'

(ii) Hyperbolic cosine of x,

$$\mathbf{cosh}\,\boldsymbol{x} = \frac{\mathbf{e}^{x} + \mathbf{e}^{-x}}{\mathbf{2}} \qquad (2)$$

'$\cosh x$' is often abbreviated to '$\mathrm{ch}\,x$' and is pronounced as 'kosh x'

(iii) Hyperbolic tangent of x,

$$\mathbf{tanh}\,\boldsymbol{x} = \frac{\mathbf{sinh}\,\boldsymbol{x}}{\mathbf{cosh}\,\boldsymbol{x}} = \frac{\mathbf{e}^{x} - \mathbf{e}^{-x}}{\mathbf{e}^{x} + \mathbf{e}^{-x}} \qquad (3)$$

'$\tanh x$' is often abbreviated to '$\mathrm{th}\,x$' and is pronounced as 'than x'

(iv) Hyperbolic cosecant of x,

$$\mathbf{cosech}\,\boldsymbol{x} = \frac{\mathbf{1}}{\mathbf{sinh}\,\boldsymbol{x}} = \frac{\mathbf{2}}{\mathbf{e}^{x} - \mathbf{e}^{-x}} \qquad (4)$$

'cosech x' is pronounced as 'coshec x'

(v) Hyperbolic secant of x,

$$\mathbf{sech}\,\boldsymbol{x} = \frac{\mathbf{1}}{\mathbf{cosh}\,\boldsymbol{x}} = \frac{\mathbf{2}}{\mathbf{e}^{x} + \mathbf{e}^{-x}} \qquad (5)$$

'sech x' is pronounced as 'shec x'

(vi) Hyperbolic cotangent of x,

$$\mathbf{coth}\,\boldsymbol{x} = \frac{\mathbf{1}}{\mathbf{tanh}\,\boldsymbol{x}} = \frac{\mathbf{e}^{x} + \mathbf{e}^{-x}}{\mathbf{e}^{x} - \mathbf{e}^{-x}} \qquad (6)$$

'$\coth x$' is pronounced as 'koth x'

Some properties of hyperbolic functions

Replacing x by 0 in equation (1) gives:

$$\sinh 0 = \frac{e^{0} - e^{-0}}{2} = \frac{1-1}{2} = 0$$

Replacing x by 0 in equation (2) gives:

$$\cosh 0 = \frac{e^{0} + e^{-0}}{2} = \frac{1+1}{2} = 1$$

If a function of x, $f(-x) = -f(x)$, then $f(x)$ is called an **odd function** of x. Replacing x by $-x$ in equation (1) gives:

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = -\left(\frac{e^{x} - e^{-x}}{2}\right) = -\sinh x$$

Replacing x by $-x$ in equation (3) gives:

$$\tanh(-x) = \frac{e^{-x} - e^{-(-x)}}{e^{-x} + e^{-(-x)}} = \frac{e^{-x} - e^{x}}{e^{-x} + e^{x}} = -\left(\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\right) = -\tanh x$$

Hence **sinh x and tanh x are both odd functions** (see Section 5.2), as also are $\mathrm{cosech}\,x \left(=\dfrac{1}{\sinh x}\right)$ and $\coth x \left(=\dfrac{1}{\tanh x}\right)$

If a function of x, $f(-x) = f(x)$, then $f(x)$ is called an **even function** of x. Replacing x by $-x$ in equation (2) gives:

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x$$

Hence **cosh x is an even function** (see Section 5.2), as also is $\mathrm{sech}\,x \left(=\dfrac{1}{\cosh x}\right)$

Hyperbolic functions may be evaluated easiest using a calculator. Many scientific notation calculators actually possess sinh and cosh functions; however, if a calculator does not contain these functions, then the definitions given above may be used. (Tables of hyperbolic functions are available, but are now rarely used)

> Problem 1. Evaluate sinh 5.4, correct to 4 significant figures.

$$\begin{aligned}\sinh 5.4 &= \tfrac{1}{2}(e^{5.4} - e^{-5.4})\\ &= \tfrac{1}{2}(221.406416\ldots - 0.00451658\ldots)\\ &= \tfrac{1}{2}(221.401899\ldots)\\ &= \mathbf{110.7}, \text{ correct to 4 significant figures}\end{aligned}$$

> Problem 2. Determine the value of cosh 1.86, correct to 3 decimal places.

$$\begin{aligned}\cosh 1.86 &= \tfrac{1}{2}(e^{1.86} + e^{-1.86})\\ &= \tfrac{1}{2}(6.42373677\ldots + 0.1556726\ldots)\\ &= \tfrac{1}{2}(6.5794093\ldots) = 3.289704\ldots\\ &= \mathbf{3.290}, \text{ correct to 3 decimal places}\end{aligned}$$

> Problem 3. Evaluate, correct to 4 significant figures,
> (a) th 0.52 (b) cosech 1.4
> (c) sech 0.86 (d) coth 0.38

(a) $$\begin{aligned}\text{th}\,0.52 &= \frac{\text{sh}\,0.52}{\text{ch}\,0.52} = \frac{\frac{1}{2}(e^{0.52} - e^{-0.52})}{\frac{1}{2}(e^{0.52} + e^{-0.52})}\\ &= \frac{e^{0.52} - e^{-0.52}}{e^{0.52} + e^{-0.52}}\\ &= \frac{(1.6820276\ldots - 0.59452054\ldots)}{(1.6820276\ldots + 0.59452054\ldots)}\\ &= \frac{1.0875070\ldots}{2.27654814\ldots}\\ &= \mathbf{0.4777}\end{aligned}$$

(b) $$\begin{aligned}\text{cosech}\,1.4 &= \frac{1}{\sinh 1.4} = \frac{1}{\frac{1}{2}(e^{1.4} - e^{-1.4})}\\ &= \frac{2}{(4.05519996\ldots - 0.24659696\ldots)}\\ &= \frac{2}{3.808603} = \mathbf{0.5251}\end{aligned}$$

(c) $$\begin{aligned}\text{sech}\,0.86 &= \frac{1}{\cosh 0.86} = \frac{1}{\frac{1}{2}(e^{0.86} + e^{-0.86})}\\ &= \frac{2}{(2.36316069\ldots + 0.42316208\ldots)}\\ &= \frac{2}{2.78632277\ldots} = \mathbf{0.7178}\end{aligned}$$

(d) $$\begin{aligned}\coth 0.38 &= \frac{1}{\text{th}\,0.38} = \frac{\text{ch}\,0.38}{\text{sh}\,0.38}\\ &= \frac{\frac{1}{2}(e^{0.38} + e^{-0.38})}{\frac{1}{2}(e^{0.38} - e^{-0.38})}\\ &= \frac{1.46228458\ldots + 0.68386140\ldots}{1.46228458\ldots - 0.68386140\ldots}\\ &= \frac{2.1461459\ldots}{0.7784231\ldots} = \mathbf{2.757}\end{aligned}$$

Now try the following exercise.

Exercise 24 Further problems on evaluating hyperbolic functions

In Problems 1 to 6, evaluate correct to 4 significant figures.

1. (a) sh 0.64 (b) sh 2.182
[(a) 0.6846 (b) 4.376]

2. (a) ch 0.72 (b) ch 2.4625
[(a) 1.271 (b) 5.910]

3. (a) th 0.65 (b) th 1.81
[(a) 0.5717 (b) 0.9478]

4. (a) cosech 0.543 (b) cosech 3.12
[(a) 1.754 (b) 0.08849]

5. (a) sech 0.39 (b) sech 2.367
[(a) 0.9285 (b) 0.1859]

6. (a) coth 0.444 (b) coth 1.843
[(a) 2.398 (b) 1.051]

A

7. A telegraph wire hangs so that its shape is described by $y = 50\,\text{ch}\,\frac{x}{50}$. Evaluate, correct to 4 significant figures, the value of y when $x = 25$. [56.38]

8. The length l of a heavy cable hanging under gravity is given by $l = 2c\,\text{sh}\,(L/2c)$. Find the value of l when $c = 40$ and $L = 30$. [30.71]

9. $V^2 = 0.55L\tanh(6.3\,d/L)$ is a formula for velocity V of waves over the bottom of shallow water, where d is the depth and L is the wavelength. If $d = 8.0$ and $L = 96$, calculate the value of V. [5.042]

5.2 Graphs of hyperbolic functions

A graph of $\boldsymbol{y = \sinh x}$ may be plotted using calculator values of hyperbolic functions. The curve is shown in Fig. 5.1. Since the graph is symmetrical about the origin, $\sinh x$ is an **odd function** (as stated in Section 5.1).

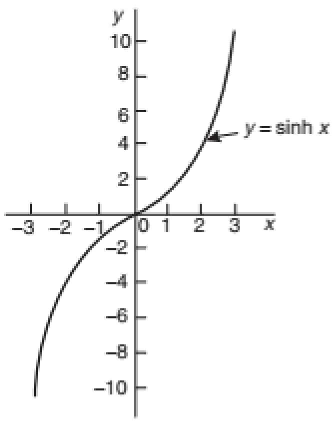

Figure 5.1

A graph of $\boldsymbol{y = \cosh x}$ may be plotted using calculator values of hyperbolic functions. The curve is shown in Fig. 5.2. Since the graph is symmetrical about the y-axis, $\cosh x$ is an **even function** (as stated in Section 5.1). The shape of $y = \cosh x$ is that of a heavy rope or chain hanging freely under gravity and is called a **catenary**. Examples include transmission lines, a telegraph wire or a fisherman's line, and is used in the design of roofs and arches. Graphs of $y = \tanh x$, $y = \text{cosech}\,x$, $y = \text{sech}\,x$ and $y = \coth x$ are deduced in Problems 4 and 5.

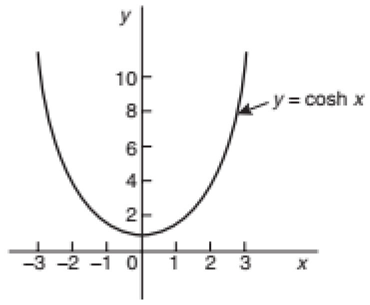

Figure 5.2

Problem 4. Sketch graphs of (a) $y = \tanh x$ and (b) $y = \coth x$ for values of x between -3 and 3.

A table of values is drawn up as shown below

x	-3	-2	-1
$\text{sh}\,x$	-10.02	-3.63	-1.18
$\text{ch}\,x$	10.07	3.76	1.54
$y = \text{th}\,x = \dfrac{\text{sh}\,x}{\text{ch}\,x}$	-0.995	-0.97	-0.77
$y = \coth x = \dfrac{\text{ch}\,x}{\text{sh}\,x}$	-1.005	-1.04	-1.31

x	0	1	2	3
$\text{sh}\,x$	0	1.18	3.63	10.02
$\text{ch}\,x$	1	1.54	3.76	10.07
$y = \text{th}\,x = \dfrac{\text{sh}\,x}{\text{ch}\,x}$	0	0.77	0.97	0.995
$y = \coth x = \dfrac{\text{ch}\,x}{\text{sh}\,x}$	$\pm\infty$	1.31	1.04	1.005

(a) A graph of $y = \tanh x$ is shown in Fig. 5.3(a)
(b) A graph of $y = \coth x$ is shown in Fig. 5.3(b)

Both graphs are symmetrical about the origin thus $\tanh x$ and $\coth x$ are odd functions.

Problem 5. Sketch graphs of (a) $y = \text{cosech}\,x$ and (b) $y = \text{sech}\,x$ from $x = -4$ to $x = 4$, and, from the graphs, determine whether they are odd or even functions.

A

7. A telegraph wire hangs so that its shape is described by $y = 50\,\text{ch}\,\dfrac{x}{50}$. Evaluate, correct to 4 significant figures, the value of y when $x = 25$. [56.38]

8. The length l of a heavy cable hanging under gravity is given by $l = 2c\,\text{sh}\,(L/2c)$. Find the value of l when $c = 40$ and $L = 30$. [30.71]

9. $V^2 = 0.55L \tanh(6.3\,d/L)$ is a formula for velocity V of waves over the bottom of shallow water, where d is the depth and L is the wavelength. If d = 8.0 and $L = 96$, calculate the value of V. [5.042]

5.2 Graphs of hyperbolic functions

A graph of $\boldsymbol{y = \sinh x}$ may be plotted using calculator values of hyperbolic functions. The curve is shown in Fig. 5.1. Since the graph is symmetrical about the origin, $\sinh x$ is an **odd function** (as stated in Section 5.1).

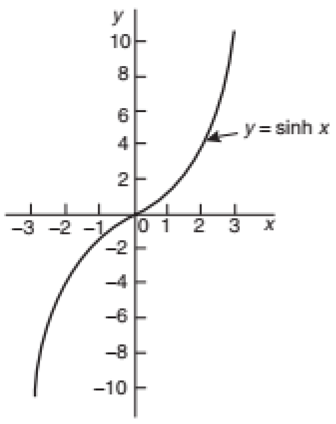

Figure 5.1

A graph of $\boldsymbol{y = \cosh x}$ may be plotted using calculator values of hyperbolic functions. The curve is shown in Fig. 5.2. Since the graph is symmetrical about the y-axis, $\cosh x$ is an **even function** (as stated in Section 5.1). The shape of $y = \cosh x$ is that of a heavy rope or chain hanging freely under gravity and is called a **catenary**. Examples include transmission lines, a telegraph wire or a fisherman's line, and is used in the design of roofs and arches. Graphs of $y = \tanh x$, $y = \text{cosech}\, x$, $y = \text{sech}\, x$ and $y = \coth x$ are deduced in Problems 4 and 5.

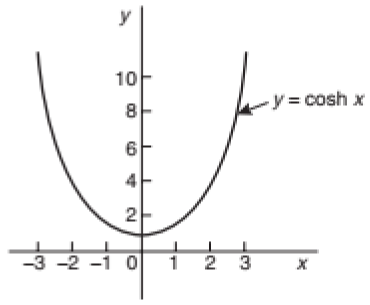

Figure 5.2

Problem 4. Sketch graphs of (a) $y = \tanh x$ and (b) $y = \coth x$ for values of x between -3 and 3.

A table of values is drawn up as shown below

x	-3	-2	-1
$\text{sh}\,x$	-10.02	-3.63	-1.18
$\text{ch}\,x$	10.07	3.76	1.54
$y = \text{th}\,x = \dfrac{\text{sh}\,x}{\text{ch}\,x}$	-0.995	-0.97	-0.77
$y = \coth x = \dfrac{\text{ch}\,x}{\text{sh}\,x}$	-1.005	-1.04	-1.31

x	0	1	2	3
$\text{sh}\,x$	0	1.18	3.63	10.02
$\text{ch}\,x$	1	1.54	3.76	10.07
$y = \text{th}\,x = \dfrac{\text{sh}\,x}{\text{ch}\,x}$	0	0.77	0.97	0.995
$y = \coth x = \dfrac{\text{ch}\,x}{\text{sh}\,x}$	$\pm\infty$	1.31	1.04	1.005

(a) A graph of $y = \tanh x$ is shown in Fig. 5.3(a)
(b) A graph of $y = \coth x$ is shown in Fig. 5.3(b)

Both graphs are symmetrical about the origin thus $\tanh x$ and $\coth x$ are odd functions.

Problem 5. Sketch graphs of (a) $y = \text{cosech}\, x$ and (b) $y = \text{sech}\, x$ from $x = -4$ to $x = 4$, and, from the graphs, determine whether they are odd or even functions.

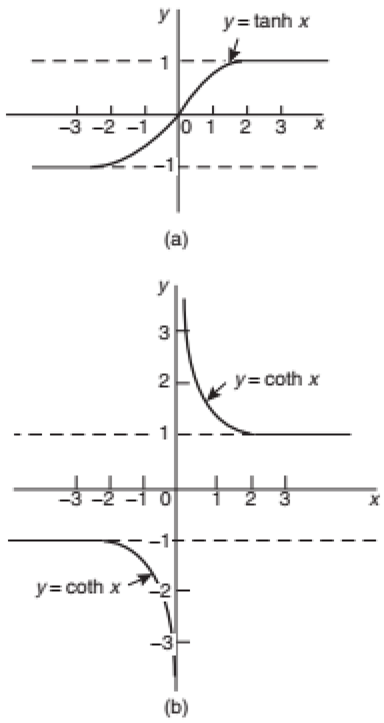

Figure 5.3

A table of values is drawn up as shown below

x	-4	-3	-2	-1
$\text{sh}\,x$	-27.29	-10.02	-3.63	-1.18
$\text{cosech}\,x = \dfrac{1}{\text{sh}\,x}$	-0.04	-0.10	-0.28	-0.85
$\text{ch}\,x$	27.31	10.07	3.76	1.54
$\text{sech}\,x = \dfrac{1}{\text{ch}\,x}$	0.04	0.10	0.27	0.65

x	0	1	2	3	4
$\text{sh}\,x$	0	1.18	3.63	10.02	27.29
$\text{cosech}\,x = \dfrac{1}{\text{sh}\,x}$	$\pm\infty$	0.85	0.28	0.10	0.04
$\text{ch}\,x$	1	1.54	3.76	10.07	27.31
$\text{sech}\,x = \dfrac{1}{\text{ch}\,x}$	1	0.65	0.27	0.10	0.04

(a) A graph of $y = \text{cosech}\,x$ is shown in Fig. 5.4(a). The graph is symmetrical about the origin and is thus an **odd function**.

(b) A graph of $y = \text{sech}\,x$ is shown in Fig. 5.4(b). The graph is symmetrical about the y-axis and is thus an **even function**.

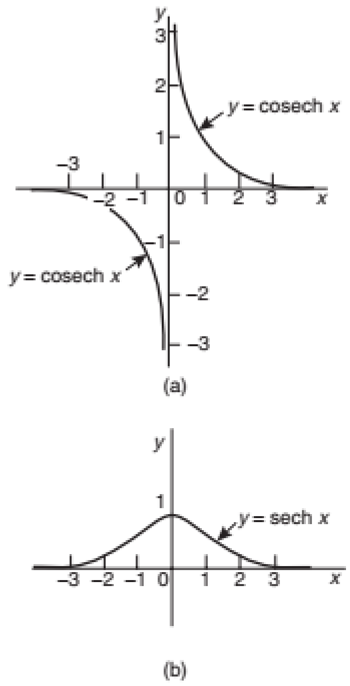

Figure 5.4

5.3 Hyperbolic identities

For every trigonometric identity there is a corresponding hyperbolic identity. **Hyperbolic identities** may be proved by either

(i) replacing $\text{sh}\,x$ by $\dfrac{e^x - e^{-x}}{2}$ and $\text{ch}\,x$ by $\dfrac{e^x + e^{-x}}{2}$, or

(ii) by using **Osborne's rule**, which states: *'the six trigonometric ratios used in trigonometrical identities relating general angles may be replaced by their corresponding hyperbolic functions, but the sign of any direct or implied product of two sines must be changed'.*

For example, since $\cos^2 x + \sin^2 x = 1$ then, by Osborne's rule, $\text{ch}^2\,x - \text{sh}^2\,x = 1$, i.e. the trigonometric functions have been changed to their corresponding hyperbolic functions and since $\sin^2 x$ is a product of two sines the sign is changed from $+$ to $-$.

A

Table 5.1 shows some trigonometric identities and their corresponding hyperbolic identities.

Problem 6. Prove the hyperbolic identities (a) $\text{ch}^2 x - \text{sh}^2 x = 1$ (b) $1 - \text{th}^2 x = \text{sech}^2 x$ (c) $\coth^2 x - 1 = \text{cosech}^2 x$.

(a) $$\text{ch}\,x + \text{sh}\,x = \left(\frac{e^x + e^{-x}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right) = e^x$$

$$\text{ch}\,x - \text{sh}\,x = \left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right) = e^{-x}$$

$$(\text{ch}\,x + \text{sh}\,x)(\text{ch}\,x - \text{sh}\,x) = (e^x)(e^{-x}) = e^0 = 1$$

i.e. $\mathbf{ch^2 x - sh^2 x = 1}$ (1)

(b) Dividing each term in equation (1) by $\text{ch}^2 x$ gives:

$$\frac{\text{ch}^2 x}{\text{ch}^2 x} - \frac{\text{sh}^2 x}{\text{ch}^2 x} = \frac{1}{\text{ch}^2 x},$$

i.e. $\mathbf{1 - th^2 x = sech^2 x}$

(c) Dividing each term in equation (1) by $\text{sh}^2 x$ gives:

$$\frac{\text{ch}^2 x}{\text{sh}^2 x} - \frac{\text{sh}^2 x}{\text{sh}^2 x} = \frac{1}{\text{sh}^2 x}$$

i.e. $\mathbf{coth^2 x - 1 = cosech^2 x}$

Problem 7. Prove, using Osborne's rule (a) $\text{ch}\,2A = \text{ch}^2 A + \text{sh}^2 A$ (b) $1 - \text{th}^2 x = \text{sech}^2 x$.

(a) From trigonometric ratios,

$$\cos 2A = \cos^2 A - \sin^2 A \qquad (1)$$

Osborne's rule states that trigonometric ratios may be replaced by their corresponding hyperbolic functions but the sign of any product of two sines has to be changed. In this case, $\sin^2 A = (\sin A)(\sin A)$, i.e. a product of two sines, thus the sign of the corresponding hyperbolic function, $\text{sh}^2 A$, is changed from + to −. Hence, from (1), $\mathbf{ch\,2A = ch^2 A + sh^2 A}$

(b) From trigonometric ratios,

$$1 + \tan^2 x = \sec^2 x \qquad (2)$$

and $\tan^2 x = \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{(\sin x)(\sin x)}{\cos^2 x}$

i.e. a product of two sines.

Hence, in equation (2), the trigonometric ratios are changed to their equivalent hyperbolic function and the sign of $\text{th}^2 x$ changed + to −, i.e. $\mathbf{1 - th^2 x = sech^2 x}$

Problem 8. Prove that $1 + 2\,\text{sh}^2 x = \text{ch}\,2x$.

Table 5.1

Trigonometric identity	Corresponding hyperbolic identity
$\cos^2 x + \sin^2 x = 1$	$\text{ch}^2 x - \text{sh}^2 x = 1$
$1 + \tan^2 x = \sec^2 x$	$1 - \text{th}^2 x = \text{sech}^2 x$
$\cot^2 x + 1 = \text{cosec}^2 x$	$\coth^2 x - 1 = \text{cosech}^2 x$
Compound angle formulae	
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\text{sh}(A \pm B) = \text{sh}\,A\,\text{ch}\,B \pm \text{ch}\,A\,\text{sh}\,B$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\text{ch}(A \pm B) = \text{ch}\,A\,\text{ch}\,B \pm \text{sh}\,A\,\text{sh}\,B$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\text{th}(A \pm B) = \dfrac{\text{th}\,A \pm \text{th}\,B}{1 \pm \text{th}\,A\,\text{th}\,B}$
Double angles	
$\sin 2x = 2 \sin x \cos x$	$\text{sh}\,2x = 2\,\text{sh}\,x\,\text{ch}\,x$
$\cos 2x = \cos^2 x - \sin^2 x$	$\text{ch}\,2x = \text{ch}^2 x + \text{sh}^2 x$
$= 2\cos^2 x - 1$	$= 2\,\text{ch}^2 x - 1$
$= 1 - 2\sin^2 x$	$= 1 + 2\text{sh}^2 x$
$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$	$\text{th}\,2x = \dfrac{2\,\text{th}\,x}{1 + \text{th}^2 x}$

Left hand side (L.H.S.)

$$= 1 + 2\,\text{sh}^2 x = 1 + 2\left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= 1 + 2\left(\frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}\right)$$

$$= 1 + \frac{e^{2x} - 2 + e^{-2x}}{2}$$

$$= 1 + \left(\frac{e^{2x} + e^{-2x}}{2}\right) - \frac{2}{2}$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \text{ch}\,2x = \text{R.H.S.}$$

Problem 9. Show that $\text{th}^2 x + \text{sech}^2 x = 1$.

$$\text{L.H.S.} = \text{th}^2 x + \text{sech}^2 x = \frac{\text{sh}^2 x}{\text{ch}^2 x} + \frac{1}{\text{ch}^2 x} = \frac{\text{sh}^2 x + 1}{\text{ch}^2 x}$$

Since $\text{ch}^2 x - \text{sh}^2 x = 1$ then $1 + \text{sh}^2 x = \text{ch}^2 x$

Thus $\dfrac{\text{sh}^2 x + 1}{\text{ch}^2 x} = \dfrac{\text{ch}^2 x}{\text{ch}^2 x} = 1 = \text{R.H.S.}$

Problem 10. Given $Ae^x + Be^{-x} \equiv 4\,\text{ch}\,x - 5\,\text{sh}\,x$, determine the values of A and B.

$$Ae^x + Be^{-x} \equiv 4\,\text{ch}\,x - 5\,\text{sh}\,x$$

$$= 4\left(\frac{e^x + e^{-x}}{2}\right) - 5\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= 2e^x + 2e^{-x} - \frac{5}{2}e^x + \frac{5}{2}e^{-x}$$

$$= -\frac{1}{2}e^x + \frac{9}{2}e^{-x}$$

Equating coefficients gives: $\boldsymbol{A = -\frac{1}{2}}$ **and** $\boldsymbol{B = 4\frac{1}{2}}$

Problem 11. If $4e^x - 3e^{-x} \equiv P\,\text{sh}\,x + Q\,\text{ch}\,x$, determine the values of P and Q.

$$4e^x - 3e^{-x} \equiv P\,\text{sh}\,x + Q\,\text{ch}\,x$$

$$= P\left(\frac{e^x - e^{-x}}{2}\right) + Q\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{P}{2}e^x - \frac{P}{2}e^{-x} + \frac{Q}{2}e^x + \frac{Q}{2}e^{-x}$$

$$= \left(\frac{P+Q}{2}\right)e^x + \left(\frac{Q-P}{2}\right)e^{-x}$$

Equating coefficients gives:

$$4 = \frac{P+Q}{2} \text{ and } -3 = \frac{Q-P}{2}$$

i.e. $P + Q = 8$ (1)

$-P + Q = -6$ (2)

Adding equations (1) and (2) gives: $2Q = 2$, i.e. $\boldsymbol{Q = 1}$

Substituting in equation (1) gives: $\boldsymbol{P = 7}$.

Now try the following exercise.

Exercise 25 Further problems on hyperbolic identities

In Problems 1 to 4, prove the given identities.

1. (a) $\text{ch}\,(P - Q) \equiv \text{ch}\,P\,\text{ch}\,Q - \text{sh}\,P\,\text{sh}\,Q$
   (b) $\text{ch}\,2x \equiv \text{ch}^2 x + \text{sh}^2 x$
2. (a) $\coth x \equiv 2\,\text{cosech}\,2x + \text{th}\,x$
   (b) $\text{ch}\,2\theta - 1 \equiv 2\,\text{sh}^2\theta$
3. (a) $\text{th}\,(A - B) \equiv \dfrac{\text{th}\,A - \text{th}\,B}{1 - \text{th}\,A\,\text{th}\,B}$
   (b) $\text{sh}\,2A \equiv 2\,\text{sh}\,A\,\text{ch}\,A$
4. (a) $\text{sh}\,(A + B) \equiv \text{sh}\,A\,\text{ch}\,B + \text{ch}\,A\,\text{sh}\,B$
   (b) $\dfrac{\text{sh}^2 x + \text{ch}^2 x - 1}{2\text{ch}^2 x \coth^2 x} \equiv \tanh^4 x$
5. Given $Pe^x - Qe^{-x} \equiv 6\,\text{ch}\,x - 2\,\text{sh}\,x$, find P and Q [$P = 2, Q = -4$]
6. If $5e^x - 4e^{-x} \equiv A\,\text{sh}\,x + B\,\text{ch}\,x$, find A and B. [$A = 9, B = 1$]

A

5.4 Solving equations involving hyperbolic functions

Equations of the form $\boldsymbol{a}\,\mathbf{ch}\,\boldsymbol{x}+\boldsymbol{b}\,\mathbf{sh}\,\boldsymbol{x}=\boldsymbol{c}$, where a, b and c are constants may be solved either by:

(a) plotting graphs of $y=a\,\text{ch}\,x+b\,\text{sh}\,x$ and $y=c$ and noting the points of intersection, or more accurately,

(b) by adopting the following procedure:

(i) Change $\text{sh}\,x$ to $\left(\dfrac{e^x-e^{-x}}{2}\right)$ and $\text{ch}\,x$ to $\left(\dfrac{e^x+e^{-x}}{2}\right)$

(ii) Rearrange the equation into the form $pe^x+qe^{-x}+r=0$, where p, q and r are constants.

(iii) Multiply each term by e^x, which produces an equation of the form $p(e^x)^2+re^x+q=0$ (since $(e^{-x})(e^x)=e^0=1$)

(iv) Solve the quadratic equation $p(e^x)^2+re^x+q=0$ for e^x by factorising or by using the quadratic formula.

(v) Given $e^x=$ a constant (obtained by solving the equation in (iv)), take Napierian logarithms of both sides to give $x=\ln(\text{constant})$

This procedure is demonstrated in Problems 12 to 14 following.

Problem 12. Solve the equation $\text{sh}\,x=3$, correct to 4 significant figures.

Following the above procedure:

(i) $\text{sh}\,x=\left(\dfrac{e^x-e^{-x}}{2}\right)=3$

(ii) $e^x-e^{-x}=6$, i.e. $e^x-e^{-x}-6=0$

(iii) $(e^x)^2-(e^{-x})(e^x)-6e^x=0$,
i.e. $(e^x)^2-6e^x-1=0$

(iv) $e^x=\dfrac{-(-6)\pm\sqrt{[(-6)^2-4(1)(-1)]}}{2(1)}$

$=\dfrac{6\pm\sqrt{40}}{2}=\dfrac{6\pm 6.3246}{2}$

Hence $e^x=6.1623$ or -0.1623

(v) $x=\ln 6.1623$ or $x=\ln(-0.1623)$ which has no solution since it is not possible in real terms to find the logarithm of a negative number. Hence $x=\ln 6.1623=\mathbf{1.818}$, correct to 4 significant figures.

Problem 13. Solve the equation

$$2.6\,\text{ch}\,x+5.1\,\text{sh}\,x=8.73,$$

correct to 4 decimal places.

Following the above procedure:

(i) $2.6\,\text{ch}\,x+5.1\,\text{sh}\,x=8.73$
i.e. $2.6\left(\dfrac{e^x+e^{-x}}{2}\right)+5.1\left(\dfrac{e^x-e^{-x}}{2}\right)=8.73$

(ii) $1.3e^x+1.3e^{-x}+2.55e^x-2.55e^{-x}=8.73$
i.e. $3.85e^x-1.25e^{-x}-8.73=0$

(iii) $3.85(e^x)^2-8.73e^x-1.25=0$

(iv) e^x

$=\dfrac{-(-8.73)\pm\sqrt{[(-8.73)^2-4(3.85)(-1.25)]}}{2(3.85)}$

$=\dfrac{8.73\pm\sqrt{95.463}}{7.70}=\dfrac{8.73\pm 9.7705}{7.70}$

Hence $e^x=2.4027$ or $e^x=-0.1351$

(v) $x=\ln 2.4027$ or $x=\ln(-0.1351)$ which has no real solution.
Hence $\boldsymbol{x}=\mathbf{0.8766}$, correct to 4 decimal places.

Problem 14. A chain hangs in the form given by $y=40\,\text{ch}\dfrac{x}{40}$. Determine, correct to 4 significant figures, (a) the value of y when x is 25 and (b) the value of x when $y=54.30$.

(a) $y=40\,\text{ch}\dfrac{x}{40}$, and when $x=25$,

$\boldsymbol{y}=40\,\text{ch}\dfrac{25}{40}=40\,\text{ch}\,0.625$

$=40\left(\dfrac{e^{0.625}+e^{-0.625}}{2}\right)$

$=20(1.8682+0.5353)=\mathbf{48.07}$

(b) When $y = 54.30$, $54.30 = 40\,\mathrm{ch}\,\dfrac{x}{40}$, from which

$$\mathrm{ch}\,\frac{x}{40} = \frac{54.30}{40} = 1.3575$$

Following the above procedure:

(i) $\dfrac{e^{\frac{x}{40}} + e^{\frac{-x}{40}}}{2} = 1.3575$

(ii) $e^{\frac{x}{40}} + e^{\frac{-x}{40}} = 2.715$, i.e. $e^{\frac{x}{40}} + e^{\frac{-x}{40}} - 2.715 = 0$

(iii) $(e^{\frac{x}{40}})^2 + 1 - 2.715e^{\frac{x}{40}} = 0$

i.e. $(e^{\frac{x}{40}})^2 - 2.715e^{\frac{x}{40}} + 1 = 0$

(iv) $e^{\frac{x}{40}} = \dfrac{-(-2.715) \pm \sqrt{[(-2.715)^2 - 4(1)(1)]}}{2(1)}$

$= \dfrac{2.715 \pm \sqrt{(3.3712)}}{2} = \dfrac{2.715 \pm 1.8361}{2}$

Hence $e^{\frac{x}{40}} = 2.2756$ or 0.43945

(v) $\dfrac{x}{40} = \ln 2.2756$ or $\dfrac{x}{40} = \ln(0.43945)$

Hence $\dfrac{x}{40} = 0.8222$ or $\dfrac{x}{40} = -0.8222$

Hence $x = 40(0.8222)$ or $x = 40(-0.8222)$;

i.e. $\boldsymbol{x = \pm 32.89}$, correct to 4 significant figures.

Now try the following exercise.

Exercise 26 Further problems on hyperbolic equations

In Problems 1 to 5 solve the given equations correct to 4 decimal places.

1. $\mathrm{sh}\,x = 1$ [0.8814]
2. $2\,\mathrm{ch}\,x = 3$ [±0.9624]
3. $3.5\,\mathrm{sh}\,x + 2.5\,\mathrm{ch}\,x = 0$ [−0.8959]
4. $2\,\mathrm{sh}\,x + 3\,\mathrm{ch}\,x = 5$ [0.6389 or −2.2484]
5. $4\,\mathrm{th}\,x - 1 = 0$ [0.2554]
6. A chain hangs so that its shape is of the form $y = 56\,\mathrm{ch}\,(x/56)$. Determine, correct to 4 significant figures, (a) the value of y when x is 35, and (b) the value of x when y is 62.35. [(a) 67.30 (b) 26.42]

5.5 Series expansions for cosh x and sinh x

By definition,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

from Chapter 4.
Replacing x by $-x$ gives:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots.$$

$$\begin{aligned}\cosh x &= \frac{1}{2}(e^x + e^{-x})\\ &= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)\right.\\ &\quad \left. + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)\right]\\ &= \frac{1}{2}\left[\left(2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots\right)\right]\end{aligned}$$

i.e. $\boldsymbol{\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots}$ (which is valid for all values of x). $\cosh x$ is an even function and contains only even powers of x in its expansion

$$\begin{aligned}\sinh x &= \frac{1}{2}(e^x - e^{-x})\\ &= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)\right.\\ &\quad \left. - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)\right]\\ &= \frac{1}{2}\left[2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots\right]\end{aligned}$$

i.e. $\boldsymbol{\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots}$ (which is valid for all values of x). $\sinh x$ is an odd function and contains only odd powers of x in its series expansion

Problem 15. Using the series expansion for ch x evaluate ch 1 correct to 4 decimal place.

$\mathrm{ch}\,x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots$ from above

Let $x = 1$,

$$\text{then ch}\,1 = 1 + \frac{1^2}{2\times 1} + \frac{1^4}{4\times 3\times 2\times 1} + \frac{1^6}{6\times 5\times 4\times 3\times 2\times 1} + \cdots$$
$$= 1 + 0.5 + 0.04167 + 0.001389 + \cdots$$

i.e. **ch 1 = 1.5431**, correct to 4 decimal places, which may be checked by using a calculator.

Problem 16. Determine, correct to 3 decimal places, the value of sh 3 using the series expansion for sh x.

$$\text{sh}\,x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \text{ from above}$$

Let $x = 3$, then

$$\text{sh}\,3 = 3 + \frac{3^3}{3!} + \frac{3^5}{5!} + \frac{3^7}{7!} + \frac{3^9}{9!} + \frac{3^{11}}{11!} + \cdots$$
$$= 3 + 4.5 + 2.025 + 0.43393 + 0.05424 + 0.00444 + \cdots$$

i.e. **sh 3 = 10.018**, correct to 3 decimal places.

Problem 17. Determine the power series for $2\,\text{ch}\left(\frac{\theta}{2}\right) - \text{sh}\,2\theta$ as far as the term in θ^5.

In the series expansion for ch x, let $x = \frac{\theta}{2}$ then:

$$2\,\text{ch}\left(\frac{\theta}{2}\right) = 2\left[1 + \frac{(\theta/2)^2}{2!} + \frac{(\theta/2)^4}{4!} + \cdots\right]$$
$$= 2 + \frac{\theta^2}{4} + \frac{\theta^4}{192} + \cdots$$

In the series expansion for sh x, let $x = 2\theta$, then:

$$\text{sh}\,2\theta = 2\theta + \frac{(2\theta)^3}{3!} + \frac{(2\theta)^5}{5!} + \cdots$$
$$= 2\theta + \frac{4}{3}\theta^3 + \frac{4}{15}\theta^5 + \cdots$$

Hence

$$\text{ch}\left(\frac{\theta}{2}\right) - \text{sh}\,2\theta = \left(2 + \frac{\theta^2}{4} + \frac{\theta^4}{192} + \cdots\right) - \left(2\theta + \frac{4}{3}\theta^3 + \frac{4}{15}\theta^5 + \cdots\right)$$
$$= \mathbf{2 - 2\theta + \frac{\theta^2}{4} - \frac{4}{3}\theta^3 + \frac{\theta^4}{192} - \frac{4}{15}\theta^5 + \cdots} \textbf{ as far the term in } \boldsymbol{\theta^5}$$

Now try the following exercise.

Exercise 27 Further problems on series expansions for cosh x and sinh x

1. Use the series expansion for ch x to evaluate, correct to 4 decimal places: (a) ch 1.5 (b) ch 0.8 [(a) 2.3524 (b) 1.3374]

2. Use the series expansion for sh x to evaluate, correct to 4 decimal places: (a) sh 0.5 (b) sh 2 [(a) 0.5211 (b) 3.6269]

3. Expand the following as a power series as far as the term in x^5: (a) sh $3x$ (b) ch $2x$
$$\left[\begin{array}{l}\text{(a) } 3x + \frac{9}{2}x^3 + \frac{81}{40}x^5 \\ \text{(b) } 1 + 2x^2 + \frac{2}{3}x^4\end{array}\right]$$

In Problems 4 and 5, prove the given identities, the series being taken as far as the term in θ^5 only.

4. $\text{sh}\,2\theta - \text{sh}\,\theta \equiv \theta + \frac{7}{6}\theta^3 + \frac{31}{120}\theta^5$

5. $2\,\text{sh}\frac{\theta}{2} - \text{ch}\frac{\theta}{2} \equiv -1 + \theta - \frac{\theta^2}{8} + \frac{\theta^3}{24} - \frac{\theta^4}{384} + \frac{\theta^5}{1920}$