PUZZLER

More than 300 years ago, Isaac Newton realized that the same gravitational force that causes apples to fall to the Earth also holds the Moon in its orbit. In recent years, scientists have used the Hubble Space Telescope to collect evidence of the gravitational force acting even farther away, such as at this protoplanetary disk in the constellation Taurus. What properties of an object such as a protoplanet or the Moon determine the strength of its gravitational attraction to another object? *(Left, Larry West/FPG International; right, Courtesy of NASA)*

web

For more information about the Hubble, visit the Space Telescope Science Institute at **http://www.stsci.edu/**

chapter 14

The Law of Gravity

Chapter Outline

Before 1687, a large amount of data had been collected on the motions of the Moon and the planets, but a clear understanding of the forces causing these motions was not available. In that year, Isaac Newton provided the key that unlocked the secrets of the heavens. He knew, from his first law, that a net force had to be acting on the Moon because without such a force the Moon would move in a straight-line path rather than in its almost circular orbit. Newton reasoned that this force was the gravitational attraction exerted by the Earth on the Moon. He realized that the forces involved in the Earth–Moon attraction and in the Sun–planet attraction were not something special to those systems, but rather were particular cases of a general and universal attraction between objects. In other words, Newton saw that the same force of attraction that causes the Moon to follow its path around the Earth also causes an apple to fall from a tree. As he put it, "I deduced that the forces which keep the planets in their orbs must be reciprocally as the squares of their distances from the centers about which they revolve; and thereby compared the force requisite to keep the Moon in her orb with the force of gravity at the surface of the Earth; and found them answer pretty nearly."

In this chapter we study the law of gravity. We place emphasis on describing the motion of the planets because astronomical data provide an important test of the validity of the law of gravity. We show that the laws of planetary motion developed by Johannes Kepler follow from the law of gravity and the concept of conservation of angular momentum. We then derive a general expression for gravitational potential energy and examine the energetics of planetary and satellite motion. We close by showing how the law of gravity is also used to determine the force between a particle and an extended object.

14.1 NEWTON'S LAW OF UNIVERSAL GRAVITATION

You may have heard the legend that Newton was struck on the head by a falling apple while napping under a tree. This alleged accident supposedly prompted him to imagine that perhaps all bodies in the Universe were attracted to each other in the same way the apple was attracted to the Earth. Newton analyzed astronomical data on the motion of the Moon around the Earth. From that analysis, he made the bold assertion that the force law governing the motion of planets was the *same* as the force law that attracted a falling apple to the Earth. This was the first time that "earthly" and "heavenly" motions were unified. We shall look at the mathematical details of Newton's analysis in Section 14.5.

In 1687 Newton published his work on the law of gravity in his treatise *Mathematical Principles of Natural Philosophy*. **Newton's law of universal gravitation** states that

every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

If the particles have masses m_1 and m_2 and are separated by a distance r, the magnitude of this gravitational force is

The law of gravity

$$F_g = G\frac{m_1 m_2}{r^2} \qquad (14.1)$$

where G is a constant, called the *universal gravitational constant,* that has been measured experimentally. As noted in Example 6.6, its value in SI units is

$$G = 6.673 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2 \quad \textbf{(14.2)}$$

The form of the force law given by Equation 14.1 is often referred to as an **inverse-square law** because the magnitude of the force varies as the inverse square of the separation of the particles.[1] We shall see other examples of this type of force law in subsequent chapters. We can express this force in vector form by defining a unit vector $\hat{\mathbf{r}}_{12}$ (Fig. 14.1). Because this unit vector is directed from particle 1 to particle 2, the force exerted by particle 1 on particle 2 is

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}_{12} \quad \textbf{(14.3)}$$

where the minus sign indicates that particle 2 is attracted to particle 1, and hence the force must be directed toward particle 1. By Newton's third law, the force exerted by particle 2 on particle 1, designated $\mathbf{F}_{21}$, is equal in magnitude to $\mathbf{F}_{12}$ and in the opposite direction. That is, these forces form an action–reaction pair, and $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

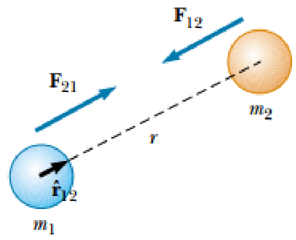


Figure 14.1 The gravitational force between two particles is attractive. The unit vector $\hat{\mathbf{r}}_{12}$ is directed from particle 1 to particle 2. Note that $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

Properties of the gravitational force

Several features of Equation 14.3 deserve mention. The gravitational force is a field force that always exists between two particles, regardless of the medium that separates them. Because the force varies as the inverse square of the distance between the particles, it decreases rapidly with increasing separation. We can relate this fact to the geometry of the situation by noting that the intensity of light emanating from a point source drops off in the same $1/r^2$ manner, as shown in Figure 14.2.

Another important point about Equation 14.3 is that **the gravitational force exerted by a finite-size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of the distribution were concentrated at the center.** For example, the force exerted by the

QuickLab

Inflate a balloon just enough to form a small sphere. Measure its diameter. Use a marker to color in a 1-cm square on its surface. Now continue inflating the balloon until it reaches twice the original diameter. Measure the size of the square you have drawn. Also note how the color of the marked area has changed. Have you verified what is shown in Figure 14.2?

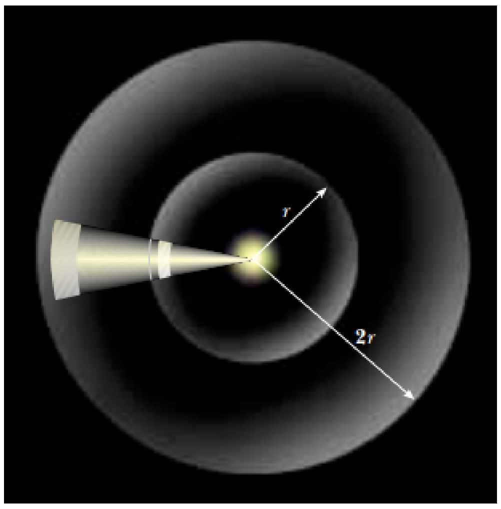


Figure 14.2 Light radiating from a point source drops off as $1/r^2$, a relationship that matches the way the gravitational force depends on distance. When the distance from the light source is doubled, the light has to cover four times the area and thus is one fourth as bright.

[1] An inverse relationship between two quantities x and y is one in which $y = k/x$, where k is a constant. A direct proportion between x and y exists when $y = kx$.

Earth on a particle of mass m near the Earth's surface has the magnitude

$$F_g = G\frac{M_E m}{R_E^2} \tag{14.4}$$

where M_E is the Earth's mass and R_E its radius. This force is directed toward the center of the Earth.

We have evidence of the fact that the gravitational force acting on an object is directly proportional to its mass from our observations of falling objects, discussed in Chapter 2. All objects, regardless of mass, fall in the absence of air resistance at the same acceleration g near the surface of the Earth. According to Newton's second law, this acceleration is given by $g = F_g/m$, where m is the mass of the falling object. If this ratio is to be the same for all falling objects, then F_g must be directly proportional to m, so that the mass cancels in the ratio. If we consider the more general situation of a gravitational force between any two objects with mass, such as two planets, this same argument can be applied to show that the gravitational force is proportional to one of the masses. We can choose *either* of the masses in the argument, however; thus, the gravitational force must be directly proportional to *both* masses, as can be seen in Equation 14.3.

14.2 MEASURING THE GRAVITATIONAL CONSTANT

The universal gravitational constant G was measured in an important experiment by Henry Cavendish (1731–1810) in 1798. The Cavendish apparatus consists of two small spheres, each of mass m, fixed to the ends of a light horizontal rod suspended by a fine fiber or thin metal wire, as illustrated in Figure 14.3. When two large spheres, each of mass M, are placed near the smaller ones, the attractive force between smaller and larger spheres causes the rod to rotate and twist the wire suspension to a new equilibrium orientation. The angle of rotation is measured by the deflection of a light beam reflected from a mirror attached to the vertical suspension. The deflection of the light is an effective technique for amplifying the motion. The experiment is carefully repeated with different masses at various separations. In addition to providing a value for G, the results show experimentally that the force is attractive, proportional to the product mM, and inversely proportional to the square of the distance r.

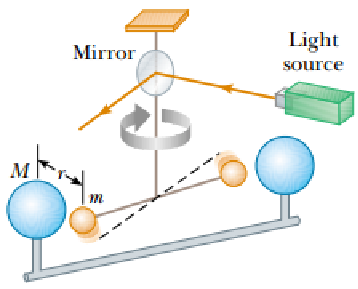


Figure 14.3 Schematic diagram of the Cavendish apparatus for measuring G. As the small spheres of mass m are attracted to the large spheres of mass M, the rod between the two small spheres rotates through a small angle. A light beam reflected from a mirror on the rotating apparatus measures the angle of rotation. The dashed line represents the original position of the rod.

EXAMPLE 14.1 Billiards, Anyone?

Three 0.300-kg billiard balls are placed on a table at the corners of a right triangle, as shown in Figure 14.4. Calculate the gravitational force on the cue ball (designated m_1) resulting from the other two balls.

Solution First we calculate separately the individual forces on the cue ball due to the other two balls, and then we find the vector sum to get the resultant force. We can see graphically that this force should point upward and toward the right. We locate our coordinate axes as shown in Figure 14.4, placing our origin at the position of the cue ball.

The force exerted by m_2 on the cue ball is directed upward and is given by

$$\mathbf{F}_{21} = G\frac{m_2 m_1}{{r_{21}}^2}\mathbf{j}$$

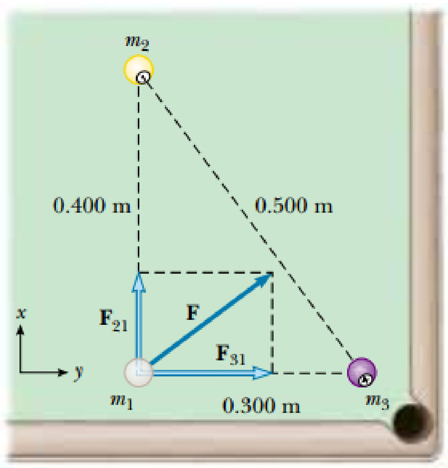


Figure 14.4 The resultant gravitational force acting on the cue ball is the vector sum $\mathbf{F}_{21} + \mathbf{F}_{31}$.

$$= \left(6.67 \times 10^{-11}\,\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\frac{(0.300\text{ kg})(0.300\text{ kg})}{(0.400\text{ m})^2}\mathbf{j}$$

$$= 3.75 \times 10^{-11}\mathbf{j}\text{ N}$$

This result shows that the gravitational forces between everyday objects have extremely small magnitudes. The force exerted by m_3 on the cue ball is directed to the right:

$$\mathbf{F}_{31} = G\frac{m_3 m_1}{{r_{31}}^2}\mathbf{i}$$

$$= \left(6.67 \times 10^{-11}\,\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\frac{(0.300\text{ kg})(0.300\text{ kg})}{(0.300\text{ m})^2}\mathbf{i}$$

$$= 6.67 \times 10^{-11}\,\mathbf{i}\text{ N}$$

Therefore, the resultant force on the cue ball is

$$\mathbf{F} = \mathbf{F}_{21} + \mathbf{F}_{31} = \boxed{(3.75\mathbf{j} + 6.67\mathbf{i}) \times 10^{-11}\text{ N}}$$

and the magnitude of this force is

$$F = \sqrt{{F_{21}}^2 + {F_{31}}^2} = \sqrt{(3.75)^2 + (6.67)^2} \times 10^{-11}$$
$$= 7.65 \times 10^{-11}\text{ N}$$

Exercise Find the direction of **F**.

Answer 29.3° counterclockwise from the positive x axis.

14.3 FREE-FALL ACCELERATION AND THE GRAVITATIONAL FORCE

In Chapter 5, when defining mg as the weight of an object of mass m, we referred to g as the magnitude of the free-fall acceleration. Now we are in a position to obtain a more fundamental description of g. Because the force acting on a freely falling object of mass m near the Earth's surface is given by Equation 14.4, we can equate mg to this force to obtain

$$mg = G\frac{M_E m}{{R_E}^2}$$

$$g = G\frac{M_E}{{R_E}^2} \qquad \textbf{(14.5)}$$

Free-fall acceleration near the Earth's surface

Now consider an object of mass m located a distance h above the Earth's surface or a distance r from the Earth's center, where $r = R_E + h$. The magnitude of the gravitational force acting on this object is

$$F_g = G\frac{M_E m}{r^2} = G\frac{M_E m}{(R_E + h)^2}$$

The gravitational force acting on the object at this position is also $F_g = mg'$, where g' is the value of the free-fall acceleration at the altitude h. Substituting this expression for F_g into the last equation shows that g' is

Variation of g with altitude

$$g' = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2} \tag{14.6}$$

Thus, it follows that g' *decreases* with *increasing altitude*. Because the weight of a body is mg', we see that as $r \rightarrow \infty$, its weight approaches zero.

EXAMPLE 14.2 Variation of g with Altitude h

The International Space Station is designed to operate at an altitude of 350 km. When completed, it will have a weight (measured at the Earth's surface) of 4.22×10^6 N. What is its weight when in orbit?

Solution Because the station is above the surface of the Earth, we expect its weight in orbit to be less than its weight on Earth, 4.22×10^6 N. Using Equation 14.6 with $h = 350$ km, we obtain

$$\begin{aligned} g' &= \frac{GM_E}{(R_E + h)^2} \\ &= \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}{(6.37 \times 10^6\ \text{m} + 0.350 \times 10^6\ \text{m})^2} \\ &= 8.83\ \text{m/s}^2 \end{aligned}$$

Because $g'/g = 8.83/9.80 = 0.901$, we conclude that the weight of the station at an altitude of 350 km is 90.1% of the value at the Earth's surface. So the station's weight in orbit is

$$(0.901)(4.22 \times 10^6\ \text{N}) = \boxed{3.80 \times 10^6\ \text{N}}$$

Values of g' at other altitudes are listed in Table 14.1.

TABLE 14.1 Free-Fall Acceleration g' at Various Altitudes Above the Earth's Surface

Altitude h (km)	g' (m/s^2)
1 000	7.33
2 000	5.68
3 000	4.53
4 000	3.70
5 000	3.08
6 000	2.60
7 000	2.23
8 000	1.93
9 000	1.69
10 000	1.49
50 000	0.13
∞	0

web The official web site for the International Space Station is **www.station.nasa.gov**

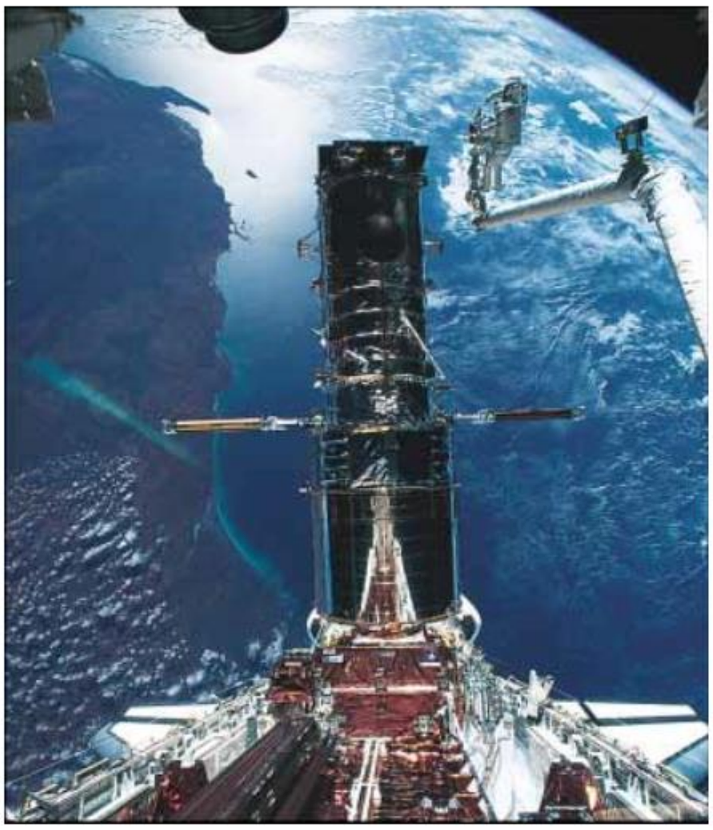
EXAMPLE 14.3 The Density of the Earth

Using the fact that $g = 9.80\ \text{m/s}^2$ at the Earth's surface, find the average density of the Earth.

Solution Using $g = 9.80\ \text{m/s}^2$ and $R_E = 6.37 \times 10^6$ m, we find from Equation 14.5 that $M_E = 5.96 \times 10^{24}$ kg. From this result, and using the definition of density from Chapter 1, we obtain

$$\begin{aligned} \rho_E &= \frac{M_E}{V_E} = \frac{M_E}{\frac{4}{3}\pi R_E^{\,3}} = \frac{5.96 \times 10^{24}\ \text{kg}}{\frac{4}{3}\pi(6.37 \times 10^6\ \text{m})^3} \\ &= \boxed{5.50 \times 10^3\ \text{kg/m}^3} \end{aligned}$$

Because this value is about twice the density of most rocks at the Earth's surface, we conclude that the inner core of the Earth has a density much higher than the average value. It is most amazing that the Cavendish experiment, which determines G (and can be done on a tabletop), combined with simple free-fall measurements of g, provides information about the core of the Earth.



Astronauts F. Story Musgrave and Jeffrey A. Hoffman, along with the Hubble Space Telescope and the space shuttle *Endeavor,* are all falling around the Earth.

14.4 KEPLER'S LAWS

People have observed the movements of the planets, stars, and other celestial bodies for thousands of years. In early history, scientists regarded the Earth as the center of the Universe. This so-called geocentric model was elaborated and formalized by the Greek astronomer Claudius Ptolemy (c. 100–c. 170) in the second century A.D. and was accepted for the next 1 400 years. In 1543 the Polish astronomer Nicolaus Copernicus (1473–1543) suggested that the Earth and the other planets revolved in circular orbits around the Sun (the heliocentric model).

The Danish astronomer Tycho Brahe (1546–1601) wanted to determine how the heavens were constructed, and thus he developed a program to determine the positions of both stars and planets. It is interesting to note that those observations of the planets and 777 stars visible to the naked eye were carried out with only a large sextant and a compass. (The telescope had not yet been invented.)

The German astronomer Johannes Kepler was Brahe's assistant for a short while before Brahe's death, whereupon he acquired his mentor's astronomical data and spent 16 years trying to deduce a mathematical model for the motion of the planets. Such data are difficult to sort out because the Earth is also in motion around the Sun. After many laborious calculations, Kepler found that Brahe's data on the revolution of Mars around the Sun provided the answer.

Johannes Kepler **German astronomer (1571–1630)** The German astronomer Johannes Kepler is best known for developing the laws of planetary motion based on the careful observations of Tycho Brahe. *(Art Resource)*

For more information about Johannes Kepler, visit our Web site at **www.saunderscollege.com/physics/**

Kepler's analysis first showed that the concept of circular orbits around the Sun had to be abandoned. He eventually discovered that the orbit of Mars could be accurately described by an **ellipse.** Figure 14.5 shows the geometric description of an ellipse. The longest dimension is called the major axis and is of length $2a$, where a is the **semimajor axis.** The shortest dimension is the minor axis, of length $2b$, where b is the **semiminor axis.** On either side of the center is a **focal point,** a distance c from the center, where $a^2 = b^2 + c^2$. The Sun is located at one of the focal points of Mars's orbit. Kepler generalized his analysis to include the motions of all planets. The complete analysis is summarized in three statements known as **Kepler's laws:**

Kepler's laws

1. All planets move in elliptical orbits with the Sun at one focal point.
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

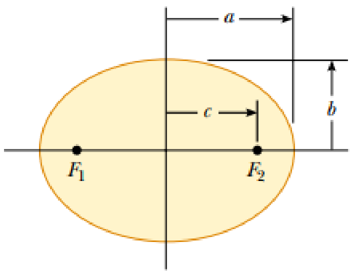


Figure 14.5 Plot of an ellipse. The semimajor axis has a length a, and the semiminor axis has a length b. The focal points are located at a distance c from the center, where $a^2 = b^2 + c^2$.

Most of the planetary orbits are close to circular in shape; for example, the semimajor and semiminor axes of the orbit of Mars differ by only 0.4%. Mercury and Pluto have the most elliptical orbits of the nine planets. In addition to the planets, there are many asteroids and comets orbiting the Sun that obey Kepler's laws. Comet Halley is such an object; it becomes visible when it is close to the Sun every 76 years. Its orbit is very elliptical, with a semiminor axis 76% smaller than its semimajor axis.

Although we do not prove it here, Kepler's first law is a direct consequence of the fact that the gravitational force varies as $1/r^2$. That is, under an inverse-square gravitational-force law, the orbit of a planet can be shown mathematically to be an ellipse with the Sun at one focal point. Indeed, half a century after Kepler developed his laws, Newton demonstrated that these laws are a consequence of the gravitational force that exists between any two masses. Newton's law of universal gravitation, together with his development of the laws of motion, provides the basis for a full mathematical solution to the motion of planets and satellites.

14.5 THE LAW OF GRAVITY AND THE MOTION OF PLANETS

In formulating his law of gravity, Newton used the following reasoning, which supports the assumption that the gravitational force is proportional to the inverse square of the separation between the two interacting bodies. He compared the acceleration of the Moon in its orbit with the acceleration of an object falling near the Earth's surface, such as the legendary apple (Fig. 14.6). Assuming that both accelerations had the same cause—namely, the gravitational attraction of the Earth—Newton used the inverse-square law to reason that the acceleration of the Moon toward the Earth (centripetal acceleration) should be proportional to $1/r_M^2$, where r_M is the distance between the centers of the Earth and the Moon. Furthermore, the acceleration of the apple toward the Earth should be proportional to $1/R_E^2$, where R_E is the radius of the Earth, or the distance between the centers of the Earth and the apple. Using the values $r_M = 3.84 \times 10^8$ m and

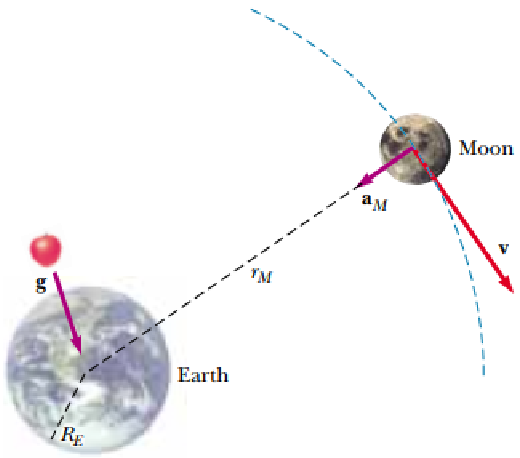


Figure 14.6 As it revolves around the Earth, the Moon experiences a centripetal acceleration $\mathbf{a}_M$ directed toward the Earth. An object near the Earth's surface, such as the apple shown here, experiences an acceleration $\mathbf{g}$. (Dimensions are not to scale.)

$R_E = 6.37 \times 10^6$ m, Newton predicted that the ratio of the Moon's acceleration a_M to the apple's acceleration g would be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37 \times 10^6 \text{ m}}{3.84 \times 10^8 \text{ m}}\right)^2 = 2.75 \times 10^{-4}$$

Therefore, the centripetal acceleration of the Moon is

Acceleration of the Moon

$$a_M = (2.75 \times 10^{-4})(9.80 \text{ m/s}^2) = 2.70 \times 10^{-3} \text{ m/s}^2$$

Newton also calculated the centripetal acceleration of the Moon from a knowledge of its mean distance from the Earth and its orbital period, $T = 27.32$ days $= 2.36 \times 10^6$ s. In a time T, the Moon travels a distance $2\pi r_M$, which equals the circumference of its orbit. Therefore, its orbital speed is $2\pi r_M/T$ and its centripetal acceleration is

$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M/T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2(3.84 \times 10^8 \text{ m})}{(2.36 \times 10^6 \text{ s})^2}$$

$$= 2.72 \times 10^{-3} \text{ m/s}^2 \approx \frac{9.80 \text{ m/s}^2}{60^2}$$

In other words, because the Moon is roughly 60 Earth radii away, the gravitational acceleration at that distance should be about $1/60^2$ of its value at the Earth's surface. This is just the acceleration needed to account for the circular motion of the Moon around the Earth. The nearly perfect agreement between this value and the value Newton obtained using g provides strong evidence of the inverse-square nature of the gravitational force law.

Although these results must have been very encouraging to Newton, he was deeply troubled by an assumption he made in the analysis. To evaluate the acceleration of an object at the Earth's surface, Newton treated the Earth as if its mass were all concentrated at its center. That is, he assumed that the Earth acted as a particle as far as its influence on an exterior object was concerned. Several years later, in 1687, on the basis of his pioneering work in the development of calculus, Newton proved that this assumption was valid and was a natural consequence of the law of universal gravitation.

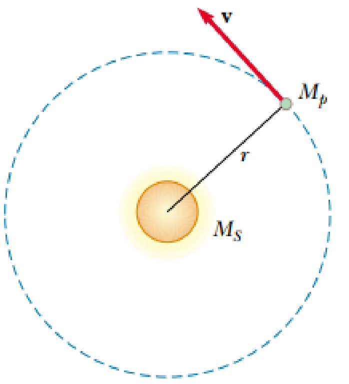


Figure 14.7 A planet of mass M_p moving in a circular orbit around the Sun. The orbits of all planets except Mercury and Pluto are nearly circular.

Kepler's third law

Kepler's Third Law

It is informative to show that Kepler's third law can be predicted from the inverse-square law for circular orbits.[2] Consider a planet of mass M_p moving around the Sun of mass M_S in a circular orbit, as shown in Figure 14.7. Because the gravitational force exerted by the Sun on the planet is a radially directed force that keeps the planet moving in a circle, we can apply Newton's second law ($\Sigma F = ma$) to the planet:

$$\frac{GM_SM_p}{r^2} = \frac{M_pv^2}{r}$$

Because the orbital speed v of the planet is simply $2\pi r/T$, where T is its period of revolution, the preceding expression becomes

$$\frac{GM_S}{r^2} = \frac{(2\pi r/T)^2}{r}$$

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right) r^3 = K_S r^3 \qquad \textbf{(14.7)}$$

where K_S is a constant given by

$$K_S = \frac{4\pi^2}{GM_S} = 2.97 \times 10^{-19}\ \mathrm{s^2/m^3}$$

Equation 14.7 is Kepler's third law. It can be shown that the law is also valid for elliptical orbits if we replace r with the length of the semimajor axis a. Note that the constant of proportionality K_S is independent of the mass of the planet. Therefore, Equation 14.7 is valid for *any* planet.[3] Table 14.2 contains a collection of useful planetary data. The last column verifies that T^2/r^3 is a constant. The small variations in the values in this column reflect uncertainties in the measured values of the periods and semimajor axes of the planets.

If we were to consider the orbit around the Earth of a satellite such as the Moon, then the proportionality constant would have a different value, with the Sun's mass replaced by the Earth's mass.

EXAMPLE 14.4 The Mass of the Sun

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.156×10^7 s and its distance from the Sun is 1.496×10^{11} m.

Solution Using Equation 14.7, we find that

$$M_S = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2(1.496 \times 10^{11}\ \mathrm{m})^3}{(6.67 \times 10^{-11}\ \mathrm{N\cdot m^2/kg^2})(3.156 \times 10^7\ \mathrm{s})^2}$$

$$= 1.99 \times 10^{30}\ \mathrm{kg}$$

In Example 14.3, an understanding of gravitational forces enabled us to find out something about the density of the Earth's core, and now we have used this understanding to determine the mass of the Sun.

[2] The orbits of all planets except Mercury and Pluto are very close to being circular; hence, we do not introduce much error with this assumption. For example, the ratio of the semiminor axis to the semimajor axis for the Earth's orbit is $b/a = 0.999\ 86$.

[3] Equation 14.7 is indeed a proportion because the ratio of the two quantities T^2 and r^3 is a constant. The variables in a proportion are not required to be limited to the first power only.

TABLE 14.2 Useful Planetary Data

Body	Mass (kg)	Mean Radius (m)	Period of Revolution (s)	Mean Distance from Sun (m)	$\frac{T^2}{r^3}$ (s^2/m^3)
Mercury	3.18×10^{23}	2.43×10^{6}	7.60×10^{6}	5.79×10^{10}	2.97×10^{-19}
Venus	4.88×10^{24}	6.06×10^{6}	1.94×10^{7}	1.08×10^{11}	2.99×10^{-19}
Earth	5.98×10^{24}	6.37×10^{6}	3.156×10^{7}	1.496×10^{11}	2.97×10^{-19}
Mars	6.42×10^{23}	3.37×10^{6}	5.94×10^{7}	2.28×10^{11}	2.98×10^{-19}
Jupiter	1.90×10^{27}	6.99×10^{7}	3.74×10^{8}	7.78×10^{11}	2.97×10^{-19}
Saturn	5.68×10^{26}	5.85×10^{7}	9.35×10^{8}	1.43×10^{12}	2.99×10^{-19}
Uranus	8.68×10^{25}	2.33×10^{7}	2.64×10^{9}	2.87×10^{12}	2.95×10^{-19}
Neptune	1.03×10^{26}	2.21×10^{7}	5.22×10^{9}	4.50×10^{12}	2.99×10^{-19}
Pluto	$\approx 1.4 \times 10^{22}$	$\approx 1.5 \times 10^{6}$	7.82×10^{9}	5.91×10^{12}	2.96×10^{-19}
Moon	7.36×10^{22}	1.74×10^{6}	—	—	—
Sun	1.991×10^{30}	6.96×10^{8}	—	—	—

Kepler's Second Law and Conservation of Angular Momentum

Consider a planet of mass M_p moving around the Sun in an elliptical orbit (Fig. 14.8). The gravitational force acting on the planet is always along the radius vector, directed toward the Sun, as shown in Figure 14.9a. When a force is directed toward or away from a fixed point and is a function of r only, it is called a **central force.** The torque acting on the planet due to this force is clearly zero; that is, because **F** is parallel to **r**,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F\hat{\mathbf{r}} = 0$$

(You may want to revisit Section 11.2 to refresh your memory on the vector product.) Recall from Equation 11.19, however, that torque equals the time rate of change of angular momentum: $\boldsymbol{\tau} = d\mathbf{L}/dt$. Therefore, **because the gravitational**

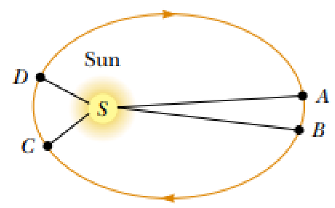


Figure 14.8 Kepler's second law is called the law of equal areas. When the time interval required for a planet to travel from A to B is equal to the time interval required for it to go from C to D, the two areas swept out by the planet's radius vector are equal. Note that in order for this to be true, the planet must be moving faster between C and D than between A and B.

Separate views of Jupiter and of Periodic Comet Shoemaker–Levy 9—both taken with the Hubble Space Telescope about two months before Jupiter and the comet collided in July 1994—were put together with the use of a computer. Their relative sizes and distances were altered. The black spot on Jupiter is the shadow of its moon Io.

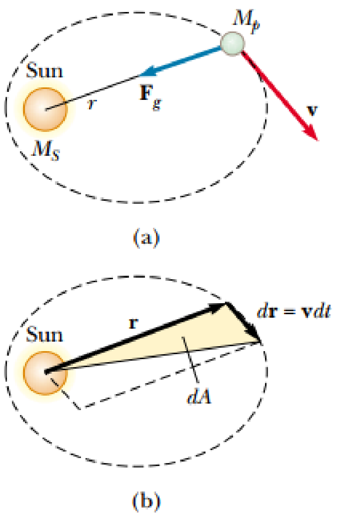


Figure 14.9 (a) The gravitational force acting on a planet is directed toward the Sun, along the radius vector. (b) As a planet orbits the Sun, the area swept out by the radius vector in a time dt is equal to one-half the area of the parallelogram formed by the vectors $\mathbf{r}$ and $d\mathbf{r} = \mathbf{v}\,dt$.

force exerted by the Sun on a planet results in no torque on the planet, the angular momentum L of the planet is constant:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times M_p\mathbf{v} = M_p\mathbf{r} \times \mathbf{v} = \text{constant} \tag{14.8}$$

Because $\mathbf{L}$ remains constant, the planet's motion at any instant is restricted to the plane formed by $\mathbf{r}$ and $\mathbf{v}$.

We can relate this result to the following geometric consideration. The radius vector $\mathbf{r}$ in Figure 14.9b sweeps out an area dA in a time dt. This area equals one-half the area $|\mathbf{r} \times d\mathbf{r}|$ of the parallelogram formed by the vectors $\mathbf{r}$ and $d\mathbf{r}$ (see Section 11.2). Because the displacement of the planet in a time dt is $d\mathbf{r} = \mathbf{v}dt$, we can say that

$$dA = \tfrac{1}{2}|\mathbf{r} \times d\mathbf{r}| = \tfrac{1}{2}|\mathbf{r} \times \mathbf{v}\,dt| = \frac{L}{2M_p}\,dt$$

$$\boxed{\frac{dA}{dt} = \frac{L}{2M_p} = \text{constant}} \tag{14.9}$$

where L and M_p are both constants. Thus, we conclude that

the radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.

It is important to recognize that this result, which is Kepler's second law, is a consequence of the fact that the force of gravity is a central force, which in turn implies that angular momentum is constant. Therefore, Kepler's second law applies to *any* situation involving a central force, whether inverse-square or not.

EXAMPLE 14.5 Motion in an Elliptical Orbit

A satellite of mass m moves in an elliptical orbit around the Earth (Fig. 14.10). The minimum distance of the satellite from the Earth is called the *perigee* (indicated by p in Fig. 14.10), and the maximum distance is called the *apogee* (indicated by a). If the speed of the satellite at p is v_p, what is its speed at a?

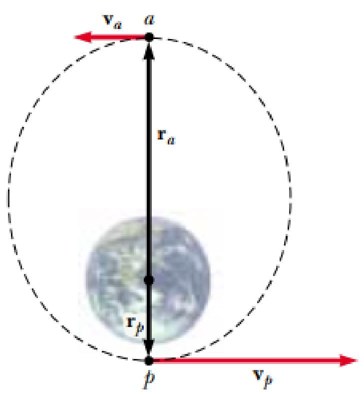


Figure 14.10 As a satellite moves around the Earth in an elliptical orbit, its angular momentum is constant. Therefore, $mv_ar_a = mv_pr_p$, where the subscripts a and p represent apogee and perigee, respectively.

Solution As the satellite moves from perigee toward apogee, it is moving farther from the Earth. Thus, a component of the gravitational force exerted by the Earth on the satellite is opposite the velocity vector. Negative work is done on the satellite, which causes it to slow down, according to the work–kinetic energy theorem. As a result, we expect the speed at apogee to be lower than the speed at perigee.

The angular momentum of the satellite relative to the Earth is $\mathbf{r} \times m\mathbf{v} = m\mathbf{r} \times \mathbf{v}$. At the points a and p, $\mathbf{v}$ is perpendicular to $\mathbf{r}$. Therefore, the magnitude of the angular momentum at these positions is $L_a = mv_ar_a$ and $L_p = mv_pr_p$. Because angular momentum is constant, we see that

$$mv_ar_a = mv_pr_p$$

$$v_a = \boxed{\frac{r_p}{r_a}\,v_p}$$

Quick Quiz 14.1

How would you explain the fact that Saturn and Jupiter have periods much greater than one year?

14.6 THE GRAVITATIONAL FIELD

When Newton published his theory of universal gravitation, it was considered a success because it satisfactorily explained the motion of the planets. Since 1687 the same theory has been used to account for the motions of comets, the deflection of a Cavendish balance, the orbits of binary stars, and the rotation of galaxies. Nevertheless, both Newton's contemporaries and his successors found it difficult to accept the concept of a force that acts through a distance, as mentioned in Section 5.1. They asked how it was possible for two objects to interact when they were not in contact with each other. Newton himself could not answer that question.

An approach to describing interactions between objects that are not in contact came well after Newton's death, and it enables us to look at the gravitational interaction in a different way. As described in Section 5.1, this alternative approach uses the concept of a **gravitational field** that exists at every point in space. When a particle of mass m is placed at a point where the gravitational field is $\mathbf{g}$, the particle experiences a force $\mathbf{F}_g = m\mathbf{g}$. In other words, the field exerts a force on the particle. Hence, the gravitational field $\mathbf{g}$ is defined as

$$\mathbf{g} \equiv \frac{\mathbf{F}_g}{m} \qquad (14.10)$$

Gravitational field

That is, the gravitational field at a point in space equals the gravitational force experienced by a *test particle* placed at that point divided by the mass of the test particle. Notice that the presence of the test particle is not necessary for the field to exist—the Earth creates the gravitational field. We call the object creating the field the *source particle* (although the Earth is clearly not a particle; we shall discuss shortly the fact that we can approximate the Earth as a particle for the purpose of finding the gravitational field that it creates). We can detect the presence of the field and measure its strength by placing a test particle in the field and noting the force exerted on it.

Although the gravitational force is inherently an interaction between two objects, the concept of a gravitational field allows us to "factor out" the mass of one of the objects. In essence, we are describing the "effect" that any object (in this case, the Earth) has on the empty space around itself in terms of the force that *would* be present *if* a second object were somewhere in that space.[4]

As an example of how the field concept works, consider an object of mass m near the Earth's surface. Because the gravitational force acting on the object has a magnitude GM_Em/r^2 (see Eq. 14.4), the field $\mathbf{g}$ at a distance r from the center of the Earth is

$$\mathbf{g} = \frac{\mathbf{F}_g}{m} = -\frac{GM_E}{r^2}\hat{\mathbf{r}} \qquad (14.11)$$

where $\hat{\mathbf{r}}$ is a unit vector pointing radially outward from the Earth and the minus

[4] We shall return to this idea of mass affecting the space around it when we discuss Einstein's theory of gravitation in Chapter 39.

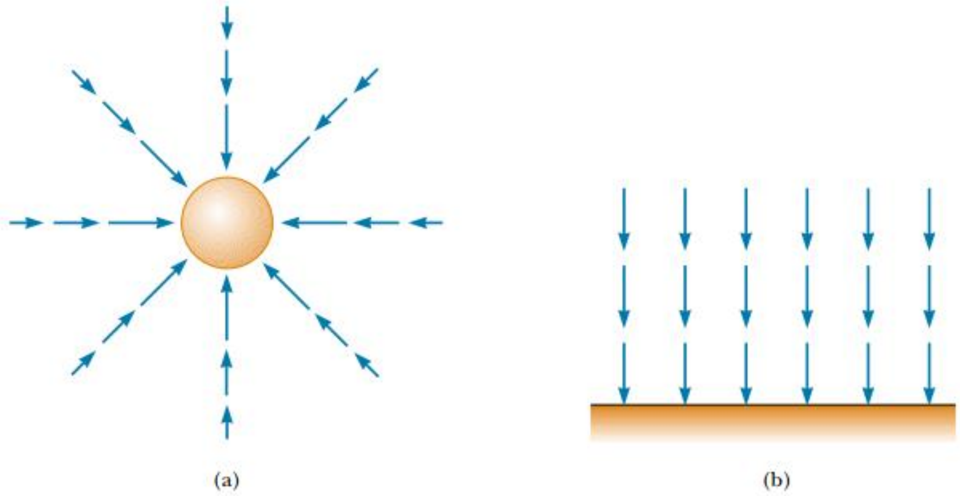


Figure 14.11 (a) The gravitational field vectors in the vicinity of a uniform spherical mass such as the Earth vary in both direction and magnitude. The vectors point in the direction of the acceleration a particle would experience if it were placed in the field. The magnitude of the field vector at any location is the magnitude of the free-fall acceleration at that location. (b) The gravitational field vectors in a small region near the Earth's surface are uniform in both direction and magnitude.

sign indicates that the field points toward the center of the Earth, as illustrated in Figure 14.11a. Note that the field vectors at different points surrounding the Earth vary in both direction and magnitude. In a small region near the Earth's surface, the downward field $\mathbf{g}$ is approximately constant and uniform, as indicated in Figure 14.11b. Equation 14.11 is valid at all points *outside* the Earth's surface, assuming that the Earth is spherical. At the Earth's surface, where $r = R_E$, $\mathbf{g}$ has a magnitude of 9.80 N/kg.

14.7 GRAVITATIONAL POTENTIAL ENERGY

In Chapter 8 we introduced the concept of gravitational potential energy, which is the energy associated with the position of a particle. We emphasized that the gravitational potential energy function $U = mgy$ is valid only when the particle is near the Earth's surface, where the gravitational force is constant. Because the gravitational force between two particles varies as $1/r^2$, we expect that a more general potential energy function—one that is valid without the restriction of having to be near the Earth's surface—will be significantly different from $U = mgy$.

Before we calculate this general form for the gravitational potential energy function, let us first verify that *the gravitational force is conservative.* (Recall from Section 8.2 that a force is conservative if the work it does on an object moving between any two points is independent of the path taken by the object.) To do this, we first note that the gravitational force is a central force. By definition, a central force is any force that is directed along a radial line to a fixed center and has a magnitude that depends only on the radial coordinate r. Hence, a central force can be represented by $F(r)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector directed from the origin to the particle, as shown in Figure 14.12.

Consider a central force acting on a particle moving along the general path P to Q in Figure 14.12. The path from P to Q can be approximated by a series of steps according to the following procedure. In Figure 14.12, we draw several thin wedges, which are shown as dashed lines. The outer boundary of our set of wedges is a path consisting of short radial line segments and arcs (gray in the figure). We select the length of the radial dimension of each wedge such that the short arc at the wedge's wide end intersects the actual path of the particle. Then we can approximate the actual path with a series of zigzag movements that alternate between moving along an arc and moving along a radial line.

By definition, a central force is always directed along one of the radial segments; therefore, the work done by **F** along any radial segment is

$$dW = \mathbf{F} \cdot d\mathbf{r} = F(r)\ dr$$

You should recall that, by definition, the work done by a force that is perpendicular to the displacement is zero. Hence, the work done in moving along any arc is zero because **F** is perpendicular to the displacement along these segments. Therefore, the total work done by **F** is the sum of the contributions along the radial segments:

$$W = \int_{r_i}^{r_f} F(r)\ dr$$

Work done by a central force

where the subscripts i and f refer to the initial and final positions. Because the integrand is a function only of the radial position, this integral depends only on the initial and final values of r. Thus, the work done is the same over *any* path from P to Q. Because the work done is independent of the path and depends only on the end points, we conclude that *any central force is conservative.* We are now assured that a potential energy function can be obtained once the form of the central force is specified.

Recall from Equation 8.2 that the change in the gravitational potential energy associated with a given displacement is defined as the negative of the work done by the gravitational force during that displacement:

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F(r)\ dr \qquad \textbf{(14.12)}$$

We can use this result to evaluate the gravitational potential energy function. Consider a particle of mass m moving between two points P and Q above the Earth's surface (Fig. 14.13). The particle is subject to the gravitational force given by Equation 14.1. We can express this force as

$$F(r) = -\frac{GM_E m}{r^2}$$

where the negative sign indicates that the force is attractive. Substituting this expression for $F(r)$ into Equation 14.12, we can compute the change in the gravitational potential energy

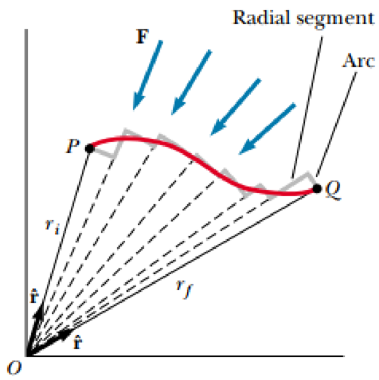


Figure 14.12 A particle moves from P to Q while acted on by a central force **F**, which is directed radially. The path is broken into a series of radial segments and arcs. Because the work done along the arcs is zero, the work done is independent of the path and depends only on r_f and r_i.

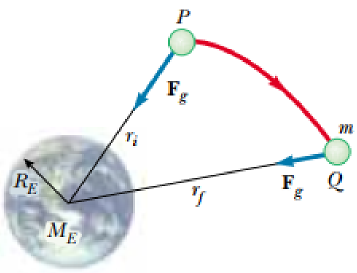


Figure 14.13 As a particle of mass m moves from P to Q above the Earth's surface, the gravitational potential energy changes according to Equation 14.12.

tional potential energy function:

$$U_f - U_i = GM_E m \int_{r_i}^{r_f} \frac{dr}{r^2} = GM_E m \left[-\frac{1}{r} \right]_{r_i}^{r_f}$$

Change in gravitational potential energy

$$\boxed{U_f - U_i = -GM_E m \left(\frac{1}{r_f} - \frac{1}{r_i} \right)} \qquad \textbf{(14.13)}$$

As always, the choice of a reference point for the potential energy is completely arbitrary. It is customary to choose the reference point where the force is zero. Taking $U_i = 0$ at $r_i = \infty$, we obtain the important result

Gravitational potential energy of the Earth–particle system for $r \geq R_E$

$$U = -\frac{GM_E m}{r} \qquad \textbf{(14.14)}$$

This expression applies to the Earth–particle system where the two masses are separated by a distance r, provided that $r \geq R_E$. The result is not valid for particles inside the Earth, where $r < R_E$. (The situation in which $r < R_E$ is treated in Section 14.10.) Because of our choice of U_i, the function U is always negative (Fig. 14.14).

Although Equation 14.14 was derived for the particle–Earth system, it can be applied to any two particles. That is, the gravitational potential energy associated with any pair of particles of masses m_1 and m_2 separated by a distance r is

$$\boxed{U = -\frac{Gm_1 m_2}{r}} \qquad \textbf{(14.15)}$$

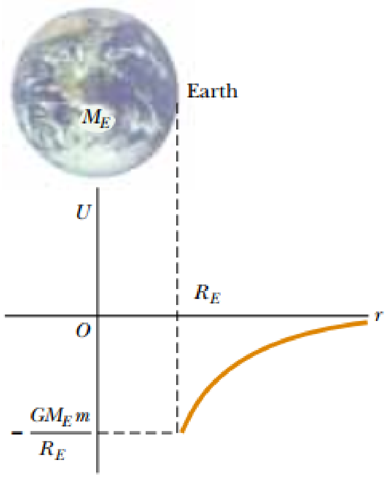


Figure 14.14 Graph of the gravitational potential energy U versus r for a particle above the Earth's surface. The potential energy goes to zero as r approaches infinity.

This expression shows that the gravitational potential energy for any pair of particles varies as $1/r$, whereas the force between them varies as $1/r^2$. Furthermore, the potential energy is negative because the force is attractive and we have taken the potential energy as zero when the particle separation is infinite. Because the force between the particles is attractive, we know that an external agent must do positive work to increase the separation between them. The work done by the external agent produces an increase in the potential energy as the two particles are separated. That is, U becomes less negative as r increases.

When two particles are at rest and separated by a distance r, an external agent has to supply an energy at least equal to $+Gm_1m_2/r$ in order to separate the particles to an infinite distance. It is therefore convenient to think of the absolute value of the potential energy as the *binding energy* of the system. If the external agent supplies an energy greater than the binding energy, the excess energy of the system will be in the form of kinetic energy when the particles are at an infinite separation.

We can extend this concept to three or more particles. In this case, the total potential energy of the system is the sum over all pairs of particles.[5] Each pair contributes a term of the form given by Equation 14.15. For example, if the system contains three particles, as in Figure 14.15, we find that

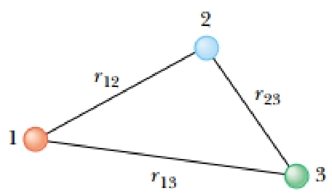


Figure 14.15 Three interacting particles.

$$U_{\text{total}} = U_{12} + U_{13} + U_{23} = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right) \qquad \textbf{(14.16)}$$

The absolute value of U_{total} represents the work needed to separate the particles by an infinite distance.

[5] The fact that potential energy terms can be added for all pairs of particles stems from the experimental fact that gravitational forces obey the superposition principle.

EXAMPLE 14.6 The Change in Potential Energy

A particle of mass m is displaced through a small vertical distance Δy near the Earth's surface. Show that in this situation the general expression for the change in gravitational potential energy given by Equation 14.13 reduces to the familiar relationship $\Delta U = mg\,\Delta y$.

Solution We can express Equation 14.13 in the form

$$\Delta U = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = GM_E m\left(\frac{r_f - r_i}{r_i r_f}\right)$$

If both the initial and final positions of the particle are close to the Earth's surface, then $r_f - r_i = \Delta y$ and $r_i r_f \approx R_E^2$. (Recall that r is measured from the center of the Earth.) Therefore, the change in potential energy becomes

$$\Delta U \approx \frac{GM_E m}{R_E^2}\,\Delta y = mg\,\Delta y$$

where we have used the fact that $g = GM_E/R_E^2$ (Eq. 14.5). Keep in mind that the reference point is arbitrary because it is the *change* in potential energy that is meaningful.

14.8 ENERGY CONSIDERATIONS IN PLANETARY AND SATELLITE MOTION

Consider a body of mass m moving with a speed v in the vicinity of a massive body of mass M, where $M \gg m$. The system might be a planet moving around the Sun, a satellite in orbit around the Earth, or a comet making a one-time flyby of the Sun. If we assume that the body of mass M is at rest in an inertial reference frame, then the total mechanical energy E of the two-body system when the bodies are separated by a distance r is the sum of the kinetic energy of the body of mass m and the potential energy of the system, given by Equation 14.15:[6]

$$E = K + U$$

$$\boxed{E = \tfrac{1}{2}mv^2 - \frac{GMm}{r}} \qquad \textbf{(14.17)}$$

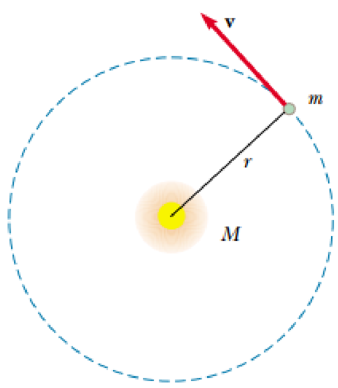


Figure 14.16 A body of mass m moving in a circular orbit about a much larger body of mass M.

This equation shows that E may be positive, negative, or zero, depending on the value of v. However, for a bound system,[7] such as the Earth–Sun system, E is necessarily *less than zero* because we have chosen the convention that $U \to 0$ as $r \to \infty$.

We can easily establish that $E < 0$ for the system consisting of a body of mass m moving in a circular orbit about a body of mass $M \gg m$ (Fig. 14.16). Newton's second law applied to the body of mass m gives

$$\frac{GMm}{r^2} = ma = \frac{mv^2}{r}$$

[6] You might recognize that we have ignored the acceleration and kinetic energy of the larger body. To see that this simplification is reasonable, consider an object of mass m falling toward the Earth. Because the center of mass of the object–Earth system is effectively stationary, it follows that $mv = M_E v_E$. Thus, the Earth acquires a kinetic energy equal to

$$\tfrac{1}{2}M_E v_E^2 = \tfrac{1}{2}\frac{m^2}{M_E}v^2 = \frac{m}{M_E}K$$

where K is the kinetic energy of the object. Because $M_E \gg m$, this result shows that the kinetic energy of the Earth is negligible.

[7] Of the three examples provided at the beginning of this section, the planet moving around the Sun and a satellite in orbit around the Earth are bound systems—the Earth will always stay near the Sun, and the satellite will always stay near the Earth. The one-time comet flyby represents an unbound system—the comet interacts once with the Sun but is not bound to it. Thus, in theory the comet can move infinitely far away from the Sun.

Multiplying both sides by r and dividing by 2 gives

$$\tfrac{1}{2}mv^2 = \frac{GMm}{2r} \tag{14.18}$$

Substituting this into Equation 14.17, we obtain

$$E = \frac{GMm}{2r} - \frac{GMm}{r}$$

Total energy for circular orbits

$$\boxed{E = -\frac{GMm}{2r}} \tag{14.19}$$

This result clearly shows that **the total mechanical energy is negative in the case of circular orbits.** Note that **the kinetic energy is positive and equal to one-half the absolute value of the potential energy.** The absolute value of E is also equal to the binding energy of the system, because this amount of energy must be provided to the system to move the two masses infinitely far apart.

The total mechanical energy is also negative in the case of elliptical orbits. The expression for E for elliptical orbits is the same as Equation 14.19 with r replaced by the semimajor axis length a. Furthermore, the total energy is constant if we assume that the system is isolated. Therefore, as the body of mass m moves from P to Q in Figure 14.13, the total energy remains constant and Equation 14.17 gives

$$E = \tfrac{1}{2}mv_i^2 - \frac{GMm}{r_i} = \tfrac{1}{2}mv_f^2 - \frac{GMm}{r_f} \tag{14.20}$$

Combining this statement of energy conservation with our earlier discussion of conservation of angular momentum, we see that **both the total energy and the total angular momentum of a gravitationally bound, two-body system are constants of the motion.**

EXAMPLE 14.7 Changing the Orbit of a Satellite

The space shuttle releases a 470-kg communications satellite while in an orbit that is 280 km above the surface of the Earth. A rocket engine on the satellite boosts it into a geosynchronous orbit, which is an orbit in which the satellite stays directly over a single location on the Earth. How much energy did the engine have to provide?

Solution First we must determine the radius of a geosynchronous orbit. Then we can calculate the change in energy needed to boost the satellite into orbit.

The period of the orbit T must be one day (86 400 s), so that the satellite travels once around the Earth in the same time that the Earth spins once on its axis. Knowing the period, we can then apply Kepler's third law (Eq. 14.7) to find the radius, once we replace K_S with $K_E = 4\pi^2/GM_E = 9.89 \times 10^{-14}\ \text{s}^2/\text{m}^3$:

$$T^2 = K_E r^3$$

$$r = \sqrt[3]{\frac{T^2}{K_E}} = \sqrt[3]{\frac{(86\,400\ \text{s})^2}{9.89 \times 10^{-14}\ \text{s}^2/\text{m}^3}} = 4.23 \times 10^7\ \text{m} = R_f$$

This is a little more than 26 000 mi above the Earth's surface.

We must also determine the initial radius (not the altitude above the Earth's surface) of the satellite's orbit when it was still in the shuttle's cargo bay. This is simply

$$R_E + 280\ \text{km} = 6.65 \times 10^6\ \text{m} = R_i$$

Now, applying Equation 14.19, we obtain, for the total initial and final energies,

$$E_i = -\frac{GM_E m}{2R_i} \qquad E_f = -\frac{GM_E m}{2R_f}$$

The energy required from the engine to boost the satellite is

$$E_{\text{engine}} = E_f - E_i = -\frac{GM_E m}{2}\left(\frac{1}{R_f} - \frac{1}{R_i}\right)$$

$$= -\frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})(470\ \text{kg})}{2}$$

$$\times \left(\frac{1}{4.23 \times 10^7\ \text{m}} - \frac{1}{6.65 \times 10^6\ \text{m}}\right)$$

$$= \boxed{1.19 \times 10^{10}\ \text{J}}$$

This is the energy equivalent of 89 gal of gasoline. NASA engineers must account for the changing mass of the spacecraft as it ejects burned fuel, something we have not done here. Would you expect the calculation that includes the effect of this changing mass to yield a greater or lesser amount of energy required from the engine?

If we wish to determine how the energy is distributed after the engine is fired, we find from Equation 14.18 that the change in kinetic energy is $\Delta K = (GM_E m/2)(1/R_f - 1/R_i) = -1.19 \times 10^{10}\text{ J}$ (a decrease), and the corresponding change in potential energy is $\Delta U = -GM_E m(1/R_f - 1/R_i) = 2.38 \times 10^{10}\text{ J}$ (an increase). Thus, the change in mechanical energy of the system is $\Delta E = \Delta K + \Delta U = 1.19 \times 10^{10}\text{ J}$, as we already calculated. The firing of the engine results in an increase in the total mechanical energy of the system. Because an increase in potential energy is accompanied by a decrease in kinetic energy, we conclude that the speed of an orbiting satellite decreases as its altitude increases.

Escape Speed

Suppose an object of mass m is projected vertically upward from the Earth's surface with an initial speed v_i, as illustrated in Figure 14.17. We can use energy considerations to find the minimum value of the initial speed needed to allow the object to escape the Earth's gravitational field. Equation 14.17 gives the total energy of the object at any point. At the surface of the Earth, $v = v_i$ and $r = r_i = R_E$. When the object reaches its maximum altitude, $v = v_f = 0$ and $r = r_f = r_{\text{max}}$. Because the total energy of the system is constant, substituting these conditions into Equation 14.20 gives

$$\frac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{r_{\text{max}}}$$

Solving for v_i^2 gives

$$v_i^2 = 2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{\text{max}}}\right) \tag{14.21}$$

Therefore, if the initial speed is known, this expression can be used to calculate the maximum altitude h because we know that

$$h = r_{\text{max}} - R_E$$

We are now in a position to calculate **escape speed,** which is the minimum speed the object must have at the Earth's surface in order to escape from the influence of the Earth's gravitational field. Traveling at this minimum speed, the object continues to move farther and farther away from the Earth as its speed asymptotically approaches zero. Letting $r_{\text{max}} \to \infty$ in Equation 14.21 and taking $v_i = v_{\text{esc}}$, we obtain

$$v_{\text{esc}} = \sqrt{\frac{2GM_E}{R_E}} \tag{14.22}$$

Escape speed

Note that this expression for v_{esc} is independent of the mass of the object. In other words, a spacecraft has the same escape speed as a molecule. Furthermore, the result is independent of the direction of the velocity and ignores air resistance.

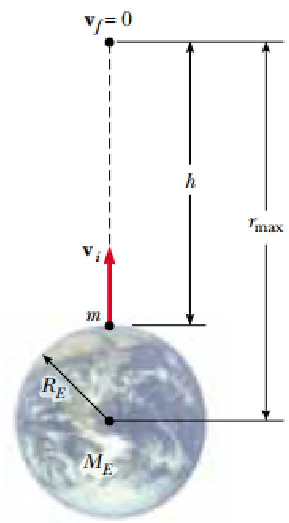


Figure 14.17 An object of mass m projected upward from the Earth's surface with an initial speed v_i reaches a maximum altitude h.

If the object is given an initial speed equal to v_{esc}, its total energy is equal to zero. This can be seen by noting that when $r \to \infty$, the object's kinetic energy and its potential energy are both zero. If v_i is greater than v_{esc}, the total energy is greater than zero and the object has some residual kinetic energy as $r \to \infty$.

EXAMPLE 14.8 Escape Speed of a Rocket

Calculate the escape speed from the Earth for a 5 000-kg spacecraft, and determine the kinetic energy it must have at the Earth's surface in order to escape the Earth's gravitational field.

Solution Using Equation 14.22 gives

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$$

$$= \sqrt{\frac{2(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}{6.37 \times 10^6\ \text{m}}}$$

$$= \boxed{1.12 \times 10^4\ \text{m/s}}$$

This corresponds to about 25 000 mi/h.

The kinetic energy of the spacecraft is

$$K = \tfrac{1}{2}mv_{esc}^2 = \tfrac{1}{2}(5.00 \times 10^3\ \text{kg})(1.12 \times 10^4\ \text{m/s})^2$$

$$= \boxed{3.14 \times 10^{11}\ \text{J}}$$

This is equivalent to about 2 300 gal of gasoline.

Equations 14.21 and 14.22 can be applied to objects projected from any planet. That is, in general, the escape speed from the surface of any planet of mass M and radius R is

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

Escape speeds for the planets, the Moon, and the Sun are provided in Table 14.3. Note that the values vary from 1.1 km/s for Pluto to about 618 km/s for the Sun. These results, together with some ideas from the kinetic theory of gases (see Chapter 21), explain why some planets have atmospheres and others do not. As we shall see later, a gas molecule has an average kinetic energy that depends on the temperature of the gas. Hence, lighter molecules, such as hydrogen and helium, have a higher average speed than heavier species at the same temperature. When the average speed of the lighter molecules is not much less than the escape speed of a planet, a significant fraction of them have a chance to escape from the planet.

This mechanism also explains why the Earth does not retain hydrogen molecules and helium atoms in its atmosphere but does retain heavier molecules, such as oxygen and nitrogen. On the other hand, the very large escape speed for Jupiter enables that planet to retain hydrogen, the primary constituent of its atmosphere.

TABLE 14.3 Escape Speeds from the Surfaces of the Planets, Moon, and Sun

Body	v_{esc} (km/s)
Mercury	4.3
Venus	10.3
Earth	11.2
Moon	2.3
Mars	5.0
Jupiter	60
Saturn	36
Uranus	22
Neptune	24
Pluto	1.1
Sun	618

Quick Quiz 14.2

If you were a space prospector and discovered gold on an asteroid, it probably would not be a good idea to jump up and down in excitement over your find. Why?

Quick Quiz 14.3

Figure 14.18 is a drawing by Newton showing the path of a stone thrown from a mountaintop. He shows the stone landing farther and farther away when thrown at higher and higher speeds (at points D, E, F, and G), until finally it is thrown all the way around the Earth. Why didn't Newton show the stone landing at B and A before it was going fast enough to complete an orbit?

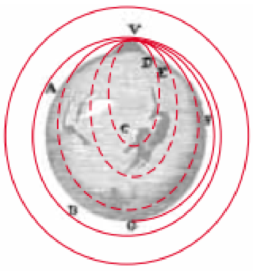


Figure 14.18 "The greater the velocity . . . with which [a stone] is projected, the farther it goes before it falls to the Earth. We may therefore suppose the velocity to be so increased, that it would describe an arc of 1, 2, 5, 10, 100, 1000 miles before it arrived at the Earth, till at last, exceeding the limits of the Earth, it should pass into space without touching." Sir Isaac Newton, *System of the World.*

Optional Section

14.9 THE GRAVITATIONAL FORCE BETWEEN AN EXTENDED OBJECT AND A PARTICLE

We have emphasized that the law of universal gravitation given by Equation 14.3 is valid only if the interacting objects are treated as particles. In view of this, how can we calculate the force between a particle and an object having finite dimensions? This is accomplished by treating the extended object as a collection of particles and making use of integral calculus. We first evaluate the potential energy function, and then calculate the gravitational force from that function.

We obtain the potential energy associated with a system consisting of a particle of mass m and an extended object of mass M by dividing the object into many elements, each having a mass ΔM_i (Fig. 14.19). The potential energy associated with the system consisting of any one element and the particle is $U = -Gm\,\Delta M_i/r_i$, where r_i is the distance from the particle to the element ΔM_i. The total potential energy of the overall system is obtained by taking the sum over all elements as $\Delta M_i \rightarrow 0$. In this limit, we can express U in integral form as

$$U = -Gm\int \frac{dM}{r} \tag{14.23}$$

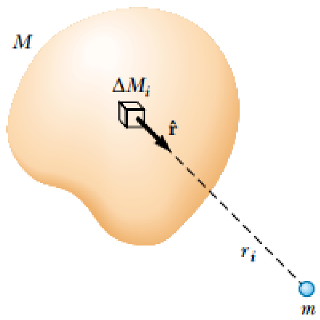


Figure 14.19 A particle of mass m interacting with an extended object of mass M. The total gravitational force exerted by the object on the particle can be obtained by dividing the object into numerous elements, each having a mass ΔM_i, and then taking a vector sum over the forces exerted by all elements.

Once U has been evaluated, we obtain the force exerted by the extended object on the particle by taking the negative derivative of this scalar function (see Section 8.6). If the extended object has spherical symmetry, the function U depends only on r, and the force is given by $-dU/dr$. We treat this situation in Section 14.10. In principle, one can evaluate U for any geometry; however, the integration can be cumbersome.

An alternative approach to evaluating the gravitational force between a particle and an extended object is to perform a vector sum over all mass elements of the object. Using the procedure outlined in evaluating U and the law of universal gravitation in the form shown in Equation 14.3, we obtain, for the total force exerted on the particle

Total force exerted on a particle by an extended object

$$\mathbf{F}_g = -Gm\int \frac{dM}{r^2}\,\hat{\mathbf{r}} \tag{14.24}$$

where $\hat{\mathbf{r}}$ is a unit vector directed from the element dM toward the particle (see Fig. 14.19) and the minus sign indicates that the direction of the force is opposite that of $\hat{\mathbf{r}}$. This procedure is not always recommended because working with a vector function is more difficult than working with the scalar potential energy function. However, if the geometry is simple, as in the following example, the evaluation of $\mathbf{F}$ can be straightforward.

EXAMPLE 14.9 Gravitational Force Between a Particle and a Bar

The left end of a homogeneous bar of length L and mass M is at a distance h from a particle of mass m (Fig. 14.20). Calculate the total gravitational force exerted by the bar on the particle.

Solution The arbitrary segment of the bar of length dx has a mass dM. Because the mass per unit length is constant, it follows that the ratio of masses dM/M is equal to the ratio of lengths dx/L, and so $dM = (M/L)\,dx$. In this problem, the variable r in Equation 14.24 is the distance x shown in Figure 14.20, the unit vector $\hat{\mathbf{r}}$ is $\hat{\mathbf{r}} = -\mathbf{i}$, and the force acting on the particle is to the right; therefore, Equation 14.24 gives us

$$\mathbf{F}_g = -Gm\int_h^{h+L} \frac{M\,dx}{L}\,\frac{1}{x^2}(-\mathbf{i}) = Gm\frac{M}{L}\int_h^{h+L}\frac{dx}{x^2}\,\mathbf{i}$$

$$\mathbf{F}_g = \frac{GmM}{L}\left[-\frac{1}{x}\right]_h^{h+L}\mathbf{i} = \boxed{\frac{GmM}{h(h+L)}\,\mathbf{i}}$$

We see that the force exerted on the particle is in the positive x direction, which is what we expect because the gravitational force is attractive.

Note that in the limit $L \rightarrow 0$, the force varies as $1/h^2$, which is what we expect for the force between two point masses. Furthermore, if $h \gg L$, the force also varies as $1/h^2$. This can be seen by noting that the denominator of the expression for $\mathbf{F}_g$ can be expressed in the form $h^2(1 + L/h)$, which is approximately equal to h^2 when $h \gg L$. Thus, when bodies are separated by distances that are great relative to their characteristic dimensions, they behave like particles.

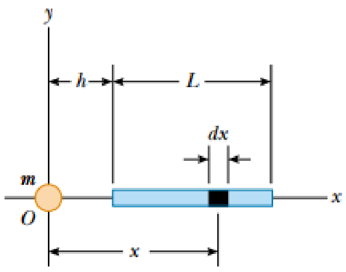


Figure 14.20 The gravitational force exerted by the bar on the particle is directed to the right. Note that the bar is *not* equivalent to a particle of mass M located at the center of mass of the bar.

Optional Section

14.10 THE GRAVITATIONAL FORCE BETWEEN A PARTICLE AND A SPHERICAL MASS

We have already stated that a large sphere attracts a particle outside it as if the total mass of the sphere were concentrated at its center. We now describe the force acting on a particle when the extended object is either a spherical shell or a solid sphere, and then apply these facts to some interesting systems.

Spherical Shell

Case 1. If a particle of mass m is located outside a spherical shell of mass M at, for instance, point P in Figure 14.21a, the shell attracts the particle as though the mass of the shell were concentrated at its center. We can show this, as Newton did, with integral calculus. Thus, as far as the gravitational force acting on a particle outside the shell is concerned, a spherical shell acts no differently from the solid spherical distributions of mass we have seen.

Case 2. If the particle is located inside the shell (at point P in Fig. 14.21b), the gravitational force acting on it can be shown to be zero.

We can express these two important results in the following way:

Force on a particle due to a spherical shell

$$\mathbf{F}_g = -\frac{GMm}{r^2}\hat{\mathbf{r}} \qquad \text{for } r \geq R \qquad \textbf{(14.25a)}$$

$$\mathbf{F}_g = 0 \qquad \text{for } r < R \qquad \textbf{(14.25b)}$$

The gravitational force as a function of the distance r is plotted in Figure 14.21c.

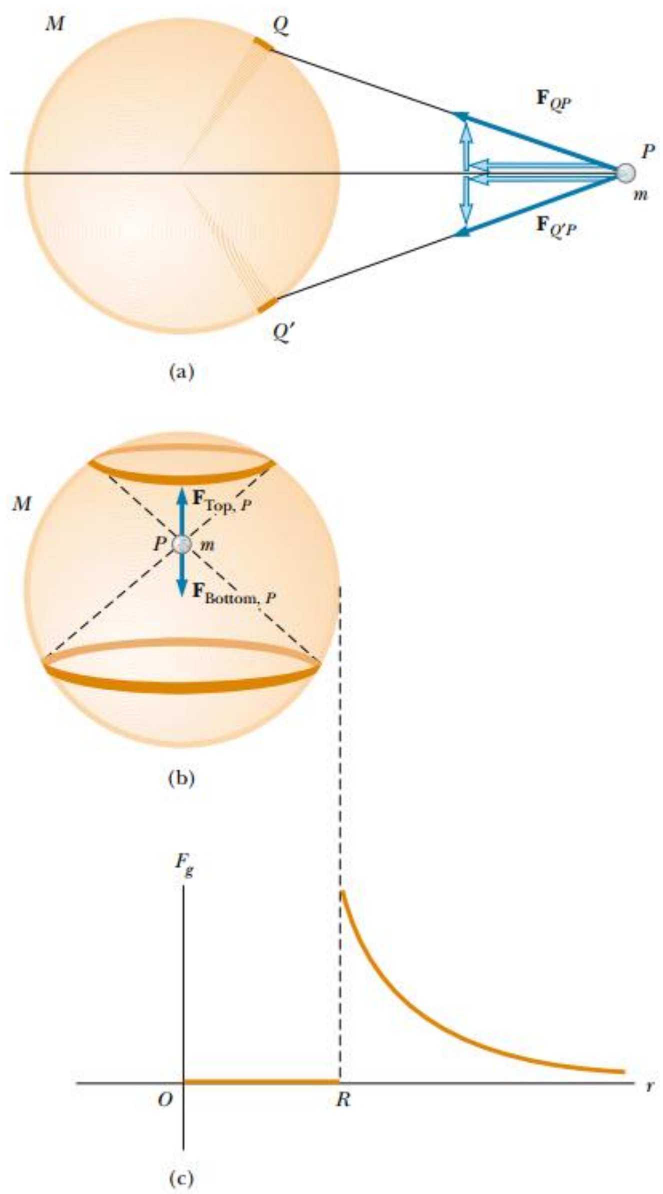


Figure 14.21 (a) The nonradial components of the gravitational forces exerted on a particle of mass m located at point P outside a spherical shell of mass M cancel out. (b) The spherical shell can be broken into rings. Even though point P is closer to the top ring than to the bottom ring, the bottom ring is larger, and the gravitational forces exerted on the particle at P by the matter in the two rings cancel each other. Thus, for a particle located at any point P inside the shell, there is no gravitational force exerted on the particle by the mass M of the shell. (c) The magnitude of the gravitational force versus the radial distance r from the center of the shell.

The shell does not act as a gravitational shield, which means that a particle inside a shell may experience forces exerted by bodies outside the shell.

Solid Sphere

Case 1. If a particle of mass m is located outside a homogeneous solid sphere of mass M (at point P in Fig. 14.22), the sphere attracts the particle as though the mass of the sphere were concentrated at its center. We have used this notion at several places in this chapter already, and we can argue it from Equation 14.25a. A solid sphere can be considered to be a collection of concentric spherical shells. The masses of all of the shells can be interpreted as being concentrated at their common center, and the gravitational force is equivalent to that due to a particle of mass M located at that center.

Case 2. If a particle of mass m is located inside a homogeneous solid sphere of mass M (at point Q in Fig. 14.22), the gravitational force acting on it is due *only* to the mass M' contained within the sphere of radius $r < R$, shown in Figure 14.22. In other words,

Force on a particle due to a solid sphere

$$\mathbf{F}_g = -\frac{GmM}{r^2}\hat{\mathbf{r}} \qquad \text{for } r \geq R \tag{14.26a}$$

$$\mathbf{F}_g = -\frac{GmM'}{r^2}\hat{\mathbf{r}} \qquad \text{for } r < R \tag{14.26b}$$

This also follows from spherical-shell Case 1 because the part of the sphere that is

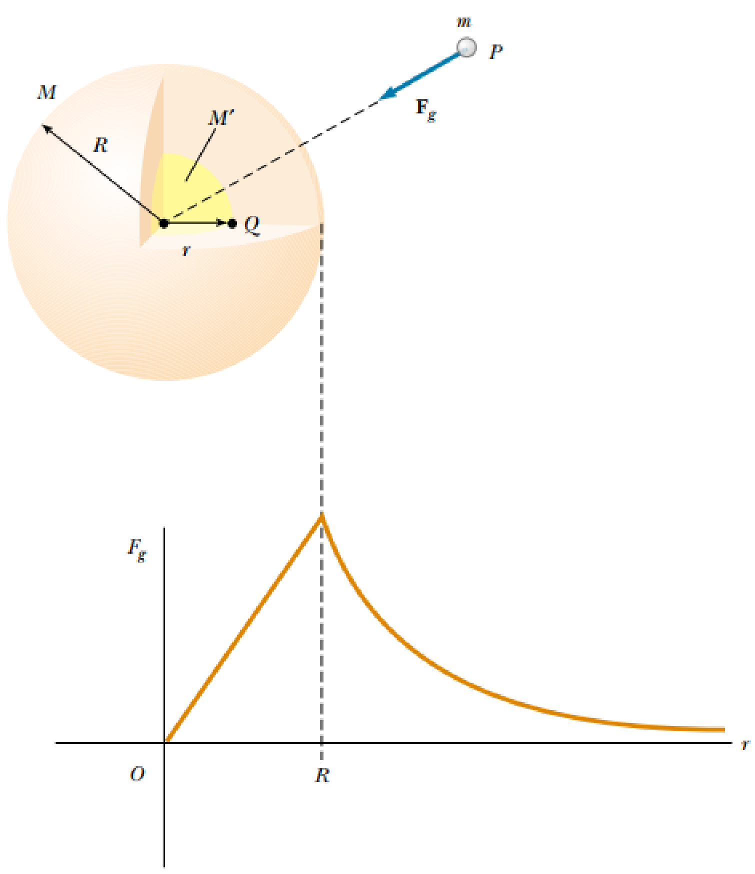


Figure 14.22 The gravitational force acting on a particle when it is outside a uniform solid sphere is GMm/r^2 and is directed toward the center of the sphere. The gravitational force acting on the particle when it is inside such a sphere is proportional to r and goes to zero at the center.

farther from the center than Q can be treated as a series of concentric spherical shells that do not exert a net force on the particle because the particle is inside them. Because the sphere is assumed to have a uniform density, it follows that the ratio of masses M'/M is equal to the ratio of volumes V'/V, where V is the total volume of the sphere and V' is the volume within the sphere of radius r only:

$$\frac{M'}{M} = \frac{V'}{V} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = \frac{r^3}{R^3}$$

Solving this equation for M' and substituting the value obtained into Equation 14.26b, we have

$$\mathbf{F}_g = -\frac{GmM}{R^3}\, r\hat{\mathbf{r}} \qquad \text{for } r < R \tag{14.27}$$

This equation tells us that at the center of the solid sphere, where $r = 0$, the gravitational force goes to zero, as we intuitively expect. The force as a function of r is plotted in Figure 14.22.

Case 3. If a particle is located inside a solid sphere having a density ρ that is spherically symmetric but not uniform, then M' in Equation 14.26b is given by an integral of the form $M' = \int \rho\, dV$, where the integration is taken over the volume contained within the sphere of radius r in Figure 14.22. We can evaluate this integral if the radial variation of ρ is given. In this case, we take the volume element dV as the volume of a spherical shell of radius r and thickness dr, and thus $dV = 4\pi r^2\, dr$. For example, if $\rho = Ar$, where A is a constant, it is left to a problem (Problem 63) to show that $M' = \pi A r^4$. Hence, we see from Equation 14.26b that F is proportional to r^2 in this case and is zero at the center.

Quick Quiz 14.4

A particle is projected through a small hole into the interior of a spherical shell. Describe

EXAMPLE 14.10 A Free Ride, Thanks to Gravity

An object of mass m moves in a smooth, straight tunnel dug between two points on the Earth's surface (Fig. 14.23). Show that the object moves with simple harmonic motion, and find the period of its motion. Assume that the Earth's density is uniform.

Solution The gravitational force exerted on the object acts toward the Earth's center and is given by Equation 14.27:

$$\mathbf{F}_g = -\frac{GmM}{R^3}\, r\hat{\mathbf{r}}$$

We receive our first indication that this force should result in simple harmonic motion by comparing it to Hooke's law, first seen in Section 7.3. Because the gravitational force on the object is linearly proportional to the displacement, the object experiences a Hooke's law force.

The y component of the gravitational force on the object is balanced by the normal force exerted by the tunnel wall, and the x component is

$$F_x = -\frac{GmM_E}{R_E^3}\, r\cos\theta$$

Because the x coordinate of the object is $x = r\cos\theta$, we can write

$$F_x = -\frac{GmM_E}{R_E^3}\, x$$

Applying Newton's second law to the motion along the x direction gives

$$F_x = -\frac{GmM_E}{R_E^3}\, x = ma_x$$

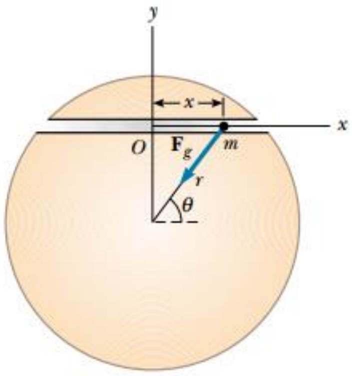


Figure 14.23 An object moves along a tunnel dug through the Earth. The component of the gravitational force $\mathbf{F}_g$ along the x axis is the driving force for the motion. Note that this component always acts toward O.

Solving for a_x, we obtain

$$a_x = -\frac{GM_E}{R_E^{\,3}}\,x$$

If we use the symbol ω^2 for the coefficient of x—$GM_E/R_E^{\,3} = \omega^2$— we see that

$$(1) \qquad a_x = -\omega^2 x$$

an expression that matches the mathematical form of Equation 13.9, which gives the acceleration of a particle in simple harmonic motion: $a_x = -\omega^2 x$. Therefore, Equation (1), which we have derived for the acceleration of our object in the tunnel, is the acceleration equation for simple harmonic motion at angular speed ω with

$$\omega = \sqrt{\frac{GM_E}{R_E^{\,3}}}$$

Thus, the object in the tunnel moves in the same way as a block hanging from a spring! The period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E^{\,3}}{GM_E}}$$

$$= 2\pi\sqrt{\frac{(6.37 \times 10^6\ \text{m})^3}{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}}$$

$$= 5.06 \times 10^3\ \text{s} = \boxed{84.3\ \text{min}}$$

This period is the same as that of a satellite traveling in a circular orbit just above the Earth's surface (ignoring any trees, buildings, or other objects in the way). Note that the result is independent of the length of the tunnel.

A proposal has been made to operate a mass-transit system between any two cities, using the principle described in this example. A one-way trip would take about 42 min. A more precise calculation of the motion must account for the fact that the Earth's density is not uniform. More important, there are many practical problems to consider. For instance, it would be impossible to achieve a frictionless tunnel, and so some auxiliary power source would be required. Can you think of other problems?

the motion of the particle inside the shell.

SUMMARY

Newton's law of universal gravitation states that the gravitational force of attraction between any two particles of masses m_1 and m_2 separated by a distance r has the magnitude

$$F_g = G\frac{m_1 m_2}{r^2} \qquad \textbf{(14.1)}$$

where $G = 6.673 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$ is the universal gravitational constant. This equation enables us to calculate the force of attraction between masses under a wide variety of circumstances.

An object at a distance h above the Earth's surface experiences a gravitational force of magnitude mg', where g' is the free-fall acceleration at that elevation:

$$g' = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2} \qquad \textbf{(14.6)}$$

In this expression, M_E is the mass of the Earth and R_E is its radius. Thus, the weight of an object decreases as the object moves away from the Earth's surface.

Kepler's laws of planetary motion state that

1. All planets move in elliptical orbits with the Sun at one focal point.
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

Kepler's third law can be expressed as

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right)r^3 \tag{14.7}$$

where M_S is the mass of the Sun and r is the orbital radius. For elliptical orbits, Equation 14.7 is valid if r is replaced by the semimajor axis a. Most planets have nearly circular orbits around the Sun.

The **gravitational field** at a point in space equals the gravitational force experienced by any test particle located at that point divided by the mass of the test particle:

$$\mathbf{g} = \frac{\mathbf{F}_g}{m} \tag{14.10}$$

The gravitational force is conservative, and therefore a potential energy function can be defined. The **gravitational potential energy** associated with two particles separated by a distance r is

$$U = -\frac{Gm_1m_2}{r} \tag{14.15}$$

where U is taken to be zero as $r \rightarrow \infty$. The total potential energy for a system of particles is the sum of energies for all pairs of particles, with each pair represented by a term of the form given by Equation 14.15.

If an isolated system consists of a particle of mass m moving with a speed v in the vicinity of a massive body of mass M, the total energy E of the system is the sum of the kinetic and potential energies:

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{r} \tag{14.17}$$

The total energy is a constant of the motion. If the particle moves in a circular orbit of radius r around the massive body and if $M \gg m$, the total energy of the system is

$$E = -\frac{GMm}{2r} \tag{14.19}$$

The total energy is negative for any bound system.

The **escape speed** for an object projected from the surface of the Earth is

$$v_{\text{esc}} = \sqrt{\frac{2GM_E}{R_E}} \tag{14.22}$$