Markov Chains and Python Simulations

Recurrent and Transient Properties

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Introduction

Stochastic processes are often used in various fields within mathematics and probability theory. In particular, Markov chains are powerful tools due to their *memorylessness*. This property allows one to predict the future based solely on the present state. Through this condition, it is possible to determine if a state is recurrent or transient in a given state-space; either the chain returns to the state *ad infinitum* or the chain will never return to the state after finitely many steps.

Not only will we provide a formal written proof utilizing the Markov property, we have also developed Python simulations to illustrate recurrence and transience in real-life scenarios. These examples are symmetric random walks on d-dimensional integer lattices.

This simulation helps us to classify recurrence further into either a null recurrence, the expected number of steps to return to where the chain started is infinite, or a positive recurrence, the expected number of steps is finite. We show by an example that in the null recurrence case, the chain returns to the starting point but the number of steps can be very large.

Research Questions

1. How can one determine whether a given state is recurrent and transient?
2. Which symmetric random walks on d-dimensional integer lattices are recurrent?
3. Use Python to simulate random walks and test whether given random walks are recurrent.

Materials and Methods

The Markov Property says that the conditional probability distribution of future states of the process depends only upon the present state and not the past history of the chain. More precisely,

$$\mathbb{P}(\mathbf{X}_{n+1} = i_{n+1} | \mathbf{X}_0 = i_0, \cdots, \mathbf{X}_n = i_n) = p_{i_n, i_{n+1}}.$$

The definition of recurrence and transience are as follows:

A state i is *recurrent if* $\mathbb{P}_i(X_n = i$ for infinitely many $n) = 1$,

and

A state i is *transient if* $\mathbb{P}_i(X_n = i$ for infinitely many $n) = 0$.

Here is a useful characterization of recurrence and transience. Let $T_i = \inf\{n \geq 1 : X_n = i\}$ be the first return time to i.

If $\mathbb{P}_i(T_i < \infty) = 1$, then i is recurrent and $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$,

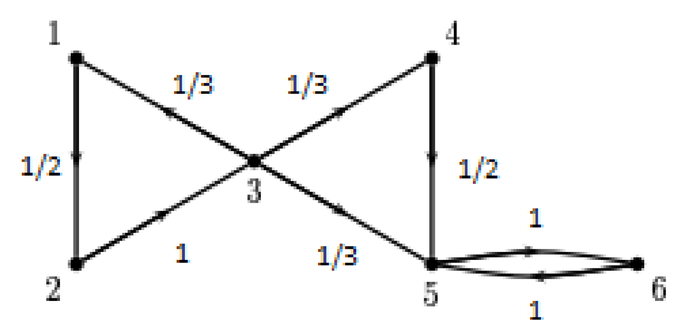
If $\mathbb{P}_i(T_i < \infty) < 1$, then i is transient and $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$.

```
def One_Dim_Random_Walk(n):
    state, count = 0, 0
    #We start at the state 0, and we want to reach 0 again
    for i in range(n):
        step = random.choice(['W', 'E'])
        #Move left or right with equal probability
        if step == 'W':
            state = state - 1
        if step == 'E':
            state = state + 1
        count += 1
        if i > 1 and state == 0:
        #Once we leave the state 0, if we return to 0, then the function will stop
            break
    return (state, count)
```

Figure 1: Code used for a one-dimensional random walk.

Finite Markov Chain

It is easy to identify recurrent and transient states in a finite, closed Markov chain.



No matter where the chain starts, eventually, after sufficiently many steps, the chain will only reside in states 5 and 6. In general, every finite closed class is recurrent.

Mathematical Proof

We will prove the case that a symmetric 1D Random Walk is recurrent. Consider we start at the state zero. Then we are interested in the probability that starting from zero, we reach zero after n-steps. Since it is only possible to return to said state in an even number of steps, we can express p as such:

$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{(n!)^2}(pq)^n.$$

Stirling's Formula provides approximations for large $n!$ values:

$$n! \approx \sqrt{2n\pi}\left(\frac{n}{e}\right)^n.$$

Hence,

$$p_{00}^{(2n)} = \frac{(2n)!}{(n!)^2}(pq)^n \approx \frac{(4pq)^n}{\sqrt{n\pi}}.$$

In the symmetric case, we have $p = q = \frac{1}{2}$. Therefore,

$$p_{00}^{(2n)} \geq \frac{(4pq)^n}{\sqrt{n\pi}} = \frac{1}{\sqrt{n\pi}}.$$

This implies that

$$\sum_{n=N}^{\infty} p_{00}^{(2n)} \geq \frac{1}{\sqrt{\pi}} \sum_{n=N}^{\infty} \frac{1}{\sqrt{n}} = \infty.$$

Hence the symmetric random walk is recurrent by definition. If p $\neq$ q, then the 1D random walk is transient because of a convergent geometric summation.

The symmetric 2D random walk is also recurrent:

$$p_{00}^{(2n)} = \left(\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right)^2 \approx \frac{1}{\pi^2 n}.$$

1D Random Walk

We have proven that a 1D random walk should be recurrent. Consider a simple case.

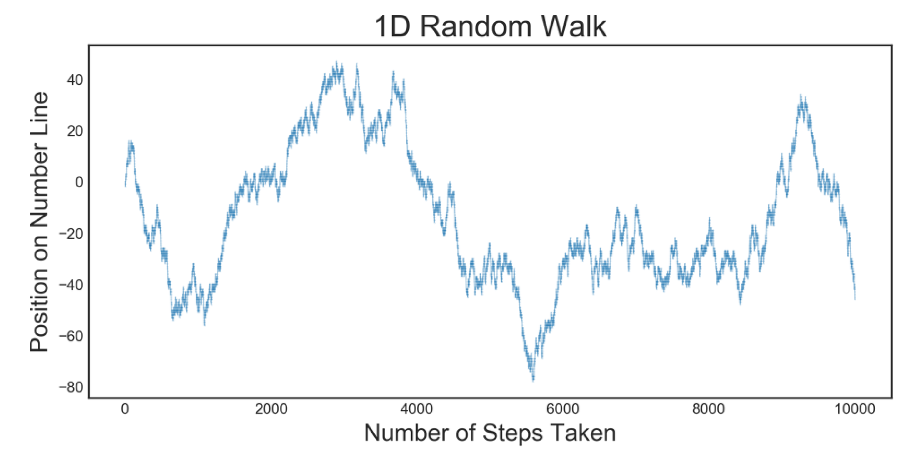


Figure 2: This is one trial over the course of a 1,000 step uninterrupted.

We can use Python to run thousands of trials to see if this 1D random walk is recurrent consistently.

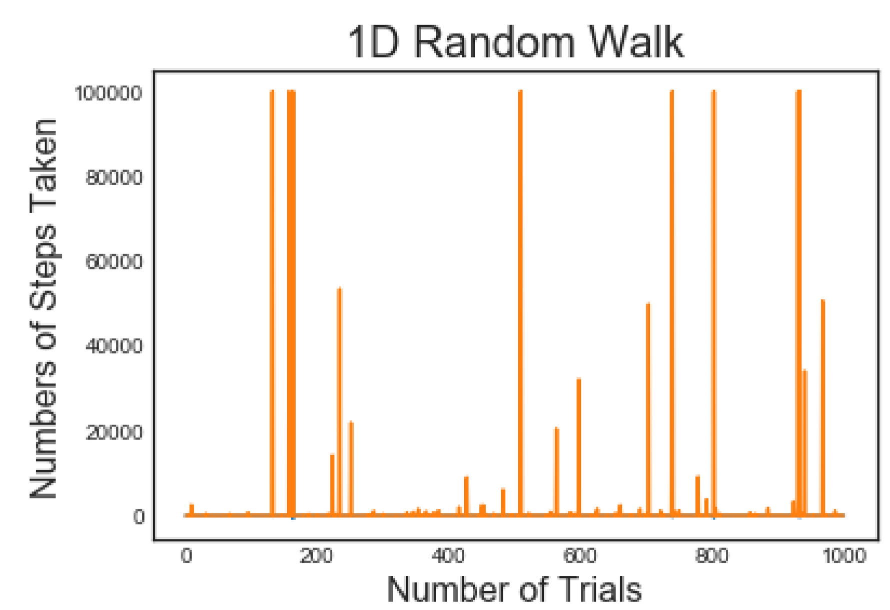


Figure 3: 1,000 Trials for a 1D random walk. Notice the spikes in particular.

This random walk is certainly recurrent; however, there are some outliers. Despite the majority of these trials returning to the origin, several do not reach the desired state. Even if we increase the number of steps, this trends continues. It seems the expectation to reach the starting state may be infinite.

2D Random Walk

A similar trend occurs in the recurrent 2D Random Walk. Due to the dimensionality, We will illustrate this property using a marginal plot using *scipy.stats* in Python.

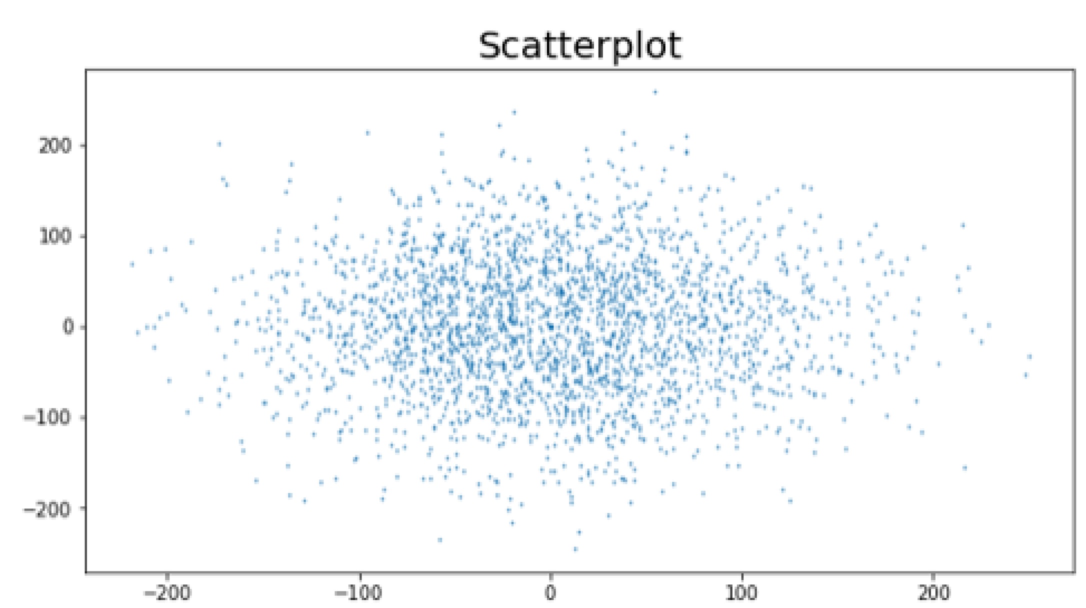


Figure 4: Based on the scatter-plot alone, it seems very few trials returned to the origin. This is why a density plot is needed to illustrate recurrence.

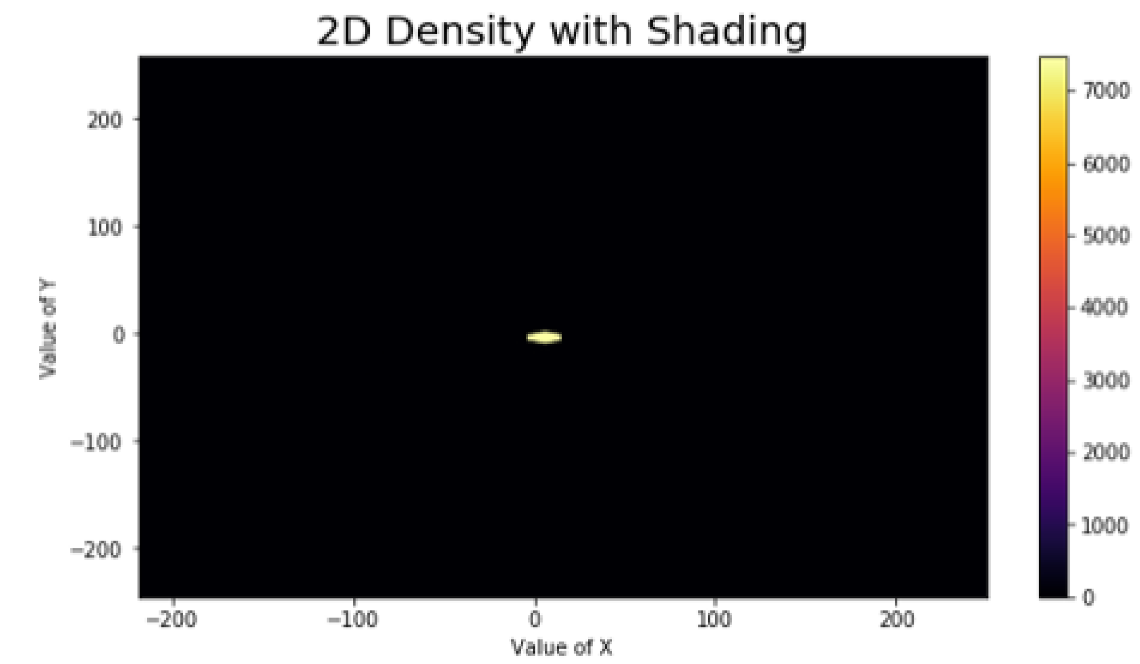


Figure 5: The color gradient indicates the density of points on the xy-plane. The lightest color represents the neighborhood around the origin. For this simulation, 7,418 trials returned to the origin.

Null and Positive Recurrence

We say a recurrent state i is *positive recurrent or null recurrent* if

$$\mathbb{E}_i[T_i] < \infty \text{ or } \mathbb{E}_i[T_i] = \infty, \text{ respectively.}$$

An irreducible Markov chain is positive recurrent if and only if it has a unique invariant distribution $\lambda P = \lambda$. In 1D Random Walk, there is no invariant distribution and this shows that the chain is *null recurrent*.

Forthcoming Research

- Study continuous time analogues (continuous time Markov chain) of discrete time Markov chains. This is important since most real world examples are modeled in a continuous time.
- If the increment of the process is independent and stationary, it is called *Lévy processes*. It is an active field of research and it is a natural object to study.

References

J. R. Norris. Markov chains. Reprint of 1997 original. Cambridge Series in Statistical and Probabilistic Mathematics, 2. Cambridge University Press, Cambridge, 1998. xvi+237 pp. ISBN: 0-521-48181-3

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