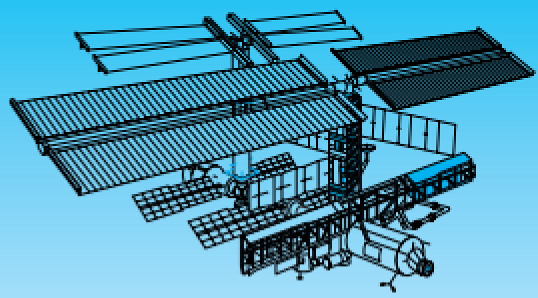
CHAPTER 4



Eigenvalues, Eigenvectors, and Diagonalization

In engineering and physics, problems involving n linear algebraic equations in n independent variables with a constant coefficient matrix $\mathbf{A}$ often arise where a solution vector $\mathbf{x}$ is required to be proportional to $\mathbf{Ax}$. Setting the constant of proportionality equal to λ, this means that $\mathbf{x}$ must be a solution of the equation $\mathbf{Ax} = \lambda\mathbf{x}$ or, equivalently, of the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. The numbers λ_i for which nonzero solutions $\mathbf{x}_i$ exist are called the eigenvalues of matrix $\mathbf{A}$, and the corresponding vectors $\mathbf{x}_i$ are called the eigenvectors of $\mathbf{A}$.

Eigenvalues and eigenvectors arise, for example, when studying vibrational problems, where the eigenvalues represent fundamental frequencies of vibration and the eigenvectors characterize the corresponding fundamental modes of vibration.

They also occur in many other ways; in mechanics, for example, the eigenvalues can represent the principal stresses in a solid body, in which case the eigenvectors then describe the corresponding principal axes of stress caused by the body being subjected to external forces. Also in mechanics, the moment of inertia of a solid body about lines through its center of gravity can be represented by an ellipsoid, with the length of a line drawn from its center to the surface of the ellipsoid proportional to the moment of inertia of the body about an axis through the center of gravity of the body drawn parallel to the line. In this case the eigenvalues represent the principal moments of inertia of the body about the principal axes of inertia, that are then determined by the eigenvectors.

More precisely, if $\mathbf{A}$ is an $n \times n$ matrix, the polynomial $P_n(\lambda)$ of degree n in the scalar λ defined as $P_n(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$ is called the characteristic polynomial of $\mathbf{A}$. The roots of the equation $P_n(\lambda) = 0$ are called the eigenvalues of matrix $\mathbf{A}$, and the column vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ satisfying the matrix equation $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{x}_i = \mathbf{0}$ are called the eigenvectors of matrix $\mathbf{A}$.

This chapter explains how eigenvalues and eigenvectors are determined and establishes important properties of eigenvectors. The eigenvectors of an $n \times n$ matrix $\mathbf{A}$ with n linearly independent eigenvectors are then used to simplify the structure of $\mathbf{A}$ by means of a process called diagonalization. An important application of diagonalization will arise later when considering the solution of linear systems of ordinary differential equations that arise from the study of mechanical, electrical, and chemical reaction problems. Diagonalization is also an important tool when working with partial differential equations, different types of which describe the temperature distribution in a metal, electromagnetic wave propagation, and diffusion processes, to name a few examples.

After a brief discussion of some special $n \times n$ matrices with complex elements, real quadratic forms are defined and the properties of eigenvectors are used to reduce a general quadratic form to a sum of squares. This is a process that finds many different applications, one of which occurs later when classifying the partial differential equations of engineering and physics in order to know the type of auxiliary conditions that must be imposed in order for them to give rise to physically meaningful solutions.

The chapter ends with the introduction of the matrix exponential $e^{\mathbf{A}}$, where $\mathbf{A}$ is a real $n \times n$ matrix, and it is shown how this enters into the solution of a linear first order matrix differential equation of the form $d\mathbf{x}/dt = \mathbf{Ax}$.

4.1 Characteristic Polynomial, Eigenvalues, and Eigenvectors

Throughout this chapter we will be considering the solutions of the homogeneous system of algebraic equations

$$\mathbf{Ax} = \lambda\mathbf{x}, \tag{1}$$

where $\mathbf{A}[a_{ij}]$ is an $n \times n$ matrix, $\mathbf{x}$ is an n element column vector with elements $x_1, x_2, \ldots, x_n$, and λ is a scalar. For $\mathbf{A}$ given we wish to find x and λ. Introducing the $n \times n$ unit matrix by $\mathbf{I}$ allows (1) to be written

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}, \tag{2}$$

showing that $\mathbf{x}$ is a solution of a homogeneous system of equations with the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$. It was seen in Chapter 3 that nontrivial solutions $\mathbf{x}$ of (2) are only possible if one or more rows of the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ are linearly dependent on its remaining rows. This means that nontrivial solutions $\mathbf{x}$ will exist if $\operatorname{rank}(\mathbf{A} - \lambda\mathbf{I}) < n$, but this, in turn, is equivalent to the more convenient condition $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. This is a polynomial equation for λ.

Let $P_n(\lambda)$ be the polynomial of degree n in λ defined by the determinant

$$P_n(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & a_{34} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} - \lambda \end{vmatrix}. \tag{3}$$

Inspection of the determinant defining $P_n(\lambda)$ shows the coefficient of λ^n is $(-1)^n$, so the polynomial is of the form

$$P_n(\lambda) = (-1)^n[\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} - \cdots + c_{n-1}\lambda + c_0]. \tag{4}$$

characteristic polynomial, equation, and eigenvalue

The polynomial $P_n(\lambda)$ is called the **characteristic polynomial** of $\mathbf{A}$ and the associated polynomial equation $P_n(\lambda) = 0$ is the **characteristic equation** of $\mathbf{A}$. As the characteristic equation of $\mathbf{A}$ is of degree n in λ, it will have n roots, some of which may be repeated. The roots of $P_n(\lambda) = 0$, or equivalently the zeros of $P_n(\lambda)$, are called the **eigenvalues** of $\mathbf{A}$ or, sometimes, the **characteristic values** of $\mathbf{A}$.

Eigenvalues (characteristic values) of A

The eigenvalues of an $n \times n$ matrix $\mathbf{A}$ are the n zeros of the polynomial $P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$, or, equivalently, the n roots of the nth degree polynomial equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

In general, a matrix with complex coefficients will have complex eigenvalues, though even when the coefficients of $\mathbf{A}$ are all real it is still possible for complex eigenvalues to arise. This is because then the characteristic equation will have real coefficients, so if complex roots occur they must do so in complex conjugate pairs.

spectrum and spectral radius

If an eigenvalue λ^* is repeated r times, corresponding to the presence of a factor $(\lambda - \lambda^*)^r$ in the characteristic polynomial $P_n(\lambda)$, the number r is called the **algebraic multiplicity** of the eigenvalue λ^*. The set of all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ is called the **spectrum** of $\mathbf{A}$, and the number $R = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$, equal to the largest of the moduli of the eigenvalues, is called the **spectral radius** of $\mathbf{A}$. The name comes from the fact that when the spectrum of $\mathbf{A}$ is plotted as points in the complex plane, they all lie inside or on a circle of radius R centered on the origin.

eigenvectors and eigenvalues

An **eigenvector** of an $n \times n$ matrix $\mathbf{A}$, corresponding to an eigenvalue $\lambda = \lambda_i$, is a nonzero n-element column vector $\mathbf{x}_i$ that satisfies the matrix equation

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

or, equivalently, that is a solution of the homogeneous system of n algebraic equations

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = \mathbf{0}. \tag{5}$$

Eigenvectors of A

The eigenvector $\mathbf{x}_i$ of the $n \times n$ matrix $\mathbf{A}$, corresponding to the eigenvalue $\lambda = \lambda_i$, is a solution of the homogeneous equation $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = \mathbf{0}$.

It is important to recognize that because system (5) is homogeneous, the elements of an eigenvector can only be determined as multiples of one of its nonzero elements as a parameter. This means that if for some choice of the parameter $\mathbf{x}$ is an eigenvalue, then $k\mathbf{x}$ will also be an eigenvalue for any $k \neq 0$.

The next theorem is fundamental to the use of eigenvectors and shows that when an $n \times n$ matrix $\mathbf{A}$ has n distinct (different) eigenvalues, its n eigenvectors form a basis for the vector space associated with the matrix $\mathbf{A}$.

THEOREM 4.1

eigenvectors are linearly independent

Linear independence of eigenvectors The eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$, corresponding to m distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, of an $n \times n$ matrix $\mathbf{A}$, are linearly independent. Furthermore, if $m = n$, the set of eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ forms a basis for the n-dimensional vector space associated with $\mathbf{A}$.

Proof The proof will be by induction, starting with two vectors, and it uses the fact that $\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$ for $i = 1, 2, \ldots, m$.

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ correspond to distinct eigenvalues λ_1 and λ_2, and let constants k_1 and k_2 be such that

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 = \mathbf{0}.$$

Then

$$\mathbf{A}(k_1\mathbf{x}_1 + k_2\mathbf{x}_2) = \mathbf{0},$$

but $\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i$, so this is equivalent to

$$k_1\lambda_1\mathbf{x}_1 + k_2\lambda_2\mathbf{x}_2 = \mathbf{0}.$$

Subtracting λ_2 times the first equation from the last result gives

$$(\lambda_1 - \lambda_2)k_1\mathbf{x}_1 = \mathbf{0}.$$

By hypothesis, $\lambda_1 \neq \lambda_2$, so as $\mathbf{x}_1 \neq \mathbf{0}$ it follows that $k_1 = 0$. Using this result in $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 = \mathbf{0}$ shows that $k_2 = 0$, so we have established the linear independence of $\mathbf{x}_1$ and $\mathbf{x}_2$.

To proceed with an inductive proof we now assume that linear independence has been proved for the first $r-1$ vectors, and show that the rth vector must also be linearly independent. To accomplish this we consider the equation

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_r\mathbf{x}_r = \mathbf{0}.$$

Premultiplying this equation by $\mathbf{A}$ and reasoning as before, we arrive at the result

$$k_1\lambda_1\mathbf{x}_1 + k_2\lambda_2\mathbf{x}_2 + \cdots + k_r\lambda_r\mathbf{x}_r = \mathbf{0}.$$

Subtracting λ_r times the first equation from the last one gives

$$(\lambda_1 - \lambda_r)k_1\mathbf{x}_1 + (\lambda_2 - \lambda_r)k_2\mathbf{x}_2 + \cdots + (\lambda_{r-1} - \lambda_r)k_{r-1}\mathbf{x}_{r-1} = \mathbf{0}.$$

By the inductive hypothesis $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{r-1}$ are linearly independent, so as $\mathbf{x}_r \neq \mathbf{0}$,

$$(\lambda_1 - \lambda_r)k_1 = (\lambda_2 - \lambda_r)k_2 = \cdots = (\lambda_{r-1} - \lambda_r)k_{r-1} = 0.$$

The eigenvalues are distinct, so the last result can only be true if $k_1 = k_2 = \cdots = k_{r-1} = 0$. Thus $k_r = 0$, and so the vector $\mathbf{x}_r$ is linearly independent of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{r-1}$. It has been shown that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, so by induction we conclude that the set of vectors $\mathbf{x}_i$ is linearly independent for $i = 1, 2, \ldots, m$.

A matrix $\mathbf{A}$ can have no more than n linearly independent eigenvectors, so when $m = n$ the set of eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ spans the n-dimensional vector space associated with matrix $\mathbf{A}$ and forms a basis for this space. The proof is complete. ■

algebraic and geometric multiplicity

It can happen that an eigenvalue with **algebraic multiplicity** $r > 1$ only has s different eigenvectors associated with it, where $s < r$, and when this occurs the number s is called the **geometric multiplicity** of the eigenvalue. The set of all eigenvectors associated with an eigenvalue with geometric multiplicity s together with the null vector $\mathbf{0}$ forms what is called the **eigenspace** associated with the eigenvalue. When one or more eigenvalues has a geometric multiplicity that is less than its algebraic multiplicity, it follows directly that the vector space associated with $\mathbf{A}$ must have dimension less than n.

EXAMPLE 4.1

Find the characteristic polynomial, the eigenvalues, and the eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{bmatrix}.$$

Solution The characteristic polynomial $P_3(\lambda)$ is given by

$$P_3(\lambda) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 3 & 2-\lambda & -3 \\ 3 & 1 & -2-\lambda \end{vmatrix},$$

and after expanding the determinant we find that

$$P_3(\lambda) = -\lambda^3 + 2\lambda^2 + \lambda - 2.$$

The characteristic equation $P_3(\lambda) = 0$ is

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0,$$

and inspection shows it has the roots 2, 1, and -1. So the *eigenvalues* of $\mathbf{A}$ are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$, and as these roots are all distinct (there are no repeated roots), each has an algebraic and geometric multiplicity of 1 (each is a single root). The set of numbers $-1, 1, 2$ forms the *spectrum* of matrix $\mathbf{A}$. As the *spectral radius* R of a matrix is defined as the largest of the moduli of the eigenvalues, we see that $R = 2$.

To find the eigenvectors $\mathbf{x}_i$ of $\mathbf{A}$ corresponding to the eigenvalues $\lambda = \lambda_i$, for $i = 1, 2, 3$, it will be necessary to solve the homogeneous system of algebraic equations

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = \mathbf{0} \quad \text{for } i = 1, 2, 3,$$

where $\mathbf{x}_i = [x_1, x_2, x_3]^{\mathrm{T}}$.

Case $\lambda_1 = 2$

The system of equations to be solved is

$$\begin{bmatrix} 2-2 & 1 & -1 \\ 3 & 2-2 & -3 \\ 3 & 1 & -2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and this matrix equation is equivalent to the set of three linear algebraic equations

$$x_2 - x_3 = 0, \quad 3x_1 - 3x_3 = 0, \quad \text{and} \quad 3x_1 + x_2 - 4x_3 = 0.$$

The first two equations are equivalent, so only one of the first two equations and the third equation are linearly independent. Solving the last two equations for x_1 and x_2 in terms of x_3, we find that $x_1 = x_2 = x_3$, so setting $x_3 = k_1$ where k_1 is an arbitrary real number (a parameter) shows that the eigenvector $\mathbf{x}_1$ corresponding to the eigenvalue $\lambda_1 = 2$ is given by

$$\mathbf{x}_1 = \begin{bmatrix} k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

As k_1 is an arbitrary parameter, for convenience we set $k_1 = 1$ and as a result obtain the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Case $\lambda_2 = 1$

This time the system of equations to be solved to find the eigenvector $\mathbf{x}_2$ is

$$\begin{bmatrix} 2-1 & 1 & -1 \\ 3 & 2-1 & -3 \\ 3 & 1 & -2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and this is equivalent to the three linear algebraic equations

$$x_1 + x_2 - x_3 = 0, \quad 3x_1 + x_2 - 3x_3 = 0, \quad \text{and} \quad 3x_1 + x_2 - 3x_3 = 0.$$

The last two equations are identical, so we must solve for x_1, x_2, and x_3 using the first two equations. It is easily seen from these two equations that $x_2 = 0$ and $x_1 = x_3$, so setting $x_1 = k_2$, where k_2 is an arbitrary real number (a parameter), gives

$$\mathbf{x}_2 = k_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Making the arbitrary choice $k_2 = 1$ shows that the eigenvector $\mathbf{x}_2$ corresponding to $\lambda_2 = 1$ is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Case $\lambda_3 = -1$

Setting $\lambda = \lambda_3$, and proceeding as before, shows that the elements of the eigenvector $\mathbf{x}_3$ must satisfy the three equations

$$3x_1 + x_2 - x_3 = 0, \quad 3x_1 + 3x_2 - 3x_3 = 0, \quad \text{and} \quad 3x_1 + x_2 - x_3 = 0,$$

with the solution $x_1 = 0, x_2 = x_3 = k_3$, where k_3 is an arbitrary real number (a parameter). Making the arbitrary choice $k_3 = 1$ allows the eigenvector $\mathbf{x}_3$ to be written as

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

We have shown that matrix $\mathbf{A}$ has the three distinct eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$, corresponding to which there are the three eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

These three eigenvectors form a basis for the three-dimensional vector space associated with $\mathbf{A}$. ■

As the eigenvectors **x** of matrix **A** satisfy the homogeneous equation (2), they can be multiplied by an arbitrary nonzero number K, which is either positive or negative, and still remain an eigenvector. This property is used to *scale* the eigenvectors of **A** to produce what are called **normalized** eigenvectors. This scaling is used in numerical calculations involving the iteration of eigenvectors, because without normalization the elements of **x** may either grow or diminish in absolute value after each stage of the calculation, leading to a progressive loss of accuracy.

Normalization of eigenvectors

a frequently used way of normalizing eigenvectors

Various normalizations are in use. The most common one for eigenvectors with real elements involves scaling the eigenvector so that the square root of the sum of the squares of its elements is 1. So, for example, if

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{the normalizing factor} \quad K = \frac{1}{(a^2+b^2+c^2)^{1/2}} \tag{6}$$

and the normalized eigenvector $\hat{\mathbf{x}}$ becomes

$$\hat{\mathbf{x}} = \begin{bmatrix} a/(a^2+b^2+c^2)^{1/2} \\ b/(a^2+b^2+c^2)^{1/2} \\ c/(a^2+b^2+c^2)^{1/2} \end{bmatrix}. \tag{7}$$

When the eigenvectors in Example 4.1 are normalized in this way, they become

$$\hat{\mathbf{x}}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad \hat{\mathbf{x}}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{x}}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

EXAMPLE 4.2

Find the characteristic polynomial, eigenvalues, and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

Solution The determinant defining the characteristic polynomial is

$$P_4(\lambda) = \begin{vmatrix} -\lambda & 0 & 1 & 1 \\ -1 & 2-\lambda & 0 & 1 \\ -1 & 0 & 2-\lambda & 1 \\ 1 & 0 & -1 & -\lambda \end{vmatrix},$$

and after the determinant is expanded the characteristic equation $P_4(\lambda) = 0$ is found to be

$$P_4(\lambda) = \lambda(\lambda^3 - 4\lambda^2 + 5\lambda - 2) = 0.$$

Clearly, $\lambda = 0$ is a root of $P_4(\lambda) = 0$, and inspection shows the other three roots to be 1, 1, and 2. So the eigenvalues of **A** are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1$, and $\lambda_4 = 2$. In this case $\lambda_2 = \lambda_3 = 1$, so the eigenvalue 1 has algebraic multiplicity 2, and the remaining two eigenvalues each have an algebraic multiplicity of 1. To find the eigenvectors corresponding to these eigenvalues we proceed as in Example 4.1.

Case $\lambda_1 = 0$

Setting $\lambda = \lambda_1 = 0$ in $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ leads to the four equations

$$x_3 + x_4 = 0, \quad -x_1 + 2x_2 + x_4 = 0, \quad -x_1 + 2x_3 + x_4 = 0, \quad \text{and} \quad x_1 - x_3 = 0.$$

Proceeding as before we find that $x_1 = x_2 = x_3 = -x_4$, so solving for x_1, x_2, and x_3 in terms of x_4, and setting $x_4 = 1$ (an arbitrary choice), shows the eigenvector $\mathbf{x}_1$ to be

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Case $\lambda_2 = \lambda_3 = 1$

The eigenvalue 1 has algebraic multiplicity 2, so we must attempt to find two *different* eigenvectors that correspond to the single eigenvalue $\lambda = 1$. Setting $\lambda = 1$ in $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ leads to the four equations

$$-x_1 + x_3 + x_4 = 0, \quad -x_1 + x_2 + x_4 = 0, \quad -x_1 + x_3 + x_4 = 0, \quad x_1 - x_3 - x_4 = 0.$$

The first, third, and fourth equations are identical, so x_1, x_2, x_3, and x_4 must be determined from the two equations

$$-x_1 + x_3 + x_4 = 0 \quad \text{and} \quad -x_1 + x_2 + x_4 = 0.$$

As there are four unknown quantities x_1, x_2, x_3, and x_4, and only two equations relating them, it will only be possible to solve for two of these quantities in terms of the remaining two. The equations show that $x_2 = x_3$ and $x_4 = x_1 - x_3$, so choosing to solve for x_3 and x_4 in terms of x_1 and x_2 by setting $x_1 = \alpha$ and $x_2 = \beta$, with α and β arbitrary constants, shows that the eigenvectors $\mathbf{x}_2$ and $\mathbf{x}_3$ are both of the form

$$\mathbf{x}_{2,3} = \begin{bmatrix} \alpha \\ \beta \\ \beta \\ \alpha - \beta \end{bmatrix}.$$

It is possible to obtain two *different* eigenvectors from this last result by choosing two different pairs of values for the arbitrary parameters α and β. We will define $\mathbf{x}_2$ by setting $\alpha = 1$ and $\beta = 1$, and $\mathbf{x}_3$ by setting $\alpha = 1$ and $\beta = 0$, and as a result we find that

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Had other choices of the parameters α and β been made, two different eigenvectors would have been produced.

Case $\lambda_4 = 2$

Setting $\lambda = \lambda_4 = 2$ in $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ leads to the four equations

$$-2x_1 + x_3 + x_4 = 0, \quad -x_1 + x_4 = 0, \quad -x_1 + x_4 = 0, \quad x_1 - x_3 - 2x_4 = 0.$$

These equations have the solution $x_1 = x_3 = x_4 = 0$, with no condition being imposed on x_2. For simplicity we choose to set $x_2 = 1$ to obtain

$$\mathbf{x}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

In this example, the eigenvalue 1 has algebraic multiplicity 2, and two different eigenvectors can be associated with it, so the geometric multiplicity of the eigenvalue is also 2. The four eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ form a basis for the four-dimensional vector space associated with matrix $\mathbf{A}$.

Had different values been used for α and β, the basis vectors for this vector space would have been different, though the vector space itself would have remained the same because linear combinations of basis vectors will produce an equivalent set of basis vectors.

The *spectrum* of $\mathbf{A}$ is the set of numbers 0, 1, 2, and the *spectral radius* of $\mathbf{A}$ is seen to be $R = 2$. ■

EXAMPLE 4.3 Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has three eigenvalues, but only two linearly independent eigenvectors.

Solution The characteristic polynomial

$$P_3(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix},$$

and after expanding the determinant the characteristic equation $P_3(\lambda) = 0$ becomes

$$P_3(\lambda) = -\lambda(1-\lambda)^2 = 0.$$

The eigenvalue $\lambda_1 = 0$ occurs with algebraic multiplicity 1 and the eigenvalue $\lambda_2 = \lambda_3 = 1$ occurs with algebraic multiplicity 2.

The equations determining the eigenvector $\mathbf{x}_1$, corresponding to the eigenvalue $\lambda = \lambda_1 = 0$, are

$$x_1 + x_2 = 0 \quad \text{and} \quad x_2 = 0,$$

so $x_1 = x_2 = 0$ and x_3 is arbitrary. Setting $x_3 = 1$ gives

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The equations determining $\mathbf{x}_2$ and $\mathbf{x}_3$, corresponding to $\lambda = \lambda_2 = \lambda_3 = 1$, are

$$x_1 = k(\text{arbitrary}) \quad \text{and} \quad x_2 = x_3 = 0,$$

so setting $k = 1$, we find that the eigenvalue $\lambda_2 = \lambda_3 = 1$ with algebraic multiplicity 2 only has associated with it the *single* eigenvector

$$\mathbf{x}_{2,3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

So the algebraic multiplicity of the eigenvalue $\lambda = 1$ is 2, but its geometric multiplicity is 1. The *spectrum* of $\mathbf{A}$ is the set of numbers 0, 1, so the *spectral radius* of $\mathbf{A}$ is $R = 1$.

The eigenvalues of a diagonal matrix can be found immediately, and the corresponding eigenvectors take on a particularly simple form. Let $\mathbf{D}$ be the $n \times n$ diagonal matrix

$$\mathbf{D} = \begin{bmatrix} a_1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & a_2 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & a_n \end{bmatrix},$$

with entries $a_1, a_2, \ldots, a_n$ on its leading diagonal, not all of which are zero, and zeros elsewhere. Then it is easily seen that the eigenvalues of $\mathbf{D}$ are $\lambda_1 = a_1, \lambda_2 = a_2, \ldots, \lambda_n = a_n$. The eigenvector $\mathbf{x}_i$ corresponding to the eigenvalue $\lambda_i = a_i$ becomes an n-element column vector in which only the ith element is nonzero. It is not difficult to show that this result remains true whatever the algebraic multiplicity of an eigenvalue, so *every* diagonal $n \times n$ matrix has n eigenvectors of this form. For convenience, the ith element in $\mathbf{x}_i$ is usually taken to be 1 so, for example, the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -5$, and $\lambda_3 = 4$ and eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly, the diagonal matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

has an eigenvalue $\lambda_1 = -2$ with multiplicity 1 and a double eigenvalue $\lambda_2 = \lambda_3 = 4$ with multiplicity 2, but the matrix still has the three distinct eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

When the degree of the characteristic equation of a matrix exceeds 2, its roots must usually be found by means of a numerical technique. In such circumstances the next theorem provides a simple and useful check for the values of the eigenvalues that have been computed.

THEOREM 4.2

a check on the sum of the eigenvectors

The sum of eigenvalues Let the $n \times n$ matrix $\mathbf{A}[a_{ij}]$ have the n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, which may be either real or complex. Then

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}) = (-1)^{n-1}\mathrm{tr}(\mathbf{A}).$$

Proof As the multiplication of a column of a matrix by a number k is equivalent to multiplication of its determinant by k, we can write

$$P_n(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n \det(\lambda\mathbf{I} - \mathbf{A}).$$

Expanding the determinant on the right in terms of the elements of the first column and separating out the factors that can give rise to the terms in λ^n and λ^{n-1}, we arrive at the result

$$P_n(\lambda) = (-1)^n\{(\lambda - a_{11})(\lambda - a_{22})\cdots(\lambda - a_{nn}) + Q_{n-2}(\lambda)\},$$

where $Q_{n-2}(\lambda)$ is a polynomial in λ of degree $n-2$.

Identifying the coefficients of λ^n and λ^{n-1} in the expression for $P_n(\lambda)$ shows that

$$P_n(\lambda) = (-1)^n\{\lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots + \text{constant} + Q_{n-2}(\lambda)\}.$$

An equivalent expression for $P_n(\lambda)$ can be obtained by expanding it in terms of its factors $(\lambda - \lambda_1), (\lambda - \lambda_2), \ldots, (\lambda - \lambda_n)$ to obtain

$$\begin{aligned} P_n(\lambda) &= (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \\ &= (-1)^n\{\lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + \text{constant}\}. \end{aligned}$$

The statement of the theorem then follows by comparing the coefficients of λ^{n-1} in the two different expressions for $P_n(\lambda)$, where it will be recalled that the **trace** of an $n \times n$ matrix $\mathbf{A}[a_{ij}]$, written $\mathrm{tr}(\mathbf{A})$, is the sum of the elements on its leading diagonal, so that $\mathrm{tr}(\mathbf{A}) = a_{11} + a_{22} + \cdots + a_{nn}$. ■

EXAMPLE 4.4

Use Theorem 4.2 to check the eigenvalues of the matrices in Examples 4.1 and 4.2.

Solution In Example 4.1, $\lambda_1 = 2, \lambda_2 = 1$, and $\lambda_3 = -1$, so $\lambda_1 + \lambda_2 + \lambda_3 = 2$, and $\mathrm{tr}(\mathbf{A}) = 2 + 2 - 2 = 2$, so the result of Theorem 4.2 is verified. Similarly, in Example 4.2, $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1$, and $\lambda_4 = 2$, so $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4$, and $\mathrm{tr}(\mathbf{A}) = 0 + 2 + 2 + 0 = 4$, showing that the result of Theorem 4.2 is again verified. ■

EXAMPLE 4.5

Find the characteristic polynomial, eigenvalues, and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -1-2i & -1-i & 2+2i \\ -4i & -i & 4i \\ -1-3i & -1-i & 2+3i \end{bmatrix},$$

and use Theorem 4.2 to check the eigenvalues.

Solution This matrix has complex elements. Expanding $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ shows that the characteristic polynomial $P_3(\lambda)$ is

$$P_3(\lambda) = \lambda^3 - \lambda^2 + \lambda - 1.$$

Inspection shows the eigenvalues determined by $P_3(\lambda) = 0$ to be $\lambda_1 = 1$, $\lambda_2 = i$, and $\lambda_3 = -i$. Finding the eigenvectors, as in Example 4.1, gives

$$(\lambda_1 = 1)\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad (\lambda_2 = i)\ \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}, \text{ and } (\lambda_3 = -i)\ \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

In this example, although the matrix $\mathbf{A}$ has complex elements, the characteristic polynomial has real coefficients, and one of its zeros (an eigenvalue) is real and its other two zeros (eigenvalues) are complex conjugates. The test in Theorem 4.2 is satisfied because $\text{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(\mathbf{A}) = 1$. ■

Complex eigenvalues arise in numerous applications of matrices, and when this happens it is often useful to have qualitative information about a region in the complex plane that contains all of the eigenvalues, without the necessity of computing their actual values. This form of approach is particularly useful when the coefficients of a polynomial are not specific, and all that is known is that they lie within given intervals or, if complex, that the modulus of each is bounded by a given number.

Another need for this type of information occurs when working with systems of linear differential equations, because it will be seen in Chapter 6 that the roots of a characteristic polynomial equation determine the form of the general solution of a homogeneous system. Roots of the form $\alpha + i\beta$ will be seen to lead to real solutions of the form $e^{\alpha t} \sin \beta t$ and $e^{\alpha t} \cos \beta t$, and these solutions will only remain bounded (stable) as $t \to +\infty$ if the real part of every root is negative. This means that the qualitative knowledge that all of the roots lie to the left of the imaginary axis will be sufficient to ensure that the solution remains finite (is stable) as $t \to +\infty$.

The theorem that follows is the simplest of many similar results that are available, all of which provide information about regions in the complex plane where all of the zeros of a characteristic polynomial are located. Two other results are to be found in the exercise set at the end of this section; the one called the **Routh–Hurwitz stability criterion** is particularly useful when working with systems of linear differential equations.

Although the theorem to be proved in this section identifies a region less precisely than many similar theorems, it has been included to illustrate how such regions can be found, and also because the derivation of the result is elementary. The proof only uses the basic properties of complex numbers extending as far as the triangle inequality.

THEOREM 4.3

finding a region that contains all the eigenvalues

The Gerschgorin circle theorem Let $\mathbf{A}[a_{ij}]$ be an $n \times n$ matrix, and define the circles $C_1, C_2, \ldots, C_n$ in the complex plane such that circle C_r has its center at a_{rr} and the radius

$$\rho_r = \sum_{j=1, j \neq r}^{n} |a_{rj}| = |a_{r1}| + |a_{r2}| + \cdots + |a_{r,r-1}| + |a_{r,r+1}| + \cdots + |a_{rn}|.$$

Then each of the eigenvalues of $\mathbf{A}$ lies in at least one of these circles.

Proof The rth equation of $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is

$$a_{r1}x_1 + \cdots + a_{r,r-1}x_{r-1} + (a_{rr} - \lambda)x_r + a_{r,r+1}x_{r+1} + \cdots + a_{rn}x_n = 0.$$

Solving for $(a_{rr} - \lambda)$, taking the modulus of the result, and making repeated use of the triangle inequality $|a + b| \leq |a| + |b|$, where a and b are arbitrary complex numbers, leads to the inequality

$$|\lambda - a_{rr}| < \sum_{j=1,\, j\neq r}^{n} |a_{rj}||x_j|/|x_r|, \quad \text{for } r = 1, 2, \ldots, n.$$

We now choose x_r to be the element of $\mathbf{x}$ with the largest modulus, so that $|x_j|/|x_r| \leq 1$ for $r = 1, 2, \ldots, n$. The statement of the theorem is obtained from the inequality involving $|\lambda - a_{rr}|$ by replacing each term $|x_j|/|x_r|$ on the right by 1, and then repeating the argument for $r = 1, 2, \ldots, n$. ■

EXAMPLE 4.6 Apply the Gerschgorin circle theorem to Example 4.1.

Solution Circle C_1 has its center at the point $a_{11} = (2, 0)$ and its radius $\rho_1 = |a_{12}| + |a_{13}| = 1 + 1 = 2$. Circle C_2 has its center at the point $a_{22} = (2, 0)$ and its radius $\rho_2 = |a_{21}| + |a_{23}| = 3 + 3 = 6$, while circle C_3 has its center at the point $a_{33} = (-2, 0)$ and its radius $\rho_3 = |a_{31}| + |a_{32}| = 3 + 1 = 4$.

Consequently, the Gerschgorin circle theorem asserts that all the eigenvalues of $\mathbf{A}$ lie in the region of the complex plane enclosed by these three circles. The circles are shown in Fig. 4.1 together with the locations of the three eigenvalues 2, 1, and −1. ■

Physical problems that give rise to matrices with real coefficients often do so in the form of real valued symmetric matrices. These matrices have a number of useful properties that we will examine after first introducing the notions of the *inner product* and *norm* of a matrix vector, and then *orthogonal* and *orthonormal* sets of matrix vectors.

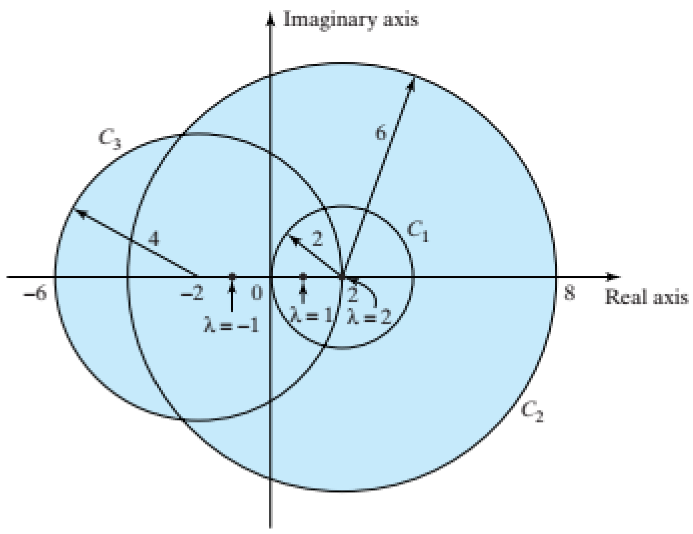


FIGURE 4.1 The Gerschgorin circles for Example 4.1.

inner products, the norm, orthogonal and orthonormal sets of vectors

Inner product of vectors

Let **u** and **v** be two n-element matrix vectors (row or column) with the respective elements $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$. Then their **dot** or **inner product**, denoted here by $\mathbf{u} \cdot \mathbf{v}$ but elsewhere often by $\langle \mathbf{u}, \mathbf{v} \rangle$, is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n. \tag{8}$$

Norm of a vector

The **norm** of an n-element vector **w** (row or column) with elements $w_1, w_2, \ldots, w_n$, written $\|\mathbf{w}\|$, is defined as $(\mathbf{w} \cdot \mathbf{w})^{1/2}$, and so is given by

$$\|\mathbf{w}\| = \left(w_1^2 + w_2^2 + \cdots + w_n^2\right)^{1/2}. \tag{9}$$

We now use the matrix norm to introduce the idea of the *orthogonality* of sets of matrix vectors, and then to show how such sets can be replaced by an equivalent *orthonormal* set of vectors.

Orthogonal and orthonormal sets of vectors

Let $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u}_n$ be a set of n-element vectors (row or column). Then the set is said to be **orthogonal** if

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 0 & \text{for } i \neq j, \\ \|\mathbf{u}_i\|^2 & \text{for } i = j, \end{cases} \tag{10}$$

and to be **orthonormal** if, in addition to being orthogonal, the norm of each vector is 1, so that $\|\mathbf{u}_i\| = 1$ for $i = 1, 2, \ldots, n$. This means that the set of vectors $\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_n$ will form an *orthonormal* set if

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 0 & \text{for } i \neq j, \\ \|\mathbf{u}_i\|^2 = 1 & \text{for } i = j. \end{cases} \tag{11}$$

EXAMPLE 4.7

Given the sets of vectors

(a)

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix},$$

and

(b)

$$\mathbf{u}_1 = [1/4, \sqrt{3}/4, \sqrt{3}/2], \quad \mathbf{u}_2 = [\sqrt{3}/2, -1/2, 0], \quad \mathbf{u}_3 = [\sqrt{3}/4, 3/4, -1/2],$$

show the vectors in set (a) are orthogonal and convert them to an orthonormal set, and that those in set in (b) are orthonormal.

Solution

(a) $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1.2 + 2.1 - 2.2 = 0$ and, similarly, $\mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, and $\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = \sqrt{9} = 3$. So the set is orthogonal but *not* orthonormal, because the vector norms are not all equal to 1. To convert the set into an orthonormal set, it is only necessary to divide each vector by its norm to arrive at the equivalent *orthonormal* set

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}, \quad \hat{\mathbf{u}}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{u}}_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}.$$

(b) Proceeding as in (a) we have $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, showing that the set is orthogonal. However, $\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1$, so the set is also orthonormal. ■

THEOREM 4.4

properties of eigenvalues and eigenvectors of symmetric matrices

Eigenvalues and eigenvectors of a symmetric matrix Let $\mathbf{A}$ be an $n \times n$ real symmetric matrix. Then

(i) the eigenvalues of $\mathbf{A}$ are all real;

(ii) the eigenvectors of $\mathbf{A}$ corresponding to distinct eigenvalues are mutually orthogonal.

Proof We start by observing that if $\mathbf{x}$ and $\mathbf{y}$ are two n-element column vectors the product $\mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ is a scalar, and so is equal to its transpose. Thus, $\mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{x} = (\mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{x})^{\mathrm{T}} = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{y}$, but as $\mathbf{A}$ is symmetric $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$, so that $\mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{y}$.

To prove (i), let λ be an eigenvalue of $\mathbf{A}$ with the corresponding eigenvector $\mathbf{x}$. Then

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

Taking the complex conjugate of this result and using the fact that $\mathbf{A}$ is real valued, so that $\overline{\mathbf{A}} = \mathbf{A}$, gives

$$\mathbf{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}.$$

This shows that $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}$ with the associated eigenvector $\overline{\mathbf{x}}$. If we now premultiply this result by $\mathbf{x}^{\mathrm{T}}$, we obtain the scalar equation

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\overline{\mathbf{x}} = \overline{\lambda}\mathbf{x}^{\mathrm{T}}\overline{\mathbf{x}},$$

but premultiplying the original eigenvalue equation by $\overline{\mathbf{x}}^{\mathrm{T}}$ gives

$$\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}.$$

Using the result $\mathbf{x}^{\mathrm{T}}\mathbf{A}\overline{\mathbf{x}} = \overline{\mathbf{x}}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ then shows that $\lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = \overline{\lambda}\mathbf{x}^{\mathrm{T}}\overline{\mathbf{x}}$, but $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\overline{\mathbf{x}}$ so $\lambda = \overline{\lambda}$, which is only possible if λ is real. This has established the first part of the theorem.

To prove (ii) we must show that if $\mathbf{x}_r$ and $\mathbf{x}_s$ are eigenvectors of $\mathbf{A}$ corresponding to the distinct eigenvalues λ_r and λ_s, with $r \neq s$, then $\mathbf{x}_r \cdot \mathbf{x}_s = 0$, which is equivalent to the condition $\mathbf{x}_r^{\mathrm{T}}\mathbf{x}_s = 0$. The eigenvalues λ_r and λ_s and the corresponding eigenvectors $\mathbf{x}_r$ and $\mathbf{x}_s$ satisfy the equations

$$\mathbf{A}\mathbf{x}_r = \lambda_r\mathbf{x}_r \quad \text{and} \quad \mathbf{A}\mathbf{x}_s = \lambda_s\mathbf{x}_s,$$

from which, after premultiplication by $\mathbf{x}_s^{\mathrm{T}}$ and $\mathbf{x}_r^{\mathrm{T}}$, respectively, we obtain the two scalar equations

$$\mathbf{x}_s^{\mathrm{T}}\mathbf{A}\mathbf{x}_r = \lambda_r \mathbf{x}_s^{\mathrm{T}}\mathbf{x}_r \quad \text{and} \quad \mathbf{x}_r^{\mathrm{T}}\mathbf{A}\mathbf{x}_s = \lambda_s \mathbf{x}_r^{\mathrm{T}}\mathbf{x}_s.$$

Again, using the fact that the transpose of a scalar leaves it unchanged, we see that the preceding results are identical, so subtracting them we arrive at the condition

$$(\lambda_r - \lambda_s)\mathbf{x}_r^{\mathrm{T}}\mathbf{x}_s = 0.$$

As $\lambda_r \neq \lambda_s$ for $r \neq s$, this is only possible if $\mathbf{x}_r^{\mathrm{T}}\mathbf{x}_s = 0$, so the eigenvectors are mutually orthogonal and the proof is complete. ■

It can be shown that even when some of the eigenvalues of a real symmetric $n \times n$ matrix $\mathbf{A}$ are repeated, the matrix $\mathbf{A}$ will still have n linearly independent eigenvectors, though this result will not be proved here. See, for example, references [2.1], [2.5], [2.8], [2.9], and [2.10].

orthogonal matrices and rotations

Orthogonal matrices

An $n \times n$ real matrix $\mathbf{Q}$ will be said to be an **orthogonal** matrix if

$$\boxed{\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}} \tag{12}$$

so, if $\mathbf{Q}$ is an orthogonal matrix, it follows that

$$\boxed{\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{-1}.}$$

When interpreted geometrically in terms of the cartesian geometry of two or three space dimensions, premultiplication of a linear transformation by an orthogonal matrix corresponds to a pure rotation (or a reflection or both; rotation only if det $Q = +1$) in space that preserves the lengths between any two points in space, and also the angles between any two straight lines.

A typical geometrical interpretation of a two-dimensional transformation performed by an orthogonal matrix has already been encountered in Section 3.2(c), where the transformation considered was $\mathbf{x}' = \mathbf{R}\mathbf{x}$, with

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad \mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

When this transformation was considered in Section 3.2(c), the column vector $\mathbf{x}$ represented a point P in the (x, y)-plane with coordinates (x, y), and $\mathbf{x}'$ represented the same point with coordinates (x', y') in the (x', y')-plane, which was obtained by rotating the O$\{x, y\}$ axes counterclockwise through an angle θ about the origin, as shown in Fig. 4.2.

The transformation (interpreted as a mapping of points) shows that every point in the O$\{x', y'\}$ plane experiences the same rotation through an angle θ about the origin. To show that lengths are preserved, let points P_1 and P_2 have coordinates (x_1, y_1) and (x_2, y_2) in the O$\{x, y\}$ plane and their image points P_1' and P_2' have the coordinates (x_1', y_1') and (x_2', y_2') in the O$\{x', y'\}$ plane. Then the square of the distance d between P_1 and P_2 is given by $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$, and the square

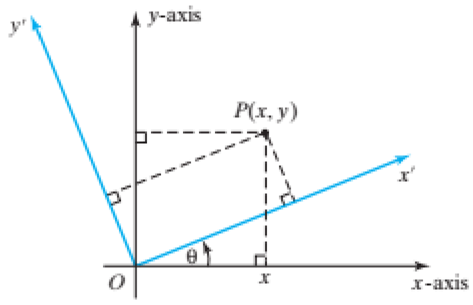


FIGURE 4.2 A rotation of axes about the origin through the angle θ.

of the distance $(d')^2$ between P_1' and P_2' is given by $(d')^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2$. However, from the linear transformation $\mathbf{x}' = \mathbf{R}\mathbf{x}$ we find that

$$x_1 = x_1' \cos\theta - y_1' \sin\theta, \quad x_2 = x_2' \cos\theta - y_2' \sin\theta$$

and

$$y_1 = x_1' \sin\theta + y_1' \cos\theta, \quad y_2 = x_2' \sin\theta + y_2' \cos\theta,$$

from which, after substituting for x_1', x_2', y_1', and y_2', it follows that $(d')^2 = d^2$, showing that distances are preserved. The angles between straight lines in the plane will be preserved because the points on each line will be rotated about the origin through the same angle without changing their distance from the origin.

EXAMPLE 4.8 Show that the matrix

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is orthogonal.

Solution We have

$$\mathbf{R}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

but $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, so $\mathbf{R}$ is orthogonal. ■

THEOREM 4.5 **Properties of orthogonal matrices**

main properties of orthogonal matrices

(i) If $\mathbf{Q}$ is orthogonal then $\det\mathbf{Q} = \pm 1$;
(ii) The product of $n \times n$ orthogonal matrices is an orthogonal matrix;
(iii) The eigenvalues of an orthogonal matrix are all of unit modulus;
(iv) The rows (columns) of an orthogonal matrix form an orthonormal set of vectors.

Proof To prove (i) we start from the fact that $\det\mathbf{Q} = \det\mathbf{Q}^{\mathrm{T}}$. This follows directly from the Laplace expansion of a determinant, because expanding $\det\mathbf{Q}$ in terms of the elements of its ith row is the same as expanding $\det\mathbf{Q}^{\mathrm{T}}$ in terms of the elements of its ith column. From (12), $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{1}$, so as $\det(\mathbf{AB}) = \det\mathbf{A}\det\mathbf{B}$ we can write $\det\mathbf{Q}\det\mathbf{Q}^{\mathrm{T}} = 1$, but $\det\mathbf{Q}^{\mathrm{T}} = \det\mathbf{Q}$ by Theorem 3.4 so $\det\mathbf{Q}\det\mathbf{Q}^{\mathrm{T}} = (\det\mathbf{Q})^2 = 1$, and so $\det\mathbf{Q} = \pm 1$. If $\det \mathbf{Q} = +1$, rotation. If $\det \mathbf{Q} = -1$, rotation plus reflection in general.

Result (ii) follows from the fact that if $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are two $n \times n$ orthogonal matrices, then $(\mathbf{Q}_1\mathbf{Q}_2)^{\mathrm{T}}\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_2^{\mathrm{T}}\mathbf{Q}_1^{\mathrm{T}}\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_2^{\mathrm{T}}\mathbf{Q}_2 = \mathbf{1}$, and the result is established.

The proof of Result (iii) is similar to the proof of (i) in Theorem 4.3. If $\mathbf{Q}$ is real, taking the complex conjugate of $\mathbf{Qx} = \lambda\mathbf{x}$ gives $\mathbf{Q}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$, so taking the transpose of this we find that $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}} = \overline{\lambda}\overline{\mathbf{x}}^{\mathrm{T}}$. Forming the product of these two results gives $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Qx} = \lambda\overline{\lambda}\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}$, but $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$, so $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = \lambda\overline{\lambda}\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}$, showing that $\lambda\overline{\lambda} = 1$. Result (iii) follows from this last result because $\lambda\overline{\lambda} = |\lambda|^2 = 1$.

Finally, Result (iv) follows from the definition of an orthogonal matrix, because $\mathbf{QQ}^{\mathrm{T}} = 1$, and if $\mathbf{u}_i$ is the ith row of $\mathbf{Q}$ and $\mathbf{v}_j$ is the jth column of $\mathbf{Q}^{\mathrm{T}}$ (the jth column of $\mathbf{Q}$), then $\mathbf{u}_i\mathbf{v}_j = 0$ for $i \neq j$, and $\mathbf{u}_i\mathbf{v}_j = 1$ for $i = j$, confirming that the vectors form an orthonormal set. ■

Summary

After definition of the eigenvalues of an $n \times n$ matrix $\mathbf{A}$ in terms of its characteristic polynomial, the associated eigenvectors were defined. An eigenvalue that is repeated r times was said to have the algebraic multiplicity r, and the set of all eigenvalues of $\mathbf{A}$ was called the spectrum of $\mathbf{A}$. The spectral radius of $\mathbf{A}$ was defined in terms of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the number $R = \max\{|\lambda_1|, |\lambda_2|, \ldots |\lambda_n|\}$, and the linear independence of the set of all eigenvectors was established. The most frequently used method of normalizing eigenvectors was introduced, and examples were worked showing how to determine eigenvectors once the eigenvalues are known.

A simple test was given to check the sum of all eigenvalues, and the Gerschgorin circle theorem was proved that determines a region inside which all eigenvalues must lie, though the region determined in this manner is far from optimal. Inner products, the norm, and systems of orthogonal and orthonormal vectors were introduced, and the most important eigenvalue and eigenvector properties of symmetric matrices and orthogonal matrices were derived.

EXERCISES 4.1

In Exercises 1 through 8, find the characteristic polynomial of the given matrix.

1. $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

2. $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$.

4. $\begin{bmatrix} 3 & 1 & 1 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

5. $\begin{bmatrix} -1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

6. $\begin{bmatrix} 4 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 2 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -3 & 3 & 0 \\ -1 & 2 & -1 & -1 \end{bmatrix}$.

8. $\begin{bmatrix} -1 & 1 & 0 & 1 \\ -1 & 2 & -1 & 1 \\ 5 & -3 & 4 & -5 \\ 3 & -2 & 3 & -3 \end{bmatrix}$.

In Exercises 9 through 24 find the eigenvalues and eigenvectors of the given matrix.

9. $\begin{bmatrix} 3 & -2 & 2 \\ 6 & -4 & 6 \\ 2 & -1 & 3 \end{bmatrix}$.

10. $\begin{bmatrix} 3 & -1 & 1 \\ 4 & -1 & 4 \\ 2 & -1 & 4 \end{bmatrix}$.

11. $\begin{bmatrix} -3 & 2 & -2 \\ 4 & -1 & 4 \\ 8 & -4 & 7 \end{bmatrix}$.

12. $\begin{bmatrix} 3 & -2 & 4 \\ -4 & 5 & -4 \\ -4 & 4 & -5 \end{bmatrix}$.

13. $\begin{bmatrix} -5 & 4 & -1 \\ -3 & 2 & -1 \\ 6 & -4 & 2 \end{bmatrix}$.

14. $\begin{bmatrix} 0 & 1 & -2 \\ 2 & -1 & 2 \\ 2 & -2 & 4 \end{bmatrix}$.

15. $\begin{bmatrix} -5 & 8 & 1 \\ -3 & 6 & 1 \\ 6 & -8 & 0 \end{bmatrix}$.

16. $\begin{bmatrix} -1 & 0 & -2 \\ -1 & 2 & -1 \\ 4 & 0 & 5 \end{bmatrix}$.

17. $\begin{bmatrix} -1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

18. $\begin{bmatrix} 6 & 0 & 4 \\ 3 & 1 & 3 \\ -8 & 0 & -6 \end{bmatrix}$.

19. $\begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix}$.

20. $\begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ -4 & 0 & 2 \end{bmatrix}$.

21. $\begin{bmatrix} 4 & 0 & -4 \\ 2 & 2 & -4 \\ 2 & 0 & -2 \end{bmatrix}$.

22. $\begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 0 \end{bmatrix}$.

23. $\begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 \\ 1 & 3 & -1 & -1 \\ -2 & 2 & -2 & 1 \end{bmatrix}$.

24. $\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & -2 & 0 & -1 \\ -3 & 3 & 0 & 2 \end{bmatrix}$.

25. Prove that the eigenvalues of upper and lower triangular matrices are equal to the elements on the leading diagonal. Show by example that, unlike the case of diagonal matrices, an eigenvalue of an upper or lower triangular matrix with algebraic multiplicity r has fewer than r eigenvectors.

26. Apply the Gerschgorin circle theorem to one or more of the matrices in Exercises 9 through 24 to verify that the eigenvalues lie within or on the circles determined by the theorem.

27. It can be shown that all the zeros of the polynomial

$$P_n(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n, \quad a_n \neq 0,$$

lie in the circle

$$|\lambda| < 1 + \max\left|\frac{a_k}{a_n}\right|, \quad k = 0, 1, 2, \ldots, n-1.$$

Verify this result by applying it to one or more of the characteristic equations associated with the matrices in Exercises 9 through 24.

The Routh–Hurwitz stability criterion

Let the real polynomial $P_n(\lambda)$ be given by

$$P_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$$

and form the determinants

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \ldots,$$

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \ldots & a_{2n-1} \\ 1 & a_2 & a_4 & \ldots & a_{2n-2} \\ 0 & a_1 & a_3 & \ldots & a_{2n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & a_n \end{vmatrix} \quad \text{with } a_k = 0 \text{ for } k > n.$$

Then, $\Delta_r > 0$ for $r = 1, 2, \ldots, n$, if and only if every zero of $P_n(\lambda)$ has a negative real part.

28.

(a) Numerical computation shows that the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 5 \\ 2 & 3 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

has the eigenvalues 5.7238, $-1.3619 + 1.9328i$, and $-1.3619 - 1.9328i$. Apply the Routh–Hurwitz stability criterion to confirm that not every zero of the characteristic polynomial has a negative real part.

(b) Numerical computation shows that the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & -2 & -3 \\ 3 & -1 & 0 \\ -4 & 0 & -3 \end{bmatrix}$$

has the eigenvalues -5.4873, $-0.2563 - 1.4564i$, and $-0.2563 + 1.4564i$. Apply the Routh–Hurwitz stability criterion to confirm that every zero of the characteristic polynomial has a negative real part.

An $n \times n$ matrix $\mathbf{A}$ is said to be **similar** to an $n \times n$ matrix $\mathbf{B}$ if there exists a nonsingular $n \times n$ matrix $\mathbf{M}$ such that $\mathbf{B} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$. The relationship between $\mathbf{A}$ and $\mathbf{B}$ is said to constitute a **similarity transformation** between the two matrices.

29. If $\mathbf{A}$ and $\mathbf{B}$ are similar, show that $\det\mathbf{A} = \det\mathbf{B}$, and by substituting $\mathbf{B} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ in $\det\mathbf{B}$ and expanding the result, show that similar matrices have the same eigenvalues.

30. Verify the result of Exercise 29 by direct calculation by using

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -1 \\ 4 & 0 & -1 \\ 4 & -2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

to show that both $\mathbf{A}$ and $\mathbf{B}$ have the eigenvalues -1, 2, and 3.

31. Let the $n \times n$ elementary matrix $\mathbf{E}$ be obtained from the unit matrix $\mathbf{I}$ by interchanging its ith and jth rows (columns). By considering the product $\mathbf{EQ}$, where $\mathbf{Q}$ is an $n \times n$ orthogonal matrix, prove that an orthogonal matrix remains orthogonal when its rows (columns) are interchanged.

4.2 Diagonalization of Matrices

Our purpose in this section will be to examine the possibility of diagonalizing an $n \times n$ matrix $\mathbf{A}$. The reason for this is to try to simplify the structure of $\mathbf{A}$ so that, in some ways, it reflects the simple properties of a diagonal matrix. Diagonalization finds many applications, some of which will be discussed later.

diagonal matrix

Let $\mathbf{D}$ be the general $n \times n$ diagonal matrix

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}. \tag{13}$$

Then, as already seen in Section 4.1, the eigenvalues of $\mathbf{D}$ are the entries $\lambda_1, \lambda_2, \dots, \lambda_n$ on its leading diagonal, and the corresponding n linearly independent eigenvectors can be taken to be

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \dots, \mathbf{x}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}. \tag{14}$$

The rule for matrix multiplication shows that

$$\mathbf{D}^m = \begin{bmatrix} \lambda_1^m & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^m & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_n^m \end{bmatrix}, \tag{15}$$

for any positive integer m, so $\mathbf{D}^m$ is easily computed and will have the same set of eigenvectors as $\mathbf{D}$, though its eigenvalues will be $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$.

In addition to these properties, it is obvious that $\det\mathbf{D} = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$, so $\mathbf{D}$ will be nonsingular provided no entry on its leading diagonal is zero. As a result, when $\mathbf{D}$ is nonsingular, the rule for matrix multiplication shows that $\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$, where

$$\mathbf{D}^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & 1/\lambda_n \end{bmatrix}. \tag{16}$$

We now state and prove the fundamental theorem on the diagonalization of $n \times n$ matrices.

THEOREM 4.6

how to diagonalize a matrix

Diagonalization of an $n \times n$ matrix Let the $n \times n$ matrix $\mathbf{A}$ have n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, not all of which need be distinct, and let there be n corresponding distinct eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, so that

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \quad i = 1, 2, \dots, n.$$

Define the matrix $\mathbf{P}$ to be the $n \times n$ matrix in which the ith column is the eigenvector $\mathbf{x}_i$, with $i = 1, 2, \ldots, n$, so that in partitioned form $\mathbf{P} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n]$, and let $\mathbf{D}$ be the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix},$$

where the eigenvalue λ_i is in the ith position in the ith row. Then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}.$$

Proof Consider the product $\mathbf{B} = \mathbf{A}\mathbf{P}$. Then, by expressing $\mathbf{P}$ in partitioned form, we can write $\mathbf{B}$ as

$$\mathbf{B} = [\mathbf{A}\mathbf{x}_1 \quad \mathbf{A}\mathbf{x}_2 \quad \ldots \quad \mathbf{A}\mathbf{x}_n].$$

Using the fact that $\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$ allows this to be rewritten as

$$\mathbf{B} = [\lambda_1\mathbf{x}_1 \quad \lambda_2\mathbf{x}_2 \quad \ldots \quad \lambda_n\mathbf{x}_n] = \mathbf{P}\mathbf{D},$$

showing that

$$\mathbf{P}\mathbf{D} = \mathbf{A}\mathbf{P}.$$

As the columns of $\mathbf{P}$ are linearly independent, $\mathbf{P}$ is nonsingular, so $\mathbf{P}^{-1}$ exists and we can premultiply by $\mathbf{P}^{-1}$ to obtain

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P},$$

and the theorem is proved. ∎

General Remarks About Diagonalization

(i) An $n \times n$ matrix can be diagonalized provided it possesses n linearly independent eigenvectors.

(ii) A symmetric matrix can always be diagonalized.

(iii) The diagonalizing matrix for a real $n \times n$ matrix $\mathbf{A}$ may contain complex elements. This is because although the characteristic polynomial of $\mathbf{A}$ has real coefficients, its zeros either will be real or will occur in complex conjugate pairs.

(iv) A diagonalizing matrix is not unique, because its form depends on the order in which the eigenvectors of $\mathbf{A}$ are used to form its columns.

A useful consequence of the diagonalized form of a matrix is that it enables it to be raised to a positive integral power with the minimum of effort. This property will be used later when the matrix exponential is introduced.

To see the ease with which an $n \times n$ matrix can be raised to a power when it is diagonalizable, we start by writing $\mathbf{A}$ in the form $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. We then have

$$\mathbf{A}^2 = (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}) = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}^2\mathbf{P}^{-1},$$

so that, in general,

$$\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}, \quad \text{for } m = 1, 2, \ldots.$$

As evaluating $\mathbf{D}^m$ simply involves raising each entry on its leading diagonal to the power m, the evaluation of $\mathbf{A}^m$ only involves three matrix multiplications.

This last result was used without justification in Section 3.2(f) when a stochastic matrix was raised to the power m (do not confuse the stochastic matrix $\mathbf{P}$ in that section with the orthogonalizing matrix $\mathbf{P}$ just defined).

EXAMPLE 4.9 Diagonalize the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{bmatrix},$$

and use the result to find $\mathbf{A}^5$.

Solution Matrix $\mathbf{A}$ was examined in Example 4.1 and shown to have the eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$, and the corresponding eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Theorem 4.5 shows that a diagonalizing matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and a routine calculation shows that

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Before finding $\mathbf{A}^5$, and although it is unnecessary for what is to follow, it is instructive to check that when the matrix $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is formed, the eigenvalues appearing in the diagonal matrix $\mathbf{D}$ do so in the order in which the corresponding eigenvectors of $\mathbf{A}$ have been used to form the columns of $\mathbf{P}$. This is seen to be so in this case because

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Returning to the calculation of $\mathbf{A}^5$ and using the expressions for $\mathbf{P}$, $\mathbf{P}^{-1}$, and $\mathbf{D}$ in $\mathbf{A}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$ gives

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 1^5 & 0 \\ 0 & 0 & (-1)^5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 32 & 31 & -31 \\ 33 & 32 & -33 \\ 33 & 31 & -32 \end{bmatrix}.$$

Had the eigenvectors been arranged in a different order when constructing $\mathbf{P}$, a different but equivalent diagonal matrix would have been obtained. For example, if $\mathbf{P}$ had been written

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$\mathbf{D}$ would have become

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

though after $\mathbf{P}^{-1}$ was found and $\mathbf{A}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$ was computed, the matrix $\mathbf{A}^5$ would, of course, remain the same.

EXAMPLE 4.10 Diagonalize the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

Solution Matrix $\mathbf{A}$ was considered in Example 4.2, which showed that it had the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1$, and $\lambda_4 = 2$, and that although the eigenvalue 1 occurred with algebraic multiplicity 2, the matrix still had the four linearly independent eigenvectors

$$(\lambda_1 = 0) \quad \mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad (\lambda_2 = 1) \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad (\lambda_3 = 1) \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and

$$(\lambda_4 = 2) \quad \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Using these eigenvectors to form $\mathbf{P}$ gives

$$\mathbf{P} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

from which it follows that

$$\mathbf{P}^{-1} = \begin{bmatrix} -1 & 0 & 1 & 1 \\ -1 & 0 & 2 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

Because of the ordering of the eigenvectors, the diagonal matrix **D** will be

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

where

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}.$$

We saw in Theorem 4.4 that a real symmetric $n \times n$ matrix **A** with distinct eigenvalues has a set of n mutually orthogonal linearly independent eigenvectors. It follows at once that if when constructing the diagonalizing matrix for **A** the normalized eigenvectors of **A** are used to form the columns of **P**, the resulting diagonalizing matrix will be an *orthogonal* matrix. This is often advantageous, because the properties of orthogonal matrices can simplify subsequent calculations that may arise. However, if an eigenvalue is repeated, the corresponding eigenvectors will not, in general, be orthogonal to the other eigenvectors, so although there will still be a set of n linearly independent eigenvectors, the set will no longer form an orthogonal set.

Because of the frequency with which symmetric matrices arise in applications, and the fact that symmetric matrices with repeated eigenvalues are not unusual, it is reasonable to ask if it is possible for symmetric matrices always to be diagonalized by an orthogonal matrix and, if so, how this can be achieved. The answer to the question about the possibility of diagonalization by an orthogonal matrix is in the affirmative. The method of arriving at an orthonormal set of vectors to be used when constructing **P** involves using a generalization of the Gram–Schmidt orthogonalization process introduced in Section 2.7 in the context of geometrical vectors in R^3.

As an n element matrix vector is simply a vector in a vector space, an extension of the Gram–Schmidt orthogonalization process to include n-element matrix vectors can be used to construct an *orthonormal* set of n vectors from any set of n linearly independent eigenvectors that are always associated with an $n \times n$ symmetric matrix **A**. The required generalization of the orthogonalization process that leads to an **orthonormal system** is an immediate extension of the one derived in Section 2.7, so the details of its derivation will be omitted.

orthogonalization of a set of linearly independent vectors

Rule for the Gram–Schmidt orthogonalization process for matrix vectors

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a set of n element linearly independent nonorthogonal matrix column vectors. Then an equivalent **orthonormal set** of vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ can be constructed from the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, via an intermediate set of orthogonal nonnormalized vectors $\mathbf{v}_2, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The steps involved in the determination of the vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are as follows:

$$\begin{aligned}
&\mathbf{p}_1 = \mathbf{x}_1/\|\mathbf{x}_1\|,\\
&\mathbf{v}_2 = \mathbf{x}_2 - (\mathbf{p}_1 \cdot \mathbf{x}_2)\mathbf{p}_1,\\
&\mathbf{p}_2 = \mathbf{v}_2/\|\mathbf{v}_2\|,\\
&\mathbf{v}_r = \mathbf{x}_r - \{(\mathbf{p}_1 \cdot \mathbf{x}_r)\mathbf{p}_1 + (\mathbf{p}_2 \cdot \mathbf{x}_r)\mathbf{p}_2 + \cdots + (\mathbf{p}_{r-1} \cdot \mathbf{x}_r)\mathbf{p}_{r-1}\}\\
&\mathbf{p}_r = \mathbf{v}_r/\|\mathbf{v}_r\|, \quad \text{for } r = 2, 3, \ldots, n.
\end{aligned}$$

When the Gram–Schmidt orthogonalization process is applied to the eigenvectors of a real symmetric matrix $\mathbf{A}$ with repeated eigenvalues, the diagonalizing matrix $\mathbf{P}$ is constructed by using the vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, obtained from the preceding scheme after starting with any linearly independent set of eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ of $\mathbf{A}$. Then, in partitioned form,

$$\mathbf{P} = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \ldots \quad \mathbf{p}_n]$$

and, as before,

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P},$$

where $\mathbf{D}$ is again a diagonal matrix with its diagonal elements equal to the eigenvalues of $\mathbf{A}$ arranged in the same order as the corresponding columns of $\mathbf{P}$. This time, however, entries on the leading diagonal will be repeated as many times as the multiplicity of the eigenvalues concerned.

EXAMPLE 4.11

Use the Gram–Schmidt orthogonalization process to construct an orthonormal set of vectors from the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Solution In this case the Gram–Schmidt orthogonalization process involves the three vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, so a set of orthonormal vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ is given by the scheme

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{x}_1/\|\mathbf{x}_1\| \\
\mathbf{v}_2 &= \mathbf{x}_2 - (\mathbf{p}_1 \cdot \mathbf{x}_2)\mathbf{p}_1 \\
\mathbf{p}_2 &= \mathbf{v}_2/\|\mathbf{v}_2\| \\
\mathbf{v}_3 &= \mathbf{x}_3 - \{(\mathbf{p}_1 \cdot \mathbf{x}_3)\mathbf{p}_1 + (\mathbf{p}_2 \cdot \mathbf{x}_3)\mathbf{p}_2\} \\
\mathbf{p}_3 &= \mathbf{v}_3/\|\mathbf{v}_3\|.
\end{aligned}$$

A series of straightforward calculations gives

$$\mathbf{p}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 0\,\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \text{so } \mathbf{p}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix},$$

and, finally,

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \sqrt{3}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} - 1/\sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix},$$

so

$$\mathbf{p}_3 = \begin{bmatrix} -1/\sqrt{6} \\ \sqrt{(2/3)} \\ -1/\sqrt{6} \end{bmatrix}.$$

■

EXAMPLE 4.12 Construct an orthogonal diagonalizing matrix for the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

Solution This has the *distinct* eigenvalues $\lambda_1 = -1, \lambda_2 = 3$, and $\lambda_1 = 4$, so the corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are orthogonal. Simple calculations show that

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The normalized eigenvectors are

$$\hat{\mathbf{x}}_1 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \hat{\mathbf{x}}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{x}}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

so the diagonalizing matrix $\mathbf{P}$ and the corresponding diagonal matrix $\mathbf{D}$ are

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

EXAMPLE 4.13 Construct an orthogonal diagonalizing matrix for the real symmetric matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 4 \\ 2 & 2 & -2 \\ 4 & -2 & -1 \end{bmatrix}.$$

Solution This has the eigenvalues $\lambda_1 = -6, \lambda_2 = 3$, and $\lambda_3 = 3$, so as the eigenvalue 3 has multiplicity 2, the corresponding set of eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ will *not* be orthogonal. The eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are easily shown to be

$$\mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

Applying the Gram–Schmidt orthogonalization process to vectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$, as in Example 4.11, after some straightforward calculations we arrive at the orthonormal set

$$\mathbf{p}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{p}_3 = \begin{bmatrix} 4/(3\sqrt{5}) \\ -2/(3\sqrt{5}) \\ \sqrt{5}/3 \end{bmatrix}.$$

In this case an orthogonal diagonalizing matrix is

$$\mathbf{P} = \begin{bmatrix} -2/3 & 1/\sqrt{5} & 4/(3\sqrt{5}) \\ 1/3 & 2/\sqrt{5} & -2/(3\sqrt{5}) \\ 2/3 & 0 & \sqrt{5}/3 \end{bmatrix},$$

and the corresponding diagonal matrix is

$$\mathbf{D} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

To close this section we state the important Cayley–Hamilton theorem, which is true for *all* square matrices, though before considering the theorem we first define a matrix polynomial.

A **matrix polynomial** involving an $n \times n$ matrix $\mathbf{A}$ is an expression of the form

$$\mathbf{A}^m + b_1\mathbf{A}^{m-1} + b_2\mathbf{A}^{m-2} + \cdots + b_{m-1}\mathbf{A} + b_m\mathbf{I},$$

in which m is an integer and $b_1, b_2, \ldots, b_m$ are real or complex numbers.

THEOREM 4.7

a matrix satisfies its own characteristic equation

The Cayley–Hamilton theorem Let $P_n(\lambda)$ be the characteristic polynomial of an arbitrary $n \times n$ square matrix $\mathbf{A}$. Then $\mathbf{A}$ satisfies its own characteristic equation, and so is a solution of the matrix polynomial equation $P_n(\mathbf{A}) = \mathbf{0}$.

Proof For simplicity, we only prove the theorem for real symmetric matrices, though it is true for every $n \times n$ matrix. If $\mathbf{A}$ is a real $n \times n$ symmetric matrix, then from Theorem 4.6 we may write $\mathbf{A} = \mathbf{PDP}^{-1}$. Let the characteristic polynomial of $\mathbf{A}$ be

$$P_n(\lambda) = (-1)^n\{\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n\}.$$

Then replacing λ by $\mathbf{A}$ converts $P_n(\lambda)$ to the matrix polynomial

$$P_n(\mathbf{A}) = (-1)^n\{\mathbf{A}^n + c_1\mathbf{A}^{n-1} + \cdots + c_{n-1}\mathbf{A} + c_n\mathbf{I}\},$$

but $\mathbf{A}^r = \mathbf{PD}^r\mathbf{P}^{-1}$, so

$$P_n(\mathbf{A}) = (-1)^n\{\mathbf{P}\{\mathbf{D}^n + c_1\mathbf{D}^{n-1} + \cdots + c_{n-1}\mathbf{D} + c_n\mathbf{I}_n\}\mathbf{P}^{-1}\}.$$

The ith row of the matrix polynomial $\mathbf{D}^n + c_1\mathbf{D}^{n-1} + \cdots + c_{n-1}\mathbf{D} + c_n\mathbf{I}$ is simply $\lambda_i^n + c_1\lambda_i^{n-1} + \cdots + c_{n-1}\lambda_i + c_n$, but this is $P_n(\lambda_i)$, and it must vanish for $i = 1, 2, \ldots, n$ because λ_i is an eigenvalue of $\mathbf{A}$. Thus, $\mathbf{D}^n + c_1\mathbf{D}^{n-1} + \cdots + c_{n-1}\mathbf{D} + c_n\mathbf{I} = \mathbf{0}$, showing that $P_n(\mathbf{A}) = \mathbf{P}\{\mathbf{0}\}\mathbf{P}^{-1} = \mathbf{0}$, and the result is proved. ■

EXAMPLE 4.14

Verify the Cayley–Hamilton theorem for the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}.$$

Solution The characteristic polynomial is $P_2(\lambda) = \lambda^2 - 4\lambda - 1$, and

$$\mathbf{A}^2 = \begin{bmatrix} 9 & 4 \\ 20 & 9 \end{bmatrix}, \quad \text{so } P_2(\mathbf{A}) = \begin{bmatrix} 9 & 4 \\ 20 & 9 \end{bmatrix} - 4\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Finding $\mathbf{A}^{-1}$ from the Cayley–Hamilton theorem

If the $n \times n$ matrix $\mathbf{A}$ is nonsingular, the following interesting result can be obtained directly from the Cayley–Hamilton theorem. Let the characteristic polynomial of $\mathbf{A}$ be $P_n(\lambda) = (-1)^n\{\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n\}$, so from Theorem 4.7

$$\mathbf{A}^n + c_1\mathbf{A}^{n-1} + \cdots + c_{n-1}\mathbf{A} + c_n\mathbf{I} = 0.$$

The matrix $\mathbf{A}^{-1}$ exists because by hypothesis $\mathbf{A}$ is nonsingular, so premultiplication of the preceding equation by $\mathbf{A}^{-1}$, followed by a rearrangement of terms, allows $\mathbf{A}^{-1}$ to be expressed in terms of powers of $\mathbf{A}$ through the result

$$\mathbf{A}^{-1} = (-1/c_n)\{\mathbf{A}^{n-1} + c_1\mathbf{A}^{n-2} + \cdots + c_{n-1}\mathbf{I}\}. \tag{17}$$

EXAMPLE 4.15 Use the result of equation (17) to find $\mathbf{A}^{-1}$ for the nonsingular matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}.$$

Solution Matrix $\mathbf{A}$ was considered in Example 4.14, where it was found that the characteristic polynomial $P_2(\lambda) = \lambda^2 - 4\lambda - 1$, so in terms of (17) we see that $c_1 = -4$ and $c_2 = -1$. Thus,

$$\mathbf{A}^{-1} = -1/(-1)\left\{\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}.$$ ■

Summary

This section has described how an $n \times n$ matrix can be diagonalized when it possesses n linearly independent eigenvectors. The diagonalization was shown not to be unique, since its form depends on the order in which the eigenvectors are used to construct the diagonalizing matrix $\mathbf{P}$.

Sometimes, when a linearly independent set of n vectors has been obtained, it is desirable to replace it by an equivalent set of n orthogonal or orthonormal vectors. The section closed by showing how this can be accomplished by means of the Gram–Schmidt orthogonalization procedure.

EXERCISES 4.2

In Exercises 1 through 12, find a diagonalizing matrix $\mathbf{P}$ for the given matrix, in each case using the fact that the zeros of the characteristic polynomial are small integers that can be found by trial and error.

1. $\begin{bmatrix} -2 & -3 & -1 \\ 1 & 2 & 1 \\ 3 & 3 & 2 \end{bmatrix}$.

2. $\begin{bmatrix} 3 & 1 & 4 \\ -4 & -2 & -4 \\ -1 & -1 & 2 \end{bmatrix}$.

3. $\begin{bmatrix} 3 & 1 & -2 \\ 6 & 2 & -6 \\ 4 & 1 & -3 \end{bmatrix}$.

4. $\begin{bmatrix} -6 & -10 & -4 \\ 2 & 3 & 2 \\ 7 & 10 & 5 \end{bmatrix}$.

5. $\begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$.

6. $\begin{bmatrix} 14 & 2 & 8 \\ -8 & -3 & -4 \\ -26 & -4 & -15 \end{bmatrix}$.

7. $\begin{bmatrix} 5 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$.

8. $\begin{bmatrix} 12 & 4 & 6 \\ -6 & -2 & -3 \\ -22 & -8 & -11 \end{bmatrix}$.

9. $\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$.

10. $\begin{bmatrix} 12 & -4 & 8 \\ -6 & 2 & -4 \\ -20 & 8 & -14 \end{bmatrix}$.

11. $\begin{bmatrix} -6 & 2 & -4 \\ -4 & 0 & -4 \\ 4 & -2 & 2 \end{bmatrix}$.

12. $\begin{bmatrix} -7 & 0 & -6 \\ 3 & -1 & 3 \\ 9 & 0 & 8 \end{bmatrix}$.

In Exercises 13 through 16 use the Gram–Schmidt orthogonalization process with the given set of vectors to find (a) an equivalent set of orthogonal vectors and (b) an orthonormal set.

13. $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}.$

14. $\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}.$

15. $\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-2\\2\end{bmatrix}.$

16. $\begin{bmatrix}-1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}.$

In Exercises 17 through 22 find an orthogonal diagonalizing matrix **P** for the given symmetric matrix.

17. $\begin{bmatrix}3&0&0\\0&3&1\\0&1&3\end{bmatrix}.$

18. $\begin{bmatrix}5&1&0\\1&5&0\\0&0&2\end{bmatrix}.$

19. $\begin{bmatrix}4&1&0\\1&4&0\\0&0&3\end{bmatrix}.$

20. $\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}.$

21. $\begin{bmatrix}4&2&0\\2&4&0\\0&0&2\end{bmatrix}.$

22. $\begin{bmatrix}4&1&1\\1&4&1\\1&1&4\end{bmatrix}.$

23. Verify by direct calculation that the matrix in Exercise 1 satisfies the Cayley–Hamilton theorem.

24. Verify by direct calculation that the matrix in Exercise 7 satisfies the Cayley–Hamilton theorem.

In Exercises 25 through 28 use (17) to find $\mathbf{A}^{-1}$ and check the result by showing that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

25. $\mathbf{A} = \begin{bmatrix}2&3\\-1&4\end{bmatrix}.$

26. $\mathbf{A} = \begin{bmatrix}5&1\\3&-2\end{bmatrix}.$

27. $\mathbf{A} = \begin{bmatrix}2&1&0\\-2&1&2\\0&-1&-2\end{bmatrix}.$

28. $\mathbf{A} = \begin{bmatrix}1&0&2\\3&1&0\\0&2&4\end{bmatrix}.$

4.3 Special Matrices with Complex Elements

In the previous section it was seen that one way in which matrices with complex elements can occur is when the eigenvectors of an arbitrary $n \times n$ matrix are used to construct a diagonalizing matrix. This is not the only reason for considering $n \times n$ matrices with complex elements, because the following three special types of matrices arise naturally in applications of mathematics to physics and engineering, and elsewhere.

Hermitian, skew-Hermitian, and unitary matrices

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix with possibly complex elements. Then:

> $\mathbf{A}$ is called an **Hermitian** matrix if $\overline{\mathbf{A}}^{\mathrm{T}} = \mathbf{A}$, so that $\overline{a}_{kj} = a_{jk}$;
>
> $\mathbf{A}$ is called a **skew-Hermitian** matrix if $\overline{\mathbf{A}}^{\mathrm{T}} = -\mathbf{A}$, so that $\overline{a}_{kj} = -a_{jk}$;
>
> $\mathbf{U}$ is called a **unitary** matrix if $\overline{\mathbf{U}}^{\mathrm{T}} = \mathbf{U}^{-1}$.

The basic properties of these three types of matrices follow almost directly from their definitions.

Basic Properties of Hermitian, Skew-Hermitian, and Unitary Matrices

1. The elements on the leading diagonal of an Hermitian matrix are real, because $\overline{a}_{ii} = a_{ii}$, and this is only possible if a_{ii} is real.
2. The elements on the leading diagonal of a skew-Hermitian matrix are either purely imaginary or 0. This follows from the fact that $\overline{a}_{ii} = -a_{ii}$, so the real part of a_{ii} must equal its negative, and this is only possible if a_{ii} is purely imaginary or 0.

3. If the elements of an Hermitian matrix are real, then the matrix is a real symmetric matrix, because then $\overline{\mathbf{A}}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}$, and the definition of an Hermitian matrix reduces to the definition of a real symmetric matrix.
4. If the elements of a skew-Hermitian matrix are real, then the matrix is a skew-symmetric matrix, because then the definition of a skew-Hermitian matrix reduces to the definition of a skew-symmetric matrix.
5. Any $n \times n$ matrix $\mathbf{A}$ of the form $\mathbf{A} = \mathbf{B} + i\mathbf{C}$, where $\mathbf{B}$ is a real symmetric matrix and $\mathbf{C}$ is a real skew-symmetric matrix, is an Hermitian matrix. This follows directly from Properties 3 and 4.
6. Any $n \times n$ matrix $\mathbf{A}$ can be written in the form $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where $\mathbf{B}$ is Hermitian and $\mathbf{C}$ is a skew-Hermitian. To see this we write $\mathbf{A} = (1/2)(\mathbf{A} + \overline{\mathbf{A}}^{\mathrm{T}}) + (1/2)(\mathbf{A} - \overline{\mathbf{A}}^{\mathrm{T}})$, and then set $\mathbf{B} = (1/2)(\mathbf{A} + \overline{\mathbf{A}}^{\mathrm{T}})$ and $\mathbf{C} = (1/2)(\mathbf{A} - \overline{\mathbf{A}}^{\mathrm{T}})$. Then $\overline{\mathbf{B}}^{\mathrm{T}} = (1/2)\overline{(\mathbf{A}^{\mathrm{T}} + \overline{\mathbf{A}})} = (1/2)(\mathbf{A} + \overline{\mathbf{A}}^{\mathrm{T}}) = \mathbf{B}$ and $\mathbf{C}^{\mathrm{T}} = (1/2)\overline{(\mathbf{A}^{\mathrm{T}} - \overline{\mathbf{A}})} = -(1/2)(\mathbf{A} - \overline{\mathbf{A}}^{\mathrm{T}}) = -\mathbf{C}$, showing that $\mathbf{B}$ is Hermitian and $\mathbf{C}$ is skew-Hermitian.
7. A real unitary matrix is an orthogonal matrix, because in that case $\overline{\mathbf{A}}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}$, causing the definition of a unitary matrix to reduce to the definition of an orthogonal matrix.
8. The determinant of a unitary matrix is ± 1. This result is established in essentially the same way as the result of Theorem 4.4(i), so the argument will not be repeated.

EXAMPLE 4.16 The following are examples of Hermitian, skew-Hermitian, and unitary matrices.

Hermitian matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2+5i & -7+3i \\ 2-5i & 0 & 1-i \\ -7-3i & 1+i & 4 \end{bmatrix}.$$

Skew-Hermitian matrix:

$$\mathbf{B} = \begin{bmatrix} 4i & -3-2i & -6-4i \\ 3-2i & -2i & 5 \\ 6-4i & -5 & 0 \end{bmatrix}.$$

Unitary matrix:

$$\mathbf{U} = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} & 0 \\ \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It can be seen from Properties 3, 4, and 7 that Hermitian, skew-Hermitian, and unitary matrices are, respectively, generalizations of symmetric, skew-symmetric, and orthogonal real-valued matrices. Accordingly, it is to be expected that some of the properties exhibited by these real-valued matrices are shared by their complex generalizations, and this is indeed the case as we now show.

THEOREM 4.8 **Eigenvalues of Hermitian, skew-Hermitian, and unitary matrices**

(i) The eigenvalues of an Hermitian matrix are real.

(ii) The eigenvalues of a skew-Hermitian matrix are either purely imaginary or 0.

(iii) The eigenvalues λ of a unitary matrix are all such that $|\lambda| = 1$.

Proof

(i) Apart for the need to introduce the complex conjugate operation, the proof is essentially the same as that of Theorem 4.4 for symmetric matrices, and so it is omitted.

(ii) Let **x** be the eigenvector of **A** corresponding to the eigenvalue λ, so $\mathbf{Ax} = \lambda\mathbf{x}$. Then $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{Ax} = \lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}$, from which we have

$$\lambda = \overline{\mathbf{x}}^{\mathrm{T}}\mathbf{Ax}/\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x},$$

but $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = x_1\overline{x}_1 + x_2\overline{x}_2 + \cdots + x_n\overline{x}_n$ is real. However, $\overline{\mathbf{A}} = -\mathbf{A}^{\mathrm{T}}$, so $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{Ax} = -\overline{\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{Ax}}$, so we can write

$$\lambda = \overline{\mathbf{x}}^{\mathrm{T}}\mathbf{Ax}/\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = -\overline{\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{Ax}/\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}}.$$

The product $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}$ is real, so this last result shows that the complex number λ equals the negative of its complex conjugate, and this is only possible if λ is purely imaginary or 0, so the proof is complete.

(iii) Apart from the need to introduce the complex conjugate operation, the proof is essentially that of Theorem 4.5(iii), so it will be omitted. ■

The location of the eigenvalues of these complex matrices and of their corresponding real forms are illustrated in Fig. 4.3.

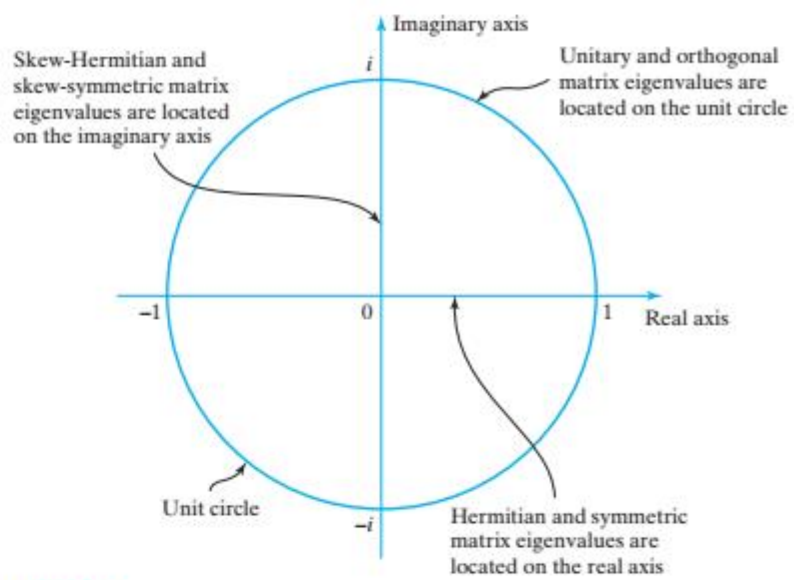


FIGURE 4.3 The location of the eigenvalues of Hermitian, skew-Hermitian, and unitary matrices in the complex plane.

If the definitions of an inner product and a norm are generalized, the concept of orthogonality can be extended to include vectors with complex elements. These generalizations have many applications, but they will only be used here to prove the orthogonality of the rows and columns of unitary matrices.

As the norm of a vector is essentially its *length* and so must be nonnegative, the previous definition of a norm in terms of an inner product must be modified in such a way that the inner product and norm of a complex vector coincide with those for a real vector when purely real vectors are considered. This is achieved by introducing the complex conjugate operation into the definition of an inner product.

Inner product of complex vectors

Let $\mathbf{w} = [w_1, w_2, \ldots, w_n]^{\mathrm{T}}$ and $\mathbf{z} = [z_1, z_2, \ldots, z_n]^{\mathrm{T}}$ be two column vectors with complex elements. Then the **inner product** of the column vectors $\mathbf{w}$ and $\mathbf{z}$, again denoted by $\mathbf{w} \cdot \mathbf{z}$, is defined as $\mathbf{w} \cdot \mathbf{z} = \overline{\mathbf{w}}^{\mathrm{T}}\mathbf{z}$, so that

$$\mathbf{w} \cdot \mathbf{z} = \overline{w}_1 z_1 + \overline{w}_2 z_2 + \cdots + \overline{w}_n z_n. \tag{18}$$

Norm of complex vectors

The **norm** of a vector $\mathbf{z}$, again denoted by $\|\mathbf{z}\|$, is defined as the nonnegative number

$$\begin{aligned}\|\mathbf{z}\| &= (\mathbf{z} \cdot \mathbf{z})^{1/2} = (\overline{\mathbf{z}}^{\mathrm{T}}\mathbf{z})^{1/2} \\ &= (\overline{z}_1 z_1 + \overline{z}_2 z_2 + \cdots + \overline{z}_n z_n)^{1/2} \\ &= (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{1/2}.\end{aligned} \tag{19}$$

It can be seen from the preceding definition that the inner product of two arbitrary complex vectors is a complex number. However, the definition of the norm of a complex vector $\mathbf{z}$ is a real nonnegative number, as would be expected.

EXAMPLE 4.17 If $\mathbf{w} = [1 + 2i,\ 3 - i,\ i]^{\mathrm{T}}$ and $\mathbf{z} = [2 + i,\ 1 - i,\ 1 + 3i]^{\mathrm{T}}$, find $\mathbf{w} \cdot \mathbf{z}$ and $\|\mathbf{z}\|$.

Solution $\mathbf{w} \cdot \mathbf{z} = \overline{(1 + 2i)}(2 + i) + \overline{(3 - i)}(1 - i) + \overline{i}(1 + 3i) = 11 - 6i$, and $\|\mathbf{z}\| = [|2 + i|^2 + |1 - i|^2 + |1 + 3i|^2]^{1/2} = 17^{1/2}$. ■

We are now in a position to generalize the concept of an orthonormal system of real vectors to a system of complex vectors that will be called a *unitary* system if the vectors satisfy the following conditions.

A unitary system

A set of complex vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$ is said to form a **unitary system** if

$$\mathbf{z}_i \cdot \mathbf{z}_j = \overline{\mathbf{z}}_i^{\mathrm{T}}\mathbf{z}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases} \tag{20}$$

THEOREM 4.9 **The eigenvectors of a unitary matrix** The rows and columns of a unitary matrix each form a unitary system of vectors.

Proof By definition the $n \times n$ matrix $\mathbf{U}$ is unitary if $\overline{\mathbf{U}}^{\mathrm{T}} = \mathbf{U}^{-1}$, so that $\overline{\mathbf{U}}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$. The element in the ith row and jth column of $\mathbf{I}$ is the inner product $\mathbf{x}_i \cdot \mathbf{x}_j = \overline{\mathbf{x}}_i^{\mathrm{T}}\mathbf{x}_j$, where $\mathbf{x}_i$ and $\mathbf{x}_j$ are the ith and jth columns of $\mathbf{U}$. Consequently,

$$\overline{\mathbf{x}}_i^{\mathrm{T}}\mathbf{x}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j, \end{cases}$$

showing that the columns of $\mathbf{U}$ form a unitary system. The rows also form a unitary system, because taking the transpose of $\overline{\mathbf{U}}^{\mathrm{T}}\mathbf{U}$ we find that $(\overline{\mathbf{U}}^{\mathrm{T}}\mathbf{U})^{\mathrm{T}} = \mathbf{U}^{\mathrm{T}}\overline{\mathbf{U}} = \mathbf{I}^{\mathrm{T}} = \mathbf{I}$. ■

Summary

Matrices with complex elements arise in a variety of different applications, and from among these matrices, the most important are Hermitian, skew-Hermitian, and unitary matrices. Hermitian and skew-Hermitian matrices are the complex analogues of real symmetric and skew-symmetric matrices, respectively, and unitary matrices are the complex analogue of real orthogonal matrices. This section derived and illustrated by means of examples the most important properties of these matrices, and then introduced the inner product and norm of matrices with complex elements.

EXERCISES 4.3

In Exercises 1 through 4 write the given matrix as the sum of an Hermitian and a skew-Hermitian matrix.

1. $\begin{bmatrix} 1+i & 3+i & 3+2i \\ -1+3i & 2 & 4+i \\ -3-2i & 2+3i & 4+2i \end{bmatrix}$.

2. $\begin{bmatrix} 0 & 3+i & 1+2i \\ 1-5i & 1+i & 2 \\ 1+4i & -2i & 3 \end{bmatrix}$.

3. $\begin{bmatrix} 4-2i & 1+i & 2+2i \\ -1-3i & 1+2i & 4 \\ 0 & 2 & 0 \end{bmatrix}$.

4. $\begin{bmatrix} 3+i & 4-i & 5+2i \\ 2+i & 1+2i & 2 \\ -1 & 2i & 4-i \end{bmatrix}$.

In Exercises 5 through 8 find the eigenvalues of the Hermitian matrices and hence confirm the result of Theorem 4.8(a) that they are real.

5. $\begin{bmatrix} 1 & 2-i \\ 2+i & 2 \end{bmatrix}$.

6. $\begin{bmatrix} 2 & 2+2i \\ 1-2i & 3 \end{bmatrix}$.

7. $\begin{bmatrix} 3 & 2-3i \\ 2+3i & 1 \end{bmatrix}$.

8. $\begin{bmatrix} -4 & 2-2i \\ 2+2i & 3 \end{bmatrix}$.

In Exercises 9 through 12 find the eigenvalues of the skew-Hermitian matrices and hence confirm the result of Theorem 4.8(b) that they are purely imaginary.

9. $\begin{bmatrix} i & 3+i \\ -3+i & 2i \end{bmatrix}$.

10. $\begin{bmatrix} 3i & 2-i \\ -2-i & 0 \end{bmatrix}$.

11. $\begin{bmatrix} 0 & 3+2i \\ -3+2i & 0 \end{bmatrix}$.

12. $\begin{bmatrix} 4i & 2+3i \\ -2+3i & i \end{bmatrix}$.

13. Show the following matrix is unitary:

$$\begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

In Exercises 14 and 15 show the matrices are unitary, find their eigenvalues and eigenvectors, and confirm that the eigenvalues all lie on the unit circle.

14. $\begin{bmatrix} (i-1)/\sqrt{2} & (1-i)/\sqrt{2} \\ (i-1)/\sqrt{2} & (i-1)/\sqrt{2} \end{bmatrix}$.

15. $\begin{bmatrix} (1+i)/\sqrt{2} & -(1+i)/\sqrt{2} \\ (1+i)/\sqrt{2} & (1+i)/\sqrt{2} \end{bmatrix}$.

4.4 Quadratic Forms

A homogeneous polynomial $P(\mathbf{x})$ of degree two of the form

$$P(\mathbf{x}) \equiv a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots + 2a_{n-1,n}x_{n-1}x_n, \tag{21}$$

real quadratic form

in which the coefficients a_{ij} and the variables in $\mathbf{x}(x_1, x_2, \ldots, x_n)$ are real numbers, is called a **real quadratic form** in the variables $x_1, x_2, \ldots, x_n$. The term *homogeneous* of degree two or, more precisely, *algebraically* homogeneous of degree two, means that each term in P is quadratic in the sense that it involves a product of precisely two of the variables $x_1, x_2, \ldots, x_n$. The terms involving the products x_ix_j with $i \neq j$ are called the **mixed product** or **cross-product terms**.

Real quadratic forms

A real quadratic form $P(\mathbf{x})$ is a homogeneous polynomial in the real variables $x_1, x_2, \ldots, x_n$ of the form shown in (21). If $\mathbf{A}$ is a real symmetric $n \times n$ matrix and $\mathbf{x}$ is an n-element column vector defined as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}, \tag{22}$$

then $P(\mathbf{x})$ can be written in the matrix form

$$P(\mathbf{x}) \equiv \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}. \tag{23}$$

There is no loss of generality in requiring $\mathbf{A}$ to be a symmetric matrix, because if the coefficient of a cross-product term x_ix_j equals b_{ij}, this can always be rewritten as $b_{ij} = 2a_{ij}$ allowing the terms a_{ij} to be positioned symmetrically about the leading diagonal, as shown in the matrix $\mathbf{A}$ in (22). Exercise 30 at the end of this section shows how the definition of a real quadratic form can be extended to any real $n \times n$ matrix.

EXAMPLE 4.18

Express the quadratic form

$$P(\mathbf{x}) \equiv 3x_1^2 - 2x_2^2 + 4x_3^2 + x_1x_2 + 3x_1x_3 - 2x_2x_3$$

as the matrix product $P(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$.

Solution By defining $\mathbf{x}$ and $\mathbf{A}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 3 & 1/2 & 3/2 \\ 1/2 & -2 & -1 \\ 3/2 & -1 & 4 \end{bmatrix},$$

we can write $P(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$. ■

Quadratic forms arise in various ways; for example, in mechanics a quadratic form can describe the ellipsoid of inertia of a solid body, the angular momentum of a solid body rotating about an axis, and the kinetic energy of a system of moving particles. Other areas in which quadratic forms occur include the geometry of conics in two space dimensions and of quadrics in three space dimensions, optimization problems, crystallography, and in the classification of partial differential equations (see Chapter 18).

We now give a general definition of a quadratic form that allows both the matrix $\mathbf{A}$ and the vector $\mathbf{x}$ to contain complex elements.

General quadratic forms

quadratic form and vectors with complex elements

Let the elements of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ and an n-element column vector $\mathbf{z}$ be complex numbers. Then a **quadratic form** $P(\mathbf{z})$ involving the variables $z_1, z_2, \ldots, z_n$ of vector $\mathbf{z}$ is an expression of the form

$$P(\mathbf{z}) = \overline{\mathbf{z}}^{\mathrm{T}}\mathbf{A}\mathbf{z} = \sum_{i=1,\, j=1}^{n} a_{ij}\overline{z}_i z_j. \tag{24}$$

This definition is seen to include real quadratic forms, because when the elements of $\mathbf{A}$ and $\mathbf{z}$ are real, result (24) reduces to the real quadratic form defined in (23).

The structure of a quadratic form becomes clearer if a change of variables is made that removes the mixed product terms, leaving only the squared terms. This is called the **reduction** of the quadratic form to its **standard form**, also known as its **canonical form**. The next theorem shows how such a simplification can be achieved.

THEOREM 4.10

how to reduce a quadratic form to a sum of squares

Reduction of a quadratic form Let the $n \times n$ real symmetric matrix $\mathbf{A}$ have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let $\mathbf{Q}$ be an orthogonal matrix that diagonalizes $\mathbf{A}$, so that $\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ as the elements on its leading diagonal. Then the change of variable $\mathbf{x} = \mathbf{Q}\mathbf{y}$, involving the column vectors $\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]^{\mathrm{T}}$, transforms the real quadratic form $P(\mathbf{x}) \equiv \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ into the **standard form**

$$P(\mathbf{x}) \equiv \sum_{i=1,\, j=1}^{n} a_{ij}x_i x_j = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

Proof The proof uses the fact that because $\mathbf{Q}$ is an orthogonal matrix, $\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q} = \mathbf{D}$. Substituting $\mathbf{x} = \mathbf{Q}\mathbf{y}$ into the real quadratic form $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ gives

$$\begin{aligned} P(\mathbf{x}) \equiv \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} &= (\mathbf{Q}\mathbf{y})^{\mathrm{T}}\mathbf{A}\mathbf{Q}\mathbf{y} \\ &= \mathbf{y}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q}\mathbf{y} \\ &= \mathbf{y}^{\mathrm{T}}\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2. \end{aligned}$$ ■

It follows immediately from Theorem 4.10 that the standard form of $P(\mathbf{x})$ is determined once the eigenvalues of $\mathbf{A}$ are known and, when needed, the transformation of coordinates between $\mathbf{x}$ and $\mathbf{y}$ is given by $\mathbf{x} = \mathbf{Q}\mathbf{y}$ or, equivalently, by $\mathbf{y} = \mathbf{Q}^{\mathrm{T}}\mathbf{x}$.

The next example provides a geometrical interpretation of Theorem 4.10 in the context of rigid body mechanics. In order to understand its implications it is necessary to know that if an origin O is taken at an arbitrary point inside a solid body, and an orthogonal set of axes $O\{x_1, x_2, x_3\}$ is located at O, nine moments and products of inertia of the body can be defined relative to these axes and displayed in the form of a 3×3 inertia matrix. The moment of inertia of the body about any line passing through the origin O is proportional to the length of the segment of the line that lies between O and the point where it intersects a three-dimensional surface defined by a quadratic form determined by the inertia matrix.

When the surface determined by the inertia matrix is scaled so the length of the line from O to its point of intersection with the surface equals the reciprocal of the moment of inertia about that line, the surface is called the **ellipsoid of inertia**. If the orientation of the $O\{x_1, x_2, x_3\}$ axes is chosen arbitrarily, the resulting quadratic form will be complicated by the presence of mixed product terms, but a suitable rotation of the axes can always remove these terms and lead to the most convenient orientation of the new system of axes $O\{y_1, y_2, y_3\}$. In the geometry of both conics and quadrics, and also in mechanics, new axes obtained in this way that lead to the elimination of mixed product terms are called the **principal axes**, and it is because of this that Theorem 4.10 is often known as the **principal axes theorem**.

quadratic forms and principal axes

EXAMPLE 4.19

The ellipsoid of inertia of a solid body is given by

$$P(\mathbf{x}) \equiv 4x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_2.$$

Find its standard form in terms of a new orthogonal set of axes $O\{y_1, y_2, y_3\}$, and find the linear transformation that connects the two sets of coordinates.

Solution The quadratic form $P(\mathbf{x})$ can be written as $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ by defining

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$, $\lambda_2 = 5$, and $\lambda_3 = 3$, so the standard form of $P(\mathbf{x})$ is

$$P(\mathbf{x}) \equiv y_1^2 + 5y_2^2 + 3y_3^2.$$

The eigenvalues and corresponding normalized eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = 1,\ \hat{\mathbf{x}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \lambda_2 = 5, \quad \hat{\mathbf{x}}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \lambda_3 = 3, \quad \hat{\mathbf{x}}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix},$$

so the orthogonal diagonalizing matrix for $\mathbf{A}$ is

$$\mathbf{Q} = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix},$$

and the change of variables between $\mathbf{x}$ and $\mathbf{y}$ determined by $\mathbf{x} = \mathbf{Q}\mathbf{y}$ becomes

$$x_1 = (-y_2 + y_3)/\sqrt{2}, \quad x_2 = (y_2 + y_3)/\sqrt{2}, \quad x_3 = y_1.$$

The equation $P(\mathbf{x}) =$ constant is seen to be an *ellipsoid* for which $O\{y_1, y_2, y_3\}$ are the *principal axes*. ■

EXAMPLE 4.20 Reduce the quadratic part of the following expression to its standard form involving the principal axes $O\{y_1, y_2\}$, and hence find the form taken by the complete expression in terms of y_1 and y_2:

$$x_1^2 + 4x_1x_2 + 4x_2^2 + x_1 - 2x_2.$$

Solution The quadratic part of the expression is $x_1^2 + 4x_1x_2 + 4x_2^2$, and this can be expressed in the form $\mathbf{x}^{\mathrm{T}}\mathbf{Ax}$ by setting

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

The eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = 5, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 0, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

so the orthogonal diagonalizing matrix is

$$\mathbf{Q} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}.$$

Making the variable change $\mathbf{x} = \mathbf{Qy}$ shows the standard form of the quadratic terms to be $5y_1^2$. The variables x_1 and x_2 are related to y_1 and y_2 by the expressions $\mathbf{x}_1 = y_1/\sqrt{5} - 2y_2/\sqrt{5}$ and $x_2 = 2y_1/\sqrt{5} + y_2/\sqrt{5}$, so $x_1 - 2x_2 = -(3y_1 + 4y_2)/\sqrt{5}$. In terms of the principal axes involving the coordinates y_1 and y_2, the complete expression $x_1^2 + 4x_1x_2 + 4x_2^2 + x_1 - 2x_2$ reduces to

$$x_1^2 + 4x_1x_2 + 4x_2^2 + x_1 - 2x_2 = 5y_1^2 - (3y_1 + 4y_2)/\sqrt{5}.$$

Quadratic forms $P(\mathbf{x})$ are classified according to the behavior of the sign of $P(\mathbf{x})$ when $\mathbf{x}$ is allowed to take all possible values. In terms of vector spaces, this amounts to saying that if the vector $\mathbf{x}$ in $P(\mathbf{x})$ is an n vector, then $\mathbf{x} \in R^n$.

how to classify quadratic forms

Classification of quadratic forms

Let $P(\mathbf{x})$ be a quadratic form. Then:

1. $P(\mathbf{x})$ is said to be **positive definite** if $P(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in R^n, with $P(\mathbf{x}) = 0$ if, and only if, $\mathbf{x} = \mathbf{0}$. $P(\mathbf{x})$ is said to be **negative definite** if in this definition the inequality sign $>$ is replaced by $<$.
2. $P(\mathbf{x})$ is said to be **positive semidefinite** if $P(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in R^n, and to be **negative semidefinite** if in this definition the inequality sign $\geq$ is replaced by $\leq$.
3. $P(\mathbf{x})$ is said to be **indefinite** if it satisfies none of the above conditions.

It is an immediate consequence of Theorem 4.10 that if $P(\mathbf{x})$ is associated with a real symmetric matrix $\mathbf{A}$, then:

(a) $P(\mathbf{x})$ is positive definite if all the eigenvalues of $\mathbf{A}$ are positive, and it is negative definite if all the eigenvalues of $\mathbf{A}$ are negative.

(b) $P(\mathbf{x})$ is positive semidefinite if all the eigenvalues of $\mathbf{A}$ are nonnegative, and it is negative semidefinite if all the eigenvalues of $\mathbf{A}$ are nonpositive. So, in each semidefinite case, one or more of the eigenvalues may be zero.

(c) $P(\mathbf{x})$ is indefinite if at least one eigenvalue is opposite in sign to the others. In this case, depending on the choice of $\mathbf{x}$, $P(\mathbf{x})$ may be either positive or negative.

EXAMPLE 4.21

The following are examples of different types of standard forms associated with a 3×3 matrix:

$x_1^2 + 2x_2^2 + 5x_3^2$ is positive definite;
$-(2x_1^2 + 7x_2^2 + 4x_3^2)$ is negative definite;
$4x_1^2 + 3x_3^2$ is positive semidefinite (it is positive, but irrespective of the value of $x_2 \neq 0$ it can vanish when $\mathbf{x} \neq \mathbf{0}$);
$-(2x_1^2 + x_3^2)$ is negative semidefinite (it is negative, but irrespective of the value of $x_2 \neq 0$ it can vanish when $\mathbf{x} \neq \mathbf{0}$);
$3x_1^2 - 2x_2^2 + x_3^2$ is indefinite (it can be positive or negative). ■

Further, and more detailed, information relating to the material in Sections 4.1 to 4.4 is to be found in the appropriate chapters of references [2.1] and [2.5] to [2.12].

Summary

A real quadratic form involving the n real variables $x_1, x_2, \ldots, x_n$ is a homogeneous polynomial of degree two in these variables. Such forms arise in many different ways, one of which occurs in optimization problems where a reduction to a sum of squares simplifies the task of finding an optimum least squares solution. In this section it was shown that a real quadratic form arises when studying the mechanics of solid bodies, since there a set of principal axes $O\{x_1, x_2, x_3\}$ is used to simplify the description of the body in terms of its inertia about each of the three axes. The reduction of a quadratic form to a sum of squares both simplifies the analysis of its properties and also enables it to be classified as being positive or negative definite, semipositive or seminegative, or of indefinite type, all of which classifications have important implications in applications.

EXERCISES 4.4

In Exercises 1 through 6 find the symmetric matrix $\mathbf{A}$ that is associated with the given quadratic form.

1. $x_1^2 + 4x_1x_3 - 6x_2x_3 + 3x_2^2 - 2x_3^2$.
2. $5x_1^2 - 2x_2^2 - 5x_3^2 - 4x_2x_3$.
3. $-2x_1^2 + 3x_2^2 - 2x_1x_3 + 4x_2x_3$.
4. $x_1^2 + 3x_2^2 - 2x_1x_2 + 4x_2x_4 - 2x_3x_4 + x_3^2 + 6x_4^2$.
5. $3x_1^2 - 4x_1x_2 - 6x_2x_3 - 2x_2x_4 + 2x_3^2 + 8x_4^2$.
6. $x_1^2 + x_2^2 + 4x_3^2 - 3x_4^2 - x_1x_2 + 2x_2x_4 + 2x_3x_4$.

In Exercises 7 through 10 write down the quadratic form associated with the given matrix.

7. $\begin{bmatrix} 2 & 4 & 4 & 0 \\ 4 & 1 & 2 & 1 \\ 4 & 2 & -1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

8. $\begin{bmatrix} 1 & -3 & 2 & 1 \\ -3 & 2 & 0 & 2 \\ 2 & 0 & -3 & 0 \\ 1 & 2 & 0 & 4 \end{bmatrix}$.

9. $\begin{bmatrix} 0 & 2 & -4 & 2 \\ 2 & 3 & 1 & 0 \\ -4 & 1 & 2 & 1 \\ 2 & 0 & 1 & 7 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & -2 & 4 & 3 \\ -2 & 3 & 1 & 2 \\ 4 & 1 & 5 & 0 \\ 3 & 2 & 0 & 3 \end{bmatrix}$.

In Exercises 11 through 18 use hand computation to reduce the quadratic form to its standard form, and use the reduction to classify it. Confirm the reduction by using computer algebra.

11. $(5/2)x_1^2 + x_1x_3 + x_2^2 + (5/2)x_3^2$.
12. $4x_1^2 + x_2^2 + 2x_2x_3 + x_3^2$.
13. $4x_1^2 + 4x_2^2 + 2x_2x_3 + 4x_3^2$.
14. $(3/2)x_1^2 - x_1x_3 + x_2^2 + (3/2)x_3^2$.
15. $(3/2)x_1^2 + x_1x_3 - x_2^2 + (3/2)x_3^2$.
16. $(1/2)x_1^2 + x_1x_3 + 2x_2^2 + (1/2)x_3^2$.
17. $2x_1^2 + x_2^2 - 4x_2x_3 + x_3^2$.
18. $2x_1^2 + 2x_2^2 + 2x_2x_3 + 2x_3^2$.

In Exercises 19 through 24 use computer algebra to reduce the quadratic form on the left to its standard form. Use the result to identify the conic section described by the equation as a circle, an ellipse, or a hyperbola.

19. $3x_1^2 - 6x_1x_2 + 9x_2^2 = 3$.
20. $8x_2^2 - x_1^2 + 20x_1x_2 = 12$.

21. $5x_1^2 + 4x_1x_2 - 10x_2^2 = 1$.

22. $10x_1^2 + 2x_1x_2 + 5x_2^2 = 4$.

23. $13x_1^2 + 18x_1x_2 + 10x_2^2 = 9$.

24. $2x_1^2 + 16x_1x_2 + 5x_2^2 = 4$.

In Exercises 25 through 29 use hand computation to reduce the quadratic part of the expression to its standard form involving the principal axes $O\{y_1, y_2\}$, and find the form taken by the complete expression in terms of y_1 and y_2. Confirm the reduction by using computer algebra.

25. $x_1^2 + 8x_1x_2 + x_2^2 + 3x_1 - 2x_2$.

26. $x_1^2 - 8x_1x_2 + x_2^2 + 2x_1 + 3x_2$.

27. $-2x_1^2 + 4x_1x_2 + x_2^2 + 4x_1 - x_2$.

28. $(8/5)x_1^2 - (8/5)x_1x_2 + (2/5)x_2^2 + 2x_1 + 4x_2$.

29. $(35/17)x_1^2 + (8/17)x_1x_2 + (50/17)x_2^2 + 4x_2$.

30. By using the definitions of a symmetric and a skew-symmetric matrix, generalize the definition of a quadratic form by proving that the quadratic form associated with any real $n \times n$ matrix $\mathbf{A}$ can be written $\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x}$, where $\mathbf{B}$ is the symmetric part of $\mathbf{A}$.

4.5 The Matrix Exponential

It is shown in Chapter 6 that the matrix exponential can be used when solving systems of linear first order differential equations. As this approach uses matrix diagonalization when determining what is called the *matrix exponential* involving an arbitrary $n \times n$ diagonalizable matrix, it is convenient to introduce the matrix exponential in this chapter.

To motivate what is to follow, we notice that the first order homogeneous linear differential equation

$$dx/dt = ax \quad (a = \text{constant}) \tag{25}$$

has the general solution

$$x = ce^{at} \tag{26}$$

where c is an arbitrary constant.

Let us now consider the system of n linear first order homogeneous differential equations

$$\begin{aligned} dx_1/dt &= a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ dx_2/dt &= a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ dx_n/dt &= a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n \end{aligned} \tag{27}$$

Setting

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}$$

allows the system of differential equations in (27) to be written in the matrix form

$$d\mathbf{x}/dt = \mathbf{A}\mathbf{x}, \tag{28}$$

where $d\mathbf{x}/dt = [dx_1/dt, dx_2/dt, \ldots, dx_n/dt]^{\mathrm{T}}$ (see Section 3.2(d)).

As the single differential equation (25) has the solution (26), it is reasonable to ask whether it is possible to express the solution of the system of differential equations in (28) in the form

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{C}. \tag{29}$$

the matrix exponential

For this to be possible it is necessary to give meaning to the expression $e^{\mathbf{A}t}$, which is called the **matrix exponential**, with t as a parameter. Our objective in the remainder of this section will be to give a brief introduction to the matrix exponential and to use the definition to determine its most important properties in preparation for their use in Chapter 6.

The starting point for this generalization of the exponential function is the familiar result

$$\begin{aligned} e^{at} &= \sum_{m=0}^{\infty} \frac{a^m t^m}{m!} \\ &= 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \cdots. \end{aligned} \tag{30}$$

If $\mathbf{A}$ is an $n \times n$ constant matrix with real coefficients we take as an intuitive definition of the matrix exponential e^{At} the infinite series of matrices

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \cdots. \tag{31}$$

In adopting (31) as a possible definition of the matrix exponential, we have set $\mathbf{A}^0 = \mathbf{I}$ and chosen to vary the convention that a scalar multiplier of a matrix is placed in front of the matrix by writing $\mathbf{A}t$, $\mathbf{A}^2t^2, \ldots,$ instead of $t\mathbf{A}$, $t^2\mathbf{A}^2, \ldots$. This notation has been adopted to make the appearance of the arguments that follow parallel as closely as possible those for the familiar single real variable case. Some books adopt this convention but make no mention of it, while others adhere strictly to the convention that a scalar multiplier is placed before a matrix and write

$$e^{t\mathbf{A}} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots.$$

The matrix exponential in (31) is an $n \times n$ matrix, each element of which is an ordinary infinite series. So to show that $e^{\mathbf{A}t}$ is convergent, it will be sufficient to show that an infinite sum of the required form containing the term of greatest absolute value in $\mathbf{A}$ is convergent. Let us consider the matrix product $\mathbf{A}^2$. Then the term $c_{rs}^{(2)}$ in the rth row and sth column of $\mathbf{A}^2$ is $c_{rs}^{(2)} = a_{r1}a_{1s} + a_{r2}a_{2s} + \cdots + a_{rn}a_{ns}$, so if the magnitude of the largest term in $\mathbf{A}$ is M, it follows that $|a_{rs}| \leq M$, and $|c_{rs}^{(2)}| \leq nM^2$. A similar argument shows that if $|c_{rs}^{(3)}|$ is the corresponding term in the matrix $\mathbf{A}^3$, then $c_{rs}^{(3)} = c_{r1}^{(2)}a_{1s} + c_{r2}^{(2)}a_{2s} + \cdots + c_{rn}^{(2)}a_{ns}$ and so $|c_{rs}^{(3)}| \leq n^2M^3$. Either by induction or by inspection, we see that the magnitude of the term $c_{rs}^{(m)}$ in the rth row and sth column of $\mathbf{A}^m$ obeys the inequality $|c_{rs}^{(m)}| \leq n^{m-1}M^m$.

An overestimate of the magnitude of the term in the rth row and sth column of $e^{\mathbf{A}t}$ is provided by the series

$$1 + tM + t^2nM^2/2! + t^3n^2M^3/3! + \cdots + t^mn^{m-1}M^m/m! + \cdots.$$

Setting $u_m = t^m n^{m-1} M^m / m!$ and applying the ratio test shows that for all fixed t

$$L = \lim_{m\to\infty} |u_{m+1}/u_m| = \lim_{m\to\infty} tnM/(m+1) = 0,$$

so the series is absolutely convergent for all fixed t. Thus, (26) serves as a satisfactory definition of the matrix exponential, and because it is absolutely convergent for all fixed t the series can be differentiated and integrated term by term with respect to t.

the formal definition of $e^{\mathbf{A}t}$ and its properties

The matrix exponential

If $\mathbf{A}$ is an $n \times n$ constant matrix with real coefficients, the **matrix exponential** $e^{\mathbf{A}t}$ is defined by the infinite series

$$e^{\mathbf{A}t} = \mathbf{I}_n + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \cdots, \tag{32}$$

which is absolutely convergent for all fixed t.

The absolute convergence of the infinite series defining the matrix exponential allows it to be differentiated term by term, so

$$\begin{aligned} d[e^{\mathbf{A}t}]/dt &= \mathbf{A} + \mathbf{A}^2 t + \mathbf{A}^3\frac{t^2}{2!} + \cdots = \mathbf{A}\left\{\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \cdots\right\} \\ &= \mathbf{A}e^{\mathbf{A}t}. \end{aligned}$$

We have established the fundamental result that

$$d[e^{\mathbf{A}t}]/dt = \mathbf{A}e^{\mathbf{A}t}, \tag{33}$$

and hence by repeated differentiation that

$$d^n[e^{\mathbf{A}t}]/dt^n = \mathbf{A}^n e^{\mathbf{A}t}. \tag{34}$$

Setting $t = 1$ in (33) shows that

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2\frac{1}{2!} + \mathbf{A}^3\frac{1}{3!} + \cdots, \tag{35}$$

whereas setting $t = 0$ shows that $e^{\mathbf{0}} = \mathbf{I}$.

EXAMPLE 4.22 Find $e^{\mathbf{A}t}$ given that

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution As $\mathbf{A}$ is a diagonal matrix

$$\mathbf{A}^m = \begin{bmatrix} 3^m & 0 & 0 \\ 0 & (-2)^m & 0 \\ 0 & 0 & 4^m \end{bmatrix},$$

so substituting into (32) gives

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} t + \begin{bmatrix} 3^2 & 0 & 0 \\ 0 & (-2)^2 & 0 \\ 0 & 0 & 4^2 \end{bmatrix} \frac{t^2}{2!} + \cdots,$$

showing that

$$e^{\mathbf{A}t} = \begin{bmatrix} \sum_{m=0}^{\infty} \frac{3^m t^m}{m!} & 0 & 0 \\ 0 & \sum_{m=0}^{\infty} \frac{(-2)^m t^m}{m!} & 0 \\ 0 & 0 & \sum_{m=0}^{\infty} \frac{4^m t^m}{m!} \end{bmatrix} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}.$$

EXAMPLE 4.23 Find $e^{\mathbf{A}}$ and $e^{\mathbf{A}t}$, and show by direct differentiation that $d[e^{\mathbf{A}t}]/dt = \mathbf{A}e^{\mathbf{A}t}$, given that

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 0 & 6 & -3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{A}^n = \mathbf{0} \text{ for } n > 3.$$

Substituting into (32) and adding the scaled matrices gives

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 2t & t + 3t^2 & t - (3/2)t^2 + t^3 \\ 0 & 1 & 3t & -2t + (3/2)t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Setting $t = 1$ in this result, we find that

$$e^{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 4 & 1/2 \\ 0 & 1 & 3 & -1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Differentiation of the terms in the matrix $e^{\mathbf{A}t}$ gives

$$d[e^{\mathbf{A}t}]/dt = \begin{bmatrix} 0 & 2 & 1 + 6t & 1 - 3t + 3t^2 \\ 0 & 0 & 3 & -2 + 3t \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and as this is equal to $Ae^{\mathbf{A}t}$, it confirms the result $d[e^{\mathbf{A}t}]/dt = \mathbf{A}e^{\mathbf{A}t}$. ■

It was possible to sum the infinite series of matrices in Example 4.22 because only a diagonal matrix was involved, so its powers could be determined immediately. The situation was different in Example 4.23 because $\mathbf{A}^n = \mathbf{0}$ for $n > 3$ so that only a finite sum of matrices was involved. Matrices such as those in Example 4.23, which vanish when raised to a finite power, are called **nilpotent** matrices.

If $\mathbf{A}$ is neither diagonal nor nilpotent, but is diagonalizable, in order to determine $\mathbf{A}^m$ it is first necessary to find the diagonalizing matrix $\mathbf{P}$ for $\mathbf{A}$. Then, if $\mathbf{D}$ is the diagonalized form of $\mathbf{A}$, so that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, it follows that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and

$$\mathbf{A}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}^2\mathbf{P}^{-1}, \quad \mathbf{A}^3 = \mathbf{AA}^2 = (\mathbf{PDP}^{-1})(\mathbf{PD}^2\mathbf{P}^{-1}) = \mathbf{PD}^3\mathbf{P}^{-1},$$

so that in general,

$$\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}.$$

Using this result in the matrix exponential gives

$$e^{\mathbf{A}t} = \mathbf{I} + (\mathbf{PDP}^{-1})t + \mathbf{PD}^2\mathbf{P}^{-1}\frac{t^2}{2!} + \cdots,$$

and writing $\mathbf{I} = \mathbf{PP}^{-1}$ reduces this to

$$e^{\mathbf{A}t} = \mathbf{P}\left\{\mathbf{I}_n + \mathbf{D}t + \mathbf{D}^2\frac{t^2}{2!} + \mathbf{D}^3\frac{t^3}{3!} + \cdots\right\}\mathbf{P}^{-1}. \tag{36}$$

The form of $e^{\mathbf{A}}$ follows directly from this by setting $t = 1$.

EXAMPLE 4.24 Determine $e^{\mathbf{A}t}$ given that

$$\mathbf{A} = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix},$$

and use the result to find $e^{\mathbf{A}}$.

Solution The eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = 1, \quad \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 4, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

so the diagonalizing matrix

$$\mathbf{P} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \text{ and } \mathbf{P}^{-1} = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}, \text{ while } \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

Substituting these matrices into (36) gives

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{P}\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}t + \begin{bmatrix} 1 & 0 \\ 0 & 4^2 \end{bmatrix}\frac{t^2}{2!} + \begin{bmatrix} 1 & 0 \\ 0 & 4^3 \end{bmatrix}\frac{t^3}{3!} + \cdots\right]\mathbf{P}^{-1} \\ &= \mathbf{P}\begin{bmatrix} e^t & 0 \\ 0 & e^{4t} \end{bmatrix}\mathbf{P}^{-1} = \begin{bmatrix} (2e^t - e^{4t}) & (e^t - e^{4t}) \\ (2e^{4t} - 2e^t) & (2e^{4t} - e^t) \end{bmatrix}. \end{aligned}$$

Finally, setting $t = 1$ we find that

$$e^{\mathbf{A}} = \begin{bmatrix} (2e - e^4) & (e - e^4) \\ (2e^4 - 2e) & (2e^4 - e) \end{bmatrix}. \qquad ■$$

So far, the properties of the matrix exponential have closely paralleled those of the ordinary exponential, but there are significant differences, one of the most important being that in general, even when $\mathbf{A} + \mathbf{B}$ is defined, $e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{(\mathbf{A}+\mathbf{B})}$. To determine under what conditions the equality is true, we consider the matrix exponentials $e^{\mathbf{A}t}e^{\mathbf{B}t}$ and $e^{(\mathbf{A}+\mathbf{B})t}$ and require their derivatives to be equal when $t = 0$.

Differentiating each expression once with respect to t gives

$$d[e^{\mathbf{A}t}e^{\mathbf{B}t}]/dt = \mathbf{A}e^{\mathbf{A}t}e^{\mathbf{B}t} + e^{\mathbf{A}t}\mathbf{B}e^{\mathbf{B}t} \quad \text{and} \quad d\left[e^{(\mathbf{A}+\mathbf{B}t)}\right]/dt = (\mathbf{A} + \mathbf{B})e^{(\mathbf{A}+\mathbf{B})t},$$

and these are seen to be equal when $t = 0$. Next, computing $d^2[e^{\mathbf{A}t}e^{\mathbf{B}t}]/dt^2$ and $d^2[e^{(\mathbf{A}+\mathbf{B})t}]/dt^2$, we obtain

$$d^2[e^{\mathbf{A}t}e^{\mathbf{B}t}]/dt^2 = \mathbf{A}^2e^{\mathbf{A}t}e^{\mathbf{B}t} + 2\mathbf{A}e^{\mathbf{A}t}\mathbf{B}e^{\mathbf{B}t} + e^{\mathbf{A}t}\mathbf{B}^2e^{\mathbf{B}t}$$

and

$$d^2\left[e^{(\mathbf{A}+\mathbf{B})t}\right]/dt^2 = (\mathbf{A} + \mathbf{B})^2e^{(\mathbf{A}+\mathbf{B})t} = (\mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2)e^{(\mathbf{A}+\mathbf{B})t}.$$

Setting $t = 0$ shows that these two expressions are only equal if $\mathbf{AB} = \mathbf{BA}$; that is, the matrices $\mathbf{A}$ and $\mathbf{B}$ must *commute*, and the same condition applies when all higher order derivatives are considered. This has established the fundamental result that

when does $e^{\mathbf{A}}e^{\mathbf{B}} = e^{(\mathbf{A}+\mathbf{B})}$

$$e^{\mathbf{A}}e^{\mathbf{B}} = e^{(\mathbf{A}+\mathbf{B})} \quad \text{if, and only if, } \mathbf{AB} = \mathbf{BA}. \tag{37}$$

Replacing $\mathbf{B}$ by $-\mathbf{A}$ in (37) gives

$$e^{\mathbf{A}}e^{-\mathbf{A}} = e^{\mathbf{0}} = \mathbf{I}, \tag{38}$$

from which we see, as would be expected, that $e^{-\mathbf{A}}$ is the inverse of $e^{\mathbf{A}}$, and also that as $e^{-\mathbf{A}}$ is nonsingular it always exists. This parallels the real variable situation, because e^{-x} exists for all finite x.

Having arrived at a satisfactory definition of $e^{\mathbf{A}t}$ and determined its derivatives, we are now in a position to define the **antiderivative** $\int e^{\mathbf{A}t}\,dt$ as the matrix obtained by integrating each element of $e^{\mathbf{A}t}$ with respect to t, it being understood that when this is done an arbitrary constant $n \times n$ matrix must always be added to the result representing the arbitrary additative constant of integration that arises when each term of $e^{\mathbf{A}t}$ is integrated.

EXAMPLE 4.25 Find $\int e^{\mathbf{A}t}\,dt$ given that $\mathbf{A}$ is the matrix in Example 4.21.

Solution It was shown in Example 4.21 that if

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{then } e^{\mathbf{A}t} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix},$$

so that

$$\int e^{\mathbf{A}t}\,dt = \begin{bmatrix} e^{3t}/3 + c_1 & 0 & 0 \\ 0 & -e^{-2t}/2 + c_2 & 0 \\ 0 & 0 & e^{4t}/4 + c_3 \end{bmatrix}$$

$$= \begin{bmatrix} e^{3t}/3 & 0 & 0 \\ 0 & -e^{-2t}/2 & 0 \\ 0 & 0 & e^{4t}/4 \end{bmatrix} + \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix},$$

where c_1, c_2, and c_3 are arbitrary constants. ■

Applications of the matrix exponential to ordinary differential equations are to be found in reference [3.15].

Summary

The matrix exponential $e^{\mathbf{A}t}$ arises as the natural extension of the exponential function when solving a system of linear first order constant coefficient differential equations in the matrix form $d\mathbf{x}/dt = \mathbf{Ax}$. This section has described how $e^{\mathbf{A}t}$ can be calculated in simple cases and shown that $e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{A}+\mathbf{B}}$ if, and only if, $\mathbf{AB} = \mathbf{BA}$. A different way of finding $e^{\mathbf{A}t}$ using the Laplace transform is given later in Section 7.3(b).

EXERCISES 4.5

1. Given that

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

show that it is nilpotent and find the smallest power for which $\mathbf{A}^n = \mathbf{0}$.

2. Given that

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

find $e^{\mathbf{A}t}$.

3. Given that

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix},$$

show that $\mathbf{A}$ and $\mathbf{B}$ do not commute, and by finding $e^{\mathbf{A}t}$, $e^{\mathbf{B}t}$, and $e^{(\mathbf{A}+\mathbf{B})t}$, verify that $e^{\mathbf{A}t}e^{\mathbf{B}t} \neq e^{(\mathbf{A}+\mathbf{B})t}$.

In Exercises 4 through 9, find $e^{\mathbf{A}t}$.

4. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

5. $\mathbf{A} = \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix}$.

6. $\mathbf{A} = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}$.

7. $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$.

8. $\mathbf{A} = \begin{bmatrix} 3 & -2 & 2 \\ 6 & -4 & 6 \\ 2 & -1 & 3 \end{bmatrix}$.

9. $\mathbf{A} = \begin{bmatrix} 0 & 1 & -2 \\ 2 & -1 & 2 \\ 2 & -2 & 4 \end{bmatrix}$.

10. By considering the definition of $e^{\mathbf{A}t}$ show, provided the square matrices $\mathbf{A}$ and $\mathbf{B}$ commute, that

$$\mathbf{A}e^{\mathbf{B}t} = e^{\mathbf{B}t}\mathbf{A}.$$

11. By considering the definition of $e^{\mathbf{A}t}$ show that $\int e^{-\mathbf{A}t}\,dt = -\mathbf{A}^{-1}e^{-\mathbf{A}t} + \mathbf{C} = e^{-\mathbf{A}t}\mathbf{A}^{-1} + \mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant matrix that is conformable for addition with $\mathbf{A}$.

12. Show that if the square matrices $\mathbf{A}$ and $\mathbf{B}$ commute, then the binomial theorem takes the form

$$(\mathbf{A} + \mathbf{B})^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{A}^k\mathbf{B}^{n-k}.$$