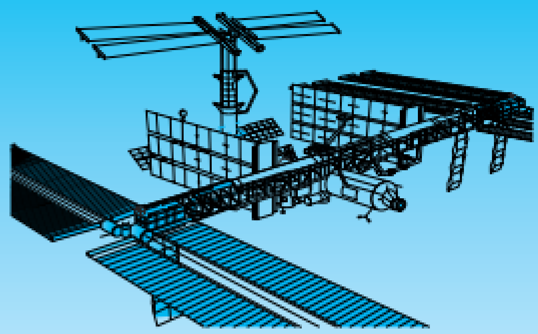
CHAPTER 12



Vector Integral Calculus

When working with the fundamental conservation laws governing engineering and physics, problems often arise that lead to the integral of the divergence of a vector function **F** over a volume V. The Gauss divergence theorem enables the integral of div **F** over volume V to be replaced by the integral of the normal component of **F** over the surface S enclosing V. This result simplifies calculations, because **F** is usually only known in general terms, whereas in physical problems the value of the normal component of **F** on S is known from the conditions of the problem.

Another vector quantity that arises naturally in engineering and physics is the vector function curl **F**, and when this occurs it is often necessary to integrate the normal component of curl **F** over an open surface S. This happens, for example, in fluid mechanics when working with the vorticity and circulation of a fluid. Stokes' theorem replaces the evaluation of the integral of the normal component of curl **F** over the open surface S by a directed line integral of **F** around the curve Γ forming the boundary of S. Here also a simplification results, because once again the vector function **F** on surface S is usually only known in general terms, whereas in physical problems its value on Γ is specified. Green's theorem in the plane is a two-dimensional form of Stokes' theorem, and it has many uses throughout engineering, physics, and mathematics.

The three most important vector integral theorems due to Gauss, Green, and Stokes are derived, followed by the derivation of two important integral transport theorems that play an essential role in mechanics, fluid mechanics, chemical engineering, electromagnetism, and elsewhere. After a review of the background of the vector integral calculus, and an introduction to the concept of an orientable surface, the Gauss divergence theorem and the theorems due to Green and Stokes are proved and applied.

The two fundamental integral transport theorems that are derived and applied are the flux transport theorem, which determines the rate of change of flux passing through an open surface bounded by a moving space curve, and Reynold's transport theorem, which concerns the rate of change of a volume integral when the volume is contained within a moving surface.

12.1 Background to Vector Integral Theorems

Information Provided by Vector Integral Theorems

Physical problems in two and three space dimensions often give rise to integrals with integrands that are determined by a vector field **F** defined over the region of integration. The most important of these integrals involves either the integration of div **F** over a finite volume V, or the integral over a finite open surface S in space of the component of curl **F** normal to S. The objective of this chapter will be to prove some fundamental integral theorems of this type due to Gauss, Stokes, and Green called, respectively, the *Gauss divergence theorem, Stokes' theorem*, and *Green's theorems*. In addition, as optional material, what is called the *flux transport theorem* and the *volume transport theorem* will be proved and, as applications, used to derive some fundamental properties of fluid mechanics.

three important theorems

It will be shown that the **Gauss divergence theorem**, often abbreviated to the **divergence theorem** or **Gauss' theorem**, relates the integral of div **F** over a volume V to the integral over the closed surface S enclosing V of the component of **F** normal to S. Thus, Gauss' theorem allows a volume integral of this type to be replaced by a simpler surface integral. **Stokes' theorem**, which will also be proved in Section 12.2, is of a different nature, in that it relates the integral of the normal component of curl **F** over an open surface S in space bounded by a closed space curve Γ to the line integral of the tangential component of **F** around Γ. So, in the case of Stokes' theorem, a surface integral of a special type over S is related to a simpler line integral around the closed space curve Γ that forms the boundary of S. **Green's theorem in the plane** is the two-dimensional form of Stokes' theorem, and a typical application is to be found in Chapter 14, where it is used in the proof of the Cauchy integral theorem for the integration of complex analytic functions.

Also proved will be two other theorems known as **Green's theorems**, though these results are also known as **Green's identities** or **Green's formulas**. They relate integrals of Laplacians of scalar functions Φ and Ψ over a volume V to the integral over the surface S enclosing V of the derivatives of these functions normal to S. Green's theorems are used extensively when working with partial differential equations involving the Laplacian operator, because they can be used to replace the integral over a volume V of a solution of Laplace's equation that is to be determined by the integral of the normal derivatives of the solution over S that occur as a prescribed boundary condition that must be satisfied by the solution.

A common feature of these theorems is that each frequently replaces an integral of a special type over a region (a volume or an open surface) by a simpler integral over the boundary of the region (a closed surface or a closed space curve), thereby reducing by one the number of dimensions involved in the integration. The integral can then be evaluated by using whichever of the two equivalent expressions is easier. When used with partial differential equations involving the Laplacian operator, Green's theorems typically allow integrals of unknown functions over a region to be replaced by simpler integrals of known functions over the boundary of the region.

The two transport theorems proved in Section 12.3 relate to the determination of the derivative with respect to time of surface and volume integrals of time-dependent integrands when the surface or volume involved moves with time. The *flux* of a vector **F** across a surface S is the integral over S of the component of

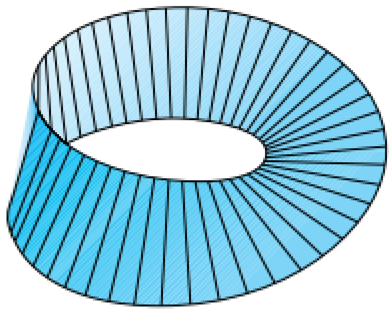


FIGURE 12.1 A Möbius strip.

F normal to S. The flux transport theorem describes the rate of change of the flux of **F** across S, taking into account the time dependence of **F** and the motion of S. A typical example of this type occurs when current is induced in a coil of wire moving in a magnetic field, because the current depends on the rate of change of magnetic flux through the moving coil.

The volume transport theorem describes the time rate of change of a volume integral due to the time dependence of the integrand and the motion of the volume over which integration takes place. A typical application of this theorem arises in fluid mechanics where the boundary of a volume of interest relating to a certain feature of the fluid flow does not move in the same way as the fluid, so that a flow takes place through the surface that encloses the volume.

Surfaces and Orientation

Section 12.2 is concerned with surfaces that have *two* sides and makes use of the normal at each point on such surfaces. It might seem unnecessary to define two-sided surfaces, but it is necessary because pathological surfaces exist that only have *one* side, and these must be excluded from the theorems of Section 12.2.

An example of a one-sided surface is provided by the **Möbius strip** shown in Fig. 12.1. This strip can be considered to be formed from a long strip of paper, the ends of which are joined after making a 180° twist in the paper about its longitudinal center line. Its one-sided nature can easily be verified by drawing a pencil line around the center line of the strip, because eventually the line will connect with the starting point, and if the strip is cut and opened out, examination will show a pencil line on both sides of the paper.

When deriving the Gauss divergence theorem, it will be necessary to work with a closed two-sided surface S, the *interior* of which contains the volume V of space that will concern us. A vector element of area of such a surface will have magnitude dS and an associated unit vector $\mathbf{n}$ normal to dS. As the normal $\mathbf{n}$ at a point on a two-sided surface S enclosing a volume V may be directed away from either side of S, it is necessary to adopt a standard convention for the direction of $\mathbf{n}$ and the vector element of area $d\mathbf{S} = \mathbf{n}dS$ on S. The normal $\mathbf{n}$ at a point on such a surface will always be chosen to be directed *out* of V. So if, for example, V is a sphere, the normal $\mathbf{n}$ at any point of its surface will be along a radial line drawn *outward* from the center of the sphere.

open surfaces and orientable surfaces

A two-sided **open surface** S bounded by a non-self-intersecting space curve Γ is a surface that does *not* have an interior, and so does *not* enclose a volume V. When

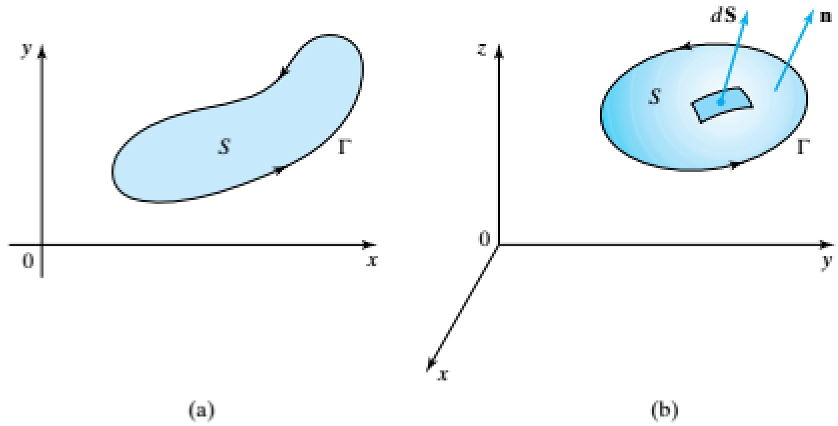


FIGURE 12.2 (a) A plane oriented surface. (b) A general oriented surface in space.

deriving Stokes' theorem it will be necessary to work with a two-sided open surface S bounded by a closed non-self-intersecting space curve Γ around which there is a given sense of direction. The normal at each point of S will be always be chosen in such a way that it points in the direction in which a right-handed screw would advance were it to be rotated in the sense of direction that is specified around the boundary curve Γ. Surfaces S of this type are called **oriented surfaces**. Pathological one-sided surfaces such as Möbius strips are said to be **nonorientable**, and they will not be considered here.

A simple but typical example of an open orientable surface S is an area in the (x, y)-plane contained within a closed curve Γ. If the sense of direction around Γ is chosen to be counterclockwise, the normal $\mathbf{n}$ to S will point in the direction of the unit vector $\mathbf{k}$. A reversal of the sense of direction around Γ will reverse the sense of $\mathbf{n}$, which will then point in the direction of $-\mathbf{k}$. Examples of *oriented surfaces* are illustrated in Fig. 12.2, where Fig. 12.2(a) shows an open oriented surface S in the (x, y)-plane and Fig. 12.2(b) shows a general open oriented surface in space.

Let S be a two-sided surface with a boundary curve Γ around which a sense of direction is prescribed, and at each point of S let $\mathbf{n}$ be the unit normal to S pointing in the direction determined by the sense of direction around Γ, as described above. Then if dS is an element of area of S, the vector element of area on the oriented surface S is $d\mathbf{S} = \mathbf{n}dS$.

Summary This brief section introduced the important concept of an open surface that is orientable, and established the right-handed screw convention by which the direction of the normal to an orientable surface is determined.

12.2 Integral Theorems

The first integral theorem to be established is the Gauss divergence theorem, which relates volume integrals and surface integrals. It is possible to formulate a more general statement of the theorem than the one given here, but to do so involves a lengthy argument, and Theorem 12.1 is sufficient for all practical purposes.

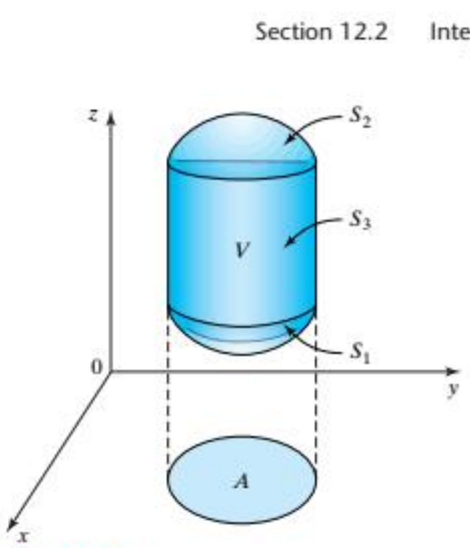


FIGURE 12.3 The volume V.

THEOREM 12.1

a theorem relating the integral of div F over a volume to the integral of the normal component of F over the surface bounding the volume

The Gauss divergence theorem Let $\mathbf{F}$ be a vector field defined throughout a volume V enclosed within a piecewise smooth surface S on which the outward drawn unit normal is $\mathbf{n}$. Then, if the components of $\mathbf{F}$ and its first order partial derivatives are continuous throughout V and on S, dV is an element of volume of V, and dS is an element of area of S,

$$\iiint_V \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $d\mathbf{S} = \mathbf{n}dS$ is a vector surface element of area on S.

Proof Consider a volume V in the form of a cylinder with its sides parallel to the z-axis, a lower surface $z = z_1(x, y)$, and an upper surface $z = z_2(x, y)$, and let A be the projection of the cross-section of the cylinder onto the (x, y)-plane, as shown in Fig. 12.3.

The lower surface in Fig. 12.3 will be denoted by S_1, the upper surface by S_2, and the cylindrical side surface by S_3, so the surface S enclosing volume V is piecewise smooth and comprises these three surfaces.

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, where the components of $\mathbf{F}$ and its first order partial derivatives are continuous in V and on S. The integral of $\partial F_3/\partial z$ with respect to z along a line in V drawn parallel to the z-axis is

$$\int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial F_3}{\partial z} dz = F_3(x, y, z_2(x, y)) - F_3(x, y, z_1(x, y)).$$

The integral of this result over the area A that is the projection of V onto the (x, y)-plane is given by

$$\iiint_V \frac{\partial F_3}{\partial z} dV = \iint_A F_3(x, y, z_2(x, y))dxdy - \iint_A F_3(x, y, z_1(x, y))dxdy.$$

The first term on the right is the integral of F_3 over the *top* of the upper two-sided surface S_2, while the second term is the integral F_3 over the *top* of the lower two-sided surface S_1. As the normals to surfaces bounding the volume V are chosen to point *outward* from V, and the normal in the last term is directed *into* volume V, the sign of the last term can be reversed and the resulting equation written as

$$\iiint_V \frac{\partial F_3}{\partial z} dV = \iint_{S_2} F_3 dxdy + \iint_{S_1} F_3 dxdy.$$

To express the integrals on the right as a single integral over the complete surface S, it is necessary to take into account the integral of F_3 over the cylindrical surface S_3. The unit normal to the element of area $dxdy$ of A is perpendicular to the (x, y)-plane in the direction $\mathbf{k}$, but $\mathbf{k}$ is orthogonal to all outward drawn normals to the cylindrical surface, so the integral of F_3 over the cylindrical surface S_3 must vanish, giving $\iint_{S_3} F_3 dxdy = 0$. Adding this integral to the preceding equation, and recognizing that the piecewise smooth surface S comprises the sum of the three surfaces S_1, S_2, and S_3, we arrive at the result

$$\iiint_V \frac{\partial F_3}{\partial z} dV = \iint_S F_3 dxdy.$$

Corresponding results involving F_1 and F_2 that can be derived in similar fashion are

$$\iiint_V \frac{\partial F_1}{\partial x} dV = \iint_S F_1 dydz$$

and

$$\iiint_V \frac{\partial F_2}{\partial y} dV = \iint_S F_2 dxdz.$$

Addition of these three integrals gives

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV = \iint_S F_1 dydz + F_2 dxdz + F_3 dxdy,$$

or equivalently,

$$\iiint_V \operatorname{div} \mathbf{F}\, dV = \iint_S F_1 dydz + F_2 dxdz + F_3 dxdy.$$

Let dS with the outward drawn unit normal $\mathbf{n}$ be an element of area of the bounding surface S, and let its projection onto the (y, z)-plane be the element of area $dydz$. Then if the angle between $\mathbf{n}$ and the normal to the (y, z)-plane is γ, it follows that $dydz = dS \cos \gamma$. However, the unit normal to the (y, z)-plane is the vector $\mathbf{i}$, so $\cos \gamma = \mathbf{i} \cdot \mathbf{n}$, and consequently $dydz = \mathbf{i} \cdot \mathbf{n} dS = \mathbf{i} \cdot d\mathbf{S}$. Similar arguments lead to the corresponding results $dxdz = \mathbf{j} \cdot d\mathbf{S}$ and $dxdy = \mathbf{k} \cdot d\mathbf{S}$.

Using these expressions in the preceding integral allows it to be written as

$$\iiint_V \operatorname{div} \mathbf{F}\, dV = \iint_S (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \cdot \mathbf{n} dS$$

or as

$$\iiint_V \operatorname{div} \mathbf{F}\, dV = \iint_S F \cdot d\mathbf{S},$$

and the theorem is proved for a volume V with sides parallel to the z-axis. ■

Modifications to the preceding form of argument that we will not detail show the theorem to be true for volumes V with boundaries formed by finitely many piecewise smooth parts, and also for boundaries on which the partial derivatives of

F_i are not differentiable at every point. The theorem remains true for domains such as a torus that have a more complicated shape. This follows because such domains can be subdivided into domains of the type covered by Theorem 12.1, and as the outward-drawn normals to each side of a dividing surface are oppositely directed, the integrals over the two sides of each such surface cancel, leaving only the integral over S of the component of $\mathbf{F}$ normal to S.

CARL FRIEDRICH GAUSS (1777–1855)
A German mathematician of truly outstanding ability who is universally regarded as the greatest mathematician of the nineteenth century. He ranks with Isaac Newton as one of the greatest mathematicians of all time. He was appointed to the directorship of the observatory in Göttingen and spent the remainder of his life there. His contributions spanned all aspects of mathematics and science, in addition to his interest in astronomy. He also made important contributions to number theory, algebra, and geometry.

The divergence theorem provides an alternative definition of div $\mathbf{F}$, because if the result of the theorem is divided by the volume V with bounding surface S over which integration is performed, and the limit is taken as $V \to 0$ about a fixed point P in space, we obtain

$$(\text{div }\mathbf{F})_P = \lim_{V\to 0} \frac{1}{V} \iint_S \mathbf{F}\cdot d\mathbf{S}. \tag{1}$$

However, $\mathbf{F}\cdot d\mathbf{S} = \mathbf{F}\cdot\mathbf{n}\,dS$ and $\mathbf{F}\cdot\mathbf{n} = F_n$ is the component of $\mathbf{F}$ normal to dS, so $\iint_S \mathbf{F}\cdot d\mathbf{S}$ is the *flux* of $\mathbf{F}$ across S at the point P. Consequently, $(\text{div }\mathbf{F})_P$ is seen to be the flux of $\mathbf{F}$ per unit volume at P.

A physical interpretation of this last result is provided by the flow of a fluid with velocity $\mathbf{q}$, because

$$(\text{div }\mathbf{q})_P = \lim_{V\to 0} \frac{1}{V} \iint_S \mathbf{q}\cdot d\mathbf{S} \tag{2}$$

an application to incompressible flow with sources and sinks

is seen to be the amount of fluid leaving an infinitesimal surface surrounding P in a unit time. If the fluid is **incompressible**, there can be no net flow either into or out of any volume, so in an incompressible fluid div $\mathbf{q} = 0$ throughout the fluid. If, however, there is a **source** of fluid at P causing fluid to flow into volume V and onward out of S, then $(\text{div }\mathbf{q})_P$ will be positive, whereas if there is removal of fluid from volume V at P due to the presence of a **sink** at P, then $(\text{div }\mathbf{q})_P$ will be negative. In a fluid that is **compressible**, div $\mathbf{q}$ may be either positive or negative at a point in the fluid without any source or sink being present.

Any vector $\mathbf{F}$ such that

$$\text{div }\mathbf{F} \equiv 0 \tag{3}$$

a solenoidal vector

is said to be a **solenoidal** vector. So as div(curl $\mathbf{F}$) $\equiv 0$, it follows that provided $\mathbf{F}$ has continuous second order partial derivatives, the vector curl $\mathbf{F}$ is a solenoidal vector.

The following examples illustrate how the divergence theorem can be used to simplify the evaluation of integrals, though more important applications arise in the formulation and solution of partial differential equations.

EXAMPLE 12.1 Evaluate

$$\iint_S 3xdydz + 2ydxdz - 5zdxdy$$

where S is a smooth surface bounding an arbitrary volume V.

Solution The integral can be written

$$\iint_S 3xdydz + 2ydxdz - 5zdxdy = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} = 3x\mathbf{i} + 2y\mathbf{j} - 5z\mathbf{k}$. So as the conditions of Theorem 12.1 are satisfied and div $\mathbf{F} = 0$, it follows from the divergence theorem that

$$\iint_S 3xdydz + 2ydxdz - 5zdxdy = \iiint_V \text{div}\,\mathbf{F}\,dV = 0.$$

EXAMPLE 12.2 Evaluate

$$\iint_S x^3dydz + y^3dxdz + z^3dxdy,$$

where the surface S is the boundary of the volume V occupying the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and above the plane $z = 0$.

Solution The volume V is a hemispherical shell between spheres of radii 1 and 2 centered on the origin and above the plane $z = 0$, so its surface S is formed by the surfaces of two hemispheres above the $z = 0$ plane and the annulus $1 \leq r \leq 2$ in the plane $z = 0$. The required integral can be written

$$I = \iint_S x^3dydz + y^3dxdz + z^3dxdy = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$. As $\mathbf{F}$ is differentiable and the surface S is piecewise smooth, the divergence theorem can be used to replace the surface integral by the triple volume integral of div $\mathbf{F}$ over V, showing that

$$I = 3\iiint_V (x^2 + y^2 + z^2)dxdydz.$$

The spherical symmetry of volume V suggests that integral I will be simplified if spherical polar coordinates are used. In terms of these coordinates, the volume V becomes $1 \leq r \leq 2, 0 \leq \phi < 2\pi$, and $0 \leq \theta \leq \pi/2$, and the integrand becomes $x^2 + y^2 + z^2 = r^2$, so as the volume element of the transformation is given by $dV = r^2 \sin\phi dr d\theta d\phi$, the integral for I becomes

$$\begin{aligned} I &= 3\int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \int_1^2 r^4 \sin\theta dr \\ &= 3\int_0^{2\pi} d\phi \int_0^{\pi/2} \frac{31}{5} \sin\theta d\theta \\ &= \frac{93}{5}\int_0^{2\pi} d\phi = \frac{186}{5}\pi. \end{aligned}$$

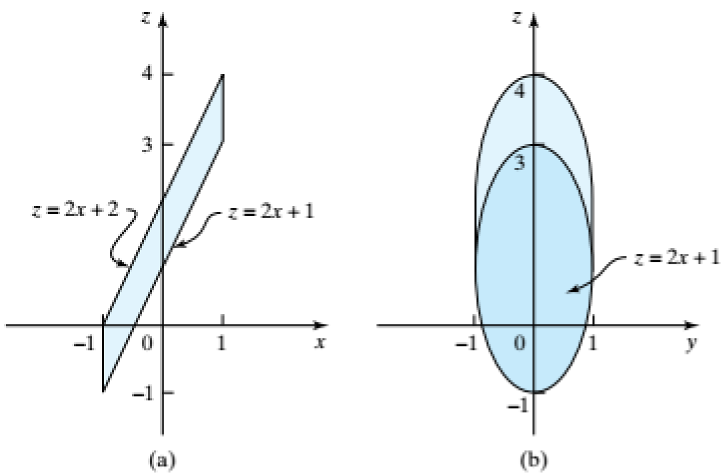


FIGURE 12.4 Cylinder with parallel oblique ends. (a) Side view; (b) front view.

EXAMPLE 12.3

Let the vector function $\mathbf{F} = (x^2 + 3y)\mathbf{i} - (3y^2 + \sin z)\mathbf{j} + 2z^2\mathbf{k}$ be defined throughout the volume V interior to the cylindrical volume with parallel oblique ends bounded by the surface S that is shown in Fig. 12.4, where the cylinder cross-section has the equation $x^2 + y^2 = 1$ and the cylinder ends are formed by the intersection of the cylinder with the planes $z = 2x + 1$ and $2x + 2$. Find the integral over S of F_n, the component of $\mathbf{F}$ normal to the surface S.

Solution The function $\mathbf{F}$ and the surface S satisfy the conditions of the divergence theorem, so as div $\mathbf{F} = 2x - 6y + 4z$, the result of applying the theorem to volume V is

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_V (2x - 6y + 4z)dV \\ &= \iint_{x^2+y^2\le 1} \left(\int_{1+2x}^{2+2x} (2x - 6y + 4z)dz \right) dxdy \\ &= \iint_{x^2+y^2\le 1} (10x - 6y + 6)dxdy.\end{aligned}$$

To proceed further, we change to plane polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ for which the Jacobian $J(r, \theta) = r$, and the area $x^2 + y^2 \le 1$ becomes $0 \le r \le 1$ with $0 \le \theta \le 2\pi$. As a result,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} d\theta \int_0^1 (10r\cos\theta - 6r\sin\theta + 6)r\,dr \\ &= \int_0^{2\pi} \left(\frac{10}{3}\cos\theta - 2\sin\theta + 3 \right) d\theta = 6\pi,\end{aligned}$$

so the required integral over S of the component F_n of $\mathbf{F}$ normal to S is

$$\iint_S \mathrm{F}_n dS = 6\pi.$$

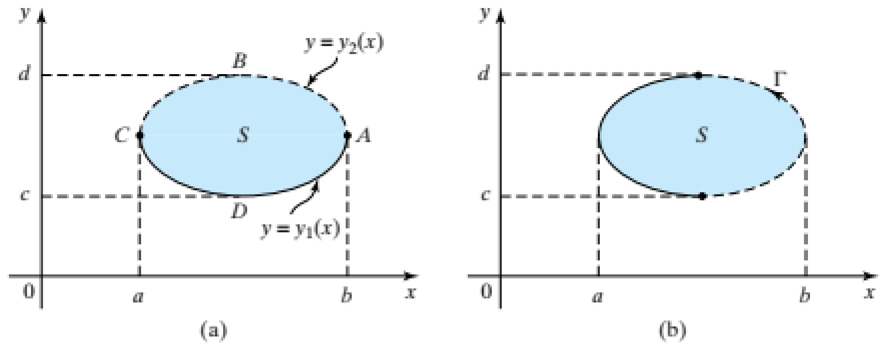


FIGURE 12.5 (a) The convex area S with lower and upper boundaries $y = y_1(x)$ and $y = y_2(x)$. (b) The convex area S with left and and right boundaries $x = x_1(y)$ and $x = x_2(y)$.

Preparatory to proving Stokes' theorem, we must prove Green's theorem in the plane that can be stated as follows.

THEOREM 12.2

a theorem relating an integral over a plane surface to an integral around its perimeter

Green's theorem in the plane Let a finite area S in (x, y)-plane be bounded by a piecewise smooth closed non-self-intersecting plane curve Γ around which a counterclockwise sense of direction is imposed. Then if $P(x, y)$ and $Q(x, y)$ and their first order partial derivatives are continuous over S and on Γ,

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\Gamma Pdx + Qdy.$$

Proof We first prove the theorem for a plane area S that is convex, which is an area S with the property that any straight line that crosses it intersects the boundary at most twice. We then show how the theorem can be applied to more complicated areas, including those with internal boundaries. A typical area S of this type is shown in Fig. 12.5.

Let us consider the integral of $\partial P/\partial y$ over the convex area S with the lower boundary $y = y_1(x)$ and upper boundary $y = y_2(x)$, as shown in Fig. 12.5(a). The integral over S can be written as the iterated integral

$$\begin{aligned}\iint_S \frac{\partial P}{\partial y} dxdy &= \int_a^b dx \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y} dy \\ &= \int_a^b P(x, y_2(x))dx - \int_a^b P(x, y_1(x))dx\end{aligned}$$

or as

$$\iint_S \frac{\partial P}{\partial y} dxdy = -\int_{ABC} P(x, y)dx - \int_{CDA} P(x, y)dx,$$

where the sign of the first integral on the right has been reversed because integration from $x = a$ to $x = b$ is in the opposite sense to the counterclockwise direction of integration required along ABC. The two arcs ABC and CDA form the closed

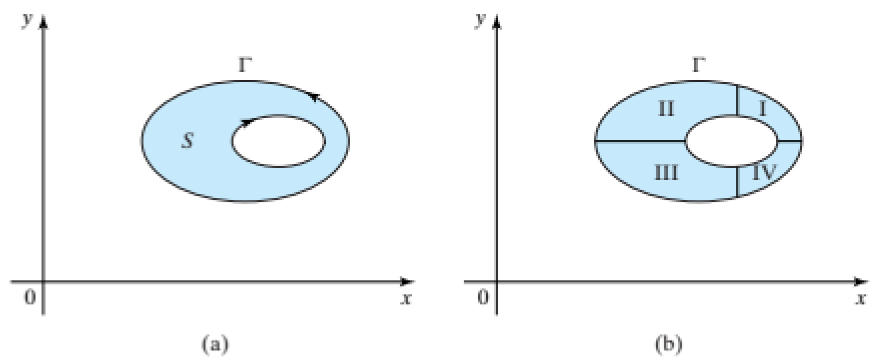


FIGURE 12.6 (a) S with an internal boundary. (b) The partitioning of S.

contour Γ, so the preceding result simplifies to

$$\iint_S \frac{\partial P}{\partial y} dxdy = -\int_\Gamma P(x, y)dx.$$

When the foregoing argument is repeated, but this time using the left and right boundaries in Fig. 12.5(b), and the integral of $\partial Q/\partial x$ over S is calculated we obtain

$$\iint_S \frac{\partial Q}{\partial x} dxdy = \int_\Gamma Q(x, y)dx.$$

However, as S is convex, each of these results is true, so subtracting them we arrive at the statement of Green's theorem

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\Gamma Pdx + Qdy.$$

We need to show this result remains true for areas S that are not convex, and also for areas with internal boundaries. It will be sufficient to consider the area S shown in Fig. 12.6(a), in which there is a single internal boundary γ, because the argument extends immediately to arbitrary areas with finitely many internal boundaries, and to areas that are not convex.

Let S be partitioned into the four areas shown in Fig. 12.6(b), to each of which Green's theorem applies. Applying the theorem to each area and adding the integrals, we see that integrals along the adjacent straight line segments will cancel, because of the continuity of P, Q, and their first order partial derivatives in S, and the fact that the integrations take place in *opposite* directions. As a result only the integrals around the boundaries Γ and γ remain, so the theorem holds, provided the sense of integration around all boundaries (both external and internal) is such that the area S always lies to the *left* as each boundary is traversed. This argument also applies to finitely many internal boundaries, so Green's theorem in the plane is proved for this more general case. ■

The sense in which integration must be performed when applying Green's theorem to an area S with internal boundaries is illustrated in Fig. 12.7.

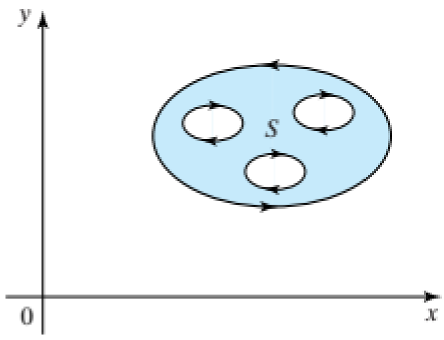


FIGURE 12.7 Direction of integration around a domain D with internal boundaries.

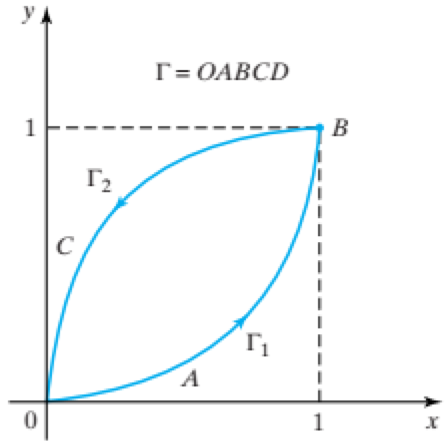


FIGURE 12.8 The curve Γ formed from two circular arcs Γ_1 and Γ.

GEORGE GREEN (1793–1841)
A self-taught English mathematical physicist who was born in Nottingham where he first worked as a baker. His contributions to electricity and magnetism, where he introduced the theorems now named after him, were first published privately in 1828, and so attracted little attention. It was not until William Thompson (Lord Kelvin) discovered his results and caused them to be republished in 1846 that their significance was recognized. Due to the limited circulation of the first published version of his work his main results were rediscovered, independently, by Lord Kelvin, Gauss, and others. He made significant contributions to the theory of optics and sound waves, and just prior to his death he was elected to a fellowship of Caius College, Cambridge.

SIR GEORGE GABRIEL STOKES (1819–1903)
A major applied mathematician and physicist who was born in County Sligo, Ireland, but spent his entire working life in Cambridge, where he was made professor of mathematics in 1849. He made fundamental contributions to the study of the flow of viscous fluids, leading to what are now called the Navier–Stokes equations, to elasticity, the propagation of sound, optics, and asymptotic series.

EXAMPLE 12.4 Evaluate

$$\int_\Gamma xy^2dx - 2x^2ydy$$

where Γ is the curve shown in Fig. 12.8, in which Γ_1 is an arc of a unit circle centered on the point $(0, 1)$, and Γ_2 is an arc of a unit circle centered on the point $(1, 0)$, and integration is in the counterclockwise sense around Γ.

Solution The equation of a unit circle with its center at $(1, 0)$ is $x^2 + (y - 1)^2 = 1$, so the equation of the arc Γ_1 is $y = 1 - 1\sqrt{1 - x^2}$ for $0 \leq x \leq 1$. The equation of a unit circle with its center at $(1, 0)$ is $(x-1)^2+y^2=1$, so the equation of arc Γ_2 is $y=\sqrt{2x-x^2}$ for $0 \le x \le 1$.

Making the identifications $P = xy^2$ and $Q = -2x^2y$ we have $\partial P/\partial y = 2xy$ and $\partial Q/\partial x = -4xy$, so substituting into Green's theorem shows that

$$\int_\Gamma xy^2dx - 2x^2ydy = \int_0^1 dx \int_{1-\sqrt{1-x^2}}^{\sqrt{2x-x^2}} (-6xy)dy$$
$$= \int_0^1 [-6x^2+6x-6x\sqrt{1-x^2}]dx = -1. \quad \blacksquare$$

THEOREM 12.3

a theorem relating an integral of the normal component of curl F over an orientable surface to the line integral of F around its perimeter

Stokes' theorem Let S be an open piecewise smooth orientable surface bounded by a closed space curve Γ around which a sense of direction is specified. At every point of the surface, let the unit normal $\mathbf{n}$ to S point in the direction specified for orientable surfaces relative to the sense around Γ. Then, if $\mathbf{F}$ is a differentiable vector function over the surface S,

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F}\cdot d\mathbf{S},$$

where $\mathbf{r}$ is the position vector of a general point on Γ.

Proof Consider Fig. 12.9, in which S is an open orientable surface $z = z(x, y)$, Γ is its bounding space curve, A is the projection of S onto the (x, y)-plane, and C is the boundary curve of A.

The proof will involve the following three steps:

(I) The line integral around Γ will be transformed into the line integral around C
(II) The line integral around C will be transformed into a double integral over A
(III) The double integral over A will be transformed into an integral over S

STEP I Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Then the line integral of F_1 around Γ is

$$\int_\Gamma F_1(x, y, z)dx = \int_C F_1(x, y, z(x, y))dx,$$

because $z = z(x, y)$ on C.

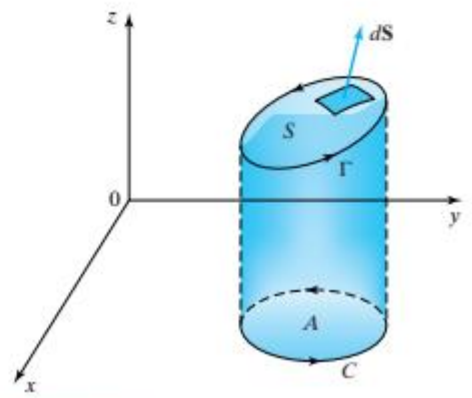


FIGURE 12.9 An orientable surface S bounded by the space curve Γ.

STEP II In the line integral on the right $z = z(x, y)$, so

$$\frac{\partial G_1}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y}, \quad \text{where } G_1(x, y) \equiv F_1(x, y, z(x, y)).$$

Applying Green's theorem in the plane to the integral in Step I and using this last result gives

$$\int_C F_1(x, y, z(x, y))dx = -\iint_A \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y}\right) dA,$$

where dA is the area element in the (x, y)-plane.

Setting $\phi = z - z(x, y)$, the surface S has the equation $\phi = 0$, so as a normal $\mathbf{N}$ to S is given by $\mathbf{N} = \operatorname{grad}\phi$

$$\mathbf{N} = \pm\left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right).$$

For $\mathbf{N}$ to have the correct *upward* direction relative to S, as required by the sense of direction of integration around the oriented surface S, it is necessary that the z-component of $\mathbf{N}$ be positive. Consequently, if we take the positive sign, the unit vector $\mathbf{n}$ normal to S is

$$\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k},$$

where the direction cosines n_1, n_2, and n_3 are given by

$$n_1 = -\frac{\partial z}{\partial x}\Big/|\mathbf{N}|, \quad n_2 = -\frac{\partial z}{\partial y}\Big/|\mathbf{N}|, \quad n_3 = 1/|\mathbf{N}| \quad \text{with}$$

$$|\mathbf{N}| = \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right)^{1/2}.$$

It now follows from these results that

$$\frac{\partial z}{\partial y} = -\frac{n_2}{n_3}.$$

If we substitute this expression for $\partial z/\partial y$ in the double integral over A, it becomes

$$\int_C F_1 dx = -\iint_A \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{n_2}{n_3}\right) dA.$$

STEP III If dA is the projection of dS onto the (x, y)-plane, we have $dA = n_3\, dS$, so the last result in Step II can be written as the double integral over S

$$\begin{aligned}\int_C F_1 dx &= -\iint_S \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{n_2}{n_3}\right) n_3 dS \\ &= \iint_S \left(\frac{\partial F_1}{\partial z}n_2 - \frac{\partial F_1}{\partial y}n_3\right) dS.\end{aligned}$$

Similar arguments show that

$$\int_C F_2 dy = \iint_S \left(\frac{\partial F_2}{\partial x}n_3 - \frac{\partial F_2}{\partial z}n_1\right) dS$$

and

$$\int_C F_3 dz = \iint_S \left(\frac{\partial F_3}{\partial y}n_1 - \frac{\partial F_3}{\partial x}n_2\right) dS.$$

Finally, the addition of these three integrals gives

$$\int_C F_1 dx + F_2 dy + F_3 dz = \iint_S \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) n_1 dS + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) n_2 dS + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) n_3 dS,$$

or equivalently,

$$\int_\Gamma F_1 dx + F_2 dy + F_3 dz = \iint_S \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dydz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dxdz + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy,$$

which is one form of Stokes' theorem. To arrive at the form given in the statement of the theorem it is only necessary to write $d\mathbf{S} = \mathbf{n}dS$, and then to recognize that

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k},$$

for the integral to become

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}. \qquad \blacksquare$$

Stokes' theorem is a generalization of Green's theorem in the plane that was used in its proof, so it is to be expected that Stokes' theorem must reduce to Green's theorem in the plane when the surface S is an area in the (x, y)-plane. That this is the case can be seen by taking $\mathbf{F}$ to be only a function of x and y, so that $\mathbf{F} = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$, because then the first form of Stokes' theorem that was proved reduces to

$$\int_\Gamma F_1 dx + F_2 dy = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy,$$

and apart from a change of notation, this is the result of Theorem 12.2.

Stokes' theorem provides a physical interpretation of curl $\mathbf{F}$ that is most easily understood in the context of a fluid flow with $\mathbf{F}$ representing the fluid velocity vector. Consider a small disc of fluid of radius ρ centered at $\mathbf{r} = \mathbf{r}_0$, as shown in Fig. 12.10,

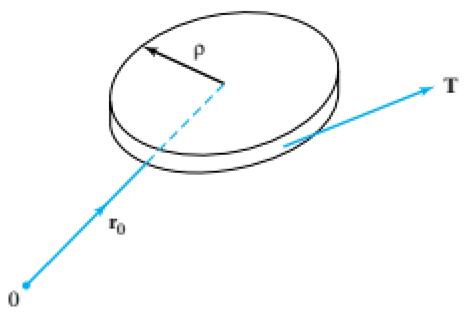


FIGURE 12.10 A disc of fluid of radius ρ with fluid velocity $\mathbf{F}$.

where S is the area of the disc and T is the unit tangent vector to the perimeter of the disc. Then $\mathbf{F}\cdot\mathbf{T}$ is the tangential component of the fluid velocity at the perimeter Γ of the disc around which the arc length is s, so the integral

$$\kappa(\mathbf{r}_0) = \int_\Gamma \mathbf{F}\cdot\mathbf{T}ds$$

is a measure of the tendency of the fluid to *rotate* around the point $\mathbf{r}_0$. This will be recognized as the *circulation* of $\mathbf{F}$ around a curve Γ introduced previously in connection with line integrals.

If the disc is small and taken on an open surface S in the fluid, and $\mathbf{N}$ is a unit normal to an element dS of the surface at $\mathbf{r} = \mathbf{r_0}$, the scalar product (curl $\mathbf{F}$) $\cdot\,\mathbf{N}$ can be regarded as a constant over the disc, so from Stokes' theorem

$$\int_\Gamma \mathbf{F}\cdot\mathbf{T}ds = \iint_S (\text{curl}\,\mathbf{F})\cdot\mathbf{N}dS \approx [(\text{curl}\,\mathbf{F})\cdot\mathbf{N}]_{\mathbf{r}_0}(\pi\rho^2),$$

and so

$$[(\text{curl}\,\mathbf{F})\cdot\mathbf{N}]_{\mathbf{r}_0} = \lim_{\rho\to 0}\frac{1}{\pi\rho^2}\int_\Gamma \mathbf{F}\cdot\mathbf{T}ds.$$

Clearly, (curl $\mathbf{F}$) $\cdot$ $\mathbf{N}$ attains its greatest value when curl $\mathbf{F}$ is parallel to $\mathbf{N}$, and it is because curl $\mathbf{F}$ is a measure of rotation that some books use the notation rot $\mathbf{F}$ in place of curl $\mathbf{F}$. Although the circulation around Γ has been illustrated by means of a fluid flow, the general concept of the circulation of a vector $\mathbf{F}$ around a curve Γ has useful physical interpretations in other situations. Another example occurs in connection with the generation of current when a wire in the form of a closed curve Γ moves in a magnetic field. Inspection of the definition of (curl $\mathbf{F}$) $\cdot$ $\mathbf{N}$ at a point $\mathbf{r}_0$ as a limit shows it is the quotient of the circulation of $\mathbf{F}$ around Γ and the area of the disc, and so again measures the rate of circulation at $\mathbf{r}_0$.

EXAMPLE 12.5

Let $\mathbf{F} = x^2\mathbf{i} + z^2y\mathbf{j} + y^2z\mathbf{k}$. Show that the line integral of $\mathbf{F}$ around any space curve Γ bounding an oriented open surface S is zero.

Solution The conditions of Stokes' theorem apply and

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & yz^2 & y^2z \end{vmatrix} = \mathbf{0},$$

so

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = 0.$$

EXAMPLE 12.6

Let S be the surface of the paraboloid of revolution $z = 1 - x^2 - y^2$ with the domain of definition $x^2 + y^2 \le 1$, and let Γ be the boundary of the paraboloid. Given $\mathbf{F} = x^3\mathbf{i} + (x + y - z)\mathbf{j} + yz\mathbf{k}$, find $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$.

Solution By Stokes' theorem

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_\Gamma \mathbf{F}\cdot d\mathbf{r},$$

so the required integral can be found by evaluating the line integral on the right. As the domain of definition of the paraboloid of revolution is $x^2 + y^2 \le 1$, it follows that the curve Γ bounding the surface of the paraboloid is the circle $x^2 + y^2 = 1$ in the plane $z = 0$. To evaluate the line integral, we parametrize Γ as $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, with $0 \le t \le 2\pi$. Then $d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j})dt$ and on Γ the vector function

$$\mathbf{F}(t) = \cos^3 t\mathbf{i} + (\cos t + \sin t)\mathbf{j},$$

so substituting into the line integral gives

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi} [\cos^3 t\mathbf{i} + (\cos t + \sin t)\mathbf{j}]\cdot[-\sin t\mathbf{i} + \cos t\mathbf{j}]dt$$

$$\int_0^{2\pi} (-\sin t\cos^3 t + \cos^2 t + \sin t\cos t)dt = \pi.$$

EXAMPLE 12.7

Given $\mathbf{F} = y\mathbf{i} - z^3\mathbf{j} + x^2\mathbf{k}$, use Stokes' theorem to evaluate $\int_\Gamma \mathbf{F}\cdot d\mathbf{r}$, where Γ is the boundary of the area S formed by the part of the plane $2x + 4y + z = 4$ that lies in the first octant, and integration around the boundary Γ is in the clockwise direction.

Solution The required integral will be determined by evaluating the integral on the right of

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}.$$

The surface S over which integration is to be performed is the plane triangular area shown in Fig. 12.11, where the boundary of S in the plane $z = 0$ is the line $x + 2y = 2$

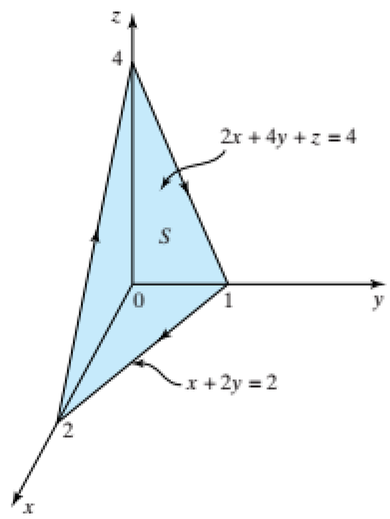


FIGURE 12.11 Plane triangular area S with clockwise direction around boundary Γ.

for $0 \leq x \leq 2$.

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -z^3 & x^2 \end{vmatrix} = 3z^2\mathbf{i} - 2x\mathbf{j} - \mathbf{k}.$$

If we set $\phi = 4 - 2x - 4y - z$, the equation of the plane is $\phi = 0$, so two possible normals $\mathbf{N}$ to the surface S of the plane are

$$\mathbf{N} = \pm\text{grad}\,\phi = \pm(-2\mathbf{i} - 4\mathbf{j} - \mathbf{k}).$$

As the direction of integration around the boundary Γ is taken to be *clockwise*, when viewed as in Fig 12.11, the normal to S must be directed away from S toward the origin, showing that the $\mathbf{k}$ component of $\mathbf{N}$ must be *negative*. Thus, the foregoing expression for $\mathbf{N}$ must be chosen with the positive sign leading to the result $\mathbf{N} = -2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$, so the unit vector $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$ with the required sense normal to the plane is

$$\mathbf{n} = \frac{1}{\sqrt{21}}(-2\mathbf{i} - 4\mathbf{j} - \mathbf{k}).$$

The line of intersection of the plane $2x + 4y + z = 4$ and the plane $z = 0$ is $x + 2y = 2$, so the base of the triangular plane surface S has the equation $x + 2y = 2$ for $0 \leq x \leq 2$.

We now have sufficient information to compute $\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$:

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \iint_S (3z^2\mathbf{i} - 2x\mathbf{j} - \mathbf{k}) \cdot d\mathbf{S},$$

but $d\mathbf{S} = \mathbf{n}dS$, so if A is the projection of S onto the plane $z = 0$, the integral over S can be replaced by the integral over A, giving

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_S (3z^2\mathbf{i} - 2x\mathbf{j} - \mathbf{k}) \cdot d\mathbf{S} = \frac{1}{\sqrt{21}} \iint_S (-6z^2 + 8x + 1)dS.$$

However, if n_3 is the $\mathbf{k}$ component of $\mathbf{n}$, $dA/dS = |n_3| = 1\sqrt{21}$ and so $dS = \sqrt{21}dA$. Using this result in the integral on the right with $z = 4 - 2x - 4y$ shows that

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_A [-6(4 - 2x - 4y)^2 + 8x + 1]dA.$$

Writing the double integral over A as a repeated integral gives

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int_0^1 dy \int_0^{-2-2y} [-6(4 - 2x - 4y)^2 + 8x + 1]dx = -\frac{29}{3}.$$ ■

The results of the next theorem, called **Green's formulas** or sometimes **Green's identities**, are used extensively in the study of partial differential equations.

THEOREM 12.4

Green's formulas Let Φ and Ψ be scalar fields such that the Laplacians $\Delta\Phi$ and $\Delta\Psi$ are defined inside a volume V enclosed in a closed piecewise smooth surface S, and if the second order partial derivatives of Φ and Ψ have any discontinuities, let them be bounded and occur only along lines on S or across finitely many surfaces in V. Then:

two useful formulas due to Green

(I) Green's first formula is

$$\iint_S \Phi \frac{\partial \Psi}{\partial n} dS = \iiint_V \{\Phi \Delta \Psi + (\text{grad } \Phi) \cdot (\text{grad } \Psi)\} dV,$$

where dV is a volume element of V.

(II) Green's second formula is

$$\iint_S \left(\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) dS = \iiint_V (\Phi \Delta \Psi - \Psi \Delta \Phi) dV.$$

Proof The proof is straightforward, but for simplicity it will only be offered for functions Φ and Ψ that have continuous second order partial derivatives inside a finite volume V and on its bounding surface S.

Setting $\mathbf{G} = \Phi(\text{grad } \Psi)$, it follows that

$$\text{div } \mathbf{G} = \Phi \text{ div}(\text{grad } \Psi) + (\text{grad } \Phi) \cdot (\text{grad } \Psi),$$

so applying the divergence theorem we have

$$\iint_S \Phi(\text{grad } \Psi) \cdot d\mathbf{S} = \iiint_V \{\Phi \Delta \Psi + (\text{grad } \Phi) \cdot (\text{grad } \Psi)\} dV.$$

However, $\Phi(\text{grad } \Psi) \cdot d\mathbf{S} = \Phi \mathbf{n} \cdot (\text{grad } \Psi) dS$, but $\mathbf{n} \cdot (\text{grad } \Psi)$ is simply the directional derivative of Ψ in the direction of the unit outward normal $\mathbf{n}$ that will be denoted by $\partial \Psi / \partial n$, so

$$\Phi(\text{grad}\Psi) \cdot d\mathbf{S} = \Phi \, \partial \Psi / \partial n dS.$$

Using this in the last result gives Green's first formula,

$$\iint_S \Phi \frac{\partial \Psi}{\partial n} dS = \iiint_V \{\Phi \Delta \Psi + (\text{grad } \Phi) \cdot (\text{grad } \Psi)\} dV.$$

Green's second formula follows directly from this by interchanging Φ and Ψ and subtracting the new result from the Green's first formula. ■

showing the uniqueness of the solution of $\Delta\phi = 0$ in a volume, on the surface of which ϕ is specified

In anticipation of Chapter 18, and as an illustration of the use of Green's first formula in the study of partial differential equations, we will prove the **uniqueness** of the solution ϕ of Laplace's equation

$$\Delta \phi = 0$$

in a volume V enclosed within a surface S on which the value of ϕ is specified at every point. Here, the *Laplacian* Δ can be considered to be expressed in terms of any system of orthogonal curvilinear coordinates, the simplest of which is, of course, the cartesian coordinate system where

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

By the *uniqueness* of the solution of Laplace's equation, we mean that when ϕ is specified over the surface S enclosing a volume V, there is only *one* function ϕ that satisfies both Laplace's equation throughout V and the specified conditions for ϕ on the surface S. A typical physical example illustrating the interpretation of this situation is provided by considering the steady state temperature distribution $T(x, y, z)$ throughout a cube of metal where the temperature is governed by the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0.$$

It is to be expected from a physical understanding of steady state heat conduction that the specification of a time-independent temperature distribution T over each face of the cube of metal will determine the temperature at each internal point of the metal, and that every time the surfaces of the same metal block are heated in the same way, the same internal temperature distribution will result. This is simply another way of saying that the solution of Laplace's equation subject to specified boundary conditions on S is expected to be *unique*.

The proof of this result is simple. Suppose, if possible, that two different solutions ϕ_1 and ϕ_2 exist that satisfy the *same* prescribed temperature conditions on S. Then, because Laplace's equation is linear, the function $\Phi = \phi_1 - \phi_2$ must also be a solution and, furthermore, $\Phi \equiv 0$ on S. Using this function Φ in Green's first formula and setting $\Psi = \Phi$ reduces it to

$$\iiint_D (\text{grad } \Phi) \cdot (\text{grad } \Phi) dV = 0.$$

The integrand is nonnegative, so this result can only be possible if grad $\Phi \equiv \mathbf{0}$, and this in turn implies that $\partial\Phi/\partial x = \partial\Phi/\partial y = \partial\Phi/\partial z = 0$, and so $\Phi =$ constant. However, as $\Phi = 0$ on the bounding surface S, this shows that $\Phi = 0$ throughout D, and so $\phi_1 \equiv \phi_2$ and the result is proved.

The theory and application of the vector integral calculus are developed in standard calculus and analytic geometry texts like those in references [1.1], [1.2], [1.5], [1.6], and [1.7]. More advanced and detailed accounts, with emphasis placed on a vector treatment, are to be found in references [5.1] to [5.3]. Extensive use of vector integral theorems in the study of hydrodynamics is made in reference [6.5].

Summary

The three fundamental integral theorems of Gauss, Green, and Stokes were proved, and in anticipation of the results of Chapter 18, a Green formula was used to establish the uniqueness of the solution of the Laplace equation $\Delta\phi = 0$ in a volume on the surface of which ϕ is specified. It will be seen later in Chapter 18 that this is called a *Dirichlet problem* for the Laplace equation, and it arises in many physical situations, such as the steady state temperature distribution in a solid, the electrostatic potential in a vacuum enclosed in a cavity, in problems of groundwater flow, and elsewhere.

EXERCISES 12.2

1. By setting $\mathbf{F} = \mathbf{a} \times \mathbf{G}$ in the divergence theorem, where $\mathbf{a}$ is an arbitrary constant vector and $\mathbf{G}$ is a differentiable vector function defined in a volume V in a closed surface S, prove by using the properties of the scalar triple product that

$$\iint_S \mathbf{G} \times d\mathbf{S} = -\iiint_V \text{curl } \mathbf{G} dV.$$

2. Given a differentiable scalar function ϕ defined in a volume V contained in a closed surface S, prove that

$$\iiint_V (\text{grad } \phi) \times d\mathbf{S} \equiv \mathbf{0}.$$

3. Given the differentiable scalar and vector functions ϕ and $\mathbf{G}$, respectively, defined in a volume V in a closed surface S, prove that

$$\iint_S \phi\mathbf{G}\cdot d\mathbf{S} = \iiint_V (\text{grad}\,\phi)\cdot\mathbf{G}\,dV + \iiint_V \phi\,\text{div}\,\mathbf{G}\,dV.$$

4. Given the differentiable vector functions $\mathbf{P}$ and $\mathbf{Q}$ defined in a volume V bounded by a closed surface S, prove that

$$\iint_S \mathbf{P}\times\mathbf{Q}\cdot d\mathbf{S} = \iiint_V \mathbf{Q}\cdot\text{curl}\,\mathbf{P}\,dV - \iiint_V \mathbf{P}\cdot\text{curl}\,\mathbf{Q}\,dV.$$

5. The time-dependent heat equation can be written

$$\mu\rho\frac{\partial T}{\partial t} = \text{div}(\kappa\,\text{grad}\,T),$$

where μ, ρ, and κ are material constants that may vary with position, t is the time, and T the temperature at a position $\mathbf{r}$ in a material occupying a volume V enclosed in a surface S. Prove that

$$\iint_S \kappa T(\text{grad}\,T)\cdot d\mathbf{S} = \iiint_V \kappa(\text{grad}\,T)\cdot(\text{grad}\,T)\,dV + \iiint_V \mu\rho T\frac{\partial T}{\partial t}dV.$$

6. Given that $\mathbf{R} = \text{curl}\,\mathbf{Q}$ and $\mathbf{Q} = \text{curl}\,\mathbf{P}$ are defined in a volume V enclosed in a surface S, prove that

$$\iiint_V \mathbf{Q}\cdot\mathbf{Q}\,dV = \iint_S \mathbf{P}\times\mathbf{Q}\cdot d\mathbf{S} + \iiint_V \mathbf{P}\cdot\mathbf{R}\,dV.$$

7. By using Stokes' theorem and considering curl $(\phi\mathbf{F})$, where ϕ and $\mathbf{F}$ are differentiable scalar and vector functions, respectively, both of which are defined over an open surface S with closed boundary curve Γ, prove that

$$\int_\Gamma \phi\mathbf{F}\cdot d\mathbf{R} = \iint_S (\text{grad}\,\phi)\times\mathbf{F}\cdot d\mathbf{S} + \iint_S \phi\,\text{curl}\,\mathbf{F}\cdot d\mathbf{S}.$$

8. Given that ϕ and ψ are differentiable scalar functions defined over an open surface S with the closed boundary curve Γ, prove that

$$\int_\Gamma \phi(\text{grad}\,\Psi)\cdot d\mathbf{r} = \iint_S (\text{grad}\,\phi)\times(\text{grad}\,\psi)\cdot d\mathbf{S}.$$

9. Let $\mathbf{F} = -y^2\mathbf{i} + xz\mathbf{j} + z^2\mathbf{k}$ and S be the surface of the plane $x + y + 2z = 2$ lying in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with a clockwise sense of direction around its triangular boundary Γ. Verify Stokes' theorem by computing $\int_\Gamma \mathbf{F}\cdot d\mathbf{r}$ and $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ and showing they are equal.

10. Given that $\mathbf{F} = yz\mathbf{i} + xy\mathbf{j} + x^2\mathbf{k}$ and S is the surface of the plane $x + 3y + z = 3$ lying in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with a clockwise sense of direction around its triangular boundary Γ when seen from 0, verify Stokes' theorem by computing $\int_\Gamma \mathbf{F}\cdot d\mathbf{r}$ and $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ and showing they are equal.

12.3 Transport Theorems

In many applications the derivative with respect to time of surface and volume integrals is required where the integrand is a time-dependent field quantity and the surface or volume over which integration is to be performed moves with time. This situation arises, for example, when the rate of change of flux of a vector quantity $\mathbf{F}(\mathbf{r}, t)$ is required through an open surface $S(t)$ bounded by a moving closed space curve $\Gamma(t)$, or when the rate of change of a scalar quantity $f(\mathbf{r}, t)$ is required in a volume $V(t)$ that is enclosed in a moving surface $S(t)$. When computing the time derivative in the first case, it is necessary to take into account not only the time variation of the integrand, but also the effect of the moving boundary $\Gamma(t)$ of the surface $S(t)$ over which the time derivative of the flux is to be determined, whereas in the second case, in addition to the time dependence of $f(\mathbf{r}, t)$, the effect of the change in volume $V(t)$ must be considered.

Situations of this type occur when determining the generation of an electric current in a moving coil of wire in a magnetic field, in fluid mechanics when the energy content of a moving volume of fluid is considered and also in the study of shock waves, and in chemically reacting fluids where the chemical composition of a moving volume of fluid changes with time.

In this section two results called **transport theorems** will be derived. The first involves the rate of change of flux of a vector field across an open moving surface, whereas the second concerns the rate of change of a volume integral of a scalar quantity when the volume involved is swept out by a moving open surface.

The first result involves computing the time derivative of the flux $\Phi(t)$ of a vector function $\mathbf{F}(\mathbf{r}, t)$ through an open surface $S(t)$ bounded by a closed time-dependent space curve $\Gamma(t)$. When deriving this result it will be assumed that the points on $S(t)$ and $\Gamma(t)$ move with a specified velocity $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ that is defined throughout the region of space involved. The **flux** $\Phi(t)$ at time t is defined as the integral of the component of $\mathbf{F}(\mathbf{r}, t)$ normal to the surface $S(t)$, and so is given by

$$\Phi(t) = \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S}, \tag{4}$$

where $d\mathbf{S}$ is an element of area of $S(t)$.

THEOREM 12.5

a transport theorem for the rate of change of flux

The flux transport theorem Let a vector field $\mathbf{F}(\mathbf{r}, t)$ be defined and differentiable in some region of space in which the points on an open surface $S(t)$ with a closed boundary curve $\Gamma(t)$ move with a prescribed velocity $\mathbf{q}(\mathbf{r}, t)$. Then the rate of change of the flux $\Phi(t)$ of the vector field $\mathbf{F}(\mathbf{r}, t)$ through $S(t)$ is given by

$$\frac{d\Phi}{dt} = \iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + (\operatorname{div} \mathbf{F})\mathbf{q}\right] \cdot d\mathbf{S} + \int_{\Gamma(t)} \mathbf{F} \times \mathbf{q} \cdot d\mathbf{r}.$$

Proof Consider the surface $S(t)$ at time t and the surface $S(t+h)$ at a subsequent time $t+h$ shown in Fig. 12.12, where the points of $S(t)$ move with the given velocity $\mathbf{v}(\mathbf{r}, t)$. Then $S(t)$ sweeps out the cylindrical volume $V(t)$ shown in the diagram, where the line AB on the side surface of the cylinder shows the path followed by point A on $\Gamma(t)$ as it moves to the corresponding point B on $\Gamma(t+h)$. Correspondingly, a typical point P on $S(t)$ will move to the point Q on $S(t+h)$ along the line PQ, where for a small time increment h the vector $\underline{AB} \approx \mathbf{v}(\mathbf{r}_A, t)h$, and the vector $\underline{PQ} \approx \mathbf{v}(\mathbf{r}_P, t)h$, where $\mathbf{r}_A$ and $\mathbf{r}_P$ are the position vectors of A and P.

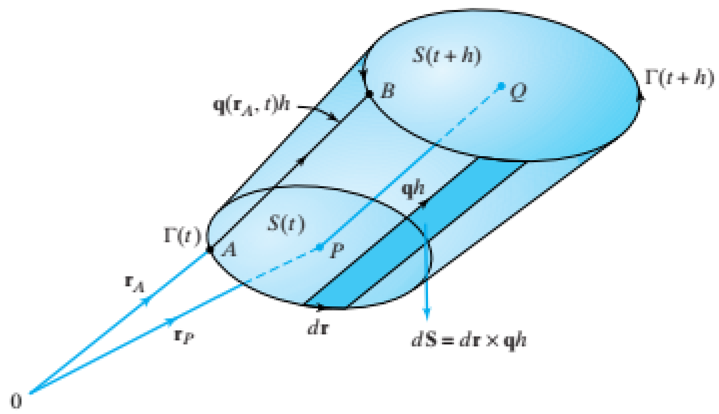


FIGURE 12.12 The surfaces S at times t and $t+h$ and the bounding curves $\Gamma(t)$ and $\Gamma(t+h)$.

It follows from the definition of a derivative that the time derivative of the flux $\Phi(t)$ is given by the limit

$$\frac{d\Phi}{dt} = \lim_{h\to 0}\left\{\frac{1}{h}\left[\iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t+h)\cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S}\right]\right\}. \tag{5}$$

In order to compute this limit, we first consider the difference

$$\iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t+h)\cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S},$$

and for small h use the Taylor approximation

$$\mathbf{F}(\mathbf{r}, t+h) \approx \mathbf{F}(\mathbf{r}, t) + h\frac{\partial \mathbf{F}}{\partial t}$$

to rewrite it as

$$\begin{aligned}&\iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t+h)\cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S}\\ &\quad\approx \iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S} + h\iint_{S(t)} \frac{\partial \mathbf{F}}{\partial t}\cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S}.\end{aligned} \tag{6}$$

To proceed further, if V is the volume swept out by $S(t)$ in time increment h, then the *outward*-drawn normal to V at $S(t+h)$ is $d\mathbf{S}$, while the *outward*-drawn normal to V at $S(t)$ is $-d\mathbf{S}$. Denoting the side of the cylindrical volume by Σ and applying the divergence theorem to $\mathbf{F}(\mathbf{r}, t)$ in V gives

$$\iiint_V \operatorname{div} \mathbf{F}(\mathbf{r}, t)dV = \iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S} + \iint_{\Sigma} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S}. \tag{7}$$

Using (7) to eliminate $\iint_{S(t)} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S}$ from (6) leads to the result

$$\begin{aligned}&\iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t+h)\cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S}\\ &\quad\approx h\iint_{S(t)} \frac{\partial \mathbf{F}}{\partial t}\cdot d\mathbf{S} + \iiint_V \operatorname{div} \mathbf{F}(\mathbf{r}, t)dV - \iint_{\Sigma} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S}.\end{aligned} \tag{8}$$

Now on the side Σ of the cylindrical surface the outward-drawn surface element $d\mathbf{S} = d\mathbf{r}\times\mathbf{q}h$, where $d\mathbf{r}$ is a vector element along $\Gamma(t)$ directed in the counterclockwise direction. The volume element dV swept out by $d\mathbf{S}$ in time increment h is the product of the area $|d\mathbf{S}|$ of $d\mathbf{S}$ and the perpendicular distance l between $S(t+h)$ and $S(t)$ given by $l = |\mathbf{q}h\cdot\mathbf{n}|$, where $\mathbf{n}$ is the unit normal to $d\mathbf{S}$, so that $dV = d\mathbf{S}\cdot\mathbf{q}h$. When these results are used to simplify (8) and h is small, it becomes

$$\begin{aligned}&\iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t+h)\cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{S}\\ &\quad\approx h\iint_{S(t)} \frac{\partial \mathbf{F}}{\partial t}\cdot d\mathbf{S} + h\iint_{S(t)} \operatorname{div} \mathbf{F}(\mathbf{r}, t)\mathbf{q}\cdot d\mathbf{S} + h\int_{\Gamma(t)} \mathbf{F}(\mathbf{r}, t)\times\mathbf{q}\cdot d\mathbf{r},\end{aligned} \tag{9}$$

where the sign of the last term has been changed by using the result $\mathbf{F}\cdot d\mathbf{r}\times\mathbf{q} = -\mathbf{F}\times\mathbf{q}\cdot d\mathbf{r}$.

Using (9) in the difference quotient (5) and proceeding to the limit as $h \to 0$ brings us to the statement of the theorem:

$$\frac{d\Phi}{dt} = \iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + (\text{div } \mathbf{F})\mathbf{q}\right] \cdot d\mathbf{S} + \int_{\Gamma(t)} \mathbf{F} \times \mathbf{q} \cdot d\mathbf{r}. \quad \blacksquare$$

EXAMPLE 12.8 Let $S(t)$ be a plane rectangular area with its corners at the points $(0, 0, z)$, $(x, 0, z)$, $(x, 1, z)$, and $(0, 1, z)$, where $x = vt$, $z = ut$, t is the time, and u and v are constant speeds. Verify the flux transport theorem in the case that $\mathbf{F} = xz\mathbf{k}$, where $\mathbf{k}$ is the unit vector in the z-direction.

Solution To verify Theorem 12.5 it will first be necessary to compute $\Phi(t)$ in order to find $d\Phi/dt$ directly. The theorem will be verified in this case if this expression for $d\Phi/dt$ can be shown to equal the sum of the surface and line integrals on the right of the statement of the theorem when each has been computed separately.

The geometry of the problem is shown in Fig. 12.13(a), and the projection of $S(t)$ onto the (x, y)-plane is shown in Fig. 12.13(b). It can be seen from the statement of the problem that the rectangular area remains parallel to the (x, y)-plane while moving along the z-axis with the constant speed u, and that its length increases with constant speed v in the positive x-direction.

We have $\mathbf{F} = xz\mathbf{k}$, $z = ut$, $x = vt$, so as the motion is uniform in the x- and z-directions, each point of $S(t)$ must move with the velocity $\mathbf{q} = v\mathbf{i} + u\mathbf{k}$. The flux $\Phi(t)$ is given by

$$\Phi(t) = \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S} = \int_0^1 \int_0^{vt} xz\mathbf{k} \cdot \mathbf{k}\, dx\, dy = \int_0^1 \int_0^{vt} xz\, dx\, dy.$$

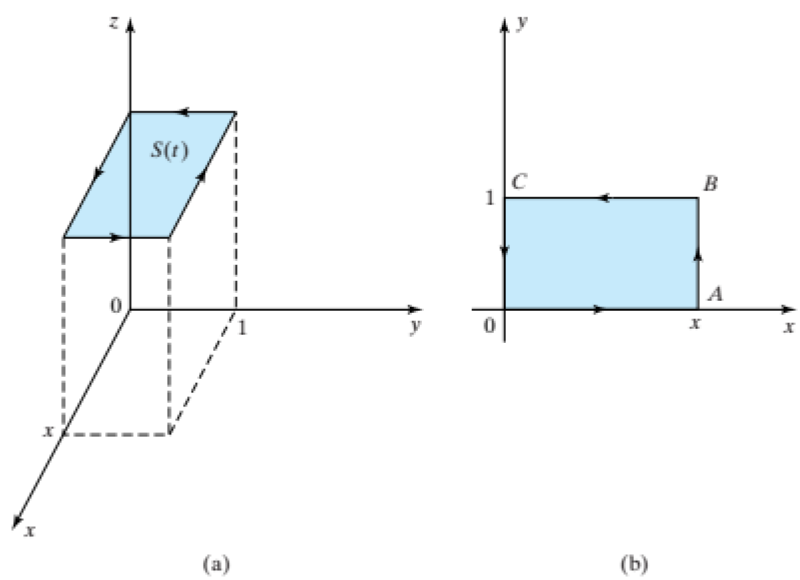


FIGURE 12.13 (a) The moving planar rectangle $S(t)$. (b) The projection of $S(t)$ onto the (x, y)-plane.

So as $z = ut$ is not involved in the integration, it can be removed as a factor to give

$$\Phi(t) = ut \int_0^1 \int_0^{vt} x dx dy = \frac{1}{2} uv^2 t^3,$$

so the rate of change of flux when computed directly is given by

$$\frac{d\Phi}{dt} = \frac{3}{2} uv^2 t^2.$$

Now $\partial \mathbf{F}/\partial t = \mathbf{0}$, div $\mathbf{F} = x$, and $d\mathbf{S} = dxdy\mathbf{k}$, so as

$$\left[\frac{\partial \mathbf{F}}{\partial \mathbf{t}} + (\text{div } \mathbf{F})\right] \mathbf{q} = xv\mathbf{i} + xu\mathbf{k},$$

$$\iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial \mathbf{t}} + \text{div } \mathbf{F}\right] \mathbf{q} \cdot d\mathbf{S} = \int_0^1 \int_0^{vt} (xv\mathbf{i} + xu\mathbf{k}) \cdot \mathbf{k} dx dy$$

$$= u \int_0^1 \int_0^{vt} x dx dy = \frac{1}{2} uv^2 t^2.$$

A simple calculation shows that $\mathbf{F} \times \mathbf{q} = xvz\mathbf{j}$, and so

$$\int_\Gamma \mathbf{F} \times \mathbf{q} \cdot d\mathbf{r} = \int_{\Gamma(t)} xvz\mathbf{j} \cdot d\mathbf{r} = uvt \int_{\Gamma(t)} x\mathbf{j} \cdot d\mathbf{r}.$$

Inspection of Fig. 12.13(b) shows that on OA, $d\mathbf{r} = dx\mathbf{i}$, on AB, $d\mathbf{r} = dy\mathbf{j}$, on BC, $d\mathbf{r} = -dx\mathbf{i}$, and on CO, $d\mathbf{r} = -dy\mathbf{j}$. The orthogonality of $\mathbf{i}$ and $\mathbf{j}$ means there are no contributions from the line integrals along OA and BC, and as $x = 0$ on OC there no contribution from the line integral along CO, so that

$$\int_{\Gamma(t)} \mathbf{F} \times \mathbf{q} \cdot d\mathbf{r} = uvtx \int_0^1 dy = uv^2 t^2.$$

We see from this that

$$\iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial \mathbf{t}} + (\text{div } \mathbf{F})\mathbf{q}\right] \cdot d\mathbf{S} + \int_{\Gamma(t)} \mathbf{F} \times \mathbf{q} \cdot d\mathbf{r} = \frac{1}{2} uv^2 t^2 + uv^2 t^2 = \frac{3}{2} uv^2 t^2.$$

This result equals the expression for $d\Phi/dt$ found previously by direct computation, so the theorem has been verified in this case. ■

a theorem determining the rate of change of an integral over a volume *V*(*t*) of a function of position and time when the surface bounding *V*(*t*) is moving

The second transport theorem concerns the rate of change of a volume integral of a differentiable scalar function $f(\mathbf{r}, t)$ when the volume $V(t)$ over which integration is performed is bounded by a closed moving surface $S(t)$, so for this reason it is called the **volume transport theorem**. Because of the importance of this theorem in fluid mechanics, where it was first derived by Reynolds, it is also known as the **Reynolds transport theorem**.

THEOREM 12.6

The Reynolds transport theorem Let the scalar function $f(\mathbf{r}, t)$ be defined and differentiable in a region of space $V(t)$ through which the points inside and on a closed surface $S(t)$ move with a prescribed velocity $\mathbf{q}(\mathbf{r}, t)$. Then

$$\frac{d}{dt} \iiint_{V(t)} f(\mathbf{r}, t) dV = \iiint_{V(t)} \frac{\partial f}{\partial t} dV + \iint_{S(t)} f(\mathbf{r}, t) \mathbf{q} \cdot d\mathbf{S}.$$

Osborne Reynolds (1842–1912)
An Irish scientist and engineer, born in Belfast into a clerical family and educated in his early years by his father. After a year spent in the workshop of the inventor and mechanical engineer Edward Hayes he studied mathematics at Cambridge University and graduated in 1867. Shortly afterwards he was appointed to the newly established Chair of Engineering in Manchester University where he remained until his death. He made many important contributions to mechanical engineering and to fluid mechanics, where he introduced the nondimensional quantity (number) now called the Reynolds' number that determines when a fluid flow is smooth or turbulent. During his lifetime he received many awards.

Proof For simplicity we only offer an intuitive derivation of the theorem. Let a scalar function $f(\mathbf{r}, t)$ be defined and differentiable throughout some region in which a volume $V(t)$ enclosed in a closed surface $S(t)$ moves, and let the points of $V(t)$ and $S(t)$ move with a prescribed velocity $\mathbf{q}(r, t)$. Then our objective will be to compute

$$\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV,$$

where dV is the volume element in $V(t)$. To accomplish this we start from the definition of a derivative in terms of a limit

$$\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV = \lim_{h\to 0}\frac{1}{h}\left[\iiint_{V(t+h)} f(\mathbf{r}, t+h)dV - \iiint_{V(t)} f(\mathbf{r}, t)dV\right], \tag{10}$$

and write $V(t+h) = V(t) + \Delta(t, h)$, where $\Delta(t, h)$ represents the change in volume $V(t)$ in the time increment h. As a result of this (10) becomes

$$\begin{aligned}
&\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV \\
&= \lim_{h\to 0}\frac{1}{h}\left[\iiint_{V(t)} f(\mathbf{r}, t+h)dV - \iiint_{V(t)} f(\mathbf{r}, t)dV + \iiint_{\Delta(t,h)} f(\mathbf{r}, t)dV\right] \\
&= \lim_{h\to 0}\iiint_{V(t)} \frac{1}{h}[f(\mathbf{r}, t+h) - f(\mathbf{r}, t)]dV + \lim_{h\to 0}\frac{1}{h}\left[\iiint_{\Delta(t,h)} f(\mathbf{r}, t+h)dV\right] \\
&= \iiint_{V(t)} \frac{\partial f(\mathbf{r}, t)}{\partial t} + \lim_{h\to 0}\frac{1}{h}\left[\iiint_{\Delta(t,h)} f(\mathbf{r}, t+h)dV\right].
\end{aligned} \tag{11}$$

The volume $\Delta(t, h)$ is the change in volume of $V(t)$ in the time increment h, but in this time a surface element $d\mathbf{S}$ of $S(t)$ is displaced by the vector $\mathbf{q}h$, so the corresponding volume element swept out by $d\mathbf{S}$ in $\Delta(t, h)$ in this time interval is $dV \approx h\mathbf{q}\cdot d\mathbf{S}$. Consequently, (11) becomes

$$\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV = \iiint_{V(t)} \frac{\partial f(\mathbf{r}, t)}{\partial t}dV + \lim_{h\to 0}\frac{1}{h}\left[\iint_{S(t)} hf(\mathbf{r}, t+h)\mathbf{q}\cdot d\mathbf{S}\right].$$

If we take the limit as $h \to 0$, when $f(\mathbf{r}, t+h) \to f(\mathbf{r}, t)$, this reduces to the statement of the theorem

$$\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV = \iiint_{V(t)} \frac{\partial f}{\partial t}dV + \iint_{S(t)} f(\mathbf{r}, t)\mathbf{q}\cdot d\mathbf{S}. \qquad \blacksquare$$

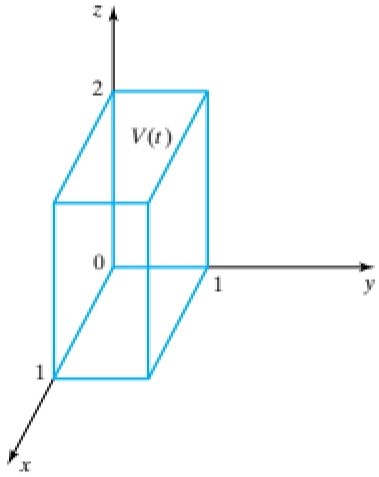


FIGURE 12.14 The rectangular parallelepiped with its top surface moving vertically with the constant speed u.

EXAMPLE 12.9 Verify the Reynolds transport theorem when $f = x^2yzt$ and the volume $V(t)$ is the rectangular parallelepiped with the corners of its base at the points $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(0, 1, 0)$, its sides normal to the (x, y)-plane, and the corners of its upper surface at the points $(0, 0, z)$, $(1, 0, z)$, $(1, 1, z)$, and $(0, 1, z)$ when $z = ut$, with t the time and u a constant speed.

Solution The geometry of the problem is shown in Fig. 12.14. To verify the Reynolds transport theorem, it is necessary first to compute the integral $\iiint_{V(t)} f(\mathbf{r}, t)dV$, and then to find its derivative with respect to time t. The theorem will be verified if this result can be shown to equal the sum of the two integrals on the right of the theorem when they are evaluated separately:

$$\iiint_{V(t)} f(\mathbf{r}, t)dV = \int_0^1 \int_0^1 \int_0^{ut} x^2yzt\,dzdydx = \frac{1}{3}\frac{1}{2}\frac{1}{2}u^2t^2t = \frac{1}{12}u^2t^3,$$

so

$$\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV = \frac{1}{4}u^2t^2.$$

We have

$$\iiint_{V(t)} \frac{\partial f}{\partial t}dV = \int_0^1 \int_0^1 \int_0^{ut} x^2yz\,dzdydx = \frac{1}{3}\frac{1}{2}\frac{1}{2}u^2t^2 = \frac{1}{12}u^2t^2,$$

and as $\mathbf{q} = u\mathbf{k}$ and $d\mathbf{S} = dxdy\mathbf{k}$,

$$\iint_{S(t)} f(\mathbf{r}, t)\mathbf{q}\cdot d\mathbf{S} = z\int_0^1 \int_0^1 x^2ytu\,dydx = \frac{1}{3}\frac{1}{2}u^2t^2 = \frac{1}{6}u^2t^2.$$

The theorem is verified, because $\frac{1}{12}u^2t^2 + \frac{1}{6}u^2t^2 = \frac{1}{4}u^2t^2$. ■

Summary

The flux transport theorem and the Reynolds' transport theorem, also known as the volume transport theorem, were proved and applied. Typical examples of the application of these theorems is the use of the first theorem to determine the rate of change of electric flux through a moving coil of wire in a generator, and the use of the second theorem when considering the continuity equation in fluid mechanics.

EXERCISES 12.3

1. Verify the rate of change of flux theorem given that $\mathbf{F} = xz\mathbf{k}$ and $S(t)$ is the plane rectangular surface with its corners at the points $(0, 0, z)$, $(x, 0, z)$, (x, y, z), and $(0, y, z)$, where $x = ut$, $y = vt$, and $z = wt$, with t the time and $u > 0$, $v > 0$, $w > 0$ a constant speed.
2. Verify the rate of change of flux theorem given that $\mathbf{F} = xz\mathbf{k}$ and $S(t)$ is the plane rectangular surface with its corners at the points $(0, 0, z)$, $(1, 0, z)$, $(1, y, z)$, and $(0, y, z)$, where $y = vt$ and $z = \alpha t^2$, with t the time and $v > 0$ a constant speed.
3.* A volume $V(t)$ in the form of a rectangular parallelepiped has the corners of its base at the points $(0, 0, z_1)$, $(1, 0, z_1)$, $(1, 1, z_1)$, and $(0, 1, z_1)$ with its sides perpendicular to the (x, y)-plane and the corners of its top surface at the points $(0, 0, z_2)$, $(1, 0, z_2)$, $(1, 1, z_2)$, and $(0, 1, z_2)$, where $z_1 = ut$ and $z_2 = vt$, with t the time and u, v constant speeds such that $u > 0$, $v > 0$. Verify the Reynolds transport theorem for the case in which $f(\mathbf{r}, t) = xyt$.
4.* A volume $V(t)$ in the form of a rectangular parallelepiped has the corners of its base at the points $(0, -\pi/2, 0)$, $(\pi, -\pi/2, 0)$, $(\pi, \pi/2, 0)$, and $(0, \pi/2, 0)$ with its sides perpendicular to the (x, y)-plane and the corners of its top surface at the points $(0, -\pi/2, z)$, $(\pi, -\pi/2, z)$, $(\pi, \pi/2, z)$, and $(0, \pi/2, z)$, where $z = ut$, with t the time and $u > 0$ a constant speed. Verify the Reynolds transport theorem for the case in which $f(\mathbf{r}, t) = \sin x \cos y e^z t^2$.
5.* A cylindrical volume $V(t)$ of height h has the center of its circular base located at the origin on the plane $z = 0$ and a radius $r = ut$, where t is the time and $u > 0$ is a constant speed. Verify the Reynolds transport theorem given that $f = r^2 t$.
6.* A hemispherical volume $V(t)$ lies in the region $z > 0$ with its center located at the origin in the plane $z = 0$ and a radius $r = ut$, where t is the time and $u > 0$ is a constant speed. Verify the Reynolds transport theorem given that $f = r^3 t$.

12.4 Fluid Mechanics Applications of Transport Theorems

When using the transport theorems, in fluid mechanics and elsewhere, two different types of time derivative occur, and for what is to follow it is important to distinguish between them. Consider a moving continuous medium, like a fluid, that has a property f associated with it, say its density, that depends on position $\mathbf{r}$ and the time t so that $f = f(\mathbf{r}, t)$. One way of finding the time derivative of f is to regard $\mathbf{r}$ as a fixed point, and then to find the time rate of change of f as seen by an observer fixed at point $\mathbf{r}$. This time derivative is denoted by $\partial f/\partial t$, and it is evaluated by differentiating f with respect to t while keeping $\mathbf{r}$ fixed. The other physically important time derivative of f involves letting the position vector $\mathbf{r}$ be a point that moves with the medium, so that $\mathbf{r} = \mathbf{r}(t)$, and then finding the time derivative of f at the moving point $\mathbf{r}$. This time derivative of f is denoted by df/dt, and in continuum mechanics it is called the **material derivative** of f, or sometimes the **convected derivative** of f, in which case it is often represented by Df/Dt.

To find the connection between the derivatives $\partial/\partial t$ and d/dt, when finding df/dt it is necessary to allow for the fact that the position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, so that $f = f(\mathbf{r}(t), t)$. Thus, allowing for the time variation in $\mathbf{r}(t)$, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \quad \text{or} \quad \frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{q}\cdot\nabla)f,$$

where $\mathbf{q} = (dx/dt)\mathbf{i} + (dy/dt)\mathbf{j} + (dz/dt)\mathbf{k}$ is the velocity of the moving point $\mathbf{r}(t)$. This shows that the material derivative operation can be written

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{q}\cdot\nabla). \tag{12}$$

Before proceeding further, notice that an application of the divergence theorem to the last term in Reynolds' transport theorem (Theorem 12.6) allows it to be written in the equivalent form

$$\frac{d}{dt}\iiint_{V(t)} f dV = \iiint_{V(t)} \left\{\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{q})\right\} dV, \tag{13}$$

but from Theorem 11.6 (iii) div $(f\mathbf{q}) = f(\nabla\cdot\mathbf{q}) + (\mathbf{q}\cdot\nabla)f$, so

$$\frac{d}{dt}\iiint_{V(t)} f dV = \iiint_{V(t)} \left\{\frac{\partial f}{\partial t} + (\mathbf{q}\cdot\nabla)f + f(\nabla\cdot\mathbf{q})\right\} dV.$$

Finally, if we use (12) this becomes

$$\frac{d}{dt}\iiint_{V(t)} f dV = \iiint_{V(t)} \left\{\frac{df}{dt} + f(\nabla\cdot\mathbf{q})\right\} dV. \tag{14}$$

Let us now use this result to derive the *equation of continuity* of fluid mechanics that describes the *conservation of mass* in any volume containing fluid in which fluid is not added (by a *source*) or removed (by a *sink*). To do this we assume that $V(t)$ is an arbitrary *material* volume in a fluid, so that $V(t)$ always contains the same fluid particles and the points on the surface $S(t)$ enclosing $V(t)$ move with the fluid. If we set $f = \rho$, where $\rho(\mathbf{r}, t)$ is the density of the fluid, the mass m of fluid in $V(t)$ is

$$m = \iiint_{V(t)} \rho(\mathbf{r}, t) dV.$$

As $V(t)$ is a material volume, provided it contains neither sources, nor sinks, the mass m must remain constant, from which it follows that $dm/dt = 0$.

Setting $f = \rho$ in (14), we find that

$$\frac{dm}{dt} = \iiint_{V(t)} \left\{\frac{d\rho}{dt} + \rho(\nabla\cdot\mathbf{q})\right\} dV = 0.$$

As $V(t)$ is arbitrary, this is only possible if the integrand is identically zero, so that

$$\frac{d\rho}{dt} + \rho(\nabla\cdot\mathbf{q}) = 0, \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{q}) = 0. \tag{15}$$

These are two equivalent forms of the **equation of continuity** of a fluid, which is of fundamental importance in the study of fluid dynamics.

If the fluid velocity is such that $\nabla \cdot \mathbf{q} = 0$ (div $\mathbf{q} = 0$), setting $f = 1$ in (14) reduces it to

$$\frac{d}{dt} \iiint_{V(t)} dV = \iiint_{V(t)} \nabla \cdot \mathbf{q} dV.$$

If div $\mathbf{q} = 0$, then $\rho_t + \rho \nabla \cdot \mathbf{q} = 0$ simplifies to $d\rho/dt = 0$. So, if initially $\rho_0 = \rho|_{t=0}$ is constant, ρ must remain constant throughout the flow even when the fluid is compressible. As $\iiint_{V(t)} dV = V$, where V is the volume of the fluid, it follows from $d/dt \iiint_{V(t)} dV = \iiint_{V(t)} \nabla \cdot \mathbf{q} dV$ that $dV/dt = 0$ when $\nabla \cdot \mathbf{q} = 0$. Consequently, in this case, the fluid motion will evolve without change of volume, even though the fluid may be compressible. In fluid mechanics, a flow of a compressible fluid that takes place without a change of volume is called **isochoric** flow. Naturally this last result is true when the fluid is incompressible, because then the density ρ is an absolute constant.

Next we derive a generalization of Theorem 12.6 that allows the function $f(\mathbf{r}, t)$ to be discontinuous across some surface Σ in $V(t)$ that moves with an arbitrary velocity $\mathbf{u}$, with $f = f_1(\mathbf{r}, t)$ on one side of Σ and $f = f_2(\mathbf{r}, t)$ on the other side. Particular cases of this result are needed when a physical quantity of interest experiences a discontinuous change across a surface, as can happen, for example, in chemical engineering and fluid mechanics.

The situation is illustrated in Fig. 12.15, where a material volume $V(t)$ with bounding surface $S(t)$ is shown divided into two parts $V_1(t)$ and $V_2(t)$ by a surface Σ that moves with an arbitrary velocity $\mathbf{u}$. The volume $V_1(t)$ is bounded by the surface $S_1(t)$ that is part of $S(t)$ and Σ, where the unit normal $\mathbf{n}_1$ to Σ directed out of $V_1(t)$ is $\mathbf{n}_1 = \boldsymbol{\nu}$. Similarly, volume $V_2(t)$ is bounded by the surface $S_2(t)$ that is part of $S(t)$ and Σ, where the unit normal $\mathbf{n}_2$ to Σ directed out of $V_2(t)$ is in the opposite sense to that of $\mathbf{n}_1$ so that $\mathbf{n}_2 = -\boldsymbol{\nu}$.

Applying Theorem 12.6 to volume $V_1(t)$ gives

$$\frac{d}{dt} \iiint_{V_1(t)} f_1 dV = \iiint_{V_1(t)} \frac{\partial f_1}{\partial t} dV + \iint_{S_1(t)} f_1 \mathbf{q} \cdot d\mathbf{S} + \iint_{\Sigma(t)} f_1 \mathbf{u} \cdot \mathbf{n}_1 dS,$$

and an application of Theorem 12.6 to the volume $V_2(t)$ gives

$$\frac{d}{dt} \iiint_{V_2(t)} f_2 dV = \iiint_{V_2(t)} \frac{\partial f_2}{\partial t} dV + \iint_{S_2(t)} f_2 \mathbf{q} \cdot d\mathbf{S} + \iint_{\Sigma(t)} f_2 \mathbf{u} \cdot \mathbf{n}_2 dS.$$

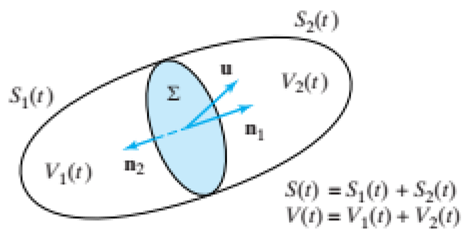


FIGURE 12.15 The material volume $V(t)$ and the surface Σ across which f is discontinuous.

Adding these two results and using the fact that $\mathbf{n}_1 = \boldsymbol{\nu}$ and $\boldsymbol{n}_2 = -\boldsymbol{\nu}$, we obtain

$$\frac{d}{dt}\iiint_{V(t)} f dV = \iiint_{V(t)} \frac{\partial f}{\partial t} dV + \iint_{S(t)} f\mathbf{q}\cdot d\mathbf{S} + \iint_{\Sigma(t)} (f_1 - f_2)\mathbf{u}\cdot d\mathbf{S}, \qquad (16)$$

which is the required generalization.

Examination of the last term in (16) shows, as would be expected, that the contribution made by the jump discontinuity $f_1 - f_2$ across the surface Σ that moves with velocity $\mathbf{u}$ depends only on the component of $\mathbf{u}$ normal to Σ, so if $\mathbf{u}$ is tangential to Σ, this term will vanish.

An extension of these ideas to allow for discontinuous solutions f in a volume $V(t)$ when f satisfies an equation of the form

$$\frac{\partial f}{\partial t} + \operatorname{div} \mathbf{h}(f) = 0,$$

called a **conservation equation**, is to be found in Chapter 18, Section 18.4, where conservation equations and shock solutions are considered. It should be noticed that an equation of this type has already been encountered in (15) when deriving the continuity equation for a fluid (the *conservation of mass equation*) in the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{q}) = 0.$$

This is a *partial differential equation*, because it is an equation relating partial derivatives of the dependent variables ρ and $\mathbf{q}$.

Let Γ be a closed curve in a fluid flow with velocity vector $\mathbf{q}$ for which $\operatorname{div}\mathbf{q} = 0$ (an *isochoric flow*), and let S be any smooth surface with boundary Γ. Then the streamlines passing through Γ define a stream tube in the fluid flow. The integral

$$\Phi = \iint_S \mathbf{q}\cdot d\mathbf{S} \qquad (17)$$

is called the **strength** of the stream tube, and it measures the flow rate through the tube. As a final application of an integral theorem, we will prove that the strength of the flow in a tube bounded by streamlines (a stream tube) remains constant along its length.

First we rewrite Theorem 12.5, which was proved in the form

$$\frac{d}{dt}\iint_{S(t)} \mathbf{F}(\mathbf{r},t)\cdot d\mathbf{S} = \iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + (\nabla\cdot\mathbf{F})\mathbf{q}\right]\cdot d\mathbf{S} + \int_{\Gamma(t)} \mathbf{F}\times\mathbf{q}\cdot d\mathbf{r}.$$

If we apply Stokes' theorem to the last integral, this becomes

$$\frac{d}{dt}\iint_{S(t)} \mathbf{F}(\mathbf{r},t)\cdot d\mathbf{S} = \iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + (\nabla\cdot\mathbf{F})\mathbf{q} + \nabla\times(\mathbf{F}\times\mathbf{q})\right]\cdot d\mathbf{S}. \qquad (18)$$

Replacing $\mathbf{F}$ by $\mathbf{q}$, we have

$$\frac{d}{dt}\iint_{S(t)} \mathbf{q}\cdot d\mathbf{S} = \iint_{S(t)} \left[\frac{\partial \mathbf{q}}{\partial t} + (\nabla\cdot\mathbf{q})\mathbf{q} + \nabla\times(\mathbf{q}\times\mathbf{q})\right]\cdot d\mathbf{S},$$

but $\mathbf{q}\times\mathbf{q} = \mathbf{0}$, and as the flow is isochoric, $(\nabla\cdot\mathbf{q}) = 0$, this result reduces to

$$\frac{d}{dt}\iint_{S(t)} \mathbf{q}\cdot d\mathbf{S} = \iint_{S(t)} \frac{\partial \mathbf{q}}{\partial t}\cdot d\mathbf{S}.$$

An application of the divergence theorem to the integral on the right, where the closed surface $V(t)$ is formed by $S(t)$, $S(t+dt)$ and streamlines through Γ, gives

$$\frac{d}{dt}\iint_{S(t)} \mathbf{q}\cdot d\mathbf{S} = \iiint_{V(t)} \nabla\cdot(\partial\mathbf{q}/\partial t)dV = \iiint_{V(t)} \partial/\partial t(\nabla\cdot\mathbf{q})dV = 0,$$

showing that the strength $\Phi = \iint_S \mathbf{q}\cdot d\mathbf{S}$ remains constant along a stream tube.

Summary

The applications considered in this section were to fluid mechanics, and they made use of the so-called material, or convected, derivative of a function f of both position and time. The determination of this derivative was seen to involve letting a position vector move with the fluid and then finding the time derivative of f at the moving point. One result obtained by means of the transport theorems was the equation of continuity of fluid mechanics. Another result used the notion of a conservation equation to establish the invariance of the flow rate (strength) in a stream tube, the walls of which are bounded by streamlines.

EXERCISES 12.4

1. Prove the **Euler expansion formula**

$$\frac{d}{dt}\iiint_{V(t)} dV = \iint_{S(t)} \mathbf{q}\cdot d\mathbf{S}.$$

2. Show that the flux transport theorem given in (18) can also be written as

$$\frac{d}{dt}\iint_{S(t)} \mathbf{F}(\mathbf{r},t)\cdot d\mathbf{S}$$
$$= \iint_{S(t)} \left[\frac{d\mathbf{F}}{dt} + (\nabla\cdot\mathbf{q})\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{q}\right]\cdot d\mathbf{S}.$$

3.* Show that if

$$\frac{\partial\mathbf{F}}{\partial t} + (\nabla\cdot\mathbf{F})\mathbf{q} + \nabla\times(\mathbf{F}\times\mathbf{q}) = \mathbf{0},$$

the strength of flow through any stream tube remains constant along its length.