4

Logarithms and exponential functions

4.1 Introduction to logarithms

With the use of calculators firmly established, logarithmic tables are now rarely used for calculation. However, the theory of logarithms is important, for there are several scientific and engineering laws that involve the rules of logarithms.

If a number y can be written in the form a^x, then the index x is called the 'logarithm of y to the base of a',

i.e. $$\boxed{\textbf{if } y = a^x \textbf{ then } x = \log_a y}$$

Thus, since $1000 = 10^3$, then $3 = \log_{10} 1000$.

Check this using the 'log' button on your calculator.

(a) Logarithms having a base of 10 are called **common logarithms** and $\log_{10}$ is usually abbreviated to lg. The following values may be checked by using a calculator:

$\lg 17.9 = 1.2528\ldots$, $\lg 462.7 = 2.6652\ldots$ and $\lg 0.0173 = -1.7619\ldots$

(b) Logarithms having a base of e (where 'e' is a mathematical constant approximately equal to 2.7183) are called **hyperbolic, Napierian** or **natural logarithms**, and $\log_e$ is usually abbreviated to ln. The following values may be checked by using a calculator:

$\ln 3.15 = 1.1474\ldots$, $\ln 362.7 = 5.8935\ldots$ and $\ln 0.156 = -1.8578\ldots$.

4.2 Laws of logarithms

There are three laws of logarithms, which apply to any base:

(i) To multiply two numbers:

$$\boxed{\log(A \times B) = \log A + \log B}$$

The following may be checked by using a calculator:

$$\lg 10 = 1, \text{ also } \lg 5 + \lg 2$$
$$= 0.69897\ldots + 0.301029\ldots = 1$$

Hence $\lg(5 \times 2) = \lg 10 = \lg 5 + \lg 2$

(ii) To divide two numbers:

$$\boxed{\log\left(\frac{A}{B}\right) = \log A - \log B}$$

The following may be checked using a calculator:

$$\ln\left(\frac{5}{2}\right) = \ln 2.5 = 0.91629\ldots$$

Also $\ln 5 - \ln 2 = 1.60943\ldots - 0.69314\ldots$
$= 0.91629\ldots$

Hence $\ln\left(\frac{5}{2}\right) = \ln 5 - \ln 2$

(iii) To raise a number to a power:

$$\boxed{\lg A^n = n \log A}$$

The following may be checked using a calculator:

$$\lg 5^2 = \lg 25 = 1.39794\ldots$$

Also $2\lg 5 = 2 \times 0.69897\ldots$
$= 1.39794\ldots$

Hence $\lg 5^2 = 2\lg 5$

Problem 1. Evaluate (a) $\log_3 9$ (b) $\log_{10} 10$ (c) $\log_{16} 8$.

(a) Let $x = \log_3 9$ then $3^x = 9$ from the definition of a logarithm, i.e. $3^x = 3^2$, from which $x = 2$

Hence $\mathbf{\log_3 9 = 2}$

A

(b) Let $x = \log_{10} 10$ then $10^x = 10$ from the definition of a logarithm, i.e. $10^x = 10^1$, from which $x = 1$
Hence $\mathbf{\log_{10} 10 = 1}$ (which may be checked by a calculator)

(c) Let $x = \log_{16} 8$ then $16^x = 8$, from the definition of a logarithm, i.e. $(2^4)^x = 2^3$, i.e. $2^{4x} = 2^3$ from the laws of indices, from which, $4x = 3$ and $x = \frac{3}{4}$
Hence $\mathbf{\log_{16} 8 = \frac{3}{4}}$

Problem 2. Evaluate (a) lg 0.001 (b) ln e (c) $\log_3 \frac{1}{81}$.

(a) Let $x = \lg 0.001 = \log_{10} 0.001$ then $10^x = 0.001$, i.e. $10^x = 10^{-3}$, from which $x = -3$
Hence $\mathbf{\lg 0.001 = -3}$ (which may be checked by a calculator)

(b) Let $x = \ln e = \log_e e$ then $e^x = e$, i.e. $e^x = e^1$ from which $x = 1$. Hence $\mathbf{\ln e = 1}$ (which may be checked by a calculator)

(c) Let $x = \log_3 \frac{1}{81}$ then $3^x = \frac{1}{81} = \frac{1}{3^4} = 3^{-4}$, from which $x = -4$
Hence $\mathbf{\log_3 \frac{1}{81} = -4}$

Problem 3. Solve the following equations: (a) $\lg x = 3$ (b) $\log_2 x = 3$ (c) $\log_5 x = -2$.

(a) If $\lg x = 3$ then $\log_{10} x = 3$ and $x = 10^3$, i.e. $\mathbf{x = 1000}$

(b) If $\log_2 x = 3$ then $\mathbf{x = 2^3 = 8}$

(c) If $\log_5 x = -2$ then $x = 5^{-2} = \frac{1}{5^2} = \mathbf{\frac{1}{25}}$

Problem 4. Write (a) log 30 (b) log 450 in terms of log 2, log 3 and log 5 to any base.

(a) $\log 30 = \log(2 \times 15) = \log(2 \times 3 \times 5)$
$= \mathbf{\log 2 + \log 3 + \log 5}$,
by the first law of logarithms

(b) $\log 450 = \log(2 \times 225) = \log(2 \times 3 \times 75)$
$= \log(2 \times 3 \times 3 \times 25)$
$= \log(2 \times 3^2 \times 5^2)$
$= \log 2 + \log 3^2 + \log 5^2$,
by the first law of logarithms

i.e. $\log 450 = \mathbf{\log 2 + 2\log 3 + 2\log 5}$,
by the third law of logarithms

Problem 5. Write $\log\left(\frac{8 \times \sqrt[4]{5}}{81}\right)$ in terms of log 2, log 3 and log 5 to any base.

$\log\left(\frac{8 \times \sqrt[4]{5}}{81}\right) = \log 8 + \log \sqrt[4]{5} - \log 81$,
by the first and second laws of logarithms

$= \log 2^3 + \log 5^{\frac{1}{4}} - \log 3^4$,
by the laws of indices,

i.e.

$\log\left(\frac{8 \times \sqrt[4]{5}}{81}\right) = \mathbf{3\log 2 + \frac{1}{4}\log 5 - 4\log 3}$,
by the third law of logarithms

Problem 6. Evaluate

$$\frac{\log 25 - \log 125 + \frac{1}{2}\log 625}{3\log 5}.$$

$$\frac{\log 25 - \log 125 + \frac{1}{2}\log 625}{3\log 5}$$

$$= \frac{\log 5^2 - \log 5^3 + \frac{1}{2}\log 5^4}{3\log 5}$$

$$= \frac{2\log 5 - 3\log 5 + \frac{4}{2}\log 5}{3\log 5}$$

$$= \frac{1\log 5}{3\log 5} = \mathbf{\frac{1}{3}}$$

Problem 7. Solve the equation:
$\log(x-1) + \log(x+1) = 2\log(x+2)$.

$$\log(x-1)+\log(x+1)=\log(x-1)(x+1),$$ from the first law of logarithms

$$=\log(x^2-1)$$

$$2\log(x+2)=\log(x+2)^2$$

$$=\log(x^2+4x+4)$$

Hence if $\log(x^2-1)=\log(x^2+4x+4)$

then $x^2-1=x^2+4x+4$

i.e. $-1=4x+4$

i.e. $-5=4x$

i.e. $\mathbf{x=-\frac{5}{4}}$ or $\mathbf{-1\frac{1}{4}}$

Now try the following exercise.

Exercise 16 Further problems on the laws of logarithms

In Problems 1 to 8, evaluate the given expression:

1. $\log_{10} 10000$ [4]
2. $\log_2 16$ [4]
3. $\log_5 125$ [3]
4. $\log_2 \frac{1}{8}$ [−3]
5. $\log_8 2$ $\left[\frac{1}{3}\right]$
6. $\lg 100$ [2]
7. $\log_4 8$ $\left[1\frac{1}{2}\right]$
8. $\ln e^2$ [2]

In Problems 9 to 14 solve the equations:

9. $\log_{10} x=4$ [10000]
10. $\log_3 x=2$ [9]
11. $\log_4 x=-2\frac{1}{2}$ $\left[\pm\frac{1}{32}\right]$
12. $\lg x=-2$ [0.01]
13. $\log_8 x=-\frac{4}{3}$ $\left[\frac{1}{16}\right]$
14. $\ln x=3$ $[e^3]$

In Problems 15 to 17 write the given expressions in terms of log 2, log 3 and log 5 to any base:

15. $\log 60$ $[2\log 2+\log 3+\log 5]$
16. $\log\left(\frac{16\times\sqrt[4]{5}}{27}\right)$ $\left[4\log 2+\frac{1}{4}\log 5-3\log 3\right]$
17. $\log\left(\frac{125\times\sqrt[4]{16}}{\sqrt[4]{81^3}}\right)$ $[\log 2-3\log 3+3\log 5]$

Simplify the expressions given in Problems 18 and 19:

18. $\log 27-\log 9+\log 81$ [5 log 3]
19. $\log 64+\log 32-\log 128$ [4 log 2]
20. Evaluate $\dfrac{\frac{1}{2}\log 16-\frac{1}{3}\log 8}{\log 4}$ $\left[\frac{1}{2}\right]$

Solve the equations given in Problems 21 and 22:

21. $\log x^4-\log x^3=\log 5x-\log 2x$ $\left[x=2\frac{1}{2}\right]$
22. $\log 2t^3-\log t=\log 16+\log t$ $[t=8]$

4.3 Indicial equations

The laws of logarithms may be used to solve certain equations involving powers—called **indicial equations**. For example, to solve, say, $3^x=27$, logarithms to a base of 10 are taken of both sides,

i.e. $\log_{10} 3^x=\log_{10} 27$

and $x\log_{10} 3=\log_{10} 27$, by the third law of logarithms

Rearranging gives

$$x=\frac{\log_{10} 27}{\log_{10} 3}=\frac{1.43136\ldots}{0.4771\ldots}=3$$

which may be readily checked

$\left(\text{Note, }\left(\frac{\log 8}{\log 2}\right)\text{ is }\textbf{not}\text{ equal to }\lg\left(\frac{8}{2}\right)\right)$

Problem 8. Solve the equation $2^x = 3$, correct to 4 significant figures.

Taking logarithms to base 10 of both sides of $2^x = 3$ gives:

$$\log_{10} 2^x = \log_{10} 3$$

i.e. $x \log_{10} 2 = \log_{10} 3$

Rearranging gives:

$$x = \frac{\log_{10} 3}{\log_{10} 2} = \frac{0.47712125\ldots}{0.30102999\ldots}$$

$= \mathbf{1.585}$, correct to 4 significant figures

Problem 9. Solve the equation $2^{x+1} = 3^{2x-5}$ correct to 2 decimal places.

Taking logarithms to base 10 of both sides gives:

$$\log_{10} 2^{x+1} = \log_{10} 3^{2x-5}$$

i.e. $(x+1)\log_{10} 2 = (2x-5)\log_{10} 3$

$$x \log_{10} 2 + \log_{10} 2 = 2x \log_{10} 3 - 5 \log_{10} 3$$

$$x(0.3010) + (0.3010) = 2x(0.4771) - 5(0.4771)$$

i.e. $0.3010x + 0.3010 = 0.9542x - 2.3855$

Hence

$$2.3855 + 0.3010 = 0.9542x - 0.3010x$$

$$2.6865 = 0.6532x$$

from which $x = \frac{2.6865}{0.6532} = \mathbf{4.11}$, correct to 2 decimal places

Problem 10. Solve the equation $x^{3.2} = 41.15$, correct to 4 significant figures.

Taking logarithms to base 10 of both sides gives:

$$\log_{10} x^{3.2} = \log_{10} 41.15$$

$$3.2 \log_{10} x = \log_{10} 41.15$$

Hence $\log_{10} x = \frac{\log_{10} 41.15}{3.2} = 0.50449$

Thus $\boldsymbol{x} = \text{antilog}\, 0.50449 = 10^{0.50449} = \mathbf{3.195}$ correct to 4 significant figures.

Now try the following exercise.

Exercise 17 Indicial equations

Solve the following indicial equations for x, each correct to 4 significant figures:

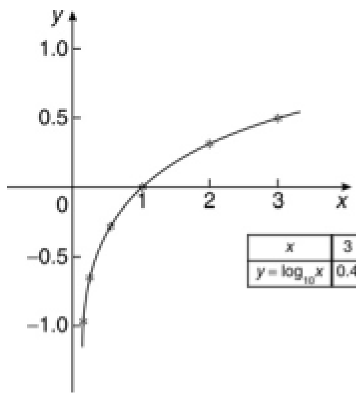
1. $3^x = 6.4$ [1.690]
2. $2^x = 9$ [3.170]
3. $2^{x-1} = 3^{2x-1}$ [0.2696]
4. $x^{1.5} = 14.91$ [6.058]
5. $25.28 = 4.2^x$ [2.251]
6. $4^{2x-1} = 5^{x+2}$ [3.959]
7. $x^{-0.25} = 0.792$ [2.542]
8. $0.027^x = 3.26$ [−0.3272]
9. The decibel gain n of an amplifier is given by:

$$n = 10 \log_{10}\left(\frac{P_2}{P_1}\right)$$

where P_1 is the power input and P_2 is the power output. Find the power gain $\frac{P_2}{P_1}$ when $n = 25$ decibels. [316.2]

4.4 Graphs of logarithmic functions

A graph of $y = \log_{10} x$ is shown in Fig. 4.1 and a graph of $y = \log_e x$ is shown in Fig. 4.2. Both are seen to be of similar shape; in fact, the same general shape occurs for a logarithm to any base.



x	3	2	1	0.5	0.2	0.1
$y = \log_{10} x$	0.48	0.30	0	−0.30	−0.70	−1.0

Figure 4.1

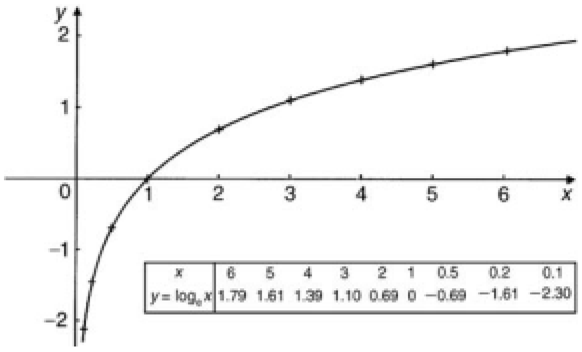


Figure 4.2

In general, with a logarithm to any base a, it is noted that:

(i) $\mathbf{log_a 1 = 0}$
Let $\log_a = x$, then $a^x = 1$ from the definition of the logarithm.
If $a^x = 1$ then $x = 0$ from the laws of indices.
Hence $\log_a 1 = 0$. In the above graphs it is seen that $\log_{10} 1 = 0$ and $\log_e 1 = 0$

(ii) $\mathbf{log_a a = 1}$
Let $\log_a a = x$ then $a^x = a$ from the definition of a logarithm.
If $a^x = a$ then $x = 1$.
Hence $\log_a a = 1$. (Check with a calculator that $\log_{10} 10 = 1$ and $\log_e e = 1$)

(iii) $\mathbf{log_a 0 \to -\infty}$
Let $\log_a 0 = x$ then $a^x = 0$ from the definition of a logarithm.
If $a^x = 0$, and a is a positive real number, then x must approach minus infinity. (For example, check with a calculator, $2^{-2} = 0.25$, $2^{-20} = 9.54 \times 10^{-7}$, $2^{-200} = 6.22 \times 10^{-61}$, and so on)
Hence $\log_a 0 \to -\infty$

4.5 The exponential function

An exponential function is one which contains e^x, e being a constant called the exponent and having an approximate value of 2.7183. The exponent arises from the natural laws of growth and decay and is used as a base for natural or Napierian logarithms. The value of e^x may be determined by using:

(a) a calculator, or
(b) the power series for e^x (see Section 4.6), or
(c) tables of exponential functions.

The most common method of evaluating an exponential function is by using a scientific notation **calculator**, this now having replaced the use of tables. Most scientific notation calculators contain an e^x function which enables all practical values of e^x and e^{-x} to be determined, correct to 8 or 9 significant figures. For example,

$$e^1 = 2.7182818 \quad e^{2.4} = 11.023176$$

$$e^{-1.618} = 0.19829489 \text{ correct to 8 significant figures}$$

In practical situations the degree of accuracy given by a calculator is often far greater than is appropriate. The accepted convention is that the final result is stated to one significant figure greater than the least significant measured value. Use your calculator to check the following values:

$e^{0.12} = 1.1275$, correct to 5 significant figures

$e^{-0.431} = 0.6499$, correct to 4 decimal places

$e^{9.32} = 11159$, correct to 5 significant figures

Problem 11. Use a calculator to determine the following, each correct to 4 significant figures:

(a) $3.72\,e^{0.18}$ (b) $53.2\,e^{-1.4}$ (c) $\dfrac{5}{122}e^7$.

(a) $3.72\,e^{0.18} = (3.72)(1.197217\ldots) = \mathbf{4.454}$, correct to 4 significant figures

(b) $53.2\,e^{-1.4} = (53.2)(0.246596\ldots) = \mathbf{13.12}$, correct to 4 significant figures

(c) $\dfrac{5}{122}e^7 = \dfrac{5}{122}(1096.6331\ldots) = \mathbf{44.94}$, correct to 4 significant figures

Problem 12. Evaluate the following correct to 4 decimal places, using a calculator:

(a) $0.0256(e^{5.21} - e^{2.49})$

(b) $5\left(\dfrac{e^{0.25} - e^{-0.25}}{e^{0.25} + e^{-0.25}}\right)$

(a) $0.0256(e^{5.21} - e^{2.49})$
$= 0.0256(183.094058\ldots - 12.0612761\ldots)$
$= \mathbf{4.3784}$, correct to 4 decimal places

A

(b) $5\left(\dfrac{e^{0.25}-e^{-0.25}}{e^{0.25}+e^{-0.25}}\right)$

$$= 5\left(\frac{1.28402541\ldots - 0.77880078\ldots}{1.28402541\ldots + 0.77880078\ldots}\right)$$

$$= 5\left(\frac{0.5052246\ldots}{2.0628261\ldots}\right)$$

$= \mathbf{1.2246}$, correct to 4 decimal places

Problem 13. The instantaneous voltage v in a capacitive circuit is related to time t by the equation $v = V\,e^{\frac{-t}{CR}}$ where V, C and R are constants. Determine v, correct to 4 significant figures, when $t = 30 \times 10^{-3}$ seconds, $C = 10 \times 10^{-6}$ farads, $R = 47 \times 10^3$ ohms and $V = 200$ V.

$$v = V\,e^{\frac{-t}{CR}} = 200\,e^{\frac{(-30\times10^{-3})}{(10\times10^{-6}\times47\times10^{3})}}$$

Using a calculator,

$$v = 200\,e^{-0.0638297\ldots} = 200(0.9381646\ldots)$$

$= \mathbf{187.6\,V}$

Now try the following exercise.

Exercise 18 Further problems on evaluating exponential functions

1. Evaluate, correct to 5 significant figures:
 (a) $3.5\,e^{2.8}$ (b) $-\dfrac{6}{5}\,e^{-1.5}$ (c) $2.16\,e^{5.7}$

 [(a) 57.556 (b) −0.26776 (c) 645.55]

 In Problems 2 and 3, evaluate correct to 5 decimal places.

2. (a) $\dfrac{1}{7}\,e^{3.4629}$ (b) $8.52\,e^{-1.2651}$
 (c) $\dfrac{5\,e^{2.6921}}{3\,e^{1.1171}}$

 [(a) 4.55848 (b) 2.40444 (c) 8.05124]

3. (a) $\dfrac{5.6823}{e^{-2.1347}}$ (b) $\dfrac{e^{2.1127}-e^{-2.1127}}{2}$
 (c) $\dfrac{4(e^{-1.7295}-1)}{e^{3.6817}}$

 [(a) 48.04106 (b) 4.07482 (c) −0.08286]

4. The length of a bar, l, at a temperature θ is given by $l = l_0\,e^{\alpha\theta}$, where l_0 and α are constants. Evaluate l, correct to 4 significant figures, when $l_0 = 2.587$, $\theta = 321.7$ and $\alpha = 1.771 \times 10^{-4}$. [2.739]

4.6 The power series for e^x

The value of e^x can be calculated to any required degree of accuracy since it is defined in terms of the following **power series:**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

(where $3! = 3 \times 2 \times 1$ and is called 'factorial 3')
The series is valid for all values of x.

The series is said to **converge**, i.e. if all the terms are added, an actual value for e^x (where x is a real number) is obtained. The more terms that are taken, the closer will be the value of e^x to its actual value. The value of the exponent e, correct to say 4 decimal places, may be determined by substituting $x = 1$ in the power series of equation (1). Thus,

$$e^1 = 1 + 1 + \frac{(1)^2}{2!} + \frac{(1)^3}{3!} + \frac{(1)^4}{4!} + \frac{(1)^5}{5!} + \frac{(1)^6}{6!} + \frac{(1)^7}{7!} + \frac{(1)^8}{8!} + \cdots$$

$$= 1 + 1 + 0.5 + 0.16667 + 0.04167 + 0.00833 + 0.00139 + 0.00020 + 0.00002 + \cdots$$

i.e. $e = 2.71828 = 2.7183$, correct to 4 decimal places

The value of $e^{0.05}$, correct to say 8 significant figures, is found by substituting $x = 0.05$ in the power series

$$e^{0.05} = 1 + 0.05 + \frac{(0.05)^2}{2!} + \frac{(0.05)^3}{3!} + \frac{(0.05)^4}{4!} + \frac{(0.05)^5}{5!} + \cdots$$
$$= 1 + 0.05 + 0.00125 + 0.000020833 + 0.000000260 + 0.000000003$$

and by adding,

$e^{0.05} = 1.0512711$, correct to 8 significant figures

In this example, successive terms in the series grow smaller very rapidly and it is relatively easy to determine the value of $e^{0.05}$ to a high degree of accuracy. However, when x is nearer to unity or larger than unity, a very large number of terms are required for an accurate result.

If in the series of equation (1), x is replaced by $-x$, then,

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots$$

i.e. $$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

In a similar manner the power series for e^x may be used to evaluate any exponential function of the form $a\,e^{kx}$, where a and k are constants. In the series of equation (1), let x be replaced by kx. Then,

$$a\,e^{kx} = a\left\{1 + (kx) + \frac{(kx)^2}{2!} + \frac{(kx)^3}{3!} + \cdots\right\}$$

Thus $$5\,e^{2x} = 5\left\{1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots\right\}$$
$$= 5\left\{1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \cdots\right\}$$

i.e. $$5\,e^{2x} = 5\left\{1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots\right\}$$

Problem 14. Determine the value of $5\,e^{0.5}$, correct to 5 significant figures by using the power series for e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Hence $$e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{(2)(1)} + \frac{(0.5)^3}{(3)(2)(1)} + \frac{(0.5)^4}{(4)(3)(2)(1)} + \frac{(0.5)^5}{(5)(4)(3)(2)(1)} + \frac{(0.5)^6}{(6)(5)(4)(3)(2)(1)}$$
$$= 1 + 0.5 + 0.125 + 0.020833 + 0.0026042 + 0.0002604 + 0.0000217$$

i.e. $e^{0.5} = 1.64872$, correct to 6 significant figures

Hence $\mathbf{5e^{0.5}} = 5(1.64872) = \mathbf{8.2436}$, correct to 5 significant figures

Problem 15. Expand $e^x(x^2 - 1)$ as far as the term in x^5.

The power series for e^x is,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Hence $e^x(x^2 - 1)$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)(x^2 - 1)$$
$$= \left(x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots\right) - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)$$

Grouping like terms gives:

$e^x(x^2 - 1)$

$$= -1 - x + \left(x^2 - \frac{x^2}{2!}\right) + \left(x^3 - \frac{x^3}{3!}\right) + \left(\frac{x^4}{2!} - \frac{x^4}{4!}\right) + \left(\frac{x^5}{3!} - \frac{x^5}{5!}\right) + \cdots$$
$$\mathbf{= -1 - x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{11}{24}x^4 + \frac{19}{120}x^5}$$

when expanded as far as the term in x^5.

Now try the following exercise.

Exercise 19 Further problems on the power series for e^x

1. Evaluate $5.6\,e^{-1}$, correct to 4 decimal places, using the power series for e^x. [2.0601]

2. Use the power series for e^x to determine, correct to 4 significant figures, (a) e^2 (b) $e^{-0.3}$ and check your result by using a calculator. [(a) 7.389 (b) 0.7408]

3. Expand $(1-2x)\,e^{2x}$ as far as the term in x^4.
$$\left[1-2x^2-\frac{8x^3}{3}-2x^4\right]$$

4. Expand $(2\,e^{x^2})(x^{\frac{1}{2}})$ to six terms.
$$\left[\begin{array}{l}2x^{\frac{1}{2}}+2x^{\frac{5}{2}}+x^{\frac{9}{2}}+\frac{1}{3}x^{\frac{13}{2}}\\ \quad +\frac{1}{12}x^{\frac{17}{2}}+\frac{1}{60}x^{\frac{21}{2}}\end{array}\right]$$

4.7 Graphs of exponential functions

Values of e^x and e^{-x} obtained from a calculator, correct to 2 decimal places, over a range $x=-3$ to $x=3$, are shown in the following table.

x	-3.0	-2.5	-2.0	-1.5	-1.0	-0.5	0
e^x	0.05	0.08	0.14	0.22	0.37	0.61	1.00
e^{-x}	20.09	12.18	7.39	4.48	2.72	1.65	1.00

x	0.5	1.0	1.5	2.0	2.5	3.0
e^x	1.65	2.72	4.48	7.39	12.18	20.09
e^{-x}	0.61	0.37	0.22	0.14	0.08	0.05

Figure 4.3 shows graphs of $y=e^x$ and $y=e^{-x}$

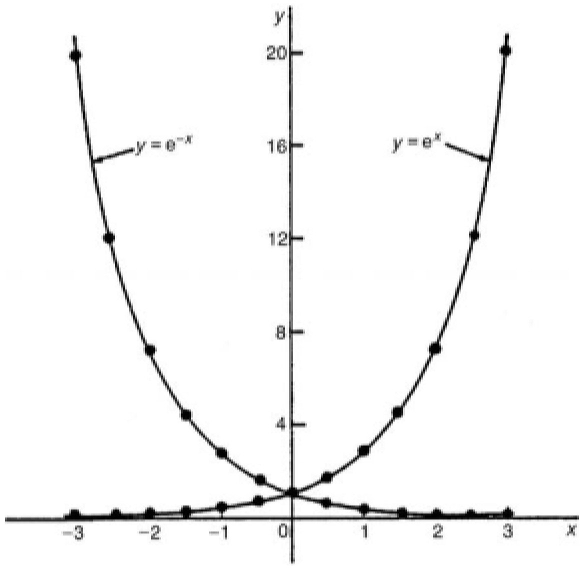


Figure 4.3

Problem 16. Plot a graph of $y=2\,e^{0.3x}$ over a range of $x=-2$ to $x=3$. Hence determine the value of y when $x=2.2$ and the value of x when $y=1.6$.

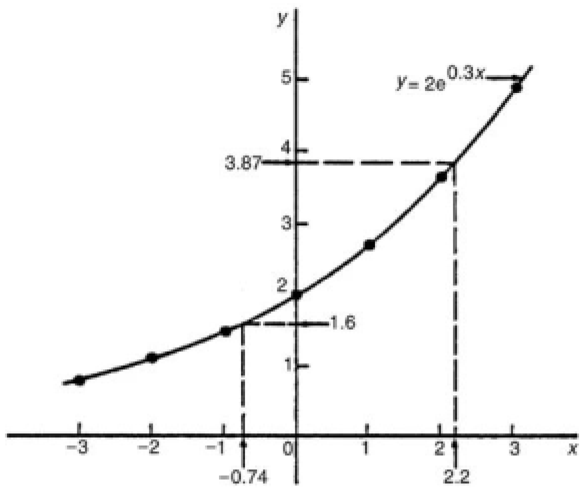


Figure 4.4

A table of values is drawn up as shown below.

x	-3	-2	-1	0	1	2	3
$0.3x$	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
$e^{0.3x}$	0.407	0.549	0.741	1.000	1.350	1.822	2.460
$2\,e^{0.3x}$	0.81	1.10	1.48	2.00	2.70	3.64	4.92

A graph of $y=2\,e^{0.3x}$ is shown plotted in Fig. 4.4.

From the graph, **when $x = 2.2, y = 3.87$** and **when $y = 1.6, x = -0.74$**.

Problem 17. Plot a graph of $y = \frac{1}{3}e^{-2x}$ over the range $x = -1.5$ to $x = 1.5$. Determine from the graph the value of y when $x = -1.2$ and the value of x when $y = 1.4$.

A table of values is drawn up as shown below.

x	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
$-2x$	3	2	1	0	-1	-2	-3
e^{-2x}	20.086	7.389	2.718	1.00	0.368	0.135	0.050
$\frac{1}{3}e^{-2x}$	6.70	2.46	0.91	0.33	0.12	0.05	0.02

A graph of $\frac{1}{3}e^{-2x}$ is shown in Fig. 4.5.

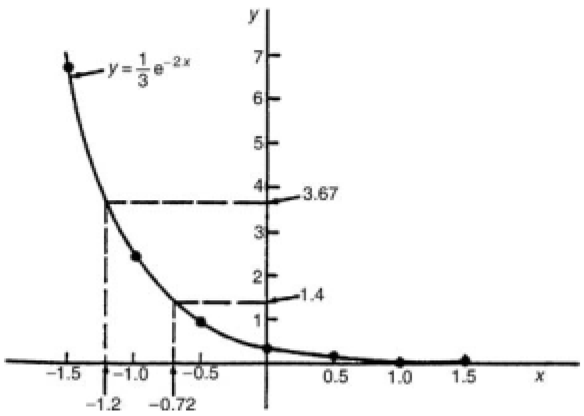


Figure 4.5

From the graph, **when $x = -1.2$, $y = 3.67$** and **when $y = 1.4$, $x = -0.72$**.

Problem 18. The decay of voltage, v volts, across a capacitor at time t seconds is given by $v = 250e^{\frac{-t}{3}}$. Draw a graph showing the natural decay curve over the first 6 seconds. From the graph, find (a) the voltage after 3.4 s, and (b) the time when the voltage is 150 V.

A table of values is drawn up as shown below.

t	0	1	2	3
$e^{\frac{-t}{3}}$	1.00	0.7165	0.5134	0.3679
$v = 250e^{\frac{-t}{3}}$	250.0	179.1	128.4	91.97

t	4	5	6
$e^{\frac{-t}{3}}$	0.2636	0.1889	0.1353
$v = 250e^{\frac{-t}{3}}$	65.90	47.22	33.83

The natural decay curve of $v = 250e^{\frac{-t}{3}}$ is shown in Fig. 4.6.

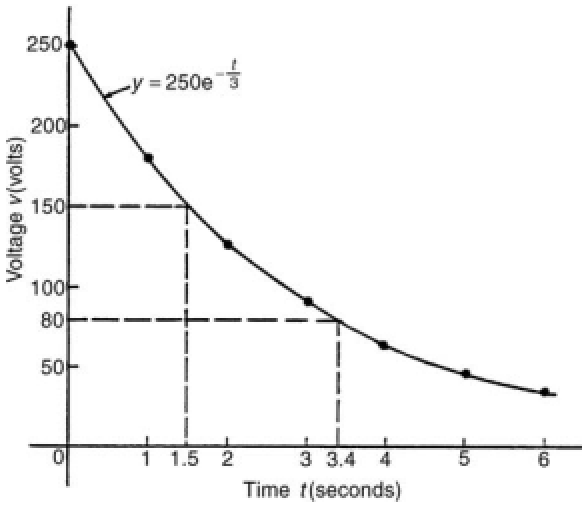


Figure 4.6

From the graph:

(a) **when time $t = 3.4$ s, voltage $v = 80$ V** and
(b) **when voltage $v = 150$ V, time $t = 1.5$ s**.

Now try the following exercise.

Exercise 20 Further problems on exponential graphs

1. Plot a graph of $y = 3e^{0.2x}$ over the range $x = -3$ to $x = 3$. Hence determine the value of y when $x = 1.4$ and the value of x when $y = 4.5$. [3.95, 2.05]

2. Plot a graph of $y = \frac{1}{2}e^{-1.5x}$ over a range $x = -1.5$ to $x = 1.5$ and hence determine the value of y when $x = -0.8$ and the value of x when $y = 3.5$. [1.65, −1.30]

3. In a chemical reaction the amount of starting material $C\,\text{cm}^3$ left after t minutes is given by $C = 40\,e^{-0.006t}$. Plot a graph of C against t and determine (a) the concentration C after 1 hour, and (b) the time taken for the concentration to decrease by half.
[(a) $28\,\text{cm}^3$ (b) 116 min]

4. The rate at which a body cools is given by $\theta = 250\,e^{-0.05t}$ where the excess of temperature of a body above its surroundings at time t minutes is θ°C. Plot a graph showing the natural decay curve for the first hour of cooling. Hence determine (a) the temperature after 25 minutes, and (b) the time when the temperature is 195°C.
[(a) 70°C (b) 5 min]

4.8 Napierian logarithms

Logarithms having a base of e are called **hyperbolic, Napierian** or **natural logarithms** and the Napierian logarithm of x is written as $\log_e x$, or more commonly, $\ln x$.

The value of a Napierian logarithm may be determined by using:

(a) a calculator, or
(b) a relationship between common and Napierian logarithms, or
(c) Napierian logarithm tables

The most common method of evaluating a Napierian logarithm is by a scientific notation **calculator**, this now having replaced the use of four-figure tables, and also the relationship between common and Napierian logarithms,

$$\log_e y = 2.3026 \log_{10} y$$

Most scientific notation calculators contain a '$\ln x$' function which displays the value of the Napierian logarithm of a number when the appropriate key is pressed.

Using a calculator,

$$\ln 4.692 = 1.5458589\ldots$$
$$= 1.5459, \text{ correct to 4 decimal places}$$

$$\text{and } \ln 35.78 = 3.57738907\ldots$$
$$= 3.5774, \text{ correct to 4 decimal places}$$

Use your calculator to check the following values:

$\ln 1.732 = 0.54928$, correct to 5 significant figures
$\ln 1 = 0$
$\ln 0.52 = -0.6539$, correct to 4 decimal places
$\ln e^3 = 3, \qquad \ln e^1 = 1$

From the last two examples we can conclude that

$$\mathbf{\log_e e^x = x}$$

This is useful when solving equations involving exponential functions. For example, to solve $e^{3x} = 8$, take Napierian logarithms of both sides, which gives:

$$\ln e^{3x} = \ln 8$$
i.e. $$3x = \ln 8$$
from which $$x = \tfrac{1}{3}\ln 8 = \mathbf{0.6931}, \text{ correct to 4 decimal places}$$

Problem 19. Use a calculator to evaluate the following, each correct to 5 significant figures:

(a) $\dfrac{1}{4}\ln 4.7291$ (b) $\dfrac{\ln 7.8693}{7.8693}$

(c) $\dfrac{5.29 \ln 24.07}{e^{-0.1762}}$

(a) $$\frac{1}{4}\ln 4.7291 = \frac{1}{4}(1.5537349\ldots) = \mathbf{0.38843}, \text{ correct to 5 significant figures}$$

(b) $$\frac{\ln 7.8693}{7.8693} = \frac{2.06296911\ldots}{7.8693} = \mathbf{0.26215}, \text{ correct to 5 significant figures}$$

(c) $$\frac{5.29 \ln 24.07}{e^{-0.1762}} = \frac{5.29(3.18096625\ldots)}{0.83845027\ldots} = \mathbf{20.070}, \text{ correct to 5 significant figures}$$

Problem 20. Evaluate the following:

(a) $\dfrac{\ln e^{2.5}}{\lg 10^{0.5}}$ (b) $\dfrac{4\,e^{2.23}\lg 2.23}{\ln 2.23}$ (correct to 3 decimal places)

(a) $\dfrac{\ln e^{2.5}}{\lg 10^{0.5}} = \dfrac{2.5}{0.5} = \mathbf{5}$

(b) $\dfrac{4\,e^{2.23}\lg 2.23}{\ln 2.23}$

$$= \frac{4(9.29986607\ldots)(0.34830486\ldots)}{0.80200158\ldots}$$

$= \mathbf{16.156}$, correct to 3 decimal places

> Problem 21. Solve the equation $7 = 4\,e^{-3x}$ to find x, correct to 4 significant figures.

Rearranging $7 = 4\,e^{-3x}$ gives:

$$\frac{7}{4} = e^{-3x}$$

Taking the reciprocal of both sides gives:

$$\frac{4}{7} = \frac{1}{e^{-3x}} = e^{3x}$$

Taking Napierian logarithms of both sides gives:

$$\ln\left(\frac{4}{7}\right) = \ln(e^{3x})$$

Since $\log_e e^{\alpha} = \alpha$, then $\ln\left(\dfrac{4}{7}\right) = 3x$.

Hence

$$x = \frac{1}{3}\ln\left(\frac{4}{7}\right) = \frac{1}{3}(-0.55962)$$

$= \mathbf{-0.1865}$, correct to 4 significant figures

> Problem 22. Given $20 = 60(1 - e^{\frac{-t}{2}})$ determine the value of t, correct to 3 significant figures.

Rearranging $20 = 60(1 - e^{\frac{-t}{2}})$ gives:

$$\frac{20}{60} = 1 - e^{\frac{-t}{2}}$$

and

$$e^{\frac{-t}{2}} = 1 - \frac{20}{60} = \frac{2}{3}$$

Taking the reciprocal of both sides gives:

$$e^{\frac{t}{2}} = \frac{3}{2}$$

Taking Napierian logarithms of both sides gives:

$$\ln e^{\frac{t}{2}} = \ln\frac{3}{2} \quad \text{i.e.} \quad \frac{t}{2} = \ln\frac{3}{2}$$

from which, $\boldsymbol{t} = 2\ln\dfrac{3}{2} = \mathbf{0.811}$, correct to 3 significant figures

> Problem 23. Solve the equation $3.72 = \ln\left(\dfrac{5.14}{x}\right)$ to find x.

From the definition of a logarithm, since

$$3.72 = \ln\left(\frac{5.14}{x}\right) \quad \text{then} \quad e^{3.72} = \frac{5.14}{x}$$

Rearranging gives:

$$x = \frac{5.14}{e^{3.72}} = 5.14\,e^{-3.72}$$

i.e. $\boldsymbol{x} = \mathbf{0.1246}$, correct to 4 significant figures

Now try the following exercise.

Exercise 21 Further problems on evaluating Napierian logarithms

1. Evaluate, correct to 4 decimal places

   (a) $\ln 1.73$ (b) $\ln 541.3$ (c) $\ln 0.09412$
   [(a) 0.5481 (b) 6.2940 (c) −2.3632]

2. Evaluate, correct to 5 significant figures.

   (a) $\dfrac{2.946\ln e^{1.76}}{\lg 10^{1.41}}$ (b) $\dfrac{5\,e^{-0.1629}}{2\ln 0.00165}$

   (c) $\dfrac{\ln 4.8629 - \ln 2.4711}{5.173}$
   [(a) 3.6773 (b) −0.33154 (c) 0.13087]

In Problems 3 to 7 solve the given equations, each correct to 4 significant figures.

3. $1.5 = 4\,e^{2t}$ [−0.4904]
4. $7.83 = 2.91\,e^{-1.7x}$ [−0.5822]
5. $16 = 24(1 - e^{\frac{-t}{2}})$ [2.197]
6. $5.17 = \ln\left(\dfrac{x}{4.64}\right)$ [816.2]
7. $3.72\ln\left(\dfrac{1.59}{x}\right) = 2.43$ [0.8274]

8. The work done in an isothermal expansion of a gas from pressure p_1 to p_2 is given by:
$$w = w_0 \ln\left(\frac{p_1}{p_2}\right)$$
If the initial pressure $p_1 = 7.0$ kPa, calculate the final pressure p_2 if $w = 3\,w_0$
[$p_2 = 348.5$ Pa]

4.9 Laws of growth and decay

The laws of exponential growth and decay are of the form $y = A\,e^{-kx}$ and $y = A(1 - e^{-kx})$, where A and k are constants. When plotted, the form of each of these equations is as shown in Fig. 4.7. The laws occur frequently in engineering and science and examples of quantities related by a natural law include:

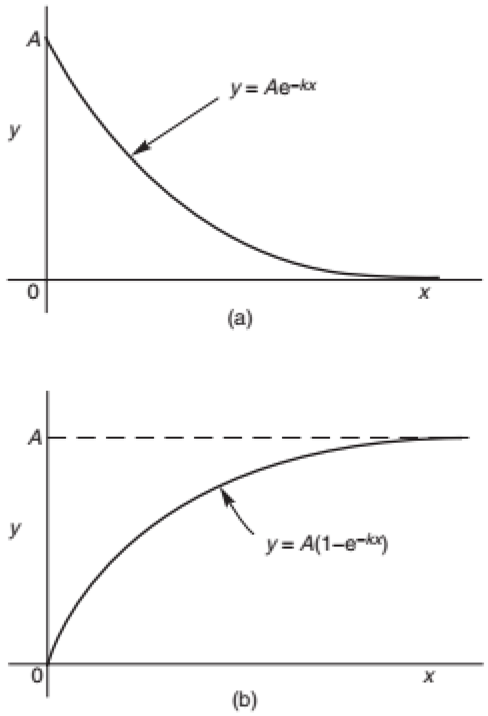


Figure 4.7

(i) Linear expansion $l = l_0\,e^{\alpha\theta}$
(ii) Change in electrical resistance with temperature $R_\theta = R_0\,e^{\alpha\theta}$
(iii) Tension in belts $T_1 = T_0\,e^{\mu\theta}$
(iv) Newton's law of cooling $\theta = \theta_0\,e^{-kt}$
(v) Biological growth $y = y_0\,e^{kt}$
(vi) Discharge of a capacitor $q = Q\,e^{-t/CR}$
(vii) Atmospheric pressure $p = p_0\,e^{-h/c}$
(viii) Radioactive decay $N = N_0\,e^{-\lambda t}$
(ix) Decay of current in an inductive circuit $i = I\,e^{-Rt/L}$
(x) Growth of current in a capacitive circuit $i = I(1 - e^{-t/CR})$

Problem 24. The resistance R of an electrical conductor at temperature θ°C is given by $R = R_0\,e^{\alpha\theta}$, where α is a constant and $R_0 = 5 \times 10^3$ ohms. Determine the value of α, correct to 4 significant figures, when $R = 6 \times 10^3$ ohms and $\theta = 1500$°C. Also, find the temperature, correct to the nearest degree, when the resistance R is 5.4×10^3 ohms.

Transposing $R = R_0\,e^{\alpha\theta}$ gives $\dfrac{R}{R_0} = e^{\alpha\theta}$.

Taking Napierian logarithms of both sides gives:
$$\ln\frac{R}{R_0} = \ln e^{\alpha\theta} = \alpha\theta$$

Hence
$$\alpha = \frac{1}{\theta}\ln\frac{R}{R_0} = \frac{1}{1500}\ln\left(\frac{6 \times 10^3}{5 \times 10^3}\right) = \frac{1}{1500}(0.1823215\ldots) = 1.215477\cdots \times 10^{-4}$$

Hence $\boldsymbol{\alpha = 1.215 \times 10^{-4}}$, correct to 4 significant figures

From above, $\ln\dfrac{R}{R_0} = \alpha\theta$

hence
$$\theta = \frac{1}{\alpha}\ln\frac{R}{R_0}$$

When $R = 5.4 \times 10^3$, $\alpha = 1.215477\ldots \times 10^{-4}$ and $R_0 = 5 \times 10^3$
$$\theta = \frac{1}{1.215477\ldots \times 10^{-4}}\ln\left(\frac{5.4 \times 10^3}{5 \times 10^3}\right) = \frac{10^4}{1.215477\ldots}(7.696104\ldots \times 10^{-2})$$
$= \mathbf{633°C}$, correct to the nearest degree

Problem 25. In an experiment involving Newton's law of cooling, the temperature $\theta(°C)$ is given by $\theta = \theta_0 \mathrm{e}^{-kt}$. Find the value of constant k when $\theta_0 = 56.6°C$, $\theta = 16.5°C$ and $t = 83.0$ seconds.

Transposing $\theta = \theta_0 \mathrm{e}^{-kt}$ gives

$$\frac{\theta}{\theta_0} = \mathrm{e}^{-kt}$$

from which $\dfrac{\theta_0}{\theta} = \dfrac{1}{\mathrm{e}^{-kt}} = \mathrm{e}^{kt}$

Taking Napierian logarithms of both sides gives:

$$\ln \frac{\theta_0}{\theta} = kt$$

from which,

$$k = \frac{1}{t} \ln \frac{\theta_0}{\theta} = \frac{1}{83.0} \ln \left(\frac{56.6}{16.5}\right) = \frac{1}{83.0}(1.2326486\ldots)$$

Hence $\boldsymbol{k = 1.485 \times 10^{-2}}$

Problem 26. The current i amperes flowing in a capacitor at time t seconds is given by $i = 8.0(1 - \mathrm{e}^{\frac{-t}{CR}})$, where the circuit resistance R is 25×10^3 ohms and capacitance C is 16×10^{-6} farads. Determine (a) the current i after 0.5 seconds and (b) the time, to the nearest millisecond, for the current to reach 6.0 A. Sketch the graph of current against time.

(a) Current $i = 8.0(1 - \mathrm{e}^{\frac{-t}{CR}})$

$$= 8.0[1 - \mathrm{e}^{\frac{-0.5}{(16 \times 10^{-6})(25 \times 10^3)}}] = 8.0(1 - \mathrm{e}^{-1.25})$$
$$= 8.0(1 - 0.2865047\ldots) = 8.0(0.7134952\ldots)$$
$$= \mathbf{5.71\ amperes}$$

(b) Transposing $i = 8.0(1 - \mathrm{e}^{\frac{-t}{CR}})$

gives $\dfrac{i}{8.0} = 1 - \mathrm{e}^{\frac{-t}{CR}}$

from which, $\mathrm{e}^{\frac{-t}{CR}} = 1 - \dfrac{i}{8.0} = \dfrac{8.0 - i}{8.0}$

Taking the reciprocal of both sides gives:

$$\mathrm{e}^{\frac{t}{CR}} = \frac{8.0}{8.0 - i}$$

Taking Napierian logarithms of both sides gives:

$$\frac{t}{CR} = \ln\left(\frac{8.0}{8.0 - i}\right)$$

Hence

$$t = CR \ln\left(\frac{8.0}{8.0 - i}\right) = (16 \times 10^{-6})(25 \times 10^3) \ln\left(\frac{8.0}{8.0 - 6.0}\right)$$

when $i = 6.0$ amperes,

i.e. $\boldsymbol{t} = \dfrac{400}{10^3} \ln\left(\dfrac{8.0}{2.0}\right) = 0.4 \ln 4.0$

$= 0.4(1.3862943\ldots) = 0.5545$ s

$=$ **555 ms**, to the nearest millisecond

A graph of current against time is shown in Fig. 4.8.

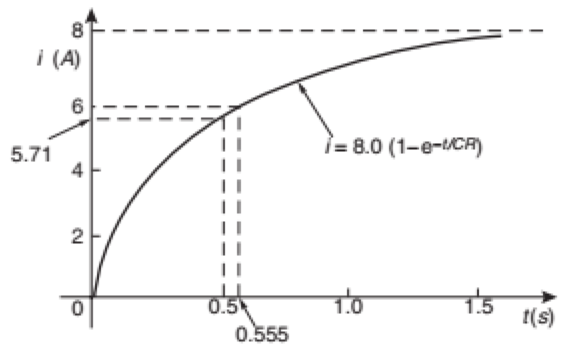


Figure 4.8

Problem 27. The temperature θ_2 of a winding which is being heated electrically at time t is given by: $\theta_2 = \theta_1(1 - \mathrm{e}^{\frac{-t}{\tau}})$ where θ_1 is the temperature (in degrees Celsius) at time $t = 0$ and τ is a constant. Calculate,

(a) θ_1, correct to the nearest degree, when θ_2 is 50°C, t is 30 s and τ is 60 s

(b) the time t, correct to 1 decimal place, for θ_2 to be half the value of θ_1.

(a) Transposing the formula to make θ_1 the subject gives:

$$\theta_1 = \frac{\theta_2}{(1 - e^{\frac{-t}{T}})} = \frac{50}{1 - e^{\frac{-30}{60}}}$$
$$= \frac{50}{1 - e^{-0.5}} = \frac{50}{0.393469\ldots}$$

i.e. $\boldsymbol{\theta_1 = 127°C}$, correct to the nearest degree

(b) Transposing to make t the subject of the formula gives:

$$\frac{\theta_2}{\theta_1} = 1 - e^{\frac{-t}{\tau}}$$

from which, $e^{\frac{-t}{\tau}} = 1 - \frac{\theta_2}{\theta_1}$

Hence $-\frac{t}{\tau} = \ln\left(1 - \frac{\theta_2}{\theta_1}\right)$

i.e. $t = -\tau \ln\left(1 - \frac{\theta_2}{\theta_1}\right)$

Since $\theta_2 = \frac{1}{2}\theta_1$

$$t = -60 \ln\left(1 - \frac{1}{2}\right)$$
$$= -60 \ln 0.5 = 41.59\,\text{s}$$

Hence the time for the temperature $\boldsymbol{\theta_2}$ to be one half of the value of $\boldsymbol{\theta_1}$ is 41.6 s, correct to 1 decimal place

Now try the following exercise.

Exercise 22 Further problems on the laws of growth and decay

1. The pressure p pascals at height h metres above ground level is given by $p = p_0 e^{\frac{-h}{C}}$, where p_0 is the pressure at ground level and C is a constant. Find pressure p when $p_0 = 1.012 \times 10^5$ Pa, height $h = 1420$ m, and $C = 71500$. [99210]

2. The voltage drop, v volts, across an inductor L henrys at time t seconds is given by $v = 200 e^{\frac{-Rt}{L}}$, where $R = 150\,\Omega$ and $L = 12.5 \times 10^{-3}$ H. Determine (a) the voltage when $t = 160 \times 10^{-6}$ s, and (b) the time for the voltage to reach 85 V. [(a) 29.32 volts (b) 71.31×10^{-6} s]

3. The length l metres of a metal bar at temperature t°C is given by $l = l_0 e^{\alpha t}$, where l_0 and α are constants. Determine (a) the value of α when $l = 1.993$ m, $l_0 = 1.894$ m and $t = 250$°C, and (b) the value of l_0 when $l = 2.416$, $t = 310$°C and $\alpha = 1.682 \times 10^{-4}$. [(a) 2.038×10^{-4} (b) 2.293 m]

4. A belt is in contact with a pulley for a sector of $\theta = 1.12$ radians and the coefficient of friction between these two surfaces is $\mu = 0.26$. Determine the tension on the taut side of the belt, T newtons, when tension on the slack side $T_0 = 22.7$ newtons, given that these quantities are related by the law $T = T_0 e^{\mu\theta}$. Determine also the value of θ when $T = 28.0$ newtons. [30.4 N, 0.807 rad]

5. The instantaneous current i at time t is given by: $i = 10 e^{\frac{-t}{CR}}$ when a capacitor is being charged. The capacitance C is 7×10^{-6} farads and the resistance R is 0.3×10^6 ohms. Determine:
   (a) the instantaneous current when t is 2.5 seconds, and
   (b) the time for the instantaneous current to fall to 5 amperes

   Sketch a curve of current against time from $t = 0$ to $t = 6$ seconds. [(a) 3.04 A (b) 1.46 s]

6. The amount of product x (in mol/cm^3) found in a chemical reaction starting with 2.5 mol/cm^3 of reactant is given by $x = 2.5(1 - e^{-4t})$ where t is the time, in minutes, to form product x. Plot a graph at 30 second intervals up to 2.5 minutes and determine x after 1 minute. [2.45 mol/cm^3]

7. The current i flowing in a capacitor at time t is given by:

   $$i = 12.5(1 - e^{\frac{-t}{CR}})$$

   where resistance R is 30 kilohms and the capacitance C is 20 micro-farads. Determine:
   (a) the current flowing after 0.5 seconds, and
   (b) the time for the current to reach 10 amperes [(a) 7.07 A (b) 0.966 s]

4.10 Reduction of exponential laws to linear form

Frequently, the relationship between two variables, say x and y, is not a linear one, i.e. when x is plotted against y a curve results. In such cases the non-linear equation may be modified to the linear form, $y = mx + c$, so that the constants, and thus the law relating the variables can be determined. This technique is called **'determination of law'**.

Graph paper is available where the scale markings along the horizontal and vertical axes are proportional to the logarithms of the numbers. Such graph paper is called **log-log graph paper**.

A **logarithmic scale** is shown in Fig. 4.9 where the distance between, say 1 and 2, is proportional to $\lg 2 - \lg 1$, i.e. 0.3010 of the total distance from 1 to 10. Similarly, the distance between 7 and 8 is proportional to $\lg 8 - \lg 7$, i.e. 0.05799 of the total distance from 1 to 10. Thus the distance between markings progressively decreases as the numbers increase from 1 to 10.

Figure 4.9

With log-log graph paper the scale markings are from 1 to 9, and this pattern can be repeated several times. The number of times the pattern of markings is repeated on an axis signifies the number of **cycles**. When the vertical axis has, say, 3 sets of values from 1 to 9, and the horizontal axis has, say, 2 sets of values from 1 to 9, then this log-log graph paper is called 'log 3 cycle $\times$ 2 cycle'. Many different arrangements are available ranging from 'log 1 cycle $\times$ 1 cycle' through to 'log 5 cycle $\times$ 5 cycle'.

To depict a set of values, say, from 0.4 to 161, on an axis of log-log graph paper, 4 cycles are required, from 0.1 to 1, 1 to 10, 10 to 100 and 100 to 1000.

Graphs of the form $y = a\,e^{kx}$

Taking logarithms to a base of e of both sides of $y = a\,e^{kx}$ gives:

$$\ln y = \ln(a\,e^{kx}) = \ln a + \ln e^{kx} = \ln a + kx \ln e$$

i.e. $\ln y = kx + \ln a$ (since $\ln e = 1$)

which compares with $Y = mX + c$

Thus, by plotting $\ln y$ vertically against x horizontally, a straight line results, i.e. the equation $y = a\,e^{kx}$ is reduced to linear form. In this case, graph paper having a linear horizontal scale and a logarithmic vertical scale may be used. This type of graph paper is called **log-linear graph paper**, and is specified by the number of cycles on the logarithmic scale.

Problem 28. The data given below is believed to be related by a law of the form $y = a\,e^{kx}$, where a and b are constants. Verify that the law is true and determine approximate values of a and b. Also determine the value of y when x is 3.8 and the value of x when y is 85.

x	−1.2	0.38	1.2	2.5	3.4	4.2	5.3
y	9.3	22.2	34.8	71.2	117	181	332

Since $y = a\,e^{kx}$ then $\ln y = kx + \ln a$ (from above), which is of the form $Y = mX + c$, showing that to produce a straight line graph $\ln y$ is plotted vertically against x horizontally. The value of y ranges from 9.3 to 332 hence 'log 3 cycle $\times$ linear' graph paper is used. The plotted co-ordinates are shown in Fig. 4.10 and since a straight line passes through the points the law $y = a\,e^{kx}$ is verified.

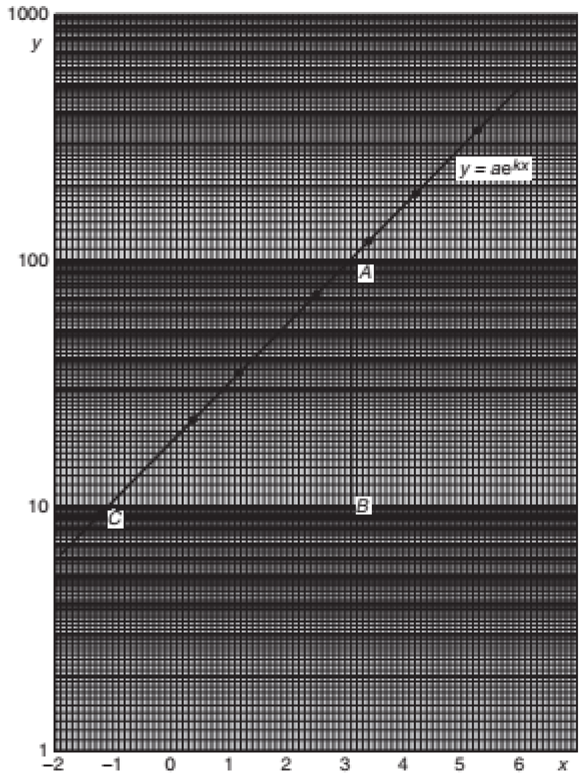


Figure 4.10

Gradient of straight line,

$$k = \frac{AB}{BC} = \frac{\ln 100 - \ln 10}{3.12 - (-1.08)} = \frac{2.3026}{4.20}$$

$= 0.55$, correct to 2 significant figures

Since $\ln y = kx + \ln a$, when $x = 0$, $\ln y = \ln a$, i.e. $y = a$

The vertical axis intercept value at $x = 0$ is 18, hence $a = 18$

The law of the graph is thus $y = 18\,e^{0.55x}$

When x is 3.8, $\quad y = 18\,e^{0.55(3.8)} = 18\,e^{2.09}$

$$= 18(8.0849) = \mathbf{146}$$

When y is 85, $\quad 85 = 18\,e^{0.55x}$

Hence, $\quad e^{0.55x} = \dfrac{85}{18} = 4.7222$

and $\quad 0.55x = \ln 4.7222 = 1.5523$

Hence $\quad x = \dfrac{1.5523}{0.55} = \mathbf{2.82}$

Problem 29. The voltage, v volts, across an inductor is believed to be related to time, t ms, by the law $v = V\,e^{\frac{t}{T}}$, where V and T are constants. Experimental results obtained are:

v volts	883	347	90	55.5	18.6	5.2
t ms	10.4	21.6	37.8	43.6	56.7	72.0

Show that the law relating voltage and time is as stated and determine the approximate values of V and T. Find also the value of voltage after 25 ms and the time when the voltage is 30.0 V.

Since $v = V\,e^{\frac{t}{T}}$ then $\ln v = \frac{1}{T}t + \ln V$ which is of the form $Y = mX + c$.

Using 'log 3 cycle × linear' graph paper, the points are plotted as shown in Fig. 4.11.

Since the points are joined by a straight line the law $v = V\,e^{\frac{t}{T}}$ is verified.

Gradient of straight line,

$$\frac{1}{T} = \frac{AB}{BC} = \frac{\ln 100 - \ln 10}{36.5 - 64.2}$$

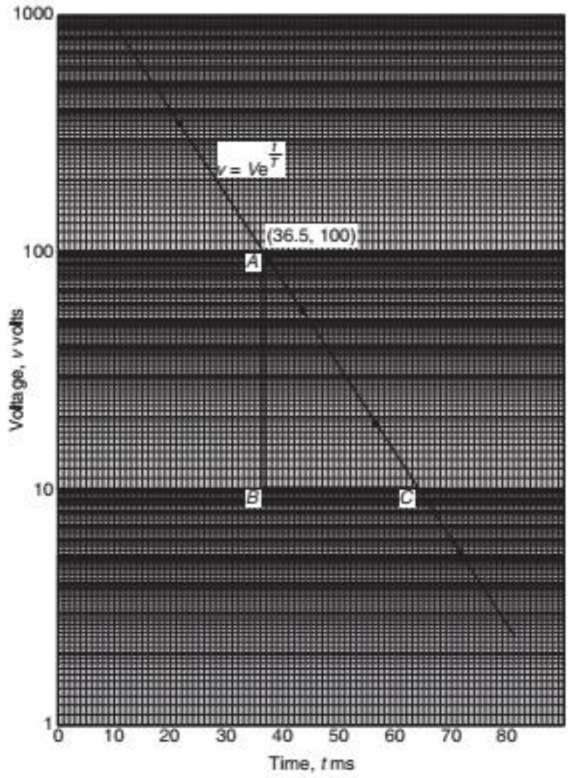


Figure 4.11

$$= \frac{2.3026}{-27.7}$$

Hence $\boldsymbol{T} = \dfrac{-27.7}{2.3026}$

$= \mathbf{-12.0}$, correct to 3 significant figures

Since the straight line does not cross the vertical axis at $t = 0$ in Fig. 4.11, the value of V is determined by selecting any point, say A, having co-ordinates (36.5,100) and substituting these values into $v = V\,e^{\frac{t}{T}}$.

Thus $\quad 100 = V\,e^{\frac{36.5}{-12.0}}$

i.e. $\quad V = \dfrac{100}{e^{\frac{-36.5}{12.0}}}$

$= \mathbf{2090\ volts}$, correct to 3 significant figures

Hence the law of the graph is $v = 2090\,e^{\frac{-t}{12.0}}$.

When time $\quad t = 25$ ms,

voltage $\quad v = 2090\,e^{\frac{-25}{12.0}} = \mathbf{260\ V}$

When the voltage is 30.0 volts, $30.0 = 2090\,e^{\frac{-t}{12.0}}$,

hence $\quad e^{\frac{-t}{12.0}} = \dfrac{30.0}{2090}$

and $\quad e^{\frac{t}{12.0}} = \dfrac{2090}{30.0} = 69.67$

Taking Napierian logarithms gives:

$$\frac{t}{12.0} = \ln 69.67 = 4.2438$$

from which, time $\boldsymbol{t} = (12.0)(4.2438) = \mathbf{50.9\,ms}$

Now try the following exercise.

Exercise 23 Further problems on reducing exponential laws to linear form

1. Atmospheric pressure p is measured at varying altitudes h and the results are as shown below:

Altitude, h m	pressure, p cm
500	73.39
1500	68.42
3000	61.60
5000	53.56
8000	43.41

Show that the quantities are related by the law $p = a\,e^{kh}$, where a and k are constants. Determine the values of a and k and state the law. Find also the atmospheric pressure at 10 000 m.

$$\left[\begin{array}{l} a = 76, k = -7 \times 10^{-5}, \\ p = 76\,e^{-7\times10^{-5}h}, 37.74\,\text{cm} \end{array}\right]$$

2. At particular times, t minutes, measurements are made of the temperature, θ°C, of a cooling liquid and the following results are obtained:

Temperature θ°C	Time t minutes
92.2	10
55.9	20
33.9	30
20.6	40
12.5	50

Prove that the quantities follow a law of the form $\theta = \theta_0\,e^{kt}$, where θ_0 and k are constants, and determine the approximate value of θ_0 and k.

$[\theta_0 = 152, k = -0.05]$