12

Introduction to trigonometry

12.1 Trigonometry

Trigonometry is the branch of mathematics which deals with the measurement of sides and angles of triangles, and their relationship with each other. There are many applications in engineering where a knowledge of trigonometry is needed.

12.2 The theorem of Pythagoras

With reference to Fig. 12.1, the side opposite the right angle (i.e. side b) is called the **hypotenuse**. The **theorem of Pythagoras** states:

'In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.'

Hence $\boldsymbol{b^2 = a^2 + c^2}$

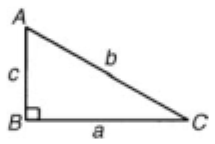

Figure 12.1

Problem 1. In Fig. 12.2, find the length of EF.

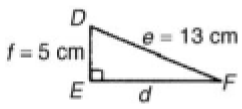

Figure 12.2

By Pythagoras' theorem:

$$e^2 = d^2 + f^2$$

Hence $13^2 = d^2 + 5^2$

$$169 = d^2 + 25$$

$$d^2 = 169 - 25 = 144$$

Thus $d = \sqrt{144} = 12\,\text{cm}$

i.e. $\boldsymbol{EF = 12\,\text{cm}}$

Problem 2. Two aircraft leave an airfield at the same time. One travels due north at an average speed of 300 km/h and the other due west at an average speed of 220 km/h. Calculate their distance apart after 4 hours.

After 4 hours, the first aircraft has travelled $4 \times 300 = 1200$ km, due north, and the second aircraft has travelled $4 \times 220 = 880$ km due west, as shown in Fig. 12.3. Distance apart after 4 hours $= BC$.

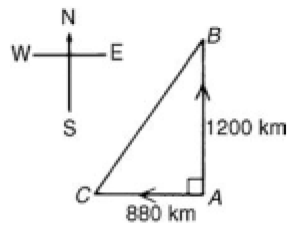

Figure 12.3

From Pythagoras' theorem:

$$BC^2 = 1200^2 + 880^2 = 1\,440\,000 + 774\,400$$

and $BC = \sqrt{(2\,214\,400)}$

Hence distance apart after 4 hours = 1488 km.

Now try the following exercise.

Exercise 52 Further problems on the theorem of Pythagoras

1. In a triangle CDE, $D = 90°$, $CD = 14.83$ mm and $CE = 28.31$ mm. Determine the length of DE. [24.11 mm]

2. Triangle PQR is isosceles, Q being a right angle. If the hypotenuse is 38.47 cm find (a) the lengths of sides PQ and QR, and

(b) the value of $\angle QPR$.
[(a) 27.20 cm each (b) 45°]

3. A man cycles 24 km due south and then 20 km due east. Another man, starting at the same time as the first man, cycles 32 km due east and then 7 km due south. Find the distance between the two men. [20.81 km]

4. A ladder 3.5 m long is placed against a perpendicular wall with its foot 1.0 m from the wall. How far up the wall (to the nearest centimetre) does the ladder reach? If the foot of the ladder is now moved 30 cm further away from the wall, how far does the top of the ladder fall? [3.35 m, 10 cm]

5. Two ships leave a port at the same time. One travels due west at 18.4 km/h and the other due south at 27.6 km/h. Calculate how far apart the two ships are after 4 hours. [132.7 km]

12.3 Trigonometric ratios of acute angles

(a) With reference to the right-angled triangle shown in Fig. 12.4:

(i) $\text{sine}\,\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}}$

i.e. $\boldsymbol{\sin\theta = \dfrac{b}{c}}$

(ii) $\text{cosine}\,\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$

i.e. $\boldsymbol{\cos\theta = \dfrac{a}{c}}$

(iii) $\text{tangent}\,\theta = \dfrac{\text{opposite side}}{\text{adjacent side}}$

i.e. $\boldsymbol{\tan\theta = \dfrac{b}{a}}$

(iv) $\text{secant}\,\theta = \dfrac{\text{hypotenuse}}{\text{adjacent side}}$

i.e. $\boldsymbol{\sec\theta = \dfrac{c}{a}}$

(v) $\text{cosecant}\,\theta = \dfrac{\text{hypotenuse}}{\text{opposite side}}$

i.e. $\boldsymbol{\text{cosec}\,\theta = \dfrac{c}{b}}$

(vi) $\text{cotangent}\,\theta = \dfrac{\text{adjacent side}}{\text{opposite side}}$

i.e. $\boldsymbol{\cot\theta = \dfrac{a}{b}}$

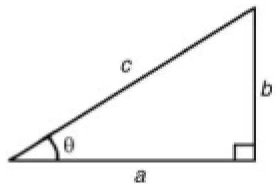

Figure 12.4

(b) From above,

(i) $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{b}{c}}{\frac{a}{c}} = \dfrac{b}{a} = \tan\theta$,

i.e. $\boldsymbol{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$

(ii) $\dfrac{\cos\theta}{\sin\theta} = \dfrac{\frac{a}{c}}{\frac{b}{c}} = \dfrac{a}{b} = \cot\theta$,

i.e. $\boldsymbol{\cot\theta = \dfrac{\cos\theta}{\sin\theta}}$

(iii) $\boldsymbol{\sec\theta = \dfrac{1}{\cos\theta}}$

(iv) $\boldsymbol{\text{cosec}\,\theta = \dfrac{1}{\sin\theta}}$

(Note '*s*' and '*c*' go together)

(v) $\boldsymbol{\cot\theta = \dfrac{1}{\tan\theta}}$

Secants, cosecants and cotangents are called the **reciprocal ratios**.

Problem 3. If $\cos X = \dfrac{9}{41}$ determine the value of the other five trigonometry ratios.

Fig. 12.5 shows a right-angled triangle XYZ.

Since $\cos X = \dfrac{9}{41}$, then $XY = 9$ units and $XZ = 41$ units.

Using Pythagoras' theorem: $41^2 = 9^2 + YZ^2$ from which $YZ = \sqrt{(41^2 - 9^2)} = 40$ units.

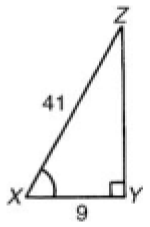

Figure 12.5

Thus

$$\mathbf{\sin X = \frac{40}{41}, \tan X = \frac{40}{9} = 4\frac{4}{9},}$$

$$\mathbf{\operatorname{cosec} X = \frac{41}{40} = 1\frac{1}{40},}$$

$$\mathbf{\sec X = \frac{41}{9} = 4\frac{5}{9}} \text{ and } \mathbf{\cot X = \frac{9}{40}}$$

Problem 4. If $\sin\theta = 0.625$ and $\cos\theta = 0.500$ determine, without using trigonometric tables or calculators, the values of $\operatorname{cosec}\theta, \sec\theta, \tan\theta$ and $\cot\theta$.

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{0.625} = \mathbf{1.60}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{0.500} = \mathbf{2.00}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.625}{0.500} = \mathbf{1.25}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{0.500}{0.625} = \mathbf{0.80}$$

Problem 5. Point A lies at co-ordinate (2, 3) and point B at (8, 7). Determine (a) the distance AB, (b) the gradient of the straight line AB, and (c) the angle AB makes with the horizontal.

(a) Points A and B are shown in Fig. 12.6(a).

In Fig. 12.6(b), the horizontal and vertical lines AC and BC are constructed.

Since ABC is a right-angled triangle, and $AC = (8 - 2) = 6$ and $BC = (7 - 3) = 4$, then by Pythagoras' theorem

$$AB^2 = AC^2 + BC^2 = 6^2 + 4^2$$

and $\mathbf{AB} = \sqrt{(6^2 + 4^2)} = \sqrt{52} = \mathbf{7.211,}$ correct to 3 decimal places

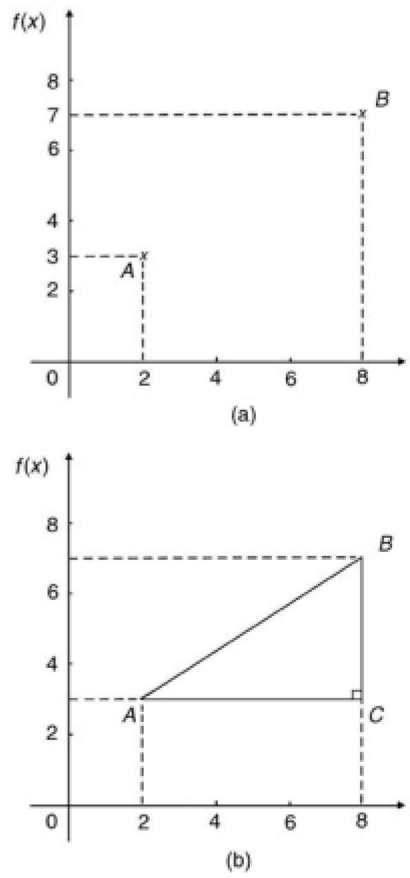

Figure 12.6

(b) The gradient of AB is given by $\tan A$,

i.e. **gradient** $= \tan A = \dfrac{BC}{AC} = \dfrac{4}{6} = \mathbf{\dfrac{2}{3}}$

(c) **The angle AB makes with the horizontal** is given by $\tan^{-1}\frac{2}{3} = \mathbf{33.69°}$.

Now try the following exercise.

Exercise 53 Further problems on trigonometric ratios of acute

1. In triangle ABC shown in Fig. 12.7, find $\sin A, \cos A, \tan A, \sin B, \cos B$ and $\tan B$.
$$\left[\begin{array}{l}\sin A = \frac{3}{5}, \cos A = \frac{4}{5}, \tan A = \frac{3}{4} \\ \sin B = \frac{4}{5}, \cos B = \frac{3}{5}, \tan B = \frac{4}{3}\end{array}\right]$$

2. If $\cos A = \dfrac{15}{17}$ find $\sin A$ and $\tan A$, in fraction form. $\left[\sin A = \dfrac{8}{17}, \tan A = \dfrac{8}{15}\right]$

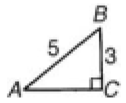

Figure 12.7

3. For the right-angled triangle shown in Fig. 12.8, find:

   (a) $\sin\alpha$ (b) $\cos\theta$ (c) $\tan\theta$

   $$\left[\text{(a) } \frac{15}{17} \quad \text{(b) } \frac{15}{17} \quad \text{(c) } \frac{8}{15}\right]$$

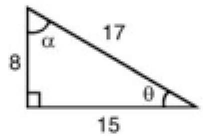

Figure 12.8

4. Point P lies at co-ordinate $(-3, 1)$ and point Q at $(5, -4)$. Determine

   (a) the distance PQ
   (b) the gradient of the straight line PQ and
   (c) the angle PQ makes with the horizontal

   [(a) 9.434 (b) -0.625 (c) 32°]

12.4 Solution of right-angled triangles

To 'solve a right-angled triangle' means 'to find the unknown sides and angles'. This is achieved by using (i) the theorem of Pythagoras, and/or (ii) trigonometric ratios. This is demonstrated in the following problems.

Problem 6. In triangle PQR shown in Fig. 12.9, find the lengths of PQ and PR.

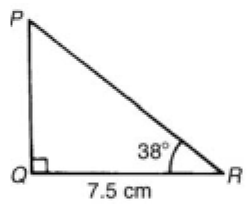

Figure 12.9

$$\tan 38° = \frac{PQ}{QR} = \frac{PQ}{7.5}$$

hence $PQ = 7.5 \tan 38° = 7.5(0.7813)$

$= \mathbf{5.860\,cm}$

$$\cos 38° = \frac{QR}{PR} = \frac{7.5}{PR}$$

hence $PR = \dfrac{7.5}{\cos 38°} = \dfrac{7.5}{0.7880} = \mathbf{9.518\,cm}$

[Check: Using Pythagoras' theorem

$(7.5)^2 + (5.860)^2 = 90.59 = (9.518)^2$]

Problem 7. Solve the triangle ABC shown in Fig. 12.10.

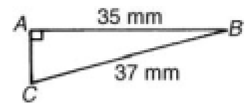

Figure 12.10

To 'solve triangle ABC' means 'to find the length AC and angles B and C'

$$\sin C = \frac{35}{37} = 0.94595$$

hence $\angle \boldsymbol{C} = \sin^{-1} 0.94595 = 71.08° = \mathbf{71°5'}$.
$\angle \boldsymbol{B} = 180° - 90° - 71°5' = \mathbf{18°55'}$ (since angles in a triangle add up to 180°)

$$\sin B = \frac{AC}{37}$$

hence $\boldsymbol{AC} = 37 \sin 18°55' = 37(0.3242)$

$= \mathbf{12.0\,mm}$

or, using Pythagoras' theorem, $37^2 = 35^2 + AC^2$, from which, $\boldsymbol{AC} = \sqrt{(37^2 - 35^2)} = \mathbf{12.0\,mm}$.

Problem 8. Solve triangle XYZ given $\angle X = 90°$, $\angle Y = 23°17'$ and $YZ = 20.0$ mm. Determine also its area.

It is always advisable to make a reasonably accurate sketch so as to visualize the expected magnitudes of unknown sides and angles. Such a sketch is shown in Fig. 12.11.

$$\angle Z = 180° - 90° - 23°17' = \mathbf{66°43'}$$

$$\sin 23°17' = \frac{XZ}{20.0}$$

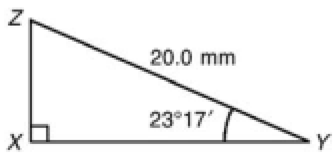

Figure 12.11

hence $\quad XZ = 20.0 \sin 23°17'$

$= 20.0(0.3953) = \mathbf{7.906\,mm}$

$$\cos 23°17' = \frac{XY}{20.0}$$

hence $\quad XY = 20.0 \cos 23°17'$

$= 20.0(0.9186) = \mathbf{18.37\,mm}$

[Check: Using Pythagoras' theorem

$(18.37)^2 + (7.906)^2 = 400.0 = (20.0)^2$]

Area of triangle XYZ

$= \frac{1}{2}$ (base) (perpendicular height)

$= \frac{1}{2}(XY)(XZ) = \frac{1}{2}(18.37)(7.906)$

$= \mathbf{72.62\,mm^2}$

Now try the following exercise.

Exercise 54 Further problems on the solution of right-angled triangles

1. Solve triangle ABC in Fig. 12.12(i).
$\left[BC = 3.50\,\text{cm}, AB = 6.10\,\text{cm}, \angle B = 55°\right]$

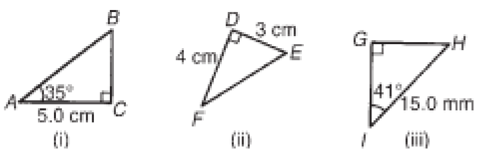

Figure 12.12

2. Solve triangle DEF in Fig. 12.12(ii)
$[FE = 5\,\text{cm}, \angle E = 53°8', \angle F = 36°52']$

3. Solve triangle GHI in Fig. 12.12(iii)
$\left[GH = 9.841\,\text{mm}, GI = 11.32\,\text{mm}, \angle H = 49°\right]$

4. Solve the triangle JKL in Fig. 12.13(i) and find its area $\left[KL = 5.43\,\text{cm}, JL = 8.62\,\text{cm}, \angle J = 39°, \text{ area } = 18.19\,\text{cm}^2\right]$

5. Solve the triangle MNO in Fig. 12.13(ii) and find its area
$\left[MN = 28.86\,\text{mm}, NO = 13.82\,\text{mm}, \angle O = 64°25', \text{ area } = 199.4\,\text{mm}^2\right]$

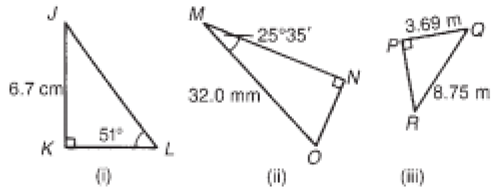

Figure 12.13

6. Solve the triangle PQR in Fig. 12.13(iii) and find its area
$\left[PR = 7.934\,\text{m}, \angle Q = 65°3', \angle R = 24°57', \text{ area } = 14.64\,\text{m}^2\right]$

7. A ladder rests against the top of the perpendicular wall of a building and makes an angle of 73° with the ground. If the foot of the ladder is 2 m from the wall, calculate the height of the building. [6.54 m]

12.5 Angles of elevation and depression

(a) If, in Fig. 12.14, BC represents horizontal ground and AB a vertical flagpole, then the **angle of elevation** of the top of the flagpole, A, from the point C is the angle that the imaginary straight line AC must be raised (or elevated) from the horizontal CB, i.e. angle θ.

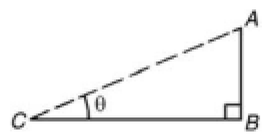

Figure 12.14

(b) If, in Fig. 12.15, PQ represents a vertical cliff and R a ship at sea, then the **angle of depression** of the ship from point P is the angle through which the imaginary straight line PR must be lowered (or depressed) from the horizontal to the ship, i.e. angle ϕ.

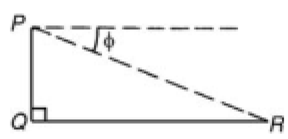

Figure 12.15

B

(Note, $\angle PRQ$ is also ϕ—alternate angles between parallel lines.)

> Problem 9. An electricity pylon stands on horizontal ground. At a point 80 m from the base of the pylon, the angle of elevation of the top of the pylon is 23°. Calculate the height of the pylon to the nearest metre.

Figure 12.16 shows the pylon AB and the angle of elevation of A from point C is 23°

$$\tan 23° = \frac{AB}{BC} = \frac{AB}{80}$$

Hence height of pylon AB

$$= 80 \tan 23° = 80(0.4245) = 33.96\,\text{m}$$
$$= \mathbf{34\,m\ to\ the\ nearest\ metre}$$

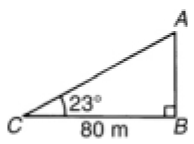

Figure 12.16

> Problem 10. A surveyor measures the angle of elevation of the top of a perpendicular building as 19°. He moves 120 m nearer the building and finds the angle of elevation is now 47°. Determine the height of the building.

The building PQ and the angles of elevation are shown in Fig. 12.17.

In triangle PQS,

$$\tan 19° = \frac{h}{x+120}$$

hence $h = \tan 19°(x + 120)$,

i.e. $h = 0.3443(x + 120)$ (1)

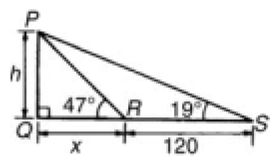

Figure 12.17

In triangle PQR, $\tan 47° = \frac{h}{x}$

hence $h = \tan 47°(x)$, i.e. $h = 1.0724x$ (2)

Equating equations (1) and (2) gives:

$$0.3443(x + 120) = 1.0724x$$
$$0.3443x + (0.3443)(120) = 1.0724x$$
$$(0.3443)(120) = (1.0724 - 0.3443)x$$
$$41.316 = 0.7281x$$
$$x = \frac{41.316}{0.7281} = 56.74\,\text{m}$$

From equation (2), **height of building**,

$\boldsymbol{h} = 1.0724x = 1.0724(56.74) = \mathbf{60.85\,m}$.

> Problem 11. The angle of depression of a ship viewed at a particular instant from the top of a 75 m vertical cliff is 30°. Find the distance of the ship from the base of the cliff at this instant. The ship is sailing away from the cliff at constant speed and 1 minute later its angle of depression from the top of the cliff is 20°. Determine the speed of the ship in km/h.

Figure 12.18 shows the cliff AB, the initial position of the ship at C and the final position at D. Since the angle of depression is initially 30° then $\angle ACB = 30°$ (alternate angles between parallel lines).

$$\tan 30° = \frac{AB}{BC} = \frac{75}{BC}$$

hence $BC = \dfrac{75}{\tan 30°} = \dfrac{75}{0.5774} = \mathbf{129.9\,m}$

= **initial position of ship from base of cliff**

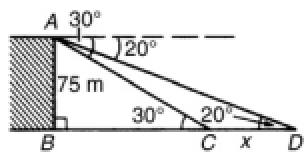

Figure 12.18

In triangle ABD,

$$\tan 20° = \frac{AB}{BD} = \frac{75}{BC + CD} = \frac{75}{129.9 + x}$$

Hence $\quad 129.9 + x = \dfrac{75}{\tan 20°} = \dfrac{75}{0.3640}$

$= 206.0\text{ m}$

from which $\quad x = 206.0 - 129.9 = 76.1\text{ m}$

Thus the ship sails 76.1 m in 1 minute, i.e. 60 s, hence speed of ship

$$= \frac{\text{distance}}{\text{time}} = \frac{76.1}{60}\text{ m/s}$$

$$= \frac{76.1 \times 60 \times 60}{60 \times 1000}\text{ km/h} = \mathbf{4.57\ km/h}$$

Now try the following exercise.

Exercise 55 Further problems on angles of elevation and depression

1. If the angle of elevation of the top of a vertical 30 m high aerial is 32°, how far is it to the aerial? [48 m]
2. From the top of a vertical cliff 80.0 m high the angles of depression of two buoys lying due west of the cliff are 23° and 15°, respectively. How far are the buoys apart? [110.1 m]
3. From a point on horizontal ground a surveyor measures the angle of elevation of the top of a flagpole as 18°40′. He moves 50 m nearer to the flagpole and measures the angle of elevation as 26°22′. Determine the height of the flagpole. [53.0 m]
4. A flagpole stands on the edge of the top of a building. At a point 200 m from the building the angles of elevation of the top and bottom of the pole are 32° and 30° respectively. Calculate the height of the flagpole. [9.50 m]
5. From a ship at sea, the angles of elevation of the top and bottom of a vertical lighthouse standing on the edge of a vertical cliff are 31° and 26°, respectively. If the lighthouse is 25.0 m high, calculate the height of the cliff. [107.8 m]
6. From a window 4.2 m above horizontal ground the angle of depression of the foot of a building across the road is 24° and the angle of elevation of the top of the building is 34°. Determine, correct to the nearest centimetre, the width of the road and the height of the building. [9.43 m, 10.56 m]
7. The elevation of a tower from two points, one due east of the tower and the other due west of it are 20° and 24°, respectively, and the two points of observation are 300 m apart. Find the height of the tower to the nearest metre. [60 m]

12.6 Evaluating trigonometric ratios

Four-figure tables are available which gives sines, cosines, and tangents, for angles between 0° and 90°. However, the easiest method of evaluating trigonometric functions of any angle is by using a **calculator**.

The following values, correct to 4 decimal places, may be checked:

sine 18° = 0.3090, cosine 56° = 0.5592

sine 172° = 0.1392 cosine 115° = −0.4226,

sine 241.63° = −0.8799, cosine 331.78° = 0.8811

tangent 29° = 0.5543,

tangent 178° = −0.0349

tangent 296.42° = −2.0127

To evaluate, say, sine 42°23′ using a calculator means finding sine $42\dfrac{23°}{60}$ since there are 60 minutes in 1 degree.

$$\frac{23}{60} = 0.383\dot{3} \text{ thus } 42°23' = 42.38\dot{3}°$$

Thus sine $42°23' = \text{sine } 42.38\dot{3}° = 0.6741$, correct to 4 decimal places.

Similarly, cosine $72°38' = \text{cosine } 72\dfrac{38°}{60} = 0.2985$, correct to 4 decimal places.

Most calculators contain only sine, cosine and tangent functions. Thus to evaluate secants, cosecants and cotangents, reciprocals need to be used. The following values, correct to 4 decimal places, may be checked:

$$\text{secant } 32° = \frac{1}{\cos 32°} = 1.1792$$

$$\text{cosecant } 75° = \frac{1}{\sin 75°} = 1.0353$$

$$\text{cotangent } 41° = \frac{1}{\tan 41°} = 1.1504$$

$$\text{secant } 215.12° = \frac{1}{\cos 215.12°} = -1.2226$$

$$\text{cosecant } 321.62° = \frac{1}{\sin 321.62°} = -1.6106$$

$$\text{cotangent } 263.59° = \frac{1}{\tan 263.59°} = 0.1123$$

Problem 12. Evaluate correct to 4 decimal places:
(a) sine 168°14′ (b) cosine 271.41°
(c) tangent 98°4′

(a) $\text{sine } 168°14' = \text{sine } 168\frac{14°}{60} = \mathbf{0.2039}$

(b) $\text{cosine } 271.41° = \mathbf{0.0246}$

(c) $\text{tangent } 98°4' = \tan 98\frac{4°}{60} = \mathbf{-7.0558}$

Problem 13. Evaluate, correct to 4 decimal places: (a) secant 161° (b) secant 302°29′

(a) $\sec 161° = \dfrac{1}{\cos 161°} = \mathbf{-1.0576}$

(b) $\sec 302°29' = \dfrac{1}{\cos 302°29'} = \dfrac{1}{\cos 302\frac{29°}{60}} = \mathbf{1.8620}$

Problem 14. Evaluate, correct to 4 significant figures:
(a) cosecant 279.16° (b) cosecant 49°7′

(a) $\text{cosec } 279.16° = \dfrac{1}{\sin 279.16°} = \mathbf{-1.013}$

(b) $\text{cosec } 49°7' = \dfrac{1}{\sin 49°7'} = \dfrac{1}{\sin 49\frac{7°}{60}} = \mathbf{1.323}$

Problem 15. Evaluate, correct to 4 decimal places:
(a) cotangent 17.49° (b) cotangent 163°52′

(a) $\cot 17.49° = \dfrac{1}{\tan 17.49°} = \mathbf{3.1735}$

(b) $\cot 163°52' = \dfrac{1}{\tan 163°52'} = \dfrac{1}{\tan 163\frac{52°}{60}} = \mathbf{-3.4570}$

Problem 16. Evaluate, correct to 4 significant figures:
(a) sin 1.481 (b) cos $(3\pi/5)$ (c) tan 2.93

(a) sin 1.481 means the sine of 1.481 radians. Hence a calculator needs to be on the radian function. Hence sin 1.481 = **0.9960**.
(b) $\cos(3\pi/5) = \cos 1.884955\cdots = \mathbf{-0.3090}$.
(c) $\tan 2.93 = \mathbf{-0.2148}$.

Problem 17. Evaluate, correct to 4 decimal places:
(a) secant 5.37 (b) cosecant $\pi/4$
(c) cotangent $\pi/24$

(a) Again, with no degrees sign, it is assumed that 5.37 means 5.37 radians.

Hence $\sec 5.37 = \dfrac{1}{\cos 5.37} = \mathbf{1.6361}$

(b) $\text{cosec}(\pi/4) = \dfrac{1}{\sin(\pi/4)} = \dfrac{1}{\sin 0.785398\ldots} = \mathbf{1.4142}$

(c) $\cot(5\pi/24) = \dfrac{1}{\tan(5\pi/24)} = \dfrac{1}{\tan 0.654498\ldots} = \mathbf{1.3032}$

Problem 18. Determine the acute angles:
(a) $\sec^{-1} 2.3164$ (b) $\text{cosec}^{-1} 1.1784$
(c) $\cot^{-1} 2.1273$

(a)
$$\sec^{-1} 2.3164 = \cos^{-1}\left(\frac{1}{2.3164}\right) = \cos^{-1} 0.4317\ldots = \mathbf{64.42°} \text{ or } \mathbf{64°25'} \text{ or } \mathbf{1.124 \text{ radians}}$$

(b)
$$\text{cosec}^{-1} 1.1784 = \sin^{-1}\left(\frac{1}{1.1784}\right) = \sin^{-1} 0.8486\ldots = \mathbf{58.06°} \text{ or } \mathbf{58°4'} \text{ or } \mathbf{1.013 \text{ radians}}$$

B

(c) $\cot^{-1} 2.1273 = \tan^{-1}\left(\dfrac{1}{2.1273}\right)$

$= \tan^{-1} 0.4700\ldots$
$= \mathbf{25.18°}$ or $\mathbf{25°11'}$
or **0.439 radians**

> Problem 19. Evaluate the following expression, correct to 4 significant figures:
>
> $$\frac{4\sec 32°10' - 2\cot 15°19'}{3\,\mathrm{cosec}\,63°8'\tan 14°57'}$$

By calculator:

$\sec 32°10' = 1.1813, \cot 15°19' = 3.6512$

$\mathrm{cosec}\,63°8' = 1.1210, \tan 14°57' = 0.2670$

Hence $\dfrac{4\sec 32°10' - 2\cot 15°19'}{3\,\mathrm{cosec}\,63°8'\tan 14°57'}$

$$= \frac{4(1.1813) - 2(3.6512)}{3(1.1210)(0.2670)}$$

$$= \frac{4.7252 - 7.3024}{0.8979}$$

$$= \frac{-2.5772}{0.8979} = \mathbf{-2.870},$$

correct to 4 significant figures

> Problem 20. Evaluate correct to 4 decimal places:
>
> (a) sec (–115°) (b) cosec (–95°47′)

(a) Positive angles are considered by convention to be anticlockwise and negative angles as clockwise.
Hence –115° is actually the same as 245° (i.e. 360° – 115°)

Hence $\sec(-115°) = \sec 245° = \dfrac{1}{\cos 245°}$
$= \mathbf{-2.3662}$

(b) $\mathrm{cosec}(-95°47') = \dfrac{1}{\sin\left(-95\dfrac{47°}{60}\right)} = \mathbf{-1.0051}$

Now try the following exercise.

Exercise 56 Further problems on evaluating trigonometric ratios

In Problems 1 to 8, evaluate correct to 4 decimal places:

1. (a) sine 27° (b) sine 172.41° (c) sine 302°52′
[(a) 0.4540 (b) 0.1321 (c) −0.8399]

2. (a) cosine 124° (b) cosine 21.46° (c) cosine 284°10′
[(a) −0.5592 (b) 0.9307 (c) 0.2447]

3. (a) tangent 145° (b) tangent 310.59° (c) tangent 49°16′
[(a) −0.7002 (b) −1.1671 (c) 1.1612]

4. (a) secant 73° (b) secant 286.45° (c) secant 155°41′
[(a) 3.4203 (b) 3.5313 (c) –1.0974]

5. (a) cosecant 213° (b) cosecant 15.62° (c) cosecant 311°50′
[(a) −1.8361 (b) 3.7139 (c) −1.3421]

6. (a) cotangent 71° (b) cotangent 151.62° (c) cotangent 321°23′
[(a) 0.3443 (b) –1.8510 (c) −1.2519]

7. (a) sine $\dfrac{2\pi}{3}$ (b) cos 1.681 (c) tan 3.672
[(a) 0.8660 (b) −0.1010 (c) 0.5865]

8. (a) sec $\dfrac{\pi}{8}$ (b) cosec 2.961 (c) cot 2.612
[(a) 1.0824 (b) 5.5675 (c) –1.7083]

In Problems 9 to 14, determine the acute angle in degrees (correct to 2 decimal places), degrees and minutes, and in radians (correct to 3 decimal places).

9. $\sin^{-1} 0.2341$ $\left[\begin{array}{l}13.54°, 13°32',\\ 0.236\,\text{rad}\end{array}\right]$

10. $\cos^{-1} 0.8271$ $\left[\begin{array}{l}34.20°, 34°12',\\ 0.597\,\text{rad}\end{array}\right]$

11. $\tan^{-1} 0.8106$ $\left[\begin{array}{l}39.03°, 39°2',\\ 0.681\,\text{rad}\end{array}\right]$

12. $\sec^{-1} 1.6214$ $\left[\begin{array}{l}51.92°, 51°55',\\ 0.906\,\text{rad}\end{array}\right]$

13. $\text{cosec}^{-1} 2.4891$ $\left[\begin{array}{l}23.69°, 23°41',\\ 0.413\,\text{rad}\end{array}\right]$

14. $\cot^{-1} 1.9614$ $\left[\begin{array}{l}27.01°, 27°1',\\ 0.471\,\text{rad}\end{array}\right]$

In Problems 15 to 18, evaluate correct to 4 significant figures.

15. $4\cos 56°19' - 3\sin 21°57'$ [1.097]

16. $\dfrac{11.5\tan 49°11' - \sin 90°}{3\cos 45°}$ [5.805]

17. $\dfrac{5\sin 86°3'}{3\tan 14°29' - 2\cos 31°9'}$ [−5.325]

18. $\dfrac{6.4\,\text{cosec}\,29°5' - \sec 81°}{2\cot 12°}$ [0.7199]

19. Determine the acute angle, in degrees and minutes, correct to the nearest minute, given by $\sin^{-1}\left(\dfrac{4.32\sin 42°16'}{7.86}\right)$ [21°42′]

20. If $\tan x = 1.5276$, determine $\sec x$, $\text{cosec}\, x$, and $\cot x$. (Assume x is an acute angle) [1.8258, 1.1952, 0.6546]

In Problems 21 to 23 evaluate correct to 4 significant figures

21. $\dfrac{(\sin 34°27')(\cos 69°2')}{(2\tan 53°39')}$ [0.07448]

22. $3\cot 14°15' \sec 23°9'$ [12.85]

23. $\dfrac{\text{cosec}\,27°19' + \sec 45°29'}{1 - \text{cosec}\,27°19' \sec 45°29'}$ [−1.710]

24. Evaluate correct to 4 decimal places:
(a) sine (−125°) (b) tan (−241°)
(c) cos (−49°15′)
[(a) −0.8192 (b) −1.8040
(c) 0.6528]

25. Evaluate correct to 5 significant figures:
(a) cosec (−143°) (b) cot (−252°)
(c) sec (−67°22′)
[(a) −1.6616 (b) −0.32492
(c) 2.5985]

12.7 Sine and cosine rules

To **'solve a triangle'** means 'to find the values of unknown sides and angles'. If a triangle is **right angled**, trigonometric ratios and the theorem of Pythagoras may be used for its solution, as shown in Section 12.4. However, for a **non-right-angled triangle**, trigonometric ratios and Pythagoras' theorem **cannot** be used. Instead, two rules, called the **sine rule** and the **cosine rule**, are used.

Sine rule

With reference to triangle ABC of Fig. 12.19, the **sine rule** states:

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

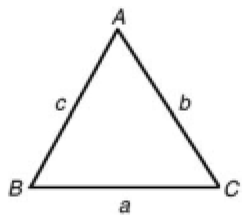

Figure 12.19

The rule may be used only when:

(i) 1 side and any 2 angles are initially given, or
(ii) 2 sides and an angle (not the included angle) are initially given.

Cosine rule

With reference to triangle ABC of Fig. 12.19, the **cosine rule** states:

B

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\text{or} \quad b^2 = a^2 + c^2 - 2ac\cos B$$
$$\text{or} \quad c^2 = a^2 + b^2 - 2ab\cos C$$

The rule may be used only when:

(i) 2 sides and the included angle are initially given, or
(ii) 3 sides are initially given.

12.8 Area of any triangle

The **area of any triangle** such as *ABC* of Fig. 12.19 is given by:

(i) $\frac{1}{2} \times \text{base} \times \text{perpendicular height}$, or
(ii) $\frac{1}{2}ab\sin C$ or $\frac{1}{2}ac\sin B$ or $\frac{1}{2}bc\sin A$, or
(iii) $\sqrt{[s(s-a)(s-b)(s-c)]}$, where $s = \dfrac{a+b+c}{2}$

12.9 Worked problems on the solution of triangles and finding their areas

Problem 21. In a triangle *XYZ*, $\angle X = 51°$, $\angle Y = 67°$ and $YZ = 15.2$ cm. Solve the triangle and find its area.

The triangle *XYZ* is shown in Fig. 12.20. Since the angles in a triangle add up to 180°, then $\mathbf{Z} = 180° - 51° - 67° = \mathbf{62°}$. Applying the sine rule:

$$\frac{15.2}{\sin 51°} = \frac{y}{\sin 67°} = \frac{z}{\sin 62°}$$

Using $\dfrac{15.2}{\sin 51°} = \dfrac{y}{\sin 67°}$ and transposing gives:

$$y = \frac{15.2 \sin 67°}{\sin 51°} = \mathbf{18.00\ cm} = \boldsymbol{XZ}$$

Using $\dfrac{15.2}{\sin 51°} = \dfrac{z}{\sin 62°}$ and transposing gives:

$$z = \frac{15.2 \sin 62°}{\sin 51°} = \mathbf{17.27\ cm} = \boldsymbol{XY}$$

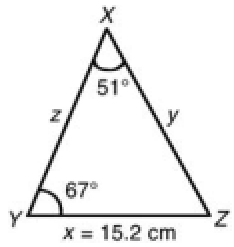

Figure 12.20

Area of triangle $\boldsymbol{XYZ} = \frac{1}{2}xy\sin Z$
$= \frac{1}{2}(15.2)(18.00)\sin 62° = \mathbf{120.8\ cm^2}$ (or area $= \frac{1}{2}xz\sin Y = \frac{1}{2}(15.2)(17.27)\sin 67° = \mathbf{120.8\ cm^2}$).

It is always worth checking with triangle problems that the longest side is opposite the largest angle, and vice-versa. In this problem, *Y* is the largest angle and *XZ* is the longest of the three sides.

Problem 22. Solve the triangle *PQR* and find its area given that $QR = 36.5$ mm, $PR = 29.6$ mm and $\angle Q = 36°$.

Triangle *PQR* is shown in Fig. 12.21.

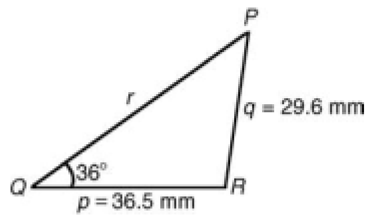

Figure 12.21

Applying the sine rule:

$$\frac{29.6}{\sin 36°} = \frac{36.5}{\sin P}$$

from which,

$$\sin P = \frac{36.5 \sin 36°}{29.6} = 0.7248$$

Hence $P = \sin^{-1} 0.7248 = 46°27'$ or $133°33'$.
When $P = 46°27'$ and $Q = 36°$ then $R = 180° - 46°27' - 36° = 97°33'$.
When $P = 133°33'$ and $Q = 36°$ then $R = 180° - 133°33' - 36° = 10°27'$.

Thus, in this problem, there are **two** separate sets of results and both are feasible solutions. Such a situation is called the **ambiguous case**.

Case 1. $P = 46°27'$, $Q = 36°$, $R = 97°33'$, $p = 36.5$ mm and $q = 29.6$ mm.
From the sine rule:

$$\frac{r}{\sin 97°33'} = \frac{29.6}{\sin 36°}$$

from which,

$$r = \frac{29.6 \sin 97°33'}{\sin 36°} = \mathbf{49.92\ mm}$$

$$\textbf{Area} = \frac{1}{2}pq \sin R = \frac{1}{2}(36.5)(29.6)\sin 97°33'$$
$$= \mathbf{535.5\ mm^2}$$

Case 2. $P = 133°33'$, $Q = 36°$, $R = 10°27'$, $p = 36.5$ mm and $q = 29.6$ mm.
From the sine rule:

$$\frac{r}{\sin 10°27'} = \frac{29.6}{\sin 36°}$$

from which,

$$r = \frac{29.6 \sin 10°27'}{\sin 36°} = \mathbf{9.134\ mm}$$

$$\textbf{Area} = \frac{1}{2}pq \sin R = \frac{1}{2}(36.5)(29.6)\sin 10°27'$$
$$= \mathbf{97.98\ mm^2}.$$

Triangle *PQR* for case 2 is shown in Fig. 12.22.

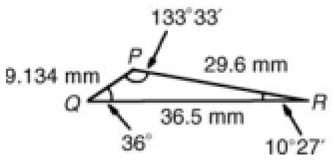

Figure 12.22

Now try the following exercise.

Exercise 57 Further problems on solving triangles and finding their areas

In Problems 1 and 2, use the sine rule to solve the triangles *ABC* and find their areas.

1. $A = 29°$, $B = 68°$, $b = 27$ mm.
$\left[\begin{array}{l} C = 83°, a = 14.1\text{ mm}, \\ c = 28.9\text{ mm, area} = 189\text{ mm}^2 \end{array}\right]$

2. $B = 71°26'$, $C = 56°32'$, $b = 8.60$ cm.
$\left[\begin{array}{l} A = 52°2', c = 7.568\text{ cm}, \\ a = 7.152\text{ cm, area} = 25.65\text{ cm}^2 \end{array}\right]$

In Problems 3 and 4, use the sine rule to solve the triangles *DEF* and find their areas.

3. $d = 17$ cm, $f = 22$ cm, $F = 26°$.
$\left[\begin{array}{l} D = 19°48', E = 134°12', \\ e = 36.0\text{ cm, area} = 134\text{ cm}^2 \end{array}\right]$

4. $d = 32.6$ mm, $e = 25.4$ mm, $D = 104°22'$.
$\left[\begin{array}{l} E = 49°0', F = 26°38', \\ f = 15.09\text{ mm, area} = 185.6\text{ mm}^2 \end{array}\right]$

In Problems 5 and 6, use the sine rule to solve the triangles *JKL* and find their areas.

5. $j = 3.85$ cm, $k = 3.23$ cm, $K = 36°$.
$\left[\begin{array}{l} J = 44°29', L = 99°31', \\ l = 5.420\text{ cm, area} = 6.132\text{ cm}^2 \text{ or} \\ J = 135°31', L = 8°29', \\ l = 0.811\text{ cm, area} = 0.916\text{ cm}^2 \end{array}\right]$

6. $k = 46$ mm, $l = 36$ mm, $L = 35°$.
$\left[\begin{array}{l} K = 47°8', J = 97°52', \\ j = 62.2\text{ mm, area} = 820.2\text{ mm}^2 \text{ or} \\ K = 132°52', J = 12°8', \\ j = 13.19\text{ mm, area} = 174.0\text{ mm}^2 \end{array}\right]$

12.10 Further worked problems on solving triangles and finding their areas

Problem 23. Solve triangle *DEF* and find its area given that $EF = 35.0$ mm, $DE = 25.0$ mm and $\angle E = 64°$.

Triangle *DEF* is shown in Fig. 12.23.

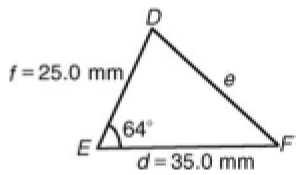

Figure 12.23

Applying the cosine rule:

$$e^2 = d^2 + f^2 - 2df \cos E$$

i.e. $$e^2 = (35.0)^2 + (25.0)^2 - [2(35.0)(25.0)\cos 64°]$$
$$= 1225 + 625 - 767.1 = 1083$$

from which, $\boldsymbol{e} = \sqrt{1083} = \mathbf{32.91\ mm}$

Applying the sine rule:

$$\frac{32.91}{\sin 64°} = \frac{25.0}{\sin F}$$

from which, $\sin F = \dfrac{25.0 \sin 64°}{32.91} = 0.6828$

Thus $\angle F = \sin^{-1} 0.6828$
$= 43°4'$ or $136°56'$

$F = 136°56'$ is not possible in this case since $136°56' + 64°$ is greater than 180°. Thus only $\boldsymbol{F = 43°4'}$ is valid

$$\angle D = 180° - 64° - 43°4' = \mathbf{72°56'}$$

Area of triangle $DEF = \frac{1}{2}\, df \sin E$
$= \frac{1}{2}(35.0)(25.0) \sin 64° = \mathbf{393.2\,mm^2}$.

Problem 24. A triangle ABC has sides $a = 9.0$ cm, $b = 7.5$ cm and $c = 6.5$ cm. Determine its three angles and its area.

Triangle ABC is shown in Fig. 12.24. It is usual first to calculate the largest angle to determine whether the triangle is acute or obtuse. In this case the largest angle is A (i.e. opposite the longest side).

Applying the cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

from which, $2bc \cos A = b^2 + c^2 - a^2$

and $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{7.5^2 + 6.5^2 - 9.0^2}{2(7.5)(6.5)}$
$= 0.1795$

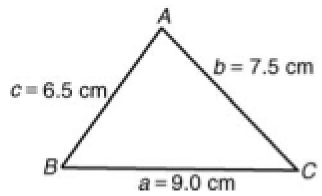

Figure 12.24

Hence $A = \cos^{-1} 0.1795 = \mathbf{79°40'}$ (or $280°20'$, which is obviously impossible). The triangle is thus acute angled since $\cos A$ is positive. (If $\cos A$ had been negative, angle A would be obtuse, i.e. lie between 90° and 180°).

Applying the sine rule:

$$\frac{9.0}{\sin 79°40'} = \frac{7.5}{\sin B}$$

from which,

$$\sin B = \frac{7.5 \sin 79°40'}{9.0} = 0.8198$$

Hence $\boldsymbol{B} = \sin^{-1} 0.8198 = \mathbf{55°4'}$

and $\boldsymbol{C} = 180° - 79°40' - 55°4' = \mathbf{45°16'}$

Area $= \sqrt{[s(s-a)(s-b)(s-c)]}$,

where $s = \dfrac{a+b+c}{2} = \dfrac{9.0 + 7.5 + 6.5}{2}$
$= 11.5$ cm

Hence **area**

$$= \sqrt{[11.5(11.5 - 9.0)(11.5 - 7.5)(11.5 - 6.5)]}$$
$$= \sqrt{[11.5(2.5)(4.0)(5.0)]} = \mathbf{23.98\,cm^2}$$

Alternatively, area $= \frac{1}{2} ab \sin C$
$= \frac{1}{2}(9.0)(7.5) \sin 45°16' = \mathbf{23.98\,cm^2}$.

Now try the following exercise.

Exercise 58 Further problems on solving triangles and finding their areas

In Problems 1 and 2, use the cosine and sine rules to solve the triangles PQR and find their areas.

1. $q = 12$ cm, $r = 16$ cm, $P = 54°$
$\left[\begin{array}{l} p = 13.2\text{ cm}, Q = 47°21', \\ R = 78°39', \text{area} = 77.7\text{ cm}^2 \end{array}\right]$

2. $q = 3.25$ m, $r = 4.42$ m, $P = 105°$
$\left[\begin{array}{l} p = 6.127\text{ m}, Q = 30°50', \\ R = 44°10', \text{area} = 6.938\text{ m}^2 \end{array}\right]$

In problems 3 and 4, use the cosine and sine rules to solve the triangles XYZ and find their areas.

3. $x = 10.0$ cm, $y = 8.0$ cm, $z = 7.0$ cm
$\left[\begin{array}{l} X = 83°20', Y = 52°37', \\ Z = 44°3', \text{area} = 27.8\text{ cm}^2 \end{array}\right]$

4. $x = 21$ mm, $y = 34$ mm, $z = 42$ mm
$\left[\begin{array}{l} Z = 29°46', Y = 53°30', \\ Z = 96°44', \text{area} = 355\,\text{mm}^2 \end{array}\right]$

12.11 Practical situations involving trigonometry

There are a number of **practical situations** where the use of trigonometry is needed to find unknown sides and angles of triangles. This is demonstrated in Problems 25 to 30.

Problem 25. A room 8.0 m wide has a span roof which slopes at 33° on one side and 40° on the other. Find the length of the roof slopes, correct to the nearest centimetre.

A section of the roof is shown in Fig. 12.25.

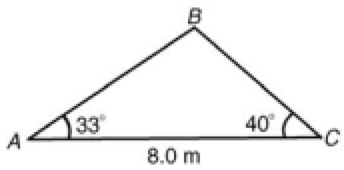

Figure 12.25

Angle at ridge, $B = 180° - 33° - 40° = 107°$
From the sine rule:

$$\frac{8.0}{\sin 107°} = \frac{a}{\sin 33°}$$

from which,

$$a = \frac{8.0 \sin 33°}{\sin 107°} = 4.556\,\text{m}$$

Also from the sine rule:

$$\frac{8.0}{\sin 107°} = \frac{c}{\sin 40°}$$

from which,

$$c = \frac{8.0 \sin 40°}{\sin 107°} = 5.377\,\text{m}$$

Hence the roof slopes are 4.56 m and 5.38 m, correct to the nearest centimetre.

Problem 26. Two voltage phasors are shown in Fig. 12.26. If $V_1 = 40$ V and $V_2 = 100$ V determine the value of their resultant (i.e. length OA) and the angle the resultant makes with V_1.

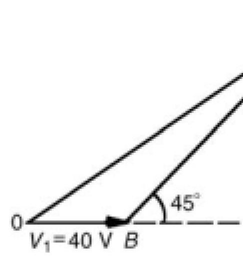

Figure 12.26

Angle $OBA = 180° - 45° = 135°$

Applying the cosine rule:

$$\begin{aligned} OA^2 &= V_1^2 + V_2^2 - 2V_1V_2 \cos OBA \\ &= 40^2 + 100^2 - \{2(40)(100)\cos 135°\} \\ &= 1600 + 10000 - \{-5657\} \\ &= 1600 + 10000 + 5657 = 17257 \end{aligned}$$

The resultant

$$OA = \sqrt{(17257)} = 131.4\,\text{V}$$

Applying the sine rule:

$$\frac{131.4}{\sin 135°} = \frac{100}{\sin AOB}$$

from which, $$\sin AOB = \frac{100 \sin 135°}{131.4} = 0.5381$$

Hence angle $AOB = \sin^{-1} 0.5381 = 32°33'$ (or $147°27'$, which is impossible in this case).

Hence the resultant voltage is 131.4 volts at 32°33′ to V_1.

Problem 27. In Fig. 12.27, PR represents the inclined jib of a crane and is 10.0 long. PQ is 4.0 m long. Determine the inclination of the jib to the vertical and the length of tie QR

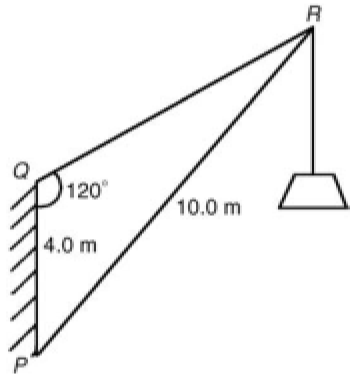

Figure 12.27

Applying the sine rule:

$$\frac{PR}{\sin 120°} = \frac{PQ}{\sin R}$$

from which,

$$\sin R = \frac{PQ \sin 120°}{PR} = \frac{(4.0)\sin 120°}{10.0} = 0.3464$$

Hence $\angle R = \sin^{-1} 0.3464 = 20°16'$ (or $159°44'$, which is impossible in this case).
$\angle P = 180° - 120° - 20°16' =$ **39°44′, which is the inclination of the jib to the vertical**.

Applying the sine rule:

$$\frac{10.0}{\sin 120°} = \frac{QR}{\sin 39°44'}$$

from which, **length of tie**,

$$QR = \frac{10.0 \sin 39°44'}{\sin 120°} = \mathbf{7.38\,m}$$

Now try the following exercise.

Exercise 59 Further problems on practical situations involving trigonometry

1. A ship P sails at a steady speed of 45 km/h in a direction of W 32° N (i.e. a bearing of 302°) from a port. At the same time another ship Q leaves the port at a steady speed of 35 km/h in a direction N 15° E (i.e. a bearing of 015°). Determine their distance apart after 4 hours. [193 km]

2. Two sides of a triangular plot of land are 52.0 m and 34.0 m, respectively. If the area of the plot is 620 m^2 find (a) the length of fencing required to enclose the plot and (b) the angles of the triangular plot. [(a) 122.6 m (b) 94°49′, 40°39′, 44°32′]

3. A jib crane is shown in Fig. 12.28. If the tie rod PR is 8.0 long and PQ is 4.5 m long determine (a) the length of jib RQ and (b) the angle between the jib and the tie rod. [(a) 11.4 m (b) 17°33′]

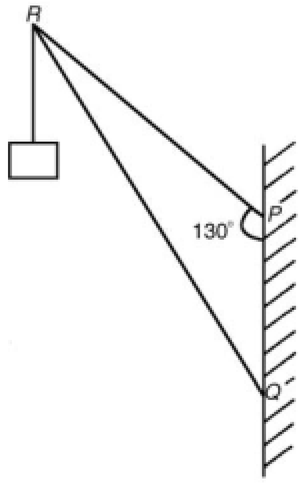

Figure 12.28

4. A building site is in the form of a quadrilateral as shown in Fig. 12.29, and its area is 1510 m^2. Determine the length of the perimeter of the site. [163.4 m]

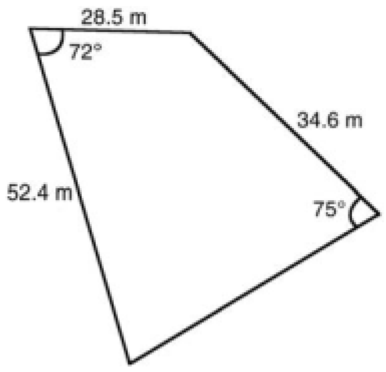

Figure 12.29

5. Determine the length of members BF and EB in the roof truss shown in Fig. 12.30. [$BF = 3.9$ m, $EB = 4.0$ m]

B

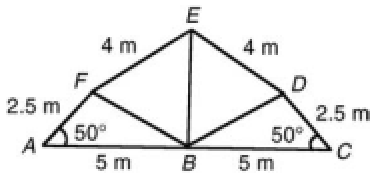

Figure 12.30

6. A laboratory 9.0 m wide has a span roof which slopes at 36° on one side and 44° on the other. Determine the lengths of the roof slopes. [6.35 m, 5.37 m]

12.12 Further practical situations involving trigonometry

Problem 28. A vertical aerial stands on horizontal ground. A surveyor positioned due east of the aerial measures the elevation of the top as 48°. He moves due south 30.0 m and measures the elevation as 44°. Determine the height of the aerial.

In Fig. 12.31, DC represents the aerial, A is the initial position of the surveyor and B his final position.

From triangle ACD, $\tan 48° = \dfrac{DC}{AC}$,

from which $AC = \dfrac{DC}{\tan 48°}$

Similarly, from triangle BCD,

$$BC = \frac{DC}{\tan 44°}$$

For triangle ABC, using Pythagoras' theorem:

$$BC^2 = AB^2 + AC^2$$

$$\left(\frac{DC}{\tan 44°}\right)^2 = (30.0)^2 + \left(\frac{DC}{\tan 48°}\right)^2$$

$$DC^2\left(\frac{1}{\tan^2 44°} - \frac{1}{\tan^2 48°}\right) = 30.0^2$$

$$DC^2(1.072323 - 0.810727) = 30.0^2$$

$$DC^2 = \frac{30.0^2}{0.261596} = 3440.4$$

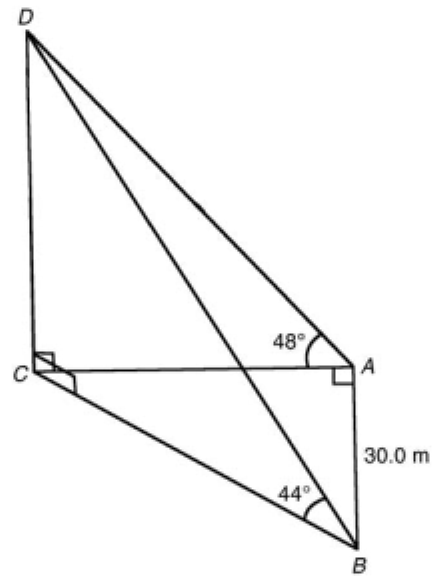

Figure 12.31

Hence, height of aerial,

$$\boldsymbol{DC = \sqrt{3440.4} = 58.65\,\text{m}}$$

Problem 29. A crank mechanism of a petrol engine is shown in Fig. 12.32. Arm OA is 10.0 cm long and rotates clockwise about O. The connecting rod AB is 30.0 cm long and end B is constrained to move horizontally.

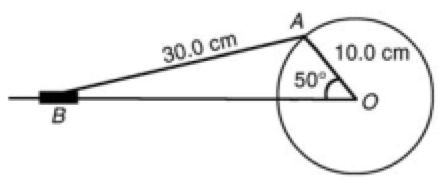

Figure 12.32

(a) For the position shown in Fig. 12.32 determine the angle between the connecting rod AB and the horizontal and the length of OB.
(b) How far does B move when angle AOB changes from 50° to 120°?

(a) Applying the sine rule:

$$\frac{AB}{\sin 50°} = \frac{AO}{\sin B}$$

from which,

$$\sin B = \frac{AO \sin 50°}{AB} = \frac{10.0 \sin 50°}{30.0} = 0.2553$$

Hence $B = \sin^{-1} 0.2553 = 14°47'$ (or $165°13'$, which is impossible in this case).

Hence the connecting rod AB makes an angle of $14°47'$ with the horizontal.

Angle $OAB = 180° - 50° - 14°47' = 115°13'$.

Applying the sine rule:

$$\frac{30.0}{\sin 50°} = \frac{OB}{\sin 115°13'}$$

from which,

$$OB = \frac{30.0 \sin 115°13'}{\sin 50°} = \mathbf{35.43\,cm}$$

(b) Figure 12.33 shows the initial and final positions of the crank mechanism. In triangle $OA'B'$, applying the sine rule:

$$\frac{30.0}{\sin 120°} = \frac{10.0}{\sin A'B'O}$$

from which,

$$\sin A'B'O = \frac{10.0 \sin 120°}{30.0} = 0.2887$$

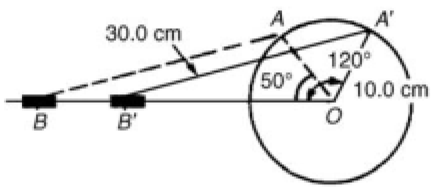

Figure 12.33

Hence $A'B'O = \sin^{-1} 0.2887 = 16°47'$ (or $163°13'$ which is impossible in this case).
Angle $OA'B' = 180° - 120° - 16°47' = 43°13'$.

Applying the sine rule:

$$\frac{30.0}{\sin 120°} = \frac{OB'}{\sin 43°13'}$$

from which,

$$OB' = \frac{30.0 \sin 43°13'}{\sin 120°} = 23.72\,\text{cm}$$

Since $OB = 35.43$ cm and $OB' = 23.72$ cm then $BB' = 35.43 - 23.72 = 11.71$ cm.

Hence B moves 11.71 cm when angle AOB changes from 50° to 120°.

Problem 30. The area of a field is in the form of a quadrilateral $ABCD$ as shown in Fig. 12.34. Determine its area.

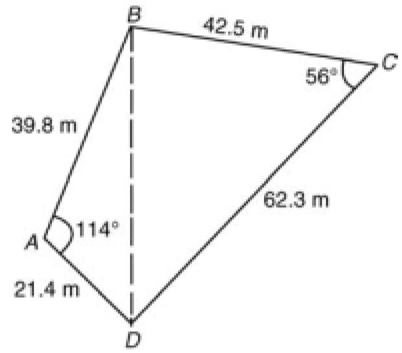

Figure 12.34

A diagonal drawn from B to D divides the quadrilateral into two triangles.

Area of quadrilateral ABCD

$$= \text{area of triangle } ABD + \text{area of triangle } BCD$$
$$= \tfrac{1}{2}(39.8)(21.4) \sin 114° + \tfrac{1}{2}(42.5)(62.3) \sin 56°$$
$$= 389.04 + 1097.5 = \mathbf{1487\,m^2}$$

Now try the following exercise.

Exercise 60 Further problems on practical situations involving trigonometry

1. PQ and QR are the phasors representing the alternating currents in two branches of a circuit. Phasor PQ is 20.0 A and is horizontal. Phasor QR (which is joined to the end of PQ to form triangle PQR) is 14.0 A and is at an angle of 35° to the horizontal. Determine the resultant phasor PR and the angle it makes with phasor PQ. [32.48 A, 14°19′]
2. Three forces acting on a fixed point are represented by the sides of a triangle of dimensions 7.2 cm, 9.6 cm and 11.0 cm. Determine the angles between the lines of action and the three forces. [80°25′, 59°23′, 40°12′]

3. Calculate, correct to 3 significant figures, the co-ordinates x and y to locate the hole centre at P shown in Fig. 12.35.
[$x = 69.3$ mm, $y = 142$ mm]

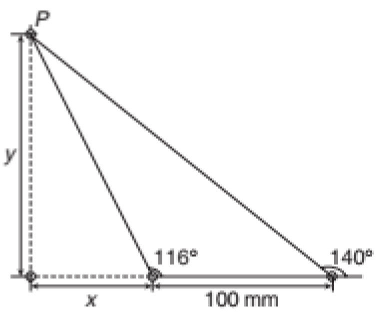

Figure 12.35

4. An idler gear, 30 mm in diameter, has to be fitted between a 70 mm diameter driving gear and a 90 mm diameter driven gear as shown in Fig. 12.36. Determine the value of angle θ between the center lines. [130°]

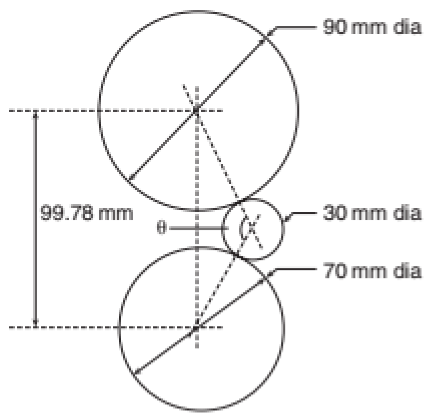

Figure 12.36

5. A reciprocating engine mechanism is shown in Fig. 12.37. The crank AB is 12.0 cm long and the connecting rod BC is 32.0 cm long. For the position shown determine the length of AC and the angle between the crank and the connecting rod. [40.25 cm, 126°3′]

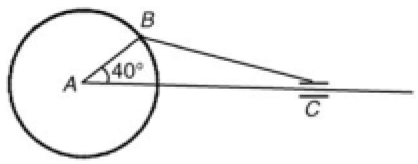

Figure 12.37

6. From Fig. 12.37, determine how far C moves, correct to the nearest millimetre when angle CAB changes from 40° to 160°, B moving in an anticlockwise direction. [19.8 cm]

7. A surveyor, standing W 25° S of a tower measures the angle of elevation of the top of the tower as 46°30′. From a position E 23° S from the tower the elevation of the top is 37°15′. Determine the height of the tower if the distance between the two observations is 75 m.
[36.2 m]

8. An aeroplane is sighted due east from a radar station at an elevation of 40° and a height of 8000 m and later at an elevation of 35° and height 5500 m in a direction E 70° S. If it is descending uniformly, find the angle of descent. Determine also the speed of the aeroplane in km/h if the time between the two observations is 45 s.
[13°57′, 829.9 km/h]