PUZZLER

A speaker for a stereo system operates even if the wires connecting it to the amplifier are reversed, that is, + for − and − for + (or red for black and black for red). Nonetheless, the owner's manual says that for best performance you should be careful to connect the two speakers properly, so that they are "in phase." Why is this such an important consideration for the quality of the sound you hear? *(George Semple)*

chapter 18

Superposition and Standing Waves

Chapter Outline

Important in the study of waves is the combined effect of two or more waves traveling in the same medium. For instance, what happens to a string when a wave traveling along it hits a fixed end and is reflected back on itself? What is the air pressure variation at a particular seat in a theater when the instruments of an orchestra sound together?

When analyzing a linear medium—that is, one in which the restoring force acting on the particles of the medium is proportional to the displacement of the particles—we can apply the principle of superposition to determine the resultant disturbance. In Chapter 16 we discussed this principle as it applies to wave pulses. In this chapter we study the superposition principle as it applies to sinusoidal waves. If the sinusoidal waves that combine in a linear medium have the same frequency and wavelength, a stationary pattern—called a *standing wave*—can be produced at certain frequencies under certain circumstances. For example, a taut string fixed at both ends has a discrete set of oscillation patterns, called *modes of vibration,* that are related to the tension and linear mass density of the string. These modes of vibration are found in stringed musical instruments. Other musical instruments, such as the organ and the flute, make use of the natural frequencies of sound waves in hollow pipes. Such frequencies are related to the length and shape of the pipe and depend on whether the pipe is open at both ends or open at one end and closed at the other.

We also consider the superposition and interference of waves having different frequencies and wavelengths. When two sound waves having nearly the same frequency interfere, we hear variations in the loudness called *beats.* The beat frequency corresponds to the rate of alternation between constructive and destructive interference. Finally, we discuss how any non-sinusoidal periodic wave can be described as a sum of sine and cosine functions.

18.1 SUPERPOSITION AND INTERFERENCE OF SINUSOIDAL WAVES

9.6 & 9.7

Imagine that you are standing in a swimming pool and that a beach ball is floating a couple of meters away. You use your right hand to send a series of waves toward the beach ball, causing it to repeatedly move upward by 5 cm, return to its original position, and then move downward by 5 cm. After the water becomes still, you use your left hand to send an identical set of waves toward the beach ball and observe the same behavior. What happens if you use both hands at the same time to send two waves toward the beach ball? How the beach ball responds to the waves depends on whether the waves work together (that is, both waves make the beach ball go up at the same time and then down at the same time) or work against each other (that is, one wave tries to make the beach ball go up, while the other wave tries to make it go down). Because it is possible to have two or more waves in the same location at the same time, we have to consider how waves interact with each other and with their surroundings.

The superposition principle states that when two or more waves move in the same linear medium, the net displacement of the medium (that is, the resultant wave) at any point equals the algebraic sum of all the displacements caused by the individual waves. Let us apply this principle to two sinusoidal waves traveling in the same direction in a linear medium. If the two waves are traveling to the right and have the same frequency, wavelength, and amplitude but differ in phase, we can express their individual wave functions as

$$y_1 = A\sin(kx - \omega t) \qquad y_2 = A\sin(kx - \omega t + \phi)$$

where, as usual, $k = 2\pi/\lambda$, $\omega = 2\pi f$, and ϕ is the phase constant, which we introduced in the context of simple harmonic motion in Chapter 13. Hence, the resultant wave function y is

$$y = y_1 + y_2 = A[\sin(kx - \omega t) + \sin(kx - \omega t + \phi)]$$

To simplify this expression, we use the trigonometric identity

$$\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

If we let $a = kx - \omega t$ and $b = kx - \omega t + \phi$, we find that the resultant wave function y reduces to

Resultant of two traveling sinusoidal waves

$$y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

This result has several important features. The resultant wave function y also is sinusoidal and has the same frequency and wavelength as the individual waves, since the sine function incorporates the same values of k and ω that appear in the original wave functions. The amplitude of the resultant wave is $2A\cos(\phi/2)$, and its phase is $\phi/2$. If the phase constant ϕ equals 0, then $\cos(\phi/2) = \cos 0 = 1$, and the amplitude of the resultant wave is $2A$—twice the amplitude of either individual wave. In this case, in which $\phi = 0$, the waves are said to be everywhere *in phase* and thus **interfere constructively.** That is, the crests and troughs of the individual waves y_1 and y_2 occur at the same positions and combine to form the red curve y of amplitude $2A$ shown in Figure 18.1a. Because the individual waves are in phase, they are indistinguishable in Figure 18.1a, in which they appear as a single blue curve. In general, constructive interference occurs when $\cos(\phi/2) = \pm 1$. This is true, for example, when $\phi = 0, 2\pi, 4\pi, \ldots$ rad—that is, when ϕ is an *even* multiple of π.

Constructive interference

When ϕ is equal to π rad or to any *odd* multiple of π, then $\cos(\phi/2) = \cos(\pi/2) = 0$, and the crests of one wave occur at the same positions as the troughs of the second wave (Fig. 18.1b). Thus, the resultant wave has *zero* amplitude everywhere, as a consequence of **destructive interference.** Finally, when the phase constant has an arbitrary value other than 0 or other than an integer multiple of π rad (Fig. 18.1c), the resultant wave has an amplitude whose value is somewhere between 0 and $2A$.

Destructive interference

Interference of Sound Waves

One simple device for demonstrating interference of sound waves is illustrated in Figure 18.2. Sound from a loudspeaker S is sent into a tube at point P, where there is a T-shaped junction. Half of the sound power travels in one direction, and half travels in the opposite direction. Thus, the sound waves that reach the receiver R can travel along either of the two paths. The distance along any path from speaker to receiver is called the **path length r.** The lower path length r_1 is fixed, but the upper path length r_2 can be varied by sliding the U-shaped tube, which is similar to that on a slide trombone. When the difference in the path lengths $\Delta r = |r_2 - r_1|$ is either zero or some integer multiple of the wavelength λ (that is, $r = n\lambda$, where $n = 0, 1, 2, 3, \ldots$), the two waves reaching the receiver at any instant are in phase and interfere constructively, as shown in Figure 18.1a. For this case, a maximum in the sound intensity is detected at the receiver. If the path length r_2 is ad-

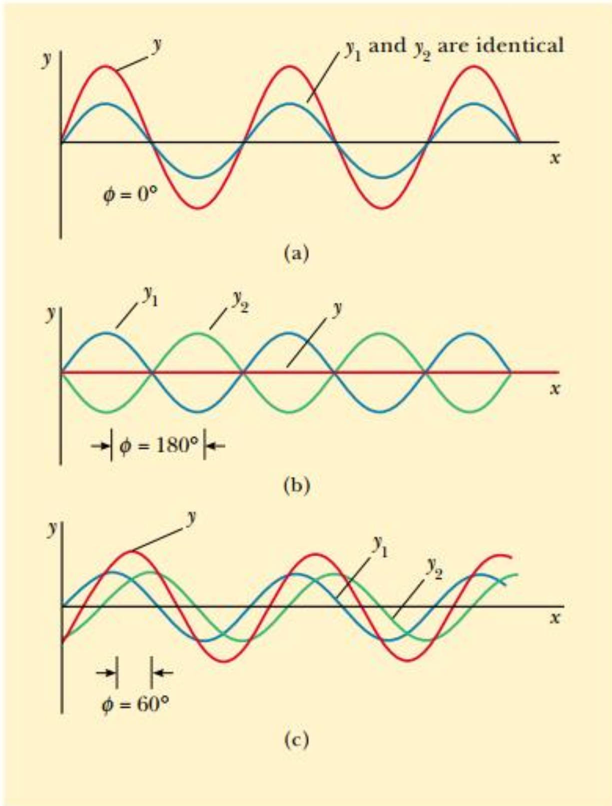


Figure 18.1 The superposition of two identical waves y_1 and y_2 (blue) to yield a resultant wave (red). (a) When y_1 and y_2 are in phase, the result is constructive interference. (b) When y_1 and y_2 are π rad out of phase, the result is destructive interference. (c) When the phase angle has a value other than 0 or π rad, the resultant wave y falls somewhere between the extremes shown in (a) and (b).

justed such that the path difference $\Delta r = \lambda/2, 3\lambda/2, \ldots, n\lambda/2$ (for n odd), the two waves are exactly π rad, or 180°, out of phase at the receiver and hence cancel each other. In this case of destructive interference, no sound is detected at the receiver. This simple experiment demonstrates that a phase difference may arise between two waves generated by the same source when they travel along paths of unequal lengths. This important phenomenon will be indispensable in our investigation of the interference of light waves in Chapter 37.

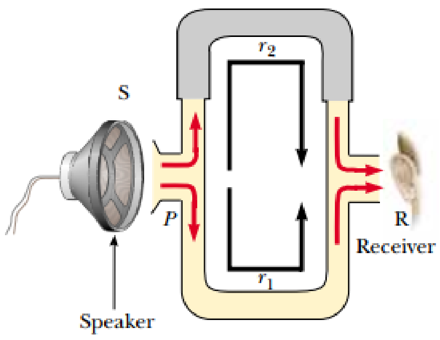


Figure 18.2 An acoustical system for demonstrating interference of sound waves. A sound wave from the speaker (S) propagates into the tube and splits into two parts at point P. The two waves, which superimpose at the opposite side, are detected at the receiver (R). The upper path length r_2 can be varied by sliding the upper section.

It is often useful to express the path difference in terms of the phase angle ϕ between the two waves. Because a path difference of one wavelength corresponds to a phase angle of 2π rad, we obtain the ratio $\phi/2\pi = \Delta r/\lambda$, or

$$\Delta r = \frac{\phi}{2\pi}\lambda \qquad (18.1)$$

Relationship between path difference and phase angle

Using the notion of path difference, we can express our conditions for constructive and destructive interference in a different way. If the path difference is any even multiple of $\lambda/2$, then the phase angle $\phi = 2n\pi$, where $n = 0, 1, 2, 3, \ldots$, and the interference is constructive. For path differences of odd multiples of $\lambda/2$, $\phi = (2n + 1)\pi$, where $n = 0, 1, 2, 3 \ldots$, and the interference is destructive. Thus, we have the conditions

$$\Delta r = (2n)\frac{\lambda}{2} \qquad \text{for constructive interference}$$

and (18.2)

$$\Delta r = (2n+1)\frac{\lambda}{2} \qquad \text{for destructive interference}$$

EXAMPLE 18.1 Two Speakers Driven by the Same Source

A pair of speakers placed 3.00 m apart are driven by the same oscillator (Fig. 18.3). A listener is originally at point O, which is located 8.00 m from the center of the line connecting the two speakers. The listener then walks to point P, which is a perpendicular distance 0.350 m from O, before reaching the *first minimum* in sound intensity. What is the frequency of the oscillator?

Solution To find the frequency, we need to know the wavelength of the sound coming from the speakers. With this information, combined with our knowledge of the speed of sound, we can calculate the frequency. We can determine the wavelength from the interference information given. The first minimum occurs when the two waves reaching the listener at point P are 180° out of phase—in other words, when their path difference Δr equals $\lambda/2$. To calculate the path difference, we must first find the path lengths r_1 and r_2.

Figure 18.3 shows the physical arrangement of the speakers, along with two shaded right triangles that can be drawn on the basis of the lengths described in the problem. From these triangles, we find that the path lengths are

$$r_1 = \sqrt{(8.00\text{ m})^2 + (1.15\text{ m})^2} = 8.08\text{ m}$$

and

$$r_2 = \sqrt{(8.00\text{ m})^2 + (1.85\text{ m})^2} = 8.21\text{ m}$$

Hence, the path difference is $r_2 - r_1 = 0.13$ m. Because we require that this path difference be equal to $\lambda/2$ for the first minimum, we find that $\lambda = 0.26$ m.

To obtain the oscillator frequency, we use Equation 16.14, $v = \lambda f$, where v is the speed of sound in air, 343 m/s:

$$f = \frac{v}{\lambda} = \frac{343\text{ m/s}}{0.26\text{ m}} = \boxed{1.3\text{ kHz}}$$

Exercise If the oscillator frequency is adjusted such that the first location at which a listener hears no sound is at a distance of 0.75 m from O, what is the new frequency?

Answer 0.63 kHz.

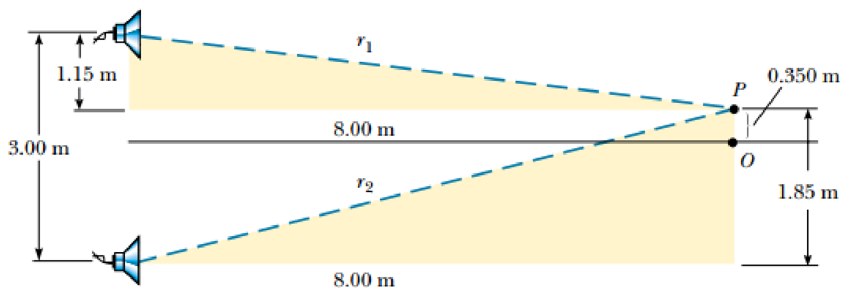


Figure 18.3

You can now understand why the speaker wires in a stereo system should be connected properly. When connected the wrong way—that is, when the positive (or red) wire is connected to the negative (or black) terminal—the speakers are said to be "out of phase" because the sound wave coming from one speaker destructively interferes with the wave coming from the other. In this situation, one speaker cone moves outward while the other moves inward. Along a line midway between the two, a rarefaction region from one speaker is superposed on a condensation region from the other speaker. Although the two sounds probably do not completely cancel each other (because the left and right stereo signals are usually not identical), a substantial loss of sound quality still occurs at points along this line.

18.2 STANDING WAVES

The sound waves from the speakers in Example 18.1 left the speakers in the forward direction, and we considered interference at a point in space in front of the speakers. Suppose that we turn the speakers so that they face each other and then have them emit sound of the same frequency and amplitude. We now have a situation in which two identical waves travel in opposite directions in the same medium. These waves combine in accordance with the superposition principle.

We can analyze such a situation by considering wave functions for two transverse sinusoidal waves having the same amplitude, frequency, and wavelength but traveling in opposite directions in the same medium:

where y_1 represents a wave traveling to the right and y_2 represents one traveling to the left. Adding these two functions gives the resultant wave function y:

$$y = y_1 + y_2 = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$

When we use the trigonometric identity $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$, this expression reduces to

Wave function for a standing wave

$$y = (2A\sin kx)\cos \omega t \qquad (18.3)$$

which is the wave function of a standing wave. A **standing wave,** such as the one shown in Figure 18.4, is an oscillation pattern with a stationary outline that results from the superposition of two identical waves traveling in opposite directions.

Notice that Equation 18.3 does not contain a function of $kx \pm \omega t$. Thus, it is not an expression for a traveling wave. If we observe a standing wave, we have no sense of motion in the direction of propagation of either of the original waves. If we compare this equation with Equation 13.3, we see that Equation 18.3 describes a special kind of simple harmonic motion. Every particle of the medium oscillates in simple harmonic motion with the same frequency ω (according to the $\cos \omega t$ factor in the equation). However, the amplitude of the simple harmonic motion of a given particle (given by the factor $2A \sin kx$, the coefficient of the cosine function) depends on the location x of the particle in the medium. We need to distinguish carefully between the amplitude A of the individual waves and the amplitude $2A \sin kx$ of the simple harmonic motion of the particles of the medium. A given particle in a standing wave vibrates within the constraints of the *envelope* function $2A \sin kx$, where x is the particle's position in the medium. This is in contrast to the situation in a traveling sinusoidal wave, in which all particles oscillate with the

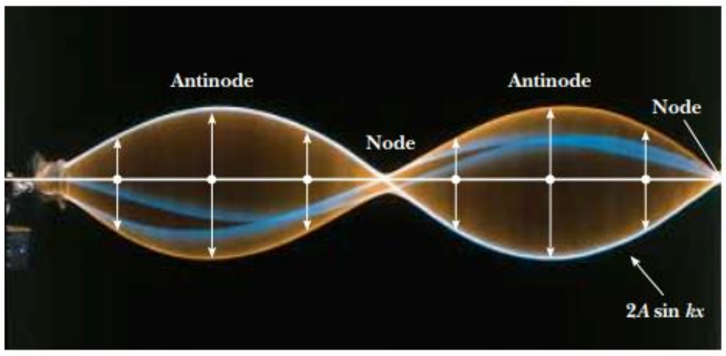


Figure 18.4 Multiflash photograph of a standing wave on a string. The time behavior of the vertical displacement from equilibrium of an individual particle of the string is given by cos ωt. That is, each particle vibrates at an angular frequency ω. The amplitude of the vertical oscillation of any particle on the string depends on the horizontal position of the particle. Each particle vibrates within the confines of the envelope function $2A \sin kx$.

same amplitude and the same frequency and in which the amplitude of the wave is the same as the amplitude of the simple harmonic motion of the particles.

The maximum displacement of a particle of the medium has a minimum value of zero when x satisfies the condition $\sin kx = 0$, that is, when

$$kx = \pi, 2\pi, 3\pi, \ldots$$

Because $k = 2\pi/\lambda$, these values for kx give

$$x = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots = \frac{n\lambda}{2} \qquad n = 0, 1, 2, 3, \ldots \qquad \textbf{(18.4)}$$

Position of nodes

These points of zero displacement are called **nodes.**

The particle with the greatest possible displacement from equilibrium has an amplitude of $2A$, and we define this as the amplitude of the standing wave. The positions in the medium at which this maximum displacement occurs are called **antinodes.** The antinodes are located at positions for which the coordinate x satisfies the condition $\sin kx = \pm 1$, that is, when

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Thus, the positions of the antinodes are given by

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots = \frac{n\lambda}{4} \qquad n = 1, 3, 5, \ldots \qquad \textbf{(18.5)}$$

Position of antinodes

In examining Equations 18.4 and 18.5, we note the following important features of the locations of nodes and antinodes:

The distance between adjacent antinodes is equal to $\lambda/2$.
The distance between adjacent nodes is equal to $\lambda/2$.
The distance between a node and an adjacent antinode is $\lambda/4$.

Displacement patterns of the particles of the medium produced at various times by two waves traveling in opposite directions are shown in Figure 18.5. The blue and green curves are the individual traveling waves, and the red curves are

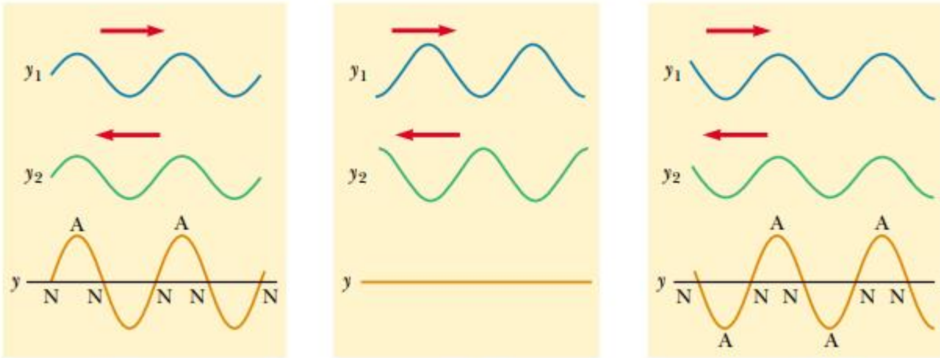


Figure 18.5 Standing-wave patterns produced at various times by two waves of equal amplitude traveling in opposite directions. For the resultant wave y, the nodes (N) are points of zero displacement, and the antinodes (A) are points of maximum displacement.

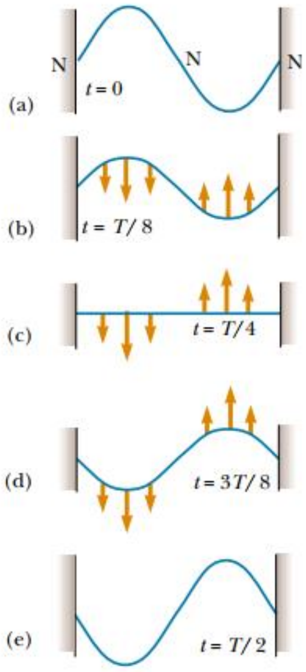


Figure 18.6 A standing-wave pattern in a taut string. The five "snapshots" were taken at half-cycle intervals. (a) At $t = 0$, the string is momentarily at rest; thus, $K = 0$, and all the energy is potential energy U associated with the vertical displacements of the string particles. (b) At $t = T/8$, the string is in motion, as indicated by the brown arrows, and the energy is half kinetic and half potential. (c) At $t = T/4$, the string is moving but horizontal (undeformed); thus, $U = 0$, and all the energy is kinetic. (d) The motion continues as indicated. (e) At $t = T/2$, the string is again momentarily at rest, but the crests and troughs of (a) are reversed. The cycle continues until ultimately, when a time interval equal to T has passed, the configuration shown in (a) is repeated.

the displacement patterns. At $t = 0$ (Fig. 18.5a), the two traveling waves are in phase, giving a displacement pattern in which each particle of the medium is experiencing its maximum displacement from equilibrium. One quarter of a period later, at $t = T/4$ (Fig. 18.5b), the traveling waves have moved one quarter of a wavelength (one to the right and the other to the left). At this time, the traveling waves are out of phase, and each particle of the medium is passing through the equilibrium position in its simple harmonic motion. The result is zero displacement for particles at all values of x—that is, the displacement pattern is a straight line. At $t = T/2$ (Fig. 18.5c), the traveling waves are again in phase, producing a displacement pattern that is inverted relative to the $t = 0$ pattern. In the standing wave, the particles of the medium alternate in time between the extremes shown in Figure 18.5a and c.

Energy in a Standing Wave

It is instructive to describe the energy associated with the particles of a medium in which a standing wave exists. Consider a standing wave formed on a taut string fixed at both ends, as shown in Figure 18.6. Except for the nodes, which are always stationary, all points on the string oscillate vertically with the same frequency but with different amplitudes of simple harmonic motion. Figure 18.6 represents snapshots of the standing wave at various times over one half of a period.

In a traveling wave, energy is transferred along with the wave, as we discussed in Chapter 16. We can imagine this transfer to be due to work done by one segment of the string on the next segment. As one segment moves upward, it exerts a force on the next segment, moving it through a displacement—that is, work is done. A particle of the string at a node, however, experiences no displacement. Thus, it cannot do work on the neighboring segment. As a result, no energy is transmitted along the string across a node, and energy does not propagate in a standing wave. For this reason, standing waves are often called **stationary waves.**

The energy of the oscillating string continuously alternates between elastic potential energy, when the string is momentarily stationary (see Fig. 18.6a), and kinetic energy, when the string is horizontal and the particles have their maximum speed (see Fig. 18.6c). At intermediate times (see Fig. 18.6b and d), the string particles have both potential energy and kinetic energy.

Quick Quiz 18.1

A standing wave described by Equation 18.3 is set up on a string. At what points on the string do the particles move the fastest?

EXAMPLE 18.2 Formation of a Standing Wave

Two waves traveling in opposite directions produce a standing wave. The individual wave functions $y = A\sin(kx - \omega t)$ are

$$y_1 = (4.0 \text{ cm}) \sin(3.0x - 2.0t)$$

and

$$y_2 = (4.0 \text{ cm}) \sin(3.0x + 2.0t)$$

where x and y are measured in centimeters. (a) Find the amplitude of the simple harmonic motion of the particle of the medium located at $x = 2.3$ cm.

Solution The standing wave is described by Equation 18.3; in this problem, we have $A = 4.0$ cm, $k = 3.0$ rad/cm, and $\omega = 2.0$ rad/s. Thus,

$$y = (2A \sin kx) \cos \omega t = [(8.0 \text{ cm}) \sin 3.0x] \cos 2.0t$$

Thus, we obtain the amplitude of the simple harmonic motion of the particle at the position $x = 2.3$ cm by evaluating the coefficient of the cosine function at this position:

$$y_{\text{max}} = (8.0 \text{ cm}) \sin 3.0x|_{x=2.3}$$
$$= (8.0 \text{ cm}) \sin(6.9 \text{ rad}) = \boxed{4.6 \text{ cm}}$$

(b) Find the positions of the nodes and antinodes.

Solution With $k = 2\pi/\lambda = 3.0$ rad/cm, we see that $\lambda = 2\pi/3$ cm. Therefore, from Equation 18.4 we find that the nodes are located at

$$x = n\frac{\lambda}{2} = \boxed{n\left(\frac{\pi}{3}\right) \text{cm}} \qquad n = 0, 1, 2, 3 \ldots$$

and from Equation 18.5 we find that the antinodes are located at

$$x = n\frac{\lambda}{4} = \boxed{n\left(\frac{\pi}{6}\right) \text{cm}} \qquad n = 1, 3, 5, \ldots$$

(c) What is the amplitude of the simple harmonic motion of a particle located at an antinode?

Solution According to Equation 18.3, the maximum displacement of a particle at an antinode is the amplitude of the standing wave, which is twice the amplitude of the individual traveling waves:

$$y_{\text{max}} = 2A = 2(4.0 \text{ cm}) = \boxed{8.0 \text{ cm}}$$

Let us check this result by evaluating the coefficient of our standing-wave function at the positions we found for the antinodes:

$$y_{\text{max}} = (8.0 \text{ cm}) \sin 3.0x|_{x=n(\pi/6)}$$
$$= (8.0 \text{ cm}) \sin\left[3.0n\left(\frac{\pi}{6}\right) \text{rad}\right]$$
$$= (8.0 \text{ cm}) \sin\left[n\left(\frac{\pi}{2}\right) \text{rad}\right] = 8.0 \text{ cm}$$

In evaluating this expression, we have used the fact that n is an odd integer; thus, the sine function is equal to unity.

18.3 STANDING WAVES IN A STRING FIXED AT BOTH ENDS

9.9 Consider a string of length L fixed at both ends, as shown in Figure 18.7. Standing waves are set up in the string by a continuous superposition of waves incident on and reflected from the ends. Note that the ends of the string, because they are fixed and must necessarily have zero displacement, are nodes by definition. The string has a number of natural patterns of oscillation, called **normal modes,** each of which has a characteristic frequency that is easily calculated.

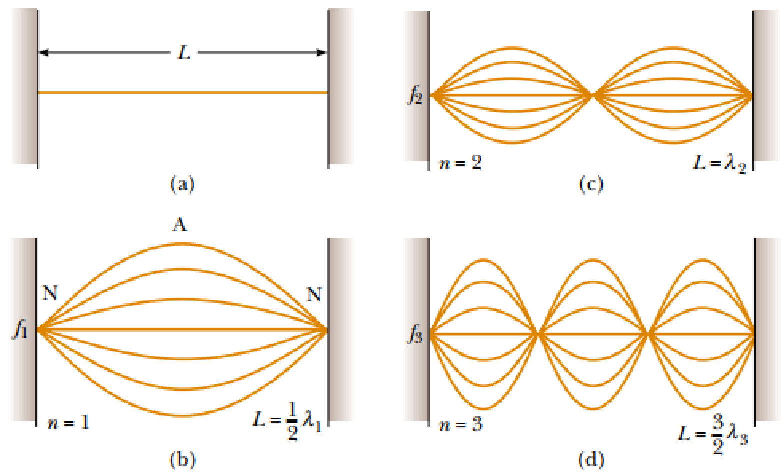


Figure 18.7 (a) A string of length L fixed at both ends. The normal modes of vibration form a harmonic series: (b) the fundamental, or first harmonic; (c) the second harmonic; (d) the third harmonic.

In general, the motion of an oscillating string fixed at both ends is described by the superposition of several normal modes. Exactly which normal modes are present depends on how the oscillation is started. For example, when a guitar string is plucked near its middle, the modes shown in Figure 18.7b and d, as well as other modes not shown, are excited.

In general, we can describe the normal modes of oscillation for the string by imposing the requirements that the ends be nodes and that the nodes and antinodes be separated by one fourth of a wavelength. The first normal mode, shown in Figure 18.7b, has nodes at its ends and one antinode in the middle. This is the longest-wavelength mode, and this is consistent with our requirements. This first normal mode occurs when the wavelength λ_1 is twice the length of the string, that is, $\lambda_1 = 2L$. The next normal mode, of wavelength λ_2 (see Fig. 18.7c), occurs when the wavelength equals the length of the string, that is, $\lambda_2 = L$. The third normal mode (see Fig. 18.7d) corresponds to the case in which $\lambda_3 = 2L/3$. In general, the wavelengths of the various normal modes for a string of length L fixed at both ends are

Wavelengths of normal modes

$$\lambda_n = \frac{2L}{n} \qquad n = 1, 2, 3, \ldots \tag{18.6}$$

where the index n refers to the nth normal mode of oscillation. These are the *possible* modes of oscillation for the string. The *actual* modes that are excited by a given pluck of the string are discussed below.

The natural frequencies associated with these modes are obtained from the relationship $f = v/\lambda$, where the wave speed v is the same for all frequencies. Using Equation 18.6, we find that the natural frequencies f_n of the normal modes are

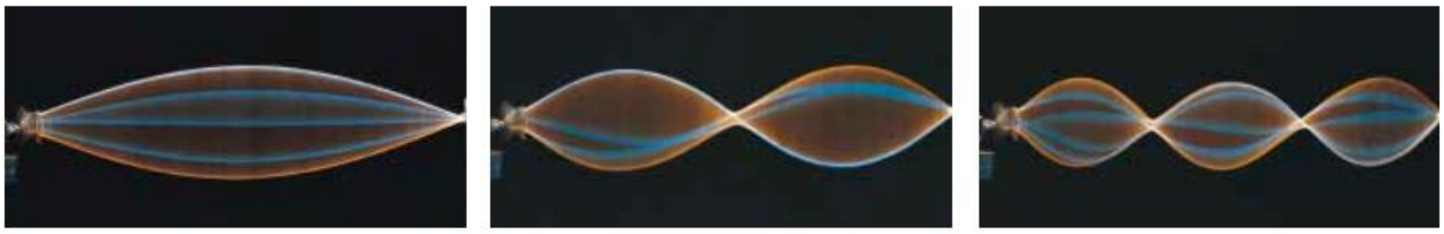
Frequencies of normal modes as functions of wave speed and length of string

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} \qquad n = 1, 2, 3, \ldots \tag{18.7}$$

Because $v = \sqrt{T/\mu}$ (see Eq. 16.4), where T is the tension in the string and μ is its linear mass density, we can also express the natural frequencies of a taut string as

Frequencies of normal modes as functions of string tension and linear mass density

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \qquad n = 1, 2, 3, \ldots \tag{18.8}$$



Multiflash photographs of standing-wave patterns in a cord driven by a vibrator at its left end. The single-loop pattern represents the first normal mode ($n = 1$). The double-loop pattern represents the second normal mode ($n = 2$), and the triple-loop pattern represents the third normal mode ($n = 3$).

The lowest frequency f_1, which corresponds to $n = 1$, is called either the **fundamental** or the **fundamental frequency** and is given by

Fundamental frequency of a taut string

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \qquad \textbf{(18.9)}$$

The frequencies of the remaining normal modes are integer multiples of the fundamental frequency. Frequencies of normal modes that exhibit an integer-multiple relationship such as this form a **harmonic series,** and the normal modes are called **harmonics.** The fundamental frequency f_1 is the frequency of the first harmonic; the frequency $f_2 = 2f_1$ is the frequency of the second harmonic; and the frequency $f_n = nf_1$ is the frequency of the nth harmonic. Other oscillating systems, such as a drumhead, exhibit normal modes, but the frequencies are not related as integer multiples of a fundamental. Thus, we do not use the term *harmonic* in association with these types of systems.

In obtaining Equation 18.6, we used a technique based on the separation distance between nodes and antinodes. We can obtain this equation in an alternative manner. Because we require that the string be fixed at $x = 0$ and $x = L$, the wave function $y(x, t)$ given by Equation 18.3 must be zero at these points for all times. That is, the *boundary conditions* require that $y(0, t) = 0$ and that $y(L, t) = 0$ for all values of t. Because the standing wave is described by $y = (2A \sin kx) \cos \omega t$, the first boundary condition, $y(0, t) = 0$, is automatically satisfied because $\sin kx = 0$ at $x = 0$. To meet the second boundary condition, $y(L, t) = 0$, we require that $\sin kL = 0$. This condition is satisfied when the angle kL equals an integer multiple of π rad. Therefore, the allowed values of k are given by[1]

$$k_nL = n\pi \qquad n = 1, 2, 3, \ldots \qquad \textbf{(18.10)}$$

Because $k_n = 2\pi/\lambda_n$, we find that

$$\left(\frac{2\pi}{\lambda_n}\right)L = n\pi \qquad \text{or} \qquad \lambda_n = \frac{2L}{n}$$

which is identical to Equation 18.6.

Let us now examine how these various harmonics are created in a string. If we wish to excite just a single harmonic, we need to distort the string in such a way that its distorted shape corresponded to that of the desired harmonic. After being released, the string vibrates at the frequency of that harmonic. This maneuver is difficult to perform, however, and it is not how we excite a string of a musical in-

QuickLab

Compare the sounds of a guitar string plucked first near its center and then near one of its ends. More of the higher harmonics are present in the second situation. Can you hear the difference?

[1] We exclude $n = 0$ because this value corresponds to the trivial case in which no wave exists ($k = 0$).

strument. If the string is distorted such that its distorted shape is not that of just one harmonic, the resulting vibration includes various harmonics. Such a distortion occurs in musical instruments when the string is plucked (as in a guitar), bowed (as in a cello), or struck (as in a piano). When the string is distorted into a non-sinusoidal shape, only waves that satisfy the boundary conditions can persist on the string. These are the harmonics.

The frequency of a stringed instrument can be varied by changing either the tension or the string's length. For example, the tension in guitar and violin strings is varied by a screw adjustment mechanism or by tuning pegs located on the neck of the instrument. As the tension is increased, the frequency of the normal modes increases in accordance with Equation 18.8. Once the instrument is "tuned," players vary the frequency by moving their fingers along the neck, thereby changing the length of the oscillating portion of the string. As the length is shortened, the frequency increases because, as Equation 18.8 specifies, the normal-mode frequencies are inversely proportional to string length.

EXAMPLE 18.3 Give Me a C Note!

Middle C on a piano has a fundamental frequency of 262 Hz, and the first A above middle C has a fundamental frequency of 440 Hz. (a) Calculate the frequencies of the next two harmonics of the C string.

Solution Knowing that the frequencies of higher harmonics are integer multiples of the fundamental frequency $f_1 = 262$ Hz, we find that

$$f_2 = 2f_1 = \boxed{524 \text{ Hz}}$$

$$f_3 = 3f_1 = \boxed{786 \text{ Hz}}$$

(b) If the A and C strings have the same linear mass density μ and length L, determine the ratio of tensions in the two strings.

Solution Using Equation 18.8 for the two strings vibrating at their fundamental frequencies gives

$$f_{1A} = \frac{1}{2L}\sqrt{\frac{T_A}{\mu}} \quad \text{and} \quad f_{1C} = \frac{1}{2L}\sqrt{\frac{T_C}{\mu}}$$

Setting up the ratio of these frequencies, we find that

$$\frac{f_{1A}}{f_{1C}} = \sqrt{\frac{T_A}{T_C}}$$

$$\frac{T_A}{T_C} = \left(\frac{f_{1A}}{f_{1C}}\right)^2 = \left(\frac{440}{262}\right)^2 = \boxed{2.82}$$

(c) With respect to a real piano, the assumption we made in (b) is only partially true. The string densities are equal, but the length of the A string is only 64 percent of the length of the C string. What is the ratio of their tensions?

Solution Using Equation 18.8 again, we set up the ratio of frequencies:

$$\frac{f_{1A}}{f_{1C}} = \frac{L_C}{L_A}\sqrt{\frac{T_A}{T_C}} = \left(\frac{100}{64}\right)\sqrt{\frac{T_A}{T_C}}$$

$$\frac{T_A}{T_C} = (0.64)^2\left(\frac{440}{262}\right)^2 = \boxed{1.16}$$

EXAMPLE 18.4 Guitar Basics

The high E string on a guitar measures 64.0 cm in length and has a fundamental frequency of 330 Hz. By pressing down on it at the first fret (Fig. 18.8), the string is shortened so that it plays an F note that has a frequency of 350 Hz. How far is the fret from the neck end of the string?

Solution Equation 18.7 relates the string's length to the fundamental frequency. With $n = 1$, we can solve for the speed of the wave on the string,

$$v = \frac{2L}{n} f_n = \frac{2(0.640 \text{ m})}{1}(330 \text{ Hz}) = 422 \text{ m/s}$$

Because we have not adjusted the tuning peg, the tension in the string, and hence the wave speed, remain constant. We can again use Equation 18.7, this time solving for L and sub-

Figure 18.8 Playing an F note on a guitar. *(Charles D. Winters)*

stituting the new frequency to find the shortened string length:

$$L = n\frac{v}{2f_n} = (1)\frac{422 \text{ m/s}}{2(350 \text{ Hz})} = 0.603 \text{ m}$$

The difference between this length and the measured length of 64.0 cm is the distance from the fret to the neck end of the string, or 3.70 cm.

18.4 RESONANCE

9.9 We have seen that a system such as a taut string is capable of oscillating in one or more normal modes of oscillation. **If a periodic force is applied to such a system, the amplitude of the resulting motion is greater than normal when the frequency of the applied force is equal to or nearly equal to one of the natural frequencies of the system.** We discussed this phenomenon, known as *resonance,* briefly in Section 13.7. Although a block–spring system or a simple pendulum has only one natural frequency, standing-wave systems can have a whole set of natural frequencies. Because an oscillating system exhibits a large amplitude when driven at any of its natural frequencies, these frequencies are often referred to as **resonance frequencies.**

Figure 18.9 shows the response of an oscillating system to various driving frequencies, where one of the resonance frequencies of the system is denoted by f_0. Note that the amplitude of oscillation of the system is greatest when the frequency of the driving force equals the resonance frequency. The maximum amplitude is limited by friction in the system. If a driving force begins to work on an oscillating system initially at rest, the input energy is used both to increase the amplitude of the oscillation and to overcome the frictional force. Once maximum amplitude is reached, the work done by the driving force is used only to overcome friction.

A system is said to be *weakly damped* when the amount of friction to be overcome is small. Such a system has a large amplitude of motion when driven at one of its resonance frequencies, and the oscillations persist for a long time after the driving force is removed. A system in which considerable friction must be overcome is said to be *strongly damped.* For a given driving force applied at a resonance frequency, the maximum amplitude attained by a strongly damped oscillator is smaller than that attained by a comparable weakly damped oscillator. Once the driving force in a strongly damped oscillator is removed, the amplitude decreases rapidly with time.

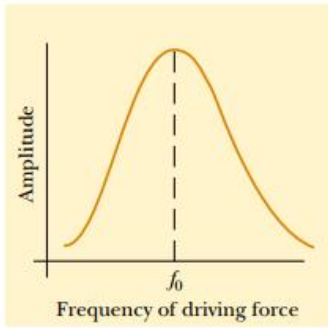


Figure 18.9 Graph of the amplitude (response) versus driving frequency for an oscillating system. The amplitude is a maximum at the resonance frequency f_0. Note that the curve is not symmetric.

Examples of Resonance

A playground swing is a pendulum having a natural frequency that depends on its length. Whenever we use a series of regular impulses to push a child in a swing, the swing goes higher if the frequency of the periodic force equals the natural frequency of the swing. We can demonstrate a similar effect by suspending pendulums of different lengths from a horizontal support, as shown in Figure 18.10. If pendulum A is set into oscillation, the other pendulums begin to oscillate as a result of the longitudinal waves transmitted along the beam. However, pendulum C, the length of which is close to the length of A, oscillates with a much greater amplitude than pendulums B and D, the lengths of which are much different from that of pendulum A. Pendulum C moves the way it does because its natural frequency is nearly the same as the driving frequency associated with pendulum A.

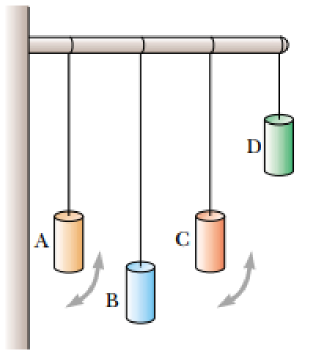


Figure 18.10 An example of resonance. If pendulum A is set into oscillation, only pendulum C, whose length matches that of A, eventually oscillates with large amplitude, or resonates. The arrows indicate motion perpendicular to the page.

Next, consider a taut string fixed at one end and connected at the opposite end to an oscillating blade, as illustrated in Figure 18.11. The fixed end is a node, and the end connected to the blade is very nearly a node because the amplitude of the blade's motion is small compared with that of the string. As the blade oscillates, transverse waves sent down the string are reflected from the fixed end. As we learned in Section 18.3, the string has natural frequencies that are determined by its length, tension, and linear mass density (see Eq. 18.8). When the frequency of the blade equals one of the natural frequencies of the string, standing waves are produced and the string oscillates with a large amplitude. In this resonance case, the wave generated by the oscillating blade is in phase with the reflected wave, and the string absorbs energy from the blade. If the string is driven at a frequency that is not one of its natural frequencies, then the oscillations are of low amplitude and exhibit no stable pattern.

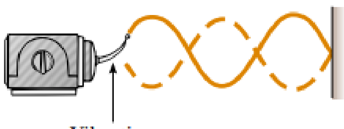
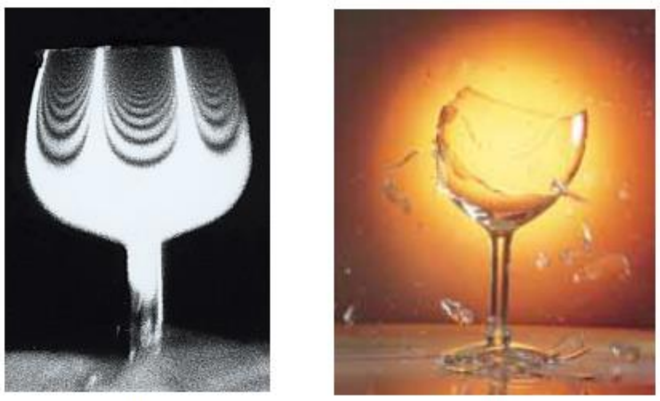


Figure 18.11 Standing waves are set up in a string when one end is connected to a vibrating blade. When the blade vibrates at one of the natural frequencies of the string, large-amplitude standing waves are created.

Once the amplitude of the standing-wave oscillations is a maximum, the mechanical energy delivered by the blade and absorbed by the system is lost because of the damping forces caused by friction in the system. If the applied frequency differs from one of the natural frequencies, energy is transferred to the string at first, but later the phase of the wave becomes such that it forces the blade to receive energy from the string, thereby reducing the energy in the string.

Quick Quiz 18.2

Some singers can shatter a wine glass by maintaining a certain frequency of their voice for several seconds. Figure 18.12a shows a side view of a wine glass vibrating because of a sound wave. Sketch the standing-wave pattern in the rim of the glass as seen from above. If an inte-



(a) (b)

Figure 18.12 (a) Standing-wave pattern in a vibrating wine glass. The glass shatters if the amplitude of vibration becomes too great.
(b) A wine glass shattered by the amplified sound of a human voice.

gral number of waves “fit” around the circumference of the vibrating rim, how many wavelengths fit around the rim in Figure 18.12a?

Quick Quiz 18.3

“Rumble strips” (Fig. 18.13) are sometimes placed across a road to warn drivers that they are approaching a stop sign, or laid along the sides of the road to alert drivers when they are drifting out of their lane. Why are these sets of small bumps so effective at getting a driver’s attention?

Figure 18.13 Rumble strips along the side of a highway.

18.5 STANDING WAVES IN AIR COLUMNS

9.9

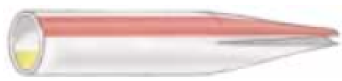
Standing waves can be set up in a tube of air, such as that in an organ pipe, as the result of interference between longitudinal sound waves traveling in opposite directions. The phase relationship between the incident wave and the wave reflected from one end of the pipe depends on whether that end is open or closed. This relationship is analogous to the phase relationships between incident and reflected transverse waves at the end of a string when the end is either fixed or free to move (see Figs. 16.13 and 16.14).

In a pipe closed at one end, **the closed end is a displacement node because the wall at this end does not allow longitudinal motion of the air molecules.** As a result, at a closed end of a pipe, the reflected sound wave is 180° out of phase with the incident wave. Furthermore, because the pressure wave is 90° out of phase with the displacement wave (see Section 17.2), **the closed end of an air column corresponds to a pressure antinode** (that is, a point of maximum pressure variation).

The open end of an air column is approximately a displacement antinode[2] and a pressure node. We can understand why no pressure variation occurs at an open end by noting that the end of the air column is open to the atmosphere; thus, the pressure at this end must remain constant at atmospheric pressure.

QuickLab

Snip off pieces at one end of a drinking straw so that the end tapers to a point. Chew on this end to flatten it, and you’ll have created a double-reed instrument! Put your lips around the tapered end, press them tightly together, and blow through the straw. When you hear a steady tone, slowly snip off pieces of the straw from the other end. Be careful to maintain a constant pressure with your lips. How does the frequency change as the straw is shortened?



[2] Strictly speaking, the open end of an air column is not exactly a displacement antinode. A condensation reaching an open end does not reflect until it passes beyond the end. For a thin-walled tube of circular cross section, this end correction is approximately $0.6R$, where R is the tube’s radius. Hence, the effective length of the tube is longer than the true length L. We ignore this end correction in this discussion.

You may wonder how a sound wave can reflect from an open end, since there may not appear to be a change in the medium at this point. It is indeed true that the medium through which the sound wave moves is air both inside and outside the pipe. Remember that sound is a pressure wave, however, and a compression region of the sound wave is constrained by the sides of the pipe as long as the region is inside the pipe. As the compression region exits at the open end of the pipe, the constraint is removed and the compressed air is free to expand into the atmosphere. Thus, there is a change in the *character* of the medium between the inside of the pipe and the outside even though there is no change in the *material* of the medium. This change in character is sufficient to allow some reflection.

The first three normal modes of oscillation of a pipe open at both ends are shown in Figure 18.14a. When air is directed against an edge at the left, longitudinal standing waves are formed, and the pipe resonates at its natural frequencies. All normal modes are excited simultaneously (although not with the same amplitude). Note that both ends are displacement antinodes (approximately). In the first normal mode, the standing wave extends between two adjacent antinodes,

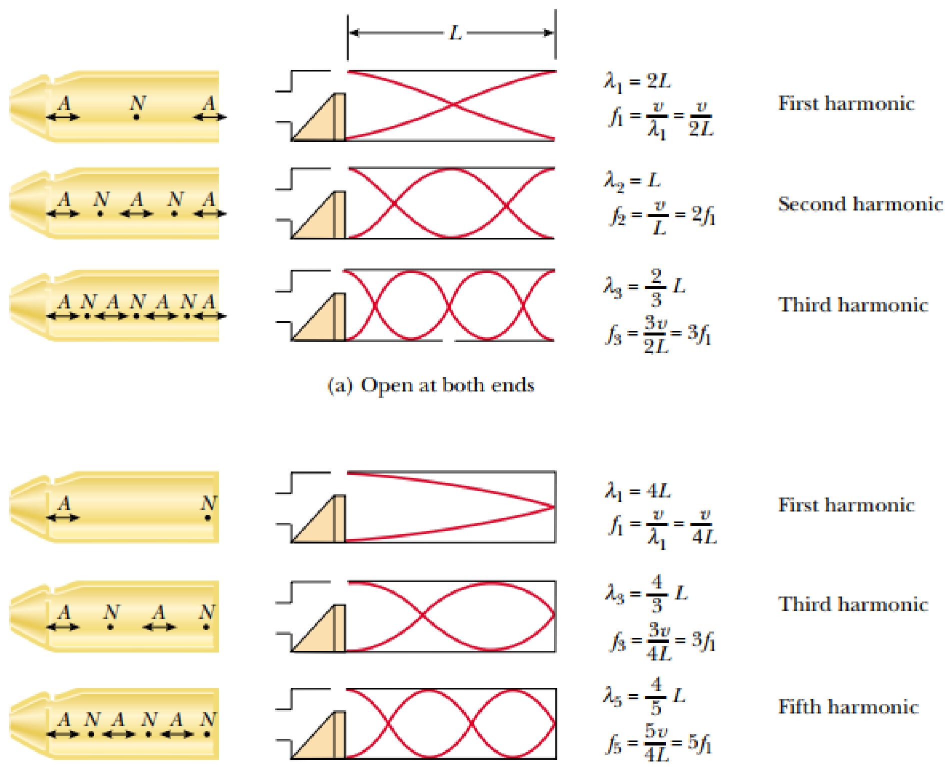


Figure 18.14 Motion of air molecules in standing longitudinal waves in a pipe, along with schematic representations of the waves. The graphs represent the displacement amplitudes, not the pressure amplitudes. (a) In a pipe open at both ends, the harmonic series created consists of all integer multiples of the fundamental frequency: $f_1, 2f_1, 3f_1, \ldots$. (b) In a pipe closed at one end and open at the other, the harmonic series created consists of only odd-integer multiples of the fundamental frequency: $f_1, 3f_1, 5f_1, \ldots$.

which is a distance of half a wavelength. Thus, the wavelength is twice the length of the pipe, and the fundamental frequency is $f_1 = v/2L$. As Figure 18.14a shows, the frequencies of the higher harmonics are $2f_1, 3f_1, \ldots$. Thus, we can say that

in a pipe open at both ends, the natural frequencies of oscillation form a harmonic series that includes all integral multiples of the fundamental frequency.

Because all harmonics are present, and because the fundamental frequency is given by the same expression as that for a string (see Eq. 18.7), we can express the natural frequencies of oscillation as

$$f_n = n\frac{v}{2L} \qquad n = 1, 2, 3 \ldots \tag{18.11}$$

Natural frequencies of a pipe open at both ends

Despite the similarity between Equations 18.7 and 18.11, we must remember that v in Equation 18.7 is the speed of waves on the string, whereas v in Equation 18.11 is the speed of sound in air.

If a pipe is closed at one end and open at the other, the closed end is a displacement node (see Fig. 18.14b). In this case, the standing wave for the fundamental mode extends from an antinode to the adjacent node, which is one fourth of a wavelength. Hence, the wavelength for the first normal mode is $4L$, and the fundamental frequency is $f_1 = v/4L$. As Figure 18.14b shows, the higher-frequency waves that satisfy our conditions are those that have a node at the closed end and an antinode at the open end; this means that the higher harmonics have frequencies $3f_1, 5f_1, \ldots$:

QuickLab

Blow across the top of an empty soda-pop bottle. From a measurement of the height of the bottle, estimate the frequency of the sound you hear. Note that the cross-sectional area of the bottle is not constant; thus, this is not a perfect model of a cylindrical air column.

In a pipe closed at one end and open at the other, the natural frequencies of oscillation form a harmonic series that includes only odd integer multiples of the fundamental frequency.

We express this result mathematically as

$$f_n = n\frac{v}{4L} \qquad n = 1, 3, 5, \ldots \tag{18.12}$$

Natural frequencies of a pipe closed at one end and open at the other

It is interesting to investigate what happens to the frequencies of instruments based on air columns and strings during a concert as the temperature rises. The sound emitted by a flute, for example, becomes sharp (increases in frequency) as it warms up because the speed of sound increases in the increasingly warmer air inside the flute (consider Eq. 18.11). The sound produced by a violin becomes flat (decreases in frequency) as the strings expand thermally because the expansion causes their tension to decrease (see Eq. 18.8).

Quick Quiz 18.4

A pipe open at both ends resonates at a fundamental frequency f_{open}. When one end is covered and the pipe is again made to resonate, the fundamental frequency is f_{closed}. Which of the following expressions describes how these two resonant frequencies compare? (a) $f_{\text{closed}} = f_{\text{open}}$ (b) $f_{\text{closed}} = \frac{1}{2}f_{\text{open}}$ (c) $f_{\text{closed}} = 2f_{\text{open}}$ (d) $f_{\text{closed}} = \frac{3}{2}f_{\text{open}}$

EXAMPLE 18.5 Wind in a Culvert

A section of drainage culvert 1.23 m in length makes a howling noise when the wind blows. (a) Determine the frequencies of the first three harmonics of the culvert if it is open at both ends. Take $v = 343$ m/s as the speed of sound in air.

Solution The frequency of the first harmonic of a pipe open at both ends is

$$f_1 = \frac{v}{2L} = \frac{343 \text{ m/s}}{2(1.23 \text{ m})} = \boxed{139 \text{ Hz}}$$

Because both ends are open, all harmonics are present; thus,

$f_2 = 2f_1 = \boxed{278 \text{ Hz}}$ and $f_3 = 3f_1 = \boxed{417 \text{ Hz.}}$

(b) What are the three lowest natural frequencies of the culvert if it is blocked at one end?

Solution The fundamental frequency of a pipe closed at one end is

$$f_1 = \frac{v}{4L} = \frac{343 \text{ m/s}}{4(1.23 \text{ m})} = \boxed{69.7 \text{ Hz}}$$

In this case, only odd harmonics are present; hence, the next two harmonics have frequencies $f_3 = 3f_1 = \boxed{209 \text{ Hz}}$ and $f_5 = 5f_1 = \boxed{349 \text{ Hz.}}$

(c) For the culvert open at both ends, how many of the harmonics present fall within the normal human hearing range (20 to 17 000 Hz)?

Solution Because all harmonics are present, we can express the frequency of the highest harmonic heard as $f_n = nf_1$, where n is the number of harmonics that we can hear. For $f_n = 17\,000$ Hz, we find that the number of harmonics present in the audible range is

$$n = \frac{17\,000 \text{ Hz}}{139 \text{ Hz}} = \boxed{122}$$

Only the first few harmonics are of sufficient amplitude to be heard.

EXAMPLE 18.6 Measuring the Frequency of a Tuning Fork

A simple apparatus for demonstrating resonance in an air column is depicted in Figure 18.15. A vertical pipe open at both ends is partially submerged in water, and a tuning fork vibrating at an unknown frequency is placed near the top of the pipe. The length L of the air column can be adjusted by moving the pipe vertically. The sound waves generated by the fork are reinforced when L corresponds to one of the resonance frequencies of the pipe.

For a certain tube, the smallest value of L for which a peak occurs in the sound intensity is 9.00 cm. What are (a) the frequency of the tuning fork and (b) the value of L for the next two resonance frequencies?

Solution (a) Although the pipe is open at its lower end to allow the water to enter, the water's surface acts like a wall at one end. Therefore, this setup represents a pipe closed at one end, and so the fundamental frequency is $f_1 = v/4L$. Taking $v = 343$ m/s for the speed of sound in air and $L = 0.090\,0$ m, we obtain

$$f_1 = \frac{v}{4L} = \frac{343 \text{ m/s}}{4(0.090\,0 \text{ m})} = \boxed{953 \text{ Hz}}$$

Because the tuning fork causes the air column to resonate at this frequency, this must be the frequency of the tuning fork.

(b) Because the pipe is closed at one end, we know from Figure 18.14b that the wavelength of the fundamental mode is $\lambda = 4L = 4(0.090\,0 \text{ m}) = 0.360$ m. Because the frequency of the tuning fork is constant, the next two normal modes (see Fig. 18.15b) correspond to lengths of $L = 3\lambda/4 = \boxed{0.270 \text{ m}}$ and $L = 5\lambda/4 = \boxed{0.450 \text{ m.}}$

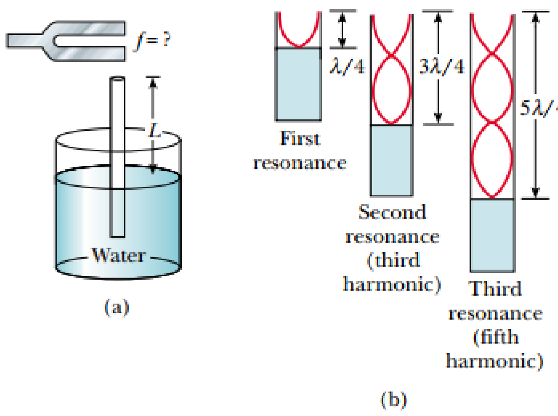


Figure 18.15 (a) Apparatus for demonstrating the resonance of sound waves in a tube closed at one end. The length L of the air column is varied by moving the tube vertically while it is partially submerged in water. (b) The first three normal modes of the system shown in part (a).

Optional Section

18.6 STANDING WAVES IN RODS AND PLATES

Standing waves can also be set up in rods and plates. A rod clamped in the middle and stroked at one end oscillates, as depicted in Figure 18.16a. The oscillations of the particles of the rod are longitudinal, and so the broken lines in Figure 18.16 represent *longitudinal* displacements of various parts of the rod. For clarity, we have drawn them in the transverse direction, just as we did for air columns. The midpoint is a displacement node because it is fixed by the clamp, whereas the ends are displacement antinodes because they are free to oscillate. The oscillations in this setup are analogous to those in a pipe open at both ends. The broken lines in Figure 18.16a represent the first normal mode, for which the wavelength is $2L$ and the frequency is $f = v/2L$, where v is the speed of longitudinal waves in the rod. Other normal modes may be excited by clamping the rod at different points. For example, the second normal mode (Fig. 18.16b) is excited by clamping the rod a distance $L/4$ away from one end.

Two-dimensional oscillations can be set up in a flexible membrane stretched over a circular hoop, such as that in a drumhead. As the membrane is struck at some point, wave pulses that arrive at the fixed boundary are reflected many times. The resulting sound is not harmonic because the oscillating drumhead and the drum's hollow interior together produce a set of standing waves having frequencies that are *not* related by integer multiples. Without this relationship, the sound may be more correctly described as *noise* than as music. This is in contrast to the situation in wind and stringed instruments, which produce sounds that we describe as musical.

Some possible normal modes of oscillation for a two-dimensional circular membrane are shown in Figure 18.17. The lowest normal mode, which has a frequency f_1, contains only one nodal curve; this curve runs around the outer edge of the membrane. The other possible normal modes show additional nodal curves that are circles and straight lines across the diameter of the membrane.

The sound from a tuning fork is produced by the vibrations of each of its prongs.

Wind chimes are usually designed so that the waves emanating from the vibrating rods blend into a harmonious sound.

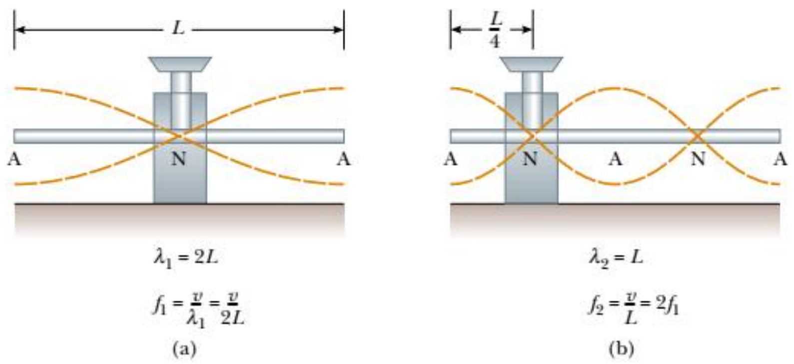


Figure 18.16 Normal-mode longitudinal vibrations of a rod of length L (a) clamped at the middle to produce the first normal mode and (b) clamped at a distance $L/4$ from one end to produce the second normal mode. Note that the dashed lines represent amplitudes parallel to the rod (longitudinal waves).

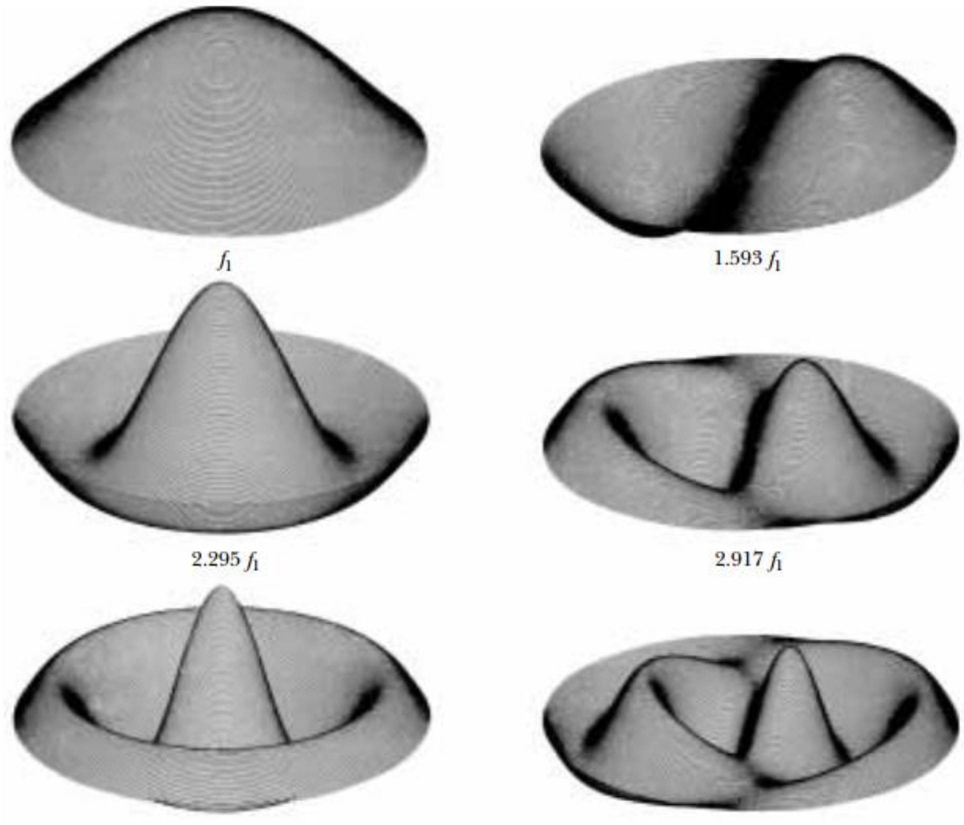


Figure 18.17 Representation of some of the normal modes possible in a circular membrane fixed at its perimeter. The frequencies of oscillation do not form a harmonic series.

18.7 BEATS: INTERFERENCE IN TIME

The interference phenomena with which we have been dealing so far involve the superposition of two or more waves having the same frequency. Because the resultant wave depends on the coordinates of the disturbed medium, we refer to the phenomenon as *spatial interference.* Standing waves in strings and pipes are common examples of spatial interference.

We now consider another type of interference, one that results from the superposition of two waves having slightly *different* frequencies. In this case, when the two waves are observed at the point of superposition, they are periodically in and out of phase. That is, there is a *temporal* (time) alternation between constructive and destructive interference. Thus, we refer to this phenomenon as *interference in time* or *temporal interference.* For example, if two tuning forks of slightly different frequencies are struck, one hears a sound of periodically varying intensity. This phenomenon is called **beating:**

Definition of beating

Beating is the periodic variation in intensity at a given point due to the superposition of two waves having slightly different frequencies.

The number of intensity maxima one hears per second, or the *beat frequency*, equals the difference in frequency between the two sources, as we shall show below. The maximum beat frequency that the human ear can detect is about 20 beats/s. When the beat frequency exceeds this value, the beats blend indistinguishably with the compound sounds producing them.

A piano tuner can use beats to tune a stringed instrument by "beating" a note against a reference tone of known frequency. The tuner can then adjust the string tension until the frequency of the sound it emits equals the frequency of the reference tone. The tuner does this by tightening or loosening the string until the beats produced by it and the reference source become too infrequent to notice.

Consider two sound waves of equal amplitude traveling through a medium with slightly different frequencies f_1 and f_2. We use equations similar to Equation 16.11 to represent the wave functions for these two waves at a point that we choose as $x = 0$:

$$y_1 = A \cos \omega_1 t = A \cos 2\pi f_1 t$$

$$y_2 = A \cos \omega_2 t = A \cos 2\pi f_2 t$$

Using the superposition principle, we find that the resultant wave function at this point is

$$y = y_1 + y_2 = A(\cos 2\pi f_1 t + \cos 2\pi f_2 t)$$

The trigonometric identity

$$\cos a + \cos b = 2 \cos\left(\frac{a - b}{2}\right) \cos\left(\frac{a + b}{2}\right)$$

allows us to write this expression in the form

$$y = \left[2 A \cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\right] \cos 2\pi\left(\frac{f_1 + f_2}{2}\right)t \qquad \textbf{(18.13)}$$

Resultant of two waves of different frequencies but equal amplitude

Graphs of the individual waves and the resultant wave are shown in Figure 18.18. From the factors in Equation 18.13, we see that the resultant sound for a listener standing at any given point has an effective frequency equal to the average frequency $(f_1 + f_2)/2$ and an amplitude given by the expression in the square

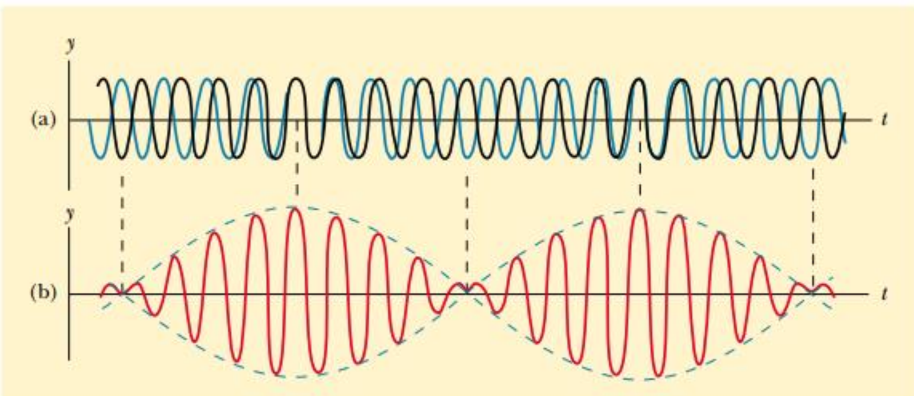


Figure 18.18 Beats are formed by the combination of two waves of slightly different frequencies. (a) The individual waves. (b) The combined wave has an amplitude (broken line) that oscillates in time.

brackets:

$$A_{\text{resultant}} = 2A \cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \qquad (18.14)$$

That is, the **amplitude and therefore the intensity of the resultant sound vary in time.** The broken blue line in Figure 18.18b is a graphical representation of Equation 18.14 and is a sine wave varying with frequency $(f_1 - f_2)/2$.

Note that a maximum in the amplitude of the resultant sound wave is detected whenever

$$\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t = \pm 1$$

This means there are *two* maxima in each period of the resultant wave. Because the amplitude varies with frequency as $(f_1 - f_2)/2$, the number of beats per second, or the beat frequency f_b, is twice this value. That is,

Beat frequency

$$f_b = |f_1 - f_2| \qquad (18.15)$$

For instance, if one tuning fork vibrates at 438 Hz and a second one vibrates at 442 Hz, the resultant sound wave of the combination has a frequency of 440 Hz (the musical note A) and a beat frequency of 4 Hz. A listener would hear a 440-Hz sound wave go through an intensity maximum four times every second.

Optional Section

18.8 NON-SINUSOIDAL WAVE PATTERNS

9.6 The sound-wave patterns produced by the majority of musical instruments are non-sinusoidal. Characteristic patterns produced by a tuning fork, a flute, and a clarinet, each playing the same note, are shown in Figure 18.19. Each instrument has its own characteristic pattern. Note, however, that despite the differences in the patterns, each pattern is periodic. This point is important for our analysis of these waves, which we now discuss.

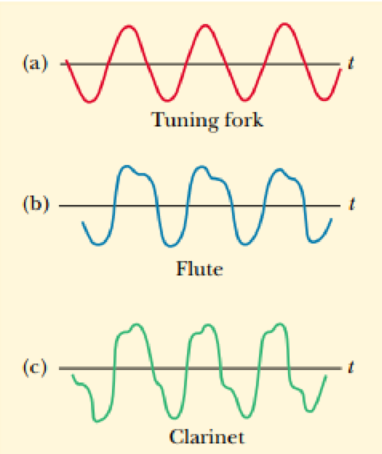


Figure 18.19 Sound wave patterns produced by (a) a tuning fork, (b) a flute, and (c) a clarinet, each at approximately the same frequency.

We can distinguish the sounds coming from a trumpet and a saxophone even when they are both playing the same note. On the other hand, we may have difficulty distinguishing a note played on a clarinet from the same note played on an oboe. We can use the pattern of the sound waves from various sources to explain these effects.

The wave patterns produced by a musical instrument are the result of the superposition of various harmonics. This superposition results in the corresponding richness of musical tones. The human perceptive response associated with various mixtures of harmonics is the *quality* or *timbre* of the sound. For instance, the sound of the trumpet is perceived to have a "brassy" quality (that is, we have learned to associate the adjective *brassy* with that sound); this quality enables us to distinguish the sound of the trumpet from that of the saxophone, whose quality is perceived as "reedy." The clarinet and oboe, however, are both straight air columns excited by reeds; because of this similarity, it is more difficult for the ear to distinguish them on the basis of their sound quality.

The problem of analyzing non-sinusoidal wave patterns appears at first sight to be a formidable task. However, if the wave pattern is periodic, it can be represented as closely as desired by the combination of a sufficiently large number of si-

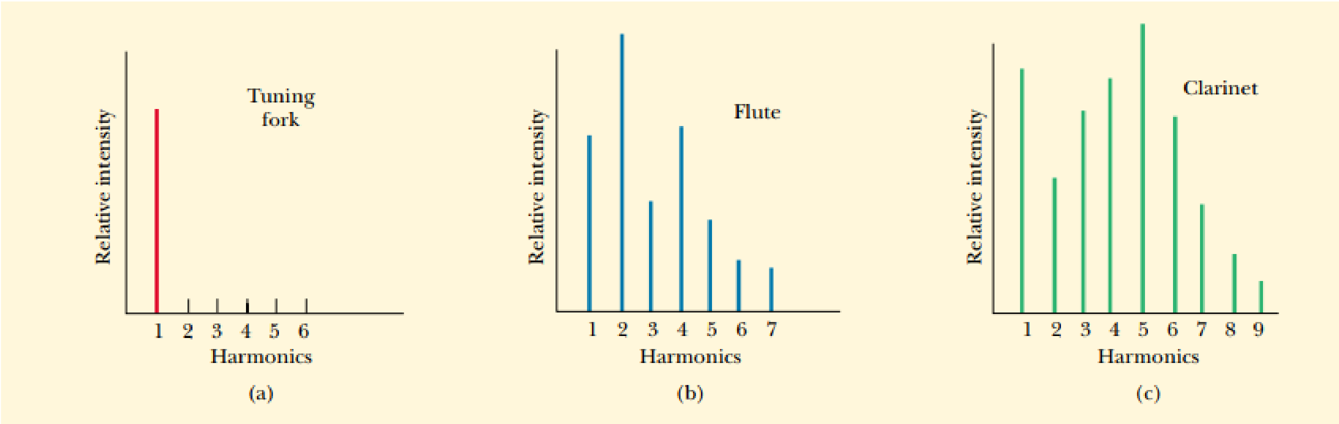


Figure 18.20 Harmonics of the wave patterns shown in Figure 18.19. Note the variations in intensity of the various harmonics.

nusoidal waves that form a harmonic series. In fact, we can represent any periodic function as a series of sine and cosine terms by using a mathematical technique based on **Fourier's theorem.**[3] The corresponding sum of terms that represents the periodic wave pattern is called a **Fourier series.**

Let $y(t)$ be any function that is periodic in time with period T, such that $y(t + T) = y(t)$. Fourier's theorem states that this function can be written as

$$y(t) = \sum_n (A_n \sin 2\pi f_n t + B_n \cos 2\pi f_n t) \qquad \textbf{(18.16)}$$

Fourier's theorem

where the lowest frequency is $f_1 = 1/T$. The higher frequencies are integer multiples of the fundamental, $f_n = nf_1$, and the coefficients A_n and B_n represent the amplitudes of the various waves. Figure 18.20 represents a harmonic analysis of the wave patterns shown in Figure 18.19. Note that a struck tuning fork produces only one harmonic (the first), whereas the flute and clarinet produce the first and many higher ones.

Note the variation in relative intensity of the various harmonics for the flute and the clarinet. In general, any musical sound consists of a fundamental frequency f plus other frequencies that are integer multiples of f, all having different intensities.

We have discussed the *analysis* of a wave pattern using Fourier's theorem. The analysis involves determining the coefficients of the harmonics in Equation 18.16 from a knowledge of the wave pattern. The reverse process, called *Fourier synthesis*, can also be performed. In this process, the various harmonics are added together to form a resultant wave pattern. As an example of Fourier synthesis, consider the building of a square wave, as shown in Figure 18.21. The symmetry of the square wave results in only odd multiples of the fundamental frequency combining in its synthesis. In Figure 18.21a, the orange curve shows the combination of f and $3f$. In Figure 18.21b, we have added $5f$ to the combination and obtained the green curve. Notice how the general shape of the square wave is approximated, even though the upper and lower portions are not flat as they should be.

[3] Developed by Jean Baptiste Joseph Fourier (1786–1830).

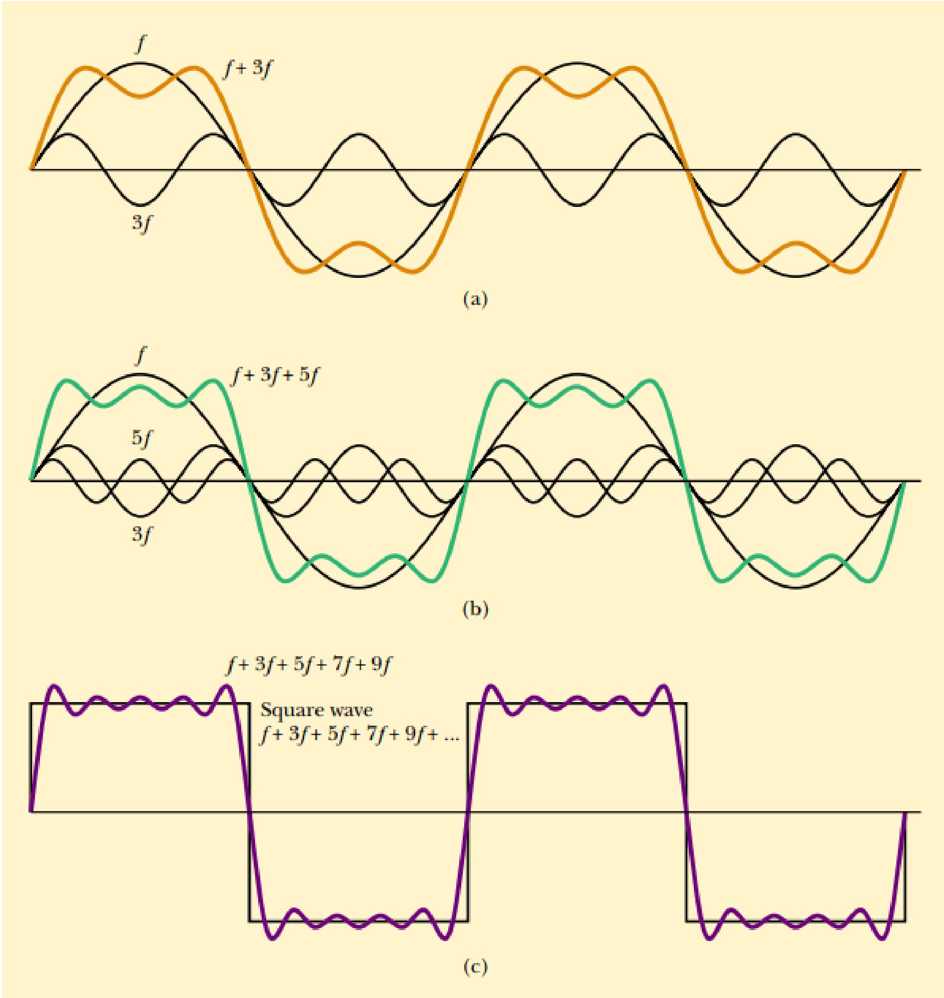


Figure 18.21 Fourier synthesis of a square wave, which is represented by the sum of odd multiples of the first harmonic, which has frequency f. (a) Waves of frequency f and $3f$ are added. (b) One more odd harmonic of frequency $5f$ is added. (c) The synthesis curve approaches the square wave when odd frequencies up to $9f$ are added.

This synthesizer can produce the characteristic sounds of different instruments by properly combining frequencies from electronic oscillators.

Figure 18.21c shows the result of adding odd frequencies up to $9f$. This approximation to the square wave (purple curve) is better than the approximations in parts a and b. To approximate the square wave as closely as possible, we would need to add all odd multiples of the fundamental frequency, up to infinite frequency.

Using modern technology, we can generate musical sounds electronically by mixing different amplitudes of any number of harmonics. These widely used electronic music synthesizers are capable of producing an infinite variety of musical tones.

SUMMARY

When two traveling waves having equal amplitudes and frequencies superimpose, the resultant wave has an amplitude that depends on the phase angle ϕ between the two waves. **Constructive interference** occurs when the two waves are in phase, corresponding to $\phi = 0, 2\pi, 4\pi, \ldots$ rad. **Destructive interference** occurs when the two waves are 180° out of phase, corresponding to $\phi = \pi, 3\pi, 5\pi, \ldots$ rad. Given two wave functions, you should be able to determine which if either of these two situations applies.

Standing waves are formed from the superposition of two sinusoidal waves having the same frequency, amplitude, and wavelength but traveling in opposite directions. The resultant standing wave is described by the wave function

$$y = (2A \sin kx) \cos \omega t \qquad \textbf{(18.3)}$$

Hence, the amplitude of the standing wave is $2A$, and the amplitude of the simple harmonic motion of any particle of the medium varies according to its position as $2A \sin kx$. The points of zero amplitude (called **nodes**) occur at $x = n\lambda/2$ ($n = 0, 1, 2, 3, \ldots$). The maximum amplitude points (called **antinodes**) occur at $x = n\lambda/4$ ($n = 1, 3, 5, \ldots$). Adjacent antinodes are separated by a distance $\lambda/2$. Adjacent nodes also are separated by a distance $\lambda/2$. You should be able to sketch the standing-wave pattern resulting from the superposition of two traveling waves.

The natural frequencies of vibration of a taut string of length L and fixed at both ends are

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \qquad n = 1, 2, 3, \ldots \qquad \textbf{(18.8)}$$

where T is the tension in the string and μ is its linear mass density. The natural frequencies of vibration $f_1, 2f_1, 3f_1, \ldots$ form a **harmonic series.**

An oscillating system is in **resonance** with some driving force whenever the frequency of the driving force matches one of the natural frequencies of the system. When the system is resonating, it responds by oscillating with a relatively large amplitude.

Standing waves can be produced in a column of air inside a pipe. If the pipe is open at both ends, all harmonics are present and the natural frequencies of oscillation are

$$f_n = n\frac{v}{2L} \qquad n = 1, 2, 3, \ldots \qquad \textbf{(18.11)}$$

If the pipe is open at one end and closed at the other, only the odd harmonics are present, and the natural frequencies of oscillation are

$$f_n = n\frac{v}{4L} \qquad n = 1, 3, 5, \ldots \qquad \textbf{(18.12)}$$

The phenomenon of **beating** is the periodic variation in intensity at a given point due to the superposition of two waves having slightly different frequencies.