9

Solving equations by iterative methods

9.1 Introduction to iterative methods

Many equations can only be solved graphically or by methods of successive approximations to the roots, called **iterative methods**. Three methods of successive approximations are (i) bisection method, introduced in Section 9.2, (ii) an algebraic method, introduction in Section 9.3, and (iii) by using the Newton-Raphson formula, given in Section 9.4.

Each successive approximation method relies on a reasonably good first estimate of the value of a root being made. One way of determining this is to sketch a graph of the function, say $y=f(x)$, and determine the approximate values of roots from the points where the graph cuts the x-axis. Another way is by using a functional notation method. This method uses the property that the value of the graph of $f(x)=0$ changes sign for values of x just before and just after the value of a root. For example, one root of the equation $x^2-x-6=0$ is $x=3$. Using functional notation:

$$f(x) = x^2 - x - 6$$
$$f(2) = 2^2 - 2 - 6 = -4$$
$$f(4) = 4^2 - 4 - 6 = +6$$

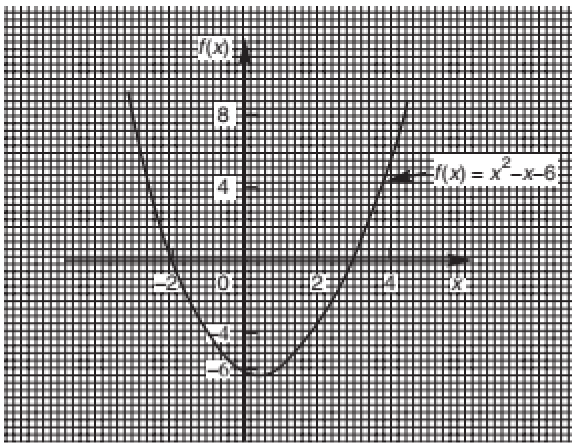

Figure 9.1

It can be seen from these results that the value of $f(x)$ changes from -4 at $f(2)$ to $+6$ at $f(4)$, indicating that a root lies between 2 and 4. This is shown more clearly in Fig. 9.1.

9.2 The bisection method

As shown above, by using functional notation it is possible to determine the vicinity of a root of an equation by the occurrence of a change of sign, i.e. if x_1 and x_2 are such that $f(x_1)$ and $f(x_2)$ have opposite signs, there is at least one root of the equation $f(x)=0$ in the interval between x_1 and x_2 (provided $f(x)$ is a continuous function). In the **method of bisection** the mid-point of the interval, i.e. $x_3 = \dfrac{x_1+x_2}{2}$, is taken, and from the sign of $f(x_3)$ it can be deduced whether a root lies in the half interval to the left or right of x_3. Whichever half interval is indicated, its mid-point is then taken and the procedure repeated. The method often requires many iterations and is therefore slow, but never fails to eventually produce the root. The procedure stops when two successive value of x are equal—to the required degree of accuracy.

The method of bisection is demonstrated in Problems 1 to 3 following.

Problem 1. Use the method of bisection to find the positive root of the equation $5x^2+11x-17=0$ correct to 3 significant figures.

Let $f(x) = 5x^2 + 11x - 17$

then, using functional notation:

$$f(0) = -17$$
$$\boldsymbol{f(1) = 5(1)^2 + 11(1) - 17 = -1}$$
$$\boldsymbol{f(2) = 5(2)^2 + 11(2) - 17 = +25}$$

Since there is a change of sign from negative to positive there must be a root of the equation between $x=1$ and $x=2$. This is shown graphically in Fig. 9.2.

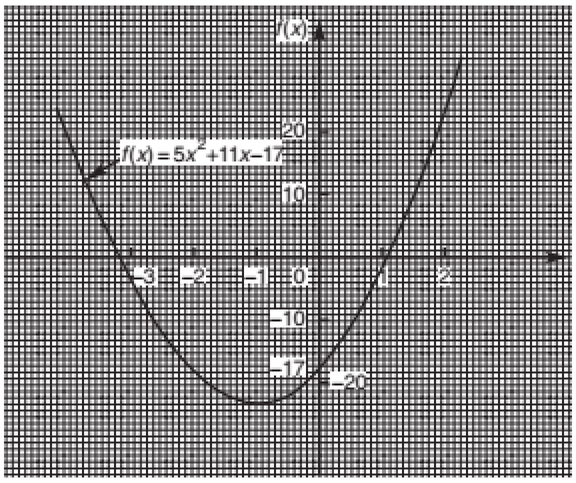

Figure 9.2

The method of bisection suggests that the root is at $\frac{1+2}{2}=1.5$, i.e. the interval between 1 and 2 has been bisected.

Hence

$$\boldsymbol{f(1.5)} = 5(1.5)^2 + 11(1.5) - 17$$
$$= \mathbf{+10.75}$$

Since $f(1)$ is negative, $f(1.5)$ is positive, and $f(2)$ is also positive, a root of the equation must lie between $x=1$ and $x=1.5$, since a **sign change** has occurred between $f(1)$ and $f(1.5)$.

Bisecting this interval gives $\frac{1+1.5}{2}$ i.e. 1.25 as the next root.

Hence

$$\boldsymbol{f(1.25)} = 5(1.25)^2 + 11x - 17$$
$$= \mathbf{+4.5625}$$

Since $f(1)$ is negative and $f(1.25)$ is positive, a root lies between $x=1$ and $x=1.25$.

Bisecting this interval gives $\frac{1+1.25}{2}$ i.e. 1.125

Hence

$$\boldsymbol{f(1.125)} = 5(1.125)^2 + 11(1.125) - 17$$
$$= \mathbf{+1.703125}$$

Since $f(1)$ is negative and $f(1.125)$ is positive, a root lies between $x=1$ and $x=1.125$.

Bisecting this interval gives $\frac{1+1.125}{2}$ i.e. 1.0625.

Hence

$$\boldsymbol{f(1.0625)} = 5(1.0625)^2 + 11(1.0625) - 17$$
$$= \mathbf{+0.33203125}$$

Since $f(1)$ is negative and $f(1.0625)$ is positive, a root lies between $x=1$ and $x=1.0625$.

Bisecting this interval gives $\frac{1+1.0625}{2}$ i.e. 1.03125.

Hence

$$\boldsymbol{f(1.03125)} = 5(1.03125)^2 + 11(1.03125) - 17$$
$$= \mathbf{-0.338867\ldots}$$

Since $f(1.03125)$ is negative and $f(1.0625)$ is positive, a root lies between $x=1.03125$ and $x=1.0625$.

Bisecting this interval gives

$$\frac{1.03125 + 1.0625}{2} \text{ i.e. } 1.046875$$

Hence

$$\boldsymbol{f(1.046875)} = 5(1.046875)^2 + 11(1.046875) - 17$$
$$= \mathbf{-0.0046386\ldots}$$

Since $f(1.046875)$ is negative and $f(1.0625)$ is positive, a root lies between $x=1.046875$ and $x=1.0625$.

Bisecting this interval gives

$$\frac{1.046875 + 1.0625}{2} \text{ i.e. } \mathbf{1.0546875}$$

The last three values obtained for the root are 1.03125, 1.046875 and 1.0546875. The last two values are both 1.05, correct to 3 significant figure. We therefore stop the iterations here.

Thus, correct to 3 significant figures, the positive root of $5x^2 + 11x - 17 = 0$ is 1.05

Problem 2. Use the bisection method to determine the positive root of the equation $x+3=e^x$, correct to 3 decimal places.

Let $f(x)=x+3-e^x$

then, using functional notation:

$$\boldsymbol{f(0)} = 0+3-e^0 = \mathbf{+2}$$
$$\boldsymbol{f(1)} = 1+3-e^1 = \mathbf{+1.2817\ldots}$$
$$\boldsymbol{f(2)} = 2+3-e^2 = \mathbf{-2.3890\ldots}$$

Since $f(1)$ is positive and $f(2)$ is negative, a root lies between $x = 1$ and $x = 2$. A sketch of $f(x) = x + 3 - e^x$, i.e. $x + 3 = e^x$ is shown in Fig. 9.3.

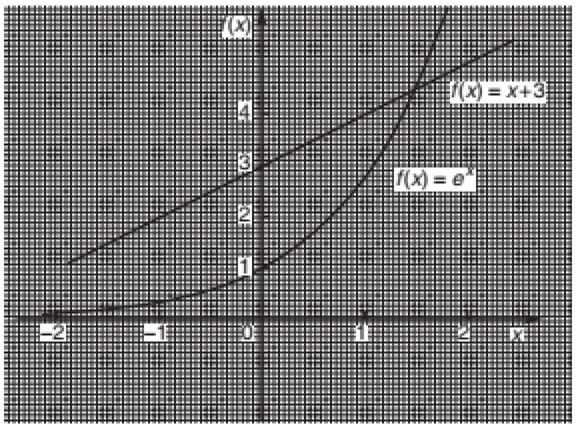

Figure 9.3

Bisecting the interval between $x = 1$ and $x = 2$ gives $\dfrac{1+2}{2}$ i.e. 1.5.

Hence

$$\boldsymbol{f(1.5)} = 1.5 + 3 - e^{1.5} = \boldsymbol{+0.01831\ldots}$$

Since $f(1.5)$ is positive and $f(2)$ is negative, a root lies between $x = 1.5$ and $x = 2$.

Bisecting this interval gives $\dfrac{1.5+2}{2}$ i.e. 1.75.

Hence

$$\boldsymbol{f(1.75)} = 1.75 + 3 - e^{1.75} = \boldsymbol{-1.00460\ldots}$$

Since $f(1.75)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.75$ and $x = 1.5$.

Bisecting this interval gives $\dfrac{1.75+1.5}{2}$ i.e. 1.625.

Hence

$$\boldsymbol{f(1.625)} = 1.625 + 3 - e^{1.625} = \boldsymbol{-0.45341\ldots}$$

Since $f(1.625)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.625$ and $x = 1.5$.

Bisecting this interval gives $\dfrac{1.625+1.5}{2}$ i.e. 1.5625.

Hence

$$\boldsymbol{f(1.5625)} = 1.5625 + 3 - e^{1.5625} = \boldsymbol{-0.20823\ldots}$$

Since $f(1.5625)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.5625$ and $x = 1.5$.

Bisecting this interval gives

$$\frac{1.5625 + 1.5}{2} \text{ i.e. } 1.53125$$

Hence

$$\boldsymbol{f(1.53125)} = 1.53125 + 3 - e^{1.53125} = \boldsymbol{-0.09270\ldots}$$

Since $f(1.53125)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.53125$ and $x = 1.5$.

Bisecting this interval gives

$$\frac{1.53125 + 1.5}{2} \text{ i.e. } 1.515625$$

Hence

$$\boldsymbol{f(1.515625)} = 1.515625 + 3 - e^{1.515625} = \boldsymbol{-0.03664\ldots}$$

Since $f(1.515625)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.515625$ and $x = 1.5$.

Bisecting this interval gives

$$\frac{1.515625 + 1.5}{2} \text{ i.e. } 1.5078125$$

Hence

$$\boldsymbol{f(1.5078125)} = 1.5078125 + 3 - e^{1.5078125} = \boldsymbol{-0.009026\ldots}$$

Since $f(1.5078125)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.5078125$ and $x = 1.5$.

Bisecting this interval gives

$$\frac{1.5078125 + 1.5}{2} \text{ i.e. } 1.50390625$$

Hence

$$\boldsymbol{f(1.50390625)} = 1.50390625 + 3 - e^{1.50390625} = \boldsymbol{+0.004676\ldots}$$

Since $f(1.50390625)$ is positive and $f(1.5078125)$ is negative, a root lies between $x = 1.50390625$ and $x = 1.5078125$.

Bisecting this interval gives

$$\frac{1.50390625 + 1.5078125}{2} \text{ i.e. } 1.505859375$$

Hence

$$\boldsymbol{f(1.505859375)} = 1.505859375 + 3 - e^{1.505859375}$$
$$= \mathbf{-0.0021666\ldots}$$

Since $f(1.50589375)$ is negative and $f(1.50390625)$ is positive, a root lies between $x = 1.50589375$ and $x = 1.50390625$.

Bisecting this interval gives

$$\frac{1.505859375 + 1.50390625}{2} \text{ i.e. } 1.504882813$$

Hence

$$\boldsymbol{f(1.504882813)} = 1.504882813 + 3 - e^{1.504882813}$$
$$= \mathbf{+0.001256\ldots}$$

Since $f(1.504882813)$ is positive and $f(1.505859375)$ is negative,

a root lies between $x = 1.504882813$ and $x = 1.505859375$.

Bisecting this interval gives

$$\frac{1.504882813 + 1.50589375}{2} \text{ i.e. } \mathbf{1.505388282}$$

The last two values of x are 1.504882813 and 1.505388282, i.e. both are equal to 1.505, correct to 3 decimal places.

Hence the root of $x + 3 = e^x$ is $x = 1.505$, correct to 3 decimal places.

The above is a lengthy procedure and it is probably easier to present the data in a table as shown in the table.

x_1	x_2	$x_3 = \frac{x_1 + x_2}{2}$	$f(x_3)$
		0	+2
		1	+1.2817...
		2	−2.3890...
1	2	1.5	+0.0183...
1.5	2	1.75	−1.0046...
1.5	1.75	1.625	−0.4534...
1.5	1.625	1.5625	−0.2082...
1.5	1.5625	1.53125	−0.0927...
1.5	1.53125	1.515625	−0.0366...
1.5	1.515625	1.5078125	−0.0090...
1.5	1.5078125	1.50390625	+0.0046...
1.50390625	1.5078125	1.505859375	−0.0021...
1.50390625	1.505859375	**1.504882813**	+0.0012...
1.504882813	1.505859375	**1.505388282**	

> Problem 3. Solve, correct to 2 decimal places, the equation $2\ln x + x = 2$ using the method of bisection.

Let $f(x) = 2\ln x + x - 2$

$f(0.1) = 2\ln(0.1) + 0.1 - 2 = -6.5051\ldots$

(Note that ln 0 is infinite that is why $x = 0$ was not chosen)

$f(1) = 2\ln 1 + 1 - 2 = -1$

$f(2) = 2\ln 2 + 2 - 2 = +1.3862\ldots$

A change of sign indicates a root lies between $x = 1$ and $x = 2$.

Since $2\ln x + x = 2$ then $2\ln x = -x + 2$; sketches of $2\ln x$ and $-x + 2$ are shown in Fig. 9.4.

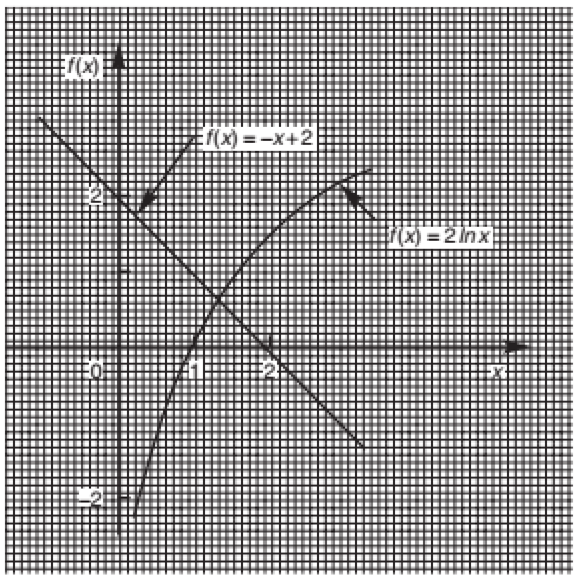

Figure 9.4

As shown in Problem 2, a table of values is produced to reduce space.

x_1	x_2	$x_3 = \frac{x_1 + x_2}{2}$	$f(x_3)$
		0.1	$-6.6051\ldots$
		1	-1
		2	$+1.3862\ldots$
1	2	1.5	$+0.3109\ldots$
1	1.5	1.25	$-0.3037\ldots$
1.25	1.5	1.375	$+0.0119\ldots$
1.25	1.375	1.3125	$-0.1436\ldots$
1.3125	1.375	1.34375	$-0.0653\ldots$
1.34375	1.375	1.359375	$-0.0265\ldots$
1.359375	1.375	**1.3671875**	$-0.0073\ldots$
1.3671875	1.375	**1.37109375**	$+0.0023\ldots$

The last two values of x_3 are both equal to 1.37 when expressed to 2 decimal places. We therefore stop the iterations.

Hence, the solution of $2\ln x + x = 2$ is $x = 1.37$, correct to 2 decimal places.

Now try the following exercise.

Exercise 39 Further problems on the bisection method

Use the method of bisection to solve the following equations to the accuracy stated.

1. Find the positive root of the equation $x^2 + 3x - 5 = 0$, correct to 3 significant figures, using the method of bisection. [1.19]

2. Using the bisection method solve $e^x - x = 2$, correct to 4 significant figures. [1.146]

3. Determine the positive root of $x^2 = 4\cos x$, correct to 2 decimal places using the method of bisection. [1.20]

4. Solve $x - 2 - \ln x = 0$ for the root near to 3, correct to 3 decimal places using the bisection method. [3.146]

5. Solve, correct to 4 significant figures, $x - 2\sin^2 x = 0$ using the bisection method. [1.849]

9.3 An algebraic method of successive approximations

This method can be used to solve equations of the form:

$$a + bx + cx^2 + dx^3 + \cdots = 0,$$

where $a, b, c, d, \ldots$ are constants.

Procedure:

First approximation

(a) Using a graphical or the functional notation method (see Section 9.1) determine an approximate value of the root required, say x_1.

Second approximation

(b) Let the true value of the root be $(x_1 + \delta_1)$.

(c) Determine x_2 the approximate value of $(x_1 + \delta_1)$ by determining the value of $f(x_1 + \delta_1) = 0$, but neglecting terms containing products of δ_1.

Third approximation

(d) Let the true value of the root be $(x_2 + \delta_2)$.

(e) Determine x_3, the approximate value of $(x_2 + \delta_2)$ by determining the value of $f(x_2 + \delta_2) = 0$, but neglecting terms containing products of δ_2.

(f) The fourth and higher approximations are obtained in a similar way.

Using the techniques given in paragraphs (b) to (f), it is possible to continue getting values nearer and nearer to the required root. The procedure is repeated until the value of the required root does not change on two consecutive approximations, when expressed to the required degree of accuracy.

Problem 4. Use an algebraic method of successive approximations to determine the value of the negative root of the quadratic equation: $4x^2 - 6x - 7 = 0$ correct to 3 significant figures. Check the value of the root by using the quadratic formula.

A first estimate of the values of the roots is made by using the functional notation method

$$f(x) = 4x^2 - 6x - 7$$
$$f(0) = 4(0)^2 - 6(0) - 7 = -7$$
$$f(-1) = 4(-1)^2 - 6(-1) - 7 = 3$$

These results show that the negative root lies between 0 and -1, since the value of $f(x)$ changes sign between $f(0)$ and $f(-1)$ (see Section 9.1). The procedure given above for the root lying between 0 and -1 is followed.

First approximation

(a) Let a first approximation be such that it divides the interval 0 to -1 in the ratio of -7 to 3, i.e. let $x_1 = -0.7$.

Second approximation

(b) Let the true value of the root, x_2, be $(x_1 + \delta_1)$.

(c) Let $f(x_1 + \delta_1) = 0$, then, since $x_1 = -0.7$,

$$4(-0.7 + \delta_1)^2 - 6(-0.7 + \delta_1) - 7 = 0$$

Hence, $$4[(-0.7)^2 + (2)(-0.7)(\delta_1) + \delta_1^2] - (6)(-0.7) - 6\delta_1 - 7 = 0$$

Neglecting terms containing products of δ_1 gives:

$$1.96 - 5.6\delta_1 + 4.2 - 6\delta_1 - 7 \approx 0$$

i.e. $$-5.6\delta_1 - 6\delta_1 = -1.96 - 4.2 + 7$$

i.e. $$\delta_1 \approx \frac{-1.96 - 4.2 + 7}{-5.6 - 6} \approx \frac{0.84}{-11.6} \approx -0.0724$$

Thus, x_2, a second approximation to the root is $[-0.7 + (-0.0724)]$,

i.e. $x_2 = -0.7724$, correct to 4 significant figures. (Since the question asked for 3 significant figure accuracy, it is usual to work to one figure greater than this).

The procedure given in (b) and (c) is now repeated for $x_2 = -0.7724$.

Third approximation

(d) Let the true value of the root, x_3, be $(x_2 + \delta_2)$.

(e) Let $f(x_2 + \delta_2) = 0$, then, since $x_2 = -0.7724$,

$$4(-0.7724 + \delta_2)^2 - 6(-0.7724 + \delta_2) - 7 = 0$$

$$4[(-0.7724)^2 + (2)(-0.7724)(\delta_2) + \delta_2^2] - (6)(-0.7724) - 6\delta_2 - 7 = 0$$

Neglecting terms containing products of δ_2 gives:

$$2.3864 - 6.1792\delta_2 + 4.6344 - 6\delta_2 - 7 \approx 0$$

i.e. $$\delta_2 \approx \frac{-2.3864 - 4.6344 + 7}{-6.1792 - 6} \approx \frac{-0.0208}{-12.1792} \approx +0.001708$$

Thus x_3, the third approximation to the root is $(-0.7724 + 0.001708)$,

i.e. $x_3 = -0.7707$, correct to 4 significant figures (or -0.771 correct to 3 significant figures).

Fourth approximation

(f) The procedure given for the second and third approximations is now repeated for

$$x_3 = -0.7707$$

Let the true value of the root, x_4, be $(x_3 + \delta_3)$.

Let $f(x_3 + \delta_3) = 0$, then since $x_3 = -0.7707$,

$$4(-0.7707 + \delta_3)^2 - 6(-0.7707 + \delta_3) - 7 = 0$$

$$4[(-0.7707)^2 + (2)(-0.7707)\delta_3 + \delta_3^2] - 6(-0.7707) - 6\delta_3 - 7 = 0$$

Neglecting terms containing products of δ_3 gives:

$$2.3759 - 6.1656\delta_3 + 4.6242 - 6\delta_3 - 7 \approx 0$$

i.e. $$\delta_3 \approx \frac{-2.3759 - 4.6242 + 7}{-6.1656 - 6} \approx \frac{-0.0001}{-12.156} \approx +0.00000822$$

Thus, x_4, the fourth approximation to the root is $(-0.7707 + 0.00000822)$, i.e. $x_4 = -0.7707$, correct to 4 significant figures, and -0.771, correct to 3 significant figures.

Since the values of the roots are the same on two consecutive approximations, when stated to the required degree of accuracy, then the negative root of $4x^2 - 6x - 7 = 0$ is **-0.771, correct to 3 significant figures**.

[Checking, using the quadratic formula:

$$x = \frac{-(-6) \pm \sqrt{[(-6)^2 - (4)(4)(-7)]}}{(2)(4)}$$
$$= \frac{6 \pm 12.166}{8} = \mathbf{-0.771 \text{ and } 2.27},$$

correct to 3 significant figures]

[**Note on accuracy and errors.** Depending on the accuracy of evaluating the $f(x+\delta)$ terms, one or two iterations (i.e. successive approximations) might be saved. However, it is not usual to work to more than about 4 significant figures accuracy in this type of calculation. If a small error is made in calculations, the only likely effect is to increase the number of iterations.]

Problem 5. Determine the value of the smallest positive root of the equation $3x^3 - 10x^2 + 4x + 7 = 0$, correct to 3 significant figures, using an algebraic method of successive approximations.

The functional notation method is used to find the value of the first approximation.

$$f(x) = 3x^3 - 10x^2 + 4x + 7$$
$$f(0) = 3(0)^3 - 10(0)^2 + 4(0) + 7 = 7$$
$$f(1) = 3(1)^3 - 10(1)^2 + 4(1) + 7 = 4$$
$$f(2) = 3(2)^3 - 10(2)^2 + 4(2) + 7 = -1$$

Following the above procedure:

First approximation

(a) Let the first approximation be such that it divides the interval 1 to 2 in the ratio of 4 to -1, i.e. let x_1 be 1.8.

Second approximation

(b) Let the true value of the root, x_2, be $(x_1 + \delta_1)$.

(c) Let $f(x_1 + \delta_1) = 0$, then since $x_1 = 1.8$,

$$3(1.8 + \delta_1)^3 - 10(1.8 + \delta_1)^2 + 4(1.8 + \delta_1) + 7 = 0$$

Neglecting terms containing products of δ_1 and using the binomial series gives:

$$3[1.8^3 + 3(1.8)^2\,\delta_1] - 10[1.8^2 + (2)(1.8)\,\delta_1] + 4(1.8 + \delta_1) + 7 \approx 0$$
$$3(5.832 + 9.720\,\delta_1) - 32.4 - 36\,\delta_1 + 7.2 + 4\,\delta_1 + 7 \approx 0$$
$$17.496 + 29.16\,\delta_1 - 32.4 - 36\,\delta_1 + 7.2 + 4\,\delta_1 + 7 \approx 0$$
$$\delta_1 \approx \frac{-17.496 + 32.4 - 7.2 - 7}{29.16 - 36 + 4}$$
$$\approx -\frac{0.704}{2.84} \approx -0.2479$$

Thus $x_2 \approx 1.8 - 0.2479 = 1.5521$

Third approximation

(d) Let the true value of the root, x_3, be $(x_2 + \delta_2)$.

(e) Let $f(x_2 + \delta_2) = 0$, then since $x_2 = 1.5521$,

$$3(1.5521 + \delta_2)^3 - 10(1.5521 + \delta_2)^2 + 4(1.5521 + \delta_2) + 7 = 0$$

Neglecting terms containing products of δ_2 gives:

$$11.217 + 21.681\,\delta_2 - 24.090 - 31.042\,\delta_2 + 6.2084 + 4\,\delta_2 + 7 \approx 0$$
$$\delta_2 \approx \frac{-11.217 + 24.090 - 6.2084 - 7}{21.681 - 31.042 + 4}$$
$$\approx \frac{-0.3354}{-5.361}$$
$$\approx 0.06256$$

Thus $x_3 \approx 1.5521 + 0.06256 \approx 1.6147$

(f) Values of x_4 and x_5 are found in a similar way.

$$f(x_3 + \delta_3) = 3(1.6147 + \delta_3)^3 - 10(1.6147 + \delta_3)^2 + 4(1.6147 + \delta_3) + 7 = 0$$

giving $\delta_3 \approx 0.003175$ and $x_4 \approx 1.618$, i.e. 1.62 correct to 3 significant figures

$$f(x_4 + \delta_4) = 3(1.618 + \delta_4)^3 - 10(1.618 + \delta_4)^2 + 4(1.618 + \delta_4) + 7 = 0$$

giving $\delta_4 \approx 0.0000417$, and $x_5 \approx 1.62$, correct to 3 significant figures.

A

Since x_4 and x_5 are the same when expressed to the required degree of accuracy, then the required root is **1.62**, correct to 3 significant figures.

Now try the following exercise.

Exercise 40 Further problems on solving equations by an algebraic method of successive approximations

Use an algebraic method of successive approximation to solve the following equations to the accuracy stated.

1. $3x^2 + 5x - 17 = 0$, correct to 3 significant figures. [−3.36, 1.69]
2. $x^3 - 2x + 14 = 0$, correct to 3 decimal places. [−2.686]
3. $x^4 - 3x^3 + 7x - 5.5 = 0$, correct to 3 significant figures. [−1.53, 1.68]
4. $x^4 + 12x^3 - 13 = 0$, correct to 4 significant figures. [−12.01, 1.000]

9.4 The Newton-Raphson method

The Newton-Raphson formula, often just referred to as **Newton's method**, may be stated as follows:

If r_1 is the approximate value of a real root of the equation $f(x) = 0$, then a closer approximation to the root r_2 is given by:

$$\boldsymbol{r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}}$$

The advantages of Newton's method over the algebraic method of successive approximations is that it can be used for any type of mathematical equation (i.e. ones containing trigonometric, exponential, logarithmic, hyperbolic and algebraic functions), and it is usually easier to apply than the algebraic method.

Problem 6. Use Newton's method to determine the positive root of the quadratic equation $5x^2 + 11x - 17 = 0$, correct to 3 significant figures.
Check the value of the root by using the quadratic formula.

The functional notation method is used to determine the first approximation to the root.

$$f(x) = 5x^2 + 11x - 17$$
$$f(0) = 5(0)^2 + 11(0) - 17 = -17$$
$$f(1) = 5(1)^2 + 11(1) - 17 = -1$$
$$f(2) = 5(2)^2 + 11(2) - 17 = 25$$

This shows that the value of the root is close to $x = 1$.

Let the first approximation to the root, r_1, be 1.

Newton's formula states that a closer approximation,

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$
$$f(x) = 5x^2 + 11x - 17,$$

$$\text{thus, } f(r_1) = 5(r_1)^2 + 11(r_1) - 17$$
$$= 5(1)^2 + 11(1) - 17 = -1$$

$f'(x)$ is the differential coefficient of $f(x)$,

i.e. $f'(x) = 10x + 11$.

Thus $f'(r_1) = 10(r_1) + 11$
$= 10(1) + 11 = 21$

By Newton's formula, a better approximation to the root is:

$$r_2 = 1 - \frac{-1}{21} = 1 - (-0.048) = 1.05,$$

correct to 3 significant figures.

A still better approximation to the root, r_3, is given by:

$$r_3 = r_2 - \frac{f(r_2)}{f'(r_2)}$$
$$= 1.05 - \frac{[5(1.05)^2 + 11(1.05) - 17]}{[10(1.05) + 11]}$$
$$= 1.05 - \frac{0.0625}{21.5}$$
$$= 1.05 - 0.003 = 1.047,$$

i.e. 1.05, correct to 3 significant figures.

Since the values of r_2 and r_3 are the same when expressed to the required degree of accuracy, the

required root is **1.05**, correct to 3 significant figures.
Checking, using the quadratic equation formula,

$$x = \frac{-11 \pm \sqrt{[121 - 4(5)(-17)]}}{(2)(5)} = \frac{-11 \pm 21.47}{10}$$

The positive root is 1.047, i.e. **1.05**, correct to 3 significant figures (This root was determined in Problem 1 using the bisection method; Newton's method is clearly quicker).

> Problem 7. Taking the first approximation as 2, determine the root of the equation $x^2 - 3\sin x + 2\ln(x+1) = 3.5$, correct to 3 significant figures, by using Newton's method.

Newton's formula states that $r_2 = r_1 - \dfrac{f(r_1)}{f'(r_1)}$, where r_1 is a first approximation to the root and r_2 is a better approximation to the root.

Since $\quad f(x) = x^2 - 3\sin x + 2\ln(x+1) - 3.5$

$$f(r_1) = f(2) = 2^2 - 3\sin 2 + 2\ln 3 - 3.5,$$

where sin2 means the sine of 2 radians

$$= 4 - 2.7279 + 2.1972 - 3.5$$
$$= -0.0307$$

$$f'(x) = 2x - 3\cos x + \frac{2}{x+1}$$

$$f'(r_1) = f'(2) = 2(2) - 3\cos 2 + \frac{2}{3}$$
$$= 4 + 1.2484 + 0.6667$$
$$= 5.9151$$

Hence, $\quad r_2 = r_1 - \dfrac{f(r_1)}{f'(r_1)}$

$$= 2 - \frac{-0.0307}{5.9151}$$

$= 2.005$ or 2.01, correct to 3 significant figures.

A still better approximation to the root, r_3, is given by:

$$r_3 = r_2 - \frac{f(r_2)}{f'(r_2)}$$

$$= 2.005 - \frac{[(2.005)^2 - 3\sin 2.005 + 2\ln 3.005 - 3.5]}{\left[2(2.005) - 3\cos 2.005 + \dfrac{2}{2.005+1}\right]}$$

$$= 2.005 - \frac{(-0.00104)}{5.9376} = 2.005 + 0.000175$$

i.e. $r_3 = 2.01$, correct to 3 significant figures.

Since the values of r_2 and r_3 are the same when expressed to the required degree of accuracy, then the required root is **2.01**, correct to 3 significant figures.

> Problem 8. Use Newton's method to find the positive root of:
> $$(x+4)^3 - e^{1.92x} + 5\cos\frac{x}{3} = 9,$$
> correct to 3 significant figures.

The functional notational method is used to determine the approximate value of the root.

$$f(x) = (x+4)^3 - e^{1.92x} + 5\cos\frac{x}{3} - 9$$

$$f(0) = (0+4)^3 - e^0 + 5\cos 0 - 9 = 59$$

$$f(1) = 5^3 - e^{1.92} + 5\cos\frac{1}{3} - 9 \approx 114$$

$$f(2) = 6^3 - e^{3.84} + 5\cos\frac{2}{3} - 9 \approx 164$$

$$f(3) = 7^3 - e^{5.76} + 5\cos 1 - 9 \approx 19$$

$$f(4) = 8^3 - e^{7.68} + 5\cos\frac{4}{3} - 9 \approx -1660$$

From these results, let a first approximation to the root be $r_1 = 3$.
Newton's formula states that a better approximation to the root,

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

$$f(r_1) = f(3) = 7^3 - e^{5.76} + 5\cos 1 - 9$$
$$= 19.35$$

$$f'(x) = 3(x+4)^2 - 1.92e^{1.92x} - \frac{5}{3}\sin\frac{x}{3}$$

$$f'(r_1) = f'(3) = 3(7)^2 - 1.92e^{5.76} - \frac{5}{3}\sin 1$$
$$= -463.7$$

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Thus, $r_2 = 3 - \dfrac{19.35}{-463.7} = 3 + 0.042$

$= 3.042 = 3.04$,

correct to 3 significant figure

Similarly, $r_3 = 3.042 - \dfrac{f(3.042)}{f'(3.042)}$

$= 3.042 - \dfrac{(-1.146)}{(-513.1)}$

$= 3.042 - 0.0022 = 3.0398 = 3.04$,

correct to 3 significant figure.

Since r_2 and r_3 are the same when expressed to the required degree of accuracy, then the required root is **3.04**, correct to 3 significant figures.

Now try the following exercise.

Exercise 41 Further problems on Newton's method

In Problems 1 to 7, use **Newton's method** to solve the equations given to the accuracy stated.

1. $x^2 - 2x - 13 = 0$, correct to 3 decimal places. [−2.742, 4.742]
2. $3x^3 - 10x = 14$, correct to 4 significant figures. [2.313]
3. $x^4 - 3x^3 + 7x = 12$, correct to 3 decimal places. [−1.721, 2.648]
4. $3x^4 - 4x^3 + 7x - 12 = 0$, correct to 3 decimal places. [−1.386, 1.491]
5. $3 \ln x + 4x = 5$, correct to 3 decimal places. [1.147]
6. $x^3 = 5 \cos 2x$, correct to 3 significant figures. [−1.693, −0.846, 0.744]
7. $300e^{-2\theta} + \dfrac{\theta}{2} = 6$, correct to 3 significant figures. [2.05]
8. Solve the equations in Problems 1 to 5, Exercise 39, page 80 and Problems 1 to 4, Exercise 40, page 83 using Newton's method.
9. A Fourier analysis of the instantaneous value of a waveform can be represented by:

   $$y = \left(t + \frac{\pi}{4}\right) + \sin t + \frac{1}{8} \sin 3t$$

   Use Newton's method to determine the value of t near to 0.04, correct to 4 decimal places, when the amplitude, y, is 0.880. [0.0399]
10. A damped oscillation of a system is given by the equation:

    $$y = -7.4e^{0.5t} \sin 3t.$$

    Determine the value of t near to 4.2, correct to 3 significant figures, when the magnitude y of the oscillation is zero. [4.19]
11. The critical speeds of oscillation, λ, of a loaded beam are given by the equation:

    $$\lambda^3 - 3.250\lambda^2 + \lambda - 0.063 = 0$$

    Determine the value of λ which is approximately equal to 3.0 by Newton's method, correct to 4 decimal places. [2.9143]