6 Arithmetic and geometric progressions

6.1 Arithmetic progressions

When a sequence has a constant difference between successive terms it is called an **arithmetic progression** (often abbreviated to AP).
Examples include:

(i) 1, 4, 7, 10, 13, ... where the **common difference** is 3 and

(ii) $a, a+d, a+2d, a+3d, \ldots$ where the common difference is d.

If the first term of an AP is 'a' and the common difference is 'd' then

$$\boxed{\text{the } n\text{'th term is: } \boldsymbol{a+(n-1)d}}$$

In example (i) above, the 7th term is given by $1+(7-1)3 = \mathbf{19}$, which may be readily checked.

The sum S of an AP can be obtained by multiplying the average of all the terms by the number of terms.

The average of all the terms $= \dfrac{a+l}{2}$, where 'a' is the first term and l is the last term, i.e. $l = a+(n-1)d$, for n terms.

Hence the sum of n terms,

$$S_n = n\left(\frac{a+l}{2}\right)$$

$$= \frac{n}{2}\{a+[a+(n-1)d]\}$$

i.e. $$\boxed{\boldsymbol{S_n = \frac{n}{2}[2a+(n-1)d]}}$$

For example, the sum of the first 7 terms of the series 1, 4, 7, 10, 13, ... is given by

$$S_7 = \frac{7}{2}[2(1)+(7-1)3], \text{ since } a=1 \text{ and } d=3$$

$$= \frac{7}{2}[2+18] = \frac{7}{2}[20] = \mathbf{70}$$

6.2 Worked problems on arithmetic progressions

> Problem 1. Determine (a) the ninth, and (b) the sixteenth term of the series 2, 7, 12, 17, ...

2, 7, 12, 17, ... is an arithmetic progression with a common difference, d, of 5.

(a) The n'th term of an AP is given by $a+(n-1)d$
Since the first term $a=2$, $d=5$ and $n=9$ then the 9th term is:
$2+(9-1)5 = 2+(8)(5) = 2+40 = \mathbf{42}$

(b) The 16th term is:
$2+(16-1)5 = 2+(15)(5) = 2+75 = \mathbf{77}$.

> Problem 2. The 6th term of an AP is 17 and the 13th term is 38. Determine the 19th term.

The n'th term of an AP is $a+(n-1)d$

The 6th term is: $a+5d = 17$ (1)

The 13th term is: $a+12d = 38$ (2)

Equation (2) − equation (1) gives: $7d = 21$, from which, $d = \dfrac{21}{7} = 3$.

Substituting in equation (1) gives: $a+15 = 17$, from which, $a = 2$.

Hence the 19th term is:
$a+(n-1)d = 2+(19-1)3 = 2+(18)(3) = 2+54 = \mathbf{56}$.

> Problem 3. Determine the number of the term whose value is 22 in the series $2\frac{1}{2}$, 4, $5\frac{1}{2}$, 7, ...

$2\frac{1}{2}, 4, 5\frac{1}{2}, 7, \ldots$ is an AP where $a = 2\frac{1}{2}$ and $d = 1\frac{1}{2}$.

Hence if the n'th term is 22 then: $a+(n-1)d=22$
i.e. $2\frac{1}{2}+(n-1)\left(1\frac{1}{2}\right)=22$
$(n-1)\left(1\frac{1}{2}\right)=22-2\frac{1}{2}=19\frac{1}{2}$.

$$n-1=\frac{19\frac{1}{2}}{1\frac{1}{2}}=13 \text{ and } n=13+1=14$$

i.e. **the 14th term of the AP is 22**.

Problem 4. Find the sum of the first 12 terms of the series 5, 9, 13, 17, ...

5, 9, 13, 17, ... is an AP where $a=5$ and $d=4$. The sum of n terms of an AP,

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

Hence the sum of the first 12 terms,

$$S_{12}=\frac{12}{2}[2(5)+(12-1)4]$$
$$=6[10+44]=6(54)=\mathbf{324}$$

Problem 5. Find the sum of the first 21 terms of the series 3.5, 4.1, 4.7, 5.3, ...

3.5, 4.1, 4.7, 5.3, ... is an AP where $a=3.5$ and $d=0.6$.

The sum of the first 21 terms,

$$S_{21}=\frac{21}{2}[2a+(n-1)d]$$
$$=\frac{21}{2}[2(3.5)+(21-1)0.6]=\frac{21}{2}[7+12]$$
$$=\frac{21}{2}(19)=\frac{399}{2}=\mathbf{199.5}$$

Now try the following exercise.

Exercise 28 Further problems on arithmetic progressions

1. Find the 11th term of the series 8, 14, 20, 26, ... [68]
2. Find the 17th term of the series 11, 10.7, 10.4, 10.1, ... [6.2]
3. The seventh term of a series is 29 and the eleventh term is 54. Determine the sixteenth term. [85.25]
4. Find the 15th term of an arithmetic progression of which the first term is 2.5 and the tenth term is 16. [23.5]
5. Determine the number of the term which is 29 in the series 7, 9.2, 11.4, 13.6, ... [11]
6. Find the sum of the first 11 terms of the series 4, 7, 10, 13, ... [209]
7. Determine the sum of the series 6.5, 8.0, 9.5, 11.0, ..., 32 [346.5]

6.3 Further worked problems on arithmetic progressions

Problem 6. The sum of 7 terms of an AP is 35 and the common difference is 1.2. Determine the first term of the series.

$n=7$, $d=1.2$ and $S_7=35$
Since the sum of n terms of an AP is given by

$$S_n=\frac{n}{2}[2a+(n-1)d], \text{ then}$$

$$35=\frac{7}{2}[2a+(7-1)1.2]=\frac{7}{2}[2a+7.2]$$

Hence $\quad \dfrac{35\times 2}{7}=2a+7.2$

$$10=2a+7.2$$

Thus $\quad 2a=10-7.2=2.8,$

from which $\quad a=\dfrac{2.8}{2}=1.4$

i.e. **the first term, $a=1.4$**

Problem 7. Three numbers are in arithmetic progression. Their sum is 15 and their product is 80. Determine the three numbers.

Let the three numbers be $(a-d)$, a and $(a+d)$

Then $(a-d)+a+(a+d)=15$, i.e. $3a=15$, from which, $a=5$

Also, $a(a-d)(a+d)=80$, i.e. $a(a^2-d^2)=80$

Since $\quad a=5, 5(5^2-d^2)=80$

$$125-5d^2=80$$

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$$125 - 80 = 5d^2$$
$$45 = 5d^2$$

from which, $d^2 = \frac{45}{5} = 9$. Hence $d = \sqrt{9} = \pm 3$.
The three numbers are thus $(5 - 3)$, 5 and $(5 + 3)$, i.e. **2, 5 and 8**.

> Problem 8. Find the sum of all the numbers between 0 and 207 which are exactly divisible by 3.

The series 3, 6, 9, 12, ..., 207 is an AP whose first term $a = 3$ and common difference $d = 3$

The last term is $a + (n - 1)d = 207$

i.e. $3 + (n - 1)3 = 207$,

from which $(n - 1) = \frac{207 - 3}{3} = 68$

Hence $n = 68 + 1 = 69$

The sum of all 69 terms is given by

$$S_{69} = \frac{n}{2}[2a + (n - 1)d]$$
$$= \frac{69}{2}[2(3) + (69 - 1)3]$$
$$= \frac{69}{2}[6 + 204] = \frac{69}{2}(210) = \mathbf{7245}$$

> Problem 9. The first, twelfth and last term of an arithmetic progression are 4, $31\frac{1}{2}$, and $376\frac{1}{2}$ respectively. Determine (a) the number of terms in the series, (b) the sum of all the terms and (c) the '80'th term.

(a) Let the AP be $a, a + d, a + 2d, \ldots, a + (n - 1)d$, where $a = 4$

The 12th term is: $a + (12 - 1)d = 31\frac{1}{2}$

i.e. $4 + 11d = 31\frac{1}{2}$,

from which, $11d = 31\frac{1}{2} - 4 = 27\frac{1}{2}$

Hence $d = \frac{27\frac{1}{2}}{11} = 2\frac{1}{2}$

The last term is $a + (n - 1)d$

i.e. $4 + (n - 1)\left(2\frac{1}{2}\right) = 376\frac{1}{2}$

$$(n - 1) = \frac{376\frac{1}{2} - 4}{2\frac{1}{2}}$$
$$= \frac{372\frac{1}{2}}{2\frac{1}{2}} = 149$$

Hence the number of terms in the series, $n = 149 + 1 = 150$

(b) Sum of all the terms,

$$S_{150} = \frac{n}{2}[2a + (n - 1)d]$$
$$= \frac{150}{2}\left[2(4) + (150 - 1)\left(2\frac{1}{2}\right)\right]$$
$$= 75\left[8 + (149)\left(2\frac{1}{2}\right)\right]$$
$$= 85[8 + 372.5]$$
$$= 75(380.5) = \mathbf{28537\frac{1}{2}}$$

(c) The 80th term is:

$$a + (n - 1)d = 4 + (80 - 1)\left(2\tfrac{1}{2}\right)$$
$$= 4 + (79)\left(2\tfrac{1}{2}\right)$$
$$= 4 + 197.5 = \mathbf{201\tfrac{1}{2}}$$

Now try the following exercise.

Exercise 29 Further problems on arithmetic progressions

1. The sum of 15 terms of an arithmetic progression is 202.5 and the common difference is 2. Find the first term of the series. [−0.5]

2. Three numbers are in arithmetic progression. Their sum is 9 and their product is 20.25. Determine the three numbers. [1.5, 3, 4.5]

3. Find the sum of all the numbers between 5 and 250 which are exactly divisible by 4. [7808]

4. Find the number of terms of the series 5, 8, 11, ... of which the sum is 1025. [25]

5. Insert four terms between 5 and 22.5 to form an arithmetic progression.
[8.5, 12, 15.5, 19]

6. The first, tenth and last terms of an arithmetic progression are 9, 40.5, and 425.5 respectively. Find (a) the number of terms, (b) the sum of all the terms and (c) the 70th term.
[(a) 120 (b) 26070 (c) 250.5]

7. On commencing employment a man is paid a salary of £7200 per annum and receives annual increments of £350. Determine his salary in the 9th year and calculate the total he will have received in the first 12 years.
[£10 000, £109 500]

8. An oil company bores a hole 80 m deep. Estimate the cost of boring if the cost is £30 for drilling the first metre with an increase in cost of £2 per metre for each succeeding metre.
[£8720]

6.4 Geometric progressions

When a sequence has a constant ratio between successive terms it is called a **geometric progression** (often abbreviated to GP). The constant is called the **common ratio, r**.
Examples include

(i) $1, 2, 4, 8, \ldots$ where the common ratio is 2 and

(ii) $a, ar, ar^2, ar^3, \ldots$ where the common ratio is r.

If the first term of a GP is 'a' and the common ratio is r, then

$$\boxed{\text{the } n\text{'th term is: } \boldsymbol{ar^{n-1}}}$$

which can be readily checked from the above examples.

For example, the 8th term of the GP $1, 2, 4, 8, \ldots$ is $(1)(2)^7 = \mathbf{128}$, since $a = 1$ and $r = 2$.

Let a GP be $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$
then the sum of n terms,

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \cdots \quad (1)$$

Multiplying throughout by r gives:

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n + \cdots \quad (2)$$

Subtracting equation (2) from equation (1) gives:

$$S_n - rS_n = a - ar^n$$
i.e. $S_n(1 - r) = a(1 - r^n)$

Thus the sum of n terms, $\boxed{\boldsymbol{S_n = \dfrac{a(1 - r^n)}{(1 - r)}}}$ which is valid when $r < 1$.
Subtracting equation (1) from equation (2) gives

$\boxed{\boldsymbol{S_n = \dfrac{a(r^n - 1)}{(r - 1)}}}$ which is valid when $r > 1$.

For example, the sum of the first 8 terms of the GP $1, 2, 4, 8, 16, \ldots$ is given by $S_8 = \dfrac{1(2^8 - 1)}{(2 - 1)}$, since $a = 1$ and $r = 2$

i.e. $S_8 = \dfrac{1(256 - 1)}{1} = \mathbf{255}$

When the common ratio r of a GP is less than unity, the sum of n terms, $S_n = \dfrac{a(1 - r^n)}{(1 - r)}$, which may be written as $S_n = \dfrac{a}{(1 - r)} - \dfrac{ar^n}{(1 - r)}$.
Since $r < 1$, r^n becomes less as n increases, i.e. $r^n \to 0$ as $n \to \infty$.
Hence $\dfrac{ar^n}{(1 - r)} \to 0$ as $n \to \infty$. Thus $S_n \to \dfrac{a}{(1 - r)}$ as $n \to \infty$.
The quantity $\dfrac{a}{(1 - r)}$ is called the **sum to infinity**, S_∞, and is the limiting value of the sum of an infinite number of terms,

i.e. $\boxed{\boldsymbol{S_\infty = \dfrac{a}{(1 - r)}}}$ which is valid when $-1 < r < 1$.

For example, the sum to infinity of the GP $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ is

$S_\infty = \dfrac{1}{1 - \frac{1}{2}}$, since $a = 1$ and $r = \frac{1}{2}$, i.e. $S_\infty = 2$.

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6.5 Worked problems on geometric progressions

Problem 10. Determine the tenth term of the series 3, 6, 12, 24, …

3, 6, 12, 24, … is a geometric progression with a common ratio r of 2. The n'th term of a GP is ar^{n-1}, where a is the first term. Hence the 10th term is: $(3)(2)^{10-1} = (3)(2)^9 = 3(512) = \mathbf{1536}$.

Problem 11. Find the sum of the first 7 terms of the series, $\frac{1}{2}$, $1\frac{1}{2}$, $4\frac{1}{2}$, $13\frac{1}{2}$, …

$\frac{1}{2}$, $1\frac{1}{2}$, $4\frac{1}{2}$, $13\frac{1}{2}$, … is a GP with a common ratio $r = 3$

The sum of n terms, $S_n = \dfrac{a(r^n - 1)}{(r-1)}$

Hence $S_7 = \dfrac{\frac{1}{2}(3^7 - 1)}{(3-1)} = \dfrac{\frac{1}{2}(2187-1)}{2} = \mathbf{546\frac{1}{2}}$

Problem 12. The first term of a geometric progression is 12 and the fifth term is 55. Determine the 8'th term and the 11'th term.

The 5th term is given by $ar^4 = 55$, where the first term $a = 12$

Hence $\quad r^4 = \dfrac{55}{a} = \dfrac{55}{12}$

and $\quad r = \sqrt[4]{\left(\dfrac{55}{12}\right)} = 1.4631719\ldots$

The 8th term is $ar^7 = (12)(1.4631719\ldots)^7 = \mathbf{172.3}$
The 11th term is $ar^{10} = (12)(1.4631719\ldots)^{10}$
$= \mathbf{539.7}$

Problem 13. Which term of the series 2187, 729, 243, … is $\frac{1}{9}$?

2187, 729, 243, … is a GP with a common ratio $r = \frac{1}{3}$ and first term $a = 2187$

The n'th term of a GP is given by: ar^{n-1}

Hence $\quad \dfrac{1}{9} = (2187)\left(\frac{1}{3}\right)^{n-1}$

from which $\quad \left(\dfrac{1}{3}\right)^{n-1} = \dfrac{1}{(9)(2187)} = \dfrac{1}{3^2 3^7}$

$$= \frac{1}{3^9} = \left(\frac{1}{3}\right)^9$$

Thus $(n-1) = 9$, from which, $n = 9 + 1 = 10$
i.e. $\mathbf{\frac{1}{9}}$ **is the 10th term of the GP**

Problem 14. Find the sum of the first 9 terms of the series 72.0, 57.6, 46.08, …

The common ratio, $r = \dfrac{ar}{a} = \dfrac{57.6}{72.0} = 0.8$

$\left(\text{also } \dfrac{ar^2}{ar} = \dfrac{46.08}{57.6} = 0.8\right)$

The sum of 9 terms,

$$S_9 = \frac{a(1-r^n)}{(1-r)} = \frac{72.0(1-0.8^9)}{(1-0.8)}$$
$$= \frac{72.0(1-0.1342)}{0.2} = \mathbf{311.7}$$

Problem 15. Find the sum to infinity of the series 3, 1, $\frac{1}{3}$, …

3, 1, $\frac{1}{3}$, … is a GP of common ratio, $r = \frac{1}{3}$
The sum to infinity,

$$S_\infty = \frac{a}{1-r} = \frac{3}{1-\frac{1}{3}} = \frac{3}{\frac{2}{3}} = \frac{9}{2} = \mathbf{4\frac{1}{2}}$$

Now try the following exercise.

Exercise 30 Further problems on geometric progressions

1. Find the 10th term of the series 5, 10, 20, 40, … [2560]
2. Determine the sum of the first 7 terms of the series $\frac{1}{4}$, $\frac{3}{4}$, $2\frac{1}{4}$, $6\frac{3}{4}$, … [273.25]
3. The first term of a geometric progression is 4 and the 6th term is 128. Determine the 8th and 11th terms. [512, 4096]

4. Find the sum of the first 7 terms of the series 2, 5, $12\frac{1}{2}$, ... (correct to 4 significant figures) [812.5]
5. Determine the sum to infinity of the series 4, 2, 1, ... [8]
6. Find the sum to infinity of the series $2\frac{1}{2}$, $-1\frac{1}{4}$, $\frac{5}{8}$, ... $\left[1\frac{2}{3}\right]$

6.6 Further worked problems on geometric progressions

Problem 16. In a geometric progression the sixth term is 8 times the third term and the sum of the seventh and eighth terms is 192. Determine (a) the common ratio, (b) the first term, and (c) the sum of the fifth to eleventh terms, inclusive.

(a) Let the GP be $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$
The 3rd term $= ar^2$ and the sixth term $= ar^5$
The 6th term is 8 times the 3rd.
Hence $ar^5 = 8ar^2$ from which, $r^3 = 8$, $r = \sqrt[3]{8}$
i.e. **the common ratio** $\boldsymbol{r = 2}$.

(b) The sum of the 7th and 8th terms is 192. Hence $ar^6 + ar^7 = 192$.

$$\text{Since}\quad r = 2,\quad \text{then } 64a + 128a = 192$$
$$192a = 192,$$

from which, $\boldsymbol{a}$, the first term, $\mathbf{= 1}$.

(c) The sum of the 5th to 11th terms (inclusive) is given by:

$$\begin{aligned} S_{11} - S_4 &= \frac{a(r^{11}-1)}{(r-1)} - \frac{a(r^4-1)}{(r-1)} \\ &= \frac{1(2^{11}-1)}{(2-1)} - \frac{1(2^4-1)}{(2-1)} \\ &= (2^{11}-1) - (2^4-1) \\ &= 2^{11} - 2^4 = 2048 - 16 = \mathbf{2032} \end{aligned}$$

Problem 17. A hire tool firm finds that their net return from hiring tools is decreasing by 10% per annum. If their net gain on a certain tool this year is £400, find the possible total of all future profits from this tool (assuming the tool lasts for ever).

The net gain forms a series:

$$£400 + £400 \times 0.9 + £400 \times 0.9^2 + \cdots,$$

which is a GP with $a = 400$ and $r = 0.9$.
The sum to infinity,

$$S_\infty = \frac{a}{(1-r)} = \frac{400}{(1-0.9)}$$
$$= \mathbf{£4000 = total\ future\ profits}$$

Problem 18. If £100 is invested at compound interest of 8% per annum, determine (a) the value after 10 years, (b) the time, correct to the nearest year, it takes to reach more than £300.

(a) Let the GP be $a, ar, ar^2, \ldots, ar^n$
The first term $a = £100$
The common ratio $r = 1.08$
Hence the second term is

$$ar = (100)(1.08) = £108,$$

which is the value after 1 year,
the third term is

$$ar^2 = (100)(1.08)^2 = £116.64,$$

which is the value after 2 years, and so on.
Thus the value after 10 years

$$= ar^{10} = (100)(1.08)^{10} = \mathbf{£215.89}$$

(b) When £300 has been reached, $300 = ar^n$

i.e. $\quad 300 = 100(1.08)^n$

and $\quad 3 = (1.08)^n$

Taking logarithms to base 10 of both sides gives:

$$\lg 3 = \lg(1.08)^n = n \lg(1.08),$$

by the laws of logarithms

from which, $n = \dfrac{\lg 3}{\lg 1.08} = 14.3$

Hence it will take 15 years to reach more than £300.

Problem 19. A drilling machine is to have 6 speeds ranging from 50 rev/min to 750 rev/min. If the speeds form a geometric progression determine their values, each correct to the nearest whole number.

Let the GP of n terms be given by a, ar, $ar^2, \ldots, ar^{n-1}$.
The first term $a = 50$ rev/min
The 6th term is given by ar^{6-1}, which is 750 rev/min,

i.e., $\quad ar^5 = 750$

from which $\quad r^5 = \dfrac{750}{a} = \dfrac{750}{50} = 15$

Thus the common ratio, $r = \sqrt[5]{15} = 1.7188$

The first term is $a = 50$ rev/min

the second term is $ar = (50)\,(1.7188) = 85.94$,

the third term is $ar^2 = (50)\,(1.7188)^2 = 147.71$,

the fourth term is $ar^3 = (50)\,(1.7188)^3 = 253.89$,

the fifth term is $ar^4 = (50)\,(1.7188)^4 = 436.39$,

the sixth term is $ar^5 = (50)\,(1.7188)^5 = 750.06$

Hence, correct to the nearest whole number, the 6 speeds of the drilling machine are **50, 86, 148, 254, 436 and 750 rev/min**.

Now try the following exercise.

Exercise 31 Further problems on geometric progressions

1. In a geometric progression the 5th term is 9 times the 3rd term and the sum of the 6th and 7th terms is 1944. Determine (a) the common ratio, (b) the first term and (c) the sum of the 4th to 10th terms inclusive.
[(a) 3 (b) 2 (c) 59022]

2. Which term of the series 3, 9, 27, … is 59049? [10^{th}]

3. The value of a lathe originally valued at £3000 depreciates 15% per annum. Calculate its value after 4 years. The machine is sold when its value is less than £550. After how many years is the lathe sold?
[£1566, 11 years]

4. If the population of Great Britain is 55 million and is decreasing at 2.4% per annum, what will be the population in 5 years time?
[48.71 M]

5. 100 g of a radioactive substance disintegrates at a rate of 3% per annum. How much of the substance is left after 11 years? [71.53 g]

6. If £250 is invested at compound interest of 6% per annum determine (a) the value after 15 years, (b) the time, correct to the nearest year, it takes to reach £750.
[(a) £599.14 (b) 19 years]

7. A drilling machine is to have 8 speeds ranging from 100 rev/min to 1000 rev/min. If the speeds form a geometric progression determine their values, each correct to the nearest whole number.
[100, 139, 193, 268, 373, 518, 720, 1000 rev/min]