1
Algebra

1.1 Introduction

In this chapter, polynomial division and the factor and remainder theorems are explained (in Sections 1.4 to 1.6). However, before this, some essential algebra revision on basic laws and equations is included.

For further Algebra revision, go to website: http://books.elsevier.com/companions/0750681527

1.2 Revision of basic laws

(a) Basic operations and laws of indices

The **laws of indices** are:

(i) $a^m \times a^n = a^{m+n}$ (ii) $\dfrac{a^m}{a^n} = a^{m-n}$

(iii) $(a^m)^n = a^{m\times n}$ (iv) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(v) $a^{-n} = \dfrac{1}{a^n}$ (vi) $a^0 = 1$

Problem 1. Evaluate $4a^2bc^3 - 2ac$ when $a = 2$, $b = \frac{1}{2}$ and $c = 1\frac{1}{2}$

$$4a^2bc^3 - 2ac = 4(2)^2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^3 - 2(2)\left(\frac{3}{2}\right)$$
$$= \frac{4\times2\times2\times3\times3\times3}{2\times2\times2\times2} - \frac{12}{2}$$
$$= 27 - 6 = \mathbf{21}$$

Problem 2. Multiply $3x + 2y$ by $x - y$.

$$\begin{array}{lr} & 3x \;+ 2y \\ & x \;- y \\ \hline \text{Multiply by } x \rightarrow & 3x^2 + 2xy \\ \text{Multiply by } -y \rightarrow & -3xy - 2y^2 \\ \hline \text{Adding gives:} & \mathbf{3x^2 - xy - 2y^2} \end{array}$$

Alternatively,

$$(3x + 2y)(x - y) = 3x^2 - 3xy + 2xy - 2y^2$$
$$= \mathbf{3x^2 - xy - 2y^2}$$

Problem 3. Simplify $\dfrac{a^3b^2c^4}{abc^{-2}}$ and evaluate when $a = 3$, $b = \frac{1}{8}$ and $c = 2$.

$$\frac{a^3b^2c^4}{abc^{-2}} = a^{3-1}b^{2-1}c^{4-(-2)} = \mathbf{a^2bc^6}$$

When $a = 3$, $b = \frac{1}{8}$ and $c = 2$,

$$a^2bc^6 = (3)^2\left(\tfrac{1}{8}\right)(2)^6 = (9)\left(\tfrac{1}{8}\right)(64) = \mathbf{72}$$

Problem 4. Simplify $\dfrac{x^2y^3 + xy^2}{xy}$

$$\frac{x^2y^3 + xy^2}{xy} = \frac{x^2y^3}{xy} + \frac{xy^2}{xy}$$
$$= x^{2-1}y^{3-1} + x^{1-1}y^{2-1}$$
$$= \mathbf{xy^2 + y} \quad \text{or} \quad \mathbf{y(xy + 1)}$$

Problem 5. Simplify $\dfrac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}}$

$$\frac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}} = \frac{x^2y^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{2}{3}}}{x^{\frac{5}{2}}y^{\frac{3}{2}}}$$
$$= x^{2+\frac{1}{2}-\frac{5}{2}}y^{\frac{1}{2}+\frac{2}{3}-\frac{3}{2}}$$
$$= x^0y^{-\frac{1}{3}}$$
$$= \mathbf{y^{-\frac{1}{3}}} \quad \text{or} \quad \mathbf{\frac{1}{y^{\frac{1}{3}}}} \quad \text{or} \quad \mathbf{\frac{1}{\sqrt[3]{y}}}$$

Now try the following exercise.

Exercise 1 Revision of basic operations and laws of indices

1. Evaluate $2ab + 3bc - abc$ when $a = 2$, $b = -2$ and $c = 4$. $[-16]$
2. Find the value of $5pq^2r^3$ when $p = \frac{2}{5}$, $q = -2$ and $r = -1$. $[-8]$
3. From $4x - 3y + 2z$ subtract $x + 2y - 3z$. $[3x - 5y + 5z]$
4. Multiply $2a - 5b + c$ by $3a + b$. $[6a^2 - 13ab + 3ac - 5b^2 + bc]$
5. Simplify $(x^2y^3z)(x^3yz^2)$ and evaluate when $x = \frac{1}{2}$, $y = 2$ and $z = 3$. $[x^5y^4z^3, 13\frac{1}{2}]$
6. Evaluate $(a^{\frac{3}{2}}bc^{-3})(a^{\frac{1}{2}}b^{-\frac{1}{2}}c)$ when $a = 3$, $b = 4$ and $c = 2$. $[\pm 4\frac{1}{2}]$
7. Simplify $\dfrac{a^2b + a^3b}{a^2b^2}$ $\left[\dfrac{1+a}{b}\right]$
8. Simplify $\dfrac{(a^3b^{\frac{1}{2}}c^{-\frac{1}{2}})(ab)^{\frac{1}{3}}}{(\sqrt{a^3}\sqrt{b}\,c)}$ $\left[a^{\frac{11}{6}}b^{\frac{1}{3}}c^{-\frac{3}{2}} \quad \text{or} \quad \dfrac{\sqrt[6]{a^{11}}\sqrt[3]{b}}{\sqrt{c^3}}\right]$

(b) Brackets, factorization and precedence

Problem 6. Simplify
$$a^2 - (2a - ab) - a(3b + a).$$

$$a^2 - (2a - ab) - a(3b + a)$$
$$= a^2 - 2a + ab - 3ab - a^2$$
$$= \mathbf{-2a - 2ab} \quad \text{or} \quad \mathbf{-2a(1 + b)}$$

Problem 7. Remove the brackets and simplify the expression:
$$2a - [3\{2(4a - b) - 5(a + 2b)\} + 4a].$$

Removing the innermost brackets gives:
$$2a - [3\{8a - 2b - 5a - 10b\} + 4a]$$

Collecting together similar terms gives:
$$2a - [3\{3a - 12b\} + 4a]$$

Removing the 'curly' brackets gives:
$$2a - [9a - 36b + 4a]$$

Collecting together similar terms gives:
$$2a - [13a - 36b]$$

Removing the square brackets gives:
$$2a - 13a + 36b = \mathbf{-11a + 36b} \quad \text{or} \quad \mathbf{36b - 11a}$$

Problem 8. Factorize (a) $xy - 3xz$ (b) $4a^2 + 16ab^3$ (c) $3a^2b - 6ab^2 + 15ab$.

(a) $xy - 3xz = \mathbf{x(y - 3z)}$

(b) $4a^2 + 16ab^3 = \mathbf{4a(a + 4b^3)}$

(c) $3a^2b - 6ab^2 + 15ab = \mathbf{3ab(a - 2b + 5)}$

Problem 9. Simplify $3c + 2c \times 4c + c \div 5c - 8c$.

The order of precedence is division, multiplication, addition and subtraction (sometimes remembered by BODMAS). Hence

$$3c + 2c \times 4c + c \div 5c - 8c$$
$$= 3c + 2c \times 4c + \left(\frac{c}{5c}\right) - 8c$$
$$= 3c + 8c^2 + \frac{1}{5} - 8c$$
$$= \mathbf{8c^2 - 5c + \frac{1}{5}} \quad \text{or} \quad \mathbf{c(8c - 5) + \frac{1}{5}}$$

Problem 10. Simplify $(2a - 3) \div 4a + 5 \times 6 - 3a$.

$$(2a - 3) \div 4a + 5 \times 6 - 3a$$
$$= \frac{2a - 3}{4a} + 5 \times 6 - 3a$$
$$= \frac{2a - 3}{4a} + 30 - 3a$$
$$= \frac{2a}{4a} - \frac{3}{4a} + 30 - 3a$$
$$= \frac{1}{2} - \frac{3}{4a} + 30 - 3a = \mathbf{30\frac{1}{2} - \frac{3}{4a} - 3a}$$

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Now try the following exercise.

Exercise 2 Further problems on brackets, factorization and precedence

1. Simplify $2(p+3q-r)-4(r-q+2p)+p$. $[-5p+10q-6r]$
2. Expand and simplify $(x+y)(x-2y)$. $[x^2-xy-2y^2]$
3. Remove the brackets and simplify: $24p-[2\{3(5p-q)-2(p+2q)\}+3q]$. $[11q-2p]$
4. Factorize $21a^2b^2-28ab$ $[7ab(3ab-4)]$.
5. Factorize $2xy^2+6x^2y+8x^3y$. $[2xy(y+3x+4x^2)]$
6. Simplify $2y+4\div 6y+3\times 4-5y$. $\left[\dfrac{2}{3y}-3y+12\right]$
7. Simplify $3\div y+2\div y-1$. $\left[\dfrac{5}{y}-1\right]$
8. Simplify $a^2-3ab\times 2a\div 6b+ab$. $[ab]$

1.3 Revision of equations

(a) Simple equations

Problem 11. Solve $4-3x=2x-11$.

Since $4-3x=2x-11$ then $4+11=2x+3x$

i.e. $15=5x$ from which, $\boldsymbol{x}=\dfrac{15}{5}=\mathbf{3}$

Problem 12. Solve

$4(2a-3)-2(a-4)=3(a-3)-1$.

Removing the brackets gives:

$$8a-12-2a+8=3a-9-1$$

Rearranging gives:

$$8a-2a-3a=-9-1+12-8$$

i.e. $3a=-6$

and $\boldsymbol{a}=\dfrac{-6}{3}=\mathbf{-2}$

Problem 13. Solve $\dfrac{3}{x-2}=\dfrac{4}{3x+4}$.

By 'cross-multiplying': $3(3x+4)=4(x-2)$

Removing brackets gives: $9x+12=4x-8$

Rearranging gives: $9x-4x=-8-12$

i.e. $5x=-20$

and $\boldsymbol{x}=\dfrac{-20}{5}=\mathbf{-4}$

Problem 14. Solve $\left(\dfrac{\sqrt{t}+3}{\sqrt{t}}\right)=2$.

$$\sqrt{t}\left(\frac{\sqrt{t}+3}{\sqrt{t}}\right)=2\sqrt{t}$$

i.e. $\sqrt{t}+3=2\sqrt{t}$

and $3=2\sqrt{t}-\sqrt{t}$

i.e. $3=\sqrt{t}$

and $\mathbf{9}=\boldsymbol{t}$

(b) Transposition of formulae

Problem 15. Transpose the formula $v=u+\dfrac{ft}{m}$ to make f the subject.

$u+\dfrac{ft}{m}=v$ from which, $\dfrac{ft}{m}=v-u$

and $m\left(\dfrac{ft}{m}\right)=m(v-u)$

i.e. $ft=m(v-u)$

and $\boldsymbol{f}=\dfrac{\boldsymbol{m}}{\boldsymbol{t}}(\boldsymbol{v}-\boldsymbol{u})$

Problem 16. The impedance of an a.c. circuit is given by $Z=\sqrt{R^2+X^2}$. Make the reactance X the subject.

$\sqrt{R^2+X^2}=Z$ and squaring both sides gives $R^2+X^2=Z^2$, from which,

$$X^2=Z^2-R^2 \text{ and } \textbf{reactance } \boldsymbol{X=\sqrt{Z^2-R^2}}$$

Problem 17. Given that $\frac{D}{d}=\sqrt{\left(\frac{f+p}{f-p}\right)}$, express p in terms of D, d and f.

Rearranging gives: $\sqrt{\left(\frac{f+p}{f-p}\right)}=\frac{D}{d}$

Squaring both sides gives: $\frac{f+p}{f-p}=\frac{D^2}{d^2}$

'Cross-multiplying' gives:

$$d^2(f+p)=D^2(f-p)$$

Removing brackets gives:

$$d^2f+d^2p=D^2f-D^2p$$

Rearranging gives: $d^2p+D^2p=D^2f-d^2f$

Factorizing gives: $p(d^2+D^2)=f(D^2-d^2)$

and $\boldsymbol{p=\dfrac{f(D^2-d^2)}{(d^2+D^2)}}$

Now try the following exercise.

Exercise 3 Further problems on simple equations and transposition of formulae

In problems 1 to 4 solve the equations

1. $3x-2-5x=2x-4$ $\quad\left[\frac{1}{2}\right]$
2. $8+4(x-1)-5(x-3)=2(5-2x)$ $\quad[-3]$
3. $\dfrac{1}{3a-2}+\dfrac{1}{5a+3}=0$ $\quad\left[-\frac{1}{8}\right]$
4. $\dfrac{3\sqrt{t}}{1-\sqrt{t}}=-6$ $\quad[4]$
5. Transpose $y=\dfrac{3(F-f)}{L}$ for f. $\quad\left[f=\dfrac{3F-yL}{3} \text{ or } f=F-\dfrac{yL}{3}\right]$
6. Make l the subject of $t=2\pi\sqrt{\dfrac{l}{g}}$ $\quad\left[l=\dfrac{t^2g}{4\pi^2}\right]$
7. Transpose $m=\dfrac{\mu L}{L+rCR}$ for L. $\quad\left[L=\dfrac{mrCR}{\mu-m}\right]$
8. Make r the subject of the formula $\dfrac{x}{y}=\dfrac{1+r^2}{1-r^2}$ $\quad\left[r=\sqrt{\left(\dfrac{x-y}{x+y}\right)}\right]$

(c) Simultaneous equations

Problem 18. Solve the simultaneous equations:

$$7x-2y=26 \qquad (1)$$
$$6x+5y=29 \qquad (2)$$

$5\times$ equation (1) gives:

$$35x-10y=130 \qquad (3)$$

$2\times$ equation (2) gives:

$$12x+10y=58 \qquad (4)$$

equation (3) + equation (4) gives:

$$47x+0=188$$

from which, $\boldsymbol{x}=\dfrac{188}{47}=\mathbf{4}$

Substituting $x=4$ in equation (1) gives:

$$28-2y=26$$

from which, $28-26=2y$ and $\boldsymbol{y=1}$

Problem 19. Solve

$$\frac{x}{8}+\frac{5}{2}=y \qquad (1)$$
$$11+\frac{y}{3}=3x \qquad (2)$$

$8\times$ equation (1) gives: $x+20=8y$ $\quad(3)$

$3\times$ equation (2) gives: $33+y=9x$ $\quad(4)$

i.e. $x-8y=-20$ $\quad(5)$

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and $\quad 9x - y = 33 \quad (6)$

$8 \times$ equation (6) gives: $72x - 8y = 264 \quad (7)$

Equation (7) − equation (5) gives:

$$71x = 284$$

from which, $\quad x = \dfrac{284}{71} = \mathbf{4}$

Substituting $x = 4$ in equation (5) gives:

$$4 - 8y = -20$$

from which, $\quad 4 + 20 = 8y$ and $\mathbf{y = 3}$

(d) Quadratic equations

> Problem 20. Solve the following equations by factorization:
> (a) $3x^2 - 11x - 4 = 0$
> (b) $4x^2 + 8x + 3 = 0$

(a) The factors of $3x^2$ are $3x$ and x and these are placed in brackets thus:
$(3x \quad)(x \quad)$
The factors of -4 are $+1$ and -4 or -1 and $+4$, or -2 and $+2$. Remembering that the product of the two inner terms added to the product of the two outer terms must equal $-11x$, the only combination to give this is $+1$ and -4, i.e.,

$$3x^2 - 11x - 4 = (3x + 1)(x - 4)$$

Thus $\quad (3x + 1)(x - 4) = 0$ hence

either $\quad (3x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{3}}$

or $\quad (x - 4) = 0$ i.e. $\boldsymbol{x = 4}$

(b) $4x^2 + 8x + 3 = (2x + 3)(2x + 1)$

Thus $\quad (2x + 3)(2x + 1) = 0$ hence

either $\quad (2x + 3) = 0$ i.e. $\boldsymbol{x = -\frac{3}{2}}$

or $\quad (2x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{2}}$

> Problem 21. The roots of a quadratic equation are $\frac{1}{3}$ and -2. Determine the equation in x.

If $\frac{1}{3}$ and -2 are the roots of a quadratic equation then,

$$(x - \tfrac{1}{3})(x + 2) = 0$$

i.e. $\quad x^2 + 2x - \frac{1}{3}x - \frac{2}{3} = 0$

i.e. $\quad x^2 + \frac{5}{3}x - \frac{2}{3} = 0$

or $\quad \mathbf{3x^2 + 5x - 2 = 0}$

> Problem 22. Solve $4x^2 + 7x + 2 = 0$ giving the answer correct to 2 decimal places.

From the quadratic formula if $ax^2 + bx + c = 0$ then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence if $4x^2 + 7x + 2 = 0$

$$\text{then}\quad x = \frac{-7 \pm \sqrt{7^2 - 4(4)(2)}}{2(4)} = \frac{-7 \pm \sqrt{17}}{8} = \frac{-7 \pm 4.123}{8} = \frac{-7 + 4.123}{8} \text{ or } \frac{-7 - 4.123}{8}$$

i.e. $\quad \mathbf{x = -0.36}$ or $\mathbf{-1.39}$

Now try the following exercise.

Exercise 4 Further problems on simultaneous and quadratic equations

In problems 1 to 3, solve the simultaneous equations

1. $8x - 3y = 51$
   $3x + 4y = 14$ $\quad [x = 6,\ y = -1]$
2. $5a = 1 - 3b$
   $2b + a + 4 = 0$ $\quad [a = 2,\ b = -3]$
3. $\dfrac{x}{5} + \dfrac{2y}{3} = \dfrac{49}{15}$
   $\dfrac{3x}{7} - \dfrac{y}{2} + \dfrac{5}{7} = 0$ $\quad [x = 3,\ y = 4]$
4. Solve the following quadratic equations by factorization:
   (a) $x^2 + 4x - 32 = 0$
   (b) $8x^2 + 2x - 15 = 0$
   [(a) 4, −8 (b) $\frac{5}{4}$, $-\frac{3}{2}$]

5. Determine the quadratic equation in x whose roots are 2 and -5.

$$[x^2 + 3x - 10 = 0]$$

6. Solve the following quadratic equations, correct to 3 decimal places:
(a) $2x^2 + 5x - 4 = 0$
(b) $4t^2 - 11t + 3 = 0$

$$\begin{bmatrix} \text{(a) } 0.637, -3.137 \\ \text{(b) } 2.443, 0.307 \end{bmatrix}$$

1.4 Polynomial division

Before looking at long division in algebra let us revise long division with numbers (we may have forgotten, since calculators do the job for us!)

For example, $\dfrac{208}{16}$ is achieved as follows:

$$\begin{array}{r@{\,}r} & 13 \\ \cline{2-2} 16\,) & 208 \\ & \underline{16} \\ & 48 \\ & \underline{48} \\ & \underline{\cdot\,\cdot} \end{array}$$

(1) 16 divided into 2 won't go
(2) 16 divided into 20 goes 1
(3) Put 1 above the zero
(4) Multiply 16 by 1 giving 16
(5) Subtract 16 from 20 giving 4
(6) Bring down the 8
(7) 16 divided into 48 goes 3 times
(8) Put the 3 above the 8
(9) $3 \times 16 = 48$
(10) $48 - 48 = 0$

Hence $\dfrac{208}{16} = \mathbf{13}$ **exactly**

Similarly, $\dfrac{172}{15}$ is laid out as follows:

$$\begin{array}{r@{\,}r} & 11 \\ \cline{2-2} 15\,) & 172 \\ & \underline{15} \\ & 22 \\ & \underline{15} \\ & \underline{7} \end{array}$$

Hence $\dfrac{172}{15} = 11$ remainder 7 or $11 + \dfrac{7}{15} = \mathbf{11\dfrac{7}{15}}$

Below are some examples of division in algebra, which in some respects, is similar to long division with numbers.
(Note that a **polynomial** is an expression of the form

$$f(x) = a + bx + cx^2 + dx^3 + \cdots$$

and **polynomial division** is sometimes required when resolving into partial fractions—see Chapter 3)

Problem 23. Divide $2x^2 + x - 3$ by $x - 1$.

$2x^2 + x - 3$ is called the **dividend** and $x - 1$ the **divisor**. The usual layout is shown below with the dividend and divisor both arranged in descending powers of the symbols.

$$\begin{array}{r@{\,}l} & 2x + 3 \\ \cline{2-2} x - 1\,) & 2x^2 + x - 3 \\ & \underline{2x^2 - 2x} \\ & \phantom{2x^2 +{}} 3x - 3 \\ & \phantom{2x^2 +{}} \underline{3x - 3} \\ & \phantom{2x^2 +{}} \underline{\cdot\cdot} \end{array}$$

Dividing the first term of the dividend by the first term of the divisor, i.e. $\dfrac{2x^2}{x}$ gives $2x$, which is put above the first term of the dividend as shown. The divisor is then multiplied by $2x$, i.e. $2x(x - 1) = 2x^2 - 2x$, which is placed under the dividend as shown. Subtracting gives $3x - 3$. The process is then repeated, i.e. the first term of the divisor, x, is divided into $3x$, giving $+3$, which is placed above the dividend as shown. Then $3(x - 1) = 3x - 3$ which is placed under the $3x - 3$. The remainder, on subtraction, is zero, which completes the process.

Thus $\mathbf{(2x^2 + x - 3) \div (x - 1) = (2x + 3)}$

[A check can be made on this answer by multiplying $(2x + 3)$ by $(x - 1)$ which equals $2x^2 + x - 3$]

Problem 24. Divide $3x^3 + x^2 + 3x + 5$ by $x + 1$.

$$\begin{array}{r l}
 & \ (1)\quad (4)\quad (7) \\
 & \ 3x^2 - 2x\ \ + 5 \\ \cline{2-2}
x+1 & \big)\ 3x^3 + \ x^2 + 3x + 5 \\
 & \ \underline{3x^3 + 3x^2} \\
 & \ \qquad -2x^2 + 3x + 5 \\
 & \ \qquad \underline{-2x^2 - 2x} \\
 & \ \qquad\qquad\quad 5x + 5 \\
 & \ \qquad\qquad\quad \underline{5x + 5} \\
 & \ \qquad\qquad\quad \underline{\ \cdot\quad \cdot\ }
\end{array}$$

(1) x into $3x^3$ goes $3x^2$. Put $3x^2$ above $3x^3$
(2) $3x^2(x + 1) = 3x^3 + 3x^2$
(3) Subtract
(4) x into $-2x^2$ goes $-2x$. Put $-2x$ above the dividend
(5) $-2x(x + 1) = -2x^2 - 2x$
(6) Subtract
(7) x into $5x$ goes 5. Put 5 above the dividend
(8) $5(x + 1) = 5x + 5$
(9) Subtract

Thus

$$\frac{3x^3 + x^2 + 3x + 5}{x + 1} = \mathbf{3x^2 - 2x + 5}$$

Problem 25. Simplify $\dfrac{x^3 + y^3}{x + y}$

$$\begin{array}{r l}
 & \ (1)\quad (4)\quad (7) \\
 & \ x^2 - \ xy\ \ + y^2 \\ \cline{2-2}
x+y & \big)\ x^3 + \ 0\ + \ 0\ + y^3 \\
 & \ \underline{x^3 + x^2y} \\
 & \ \quad -x^2y \qquad\quad + y^3 \\
 & \ \quad \underline{-x^2y - xy^2} \\
 & \ \qquad\qquad\ xy^2 + y^3 \\
 & \ \qquad\qquad\ \underline{xy^2 + y^3} \\
 & \ \qquad\qquad\ \underline{\ \cdot\quad \cdot\ }
\end{array}$$

(1) x into x^3 goes x^2. Put x^2 above x^3 of dividend
(2) $x^2(x + y) = x^3 + x^2y$
(3) Subtract
(4) x into $-x^2y$ goes $-xy$. Put $-xy$ above dividend
(5) $-xy(x + y) = -x^2y - xy^2$
(6) Subtract
(7) x into xy^2 goes y^2. Put y^2 above dividend
(8) $y^2(x + y) = xy^2 + y^3$
(9) Subtract

Thus

$$\frac{x^3 + y^3}{x + y} = \mathbf{x^2 - xy + y^2}$$

The zero's shown in the dividend are not normally shown, but are included to clarify the subtraction process and to keep similar terms in their respective columns.

Problem 26. Divide $(x^2 + 3x - 2)$ by $(x - 2)$.

$$\begin{array}{r l}
 & \ x\ + 5 \\ \cline{2-2}
x-2 & \big)\ x^2 + 3x - \ 2 \\
 & \ \underline{x^2 - 2x} \\
 & \ \qquad 5x - \ 2 \\
 & \ \qquad \underline{5x - 10} \\
 & \ \qquad\qquad \underline{\ 8\ }
\end{array}$$

Hence

$$\frac{x^2 + 3x - 2}{x - 2} = \mathbf{x + 5 + \frac{8}{x - 2}}$$

Problem 27. Divide $4a^3 - 6a^2b + 5b^3$ by $2a - b$.

$$\begin{array}{r l}
 & \ 2a^2 - 2ab - \ b^2 \\ \cline{2-2}
2a-b & \big)\ 4a^3 - 6a^2b \qquad\quad + 5b^3 \\
 & \ \underline{4a^3 - 2a^2b} \\
 & \ \qquad -4a^2b \qquad\quad + 5b^3 \\
 & \ \qquad \underline{-4a^2b + 2ab^2} \\
 & \ \qquad\qquad\quad -2ab^2 + 5b^3 \\
 & \ \qquad\qquad\quad \underline{-2ab^2 + \ b^3} \\
 & \ \qquad\qquad\qquad\qquad \underline{4b^3}
\end{array}$$

Thus

$$\frac{4a^3 - 6a^2b + 5b^3}{2a - b} = \mathbf{2a^2 - 2ab - b^2 + \frac{4b^3}{2a - b}}$$

Now try the following exercise.

Exercise 5 Further problems on polynomial division

1. Divide $(2x^2 + xy - y^2)$ by $(x + y)$. $[2x - y]$
2. Divide $(3x^2 + 5x - 2)$ by $(x + 2)$. $[3x - 1]$
3. Determine $(10x^2 + 11x - 6) \div (2x + 3)$. $[5x - 2]$
4. Find $\dfrac{14x^2 - 19x - 3}{2x - 3}$. $[7x + 1]$
5. Divide $(x^3 + 3x^2y + 3xy^2 + y^3)$ by $(x + y)$. $[x^2 + 2xy + y^2]$
6. Find $(5x^2 - x + 4) \div (x - 1)$. $\left[5x + 4 + \dfrac{8}{x - 1}\right]$
7. Divide $(3x^3 + 2x^2 - 5x + 4)$ by $(x + 2)$. $\left[3x^2 - 4x + 3 - \dfrac{2}{x + 2}\right]$
8. Determine $(5x^4 + 3x^3 - 2x + 1)/(x - 3)$. $\left[5x^3 + 18x^2 + 54x + 160 + \dfrac{481}{x - 3}\right]$

1.5 The factor theorem

There is a simple relationship between the factors of a quadratic expression and the roots of the equation obtained by equating the expression to zero.

For example, consider the quadratic equation $x^2 + 2x - 8 = 0$.

To solve this we may factorize the quadratic expression $x^2 + 2x - 8$ giving $(x - 2)(x + 4)$.

Hence $(x - 2)(x + 4) = 0$.

Then, if the product of two numbers is zero, one or both of those numbers must equal zero. Therefore,

either $(x - 2) = 0$, from which, $x = 2$

or $(x + 4) = 0$, from which, $x = -4$

It is clear then that a factor of $(x - 2)$ indicates a root of $+2$, while a factor of $(x + 4)$ indicates a root of -4.

In general, we can therefore say that:

a factor of $(x - a)$ corresponds to a root of $x = a$

In practice, we always deduce the roots of a simple quadratic equation from the factors of the quadratic expression, as in the above example. However, we could reverse this process. If, by trial and error, we could determine that $x = 2$ is a root of the equation $x^2 + 2x - 8 = 0$ we could deduce at once that $(x - 2)$ is a factor of the expression $x^2 + 2x - 8$. We wouldn't normally solve quadratic equations this way — but suppose we have to factorize a cubic expression (i.e. one in which the highest power of the variable is 3). A cubic equation might have three simple linear factors and the difficulty of discovering all these factors by trial and error would be considerable. It is to deal with this kind of case that we use the **factor theorem**. This is just a generalized version of what we established above for the quadratic expression. The factor theorem provides a method of factorizing any polynomial, $f(x)$, which has simple factors.

A statement of the **factor theorem** says:

'if $x = a$ is a root of the equation $f(x) = 0$, then $(x - a)$ is a factor of $f(x)$'

The following worked problems show the use of the factor theorem.

Problem 28. Factorize $x^3 - 7x - 6$ and use it to solve the cubic equation $x^3 - 7x - 6 = 0$.

Let $f(x) = x^3 - 7x - 6$

If $x = 1$, then $f(1) = 1^3 - 7(1) - 6 = -12$

If $x = 2$, then $f(2) = 2^3 - 7(2) - 6 = -12$

If $x = 3$, then $f(3) = 3^3 - 7(3) - 6 = 0$

If $f(3) = 0$, then $(x - 3)$ is a factor — from the factor theorem.

We have a choice now. We can divide $x^3 - 7x - 6$ by $(x - 3)$ or we could continue our 'trial and error' by substituting further values for x in the given expression — and hope to arrive at $f(x) = 0$.

Let us do both ways. Firstly, dividing out gives:

$$
\begin{array}{r|l}
 & x^2 + 3x \;\; + 2 \\
\hline
x - 3 & x^3 - 0 \;\; - 7x - 6 \\
 & x^3 - 3x^2 \\
\hline
 & \quad\;\; 3x^2 - 7x - 6 \\
 & \quad\;\; 3x^2 - 9x \\
\hline
 & \qquad\quad\;\; 2x - 6 \\
 & \qquad\quad\;\; 2x - 6 \\
\hline
 & \qquad\quad\;\; \cdot \quad \cdot
\end{array}
$$

Hence $\dfrac{x^3 - 7x - 6}{x - 3} = x^2 + 3x + 2$

i.e. $x^3 - 7x - 6 = (x - 3)(x^2 + 3x + 2)$

$x^2 + 3x + 2$ factorizes 'on sight' as $(x + 1)(x + 2)$. Therefore

$$\mathbf{x^3 - 7x - 6 = (x - 3)(x + 1)(x + 2)}$$

A second method is to continue to substitute values of x into $f(x)$.

Our expression for $f(3)$ was $3^3 - 7(3) - 6$. We can see that if we continue with positive values of x the first term will predominate such that $f(x)$ will not be zero.

Therefore let us try some negative values for x. Therefore $f(-1) = (-1)^3 - 7(-1) - 6 = 0$; hence $(x + 1)$ is a factor (as shown above). Also $f(-2) = (-2)^3 - 7(-2) - 6 = 0$; hence $(x + 2)$ is a factor (also as shown above).

To solve $x^3 - 7x - 6 = 0$, we substitute the factors, i.e.,

$$(x - 3)(x + 1)(x + 2) = 0$$

from which, $\mathbf{x = 3, x = -1}$ **and** $\mathbf{x = -2}$.

Note that the values of x, i.e. 3, -1 and -2, are all factors of the constant term, i.e. the 6. This can give us a clue as to what values of x we should consider.

Problem 29. Solve the cubic equation $x^3 - 2x^2 - 5x + 6 = 0$ by using the factor theorem.

Let $f(x) = x^3 - 2x^2 - 5x + 6$ and let us substitute simple values of x like 1, 2, 3, -1, -2, and so on.

$f(1) = 1^3 - 2(1)^2 - 5(1) + 6 = 0$, hence $(x - 1)$ is a factor

$f(2) = 2^3 - 2(2)^2 - 5(2) + 6 \neq 0$

$f(3) = 3^3 - 2(3)^2 - 5(3) + 6 = 0$, hence $(x - 3)$ is a factor

$f(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6 \neq 0$

$f(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + 6 = 0$, hence $(x + 2)$ is a factor

Hence $x^3 - 2x^2 - 5x + 6 = (x - 1)(x - 3)(x + 2)$

Therefore if $x^3 - 2x^2 - 5x + 6 = 0$
then $(x - 1)(x - 3)(x + 2) = 0$

from which, $\mathbf{x = 1, x = 3}$ **and** $\mathbf{x = -2}$

Alternatively, having obtained one factor, i.e. $(x - 1)$ we could divide this into $(x^3 - 2x^2 - 5x + 6)$ as follows:

$$\begin{array}{r|l}
 & x^2 - x - 6 \\
x - 1 & x^3 - 2x^2 - 5x + 6 \\
 & x^3 - x^2 \\
\hline
 & -x^2 - 5x + 6 \\
 & -x^2 + x \\
\hline
 & -6x + 6 \\
 & -6x + 6 \\
\hline
 & \cdot\ \cdot
\end{array}$$

Hence $x^3 - 2x^2 - 5x + 6$
$= (x - 1)(x^2 - x - 6)$
$= \mathbf{(x - 1)(x - 3)(x + 2)}$

Summarizing, the factor theorem provides us with a method of factorizing simple expressions, and an alternative, in certain circumstances, to polynomial division.

Now try the following exercise.

Exercise 6 Further problems on the factor theorem

Use the factor theorem to factorize the expressions given in problems 1 to 4.

1. $x^2 + 2x - 3$ $[(x - 1)(x + 3)]$
2. $x^3 + x^2 - 4x - 4$ $[(x + 1)(x + 2)(x - 2)]$
3. $2x^3 + 5x^2 - 4x - 7$ $[(x + 1)(2x^2 + 3x - 7)]$
4. $2x^3 - x^2 - 16x + 15$ $[(x - 1)(x + 3)(2x - 5)]$
5. Use the factor theorem to factorize $x^3 + 4x^2 + x - 6$ and hence solve the cubic equation $x^3 + 4x^2 + x - 6 = 0$.
$\left[\begin{array}{l} x^3 + 4x^2 + x - 6 \\ \quad = (x - 1)(x + 3)(x + 2) \\ x = 1, x = -3 \text{ and } x = -2 \end{array}\right]$
6. Solve the equation $x^3 - 2x^2 - x + 2 = 0$. $[x = 1, x = 2 \text{ and } x = -1]$

1.6 The remainder theorem

Dividing a general quadratic expression $(ax^2 + bx + c)$ by $(x - p)$, where p is any whole number, by long division (see section 1.3) gives:

$$
\begin{array}{r|l}
 & ax \quad + (b + ap) \\
\hline
x - p & ax^2 + bx \qquad\quad + c \\
 & \underline{ax^2 - apx} \\
 & \qquad (b + ap)x + c \\
 & \qquad \underline{(b + ap)x - (b + ap)p} \\
 & \qquad\qquad\quad \underline{c + (b + ap)p}
\end{array}
$$

The remainder, $c + (b + ap)p = c + bp + ap^2$ or $ap^2 + bp + c$. This is, in fact, what the **remainder theorem** states, i.e.,

'if $(ax^2 + bx + c)$ is divided by $(x - p)$, the remainder will be $ap^2 + bp + c$'

If, in the dividend $(ax^2 + bx + c)$, we substitute p for x we get the remainder $ap^2 + bp + c$.

For example, when $(3x^2 - 4x + 5)$ is divided by $(x - 2)$ the remainder is $ap^2 + bp + c$ (where $a = 3$, $b = -4$, $c = 5$ and $p = 2$),
i.e. the remainder is

$$3(2)^2 + (-4)(2) + 5 = 12 - 8 + 5 = \mathbf{9}$$

We can check this by dividing $(3x^2 - 4x + 5)$ by $(x - 2)$ by long division:

$$
\begin{array}{r|l}
 & 3x + 2 \\
\hline
x - 2 & 3x^2 - 4x + 5 \\
 & \underline{3x^2 - 6x} \\
 & \qquad 2x + 5 \\
 & \qquad \underline{2x - 4} \\
 & \qquad\quad\;\; \underline{9}
\end{array}
$$

Similarly, when $(4x^2 - 7x + 9)$ is divided by $(x + 3)$, the remainder is $ap^2 + bp + c$, (where $a = 4, b = -7$, $c = 9$ and $p = -3$) i.e. the remainder is $4(-3)^2 + (-7)(-3) + 9 = 36 + 21 + 9 = \mathbf{66}$.

Also, when $(x^2 + 3x - 2)$ is divided by $(x - 1)$, the remainder is $1(1)^2 + 3(1) - 2 = \mathbf{2}$.

It is not particularly useful, on its own, to know the remainder of an algebraic division. However, if the remainder should be zero then $(x - p)$ is a factor. This is very useful therefore when factorizing expressions.

For example, when $(2x^2 + x - 3)$ is divided by $(x - 1)$, the remainder is $2(1)^2 + 1(1) - 3 = 0$, which means that $(x - 1)$ is a factor of $(2x^2 + x - 3)$.

In this case the other factor is $(2x + 3)$, i.e.,

$$(2x^2 + x - 3) = (x - 1)(2x - 3)$$

The **remainder theorem** may also be stated for a **cubic equation** as:

'if $(ax^3 + bx^2 + cx + d)$ is divided by $(x - p)$, the remainder will be $ap^3 + bp^2 + cp + d$'

As before, the remainder may be obtained by substituting p for x in the dividend.

For example, when $(3x^3 + 2x^2 - x + 4)$ is divided by $(x - 1)$, the remainder is $ap^3 + bp^2 + cp + d$ (where $a = 3$, $b = 2$, $c = -1$, $d = 4$ and $p = 1$), i.e. the remainder is $3(1)^3 + 2(1)^2 + (-1)(1) + 4 = 3 + 2 - 1 + 4 = \mathbf{8}$.

Similarly, when $(x^3 - 7x - 6)$ is divided by $(x - 3)$, the remainder is $1(3)^3 + 0(3)^2 - 7(3) - 6 = 0$, which means that $(x - 3)$ is a factor of $(x^3 - 7x - 6)$.

Here are some more examples on the remainder theorem.

> Problem 30. Without dividing out, find the remainder when $2x^2 - 3x + 4$ is divided by $(x - 2)$.

By the remainder theorem, the remainder is given by $ap^2 + bp + c$, where $a = 2$, $b = -3$, $c = 4$ and $p = 2$.
Hence **the remainder is:**

$$2(2)^2 + (-3)(2) + 4 = 8 - 6 + 4 = \mathbf{6}$$

> Problem 31. Use the remainder theorem to determine the remainder when $(3x^3 - 2x^2 + x - 5)$ is divided by $(x + 2)$.

By the remainder theorem, the remainder is given by $ap^3 + bp^2 + cp + d$, where $a = 3$, $b = -2$, $c = 1$, $d = -5$ and $p = -2$.
Hence **the remainder is:**

$$
\begin{aligned}
& 3(-2)^3 + (-2)(-2)^2 + (1)(-2) + (-5) \\
& = -24 - 8 - 2 - 5 \\
& = \mathbf{-39}
\end{aligned}
$$

Problem 32. Determine the remainder when $(x^3 - 2x^2 - 5x + 6)$ is divided by (a) $(x - 1)$ and (b) $(x+2)$. Hence factorize the cubic expression.

(a) When $(x^3 - 2x^2 - 5x + 6)$ is divided by $(x - 1)$, the remainder is given by $ap^3 + bp^2 + cp + d$, where $a = 1$, $b = -2$, $c = -5$, $d = 6$ and $p = 1$,

i.e. **the remainder** $= (1)(1)^3 + (-2)(1)^2 + (-5)(1) + 6$
$= 1 - 2 - 5 + 6 = \mathbf{0}$

Hence $(x - 1)$ is a factor of $(x^3 - 2x^2 - 5x + 6)$.

(b) When $(x^3 - 2x^2 - 5x + 6)$ is divided by $(x + 2)$, **the remainder is** given by

$$(1)(-2)^3 + (-2)(-2)^2 + (-5)(-2) + 6$$
$$= -8 - 8 + 10 + 6 = \mathbf{0}$$

Hence $(x+2)$ is also a factor of (x^3-2x^2-5x+6). Therefore $(x-1)(x+2)(x \quad) = x^3-2x^2-5x+6$. To determine the third factor (shown blank) we could

(i) divide $(x^3 - 2x^2 - 5x + 6)$ by $(x - 1)(x + 2)$.

or (ii) use the factor theorem where $f(x) = x^3 - 2x^2 - 5x + 6$ and hoping to choose a value of x which makes $f(x) = 0$.

or (iii) use the remainder theorem, again hoping to choose a factor $(x - p)$ which makes the remainder zero.

(i) Dividing $(x^3 - 2x^2 - 5x + 6)$ by $(x^2 + x - 2)$ gives:

$$\begin{array}{r|l} & x \quad -3 \\ \hline x^2+x-2 & x^3 - 2x^2 - 5x + 6 \\ & x^3 + x^2 - 2x \\ \hline & \quad -3x^2 - 3x + 6 \\ & \quad -3x^2 - 3x + 6 \\ \hline & \quad\quad \cdot \quad \cdot \quad \cdot \end{array}$$

Thus $\mathbf{(x^3 - 2x^2 - 5x + 6)}$
$\mathbf{= (x - 1)(x + 2)(x - 3)}$

(ii) Using the factor theorem, we let

$f(x) = x^3 - 2x^2 - 5x + 6$

Then $f(3) = 3^3 - 2(3)^2 - 5(3) + 6$
$= 27 - 18 - 15 + 6 = 0$

Hence $(x - 3)$ is a factor.

(iii) Using the remainder theorem, when (x^3-2x^2-5x+6) is divided by $(x-3)$, the remainder is given by ap^3+bp^2+cp+d, where $a = 1$, $b = -2$, $c = -5$, $d = 6$ and $p = 3$.
Hence the remainder is:

$$1(3)^3 + (-2)(3)^2 + (-5)(3) + 6$$
$$= 27 - 18 - 15 + 6 = 0$$

Hence $(x - 3)$ is a factor.

Thus $\mathbf{(x^3 - 2x^2 - 5x + 6)}$
$\mathbf{= (x - 1)(x + 2)(x - 3)}$

Now try the following exercise.

Exercise 7 Further problems on the remainder theorem

1. Find the remainder when $3x^2 - 4x + 2$ is divided by
(a) $(x - 2)$ (b) $(x + 1)$ [(a) 6 (b) 9]

2. Determine the remainder when $x^3 - 6x^2 + x - 5$ is divided by
(a) $(x + 2)$ (b) $(x - 3)$
[(a) -39 (b) -29]

3. Use the remainder theorem to find the factors of $x^3 - 6x^2 + 11x - 6$.
[$(x - 1)(x - 2)(x - 3)$]

4. Determine the factors of $x^3 + 7x^2 + 14x + 8$ and hence solve the cubic equation $x^3 + 7x^2 + 14x + 8 = 0$.
[$x = -1$, $x = -2$ and $x = -4$]

5. Determine the value of 'a' if $(x+2)$ is a factor of $(x^3 - ax^2 + 7x + 10)$.
[$a = -3$]

6. Using the remainder theorem, solve the equation $2x^3 - x^2 - 7x + 6 = 0$.
[$x = 1$, $x = -2$ and $x = 1.5$]