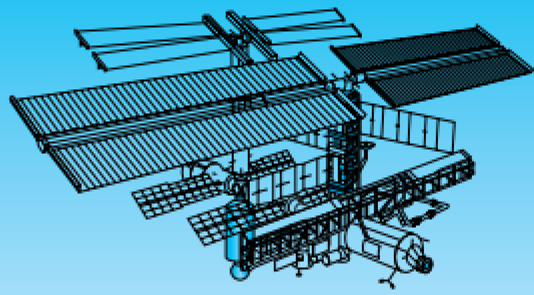
CHAPTER 7



The Laplace Transform

Many problems in engineering and physics can be described in terms of the evolution of solutions of linear differential equations subject to initial conditions. An important group of these problems involves constant coefficient differential equations, and equations like these can be solved very easily by using the Laplace transform.

The Laplace transform is an integral transform that changes a real variable function $f(t)$ into a function $F(s)$ of a variable s through

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

where in general s is a complex variable.

The importance of the Laplace transform in the study of initial value problems for linear constant coefficient differential equations is that it replaces the operation of integrating a differential equation in $f(t)$ by much simpler algebraic operations involving $F(s)$. Unlike previous methods, where first a general solution is found, and then the constants in the complementary function are chosen to match the initial conditions, when the Laplace transform method is used the initial conditions are incorporated from the start. The task of finding the function $f(t)$ from its Laplace transform $F(s)$ is called inverting the transform, and when working with constant coefficient equations we can accomplish this by appeal to tables of Laplace transform pairs—that is, to a table listing a function $f(t)$ and its corresponding Laplace transform $F(s)$.

The fundamental ideas underlying the Laplace transform are derived, along with its operational properties, which are illustrated by examples. Initial value problems for ordinary differential equations are solved by the Laplace transform, which is then applied to systems of equations and to certain variable coefficient equations. The chapter concludes with applications of the Laplace transform to a variety of problems, the last of which is the heat equation.

7.1 Laplace Transform: Fundamental Ideas

Let the real function $f(t)$ be defined for $a \le t \le b$, and let the function $K(t, s)$ of the variables t and s be defined for $a \le t \le b$ and some s. When it exists, the integral $\int_a^b f(t)K(t,s)\,dt$ is a function of the single variable s, so denoting the integral by $F(s)$ we can write

$$F(s) = \int_a^b K(t,s)f(t)\,dt. \tag{1}$$

The function $F(s)$ in (1) is called an **integral transform** of $f(t)$, the function $K(t,s)$ is the **kernel** of the transform, and s is the **transform variable**. The limits a and b may be finite or infinite, and when at least one limit is infinite the integral in (1) becomes an improper integral.

When it exists, the **Laplace transform** $F(s)$ of a real function $f(t)$ with domain of definition $0 \le t < \infty$ is defined as the integral transform (1) with the kernel $K(t,s) = e^{-st}$, the interval of integration $0 \le t < \infty$, and s a complex variable such that Re $s < c$ for some nonnegative constant c, so that

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt. \tag{2}$$

Throughout the present chapter the transform variable s will be considered to be a real variable, and c will be chosen such that the integral in (2) converges. However, when the general problem of recovering a function $f(t)$ from its Laplace transform $F(s)$ is considered in Chapter 16, it will be seen that s must be allowed to be a complex variable. The advantage of restricting s to the real variable case in this chapter is that the recovery of many useful and frequently occurring functions $f(t)$ from their Laplace transforms $F(s)$ can be accomplished in a very simple manner without the use of complex variable methods.

The reason for interest in integral transforms in general, and the Laplace transform in particular, will become clear when the solution of initial value problems for differential equations is considered. It will then be seen that the Laplace transform replaces integrations with respect to t by simple algebraic operations involving $F(s)$, so provided $f(t)$ can be recovered from $F(s)$ in a simple manner, the solution of an initial value problem can be found by means of straightforward algebraic operations.

Clearly the kernel e^{-st} will only decrease as t increases if $s > 0$, and the Laplace transform of $f(t)$ will only be defined for functions $f(t)$ that decrease sufficiently rapidly as $t \to \infty$ for the integral in (2) to exist. In general, if the function to be transformed is denoted by a lowercase letter such as f, then its Laplace transform will be denoted by the corresponding uppercase letter F, as in (2). It is convenient to denote the Laplace transform operation by the symbol $\mathcal{L}$, so that symbolically $F(s) = \mathcal{L}\{f(t)\}$.

formal definition of the Laplace transform

The Laplace transform

Let $f(t)$ be defined for $0 \le t < \infty$. Then, when the improper integral exists, the Laplace transform $F(s)$ of $f(t)$, written symbolically $F(s) = \mathcal{L}\{f(t)\}$, is defined as

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt.$$

EXAMPLE 7.1 Find $\mathcal{L}\{e^{at}\}$ where a is real.

Solution From (2) we have

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st}e^{at}dt \\ &= \left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^{t\to\infty} \\ &= \lim_{t\to\infty}\left[\frac{-e^{-(s-a)t}}{s-a}\right] + \frac{1}{s-a} \\ &= \frac{1}{s-a},\end{aligned}$$

provided $s > a$, for only then will the limit in the first term vanish. This has shown that $\mathcal{L}\{e^{at}\} = F(s) = 1/(s-a)$ for $s > a$, where it is necessary to include the inequality $s > a$ to ensure the convergence of the integral. ■

PIERRE SIMON LAPLACE (1749–1827)
A French mathematician of remarkable ability who made contributions to analysis, differential equations, probability, and celestial mechanics. He used mathematics as a tool with which to investigate physical phenomena, and made fundamental contributions to hydrodynamics, the propagation of sound, surface tension in liquids, and many other topics. His many contributions had a wide-ranging effect on the development of mathematics.

Laplace transform pair and inverse transform

The two functions $f(t)$ and $F(s)$ are called a **Laplace transform pair**, and for all ordinary functions, given $F(s)$ the corresponding function $f(t)$ is determined uniquely, just as $f(t)$ determines $F(s)$ uniquely. This relationship is expressed symbolically by using the symbol $\mathcal{L}^{-1}$ to denote the operation of finding a function $f(t)$ with a given Laplace transform $F(s)$. This process is called finding the **inverse Laplace transform** of $F(s)$. In terms of the foregoing example, we have $\mathcal{L}\{e^{at}\} = 1/(s-a)$ and $\mathcal{L}^{-1}\{1/(s-a)\} = e^{at}$. This is a particular case of the general result that, by definition, the inverse Laplace transform acting on the Laplace transform of the function returns the original function, so we can write

$$\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t).$$

how to be sure a Laplace transform exists

A sufficient condition for the existence of the Laplace transform of a function $f(t)$ is that the absolute value of $f(t)$ can be bounded for all $t \geq 0$ by

$$|f(t)| \leq Me^{kt}, \tag{3}$$

for some constants M and k. This means that if numbers M and k can be found such that

$$|e^{-st}f(t)| \leq Me^{(k-s)t},$$

then

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt \leq M\int_0^\infty e^{(k-s)t}dt = M/(s-k).$$

TABLE 7.1 Laplace Transform Pairs

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	Condition on s
1. 1	$1/s$	$s > 0$
2. t	$1/s^2$	$s > 0$
3. t^n $(n = 1, 2, \ldots)$	$n!/s^{n+1}$	$s > 0$
4. t^a $(a > -1)$	$\Gamma(a+1)/s^{a+1}$	$s > a$
5. e^{at}	$1/(s-a)$	$s > a$
6. $t^n e^{at}$ $(n = 1, 2, \ldots)$	$n!/(s-a)^{n+1}$	$s > a$
7. $H(t-a)$	e^{-as}/s	$s \geq a$
8. $\delta(t-a)$	e^{-as}	$s > 0, a > 0$
9. $\sin at$	$a/(s^2+a^2)$	$s > 0$
10. $\cos at$	$s/(s^2+a^2)$	$s > 0$
11. $t \sin at$	$2as/(s^2+a^2)^2$	$s > 0$
12. $t \cos at$	$(s^2-a^2)/(s^2+a^2)^2$	$s > 0$
13. $e^{at} \sin bt$	$b/[(s-a)^2+b^2]$	$s > a$
14. $e^{at} \cos bt$	$(s-a)/[(s-a)^2+b^2]$	$s > a$
15. $\dfrac{1}{2a^3} \sin at - \dfrac{1}{2a^2} t \cos at$	$1/(s^2+a^2)^2$	$s > 0$
16. $\dfrac{1}{2a} \sin at + \dfrac{1}{2} t \cos at$	$s^2/(s^2+a^2)^2$	$s > 0$
17. $1 - \cos at$	$a^2/[s(s^2+a^2)]$	$s > 0$
18. $at - \sin at$	$a^3/[s^2(s^2+a^2)]$	$s > 0$
19. $\sinh at$	$a/(s^2-a^2)$	$s > \|a\|$
20. $\cosh at$	$s/(s^2-a^2)$	$s > \|a\|$
21. $\dfrac{1}{2a^3} \sinh at + \dfrac{1}{2a^2} t \cosh at$	$1/(s^2-a^2)^2$	$s > \|a\|$
22. $\dfrac{1}{2a} t \sinh at$	$s/(s^2-a^2)^2$	$s > \|a\|$
23. $\dfrac{1}{2a} \sinh at + \dfrac{1}{2} t \cosh at$	$s^2/(s^2-a^2)^2$	$s > \|a\|$
24. $\sinh at - \sin at$	$2a^3/(s^4-a^4)$	$s > \|a\|$
25. $\cosh at - \cos at$	$2a^2 s/(s^4-a^4)$	$s > \|a\|$

The integral on the right will be convergent provided $s > k > 0$, so when this is true the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ will exist. It should be clearly understood that (3) is only a *sufficient* condition for the existence of a Laplace transform, and *not* a necessary one, because Laplace transforms can be found for functions that do not satisfy condition (3). For example, the function $f(t) = t^{-1/4}$ does not satisfy condition (3), but its Laplace transform exists and is a special case of entry 4 in Table 7.1.

The preceding inequality implies that when $\mathcal{L}\{f(t)\}$ exists, $F(s)$ must be such that $\lim_{s\to\infty} F(s) = 0$. In addition, the condition $\mathcal{L}\{f(t)\} \leq M/(s-k)$ implies that $F(s)$ cannot be the Laplace transform of on ordinary function $f(t)$ unless $F(s) \to 0$ as $s \to \infty$. For example, $F(s) = (s^2-1)/(s^2+1)$ is not a Laplace transform of an ordinary function. Exceptions to this condition are functions like the *delta function*, which is defined in Section 7.2, though there the delta function will be seen to involve integration, and so it is not a *function* in the usual sense.

The Laplace transform is a linear operation, and the consequence of this important and useful property is expressed in the following theorem.

THEOREM 7.1

fundamental linearity property

Linearity of the Laplace transformation Let the functions $f_1(t), f_2(t), \ldots, f_n(t)$ have Laplace transforms, and let $c_1, c_2, \ldots, c_n$ be any set of arbitrary constants. Then

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\} + \cdots + c_n\mathcal{L}\{f_n(t)\}.$$

Proof The proof is simple and follows directly from the fact that integration is a linear operation, so the integral of a sum of functions is the sum of their integrals. Thus,

$$\begin{aligned}
&\int_0^\infty e^{-st}\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\}dt \\
&= c_1\int_0^\infty f_1(t)e^{-st}dt + c_2\int_0^\infty f_2(t)e^{-st}dt + \cdots + c_n\int_0^\infty f_n(t)e^{-st}dt \\
&= c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\} + \cdots + c_n\mathcal{L}\{f_n(t)\}.
\end{aligned}$$

■

This theorem has many applications and its use is essential when working with the Laplace transform.

EXAMPLE 7.2

Find the Laplace transform of $f(t) = c_1 e^{at} + c_2 e^{-at}$, and use the result to find $\mathcal{L}\{\sinh at\}$ and $\mathcal{L}\{\cosh at\}$.

some examples

Solution Applying Theorem 7.1 and the result $\mathcal{L}\{e^{at}\} = 1/(s-a)$ from Example 7.1, we find that

$$\mathcal{L}\{c_1 e^{at} + c_2 e^{-at}\} = c_1\mathcal{L}\{e^{at}\} + c_2\mathcal{L}\{e^{-at}\} = c_1/(s-a) + c_2/(s+a).$$

As $\sinh at = (e^{at} - e^{-at})/2$ and $\cosh at = (e^{at} + e^{-at})/2$, $\mathcal{L}\{\sinh at\}$ is obtained from the preceding result by setting $c_1 = 1/2$ and $c_2 = -1/2$, and $\mathcal{L}\{\cosh at\}$ is obtained by setting $c_1 = c_2 = 1/2$, when we obtain

$$\mathcal{L}\{\sinh at\} = a/(s^2 - a^2) \quad \text{and} \quad \mathcal{L}\{\cosh at\} = s/(s^2 - a^2),$$

for $s > |a| \geq 0$. Notice that because s must be be positive, but in $\sinh at$ and $\cosh at$ the number a may be either positive or negative, the relationship between s and a necessary to ensure that the convergence of the integrals must be $s > |a| \geq 0$, and not $s > a > 0$.

The process of finding an inverse Laplace transformation involves reversing the foregoing argument and seeking a function $f(t)$ that has the required Laplace transform $F(s)$. Where possible, this is accomplished by simplifying the algebraic structure of $F(s)$ to the point at which it can be recognized as the sum of the Laplace transforms of known functions of t.

EXAMPLE 7.3

Find the inverse Laplace transform of

$$F(s) = \frac{4s + 10}{s^2 + 6s + 8}.$$

Solution Expanding the Laplace transform in terms of partial fractions gives

$$\frac{4s+10}{s^2+6s+8} = \frac{1}{s+2} + \frac{3}{s+4},$$

so

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{4s+10}{s^2+6s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}.$$

Using the result of Example 7.1 we find that

$$f(t) = \mathcal{L}^{-1}\left\{\frac{4s+10}{s^2+6s+8}\right\} = e^{-2t} + 3e^{-4t}.$$

■

EXAMPLE 7.4 Find (a) $\mathcal{L}\{1\}$ and (b) $\mathcal{L}\{t\}$.

Solution

(a) By definition,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \quad \text{for } s > 0.$$

(b) By definition,

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,dt = \left(-\frac{t}{s}e^{-st} - \frac{e^{-st}}{s^2}\right)\Bigg|_{t=0}^{\infty} = \frac{1}{s^2}, \quad \text{for } s > 0.$$

■

EXAMPLE 7.5 Find $\mathcal{L}\{\sin at\}$.

Solution By definition,

$$\begin{aligned}\mathcal{L}\{\sin at\} &= \int_0^\infty e^{-st}\sin at\,dt = \lim_{k\to\infty}\int_0^k e^{-st}\sin at\,dt \\ &= \lim_{k\to\infty}\left(\frac{-e^{-sk}(a\cos ak + s\sin ak)}{s^2+a^2}\right) + \frac{a}{s^2+a^2} \\ &= \frac{a}{s^2+a^2} \quad \text{for } s > 0,\end{aligned}$$

where the condition $s > 0$ is required to ensure that the limit is finite as $k \to 0$. This has shown that

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad \text{for } s > 0.$$

■

In the next example we find $\mathcal{L}\{t^n\}$, and in the process introduce an integral that will be useful later in Chapter 8 when finding series solutions of linear second order variable coefficient differential equations.

EXAMPLE 7.6 Find $\mathcal{L}\{t^n\}$ for $n = 1, 2, \ldots$.

Solution By definition

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}t^n\mathrm{d}t.$$

To evaluate this integral we will make use of integration by parts to establish a recursion (recurrence) relation from which the result for arbitrary positive integral n can be found.

Accordingly, we define $I(n, s)$ as

$$I(n,s) = \int_0^\infty e^{-st} t^n dt = \lim_{k\to\infty} \int_0^k \frac{-t^n}{s} \frac{d}{dt}(e^{-st}) dt$$

and use integration by parts to express this as

$$= \lim_{k\to\infty} \left[\frac{-t^n e^{-st}}{s} \right]_{t=0}^k + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt$$

$$= \left(\frac{n}{s}\right) I(n-1, s), \quad \text{for } s > 0.$$

This has established the *recursion relation*

$$I(n,s) = (n/s) I(n-1,\ s),$$

satisfied by the integral $I(n, s)$.

As $I(0, s) = \int_0^\infty e^{-st} dt = 1/s$, by setting $n = 1$ in the recursion relation we find that

$$I(1,s) = (1/s) I(0,s) = 1/s^2, \quad \text{for } s > 0.$$

Similarly, setting $n = 2$ in the recursion relation shows that

$$I(2,s) = (2/s) I(1,s) = 2 \cdot 1/s^3 = 2!/s^3, \quad \text{for } s > 0,$$

and an inductive argument shows that

$$I(n,s) = n!/s^{n+1}.$$

In terms of the Laplace transform notation, we have shown that

$$\mathcal{L}\{t^n\} = n!/s^{n+1} \quad \text{for } n = 0,\ 1,\ 2, \ldots, \quad \text{for } s > 0.$$ ■

Notice that setting $s = 1$ in the general result of Example 7.3 enables $n!$ to be expressed as the integral

$$n! = \int_0^\infty e^{-t} t^n dt, \quad \text{for } n = 0,\ 1,\ 2, \ldots.$$

first encounter with the Gamma function

This provides a way of representing factorial n in terms of an integral, and it is our first encounter with a special case of the **Gamma function** that will be required later. The gamma function, denoted by $\Gamma(x)$ for $x > 0$, is defined by the integral

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{4}$$

In terms of the earlier notation, when the restriction that n is an integer is removed, and n is replaced by a positive real variable x, we can write

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt = I(x, 1),$$

but

$$I(x,1) = x I(x-1,1) = x\Gamma(x) \quad \text{for } x > 0,$$

so combining results shows that the gamma function satisfies the fundamental relation

$$\Gamma(x+1) = x\Gamma(x) \quad \text{for } x > 0. \tag{5}$$

It is easily seen from this that

$$\Gamma(n+1) = n! \quad \text{for } n = 0, 1, 2, \ldots,$$

so as $\Gamma(x)$ is defined for all positive x the gamma function provides a generalization of the factorial function $n!$ for positive non-integer values of n. It will be seen later that the gamma function, which belongs to the general class of functions called **higher transcendental functions**, occurs frequently throughout mathematics.

Discontinuous Functions

Because the Laplace transform is defined in terms of an integral, it is possible to find Laplace transforms of discontinuous functions. Suppose, for example, that a function $g(t)$ is discontinuous at $t = a$, as in Fig. 7.1. Then, provided it converges, the integral defining the Laplace transform of $g(t)$ is given by

$$\mathcal{L}\{g(t)\} = \lim_{\varepsilon \to 0} \int_0^{a-\varepsilon} e^{-st} g(t)dt + \lim_{\delta \to 0} \int_{a+\delta}^{\infty} e^{-st} g(t)dt, \tag{6}$$

where ε and δ are both positive. For simplicity, the upper limit in the first integral is usually denoted by a_- and the lower limit in the second integral by a_+. These are, respectively, the limits of integration to the left and right of $t = a$.

An important discontinuous function that finds numerous applications in connection with the Laplace transform, and elsewhere, is the **unit step function** $f(t) = H(t-a)$ with $a \geq 0$, known also as the **Heaviside step function**. The unit step function is defined as

Heaviside step function

$$H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \quad (a \geq 0). \tag{7}$$

A related function that is also of considerable importance is the **unit pulse function**,

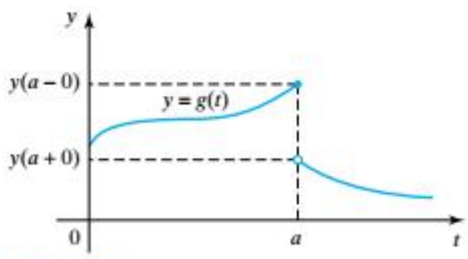


FIGURE 7.1 A discontinuous function $g(t)$.

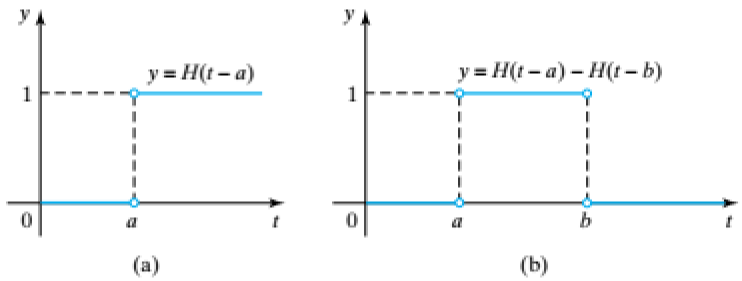


FIGURE 7.2 (a) The unit step function $y = H(t - a)$. (b) The unit pulse function $y = p(t) = H(t - a) - H(t - b)$.

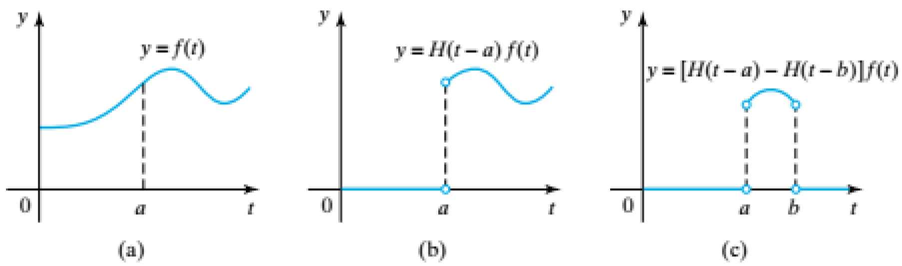


FIGURE 7.3 The effect on $f(t)$ of multiplication by $H(t - a)$ and $H(t - a) - H(t - b)$.

defined as

$$p(t) = H(t - a) - H(t - b), \quad \text{with } b > a \geq 0. \tag{8}$$

The function $p(t)$ operates like a "switch," because it switches on at $t = a$ and off at $t = b$. Graphs of these two functions are shown in Fig. 7.2.

switching functions on and off with the Heaviside step function

If a function $f(t)$ is multiplied by a unit step function, the function $f(t)$ can be considered to be "switched on" at time $t = a$, in the sense that the product $H(t - a)\, f(t)$ is zero for $t < a$ and $f(t)$ for $t > a$. Similarly, multiplication of $f(t)$ by a unit pulse function "switches on" the function $f(t)$ at time $t = a$ and "switches it off" at time $t = b$. This property is illustrated in Fig. 7.3, where Fig. 7.3(a) shows the original function $f(t)$, Fig. 7.3(b) shows the product $H(t - a)\, f(t)$, and Fig. 7.3(c) the product $\{H(t - a) - H(t - b)\}\, f(t)$.

In the next example we make use of result (6) to find the Laplace transforms of the unit step function and the unit pulse function.

EXAMPLE 7.7 Find (a) $\mathcal{L}\{H(t - a)\}$ and (b) $\mathcal{L}\{H(t - a) - H(t - b)\}$.

Solution

(a) By definition

$$\mathcal{L}\{H(t - a)\} = \int_a^\infty e^{-st} dt$$

$$= \left(-\frac{e^{-st}}{s}\right)\Bigg|_{t=a}^{\infty} = \frac{e^{-as}}{s} \quad \text{for } s > a \geq 0.$$

(b) Using result (a) we have

$$\begin{aligned}\mathcal{L}\{H(t-a)-H(t-b)\} &= \int_a^b e^{-st}\,dt \\ &= \int_a^\infty e^{-st}\,dt - \int_b^\infty e^{-st}\,dt \\ &= \frac{e^{-as}-e^{-bs}}{s} \quad \text{for } s > b > a \geq 0.\end{aligned}$$

■

EXAMPLE 7.8 Find (a) $\mathcal{L}\{t^3 - 4t + 5 + 3\sin 2t\}$ and (b) $\mathcal{L}^{-1}\{(s^4+5s^2+2)/[s^3(s^2+1)]$.

Solution

(a) Using Theorem 7.1 together with the Laplace transform pairs found in the previous examples, we have

$$\begin{aligned}\mathcal{L}\{t^3 - 4t + 5 + 2\sin 3t\} &= \mathcal{L}\{t^3\} - 4\mathcal{L}\{t\} + \mathcal{L}\{5\} + 3\mathcal{L}\{\sin 2t\} \\ &= 6/s^4 - 4/s^2 + 5/s + 6/(s^2+4) \\ &= (5s^5 + 2s^4 + 20s^3 - 10s^2 + 24)/[s^4(s^2+4)].\end{aligned}$$

(b) Simplifying the transform by means of partial fractions gives

$$\frac{s^4+5s^2+2}{s^3(s^2+1)} = \frac{2}{s^3} + \frac{3}{s} - 2\frac{s}{s^2+1}.$$

Taking the inverse Laplace transform of each term on the right and using the linearity property of the Laplace transform, we find that

$$\mathcal{L}^{-1}\left(\frac{s^4+5s^2+2}{s^3(s^2+1)}\right) = \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}.$$

Finally, using the transform pairs established in the previous examples, we have

$$\mathcal{L}^{-1}\left\{\frac{s^4+5s^2+2}{s^3(s^2+1)}\right\} = t^2 + 3 - 2\cos t.$$

■

To make further progress with the Laplace transform it is necessary to have available a table of Laplace transform pairs for the most commonly occurring functions. Theorems to be developed later will enable such a table to be extended in a straightforward manner, so that transforms and inverse Laplace transforms of more complicated functions can be found.

Table 7.1 provides a list of the most useful Laplace transform pairs involving elementary functions. All of these entries can be established either by means of routine integration, or by the combination of simpler results, with the sole exception of the *delta function* $\delta(t-a)$ in entry 8. The derivation of this result is to be found in Section 7.2 after the delta function has been defined.

The example that now follows illustrates how entry 15 can be found from entries 9 through 12.

EXAMPLE 7.9 Find $\mathcal{L}^{-1}\{1/(s^2+a^2)^2\}$ by combining related entries in Table 7.1.

Solution Our objective will be to use the linearity property of the Laplace transform to express $1/(s^2+a^2)^2$ as a linear combination of terms that we hope will be found listed in the column $F(s)$ of Table 7.1. If this is possible, the inverse Laplace transform can then be found by adding the inverse transform of each expression in partial fraction representation of $F(s)$. A routine calculation shows that $F(s)$ can be written as

$$\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}\left(\frac{a}{s^2+a^2}\right) - \frac{1}{2a^2}\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right),$$

so from using entries 9 and 12 in Table 7.1 we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}\sin at - \frac{1}{2a^2}t\cos at,$$

and this is entry 15 in the table. ■

Summary

The Laplace transform of a function $f(t)$ has been defined. A condition has been given that ensures the existence of the transform, and the concept of a Laplace transform pair has been introduced. The transform has been shown to have the fundamental property of linearity, and some simple transform pairs have been found directly from the definition. The Heaviside unit step function $H(t-a)$, which jumps from zero for $0 \le t < a$ to unity for $t > a$, has been introduced and used. The section closed with a table of useful Laplace transform pairs.

EXERCISES 7.1

In Exercises 1 through 4 use the definition of the Laplace transform to obtain the stated result.

1. Show that $\mathcal{L}\{t^2\} = 2/s^3$ for $s > 0$.
2. Show that $\mathcal{L}\{te^{at}\} = 1/(s-a)^2$ for $s > a$.
3. Find $\mathcal{L}\{e^{iat}\}$, and by equating the real and imaginary parts show that $\mathcal{L}\{\sin at\} = a/(s^2+a^2)$ and $\mathcal{L}\{\cos at\} = s/(s^2+a^2)$ for $s > 0$.
4. Show that $\mathcal{L}\{\sinh at\} = a/(s^2-a^2)$ for $s > |a|$.

In Exercises 5 through 20 use Table 7.1 of Laplace transform pairs to find $\mathcal{L}\{f(t)\}$.

5. $f(t) = te^{2t}$.
6. $f(t) = 2\sin 3t - \cos 3t$.
7. $f(t) = t - t^2 + t^3$.
8. $f(t) = e^{3t}(\sin t - \cos t)$.
9. $f(t) = e^{-2t}(\cos 2t - \sin 2t)$.
10. $f(t) = t(\sin 2t - \cos 2t)$.
11. $f(t) = t\cosh 3t - \sinh 3t$.
12. $f(t) = \sinh t - t\cos t$.
13. $f(t) = e^{-t}\cos 2t - t$.
14. $f(t) = 2t^2 - 3t + 4\cos 3t$.
15. $f(t) = H(t-\pi/2)e^t\sin t$.
16. $f(t) = H(t-3\pi/2)(\sin t - 3\cos t)$.
17. $f(t) = [H(t-\pi/2) - H(t-\pi)]t$.
18. $f(t) = [1 - H(t-\pi/2)]t$.
19. $f(t) = H(t-\pi/2)e^{-t}\cos t$.
20. $f(t) = [1 - H(t-\pi/2)]e^{3t}$.

In Exercises 21 through 30 use Table 7.1 of Laplace transform pairs to find $\mathcal{L}^{-1}\{F(s)\}$.

21. $F(s) = (s^2-1)/[s(s^2+4)]$.
22. $F(s) = (s^2+3s+1)/[s(s^2-4)]$.
23. $F(s) = (3s+5)/[s(s^2+9)]$.
24. $F(s) = (s^2-4)/[(s^2+1)(s^2-1)]$.
25. $F(s) = (s^3-1)/[(s+2)^2(s^2-9)]$.
26. $F(s) = (s^2+s+1)/[(s^2+4)(s^2-9)]$.
27. $F(s) = s^2/[(s-1)^2(s+1)]$.
28. $F(s) = s/(s-1)^3$.
29. $F(s) = (s^2+4)/[(s^2-9)(s-1)]$.
30. $F(s) = (s^2+1)/[(s+1)(s+2)(s+3)$.

In Exercises 31 through 36 find the Laplace transform of the function $f(t)$ shown in graphical form.

31.

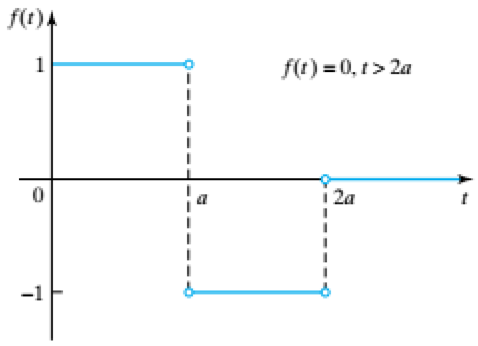


FIGURE 7.4

32.

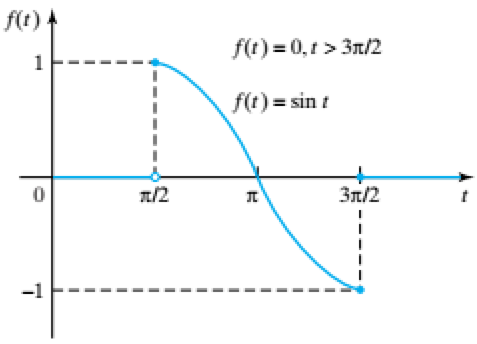


FIGURE 7.5

33.

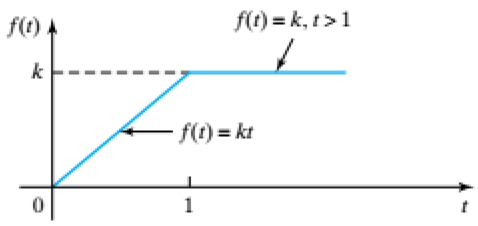


FIGURE 7.6

34.

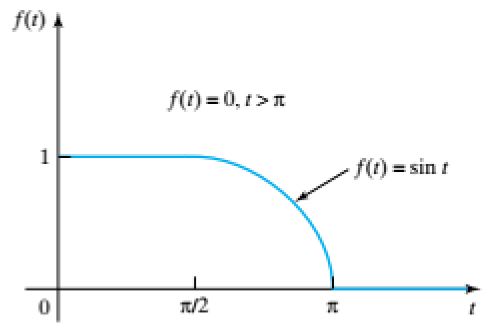


FIGURE 7.7

35.

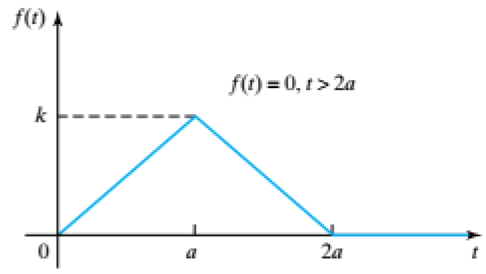


FIGURE 7.8

36.

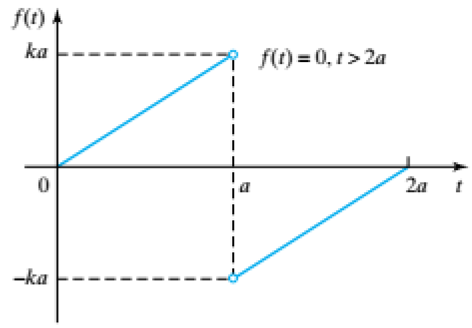


FIGURE 7.9

7.2 Operational Properties of the Laplace Transform

In the previous section the Laplace transform of a basic list of commonly occurring functions $f(t)$ was recorded as the list of Laplace transform pairs in Table 7.1. To use the Laplace transform to solve initial value problems for linear differential equations and systems it is necessary to establish a number of fundamental properties of the transform known as its **operational properties**. This name is given to properties of the transform itself that relate to the way it *operates* on any function $f(t)$ that is transformed, rather than to the effect these properties of the transform have on specific functions $f(t)$.

This means that operational properties are general properties of the Laplace transform that are not specific to any particular function $f(t)$ or to its transform

v

$F(s)$. An important example of an operational property has already been encountered in Theorem 7.1, where the linearity property of the transformation was established.

Some operational properties, such as the scaling and shift theorems that will be proved later, save effort when finding the Laplace transform of a function or inverting a transform, whereas others such as the transform of a derivative are essential when applying the Laplace transform to solve initial value problems for differential equations.

The way derivatives transform is used to find how the homogeneous part of a linear differential equation is transformed, and we will see later that it also shows how the initial conditions for the differential equation enter into the transformed equation. Table 7.1 of Laplace transform pairs is needed when transforming the nonhomogeneous term in the differential equation.

THEOREM 7.2

transforming derivatives

Transform of a derivative Let $f(t)$ be continuous on $0 \le t < \infty$, and let $f'(t)$ be piecewise continuous on every finite interval contained in $t \ge 0$. Then if $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

Proof Using integration by parts, and assuming that f satisfies the sufficiency condition for the existence of a Laplace transform, we have

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)dt = \lim_{k\to\infty} \int_0^k e^{-st} f'(t)dt \\
&= \lim_{k\to\infty} [e^{-st} f(t)]_0^k - \lim_{k\to\infty} \int_0^k -se^{-st} f(t)dt \\
&= \lim_{k\to\infty} [e^{-sk} f(k) - f(0)] + sF(s) \\
&= sF(s) - f(0),
\end{aligned}$$

where $\lim_{k\to\infty} e^{-sk} f(k) = 0$ because of condition (3). ■

THEOREM 7.3

Transform of a higher derivative Let $f(t)$ be continuous on $0 \le t < \infty$, and let $f'(t), f''(t), \ldots, f^{(n-1)}(t)$ be piecewise continuous on every finite interval contained in $t \ge 0$. Then if $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Proof The proof uses repeated integration by parts, but otherwise is analogous to the one used in Theorem 7.2, so the details are left as an exercise. ■

The two most frequently used results are those of Theorem 7.2 and the result from Theorem 7.3 corresponding to $n = 2$, so for convenience we record these here.

The Laplace transform of first and second derivatives

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0). \tag{9a}$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0). \tag{9b}$$

THEOREM 7.4

Transform of f' when f is discontinuous at $t = a$ Let $f(t)$ be continuous on $0 \le t < a$ and on $a < t < \infty$, and let it have a simple jump discontinuity at $t = a$ with the value $f_-(a)$ to the immediate left of a at $t = a-$ and the value $f_+(a)$ to the immediate right of $t = a$ at $a+$. Then if $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) + [f_-(a) - f_+(a)]e^{-as}.$$

Proof Using integration by parts, as in Theorem 7.2, we have

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^{a-} e^{-st} f'(t)dt + \lim_{k\to\infty}\int_{a+}^{\infty} e^{-st} f'(t)dt \\ &= [e^{-st} f(t)]_0^{a-} + \lim_{k\to\infty}[e^{-sk} f(k) - e^{-as} f_+(a)] + sF(s) \\ &= sF(s) - f(0) + [f_-(a) - f_+(a)]e^{-as}.\end{aligned}$$

■

The next example illustrates the application of results (8) and (9) to a simple initial value problem.

EXAMPLE 7.10

Solve the initial value problem

$$y'' + 3y' + 2y = \sin 2t, \quad \text{where } y(0) = 2 \quad \text{and} \quad y'(0) = -1.$$

Solution Because of the linearity of the equation and of the Laplace transform operation, taking the Laplace transform of the differential equation we have

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\sin 2t\}.$$

Setting $\mathcal{L}\{y(t)\} = Y(s)$, and using the initial conditions $y(0) = 2$ and $y'(0) = -1$, we find from (9a,b) that

$$\mathcal{L}\{y''\} = s^2Y(s) - 2s + 1,$$

and

$$\mathcal{L}\{y'\} = sY(s) - 2.$$

Entry 9 in Table 7.1 shows that $\mathcal{L}\{\sin 2t\} = 2/(s^2+4)$, so combining these results enables the transformed differential equation to be written

$$s^2Y(s) - 2s + 1 + 3[sY(s) - 2] + 2Y(s) = \frac{2}{s^2+4},$$

or as

$$(s^2 + 3s + 2)Y(s) = \frac{2s^3 + 5s^2 + 8s + 22}{s^2+4}.$$

Solving for the Laplace transform of the solution gives

$$Y(s) = \frac{2s^3 + 5s^2 + 8s + 22}{(s^2+4)(s^2+3s+2)}.$$

When expressed in partial fraction form, $Y(s)$ becomes

$$Y(s) = \frac{-5}{4}\frac{1}{s+2} + \frac{17}{5}\frac{1}{s+1} - \frac{1}{20}\frac{2}{s^2+4} - \frac{3}{20}\frac{s}{s^2+4}.$$

Using the linearity property when taking the inverse Laplace transform, we have

$$\begin{aligned}\mathcal{L}^{-1}\{Y(s)\} &= -\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{17}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &\quad -\frac{1}{20}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - \frac{3}{20}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\},\end{aligned}$$

so using Table 7.1 to identify the four transforms involved shows that the solution of the initial value problem is

$$y(t) = -\frac{5}{4}e^{-2t} + \frac{17}{5}e^{-t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t, \quad \text{for } t > 0.$$

This example illustrates a fundamental difference between the solution of an initial value problem obtained by using the Laplace transform and that obtained by the previous methods that have been developed. In the other methods, when solving an initial value problem, first a general solution was found, and then the arbitrary constants were matched to the initial conditions. However, in the Laplace transform approach the initial conditions are incorporated when the equation is transformed, so the inversion of $Y(s)$ gives the required solution of the initial value problem immediately.

As the *structure* of the solution in Example 7.10 is typical of the structure obtained when solving all initial value problems for ordinary differential equations by means of the Laplace transform, a closer examination of it will help understand how the solution is generated.

Returning to the point where the equation was transformed, the result can be rewritten as

$$\underbrace{(s^2+3s+2)}_{\substack{\text{Transformed homogeneous equation}\\ \text{with } y'', y', \text{ and } y \text{ replaced, respectively,}\\ \text{by } s^2, s, \text{ and } 1}} Y(s) = \underbrace{2s+5}_{\substack{\text{Transformed initial}\\ \text{conditions}}} + \underbrace{\frac{2}{s^2+2}}_{\substack{\text{Transformed nonhomogeneous}\\ \text{term}}}$$

Setting $G(s) = 1/(s^2+3s+2)$, and denoting the transformed initial conditions by $I(s)$ and the transformed nonhomogeneous term by $R(s)$, the above result can be solved for $Y(s)$ and written in the form

$$Y(s) = G(s)I(s) + G(s)R(s). \tag{10}$$

transfer function

This shows how the transform $G(s)$, called in engineering applications the **transfer function** associated with the differential equation, modifies the transform of the initial conditions and the transform of the nonhomogeneous term to arrive at the transform $Y(s)$ of the solution. The name *transfer function* comes from the fact that when all the initial conditions are zero, so $I(s) = 0$, the only term generating a solution is the forcing function (the nonhomogeneous term), so (10) describes how the effect of the *input* is *transferred* to the *output* (the solution). In terms of Example 7.10 we can write

$$\begin{aligned}G(s) &= \frac{Y(s)}{R(s)} = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{\sin 2t\}} \\ &= \frac{\mathcal{L}\{output\}}{\mathcal{L}\{input\}}.\end{aligned} \tag{11}$$

In control theory the transfer function of a system characterizes the behavior of the entire system.

We now develop the most important operational properties of the Laplace transform, starting with the first shift theorem, also called the s-shift theorem.

THEOREM 7.5

the s-shift theorem

The first shift theorem or the s-shift theorem Let $\mathcal{L}\{f(t)\} = F(s)$ for $s > \gamma$. Then the Laplace transform of $e^{at} f(t)$ is obtained from $F(s)$ by replacing s by $s - a$, where $s - a > \gamma$. Thus,

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \quad \text{for } s - a > \gamma.$$

Conversely, the inverse transform

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t).$$

Proof From the conditions of the theorem, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ for $s > \gamma$, so

$$\mathcal{L}\{e^{-at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)dt = \int_0^\infty e^{-(s-a)t} f(t)dt = F(s-a) \quad \text{for } s - a > \gamma.$$

The converse result follows by reversing this argument to arrive at the result

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t).$$ ■

EXAMPLE 7.11

Use Theorem 7.5 to find $\mathcal{L}\{e^{at}t^n\}$, $\mathcal{L}\{e^{at}\cos bt\}$, and $\mathcal{L}\{e^{at}t \sin bt\}$.

Solution Using the Laplace transforms of t^n, $\cos bt$, and $t \sin bt$ listed as entries 3, 10, and 11 in Table 7.1, with a replaced by b in entries 10 and 11, and then replacing s by $s - a$ we find that

$$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}} \quad \text{for } s > 0, \quad \mathcal{L}\{e^{at}\cos bt\} = \frac{(s-a)}{[(s-a)^2 + b^2]} \quad \text{for } s > a,$$

and

$$\mathcal{L}\{e^{at}t \sin bt\} = \frac{2b(s-a)}{[(s-a)^2 + b^2]^2} \quad \text{for } s > a.$$ ■

EXAMPLE 7.12

Use Theorem 7.5 to find $\mathcal{L}^{-1}\{1/(s^2 + 4s + 13)\}$.

Solution Completing the square in the denominator we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4s + 13}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2 + 3^2}\right\}.$$

A comparison with entry 13 in Table 7.1 shows that

$$\mathcal{L}^{-1}\{1/(s^2 + 4s + 13)\} = \frac{1}{3}e^{-2t}\sin 3t.$$ ■

We now derive the second shift theorem, also called the t-shift theorem, in which use will be made of the unit step function $H(t - a)$.

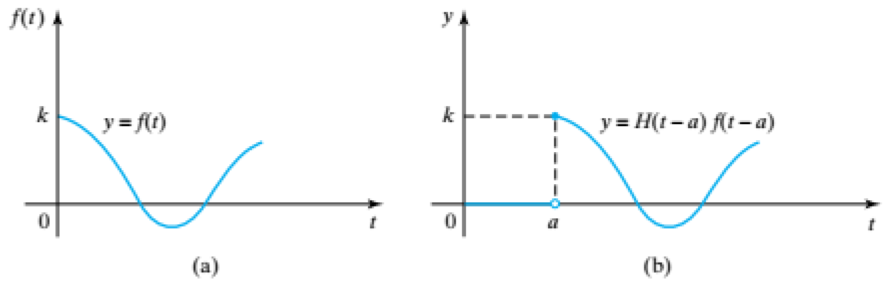


FIGURE 7.10 The relationship between $f(t)$ and $H(t-a)f(t-a)$.

THEOREM 7.6

the *t*-shift theorem

The second shift theorem or the *t*-shift theorem Let $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s)$$

and, conversely,

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)f(t-a).$$

Proof Before proving the theorem it is necessary to understand the precise meaning of $H(t-a)f(t-a)$. This can be seen by examining Fig. 7.10. The unit step function $H(t-a)$ is zero until $t = a$, when it jumps to the value 1 and thereafter remains constant for $t > a$. The function $f(t-a)$ is simply the function $f(t)$ with its origin shifted to $t = a$, so it can be considered to be the function $f(t)$ translated to the right by an amount a. Thus, $H(t-a)f(t-a)$ is a function that is zero until $t = a$, after which it reproduces the function $f(t)$ translated to the right by an amount a.

The result of the theorem is obtained as follows:

$$\mathcal{L}\{H(t-a)f(t-a)\} = \int_0^\infty e^{-st}H(t-a)f(t-a)dt = \int_a^\infty e^{-st}f(t-a)dt.$$

If we make the change of variable $\tau = t - a$, this becomes

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)d\tau$$

and so

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s).$$

The converse result follows by reversing this argument. ■

EXAMPLE 7.13

Use Theorem 7.6 to find (a) $\mathcal{L}\{H(t-4)\sin(t-4)\}$, (b) to show that $\mathcal{L}\{H(t-a)\} = e^{-as}/s$ in agreement with entry 7 in Table 7.1, and (c) to find $\mathcal{L}^{-1}\{se^{-as}/(s^2+b^2)\}$.

Solution **(a)** From entry 9 in Table 7.1 we have $\mathcal{L}\{\sin t\} = 1/(s^2+1)$, so applying Theorem 7.6 with $a = 4$ gives

$$\mathcal{L}\{H(t-4)\sin(t-4)\} = e^{-4s}/(s^2+1).$$

(b) Setting $f(t) = 1$ in Theorem 7.6 and using the fact that $\mathcal{L}\{1\} = 1/s$ gives

$$\mathcal{L}\{H(t-a)\} = e^{-as}/s.$$

(c) Entry 10 in Table 7.1 shows that $\mathcal{L}\{\cos bt\} = s/(s^2 + b^2)$, so using this in Theorem 7.6 gives

$$\mathcal{L}^{-1}\{se^{-as}/(s^2 + b^2)\} = H(t - a)\cos[b(t - a)].$$

The next example makes use of Theorem 7.6 when solving an initial value problem.

EXAMPLE 7.14 Solve the initial value problem

$$y'' + 3y' + 2y = H(t - \pi)\sin 2t \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

Solution Setting $\mathcal{L}\{y(t)\} = Y(s)$, transforming the differential equation, and incorporating the initial conditions as in Example 7.10 gives

$$s^2Y(s) - s + 3(sY(s) - 1) + 2Y(s) = \frac{2e^{-\pi s}}{s^2 + 4},$$

or

$$(s^2 + 3s + 2)Y(s) = s + 3 + \frac{2e^{-\pi s}}{s^2 + 4}.$$

As $s^2 + 3s + 2 = (s + 1)(s + 2)$, this last result can be written in the form

$$Y(s) = \frac{s + 3}{(s + 1)(s + 2)} + \frac{2e^{-\pi s}}{(s^2 + 4)(s + 1)(s + 2)}.$$

It is now necessary to invert $Y(s)$, and to accomplish this some algebraic manipulation will be necessary if we are to identify terms on the right with entries in Table 7.1. When expressed in terms of partial fractions, after a little manipulation $Y(s)$ becomes

$$Y(s) = \frac{2}{s + 1} - \frac{1}{s + 2} + e^{-\pi s}\left(\frac{2}{5}\frac{1}{s + 1} - \frac{1}{4}\frac{1}{s + 2} - \frac{1}{20}\frac{2}{s^2 + 4} - \frac{3}{20}\frac{s}{s^2 + 4}\right).$$

Each term can now be identified as the transform of an entry in Table 7.1, though as the last four terms are multiplied by $e^{-\pi s}$ their inverse Laplace transforms will need to be obtained by using Theorem 7.6. As a result, $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ becomes

$$\begin{aligned} y(t) &= 2e^{-t} - e^{-2t} + H(t - \pi) \\ &\quad \times \left(\frac{2}{5}e^{-(t-\pi)} - \frac{1}{4}e^{-2(t-\pi)} - \frac{1}{20}\sin 2(t - \pi) - \frac{3}{20}\cos 2(t - \pi)\right), \end{aligned}$$

for $t > 0$. A graph of this solution is shown in Fig. 7.11, from which it can be seen that in the interval $0 < t < \pi$ the solution $y(t)$ only involves the first two terms, and so decays exponentially. At $t = \pi$ the forcing function $\sin 2t$ is switched on, after which all the exponential terms decay to zero as $t \to \infty$, leaving only the periodic steady state solution.

THEOREM 7.7 **Differentiation of a transform: Multiplication of $f(t)$ by t^n** Let $\mathcal{L}\{f(t)\} = F(s)$. Then

differentiating a transform

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}.$$

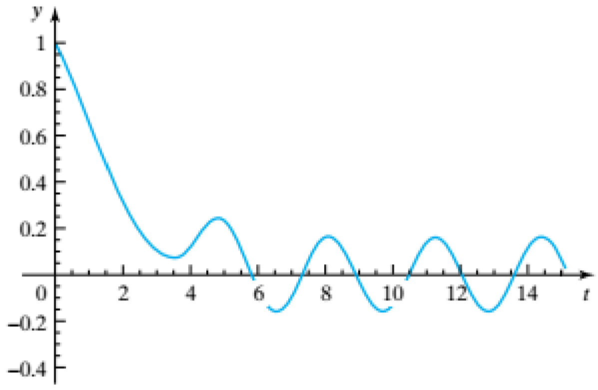


FIGURE 7.11 The solution $y(t)$ showing the influence of the forcing function after $t = \pi$.

Proof By definition

$$\int_0^{\infty} e^{-st} f(t)dt = F(s),$$

so differentiating under the integral sign with respect to s gives

$$\frac{dF(s)}{ds} = \int_0^{\infty} \frac{\partial(e^{-st})}{\partial s} f(t)dt,$$

and so

$$\frac{dF(s)}{ds} = \int_0^{\infty} (-t)e^{-st} f(t)dt = -\int_0^{\infty} e^{-st} t f(t)dt,$$

which is the result of the theorem when $n = 1$. Each subsequent differentiation will introduce a further factor $(-t)$ into the integrand, leading the general result of the theorem. ■

EXAMPLE 7.15 Use Theorem 7.7 to find (a) $\mathcal{L}\{t \sin at\}$ and (b) $\mathcal{L}\{t\ e^{at} \cos bt\}$.

Solution **(a)** Entry 9 in Table 7.1 shows that $\mathcal{L}\{\sin at\} = a/(s^2 + a^2)$ for $s > 0$, so from Theorem 7.7

$$\mathcal{L}\{t \sin at\} = (-1)\frac{d}{ds}\frac{a}{(s^2 + a^2)} = \frac{2as}{(s^2 + a^2)^2} \quad \text{for } s > 0,$$

in agreement with entry 11 in Table 7.1.
(b) Entry 14 in Table 7.1 shows that $\mathcal{L}\{e^{at} \cos bt\} = (s - a)/[(s - a)^2 + b^2]$ for $s > a$, so from Theorem 7.7

$$\begin{aligned}\mathcal{L}\{t\ e^{at} \cos bt\} &= (-1)\frac{d}{ds}\frac{(s - a)}{[(s - a)^2 + b^2]} \\ &= \frac{(s - a)^2 - b^2}{[(s - a)^2 + b^2]^2} \quad \text{for } s > a.\end{aligned}$$

These examples show that, in many cases, less effort is involved finding transforms by means of Theorem 7.7 than by direct use of the definition of the Laplace transform. ■

THEOREM 7.8

scaling a transform

Scaling theorem Let $\mathcal{L}\{f(t)\} = F(s)$. Then if $k > 0$,

$$\mathcal{L}\{f(kt)\} = \frac{1}{k}F\left(\frac{s}{k}\right).$$

Proof The result follows by setting $u = kt$ in the definition of the Laplace transform, because

$$\begin{aligned}\{f(kt)\} &= \int_0^\infty e^{-st} f(kt)dt \\ &= \frac{1}{k}\int_0^\infty e^{-s(u/k)} f(u)du \\ &= \frac{1}{k}\int_0^\infty e^{-(s/k)u} du \\ &= \frac{1}{k}F\left(\frac{s}{k}\right).\end{aligned}$$

■

EXAMPLE 7.16

If $\mathcal{L}\{f(t)\} = e^{-3s}(1-2s)/(2s^2 - s + 1)$, find $\{f(3t)\}$.

Solution In this case $k = 3 > 0$, so from Theorem 7.8, replacing s by $s/3$ in $\mathcal{L}\{f(t)\}$ and multiplying the result by 1/3 gives

$$\begin{aligned}\mathcal{L}\{f(3t)\} &= \frac{1}{3}\frac{e^{-s}(1-2s/3)}{(2(s/3)^2 - s/3 + 1)} \\ &= \frac{e^{-s}(3-2s)}{2s^2 - 3s + 9}.\end{aligned}$$

■

Many functions whose Laplace transform is required are periodic functions with period T, though they are not necessarily continuous functions for all $t > 0$. In the Laplace transform, where only the behavior of a function $f(t)$ for $t > 0$ is involved, a **periodic function** with **period** T is defined as a function $f(t)$ with the property that T is the smallest value for which

$$f(t+T) = f(t) \quad \text{for all } t > 0. \tag{12}$$

An example of a piecewise continuous function $f(t)$ with period T that is defined for $t > 0$ is shown in Fig. 7.12.

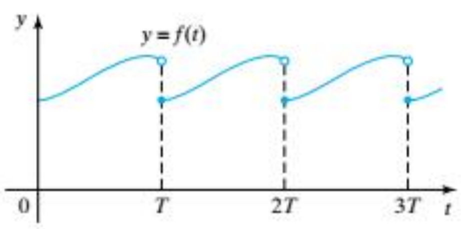


FIGURE 7.12 A function $f(t)$ with period T.

THEOREM 7.9

transforming a periodic function

Transform of a periodic function with period T Let $f(t)$ be a periodic function with period T such that $\int_0^T e^{-st} f(t)dt$ is finite. Then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)dt \quad \text{for } s > 0.$$

Proof In the definition of the Laplace transform we divide the interval of integration into subintervals of length T and write

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \cdots.$$

Then, because of the periodicity of $f(t)$, the function $f(t)$ will be the same in each integral. Consequently, changing the variable in the $(r+1)$th integral to $t = \tau + rT$ with $r = 0, 1, 2, \ldots$ gives

$$\int_0^T e^{-s(\tau+rT)} f(\tau)d\tau = e^{-rsT} \int_0^T e^{-s\tau} f(\tau)d\tau \quad \text{for } r = 0,\ 1, 2, \ldots$$
$$= e^{-rsT} \int_0^T e^{-st} f(t)dt,$$

where the dummy variable τ has been replaced by t. Substituting this result into the original integral gives

$$\mathcal{L}\{f(t)\} = [1 + e^{-Ts} + e^{-2Ts} + \cdots] \int_0^T e^{-st} f(t)dt,$$

which is finite because we have assumed that $\int_0^T e^{-st} f(t)dt$ is finite. The bracketed terms form a geometrical series with the common ratio $e^{-Ts} < 1$, so its sum is $1/(1-e^{-Ts})$, and thus

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)dt, \quad \text{for } s > 0,$$

and the proof is complete. ∎

The necessity of the condition in Theorem 7.9 that $\int_0^T e^{-st} f(t)dt$ is finite arises because periodic functions exist for which this integral is divergent.

EXAMPLE 7.17

Find the Laplace transform of the square wave shown in Fig. 7.13.

Solution As the function is discontinuous with period $2a$ we compute the integral in Theorem 7.9 in two parts as

$$\int_0^{2a} e^{-st} f(t)dt = \int_0^a ke^{-st}dt + \int_a^{2a} (-k)e^{-st}dt$$
$$= \frac{k}{s}(1 - e^{-as}) + \frac{k}{5}(e^{-2as} - e^{-as})$$
$$= \frac{k}{s}(1 + e^{-2as} - 2e^{-as}).$$

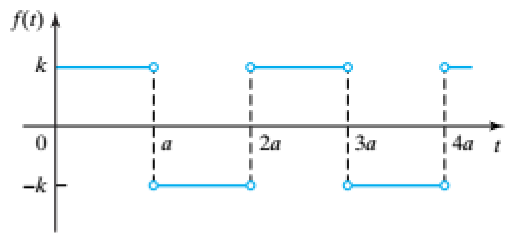


FIGURE 7.13 A square wave with period $2a$.

Then from Theorem 7.9 we have

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{k(1+e^{-2as}-2e^{-as})}{s(1-e^{-2as})}\\ &= \frac{k(1-e^{-as})}{s(1+e^{-as})}\\ &= \frac{k(e^{as/2}-e^{-as/2})}{s(e^{as/2}+e^{-as/2})}\\ &= \frac{k\sinh(as/2)}{s\cosh(as/2)} = \frac{k}{s}\tanh(as/2) \quad \text{for } s>0.\end{aligned}$$

EXAMPLE 7.18 Use Theorem 7.9 to show that $\mathcal{L}\{\sin t\} = 1/(s^2+1)$ and Theorem 7.8 to show that $\mathcal{L}\{\sin at\} = a/(s^2+a^2)$.

Solution The function $f(t) = \sin t$ is periodic with period 2π and $\int_0^{2\pi} e^{-st}\sin t\,dt$ is finite, so from Theorem 7.9 we have

$$\begin{aligned}\mathcal{L}\{\sin t\} &= \frac{1}{(1-e^{-2\pi s})}\int_0^{2\pi} e^{-st}\sin t\,dt\\ &= \frac{1}{(1-e^{-2\pi s})}\left(\frac{1}{s^2+1}-\frac{e^{-2\pi s}}{s^2+1}\right)\\ &= \frac{1}{s^2+1} \quad \text{for } s>0.\end{aligned}$$

Setting $k=a$ in Theorem 7.8 and using the preceding result gives

$$\begin{aligned}\mathcal{L}\{\sin at\} &= \frac{1}{a}\frac{1}{[(s/a)^2+1]}\\ &= \frac{a}{s^2+a^2} \quad \text{for } s>0.\end{aligned}$$

EXAMPLE 7.19 Find the Laplace transform of the solution of the initial value problem

$$y''+3y'+2y=f(t), \quad \text{where } y(0)=y'(0)=0$$

and $f(t)$ is the square wave in Example 7.17.

Solution Transforming the equation as in Examples 7.10 and 7.14 and using the result of Example 7.17 gives

$$s^2Y(s)+3sY(s)+2Y(s) = \frac{k}{s}\tanh(as/2),$$

so

$$Y(s) = \frac{k \tanh(as/2)}{s(s^2 + 3s + 2)}.$$

The convolution operation

Let the functions $f(t)$ and $g(t)$ be defined for $t \geq 0$. Then the **convolution** of the functions f and g denoted by $(f * g)(t)$, and in abbreviated form by $(f * g)$, is defined as the integral

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau.$$

convolution and the convolution theorem

The change of variable $v = t - \tau$ followed by the replacement of the dummy variable v by t shows that the convolution operation is *commutative*, so

$$(f * g)(t) = (g * f)(t). \tag{13}$$

EXAMPLE 7.20

Find $(t^2 * \cos t)$ and $(\cos t * t^2)$ and hence confirm the equality of these two convolution operations. Compare the effort required in each case.

Solution We have

$$\begin{aligned}(t^2 * \cos t) &= \int_0^t \tau^2 \cos(t - \tau)d\tau \\ &= \int_0^t \tau^2[\cos t \cos\tau + \sin t \sin\tau]d\tau \\ &= \cos t \int_0^t \tau^2 \cos\tau d\tau + \sin t \int_0^t \tau^2 \sin\tau d\tau \\ &= 2(t - \sin t).\end{aligned}$$

Similarly,

$$\begin{aligned}(\cos t * t^2) &= \int_0^t \cos\tau(t - \tau)^2 d\tau \\ &= t^2 \int_0^t \cos\tau d\tau - 2t \int_0^t \tau\cos\tau d\tau + \int_0^t \tau^2 \cos\tau d\tau \\ &= 2(t - \sin t).\end{aligned}$$

While confirming that the convolution operation is commutative, this example also shows that sometimes calculating $(f * g)(t)$ is simpler than calculating $(g * f)(t)$.

The convolution operation has various uses, one of the most important of which occurs in the following important theorem that expresses the relationship between the product of two Laplace transforms $F(s)$ and $G(s)$ and the convolution of their transform pairs $f(t)$ and $g(t)$.

THEOREM 7.10 **The convolution theorem** Let $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$$

or, equivalently,

$$\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = F(s)G(s).$$

Conversely,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Proof From the definition of the Laplace transform and the convolution operation, we have

$$\mathcal{L}\{(f * g)(t)\} = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right]dt.$$

Inspection of Fig. 7.14 shows that interchanging the order of integration allows the integral to be written as

$$\mathcal{L}\{(f * g)(t)\} = \int_0^\infty f(\tau)\left[\int_\tau^\infty e^{-st}g(t-\tau)dt\right]d\tau.$$

Using the second shift theorem reduces the inner integral to $e^{-s\tau}G(s)$, so that

$$\begin{aligned}\mathcal{L}\{(f * g)(t)\} &= \int_0^\infty G(s)e^{-s\tau}f(\tau)d\tau \\ &= G(s)\int_0^\infty e^{-s\tau}f(\tau)d\tau \\ &= G(s)F(s).\end{aligned}$$

The converse result follows if we reverse the argument to find the inverse Laplace transform of $F(s)G(s)$. ■

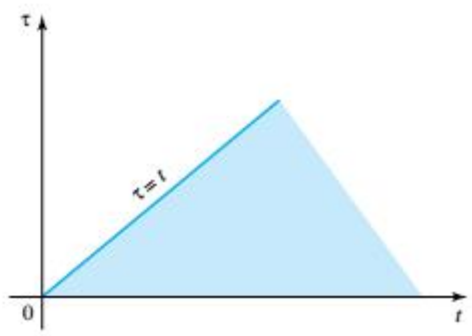


FIGURE 7.14 Region of integration for Theorem 7.10.

EXAMPLE 7.21 Use Theorem 7.10 to find (a) $\mathcal{L}\{t^2 * \cos t\}$ and (b) $\mathcal{L}^{-1}\{s/(s^2+a^2)^2\}$.

Solution

(a) $\mathcal{L}\{t^2\} = 2/s^3$ and $\mathcal{L}\{\cos t\} = s/(s^2+a^2)$, so from Theorem 7.10

$$\mathcal{L}\{t^2 * \cos t\} = \mathcal{L}\{t^2\}\,\mathcal{L}\{\cos t\} = \frac{2s}{(s^2+a^2)}.$$

(b) Writing

$$\frac{s}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)}\frac{s}{(s^2+a^2)}$$

shows that in Theorem 7.10 we may take

$$F(s) = \frac{1}{(s^2+a^2)} \quad \text{and} \quad G(s) = \frac{s}{(s^2+a^2)}.$$

So as $\mathcal{L}^{-1}\{F(s)\} = (1/a)\sin at$ and $\mathcal{L}^{-1}\{G(s)\} = \cos at$, it follows from Theorem 7.10 that

$$\begin{aligned}\mathcal{L}^{-1}\{s/(s^2+a^2)^2\} &= (1/a)(\sin at * \cos at)\\ &= \frac{1}{a}\int_0^t \sin a\tau \cos a(t-\tau)d\tau\\ &= \frac{1}{2a}t\sin at,\end{aligned}$$

in agreement with entry 11 in Table 7.1. ■

When evaluating convolution integrals of this type, instead of expanding a term such as $\cos a(t-\tau)$ and $\sin a(t-\tau)$ using integration by parts, it is often quicker to replace $\sin at$ and $\cos at$ by

$$\sin at = (e^{iat} - e^{-iat})/(2i) \quad \text{and} \quad \cos a(t-\tau) = \left(e^{i(t-\tau)} + e^{-i(t-\tau)}\right)/2$$

before performing the integrations, and again using these identities to interpret the result in terms of trigonometric functions.

EXAMPLE 7.22 Solve the initial value problem

$$y'' + 4y' + 13y = 2e^{-2t}\sin 3t \quad \text{with } y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

Solution Before we solve this initial value problem, it should be noted that the complementary function is

$$y_c(t) = e^{-2t}(C_1\cos 3t + C_2\sin 3t),$$

so the nonhomogeneous term $2e^{-2t}\sin 3t$ is contained in $y_c(t)$. It will be seen that, unlike the special cases that arise when determining a particular integral by the method of undetermined coefficients, this situation does not give rise to a special case when the solution is obtained by means of the Laplace transform.

Transforming the equation in the usual way gives

$$s^2Y(s) - s + 4(sY(s) - 1) + 13Y(s) = \frac{6}{s^2+4s+13},$$

and so

$$Y(s) = \frac{s+4}{s^2+4s+13} + \frac{6}{(s^2+4s+13)^2}.$$

Writing $s+4 = s+2+(2/3)3$ allows $Y(s)$ to be rewritten as

$$Y(s) = \frac{s+2}{(s+2)^2+3^2} + \frac{2}{3}\frac{3}{(s+2)^2+3^2} + \frac{6}{[(s+2)^2+3^2]^2}.$$

Taking the inverse Laplace transform of $Y(s)$ and using entries 13 and 14 of Table 7.1 leads to the result

$$y(t) = e^{-2t}\left[\cos 3t + \frac{2}{3}\sin 3t\right] + \mathcal{L}^{-1}\{6/[(s+2)^2+3^2]^2\}.$$

To find $\mathcal{L}^{-1}\{6/[(s+2)^2+3^2]^2\}$, we first write this as

$$\frac{6}{[(s+2)^2+3^2]^2} = \frac{2}{3}\left(\frac{3}{(s+2)^2+3^2}\right)\left(\frac{3}{(s+2)^2+3^2}\right),$$

and then, from entry 13 in Table 7.1, we find that $\mathcal{L}^{-1}\{3/[(s+2)^2+3^2]\} = e^{-2t}\sin 3t$. An application of Theorem 7.10 shows that

$$\begin{aligned}\mathcal{L}^{-1}\{6/[(s+2)^2+3^2]^2\} &= \frac{2}{3}(e^{-2t}\sin 3t * e^{-2t}\sin 3t)\\ &= \frac{2}{3}\int_0^t e^{-2\tau}\sin 3\tau e^{-2(t-\tau)}\sin 3(t-\tau)d\tau\\ &= \frac{2}{3}e^{-2t}\int_0^t \sin 3\tau \sin 3(t-\tau)d\tau\\ &= \frac{2}{3}e^{-2t}\left(\frac{1}{6}\sin 3t - \frac{1}{2}t\cos 3t\right).\end{aligned}$$

Substituting this result in the expression for $y(t)$ shows that the solution of the initial value problem is

$$y(t) = e^{-2t}\left(\cos 3t + \frac{7}{9}\sin 3t - \frac{1}{3}t\cos 3t\right), \quad \text{for } t > 0.$$

Although the previous example could have been solved by the method of undetermined coefficients, the next two examples cannot be solved in this manner. The first involves a special type of equation called an **integral equation**, and the second an **integro-differential** equation.

integral equation

An equation of the form

$$y(t) = f(t) + \lambda\int_0^t K(t,\tau)y(\tau)d\tau \tag{14}$$

is called a **Volterra integral equation**, where λ is a parameter and $K(t, \tau)$ is called the **kernel** of the integral equation. Equations of this type are often associated with the solution of initial value problems. The Laplace transform is well suited to the solution of such integral equations when the kernel $K(t,\tau)$ has a special form that depends on t and τ only through the difference $t-\tau$, because then $K(t,\tau) = K(t-\tau)$ and the integral in (14) becomes a convolution integral.

An examination of the Volterra integral equation in (14) shows it to be essentially the integral form of an initial value problem, and it relates the solution $y(t)$ at the current time t to an integral of the past history of the solution over the interval $[0, t]$.

The following is a simple example of a problem that leads to a Volterra integral equation. Determine the amount of a manufactured material contained in a store from time $t = 0$ until time t, if the only supply of material comes immediately from the manufacturer and it begins degrading exponentially with time from the moment it enters the store. Let the amount of material present at time $t = 0$ be Q and the amount present in the store at time t be $y(t)$, and suppose it degrades exponentially as e^{-kt} with $k > 0$. Then, by time t, the amount of material that entered the store at time τ but has not degraded is $e^{-k(t-\tau)}y(\tau)$. Thus the amount of material present at time t is determined by the solution of the Volterra integral equation

$$y(t) = Qe^{-kt} + \int_0^t e^{-k(t-\tau)}y(\tau)d\tau.$$

By using the method of solution explained in the next example, the solution of this problem is easily shown to be

$$y(t) = Qe^{-(k-1)t}.$$

EXAMPLE 7.23 Solve the Volterra integral equation

$$y(t) = 2e^{-t} + \int_0^t \sin(t-\tau)y(\tau)d\tau.$$

Solution The Laplace transform of the integral equation is

$$Y(s) = \frac{2}{s+1} + \mathcal{L}\int_0^t \sin(t-\tau)y(\tau)d\tau,$$

and after applying Theorem 7.10 to the last term the equation for $Y(s)$ becomes

$$Y(s) = \frac{2}{s+1} + \frac{Y(s)}{s^2+1}.$$

Solving for $Y(s)$ and expanding the result in partial fractions shows that

$$Y(s) = \frac{2(s^2+1)}{s^2(s+1)} = \frac{2}{s^2} - \frac{2}{s} + \frac{4}{s+1}.$$

Taking the inverse Laplace transform shows the solution to be

$$y(t) = 2t - 2 + 4e^{-t}, \quad \text{for } t > 0.$$

■

integro-differential equation

The next example is a differential equation of an unusual type, because the function $y(t)$ occurs not only as the dependent variable in the differential equation, but also inside a convolution integral that forms the nonhomogeneous term. Equations of this type that involve both the integral of an unknown function and its derivative are called **integro-differential equations**. These equations occur in many applications of mathematics, one of which arises in the continuum mechanics of polymers, where the dynamical response $y(t)$ of certain types of material at time t depends on a derivative of $y(t)$ and the time-weighted cumulative effect of what has happened to the material prior to time t. For obvious reasons materials of this type are called *materials with memory*.

An example of an integro-differential equation was obtained in Section 5.3(d) when considering the R–L–C circuit in Fig. 5.4, though at the time this was not recognized. When the circuit was closed, and the charge q on the capacitor was allowed to flow causing a current $i(t)$ in the circuit, the equation determining $i(t)$ was shown to be

$$L\frac{di}{dt} + Ri + \frac{q}{C} = 0.$$

To recognize that this is an integro-differential equation, we use the result that at time t we have $q = \int_0^t i(\tau)d\tau$, so the equation determining $i(t)$ becomes the integro-differential equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)d\tau.$$

In this case it was possible to reduce this to a second order constant coefficient differential equation for $i(t)$, but in other more complicated cases a reduction of this type may not be possible.

EXAMPLE 7.24 Solve the equation

$$y'' + y = \int_0^t \sin\tau\, y(t-\tau)d\tau,$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Solution Taking the Laplace transform in the usual way gives

$$s^2Y(s) - s + Y(s) = \mathcal{L}\int_0^t \sin\tau\, y(t-\tau)d\tau.$$

The last term is the Laplace transform of a convolution integral, so from Theorem 7.10 it follows that

$$\begin{aligned}\mathcal{L}\left\{\int_0^t \sin\tau\, y(t-\tau)d\tau\right\} &= \mathcal{L}\{\sin t\}\mathcal{L}\{y(t)\}\\ &= \frac{Y(s)}{s^2+1}.\end{aligned}$$

Using this result in the transformed equation, solving for $Y(s)$, and expanding the result using partial fractions gives

$$Y(s) = \frac{s^2+1}{s(s^2+2)} = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{s}{(s^2+2)}.$$

After the inverse Laplace transform is taken, the solution becomes

$$y(t) = \frac{1}{2}(1 + \cos\sqrt{2}t), \quad \text{for } t > 0.$$

■

THEOREM 7.11

transforming an integral

The transform of an integral Let $f(t)$ be a piecewise continuous function such that $|f(t)| \le Me^{kt}$ for $k > 0$ and all $t \ge 0$. Then, if $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \quad \text{for} \quad s > k,$$

and, conversely,

$$\mathcal{L}^{-1}\{F(s)/s\} = \int_0^t f(\tau)d\tau.$$

Proof The condition $|f(t)| \le Me^{kt}$ is sufficient to ensure the existence of the Laplace transform $F(s)$, so writing $h(t) = \int_0^t f(\tau)d\tau$ we have

$$|h(t)| \le \int_0^t |f(\tau)|d\tau \le M\int_0^t e^{k\tau}d\tau \le M\frac{e^{kt}}{k} \quad \text{for} \quad t \ge 0.$$

This result shows that $|h(t)|$ grows no faster than $|f(t)|$ as $t \to \infty$, so the existence of the Laplace transform $Y(s)$ ensures the existence of the Laplace transform of $h(t)$. Using the fundamental result from the calculus that $h'(t) = f(t)$ together with Theorem 7.2 means that, apart from points where $f(t)$ is discontinuous,

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{h'(t)\} = s\mathcal{L}\{h(t)\} = s\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\},$$

and so

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}.$$

The converse result follows by taking the inverse Laplace transform and the proof is complete. ■

EXAMPLE 7.25

Find (a) $\mathcal{L}\{\int_0^t \tau\cos a\tau d\tau\}$ and (b) $\mathcal{L}^{-1}\{1/[s(s^2+a^2)]\}$.

Solution **(a)** As $\mathcal{L}\{t\cos at\} = (s^2-a^2)/(s^2+a^2)^2$ for $s > 0$, an application of Theorem 7.11 shows that

$$\mathcal{L}\left\{\int_0^t \tau\cos a\tau d\tau\right\} = \frac{s^2-a^2}{s(s^2+a^2)^2} \quad \text{for } s > 0.$$

(b) We can write

$$\frac{1}{s(s^2+a^2)} = \frac{1}{s^2+a^2}\frac{1}{s}.$$

So if we set $F(s) = 1/(s^2+a^2)$, for which $f(t) = \mathcal{L}^{-1}F(s) = (1/a)\sin at$, it follows from Theorem 7.11 that

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin a\tau d\tau$$
$$= \frac{1}{a^2}(1-\cos at),$$

in agreement with entry 17 of Table 7.1. ■

THEOREM 7.12

integrating a transform

The integral of a transform Let $f(t)/t$ be piecewise continuous, defined for $t \geq 0$ and such that $|f(t)/t| \leq Me^{-kt}$ for $t \geq 0$. Then if $\mathcal{L}\{f(t)/t\} = G(s)$ for $s > k$, and $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(u)du$$

and, conversely,

$$\mathcal{L}^{-1}\{G(s)\} = \frac{-1}{t}\mathcal{L}^{-1}\{G'(s)\}.$$

Proof We have

$$G(s) = \int_0^{\infty} e^{-st}\frac{f(t)}{t} \quad \text{for} \quad s > k.$$

However, from Theorem 7.7,

$$G'(s) = \int_0^{\infty} e^{-st}(-t)\frac{f(t)}{t}dt = -\int_0^{\infty} e^{-st}f(t)dt = -F(s),$$

so after integration we have

$$\int_s^{\infty} F(u)du = -\int_s^{\infty} G'(u)du = G(s) - G(\infty)$$

To proceed further we now make use of the fact that the condition $|f(t)/t| \leq Me^{-kt}$ implies that $G(s)_{\lim s\to\infty} = 0$, showing that

$$G(s) = \mathcal{L}\{f(t)/t\} = \int_s^{\infty} F(u)du \quad \text{for } s > k.$$

The converse result follows by taking the inverse Laplace transform and using the fact that $\mathcal{L}^{-1}\{G(s)\} = f(t)/t$ together with the result $\mathcal{L}\{f(t)\} = F(s) = -G'(s)$. ■

EXAMPLE 7.26

Find

$$\text{(a) } \mathcal{L}\left\{\frac{\sin at}{t}\right\} \quad \text{and} \quad \text{(b) } \mathcal{L}^{-1}\left\{\ln\left(\frac{s+a}{s+b}\right)\right\}.$$

Solution **(a)** The function $(\sin at)/t$ is defined and finite for all $t > 0$, so Theorem 7.12 can be applied. If we use the fact that $\mathcal{L}\{\sin at\} = a/(s^2 + a^2)$, it follows from the first part of Theorem 7.12 that

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin at}{t}\right\} &= \int_s^{\infty} \frac{a}{u^2 + a^2}du \\ &= \pi/2 - \text{Arctan}\,(s/a) \\ &= \text{Arctan}\,(a/s).\end{aligned}$$

(b) If we set

$$G(s) = \ln\left(\frac{s+a}{s+b}\right),$$

differentiation gives

$$G'(s) = \frac{b-a}{(s+a)(s+b)} = \frac{1}{s+a} + \frac{1}{s+b},$$

from which we see that

$$\mathcal{L}^{-1}\{G'(s)\} = e^{-at} - e^{-bt}.$$

From the second part of Theorem 7.11 we have

$$\begin{aligned}\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\ln\left(\frac{s+a}{s+b}\right)\right\} &= \frac{-1}{t}\mathcal{L}^{-1}\{G'(s)\}\\ &= (e^{-bt} - e^{-at})/t.\end{aligned}$$

The conditions of Theorem 7.11 assert that method used to derive this result is permissible if $\mathcal{L}^{-1}\{G(s)\}$ is defined and finite for $t \geq 0$. We see from the preceding result that $\mathcal{L}^{-1}\{G(s)\}$ is defined and finite for $t > 0$ and $\lim_{t\to 0}[(e^{-bt} - e^{-at})/t] = a - b$, so the conditions of the theorem are satisfied and we have shown that

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s+a}{s+b}\right)\right\} = (e^{-bt} - e^{-at})/t.$$

The theorem that follows shows how the initial values $f(0), f'(0), \ldots,$ of a suitably differentiable function $f(t)$ can be found directly from its Laplace transform $F(s)$. An example of the use of the theorem is to be found in Section 7.3(d) when determining the Laplace transform of a function known only as the solution of a differential equation.

THEOREM 7.13

relating initial values and the transform

The initial value theorem Let $\mathcal{L}\{f(t)\} = F(s)$ be the Laplace transform of an n times differentiable function $f(t)$. Then

$$f^{(r)}(0) = \lim_{s\to\infty}\left\{s^{r+1}F(s) - s^r f(0) - s^{r-1}f'(0) - \cdots - sf^{(r-1)}(0)\right\},\qquad r = 0, 1, \ldots, n.$$

In particular,

$$f(0) = \lim_{s\to\infty}\{sF(s)\},\qquad f'(0) = \lim_{s\to\infty}\{s^2F(s) - sf(0)\}$$
$$f''(0) = \lim_{s\to\infty}\{s^3F(s) - s^2f(0) - sf'(0)\}.$$

Proof The theorem follows directly from Theorem 7.3 by first replacing n by $r+1$ and rewriting the result as

$$f^{(r)}(0) = s^{r+1}F(s) - s^r f(0) - \cdots - sf^{(r-1)}(0) - \mathcal{L}\left\{f^{(r+1)}(t)\right\}.$$

Then, provided $f^{(r+1)}(t)$ satisfies the sufficiency condition for the existence of a Laplace transform given in (3), it follows that for some $M > 0$ and $k > 0$

$$\mathcal{L}\left\{f^{(r+1)}(t)\right\} < M/(s-k)\quad\text{for}\quad s > k\quad\text{and}\quad r = 0, 1, \ldots, n.$$

As a result,

$$\lim_{s\to\infty}\left\{f^{(r+1)}(t)\right\} = 0,$$

and the theorem is proved. ■

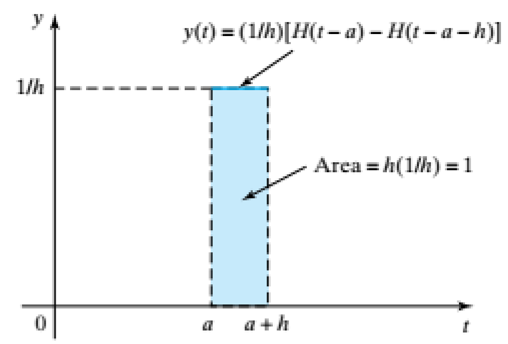


FIGURE 7.15 $\delta(t-a) = \lim_{h\to 0} y(t)$.

EXAMPLE 7.27

Given that $F(s) = 2as/(s^2+a^2)^2$, use Theorem 7.13 to find $f(0)$, $f'(0)$, and $f''(0)$. Use $f(t) = \mathcal{L}^{-1}\{F(s)\} = t\sin at$ to confirm the results by direct differentiation.

Solution From Theorem 7.13

$$f(0) = \lim_{s\to\infty}\{sF(s)\} = \lim_{s\to\infty}\frac{2as^2}{(s^2+a^2)^2} = 0,$$

$$f'(0) = \lim_{s\to\infty}\{s^2F(s) - sf(0)\} = \lim_{s\to\infty}\frac{2as^3}{(s^2+a^2)^2} = 0,$$

$$f''(0) = \lim_{s\to\infty}\{s^3F(s) - s^2f(0) - sf'(0)\} = \lim_{s\to\infty}\frac{2as^4}{(s^2+a^2)^2} = 2a.$$

These results are easily confirmed by differentiation of $f(t) = t\sin at$. ■

The last operational property to be considered concerns the **Dirac delta function**, usually abbreviated to the **delta function** and sometimes called the **unit impulse function**. The Dirac delta function, named after the Oxford University Nobel laureate mathematical physicist P. A. M. Dirac and denoted by $\delta(t-a)$, is actually a limiting mathematical *operation*, and not a function as its name implies. For our purposes the delta function can be considered to be the limit of a rectangular "pulse" of height h and width $1/h$ in the limit as $h\to\infty$. Thus the area of the graph representing the pulse remains constant at 1 as $h\to\infty$, while its height increases to infinity and its width decreases to zero. The graphical representation of such a pulse $f(t) = \delta(t-a)$ located at $t = a$, before proceeding to the limit, is shown in Fig. 7.15.

We adopt the following definition of the delta function in terms of the unit step function.

The delta function

the delta or impulse function

The **delta function** located at $t = a$ and denoted by $\delta(t-a)$ is defined as the limit

$$\delta(t-a) = \lim_{h\to 0}\frac{1}{h}[H(t-a) - H(t-a-h)].$$

The operational property of the delta function, usually called its **filtering property** and sometimes its **sifting property**, is represented by the following theorem.

THEOREM 7.14

a useful property of the delta function

Filtering property of the delta function Let $f(t)$ be defined and integrable over all intervals contained within $0 \leq t < \infty$, and let it be continuous in a neighborhood of a. Then for $a \geq 0$

$$\int_0^{\infty} f(t)\delta(t-a)dt = f(a).$$

Proof From the definition of the delta function,

$$\int_0^{\infty} f(t)\delta(t-a)dt = \lim_{h\to 0}\int_a^{a+h} \frac{f(t)}{h}dt,$$

so applying the mean value theorem for integrals we have

$$\int_0^{\infty} f(t)\delta(t-a)dt = \lim_{h\to 0}\left[h\left(\frac{1}{h}\right)f(t_h)\right],$$

where $a < t_h < a+h$. In the limit as $h \to 0$ the variable $t_h \to a$, showing that

$$\int_0^{\infty} f(t)\delta(t-a)dt = f(a),$$

and the theorem is proved. ■

Consideration of the definition of the delta function suggests that, in a sense, $\delta(t-a)$ is the derivative of the unit step function $H(t-a)$, though the justification of this conjecture requires arguments involving **generalized functions** that are beyond the scope of this account.

In mechanical problems the delta function is used to represent an *impulse*, defined as the integral of a large force applied locally for a very short time. The delta function has many other applications, such as the distribution of point masses along a supporting beam, whereas in electrical systems it can be used to represent the brief application of a very large voltage, or the sudden discharge of energy contained in a capacitor.

A purely formal derivation of the Laplace transform of the delta function proceeds as follows. By definition,

$$\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\delta(t-a)dt.$$

An application of the filtering property of Theorem 7.14 reduces this to

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}. \tag{15}$$

As a special case we have

$$\mathcal{L}\{\delta(t)\} = 1. \tag{16}$$

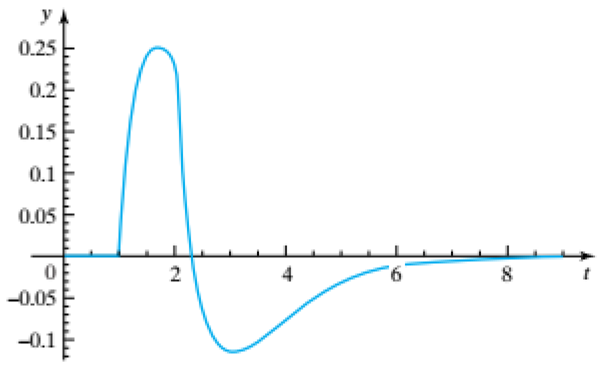


FIGURE 7.16 The solution $y(t)$ as a function of the time t.

EXAMPLE 7.28 Solve the initial value problem

$$y'' + 3y' + 2y = \delta(t-1) - \delta(t-2) \quad \text{with } y(0) = y'(0) = 0.$$

Solution Taking the Laplace transform in the usual way and using result (15) gives

$$(s^2 + 3s + 2)Y(s) = e^{-s} - e^{-2s},$$

and so

$$Y(s) = \frac{e^{-s} - e^{-2s}}{s^2 + 3s + 2} = \frac{e^{-s} - e^{-2s}}{s+1} - \frac{e^{-s} - e^{-2s}}{s+2}.$$

Inverting the transform using Theorem 7.6 (the t-shift theorem) shows that

$$y(t) = H(t-1)[e^{1-t} - e^{2-2t}] - H(t-2)[e^{2-t} - e^{4-2t}].$$

A graph of this solution is given in Fig. 7.16. The graph shows that a physical system represented by the given differential equation subject to the equilibrium initial conditions $y(0) = y'(0) = 0$ is at rest until it is excited by the delta function at time $t = 1$ and then, after peaking just before $t = 2$, it is excited in the opposite sense by the delta function at time $t = 2$, after which the solution decays to zero as t increases, corresponding to the system returning to rest.

The Laplace transform is also discussed in references [3.4], [3.8], [3.9], [3.17], and [3.20]; tables of Laplace transform pairs are to be found in references [G.1], [G.3], [3.11], and [3.14]. An advanced account of the Laplace transform is to be found in reference [3.19]. ■

PAUL ADRIEN DIRAC (1902–1984)
An English mathematical physicist who introduced the delta function in a fundamental paper on quantum mechanics presented to the Royal Society of London in 1927. Together with the German physicist **Erwin Schrodinger** he shared the Nobel Prize for physics because of contributions made to quantum mechanics.

Summary

This section has been concerned with what are known as the operational properties of the Laplace transform. These are general properties of the transform itself that can be applied to any function $f(t)$ that possesses a Laplace transform, or to any function $F(s)$ that is the Laplace transform of a function $f(t)$. It will be seen later that these properties can be used to extend the table of Laplace transforms given at the end of Section 7.1, and when using the Laplace transform to solve differential equations.

EXERCISES 7.2

Exercises involving the transformation of derivatives

1. Prove that $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$.
2. Prove that $\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$.
3. Given that $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$, find $\mathcal{L}\{f'''(t)\}$.
4. Given that $f(0) = 0$, $f'(0) = 2$, $f''(0) = 2$, $f'''(0) = -4$, find $\mathcal{L}\{f^{(4)}(t)\}$.
5. Given that $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2 \\ t = 0, & t \ge \pi/2 \end{cases}$, find $\mathcal{L}\{f(t)\}$.
6. Given that $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2 \\ 1, & t \ge \pi/2 \end{cases}$, find $\mathcal{L}\{f(t)\}$.
7. Solve $y'' - 3y' + 2y = \cos t$, with $y(0) = 1$, $y'(0) = -1$.
8. Solve $y'' + 5y' + 4y = \exp(-t)$, with $y(0) = 1$, $y'(0) = 0$.
9. Solve $y'' + 8y' - 9y = t$, with $y(0) = 2$, $y'(0) = 1$.
10. Solve $y'' + 5y' + 6y = 1 + t^2$, with $y(0) = 0$, $y'(0) = 0$.

Exercises involving the first shift theorem (s-shift)

11. Find $\mathcal{L}\{(2 + t^3)e^{-2t}\}$.
12. Find $\mathcal{L}\{e^{-3t}\cos 2t\}$.
13. Find $\mathcal{L}\{e^{-t}t\sin 2t\}$.
14. Find $\mathcal{L}\{(1 + t^2)e^{-4t}\}$.
15. Find $\mathcal{L}\{e^{2t}\sin 3t\}$.
16. Find $\mathcal{L}\{e^{-4t}\sinh 3t\}$.
17. Find $\mathcal{L}^{-1}\{1/(s^2 - 4s + 13)\}$.
18. Find $\mathcal{L}^{-1}\{s/(s^2 + 4s + 13)\}$.
19. Find $\mathcal{L}^{-1}\{(1 - 3s)/(s^2 + 2s + 5)\}$.
20. Find $\mathcal{L}^{-1}\{1/[s(s^2 - 2s + 5)]\}$.
21. Find $\mathcal{L}^{-1}\{s/[(s + 1)(s^2 - 4s + 13)]\}$.
22. Find $\mathcal{L}^{-1}\{3/(s^2 + 6s + 25)\}$.
23. Find $\mathcal{L}^{-1}\{3(s^2 + 4)/[s(s^2 + 4s + 8)]\}$.
24. Find $\mathcal{L}^{-1}\{2/[(s + 3)^2(s^2 + 8s + 20)]\}$.

Exercises involving graphing functions with a t-shift

25. Sketch $f(t) = H(t - 2)(1 + t)$.
26. Sketch $f(t) = H(t - \pi)\sin t + H(t - 2\pi)$.
27. Sketch $f(t) = [H(t - \pi) - H(t - 2\pi)]\cos t$.
28. Sketch $f(t) = \sum_{r=0}^{4} H(t - r)$.
29. Sketch $f(t) = H(t - \pi)\cos(t - \pi)$.
30. Sketch $f(t) = H(t - 1)(t - 1)^2$.
31. Sketch $f(t) = [H(t - 1) - H(t - 2)](t - 1)^2$.
32. Sketch $f(t) = H(t - \pi/2)\cos(t - \pi/2)$.

Exercises involving the second shift theorem (t-shift)

33. Find $\mathcal{L}\{H(t - 3)(t - 3)^3\}$.
34. Find $\mathcal{L}\{H(t - 1)\sin(t - 1)\}$.
35. Find $\mathcal{L}\{H(t - 3\pi/2)\sin 2(t - 3\pi/2)\}$.
36. Find $\mathcal{L}\{H(t - \pi/2)(t - \pi/2)^3 - H(t - 3\pi/2) \times (t - 3\pi/2)^3\}$.
37. Find $\mathcal{L}\{H(t - 4)\sinh 3(t - 4)\}$.
38. Find $\mathcal{L}\{H(t - 1)(t - 1)\sin(t - 1)\}$.
39. Find $\mathcal{L}^{-1}\{s\,e^{-2s}/(s^2 + 4)\}$.
40. Find $\mathcal{L}^{-1}\{e^{-\pi s/3}/(s^2 + 9)\}$.
41. Find $\mathcal{L}^{-1}\{e^{-\pi s/2}(s + 1)/(s^2 + 4s + 5)\}$.
42. Find $\mathcal{L}^{-1}\{e^{-2s}(s^2 + s + 1)/[s(s + 2)^2]\}$.
43. Find $\mathcal{L}^{-1}\{e^{-4s}(s + 3)/(s^2 + 4s + 13)\}$.
44. Find $\mathcal{L}^{-1}\{e^{-3s}s^2/[s(s^2 + 4s + 8)]\}$.
45. Solve $y'' + 5y' + 6y = H(t - \pi)\cos(t - \pi)$, with $y(0) = 1$, $y'(0) = 0$.
46. Solve $y'' - 5y' + 6y = tH(t - 1)$, with $y(0) = 0$, $y'(0) = 0$.
47. Solve $y'' - 5y' + 6y = 1 + tH(t - 2)$, with $y(0) = 0$, $y'(0) = 1$.
48. Solve $y'' - 6y' + 10y = tH(t - 3)$, with $y(0) = 1$, $y'(0) = 1$.
49. Solve $y'' + 2y' + 10y = e^{-t}H(t - 1)$, with $y(0) = -1$, $y'(0) = 0$.
50. Solve $y'' - y' - 2y = e^{-t}H(t - 1)$, with $y(0) = 1$, $y'(0) = 0$.

Exercises involving differentiation of transforms

51. Find $\mathcal{L}\{t^2e^{3t}\sin t\}$.
52. Find $\mathcal{L}\{te^{-t}\sin 4t\}$.
53. Find $\mathcal{L}\{t^3e^{2t}\sin 2t\}$.
54. Find $\mathcal{L}\{t^2e^{3t}\cos 2t\}$.

Exercises involving scaling

55. If $\mathcal{L}\{f(t)\} = e^{-3s}(s^2 - 1)/(s^4 - a^4)$, find $\mathcal{L}\{f(2t)\}$.
56. If $\mathcal{L}\{f(t)\} = (s + 1)(s^2 + 2)/(s^2 + 4)^2$, find $\mathcal{L}\{f(3t)\}$.
57. If $\mathcal{L}\{f(t)\} = 1/[s^2(s^2 + 4)]$, find $\mathcal{L}\{f(t/3)\}$.
58. If $\mathcal{L}\{f(t)\} = (s^2 - 4)/[(s^2 + 4)^2]$, find $\mathcal{L}\{f(t/2)\}$.

Exercises involving the Laplace transform of periodic functions

In Exercises 59 through 66 find the Laplace transform of the periodic function $f(t)$.

59.

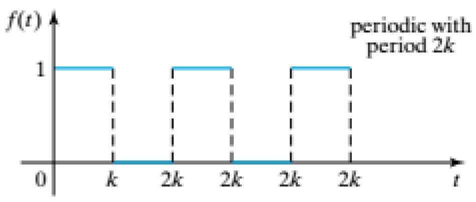


FIGURE 7.17

60.

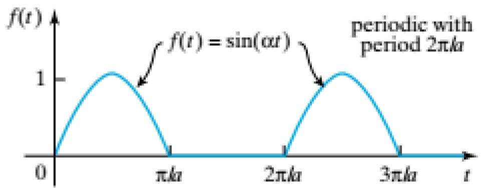


FIGURE 7.18

61.

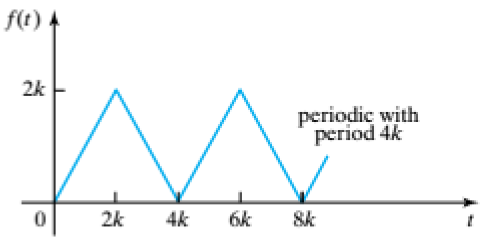


FIGURE 7.19

62.

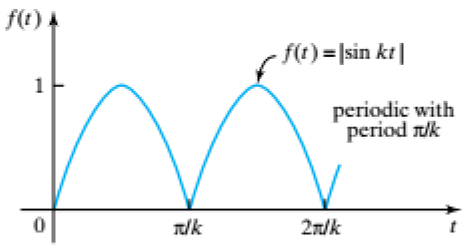


FIGURE 7.20

63.

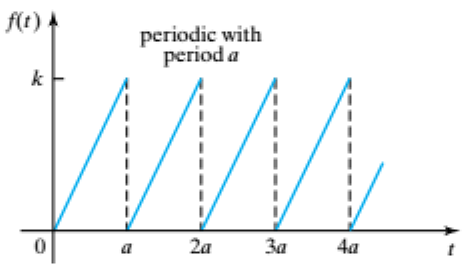


FIGURE 7.21

64.

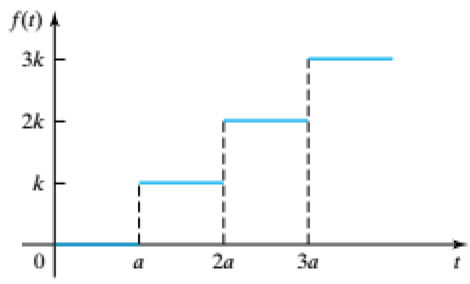


FIGURE 7.22

65.

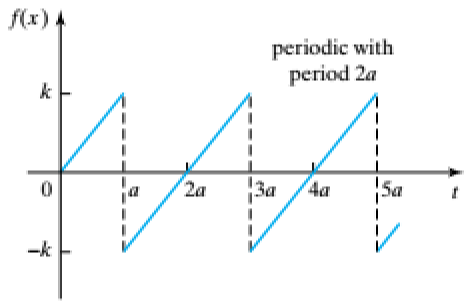


FIGURE 7.23

66.

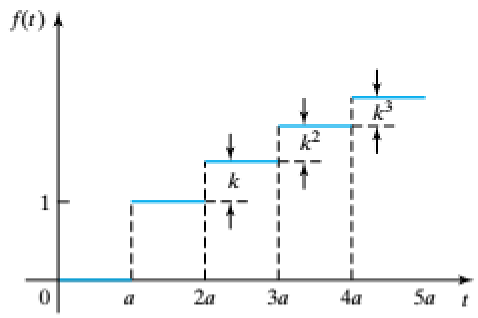


FIGURE 7.24

Exercises involving the convolution operation

67. Find $(e^{-t} * e^{-2t})$.

68. Find $(t * \sin t)$.

69. Find $(t^2 * \sin t)$.

70. Find $(t * e^{-t})$.

71. Find $(\cos t * \cos t)$.

72. Find $(\sin 2t * \sin 2t)$.

Exercises involving the convolution theorem

73. Find $\mathcal{L}\{t * e^{-2t}\}$.

74. Find $\mathcal{L}\{2t * \cos 2t\}$.

75. Find $\mathcal{L}\{e^{-t} \sin t * t\}$.

76. Find $\mathcal{L}\{e^{-2t} \cos t * e^{t}\}$.

77. Find $\mathcal{L}^{-1}\{1/[s^2(s^2+4)]\}$.

78. Find $\mathcal{L}^{-1}\{1/(s^2-9)^2\}$.

79. Find $\mathcal{L}^{-1}\{s^2/(s^2-1)^2\}$.

80. Find $\mathcal{L}^{-1}\{s/(s^2-4)^2\}$.

Exercises involving integral equations

81. Solve $y(t) = \sin t + \int_0^t \sin(t - \tau)y(\tau)d\tau$.

82. Solve $y(t) = \cos t + \int_0^t \sin[2(t - \tau)]y(\tau)d\tau$.

83. Solve $y(t) = t^2 + \int_0^t \cos(t - \tau)y(\tau)d\tau$.

84. Solve $y(t) = e^{-2t} + \int_0^t \cos(t - \tau)y(\tau)d\tau$.

Exercises involving integro-differential equations

85. Solve $y' + 4y = 4\int_0^t \sin\tau\, y(t - \tau)d\tau$, with $y(0) = 1$.

86. Solve $y' + y = \int_0^t e^{-2\tau} y(t - \tau)d\tau$, with $y(0) = 3$.

87. Solve $y'' - y = \int_0^t \sinh\tau\, y(t - \tau)d\tau$, with $y(0) = 1$, $y'(0) = 0$.

88. Solve $y'' - 4y = 2\int_0^t \sinh 2\tau\, y(t - \tau)d\tau$, with $y(0) = 1$, $y'(0) = 0$.

Exercises involving the transform of an integral

89. Find $\mathcal{L}\left\{\int_0^t \tau^2 \sin 2\tau\, d\tau\right\}$.

90. Find $\mathcal{L}\left\{\int_0^t e^{2\tau} \cos\tau\, d\tau\right\}$.

91. Find $\mathcal{L}^{-1}\{1/(s^2 + a^2)^2\}$.

92. Find $\mathcal{L}^{-1}\{s/(s^2 + a^2)\}$.

Exercises involving an integral of a transform

93. Find $\mathcal{L}\left\{\frac{\sinh 2t}{t}\right\}$.

94. Find $\mathcal{L}\left\{\frac{1 - \cos 3t}{t}\right\}$.

95. Find $\mathcal{L}^{-1}\left\{\ln\left(\frac{s^2 - a^2}{s^2}\right)\right\}$.

96. Find $\mathcal{L}^{-1}\left\{\ln\left(\frac{s^2 + a^2}{s^2}\right)\right\}$.

Exercises involving the initial value theorem

In Exercises 97 through 100 use the initial value theorem to find $f(0)$, $f'(0)$, and $f''(0)$ from $F(s)$, and verify the result by differentiation of $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

97. $F(s) = (s^2 + 6)/\{s(s^2 + 9)\}$.

98. $F(s) = s/(s^2 + 6s + 9)$.

99. $F(s) = (s - 1)/(s^2 - 4s + 4)$.

100. $F(s) = (2s^2 + s - 12)/\{s(s + 2)(s + 3)\}$.

Exercises involving the delta function

101. Evaluate $\int_0^\infty \left(\frac{1 - 3\sin^2 t}{t}\right)\delta(t - \pi/2)dt$.

102. Evaluate $\int_0^4 \sin^2 t\,\delta(t - 2\pi)dt$.

103. Evaluate $\int_0^\infty \sum_{n=1}^{3}\left\{\left(\frac{\sin nt}{t}\right)\delta\left[t - (2n + 1)\frac{\pi}{2}\right]\right\}dt$.

104. Evaluate $\int_0^\infty \{[H(t - 1) - H(t - 2)]t + \cos(t - 3\pi)\delta(t - 3\pi)\}dt$.

105. Solve $y'' + 9y = 1 + \delta(t - 1)$, with $y(0) = 0$, $y'(0) = 0$.

106. Solve $y'' + 4y' + 4y = \delta(t - 1)$, with $y(0) = 1$, $y'(0) = 1$.

107. Solve $y'' + 2y' + y = \sin t + \delta(t - \pi)$, with $y(0) = y'(0) = 0$.

108. Solve $y'' - 4y' + 3y = e^{-t} + 3\delta(t - 2)$, with $y(0) = y'(0) = 0$.

109. Solve $y'' + 4y = 1 - H(t - 1) + \delta(t - 2)$, with $y(0) = 1$, $y'(0) = 0$.

110. Solve $y'' + 3y' + 2y = \delta(t - 1)$, with $y(0) = 0$, $y'(0) = 1$.

7.3 Systems of Equations and Applications of the Laplace Transform

(a) Solution of Systems of Linear First Order Equations by the Laplace Transform

The Laplace transform can be used to solve initial value problems for systems of linear first order differential equations by introducing the Laplace transform of each dependent variable that is involved, solving the resulting algebraic equations for each transformed dependent variable, and then inverting the results.

As a system of linear higher order differential equations can always be reduced to a system of first order equations by introducing higher order derivatives as new dependent variables, the solution of a system of linear first order equations can be considered to be the most general case.

solving systems of equations

The example that follows, involving two simultaneous first order equations, illustrates the approach to be used in all cases, but by restricting the number of equations and using simple nonhomogeneous terms (forcing functions) the algebra is kept to a minimum.

EXAMPLE 7.29 Solve the initial value problem

$$x' - 2x + y = \sin t$$
$$y' + 2x - y = 1,$$

with $x(0) = 1,\ y(0) = -1$.

Solution We define the transforms of the dependent variables $x(t)$ and $y(t)$ to be

$$\mathcal{L}\{x(t)\} = X(s), \quad \mathcal{L}\{y(t)\} = Y(s).$$

Transforming the system of equations in the usual way leads to the following system of linear algebraic equations for $X(s)$ and $Y(s)$:

$$sX(s) - 1 - 2X(s) + Y(s) = 1/(s^2+1)$$
$$sY(s) + 1 + 2X(s) - Y(s) = 1/s.$$

Solving these for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{(s-1)(s^3+s^2+2s+1)}{s^2(s-3)(s^2+1)} \quad \text{and} \quad Y(s) = \frac{-(s^4-s^3+3s^2+s+2)}{s^2(s-3)(s^2+1)}.$$

Expressing these results in terms of partial fractions, we find that

$$X(s) = \frac{4}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} - \frac{1}{5}\frac{1}{s^2+1} - \frac{2}{5}\frac{s}{s^2+1} + \frac{43}{45}\frac{1}{s-3}$$

and

$$Y(s) = \frac{5}{9}\frac{1}{s} + \frac{2}{3}\frac{1}{s^2} + \frac{1}{5}\frac{1}{s^2+1} - \frac{3}{5}\frac{s}{s^2+1} - \frac{43}{45}\frac{1}{s-3}.$$

Finally, taking the inverse transform gives the solution

$$x(t) = \frac{4}{9} + \frac{1}{3}t - \frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{43}{45}e^{3t}$$

and

$$y(t) = \frac{5}{9} + \frac{2}{3}t + \frac{1}{5}\sin t - \frac{3}{5}\cos t - \frac{43}{45}e^{3t} \quad \text{for } t > 0.$$ ■

This method can be used for any number of simultaneous linear differential equations, though the complexity of both the algebraic manipulation and the associated inversion problem increases rapidly when more than two equations are involved.

A typical example of the way systems of first order equations arise in practice is provided by considering a chemical reaction that converts a raw chemical into an end product, via several intermediate reactions. The simplest situation involves chemical reactions that are irreversible, so that once a product has been produced the chemical process cannot be reversed, causing the new product to revert to a previous one.

Let us derive the system of equations governing such a process when three intermediate reactions are involved, each of which is irreversible, with each reaction proceeding at a rate that is proportional to the amount of material to be converted from one stage to the next. Denote the raw chemical by A and the end product by E, with the intermediate products denoted by B, C, and D, and let the reaction rates (the constants of proportionality) from $A \to B$, $B \to C$, $C \to D$, and $D \to E$ be k_1, k_2, k_3, and k_4, respectively. Then if the amounts of chemicals A, B, C, D, and E present at time t are x, y, u, v, and w, the production and removal of the chemical products involved is described as follows.

Reaction	Reaction Rate of Removal	Reaction Rate of Production
$A \to B$	$\left(\frac{dx}{dt}\right)_{A\to B} = -k_1 x$	$\left(\frac{dy}{dt}\right)_{A\to B} = k_1 x$
$B \to C$	$\left(\frac{dy}{dt}\right)_{B\to C} = -k_2 y$	$\left(\frac{du}{dt}\right)_{B\to C} = k_2 y$
$C \to D$	$\left(\frac{du}{dt}\right)_{C\to D} = -k_3 u$	$\left(\frac{dv}{dt}\right)_{C\to D} = k_3 u$
$D \to E$	$\left(\frac{dv}{dt}\right)_{D\to E} = -k_4 v$	$\left(\frac{dw}{dt}\right)_{D\to E} = k_4 v$

Combining these results gives

$$\begin{aligned}
\frac{dx}{dt} &= \left(\frac{dx}{dt}\right)_{A\to B} = -k_1 x \\
\frac{dy}{dt} &= \left(\frac{dy}{dt}\right)_{A\to B} + \left(\frac{dy}{dt}\right)_{B\to C} = k_1 x - k_2 y \\
\frac{du}{dt} &= \left(\frac{du}{dt}\right)_{B\to C} + \left(\frac{du}{dt}\right)_{C\to D} = k_2 y - k_3 u \\
\frac{dv}{dt} &= \left(\frac{dv}{dt}\right)_{C\to D} + \left(\frac{dv}{dt}\right)_{D\to E} = k_3 u - k_4 v.
\end{aligned}$$

If the amount of raw material A present at the start is Q, the initial conditions for the system are seen to be

$$x(0) = Q, \quad y(0) = 0, \quad u(0) = 0, \quad v(0) = 0, \quad \text{and} \quad w(0) = 0.$$

Provided no additional by-products are produced during the reactions, it follows from the conservation of mass that $x + y + u + v + w = Q$, and so

$$w = Q - x - y - u - v.$$

Taking the Laplace transform of this system of first order linear equations and using the stated initial conditions leads to the transformed system

$$\begin{aligned} sX(s) + k_1X(s) &= Q \\ sY(s) - k_1X(s) + k_2Y(s) &= 0 \\ sU(s) - k_2Y(s) + k_3U(s) &= 0 \\ sV(s) - k_3U(s) + k_4V(s) &= 0, \end{aligned}$$

where $\mathcal{L}\{x(t)\} = X(s)$, $\mathcal{L}\{y(t)\} = Y(s)$, $\mathcal{L}\{u(t)\} = U(s)$, and $\mathcal{L}\{v(t)\} = V(s)$.

Solving for the Laplace transforms, we have

$$X(s) = \frac{Q}{s+k_1}, \quad Y(s) = \frac{k_1Q}{(s+k_1)(s+k_2)}, \quad U(s) = \frac{k_1k_2Q}{(s+k_1)(s+k_2)(s+k_3)},$$

and

$$V(s) = \frac{k_1k_2k_3Q}{(s+k_1)(s+k_2)(s+k_3)(s+k_4)}.$$

After expressing these Laplace transforms in terms of partial fractions the required solutions are seen to be

$$x(t) = Qe^{-k_1t}, \quad y(t) = \frac{k_1Q}{k_1-k_2}(e^{-k_1t} - e^{-k_2t})$$

and

$$u(t) = k_1k_2Q\left(\frac{1}{(k_2-k_1)(k_3-k_1)}e^{-k_1t} + \frac{1}{(k_1-k_2)(k_3-k_2)}e^{-k_2t} + \frac{1}{(k_1-k_3)(k_2-k_3)}e^{-k_3t}\right)$$

with $v(t)$ similarly defined. The amount of the end product $w(t)$ produced at time t follows from

$$w(t) = Q - x(t) - y(t) - u(t) - v(t).$$

solving systems of equations in matrix form

We now outline a matrix method of solution of initial value problems for systems of linear first order differential equations, of which Example 7.29 is a typical case. Let us consider the system

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t), \tag{17}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & & \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}, \quad \mathbf{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \cdot \\ \cdot \\ b_n(t) \end{bmatrix},$$

subject to the initial conditions $x_1(0) = x_1, x_2(0) = x_2, \ldots, x_n(0) = x_n$.

Define $\mathcal{L}\{x_1(t)\} = X_1(s)$, $\mathcal{L}\{x_2(t)\} = X_2(s) \ldots$, $\mathcal{L}\{x_n(t)\} = X_n(s)$, $\mathcal{L}\{b_1(t)\} = B_1(s)$, $\mathcal{L}\{b_2(t)\} = B_2(s), \ldots, \mathcal{L}\{b_n(t)\} = B_n(s)$, and set

$$\mathbf{Z}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \cdot \\ \cdot \\ X_n(s) \end{bmatrix}, \quad \mathbf{c}(s) = \begin{bmatrix} B_1(s) \\ B_2(s) \\ \cdot \\ \cdot \\ B_n(s) \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}.$$

Then taking the Laplace transform of (17) and using the result $\mathcal{L}\{x_r'(t)\} = sX(s) - x_r$, for $r = 1, 2, \ldots, n$, we arrive at the system

$$s\mathbf{Z}(s) - \mathbf{v} = \mathbf{A}\mathbf{Z}(s) + \mathbf{c}(s)$$

or, equivalently,

$$(s\mathbf{I} - \mathbf{A})\mathbf{Z}(s) = \mathbf{v} + \mathbf{c}(s),$$

where $\mathbf{I}$ is the $n \times n$ unit matrix. Premultiplying this last result by $(s\mathbf{I} - \mathbf{A})^{-1}$ gives

$$\mathbf{Z}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}[\mathbf{v} + \mathbf{c}(s)]. \tag{18}$$

Finally, taking the inverse Laplace transform of (18) we obtain the solution $\mathbf{x}(t)$ of the initial value problem in the form

$$\mathbf{x}(t) = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}[\mathbf{v} + \mathbf{c}(s)]\}. \tag{19}$$

EXAMPLE 7.30 Solve the initial value problem of Example 7.29 by using result (19).

Solution Making the necessary identifications we have

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{c}(s) = \begin{bmatrix} 1/(s^2+1) \\ 1/s \end{bmatrix},$$

so (18) becomes

$$\mathbf{Z}(s) = \left[s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\right]^{-1}\left[\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1/(s^2+1) \\ 1/s \end{bmatrix}\right],$$

or

$$\mathbf{Z}(s) = \begin{bmatrix} s-2 & 1 \\ 2 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} (s^2+2)/(s^2+1) \\ (1-s)/s \end{bmatrix}.$$

The inverse of the first matrix in this product is

$$\begin{bmatrix} s-2 & 1 \\ 2 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{s-1}{s(s-3)} & \dfrac{-1}{s(s-3)} \\ \dfrac{-2}{s(s-3)} & \dfrac{s-2}{s(s-3)} \end{bmatrix},$$

so

$$\mathbf{Z}(s) = \begin{bmatrix} \dfrac{s-1}{s(s-3)} & \dfrac{-1}{s(s-3)} \\ \dfrac{-2}{s(s-3)} & \dfrac{s-2}{s(s-3)} \end{bmatrix}\begin{bmatrix} \dfrac{s^2+2}{s^2+1} \\ \dfrac{1-s}{s} \end{bmatrix}.$$

After forming the matrix product this becomes

$$\mathbf{Z}(s) = \begin{bmatrix} \dfrac{(s-1)(s^3+s^2+2s+1)}{s^2(s-3)(s^2+1)} \\ \dfrac{-(s^4-s^3+3s^2+s+2)}{s^2(s-3)(s^2+1)} \end{bmatrix}.$$

The inverse transforms involved are, of course, the same as the ones in Example 7.29, so, as would be expected, the solution is the same as before, apart from a change of notation involving the replacement of $x(t)$ and $y(t)$ by $x_1(t)$ and $x_2(t)$ giving

$$x_1(t) = \frac{4}{9} + \frac{1}{3}t - \frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{43}{45}e^{3t}$$

and

$$x_2(t) = \frac{5}{9} + \frac{2}{3}t + \frac{1}{5}\sin t - \frac{3}{5}\cos t - \frac{43}{45}e^{3t} \quad \text{for } t > 0.$$ ■

(b) Determination of $e^{t\mathbf{A}}$ by Means of the Laplace Transform

The matrix solution of system (17) given in (19) has an interesting and useful consequence, because it provides a different and efficient way of finding the matrix exponential $e^{t\mathbf{A}}$. To see how this comes about, notice that from equation (114) in Section 6.10(c) the solution of the homogeneous system of equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{20}$$

subject to the initial condition $\mathbf{x}(0) = \mathbf{v}$, can be written

$$\mathbf{x}(t) = e^{t\mathbf{A}}\,\mathbf{v}. \tag{21}$$

Setting $\mathbf{c}(s) = \mathbf{0}$ (corresponding to $\mathbf{b}(t) = \mathbf{0}$) reduces solution (19) to

$$\mathbf{x}(t) = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}\mathbf{v}, \tag{22}$$

so comparison of (21) and (22) shows that

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}. \tag{23}$$

We have established the following theorem.

THEOREM 7.15

finding the matrix exponential by the Laplace transform

Determination of $e^{t\mathbf{A}}$ by means of the Laplace transform Let $\mathbf{A}$ be a real $n \times n$ matrix with constant elements. Then the exponential matrix

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}.$$ ■

The following examples show how Theorem 7.15 determines $e^{t\mathbf{A}}$ in the cases when $\mathbf{A}$ is diagonalizable with real eigenvalues, when it is diagonalizable with complex conjugate eigenvalues, and also when it is not diagonalizable.

EXAMPLE 7.31

Use Theorem 7.15 to find $e^{t\mathbf{A}}$ when

$$\mathbf{A} = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}.$$

Solution Matrix **A** has the distinct eigenvalues 1 and 2, and so is diagonalizable.

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix}$$

so

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{s-5}{s^2-3s+2} & \dfrac{6}{s^2-3s+2} \\ \dfrac{-2}{s^2-3s+2} & \dfrac{s+2}{s^2-3s+2} \end{bmatrix}.$$

Expressing each element of this matrix in terms of partial fractions and taking the inverse Laplace transform gives

$$e^{t\mathrm{A}} = \begin{bmatrix} 4e^t - 3e^{2t} & -6e^t + 6e^{2t} \\ 2e^t - 2e^{2t} & -3e^t + 4e^{2t} \end{bmatrix},$$

in agreement with the result in Example 6.33. ■

EXAMPLE 7.32 Use Theorem 7.14 to find $e^{t\mathrm{A}}$ when

$$\mathbf{A} = \begin{bmatrix} -3 & -4 \\ 2 & 1 \end{bmatrix}.$$

Solution Matrix **A** has the complex conjugate eigenvalues $-1 \pm 2i$.

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+3 & 4 \\ -2 & s-1 \end{bmatrix},$$

so

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{s-1}{s^2+2s+5} & \dfrac{-4}{s^2+2s+5} \\ \dfrac{2}{s^2+2s+5} & \dfrac{s+3}{s^2+2s+5} \end{bmatrix}.$$

Expressing each element of this matrix in terms of partial fractions and taking the inverse Laplace transform gives

$$e^{t\mathrm{A}} = \begin{bmatrix} e^{-t}(\cos 2t - \sin 2t) & -2e^{-t}\sin 2t \\ e^{-t}\sin 2t & e^{-t}(\cos 2t + \sin 2t) \end{bmatrix},$$

in agreement with the result of Example 6.34. ■

EXAMPLE 7.33 Use Theorem 7.14 to find $e^{t\mathrm{A}}$ when

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}.$$

Solution Matrix **A** has the repeated eigenvalue 4 and is not diagonalizable.

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s-4 & -1 \\ 0 & s-4 \end{bmatrix},$$

so

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{1}{s-4} & \dfrac{1}{(s-4)^2} \\ 0 & \dfrac{1}{s-4} \end{bmatrix}.$$

Taking the inverse of the elements of this matrix, we find that

$$e^{t\mathrm{A}} = \begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix},$$

in agreement with the result of Example 6.35. ■

(c) The Weighting Function

To introduce the concept of a *weighting function*, which has important engineering applications, we consider the differential equation

$$a_0\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = f(t), \tag{24}$$

weighting function and its uses

subject to the initial conditions $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$. We shall denote by $w(t)$ the solution of equation (24) when $f(t) = \delta(t)$, and call it the **weighting function** associated with the equation. Thus the solution $w(t)$ can be regarded as the *output* from a system described by equation (24) that is produced by the impulsive *input* (nonhomogeneous term) $\delta(t)$ applied at time $t = 0$ when the system is at rest. The weighting function $w(t)$ is the solution of the equation

$$a_0\frac{d^n w}{dt^n} + a_1\frac{d^{n-1}w}{dt^{n-1}} + \cdots + a_n w = \delta(t), \tag{25}$$

with $w(t) = 0$ for $t < 0$.

Let us now consider the *output* $y(t)$ from a system described by (24) produced by an arbitrary *input* $f(t)$, subject to the homogeneous initial conditions $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$. Taking the Laplace transform of (24) we find that

$$G(s)Y(s) = F(s), \tag{26}$$

where

$$G(s) = a_0s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n,\ \ Y(s) = \mathcal{L}\{y(t)\} \quad \text{and} \quad F(s) = \mathcal{L}\{f(t)\}.$$

Setting $W(s) = \mathcal{L}\{w(t)\}$, taking the Laplace transform of (25), and using the fact that $w(t)$ and all its derivatives vanish for $t < 0$ leads to the result

$$G(s)W(s) = 1. \tag{27}$$

Eliminating $G(s)$ between (26) and (27) relates the Laplace transform of the output $Y(s)$ to the Laplace transform $F(s)$ of the input by the equation

$$Y(s) = W(s)F(s). \tag{28}$$

Taking the inverse Laplace transform of (28) and using the convolution theorem gives

$$y(t) = \int_0^t w(\tau)f(t-\tau)d\tau. \tag{29}$$

This form of the solution of (24) explains why $w(t)$ is called the *weighting function*, because (29) shows how the input $y(t-\tau)$ at time $t-\tau$ is *weighted* by the function $w(\tau)$ over the interval $0 \le \tau \le t$ in the integral determining $y(t)$.

The determination of the weighting function has the advantage that once it has been found, the solution of (24), subject to the conditions that $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$, is always expressible as result (29) for every nonhomogeneous term $f(t)$. It is instructive to compare this result, which applies to a linear differential equation of *any* order, to the one in (76) of Section 6.6, which was obtained by applying the method of variation of parameters to a second order equation with homogeneous initial conditions when $t = a$. The weighting function is also sometimes called the *Green's function* for an initial value problem for a homogeneous differential equation.

The modification that must be made to result (29) to take account of initial conditions for $y(t)$ that are not all zero at $t = 0$ is to be found in Exercise 25 at the end of this section.

EXAMPLE 7.34 Find the weighting function for the equation

$$y'' + 2y' + 5y = \sin t$$

and use it to solve the equation subject to the initial condition $y(0) = y'(0) = 0$.

Solution The weighting function $w(t)$ is the solution of

$$w'' + 2w' + 5w = \delta(t)$$

with $w(0) = w'(0) = 0$. Taking the Laplace transform and setting $\mathcal{L}\{w(t)\} = W(s)$ gives

$$s^2W(s) + 2sW(s) + 5W(s) = 1,$$

so

$$W(s) = \frac{1}{s^2+2s+5}.$$

Taking the inverse Laplace transform, we find that

$$w(t) = \mathcal{L}^{-1}\{W(s)\} = \frac{1}{2}e^{-t}\sin 2t \quad \text{for } t \ge 0.$$

The solution of the differential equation with $y(0) = y'(0) = 0$ now follows from (29) as

$$\begin{aligned} y(t) &= \int_0^t w(\tau)\sin(t-\tau)d\tau \\ &= \frac{1}{2}\int_0^t e^{-\tau}\sin 2\tau \sin(t-\tau)d\tau \\ &= \frac{1}{5}\sin t - \frac{1}{10}\cos t + \frac{e^{-t}}{20}(2\cos 2t - \sin 2t). \end{aligned}$$ ■

The concept of a weighting function can be generalized to include systems of equations, though then more than one weighting function must be introduced, and the solution of each dependent variable becomes the sum of convolution integrals of the type given in (29). The ideas involved are illustrated by considering the following system of equations involving $x(t)$ and $y(t)$:

$$\begin{aligned} x' + ax + by &= f_1(t) \\ y' + cx + dy &= f_2(t), \end{aligned} \tag{30}$$

subject to the initial conditions $x(0) = y(0) = 0$.

It is necessary to introduce a weighting function for each of the variables $x(t)$ and $y(t)$ corresponding first to $f_1(t) = \delta(t)$ and $f_2(t) = 0$, and then to $f_1(t) = 0$ and $f_2(t) = \delta(t)$. Let $w_{x1}(t)$ and $w_{y1}(t)$ be the weighting functions corresponding to

$$\begin{aligned} w'_{x1} + aw_{x1} + bw_{y1} &= \delta(t) \\ w'_{y1} + cw_{x1} + dw_{y1} &= 0, \end{aligned} \tag{31}$$

and $w_{x2}(t)$ and $w_{y2}(t)$ be the Green's functions corresponding to

$$\begin{aligned} w'_{x2} + aw_{x2} + bw_{y2} &= 0 \\ w'_{y2} + cw_{x2} + dw_{y2} &= \delta(t), \end{aligned} \tag{32}$$

where $w_{x1}(0) = w_{x2}(0) = w_{y1}(0) = w_{y2}(0) = 0$.

The notation used here indicates that $w_{x1}(t)$ is the x response and $w_{y1}(t)$ the y response to the input $f_1(t) = \delta(t)$ and $f_2(t) = 0$, and $w_{x2}(t)$ is the x response and $w_{y2}(t)$ the y response to the input $f_1(t) = 0$ and $f_2(t) = \delta(t)$. Then, because the equations are linear, to obtain the solution $x(t)$ subject to the initial conditions $x(0) = y(0) = 0$, it is necessary to add the contribution due to $w_{x1}(t)$ to the one due to $w_{x2}(t)$, and similarly for the solution $y(t)$.

This leads to the solution in the form

$$x(t) = \int_0^t w_{x1}(\tau) f_1(t-\tau)d\tau + \int_0^t w_{x2}(\tau) f_2(t-\tau)d\tau \tag{33a}$$

and

$$y(t) = \int_0^t w_{y1}(\tau) f_1(t-\tau)d\tau + \int_0^t w_{y2}(\tau) f_2(t-\tau)d\tau. \tag{33b}$$

Once the weighting functions have been found, equations (33) give the solution of system (30) for any choice of functions $f_1(t)$ and $f_2(t)$, subject to the initial conditions $x(0) = y(0) = 0$.

EXAMPLE 7.35 Find weighting functions for the equations

$$\begin{aligned} x' + 2x - y &= f_1(t) \\ y' - 2x + y &= f_2(t) \end{aligned}$$

and use them to solve the system subject to the initial conditions $x(0) = y(0) = 0$ when (a) $f_1(t) = \sin t$ and $f_2(t) = 2$ and (b) $f_1(t) = \cos t$ and $f_2(t) = 0$.

Solution **(a)** From (31) the functions $w_{x1}(t)$ and $w_{y1}(t)$ satisfy

$$\begin{aligned} w'_{x1} + 2w_{x1} - w_{y1} &= \delta(t) \\ w'_{y1} - 2w_{x1} + w_{y1} &= 0, \end{aligned}$$

so taking the Laplace transform of these equations we have

$$(s+2)\mathcal{L}\{w_{x1}(t)\} - \mathcal{L}\{w_{y1}(t)\} = 1$$
$$(s+1)\mathcal{L}\{w_{y1}(t)\} - 2\mathcal{L}\{w_{x1}(t)\} = 0.$$

Solving for $\mathcal{L}\{w_{x1}(t)\}$ and $\mathcal{L}\{w_{y1}(t)\}$ gives

$$\mathcal{L}\{w_{x1}(t)\} = \frac{s+1}{s(s+3)} \quad \text{and} \quad \mathcal{L}\{w_{y1}(t)\} = \frac{2}{s(s+3)}.$$

Taking the inverse Laplace transforms, we find that

$$w_{x1}(t) = \frac{1}{3} + \frac{2}{3}e^{-3t} \quad \text{and} \quad w_{y1}(t) = \frac{2}{3} - \frac{2}{3}e^{-3t} \quad \text{for } t \geq 0.$$

Similarly, solving the equations for $w_{x2}(t)$ and $w_{y2}(t)$ corresponding to (32), we obtain

$$w_{x2}(t) = \frac{1}{3} - \frac{1}{3}e^{-3t} \quad \text{and} \quad w_{y2}(t) = \frac{2}{3} + \frac{1}{3}e^{-3t} \quad \text{for } t \geq 0.$$

The solution of the system subject to the initial conditions $x(0) = y(0) = 0$, $f_1(t) = \sin t$, and $f_2(t) = 2$ now follows from (33) as

$$x(t) = \int_0^t w_{x1}(\tau)\sin(t-\tau)d\tau + 2\int_0^t w_{x2}(\tau)d\tau$$

and

$$y(t) = \int_0^t w_{y1}(\tau)\sin(t-\tau)d\tau + 2\int_0^t w_{y2}(\tau)d\tau.$$

After the integrations are performed, the solution is found to be

$$x(t) = \frac{1}{9} + \frac{2}{3}t + \frac{13}{45}e^{-3t} + \frac{1}{5}\sin t - \frac{2}{5}\cos t$$

and

$$y(t) = \frac{8}{9} + \frac{4}{3}t - \frac{13}{45}e^{-3t} - \frac{1}{5}\sin t - \frac{3}{5}\cos t \quad \text{for } t > 0.$$

(b) Similarly, the solution when $f_1(t) = \cos t$ and $f_2(t) = 0$ is given by

$$x(t) = \int_0^t w_{x1}(\tau)\cos(t-\tau)d\tau$$

and

$$y(t) = \int_0^t w_{y1}(\tau)\cos(t-\tau)d\tau,$$

so after performing the integrations,

$$x(t) = -\frac{1}{5}e^{-3t} + \frac{2}{5}\sin t + \frac{1}{5}\cos t$$

and

$$y(t) = \frac{1}{5}e^{-3t} + \frac{3}{5}\sin t - \frac{1}{5}\cos t \quad \text{for } t > 0.$$ ■

(d) Differential Equations with Polynomial Coefficients

special variable coefficient differential equations

The Laplace transform can be applied to linear differential equations with polynomial coefficients to find the solution of an initial value problem in the usual way, and also to deduce the Laplace transform of a function from its defining differential equation. This last situation is useful when the integral defining the Laplace transform of a function $f(t)$ cannot be evaluated directly. First, however, we use Theorems 7.3 and 7.7 to find the transform of a product of a power of t and a derivative of $f(t)$.

THEOREM 7.16 $\mathcal{L}\{t^m f^{(n)}(t)\}$ Let $f(t)$ be n times differentiable with $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\left\{t^m f^{(n)}(t)\right\} = (-1)^m \frac{d^m}{ds^m}[s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)].$$

Useful special cases are:

(i) $\mathcal{L}\{tf(t)\} = -F'(s)$

(ii) $\mathcal{L}\{tf'(t)\} = -sF'(s) - F(s)$

(iii) $\mathcal{L}\{tf''(t)\} = -s^2F'(s) - 2sF(s) + f(0)$

(iv) $\mathcal{L}\{t^2 f'(t)\} = sF'(s) + 2F(s)$

(v) $\mathcal{L}\{t^2 f''(t)\} = s^2F''(s) + 4sF'(s) + 2F(s)$

Proof The results of the theorem are direct consequences of Theorems 7.3 and 7.7. We prove the general result, from which the special cases all follow. From Theorem 7.3 we have

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0),$$

whereas from Theorem 7.7 $\mathcal{L}\{t^m g(t)\} = (-1)^m \frac{d^m}{ds^m} G(s)$, where $\mathcal{L}\{g(t)\} = G(s)$. The main result of the theorem now follows by setting $g(t) = f^{(n)}(t)$ in this last result. ■

(i) $\mathcal{L}\{\exp(-t^2)\}$ and its connection with the error function

Laplace transform of the error function

We will use the differential equation satisfied by $y(t) = \exp(-t^2)$ to show that

$$\mathcal{L}\{\exp(-t^2)\} = \frac{1}{2}\sqrt{\pi}\exp(s^2/4)[1 - \mathrm{erf}(s/2)],$$

where

$$\mathrm{erf}\, s = \frac{2}{\sqrt{\pi}} \int_0^s \exp(-u^2)du$$

is a special function called the **error function**. The error function arises in the theory of heat conduction (see Section 7.3(f) and Chapter 18), in chemical diffusion processes, statistics, and elsewhere.

An attempt to find $\mathcal{L}\{\exp(-t^2)\}$ directly from the definition fails because the integral cannot be evaluated in terms of elementary functions, so some other method must be used. If we set $y(t) = \exp(-t^2)$, it is easily shown that $y(t)$ satisfies the first order variable coefficient equation

$$\frac{dy}{dt} + 2ty = 0,$$

subject to the initial condition $y(0) = \exp(0) = 1$.

Setting $\mathcal{L}\{y(t)\} = Y(s)$ and taking the Laplace transform of the differential equation gives

$$sY(s) - y(0) + 2\mathcal{L}\{ty(t)\} = 0.$$

However, $y(0) = 1$, and from result (i) of Theorem 7.15 (or directly from Theorem 7.7) $\mathcal{L}\{ty(t)\} = -Y'(s)$, so using these results in the preceding equation shows that the Laplace transform satisfies the differential equation

$$\frac{dY}{ds} - \frac{1}{2}sY = -\frac{1}{2}.$$

The integrating factor for this linear first order equation is $\mu(s) = \exp(-s^2/4)$, so after multiplication of the equation by $\mu(s)$ the result becomes

$$\frac{d}{ds}[\exp(-s^2/4)Y(s)] = -\frac{1}{2}\exp(-s^2/4).$$

Integrating over the interval $0 \le u \le s$ gives (after the introduction of the dummy variable u)

$$\int_0^s \frac{d}{du}[\exp(-u^2/4)Y(u)]du = -\frac{1}{2}\int_0^s \exp(-u^2/4)du,$$

or

$$\exp(-s^2/4)Y(s) - Y(0) = -\frac{1}{2}\int_0^s \exp(-u^2/4)du.$$

From the definition $Y(s) = \int_0^\infty e^{-st}\exp(-t^2)dt$, we find that $Y(0) = \int_0^\infty \exp(-t^2)dt$. The integral determining $Y(0)$ is a standard result, $\int_0^\infty \exp(-t^2)dt = \sqrt{\pi}/2$, so making use of this we find that

$$Y(s) = \frac{\sqrt{\pi}}{2}\exp(s^2/4)\left[1 - \frac{1}{\sqrt{\pi}}\int_0^s \exp(-u^2/4)du\right].$$

The change of variable $u = 2v$ brings this last result into the form

$$Y(s) = \frac{\sqrt{\pi}}{2}\exp(s^2/4)\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{s/2} \exp(-v^2)dv\right].$$

If we now define the **error function as**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-v^2)dv,$$

the Laplace transform $Y(s)$ becomes

$$Y(s) = \mathcal{L}\{\exp(-t^2)\} = \frac{\sqrt{\pi}}{2}\exp(s^2/4)[1 - \operatorname{erf}(s/2)].$$

The function erfc (x), defined as

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x),$$

is called the **complementary error function**, so in terms of this function the transform $Y(s)$ becomes

$$Y(s) = \frac{\sqrt{\pi}}{2}\exp(s^2/4)\operatorname{erfc}(s/2).$$

This method of determining the Laplace transform was successful because the differential equation satisfied by $Y(s)$ happened to be simpler than the differential equation satisfied by $y(t)$.

(ii) Laplace transform of the Bessel function $J_0(t)$ and the series expansion of $J_0(t)$

Laplace transform of a Bessel function

The following linear second order differential equation, called **Bessel's equation**,

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0,$$

contains a parameter ν that is a constant. It has many applications, one of which is to be found in Chapter 18, where it enters into the solution of a vibrating circular membrane. The properties of its solutions are developed in some detail in Sections 8.6 and 8.7 of Chapter 8.

For each constant value ν, Bessel's equation has two linearly independent solutions denoted by $J_\nu(t)$ and $Y_\nu(t)$, called, respectively, Bessel functions of order ν of the first and second kind. We now use the Laplace transform to find $\mathcal{L}\{J_0(t)\}$, and then to find a power series expansion for $J_0(t)$ that will be obtained in a completely different way in Section 8.6. When $\nu = 0$, Bessel's equation reduces to

$$t\frac{d^2J_0}{dt^2} + \frac{dJ_0}{dt} + tJ_0 = 0,$$

and we will now find $\mathcal{L}\{J_0(t)\}$ subject to the initial condition $J_0(0) = 1$.

A second initial condition follows by setting $t = 0$ in the differential equation that gives $J_0'(0) = 0$, though this result will not be needed in what is to follow as the condition is implied later when the initial value Theorem 7.13 is used.

Taking the Laplace transform of Bessel's equation of order zero, setting $\mathcal{L}\{J_0(t)\} = Y(s)$, and using the results of Theorem 7.16, we obtain

$$-s^2Y'(s) - 2sY(s) + 1 + sY(s) - 1 - Y'(s) = 0,$$

and after simplification this shows that $Y(s)$ satisfies the first order differential equation

$$\frac{dY}{ds} + \frac{s}{s^2+1}Y(s) = 0.$$

Separating the variables and integrating gives

$$\int \frac{dY}{Y} = -\int \frac{s}{s^2+1} ds,$$

and so

$$Y(s) = \frac{C}{(s^2+1)^{1/2}}.$$

We now know the form of $Y(s)$, apart from the magnitude of the constant C. To find the constant we use the initial value theorem (Theorem 7.13), which shows that we must have

$$J_0(0) = \lim_{s\to\infty}[sY(s)],$$

but from the initial condition $J_0(0) = 1$, so

$$1 = \lim_{s\to\infty} \frac{sC}{(s^2+1)^{1/2}} = C,$$

and thus

$$\mathcal{L}\{J_0(t)\} = \frac{1}{(s^2+1)^{1/2}} \quad \text{for } s > 0.$$

This result can be used to obtain a series expansion for $J_0(t)$ by first writing it as

$$\mathcal{L}\{J_0(t)\} = \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2},$$

and then expanding the result by the binomial theorem to obtain

$$\mathcal{L}\{J_0(t)\} = \frac{1}{s} - \frac{1}{2}\frac{1}{s^3} + \frac{3}{8}\frac{1}{s^5} - \frac{5}{16}\frac{1}{s^7} + \cdots.$$

Finally, taking the inverse Laplace transform of each term and adding the results, we arrive at the series expansion of $J_0(t)$:

$$J_0(t) = 1 - \frac{t^2}{4} + \frac{t^4}{64} - \frac{t^6}{2304} + \cdots.$$

If the general term in the expansion of $\frac{1}{s}(1 + \frac{1}{s^2})^{-1/2}$ is found, and the result is combined with entry 3 of Table 7.1, it is not difficult to show that $J_0(t)$ can be written as

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

(iii) $\mathcal{L}\{\sin\sqrt{t}\}$

We now show how $\mathcal{L}\{\sin\sqrt{t}\} = Y(s)$ can be found from the differential equation satisfied by the function $\sin\sqrt{t}$, and how in this case a different form of argument from the one used in (ii) must be employed to determine the constant of integration in the expression for $Y(s)$. It is easily seen that $y(t) = \sin\sqrt{t}$ is a solution of

$$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0,$$

and clearly $y(0) = 0$. Writing $\mathcal{L}\{y(t)\} = Y(s)$, transforming the equation using result (iii) of Theorem 7.16, and incorporating the initial condition $y(0) = 0$ leads to the following first order differential equation for $Y(s)$:

$$\frac{dY}{ds} = \left(\frac{1-6s}{4s^2}\right)Y.$$

Integration of this variables separable equation gives

$$Y(s) = Cs^{-3/2}\exp[-1/(4s)],$$

so it only remains to determine the value of the constant C.

In this case the initial value theorem is of no help in determining C, so to accomplish this we return to the definition of the Laplace transform:

$$\mathcal{L}\{\sin\sqrt{t}\} = Y(s) = \int_0^\infty e^{-st}\sin\sqrt{t}\,dt.$$

The intuitive argument we now use can be made rigorous, but as the details of its justification are not appropriate here, they will be omitted. Inspection of the integrand shows that as $|\sin\sqrt{t}| \le 1$ for all t, when s is large and positive the exponential function will only be significant close to the origin where the function $\sin\sqrt{t}$ can be approximated by $\sqrt{t}$. So for large s the integral can be approximated by

$$\begin{aligned}\mathcal{L}\{\sin\sqrt{t}\} &\approx \int_0^\infty e^{-st}t^{1/2}dt,\\ &= \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}},\end{aligned}$$

where entry 4 of Table 7.1 has been used together with the result $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$ that will be proved later in Section 8.5 of Chapter 8.

Comparing the original expression for $Y(s)$ when s is large with this last result gives $C = \frac{1}{2}\sqrt{\pi}$, so

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}\exp[-1/(4s)], \quad \text{for } s > 0.$$

This form of argument used to determine the behavior of the integral as $s \to \infty$, where the approximation approaches arbitrarily close to the exact value as s increases, is called an *asymptotic* argument (see, for example, reference [3.3]).

(e) Two-Point Boundary Value Problems: Bending of Beams

boundary value problems and the bending of beams

The Laplace transform is ideally suited to the solution of initial value problems because of the way the initial values of a function enter into the Laplace transform of its derivatives. It can, however, also be used to solve certain types of two-point boundary value problems, as we now show. It will be helpful to use a simple physical example to illustrate the method of approach, so we will consider the case of a

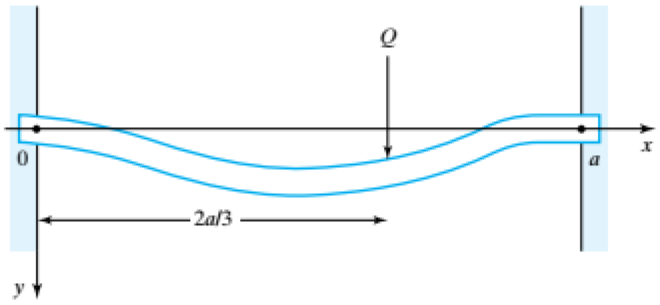


FIGURE 7.25 Clamped beam supporting a point load.

uniform horizontal beam of mass M and length a that is clamped at each end and supports a point load Q at a distance $2a/3$ from one end, as illustrated in Fig. 7.25.

The beam equation was introduced in Section 5.2(f) and is

$$EI\frac{d^4y}{dx^4} = w(x).$$

Here x is measured along the axis of the undeflected beam, $y(x)$ is the vertical deflection, E is the Young's modulus of the material of the beam, I is the second moment of the area of the beam about an axis normal to the x- and y-axes, and $w(x)$ is the transverse load per unit length of the beam, which in this case is an isolated point mass Q located at $x = 2a/3$. The boundary conditions for a clamped beam are

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(a) = y'(a) = 0,$$

because neither deflection nor bending can occur at the ends, so both $y(x)$ and $y'(x)$ vanish at $x = 0$ and $x = a$.

The function $w(x)$ can be expressed as

$$w(x) = \frac{M}{a} + Q\delta(x - 2a/3), \quad \text{for } 0 \le x \le a,$$

where the point load Q is represented by the delta function that only makes a contribution at $x = 2a/3$.

Transforming the equation, setting $\mathcal{L}\{y(x)\} = Y(s)$, and this time writing x in place of t, because it is conventional to denote a length by x, we find

$$EI[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] = \mathcal{L}\{w(x)\}.$$

However,

$$\mathcal{L}\{w(x)\} = \frac{M}{as} + Qe^{-2as/3},$$

so using this in the preceding equation, incorporating the two known initial conditions $y(0) = y'(0) = 0$, and rearranging terms, we find that

$$Y(s) = \frac{M}{a\,EI}\frac{1}{s^5} + \frac{Q}{EI}\frac{e^{-2as/3}}{s^4} + \frac{1}{s^3}y''(0) + \frac{1}{s^4}y'''(0).$$

Taking the inverse Laplace transform of this expression gives

$$y(x) = \frac{M}{24a\,EI}x^4 + \frac{Q}{6EI}(x - 2a/3)^3H(x - 2a/3) + \frac{1}{2}x^2y''(0) + \frac{1}{6}x^3y'''(0).$$

We must now solve for the unknown initial conditions $y''(0)$ and $y'''(0)$ by requiring this expression to satisfy the two remaining boundary conditions at $x = a$, namely, $y(a) = y'(a) = 0$. The condition $y(a) = 0$ gives

$$0 = \frac{Ma}{4EI} + \frac{Qa}{27EI} + 3y''(0) + ay'''(0),$$

and the condition $y'(a) = 0$ gives

$$0 = \frac{Ma^2}{6EI} + \frac{Q}{18EI} + y''(0) + \frac{1}{2}ay'''(0),$$

so solving for $y''(0)$ and $y'''(0)$, we obtain

$$y''(0) = \frac{a}{108EI}(9M + 8Q) \quad \text{and} \quad y'''(0) = -\frac{1}{54EI}(27M + 14Q).$$

The required solution is then given by

$$y(x) = \frac{M}{24a\,EI}x^4 + \frac{Q}{6EI}(x - 2a/3)^3 H(x - 2a/3) + \frac{a}{216EI}(9M + 8Q)x^2 - \frac{1}{324EI}(27M + 14Q),$$

for $0 \le x \le a$.

This same form of approach can be used for other two-point boundary value problems, but its success depends on the ability to solve for the unknown initial values in terms of the given boundary conditions.

(f) An Application of the Laplace Transform to the Heat Equation

The Laplace transform can also be used to solve certain types of partial differential equation, involving two or more independent variables. Although the solution of partial differential equations (PDEs) forms the topic of Chapter 18, it will be instructive at this early stage to introduce a simple example that illustrates how the transform can be used for this purpose, and the way the result of Section 7.3d(i) enters into the solution.

a first encounter with a partial differential equation: the heat equation

The **one-dimensional heat equation** is the partial differential equation

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2},$$

where $T(x, t)$ is the temperature in a one-dimensional heat-conducting solid at position x at time t, and κ is a constant that describes the thermal conductivity property of the solid. This is a *partial* differential equation because it is a differential equation that involves the partial derivatives of the dependent variable $T(x, t)$. The physical situation modeled by this equation can be considered to be a semi-infinite slab of metal with a plane face on which the origin of the x-axis is located, with the positive half of the axis directed into the slab. This situation is illustrated in Fig. 7.26.

We will consider the situation where for $t < 0$ all of the metal in the slab is at the temperature $T = 0$ and then, at time $t = 0$, the plane face of the slab is suddenly brought up to and maintained at the constant temperature $T = T_0$. The problem is to find the temperature inside the slab on any plane $x =$ constant at any time $t > 0$, knowing that physically the temperature must remain finite for all $x > 0$ and $t > 0$.

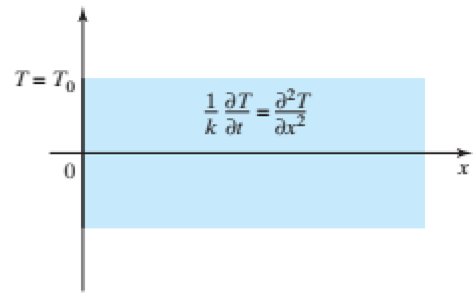


FIGURE 7.26 A semi-infinite metal slab.

The approach will be to take the Laplace transform of the dependent variable $T(x, t)$ in the heat equation with respect to the time t, as a result of which an ordinary differential equation with x as its independent variable will be obtained for the transformed variable that will then depend on both the Laplace transform variable s and x. After this ordinary differential equation has been solved for the transformed variable, the inverse Laplace transform will be used to recover the time variation, and so to arrive at the required solution as a function of x and t.

Before proceeding with this approach we notice first that if the Laplace transform is applied to the independent variable t in the function of two variables $T(x,\ t)$, the variable x will behave like a constant. Consequently, the rules for transforming derivatives of functions of a single independent variable also apply to a function of two independent variables. So, using the notation $\overline{T}(x,\ s) = {}_t\mathcal{L}\{T(x,\ t)\}$ to denote the Laplace transform of $T(x,\ t)$ with respect to the time t, it follows directly from the formula for the transform of a derivative in (9a) that

$$ {}_t\mathcal{L}\{\partial T(x, t)\} = s\overline{T}(x, s) - T(x, 0). $$

To proceed further we must now use the condition that at time $t = 0$ the material of the slab is at zero temperature, so $T(x,\ 0) = 0$, as a result of which

$$ {}_t\mathcal{L}\{\partial T(x, t)/\partial t\} = s\overline{T}(x, s). $$

Next, as x is regarded as a constant, we have

$$ {}_t\mathcal{L}\{\partial^2 T(x, t)/\partial x^2\} = \frac{\partial^2 \overline{T}(x, s)}{\partial x^2}. $$

Using these results when taking the Laplace transform of the heat equation with respect to t, and making use of the linearity property of the transform, gives

$$ s\overline{T}(x, s) = \kappa \frac{d^2 \overline{T}(x, s)}{dx^2}, $$

where we now use an ordinary derivative with respect to x because in this differential equation s appears as a parameter so x can be considered to be the only independent variable. When the differential equation is written

$$ \overline{T}'' - \frac{s}{\kappa}\overline{T} = 0, $$

using a prime to denote a derivative with respect to x, it is seen to have the general solution

$$ \overline{T}(x, s) = A\exp\left[\sqrt{\frac{s}{\kappa}}x\right] + B\exp\left[-\sqrt{\frac{s}{\kappa}}x\right]. $$

As a Laplace transform must vanish in the limit $s \to +\infty$, we must set $A = 0$, so the Laplace transform of the temperature is seen to be given by

$$\overline{T}(x, s) = B \exp\left[-\sqrt{\frac{s}{\kappa}}x\right].$$

In this case, the rejection of the term with the positive exponent in the general solution for $\overline{T}(x, s)$ corresponds to the physical requirement that the temperature remain finite for $x > 0$ and $t > 0$.

To determine B we now make use of the boundary condition on the plane face of the slab that requires $T(0, t) = T_0$, from which it follows that ${}_t\mathcal{L}\{T(0, t)\} = T_0/s$. Thus, the Laplace transform of the solution with respect to the time t is seen to be

$$\overline{T}(x, s) = \frac{T_0}{s} \exp\left[-\sqrt{\frac{s}{\kappa}}x\right].$$

To recover the time variation from this Laplace transform it is necessary to find ${}_t\mathcal{L}^{-1}\{\overline{T}(x, s)\}$. As $\overline{T}(x, s)$ is not the Laplace transform of an elementary function listed in our table of transform pairs, the solution $T(x, t)$ must be found by means of the Laplace inversion integral. In Chapter 16 on the Laplace inversion integral, it is shown in Example 16.6 that

$${}_t\mathcal{L}^{-1}\{e^{-k\sqrt{s}}\} = \frac{k}{2\sqrt{\pi t^3}} \exp\left\{-\frac{k^2}{4t}\right\}.$$

So, setting $k = x/\kappa^2$ in this result and using it with Theorem 7.11 to invert the Laplace transform $\overline{T}(x, s)$ shows that the solution is

$$\boxed{T(x, t) = T_0 \operatorname{erfc}\left\{\frac{x}{2\sqrt{\kappa t}}\right\},} \qquad \text{for } x > 0,\ t > 0.$$

The use of integral transforms is discussed in reference [4.4].

Summary

The Laplace transform has been applied to systems of differential equations, and the results extended to systems in matrix form. Various applications have been made to some useful variable coefficient ordinary differential equations, and to the important partial differential equation that describes one-dimensional unsteady heat flow.

EXERCISES 7.3

(a) Exercises involving systems of equations

1. Solve

$$x' + 5x - 2y = 1 \quad \text{and} \quad y' - 5x + 2y = 3$$
given $x(0) = 0,\ y(0) = 2$.

2. Solve

$$x' - x - y = \cos t \quad \text{and} \quad y' + x + y = \cos t$$
given $x(0) = 1,\ y(0) = 1$.

3. Solve

$$x' + x + y = 2 \quad \text{and} \quad y' + x - y = 1$$
given $x(0) = -1,\ y(0) = 1$.

4. Solve

$$x' + x + 2y = e^{-t} \quad \text{and} \quad y' + 2x + y = 1$$
given $x(0) = 0,\ y(0) = 0$.

5. Solve

$$x' - x + 3y = 1 + t \quad \text{and} \quad y' + x - y = 2$$
given $x(0) = 2,\ y(0) = -2$.

6. Solve

$$x' + x + y = \sin 2t \quad \text{and} \quad y' + x - y = 1 \quad \text{given} \quad x(0) = 0,\ y(0) = 0.$$

7. Solve

$$x' + x - z = 1, \quad y' - x + y = 1, \quad z' + y - x = 0,$$
given that $x(0) = 1,\ y(0) = 0,\ z(0) = 1$.

8. Solve

$$x' + x - y = 1, \quad y' - y + 2z = 0, \quad z' + x - y = \sin t,$$
given $x(0) = 1,\ y(0) = 0,\ z(0) = 2$.

9. Solve

$$x' - z = e^t, \quad y' - z = 2, \quad z' - x = 1, \quad \text{given } x(0) = 0,\ y(0) = 1,\ z(0) = 0.$$

10. Solve

$$x' + z = 3, \quad y' + x = 1, \quad z' - x = \sin t, \quad \text{given} \quad x(0) = 1,\ y(0) = 0,\ z(0) = 1.$$

(b) Exercises involving $e^{t\mathbf{A}}$

In Exercises 11 through 24 find $e^{t\mathbf{A}}$ for the given matrix $\mathbf{A}$.

11. $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$.

12. $\mathbf{A} = \begin{bmatrix} -2 & 4 \\ 3 & 2 \end{bmatrix}$.

13. $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix}$.

14. $\mathbf{A} = \begin{bmatrix} 3 & 7 \\ 3 & -1 \end{bmatrix}$.

15. $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 4 & 0 \end{bmatrix}$.

16. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix}$.

17. $\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix}$.

18. $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 5 & 0 \end{bmatrix}$.

19. $\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 0 & 6 \end{bmatrix}$.

20. $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$.

21. $\mathbf{A} = \begin{bmatrix} -2 & 4 \\ 0 & -2 \end{bmatrix}$.

22. $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix}$.

23. $\mathbf{A} = \begin{bmatrix} 5 & 10 & 7 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$.

24. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

(c) Exercises involving the weighting function

In Exercises 26 through 32 find the weighting function when a single equation is involved, and the four weighting functions when a pair of equations is involved. Use the weighting function(s) to solve the given differential equation(s).

25. Show that if the initial conditions for equation (24) are $y(0) = y_0,\ y'(0) = y_1,\ \ldots,\ y^{(n-1)} = y_{n-1}$, the solution can be written in the form

$$y(t) = \int_0^t w(\tau)[y_0(t-\tau) - h(t-\tau)]d\tau.$$

Here $y_0(t)$ is the solution of the equation with the initial conditions $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$, and $h(t) = \{H(s)/G(s)\}$, with $H(s)$ the polynomial produced by the nonvanishing initial values of the derivatives, so that the transformed equation corresponding to (26) becomes

$$G(s)Y(s) + H(s) = F(s).$$

26. $y'' - 4y' + 3y = \cos t$, given $y(0) = 0$ and $y'(0) = 0$.

27. $y'' + 2y' + 2y = e^{2t}$, given $y(0) = 0$ and $y'(0) = 0$.

28. $y'' + 4y' + 13y = \cos 2t$, given $y(0) = 0$ and $y'(0) = 0$.

29. $y'' + 6y' + 5y = e^{-t}$, given $y(0) = 0$ and $y'(0) = 0$.

30. Use the result of Exercise 25 to solve

$$y'' - 2y' - 3y = 1 + \sin t, \quad \text{given } y(0) = 1 \quad \text{and} \quad y'(0) = -1.$$

31. $x' - 3x + 2y = e^{-t}$, $y' + 3x - 4y = 3$, with $x(0) = y(0) = 0$.

32. $x' + 2x - y = \sin t$, $y' - 2x + y = 2$, with $x(0) = y(0) = 0$.

(d) Differential equations with polynomial coefficients

33. Use the fact that $y(x) = \sin ax$ satisfies the differential equation

$$y'' + a^2 y = 0 \quad \text{with } y(0) = 0,\ y'(0) = a$$

to derive $\mathcal{L}\{\sin ax\}$ from the differential equation.

34. Use the fact that $y(x) = 1 - \cos ax$ satisfies the differential equation

$$y'' + a^2 y = a^2 \quad \text{with } y(0) = 0,\ y'(0) = 0$$

to derive $\mathcal{L}\{1 - \cos ax\}$ from the differential equation.

35.* The **Laguerre equation**

$$xy'' + (1-x)y' + ny = 0,$$

with $n = 0,\ 1,\ 2, \ldots$ a parameter, has polynomial solutions $y(x) = L_n(x)$ called **Laguerre polynomials**. These polynomials are used in many branches of mathematics and physics, and also in connection with numerical integration. By taking the Laplace transform of the differential equation find $\mathcal{L}\{L_n(x)\}$ and hence show that

$$L_4(x) = 24 - 96x + 72x^2 - 16x^3 + x^4.$$

36.* The **Hermite equation**

$$y'' - 2xy' + 2ny = 0,$$

with $n = 0, 1, 2, \ldots$ a parameter, has polynomial solutions $y(x) = H_n(x)$ called **Hermite polynomials**. Like the Laguerre polynomials, these polynomials are also used in mathematics and physics, and in connection with numerical integration. By transforming the equation and using the initial conditions $y(0) = H_4(0) = 12$ and $y'(0) = 0$, find $\mathcal{L}\{H_4(x)\}$, and hence show that

$$H_4(x) = 16x^4 - 48x^2 + 12.$$

37.* The Bessel function $y(x) = J_0(ax)$ satisfies the differential equation

$$xy'' + y' + axy = 0$$

subject to the initial conditions $y(0) = J_0(0) = 0$. Derive $\mathcal{L}\{J_0(ax)\}$ from the differential equation and confirm the result by using $\mathcal{L}\{J_0(x)\} = 1/(s^2+1)^{1/2}$ in conjunction with the scaling theorem.

38.* The Bessel function $y(x) = J_1(x)$ satisfies the differential equation

$$x^2y'' + xy' + (x^2 - 1)y = 0 \quad \text{with } J_1(0) = 0 \text{ and } J_1{}'(0) = 1/2.$$

By taking the Laplace transform of the differential equation show that $\mathcal{L}\{J_1(x)\} = C\{1 - s/(s^2+1)^{1/2}\}$, and deduce that $C = 1$.

(e) Exercises involving two-point boundary value problems

39. Solve $x'' + x = \sin 2t$ with $x(0) = 0$ and $x(\pi/2) = 1$.

40. Using the notation of Section 7.3(e), solve the beam equation

$$EI\frac{d^4y}{dx^4} = w(x)$$

for the uniform cantilevered beam of mass M and length a shown in Fig. 7.27, where a point mass Q is located at a distance $a/3$ from the clamped end. The boundary conditions to be used are

$$y(0) = y'(0) = 0 \quad \text{and} \quad y''(a) = y'''(a) = 0.$$

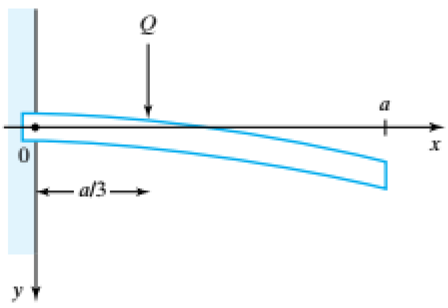


FIGURE 7.27 Cantilevered beam with a point load.

41. Using the notation of Section 7.3(e), solve the beam equation

$$EI\frac{d^4y}{dx^4} = w(x)$$

for the uniform beam of mass M and length a with clamped ends shown in Fig. 7.28, where a point mass Q is located at a distance $3a/4$ from the left-hand end. The boundary conditions to be used are

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(a) = y'(a) = 0.$$

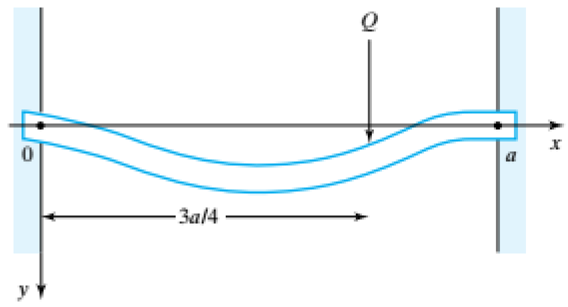


FIGURE 7.28 Supported beam with clamped ends and a point load.

42. Using the notation of Section 7.3(e), solve the beam equation

$$EI\frac{d^4y}{dx^4} = w(x)$$

for the uniform beam of mass M and length a shown in Fig. 7.29 that is clamped at the end $x = 0$ and supported at the end $x = a$, where a point mass Q is located at a distance $a/4$ from the right-hand end. The boundary conditions to be used are

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(a) = y''(a) = 0.$$

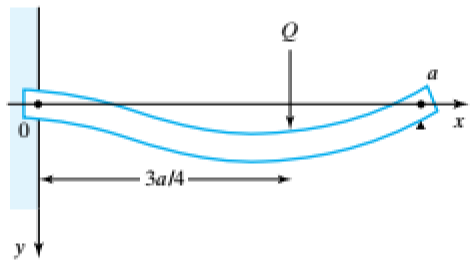


FIGURE 7.29 Beam clamped at one end and supported at the other with a point load.

(f) Physical problems to be solved by computer algebra

43. In an R–L–C circuit the current $i(t)$ and charge $q(t)$ resulting from a constant voltage E_0 applied at time

$t = 0$, when $i(0) = 0$ and $q(0) = 0$, are determined by the equations

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E_0 \quad \text{and} \quad i = \frac{dq}{dt}.$$

Find $i(t)$, and comment on its form depending on the sign of $R^2C - 4L$. Choose representative values of R, L, C corresponding to each of the foregoing cases and plot $i(t)$ in a suitable interval $0 \le t \le T$.

44. Figure 6.10 in Section 6.3 illustrates three particles of equal mass joined by identical springs that oscillate in a straight line, with each end of the system clamped. In a representative case, the nondimensional equations determining the magnitudes of the displacements $y_1(t)$, $y_2(t)$, and $y_3(t)$ are

$$3\frac{d^2y_1}{dt^2} = y_2 - 2y_1 + y_3, \quad 3\frac{d^2y_2}{dt^2} = y_3 - 2y_2 + y_1,$$
$$3\frac{d^2y_3}{dt^2} = y_1 - 2y_3 + y_2.$$

Find $y_1(t)$, $y_2(t)$, and $y_3(t)$ given that $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 2$, $y_2'(0) = 1$, $y_3(0) = 3$, $y_3'(0) = 0$.

45. If, similar to the example in Section 7.3(a), an irreversible reaction converts a molecule of chemical A into a molecule of chemical D, via molecules of chemicals B and C, the governing equations in terms of the respective reaction rates k_1, k_2, and k_3 are

$$\frac{dx}{dt} = -k_1x, \quad \frac{dy}{dt} = k_1x - k_2y, \quad \text{and} \quad \frac{dz}{dt} = k_2y - k_3z,$$

where x, y, and z are the number of molecules of A, B, and C present at time t. If Q molecules of A are present at time $t = 0$, the number of molecules of D present at time t is $w(t) = Q - x(t) - y(t) - z(t)$. Find $w(t)/Q$ as a function of t given that $k_1 = 2$, $k_2 = 3$, and $k_3 = 3$, and plot the result for $0 \le t \le 5$. Find the percentage of chemical A that has been transformed into chemical D at the instants of time $t = 1, 2$, and 3.

46. In the following nondimensional equations, $x(t)$ and $y(t)$ represent the magnitudes of the currents flowing in the primary and secondary windings of a transformer, when initially $x(0) = 0$, $y(0) = 0$ and at time $t = 0$ the primary winding is subjected to an exponentially decaying voltage of magnitude e^{-t}:

$$\frac{dx}{dt} + \frac{1}{3}\frac{dy}{dt} + 3x = e^{-t}, \quad \frac{dx}{dt} + 3\frac{dy}{dt} + 9y = 0.$$

Find $x(t)$ and $y(t)$, and by plotting the magnitudes of the currents show that $x(t)$ is always positive and after peaking decays to zero, while $y(t)$ is initially negative, but after becoming positive it decays to zero faster than $x(t)$.

7.4 The Transfer Function, Control Systems, and Time Lags

The study of engineering systems of all types whose behavior is determined by *linear* ordinary differential equations is often carried out by examining what is called the system **transfer function**. Typically, a system is governed by a linear nth order constant coefficient ordinary differential equation whose solution or **output**, also called the **response** of the system, we will denote by $u_0(t)$ and whose forcing function, or **input**, is a known function we will denote by $u_i(t)$, where t is the time.

A typical example of a simple system has already been encountered in Fig. 6.2, where the spring-mounted and damped vibrating machine has an input $F(t)$ and an output $y(t)$ that are related by

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = F(t).$$

An nth order system may be governed by the equation

$$a_n\frac{d^nu_0}{dt^n} + a_{n-1}\frac{d^{n-1}u_0}{dt^{n-1}} + \cdots + a_0u_0 = u_i,$$

which can be represented graphically as in Fig. 7.30, where F[.] is the differential operator

$$F[.] \equiv a_n\frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_0. \tag{34}$$

FIGURE 7.30 Block-diagram representation of equation (34).

More generally, in linear systems the input itself may be the solution of another linear differential equation, in which case the system relating the response $u_0(t)$ to the input $u_i(t)$ becomes

$$a_n \frac{d^n u_0}{dt^n} + a_{n-1} \frac{d^{n-1} u_0}{dt^{n-1}} + \cdots + a_0 u_0 = b_m \frac{d^m u_i}{dt^m} + b_{m-1} \frac{d^{m-1} u_i}{dt^{m-1}} + \cdots b_0 u_i, \tag{35}$$

where $n \geq m$ and the coefficients a_r and b_s are constants.

The **transfer function** of a system is defined as the quotient of the Laplace transforms of the system output and the system input, when all of the initial conditions are taken to be *zero*. This last condition means that when the Laplace transform is used to transform a differential equation we may set $\mathcal{L}\{d^r u/dt^r\} = s^r U(s)$. So, after transforming (35), we obtain

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) U_0(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots b_0) U_i(s), \tag{36}$$

where $U_0(s) = \mathcal{L}\{u_0(t)\}$ and $U_i(s) = \mathcal{L}\{u_i(t)\}$. The transfer function $G(s) = U_0(s)/U_i(s)$ becomes the rational function of the transform variable s

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_0}. \tag{37}$$

Let us now set $G(s) = N(s)/D(s)$, where $N(s)$ is the polynomial in s of degree m in the numerator of $G(s)$, and $D(s)$ is the polynomial in s of degree n in the denominator. The polynomial $D(s)$ is called the **characteristic polynomial** of the system, and $D(s) = 0$ is called the **characteristic equation** of the system. The **order** of the system in (37) is the degree n of the polynomial $D(s)$.

As the coefficients of $D(s)$ are real, it follows that the roots of the characteristic equation, called the **poles** of the transfer function $G(s)$, either are all real or, if complex, they must occur in complex conjugate pairs. When $G(s)$ is expressed in partial fraction form, this last observation implies that the system will be **stable** provided all the roots of the characteristic equation have negative real parts. Here, by **stability**, we mean that any bounded input to a system that is stable will result in an output that is also bounded for all time, and this will be the case when every root of $D(s) = 0$ has a negative real part. The requirement that $n \geq m$ imposed on (35) is necessary in order to prevent unbounded behavior of the output caused by the occurrence of delta functions.

It is important to recognize that systems describing quite different physical phenomena can have the *same* transfer function, so transfer functions provide a means of examining a class of similar systems independently of their physical origin. It follows that for any given input with Laplace transform $U_i(s)$, the Laplace transform of the output $U_0(s)$ is given by

$$U_0(s) = G(s) U_i(s). \tag{38}$$

The time variation of the output of the system then follows by taking the inverse Laplace transform of (38).

EXAMPLE 7.36 Find the transfer function of the system with input $u_i(t)$ and output $u_0(t)$ described by

$$4\frac{d^2u_0(t)}{dt^2} + 16\frac{du_0(t)}{dt} + 25u_0(t) = 3\frac{du_i(t)}{dt} + 2u_i(t),$$

and show it is stable.

Solution Taking the Laplace transform of the governing equation and assuming all initial conditions to be zero gives

$$(4s^2 + 16s + 25)U_0(s) = (3s + 2)U_i(s),$$

so the system transfer function is

$$G(s) = \frac{U_0(s)}{U_i(s)} = \frac{3s + 2}{4s^2 + 16s + 25}.$$

The system is of order 2, and its characteristic equation is

$$4s^2 + 16s + 25 = 0.$$

The characteristic equation has the roots $s_1 = -2 - \frac{3}{2}i$ and $s_2 = -2 + \frac{3}{2}i$, so as their real parts are negative, the system is stable. ■

Systems that compare the difference between an input and an output, and attempt to reduce the difference to zero to make the output *follow* the input, are called **control systems**. A typical example is a temperature control system for a chemical reactor in which the temperature is required to remain constant, but where as the reaction progresses heat is released at variable rates, causing cooling to become necessary.

A simple control system is illustrated in Fig. 7.31, where F is the system differential equation. The idea here is that an input u_i is compared with the output u_0, called the **feedback**, and the difference $\varepsilon = u_i - u_0$, called the **error signal**, is then used as an input to system F. The result is that $u_0 = u_i$ when $\varepsilon = 0$. It is often necessary to modify the feedback by passing u_0 through another system G with output $v = G[u_0]$, and then to use the the difference $v - u_i$ to drive F. The reason for this is to improve the overall performance of a system, whose physical characteristics may be difficult to alter, by using an easily modified feedback to make the system more responsive and to reduce any tendency it may have for excessive oscillation.

EXAMPLE 7.37 A steering mechanism for a small boat comprises an input heading θ_i from the helm, an amplifier for the error signal, and a servomotor to drive the rudder with moment of inertia I that produces a resisiting torque proportional to the rate of change of the output angle θ_0. Derive the differential equation governing the system and find its transfer function given that the feedback is the unmodified output θ_0.

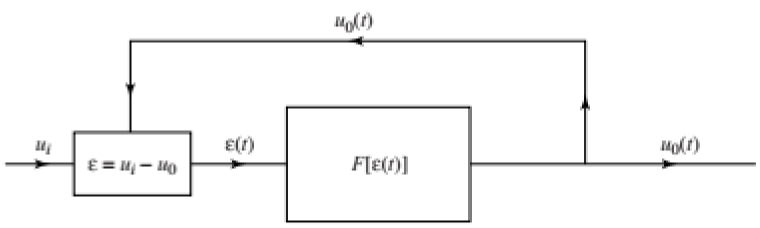


FIGURE 7.31 A typical feedback control system.

Solution If the resisting torque is $kd\theta_0/dt$ and the amplifier increases the magnitude of the error signal by a factor K, the system can be represented as in Fig. 7.31 with the governing differential equation

$$I\frac{d^2\theta_0}{dt^2} + k\frac{d\theta_0}{dt} = K(\theta_i - \theta_0).$$

Taking the Laplace transform of this equation gives

$$(Is^2 + ks + K)\mathcal{L}\{\theta_0\} = \mathcal{L}\{\theta_i\},$$

and so

$$\mathcal{L}\{\theta_0\} = \frac{1}{Is^2 + ks + K}\mathcal{L}\{\theta_i\}.$$

This result shows that the transfer function $G(s) = 1/(Is^2 + ks + K)$, so the system will be stable provided the roots of the characteristic equation $Is^2 + ks + K = 0$ have negative real parts. This will be the case since $I > 0$ and $K > 0$, but the steering will oscillate about the required heading if $4IK > k^2$.

As the design of the boat determines I and k, any improvement of the steering response can only be obtained by using a modified feedback signal instead of the direct feedback θ_0. ■

We close this section by mentioning an important consequence of the introduction of a delay into an equation governing the response of a system. Consider a vibrating system characterized by $y(t)$ in which instantaneous damping proportional to the velocity dy/dt occurs with coefficient of proportionality a_1, and where there is also present an additional time retarded damping of a similar type but with a time lag τ and a coefficient of proportionality a_2. Then, when a springlike restoring effect is present with constant of proportionality a_3, the governing equation takes the form

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_2\frac{dy(t-\tau)}{dt} + a_3y(t) = 0. \tag{39}$$

Because of the presence of the time-delayed derivative $dy(t-\tau)/dt$, an equation of this type is called a **differential-difference equation**.

If we now seek a solution of this equation by using the Laplace transform (or by seeking solutions of the form $y(t) = A\exp(\lambda t)$, where A and λ are constants) we arrive at a characteristic equation of the form

$$s^2 + a_1s + a_2s\exp(-\tau s) + a_3 = 0. \tag{40}$$

This is called an **exponential polynomial** in s, and its root will determine both the stability and response of the system.

Without going into detail, by using Rouche's theorem from complex analysis it is not difficult to prove that exponential polynomials have an infinite number of zeros. Consequently, the response of a system with a characteristic polynomial in the form of an exponential polynomial will only be stable if all of its zeros have negative real parts, and this can only be shown analytically. Methods exist that can be used to determine when all the zeros of such exponential polynomials have negative real parts. An interested reader will find a valuable discussion of this subject in Section 13 of *Differential-Difference Equations* by R. Bellman and K. Cooke, published by Academic Press in 1963.

It is necessary to ask in what way the infinite number of zeros of an exponential polynomial of degree n approximate the n zeros of the ordinary polynomial of degree n when time lags are absent. This is a simpler question, and it can be answered by appeal to Hurwitz's theorem from complex analysis, though again the arguments used go beyond this first account of the subject.

A result on exponential polynomials

Let $P_\tau(s)$ be an exponential polynomial of degree n in s with a time lag τ, and let $P_0(s)$ be the corresponding constant coefficient polynomial when $\tau = 0$. Then, as $\tau \to 0$, so each of the n zeros s_i of $P_0(s)$ is approached arbitrarily closely by a number of zeros of $P_\tau(s)$ equal in number to its multiplicity, and the remaining infinite number of zeros of $P_\tau(s)$ can be made to lie outside a circle of arbitrarily large radius centered on the origin.

As this result says nothing about how the zeros move as $\tau \to 0$, it is possible for the system to be stable when τ lies in certain intervals and unstable otherwise.

EXERCISES 7.4

1. Find the transfer function for each of the following systems. Determine the order of each system and find which is stable.

(a) $\dfrac{d^3u_0}{dt^3} + 3\dfrac{d^2u_0}{dt^2} + 16\dfrac{du_0}{dt} - 20u_0 = 2\dfrac{d^2u_i}{dt^2} + \dfrac{du_i}{dt} - 6u_i.$

(b) $\dfrac{d^3u_0}{dt^3} + 4\dfrac{d^2u_0}{dt^2} + 14\dfrac{du_0}{dt} + 20u_0 = 6\dfrac{d^2u_i}{dt^2} - 13\dfrac{du_i}{dt} + 6u_i.$

(c) $9\dfrac{d^2u_0}{dt^2} + 6\dfrac{du_0}{dt} + 10u_0 = 6\dfrac{d^2u_i}{dt^2} + 5\dfrac{du_i}{dt} - 6u_i.$

2.* For safety reasons, a control system is often duplicated, with the sensors for each system located in different positions, and in such cases the possibility of interaction between the control systems must be considered. A typical case is illustrated in Fig. 7.32, where two identical control systems are shown between which there is assumed to be linear **cross-coupling** of the error signals. This means that the respective actuating error signals are $\varepsilon_1' = a_{11}\varepsilon_1 + a_{12}\varepsilon_2$ and $\varepsilon_2' = a_{21}\varepsilon_1 + a_{22}\varepsilon_2$, with the coefficients a_{ij} constants. Derive and discuss the equations governing the response of the system when

$$F(u_0) = \frac{d^2u_0}{dt^2} + 2\zeta\Omega\frac{du_0}{dt} + \Omega^2 u_0,$$

with $\zeta > 0$ and $\Omega > 0$.

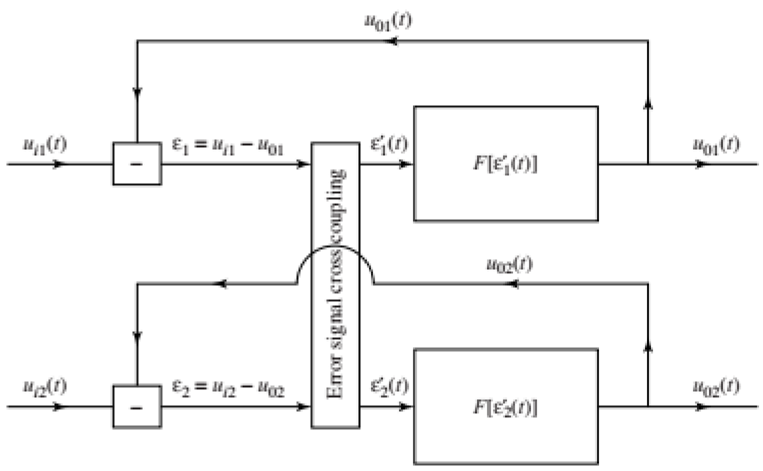


FIGURE 7.32 Two interacting control systems.