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The method of lines for solution of the one-dimensional wave equation subject to an integral conservation condition

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ABSTRACT

Hyperbolic partial differential equations with an integral condition serve as models in many branches of physics and technology. Recently, much attention has been expended in studying these equations and there has been a considerable mathematical interest in them. In this work, the solution of the one-dimensional nonlocal hyperbolic equation is presented by the method of lines. The method of lines (MOL) is a general way of viewing a partial differential equation as a system of ordinary differential equations. The partial derivatives with respect to the space variables are discretized to obtain a system of ODEs in the time variable and then a proper initial value software can be used to solve this ODE system. We propose two forms of MOL for solving the described problem. Several numerical examples and also some comparisons with finite difference methods will be investigated to confirm the efficiency of this procedure.

1. Introduction

Over the last few years, it has become increasingly apparent that many physical phenomena can be described in terms of hyperbolic partial differential equations with an integral condition replacing the classic boundary condition [1]. This type of equations arises, for example in the study of thermoelasticity [2,3], plasma physics [4], chemical heterogeneity [5,6] and etc. Growing attention is being paid to the development, analysis and implementation of numerical methods for the solution of these problems. Hyperbolic initial boundary value problems in one dimension that involve nonlocal boundary conditions have been studied by several authors [7,8,1,9–12]. For parabolic equations subject to nonlocal boundary conditions the interested reader can see [13] and the references therein. Also nonlocal problems include the problems with an integral term in initial condition [14]. In the current work we will not discuss on this group.

In this research, we consider the following problem of this family of equations

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = q(x,t), \quad 0 \le x \le l, 0 < t \le T, \tag{1.1}$$

with initial conditions

$$v(x,0) = f_1(x), \quad 0 \le x \le l, \tag{1.2}$$

and

$$v_t(x,0) = f_2(x), \quad 0 \le x \le l, \tag{1.3}$$

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and Dirichlet boundary condition

$$v(0,t) = g_1(t), \quad 0 < t \le T, \tag{1.4}$$

and the nonlocal condition

$$\int_0^l v(x,t)\mathrm{d}x = g_2(t), \quad 0 < t \le T, \tag{1.5}$$

where q, f_1, f_2, g_1 and g_2 are known functions. We assume that the functions q, f_1, f_2, g_1, and g_2 satisfy the conditions in order that the solution of this equation exists and is unique. The existence and uniqueness of the solution of this problem are discussed in [15].

Author of [1] presented several finite difference schemes for the numerical solution of problem (1.1)–(1.5). These three-level techniques are based on two second-order (one explicit and one weighted) schemes and a fourth-order technique (a weighted explicit) [1]. Also in [16] the shifted Legendre Tau technique is developed for the solution of the studied model. The approach in their work consists of reducing the problem to a set of algebraic equations by expanding the approximate solution as a shifted Legendre function with unknown coefficients. The integral and derivative operational matrices are given. These matrices together with the tau method are then utilized to evaluate the unknown coefficients of shifted Legendre functions. Author of [17] developed a numerical technique based on an integro-differential equation and local interpolating functions for solving the one-dimensional wave equation subject to a nonlocal conservation condition and suitably prescribed initial boundary conditions. Authors of [18] combined finite difference and spectral methods to solve the one-dimensional wave equation with an integral condition. The time variable is approximated using a finite difference scheme. But the spectral method is employed for discretizing the space variable. The main idea behind their approach is that high-order results can be obtained. They have tested the new method for two examples from the literature [18]. A numerical scheme for solving the second-order wave equation with given initial conditions and a boundary condition and an integral condition in place of the classical boundary condition is presented in [19]. The cubic B-spline scaling function together with boundary scaling function on interval [0, 1] are employed for solving the model. The obtained results show that this approach can solve the problem effectively. It is worth pointing out that the matrices are not sparse such as those that can be obtained when one uses the finite difference methods, but here [19] we need smaller matrices to get some results to be compared with finite difference methods.

In this work a different approach is used, the solution of the above equation is computed by the method of lines. Method of lines is an alternative computational approach which involves making an approximation to the space derivatives and by reducing the problem to that of solving a system of initial value ordinary differential equations and then using a time integrator for solving the ODE system. One can increase the accuracy of the method by the use of highly efficient reliable initial value ODE solvers which means that comparable orders of accuracy can also be achieved in the time integration without using extremely small time steps.

This paper is organized in the following way. In Section 2, we introduce the method of lines briefly and apply it to Eqs. (1.1)–(1.5) in two different ways. Some numerical results and comparisons with the finite difference method presented in [1] are given in Section 3 and finally a conclusion is drawn in Section 4.

2. Method of lines

Method of lines is a semi-discrete method [20–25] which involves reducing an initial boundary value problem to a system of ordinary differential equations (ODE) in time through the use of a discretization in space. The resulting ODE system can be solved using the standard initial value software, which may use a variable time-step/variable order approach with time local error control. The most important advantage of the MOL approach is that it has not only the simplicity of the explicit methods [26] but also the superiority (stability advantage) of the implicit ones unless a poor numerical method for solution of ODEs is employed. It is possible to achieve higher-order approximations in the discretization of spatial derivatives without significant increases in the computational complexity. This technique has the broad applicability to physical and chemical systems modeled by PDEs. The models that include the solution of mixed systems of algebraic equations, ODEs and PDEs, the resolution of steep moving fronts, parameter estimation and optimal control, other problems such as delay differential equations [27], two-dimensional sine-Gordon equation [28], the Nwogu one-dimensional extended Boussinesq equation [29], partial differential equation problems describing nonlinear wave phenomena, e.g., a fully nonlinear third-order Korteweg–de Vries (KdV) equation, the fourth-order Boussinesq equation, the fifth-order Kaup–Kupershmidt equation and an extended KdV5 equation [30], nonlinear inverse heat conduction problem [31], interface problem [32], multi-component atmospheric pollutant propagation model with pollutants phase transformation consideration [33], a Bingham problem in cylindrical pipes [34], a mathematical model for capillary formation [35], three-dimensional transient radiative transfer equation [36], elliptic partial differential equations which describe steady-state mass and energy transport in solids [37] and many other physical problems.

Author of [38] used MOL to transform the initial boundary value problem associated with the nonlinear hyperbolic Boussinesq equation, into a first-order nonlinear initial value problem. Numerical methods are developed by replacing the matrix-exponential term into a recurrence relation by rational approximants. MOL is employed in [25] to solve the

Korteweg–de Vries equation. Authors of [39] investigated the performance of various terms of upwinding to provide some guidance in the selection of upwind methods in the MOL solution of strongly convective partial differential equations. MOL is used in [40] to obtain numerical solution of a mathematical model for capillary formation in tumor angiogenesis. Authors of [41] applied moving and adaptive mesh methods for the numerical solution of an extended fifth-order Korteweg–de Vries model motivated by water waves in the presence of surface tension. Also they investigated the dynamics and interaction of embedded solitons. Authors of [42] applied MOL for solving parabolic partial differential equations. They showed (experimentally) that increasing the points in the formula in the spatial approximation cannot bring the MOL and exact solutions into close agreement(for parabolic problems but not necessarily for hyperbolic or elliptic equations). The method of lines and finite difference method were tested in [43] from the viewpoints of solution accuracy and central processing unit time by applying them to the solution of time-dependent two-dimensional Navier–Stokes equations for transient laminar flow without/with sudden expansion and comparing their results with steady-state numerical predictions and measurements previously reported in the literature. MOL was found to be superior to finite difference method with respect to CPU and set-up times and its flexibility for incorporation of other conservative equations [43]. The method of lines concept has been combined with the boundary element based elimination of the spatial derivatives to obtain a solution method for partial differential equations which are parabolic in time [44]. The boundary element method alleviates the need for spatial discretization and casts the problem in an integral format. Hence errors associated with the numerical approximation of the spatial derivatives are totally eliminated. Applications of the method of lines to the hyperbolic partial differential equations with stiff nonlinear source terms is investigated in [45]. Some of the options available for time integration when using a moving grid method of lines code is surveyed in [46]. A new technique is proposed in [47] for the numerical integration of the system of ordinary differential equations that arises in the method of lines solution of time-dependent partial differential equations. This system is usually stiff, so it is desirable for the numerical method to solve it to have good properties concerning stability.

In order to use this approach for solving (1.1)–(1.5), we discretize the coordinate x with M (M even) uniformly spaced grid points $x_i = x_{i-1}+h, x_0 = 0, x_M = l, i = 1, 2, \dots, M$. Note that $h = l/M$ and we can also write $x_i = ih$. At first, in part (I), we use a second-order difference approximation for the second derivative in x in grid points x_i, $i = 1, 2, \dots, M-1$ and in part (II), we apply a fourth-order difference approximation to the second derivative in x in grid points x_i, $i = 2, 3, \dots, M-2$.

2.1. MOL I

Let $v_i(t)$ approximate $v(x_i, t)$. In Fig. 1, the lines along which the approximations $v_i(t)$ are defined, are shown. Using the second-order central difference approximation for the second derivative in x results in

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} - \frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} = q(x_i, t), \quad i = 1, \dots, M-1, 0 < t \le T. \tag{2.1}$$

Also the conditions (1.2)–(1.4) are reduced to

$$v_i(0) = f_1(x_i), \quad 0 \le x \le l, \tag{2.2}$$

$$(v_i)_t(0) = f_2(x_i), \quad 0 \le x \le l, \tag{2.3}$$

and

$$v_0(t) = g_1(t), \quad 0 < t \le T. \tag{2.4}$$

Applying Simpson's numerical integration rule, we approximate the nonlocal condition (1.5) as follows

$$\int_0^l v(x,t)\mathrm{d}x = h\sum_{i=0}^{M} c_i v_i(t) = g_2(t), \quad 0 < t \le T, \tag{2.5}$$

where $c_0 = c_M = \frac{1}{3}, c_{2k-1} = \frac{4}{3}$ for $k = 1, \dots, \frac{M}{2}$ and $c_{2k} = \frac{2}{3}$, for $k = 1, \dots, \frac{M}{2} - 1$. Now, using (2.4) and (2.5), we can obtain the following formula for $v_M(t)$

$$v_M(t) = -\frac{1}{hc_M}\left[h\sum_{i=0}^{M-1} c_i v_i(t) - g_2(t)\right] = -\frac{1}{hc_M}\left[hc_0 g_1(t) + h\sum_{i=1}^{M-1} c_i v_i(t) - g_2(t)\right]. \tag{2.6}$$

Before we solve the system of ODE (2.1) with initial conditions (2.2) and (2.3), we apply some transformations to convert this system to a system of the first-order equations

$$u_1(t) = \frac{\mathrm{d}v_1}{\mathrm{d}t},$$

$$u_2(t) = \frac{\mathrm{d}v_2}{\mathrm{d}t},$$

$$\vdots$$

$$u_{M-1}(t) = \frac{\mathrm{d}v_{M-1}}{\mathrm{d}t},$$

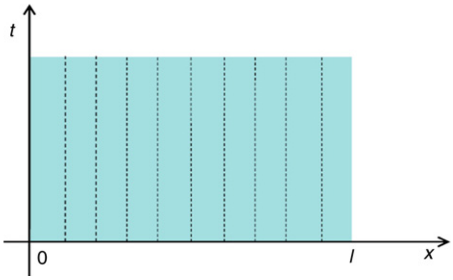

Fig. 1. The colored strip is the domain on which the PDE is defined. The approximations $v_i(t)$ are defined along the dashed lines.

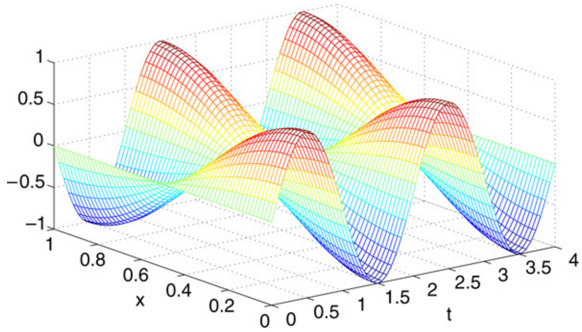

Fig. 2. Plot of the exact solution in Example 1.

which results in the following system of $2M-2$ equations

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} - \frac{v_{i-1}-2v_i+v_{i+1}}{h^2} = q(x_i,t), \quad i=1,\ldots,M-2,\, 0<t\le T,$$

$$\frac{\mathrm{d}u_{M-1}}{\mathrm{d}t} - \frac{v_{M-2}(1-\frac{c_{M-2}}{c_M}) - v_{M-1}(2+\frac{c_{M-1}}{c_M}) - \frac{1}{c_M}\sum_{i=1}^{M-3} c_i v_i(t)}{h^2} = q(x_{M-1},t) - \frac{1}{h^3 c_M}[hc_0 g_1(t) - g_2(t)],$$

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = u_i(t), \quad i=1,\ldots,M-1,\, 0<t\le T,$$

with initial conditions

$$v_i(0) = f_1(x_i), \qquad u_i(0) = f_2(x_i).$$

2.2. MOL II

In this part, for the second derivative in x in grid points x_i, $i = 2,3,\ldots,M-2$, we use the following fourth-order difference approximation

$$(v_i)_{xx} = -\frac{v_{i+2}-16v_{i+1}+30v_i-16v_{i-1}+v_{i-2}}{12h^2}, \quad i=2,\ldots,M-3,$$

$$(v_{M-2})_{xx} = \frac{\frac{1}{hc_M}[hc_0g_1(t)-g_2(t)] + v_{M-1}(16+\frac{c_{M-1}}{c_M}) - v_{M-2}(30+\frac{c_{M-2}}{c_M})}{12h^2} + \frac{v_{M-3}(16+\frac{c_{M-3}}{c_M}) - v_{M-4}(1+\frac{c_{M-4}}{c_M}) + \frac{1}{c_M}\sum_{i=1}^{M-5} c_i v_i(t)}{12h^2}.$$

Using transformations introduced in previous part, an ODE system is obtained as follows

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} - \frac{v_0-2v_1+v_2}{h^2} = q(x_1,t), \quad 0<t\le T,$$

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + \frac{v_{i+2}-16v_{i+1}+30v_i-16v_{i-1}+v_{i-2}}{12h^2} = q(x_i,t), \quad i=2,\ldots,M-3,\, 0<t\le T,$$

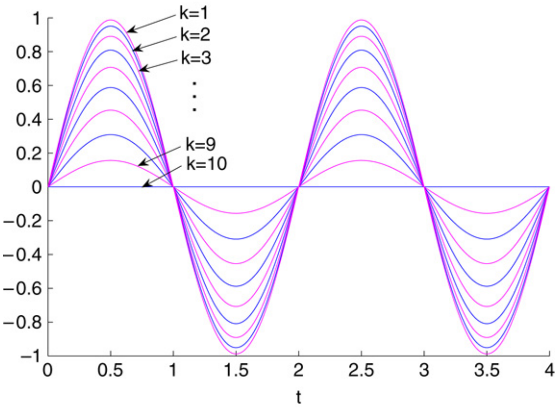

Fig. 3. Plot of the $v_i(t)$ for $i = 5k, k = 1, \ldots, 10$ in Example 1, the graphs of $v_i(t)$ for $i = 5k, k = 11, \ldots, 19$, are symmetries of $v_i(t), i = 5k, k = 9, \ldots, 1$, regarding the horizontal axis respectively.

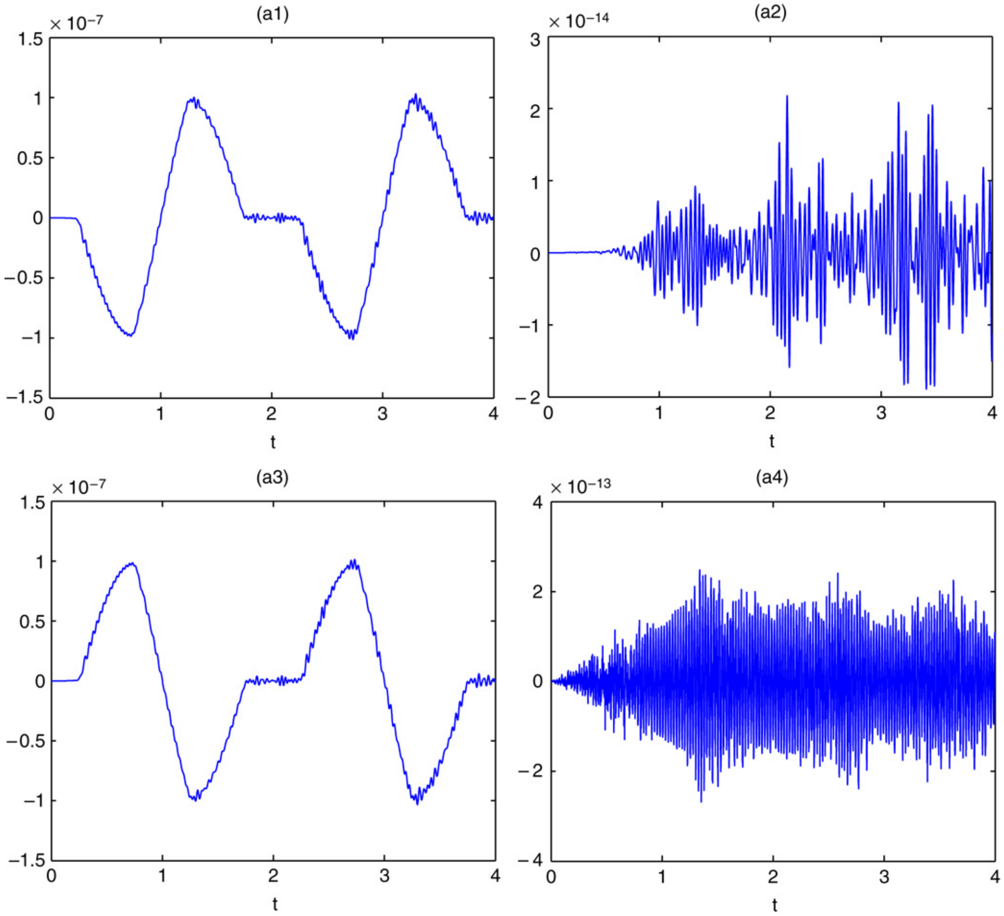

Fig. 4. Plot of the $v(x_i, t) - v_i(t)$ for (a1) $x_i = \frac{1}{4}$, (a2) $x_i = \frac{1}{2}$, (a3) $x_i = \frac{3}{4}$, (a4) $x_i = 1$ in Example 1.

$$\frac{\mathrm{d}u_{M-2}}{\mathrm{d}t} - \frac{v_{M-1}(16 - \frac{c_{M-1}}{c_M}) - v_{M-2}(30 + \frac{c_{M-2}}{c_M})}{12h^2} - \frac{v_{M-3}(16 - \frac{c_{M-3}}{c_M}) - v_{M-4}(1 + \frac{c_{M-4}}{c_M}) + \frac{1}{c_M}\sum_{i=1}^{M-5} c_i v_i(t)}{12h^2}$$
$$= q(x_{M-2}, t) + \frac{1}{12h^3 c_M}[hc_0 g_1(t) - g_2(t)], \quad 0 < t \leq T,$$

Table 1
Computational results for Example 1

x_i	Exact value	Absolute error by [1]	Absolute error by MOL I	Absolute error by MOL II
0.10	0.95105651629515	3.3×10^{-5}	$1.5970339260263 \times 10^{-5}$	$9.1695313453322 \times 10^{-8}$
0.20	0.80901699437495	3.0×10^{-5}	$2.2782454249470 \times 10^{-5}$	$8.0261219004285 \times 10^{-8}$
0.30	0.58778525229247	3.2×10^{-5}	$2.0901699376741 \times 10^{-5}$	$5.9441112476577 \times 10^{-8}$
0.40	0.30901699437495	3.1×10^{-5}	$1.2281581424911 \times 10^{-5}$	$3.3507949526168 \times 10^{-8}$
0.50	0.00000000000000	3.3×10^{-5}	$1.612449505348880 \times 10^{-17}$	$1.087626749329525 \times 10^{-16}$
0.60	−0.30901699437495	3.4×10^{-5}	$1.2281581424800 \times 10^{-5}$	$3.3507949692702 \times 10^{-8}$
0.70	−0.58778525229247	3.1×10^{-5}	$2.0901699376519 \times 10^{-5}$	$5.9441111255332 \times 10^{-8}$
0.80	−0.80901699437495	3.2×10^{-5}	$2.2782454249470 \times 10^{-5}$	$8.0261216561794 \times 10^{-8}$
0.90	−0.95105651629515	3.4×10^{-5}	$1.5970339258931 \times 10^{-5}$	$9.1695316228879 \times 10^{-8}$
1.00	−1.00000000000000	3.2×10^{-5}	$8.437694987151190 \times 10^{-15}$	$2.664535259100376 \times 10^{-15}$

$$\frac{du_{M-1}}{dt} - \frac{v_{M-2}(1-\frac{c_{M-2}}{c_M}) - v_{M-1}(2+\frac{c_{M-1}}{c_M}) - \frac{1}{c_M}\sum_{i=1}^{M-3} c_i v_i(t)}{h^2} = q(x_{M-1}, t) - \frac{1}{h^3 c_M}[hc_0 g_1(t) - g_2(t)],$$
$$0 < t \le T,$$
$$\frac{dv_i}{dt} = u_i(t), \quad i = 1, \dots, M-1, 0 < t \le T.$$

Now we can solve the resulting ODE system using an ODE solver with time local error control. As is mentioned in [23] the basic idea of the method of lines is to replace the spatial derivatives in the partial differential equation with algebraic approximations. Once this is carried out, the spatial derivatives are no longer stated explicitly in terms of the spatial independent variables. Thus, in effect only the initial value variable, typically time in a physical problem, remains. In other words, with only one remaining independent variable, we have a system of ordinary differential equations that approximate the original PDE. The challenge, then, is to formulate the approximating system of ODEs. Once this is carried out, we can apply any integration algorithm for initial value ODEs to compute an approximate numerical solution to the PDE. Thus, one of the salient features of the MOL is the use of existing, and generally well-established, numerical methods for ODEs [22]. In the current work, to solve the obtained ODE system, we use ode45 solver in MATLAB which is based on the explicit Runge–Kutta (4, 5) formula, the Dormand–Prince pair.

3. Numerical examples

In this section, some examples will be investigated to show the reliability and efficiency of the proposed schemes in this paper.

3.1. Test 1

As the first example, consider the Eqs. (1.1)–(1.5) with $l = 1$, $T = 4$,

$$q(x,t) = 0,$$
$$f_1(x) = 0, \qquad f_2(x) = \pi\cos(\pi x),$$
$$g_1(t) = \sin(\pi t), \qquad g_2(t) = 0.$$

The exact solution of this equation is

$$v(x,t) = \cos(\pi x)\sin(\pi t).$$

We compare the results obtained by the procedure in previous section with finite difference method introduced in [1] in Table 1. For comparison purpose, we set $h = 0.01$ and calculate the absolute errors for $t = \frac{1}{2}$ (the final time T in [1] is $\frac{1}{2}$).

Also the maximum norms of errors which are considered in the following are reported in Tables 2 and 3.

$$\|v_{\text{exact}} - v_{\text{MOL}}\|_{x_i,\infty} = \max_{0<t\le T} |v(x_i,t) - v_i(t)|.$$
$$\|v_{\text{exact}} - v_{\text{MOL}}\|_{\infty} = \max_{1\le i\le M}\max_{0<t\le T} |v(x_i,t) - v_i(t)|.$$

Some other numerical results are shown in Figs. 2–4. It is worth noting that since discretization in MOL is only applied to the spatial variable, increasing the final time T does not decrease the accuracy of the solution and we can obtain an improved solution by using a more accurate integrator for solving the ODE system, but in finite difference methods, there is usually limitations when final time T increases.

As it may be seen from Table 1, the results obtained using MOL I, have generally the same accuracy as those obtained from finite difference method in [1], but MOL II is more accurate than these two schemes.

Table 2
The norm $\|v_{\text{exact}} - v_{\text{MOL}}\|_{x_i,\infty}$ in Example 1

x_i	MOL I	MOL II
0.10	$1.9404299881232 \times 10^{-5}$	$1.0078024292870 \times 10^{-7}$
0.20	$2.9082139621384 \times 10^{-5}$	$1.0395382088468 \times 10^{-7}$
0.30	$2.7979999464800 \times 10^{-5}$	$9.519132848634 \times 10^{-8}$
0.40	$1.6949536840560 \times 10^{-5}$	$6.254244772075 \times 10^{-8}$
0.50	$1.109874786267721 \times 10^{-14}$	$2.177963786387436 \times 10^{-14}$
0.60	$1.6949536832678 \times 10^{-5}$	$6.254244200310 \times 10^{-8}$
0.70	$2.7979999469574 \times 10^{-5}$	$9.519131310975 \times 10^{-8}$
0.80	$2.9082139622272 \times 10^{-5}$	$1.0395382143980 \times 10^{-7}$
0.90	$1.9404299884340 \times 10^{-5}$	$1.0078024492710 \times 10^{-7}$
1.00	$5.933031843596837 \times 10^{-13}$	$2.680078381445128 \times 10^{-13}$

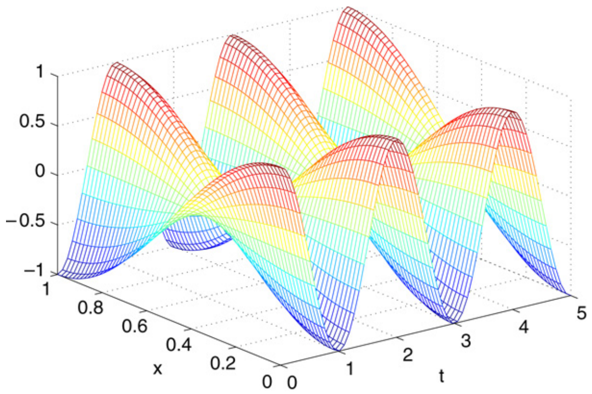

Fig. 5. Plot of the exact solution in Example 2.

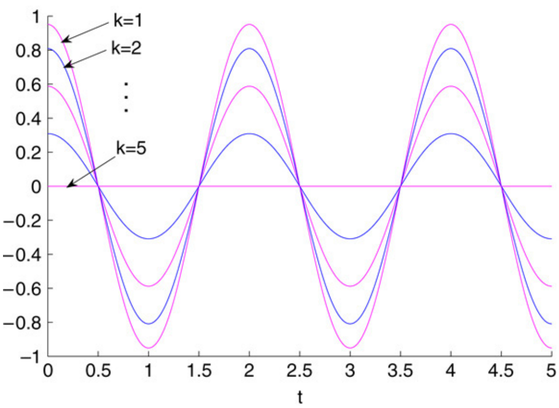

Fig. 6. Plot of the $v_i(t)$ for $i = 10k, k = 1, \ldots, 5$ in Example 2, the graphs of $v_i(t)$ for $i = 10k, k = 6, \ldots, 9$, are symmetries of $v_i(t), i = 10k, k = 4, \ldots, 1$, regarding the horizontal axis respectively.

3.2. Test 2

Consider the problem (1.1)–(1.5) with $l = 1, T = 5$,

$$q(x,t) = 0,$$
$$f_1(x) = \cos(\pi x), \qquad f_2(x) = 0,$$
$$g_1(t) = \cos(\pi t), \qquad g_2(t) = 0.$$

$v(x,t) = \frac{1}{2}(\cos(\pi(x+t)) + \cos(\pi(x-t)))$ is the exact solution of this system. The computational results obtained by MOL I and MOL II with $h = 0.01$ are compared with those investigated in [1] in Table 4. Because of the comparison, the absolute errors are tabulated at $t = \frac{1}{4}$. From this table, we see that the solution based on the MOL II is more accurate as compared to MOL I or the finite difference method developed in [1]. The norm of errors resulted by MOL I and MOL II, graphs of the

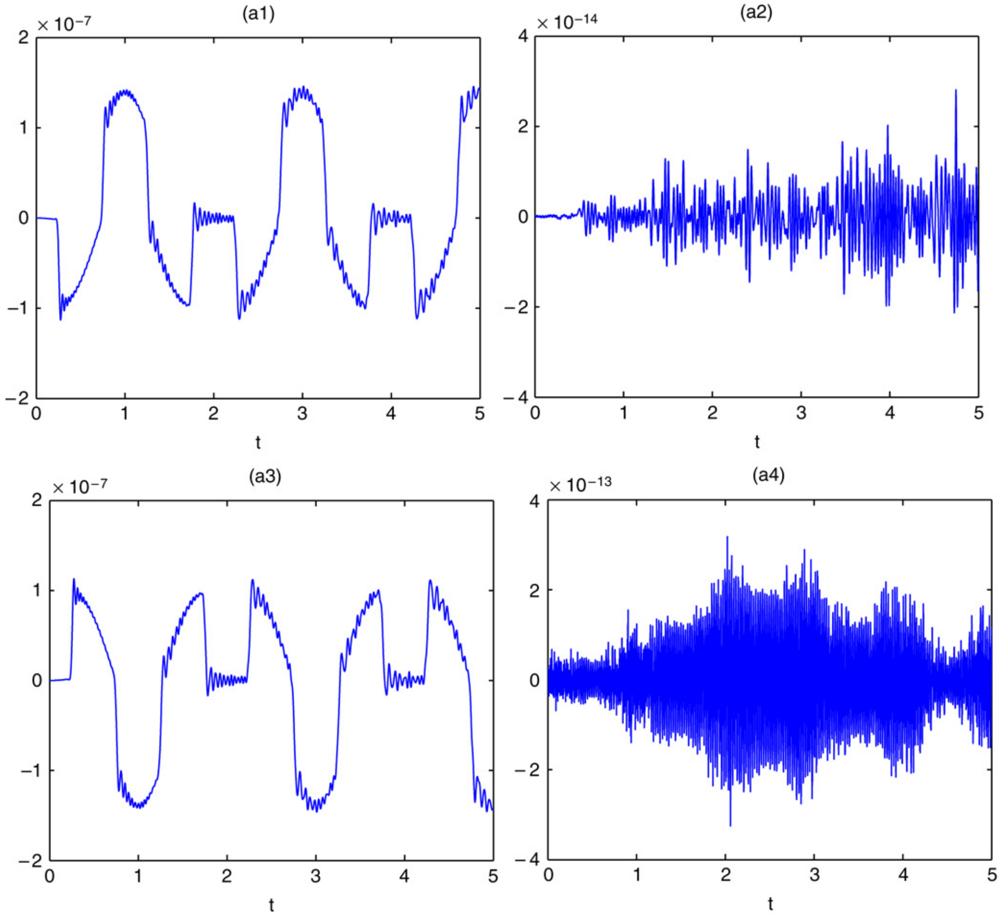

Fig. 7. Plot of the $v(x_i, t) - v_i(t)$ for (a1) $x_i = \frac{1}{4}$, (a2) $x_i = \frac{1}{2}$, (a3) $x_i = \frac{3}{4}$, (a4) $x_i = 1$ in Example 2.

Table 3
The norm $\|v_{\text{exact}} - v_{\text{MOL}}\|_\infty$ in Example 1

MOL I	MOL II
$2.9082139622272 \times 10^{-5}$	$1.0395382143980 \times 10^{-7}$

Table 4
Computational results for Example 2

x_i	Exact value	Absolute error by [1]	Absolute error by MOL I	Absolute error by MOL II
0.10	0.67249851196396	5.2×10^{-5}	$1.2923596914405 \times 10^{-5}$	$8.6007681532330 \times 10^{-8}$
0.20	0.57206140281768	5.1×10^{-5}	$1.7467564661144 \times 10^{-5}$	$8.9389747826019 \times 10^{-8}$
0.30	0.41562693777745	5.1×10^{-5}	$1.3423564604209 \times 10^{-5}$	$1.731384369208 \times 10^{-9}$
0.40	0.21850801222441	5.3×10^{-5}	$7.057215508699 \times 10^{-6}$	$9.28636417763 \times 10^{-10}$
0.50	0.00000000000000	5.0×10^{-5}	$1.312806175591734 \times 10^{-17}$	$6.396931098296924 \times 10^{-17}$
0.60	−0.21850801222441	5.2×10^{-5}	$7.057215509060 \times 10^{-6}$	$9.28636584296 \times 10^{-10}$
0.70	−0.41562693777745	5.4×10^{-5}	$1.3423564604320 \times 10^{-5}$	$1.731384868808 \times 10^{-9}$
0.80	−0.57206140281768	5.3×10^{-5}	$1.7467564658924 \times 10^{-5}$	$8.9389752599978 \times 10^{-8}$
0.90	−0.67249851196396	5.5×10^{-5}	$1.2923596913073 \times 10^{-5}$	$8.6007680755174 \times 10^{-8}$
1.00	−0.70710678118655	5.4×10^{-5}	$3.952393967665557 \times 10^{-14}$	$2.198241588757810 \times 10^{-14}$

exact solution, approximate solution $v_i(t)$ for several values of i and errors obtained by MOL II are given in Tables 5 and 6 and Figs. 5–7 respectively.

Table 5
The norm $\|v_{\text{exact}} - v_{\text{MOL}}\|_{x_i,\infty}$ in Example 2

x_i	MOL I	MOL II
0.10	$3.204176544324699 \times 10^{-5}$	$1.984972012314401 \times 10^{-7}$
0.20	$4.558331840376351 \times 10^{-5}$	$1.679581814739706 \times 10^{-7}$
0.30	$4.182611406278181 \times 10^{-5}$	$1.219577727695764 \times 10^{-7}$
0.40	$2.458643023162122 \times 10^{-5}$	$1.137402102918683 \times 10^{-7}$
0.50	$1.242518910410138 \times 10^{-14}$	$2.811945523278168 \times 10^{-14}$
0.60	$2.458643024139118 \times 10^{-5}$	$1.137402084530614 \times 10^{-7}$
0.70	$4.182611405878500 \times 10^{-5}$	$1.219577708821973 \times 10^{-7}$
0.80	$4.558331840875951 \times 10^{-5}$	$1.679581880242864 \times 10^{-7}$
0.90	$3.204176544702175 \times 10^{-5}$	$1.984971881308084 \times 10^{-7}$
1.00	$3.066435994014682 \times 10^{-13}$	$3.248512570053208 \times 10^{-13}$

Table 6
The norm $\|v_{\text{exact}} - v_{\text{MOL}}\|_{\infty}$ in Example 2

MOL I	MOL II
$4.558331840875951 \times 10^{-5}$	$1.984972012314401 \times 10^{-7}$

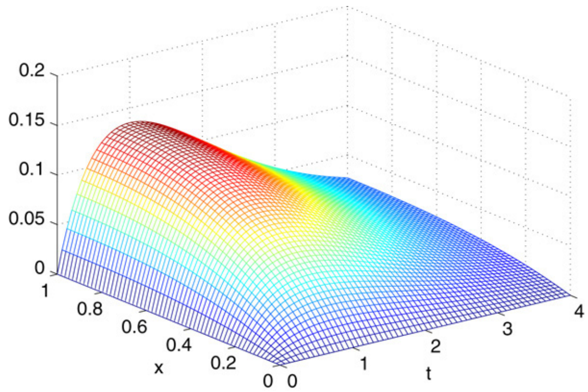

Fig. 8. Plot of the exact solution in Example 3.

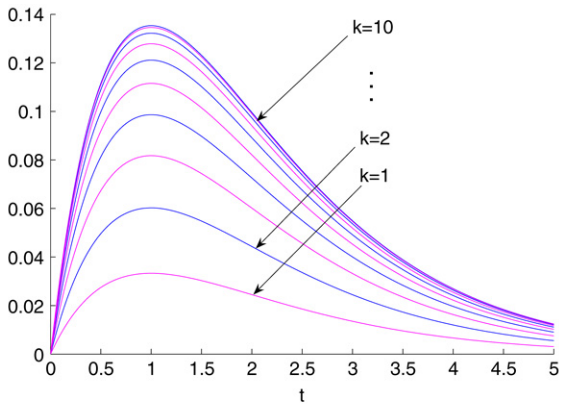

Fig. 9. Plot of the $v_i(t)$ for $i = 10k$, $k = 1, \ldots, 10$ in Example 3.

3.3. Test 3

In this example, we apply the proposed methods in this paper to find the solution of (1.1)–(1.5) with $l = 1$, $T = 4$,

$$q(x,t) = -2(x-t)\exp(-x-t),$$
$$f_1(x) = 0, \qquad f_2(x) = x\exp(-x),$$
$$g_1(t) = 0, \qquad g_2(t) = -2t\exp(-t-1) + t\exp(-t).$$

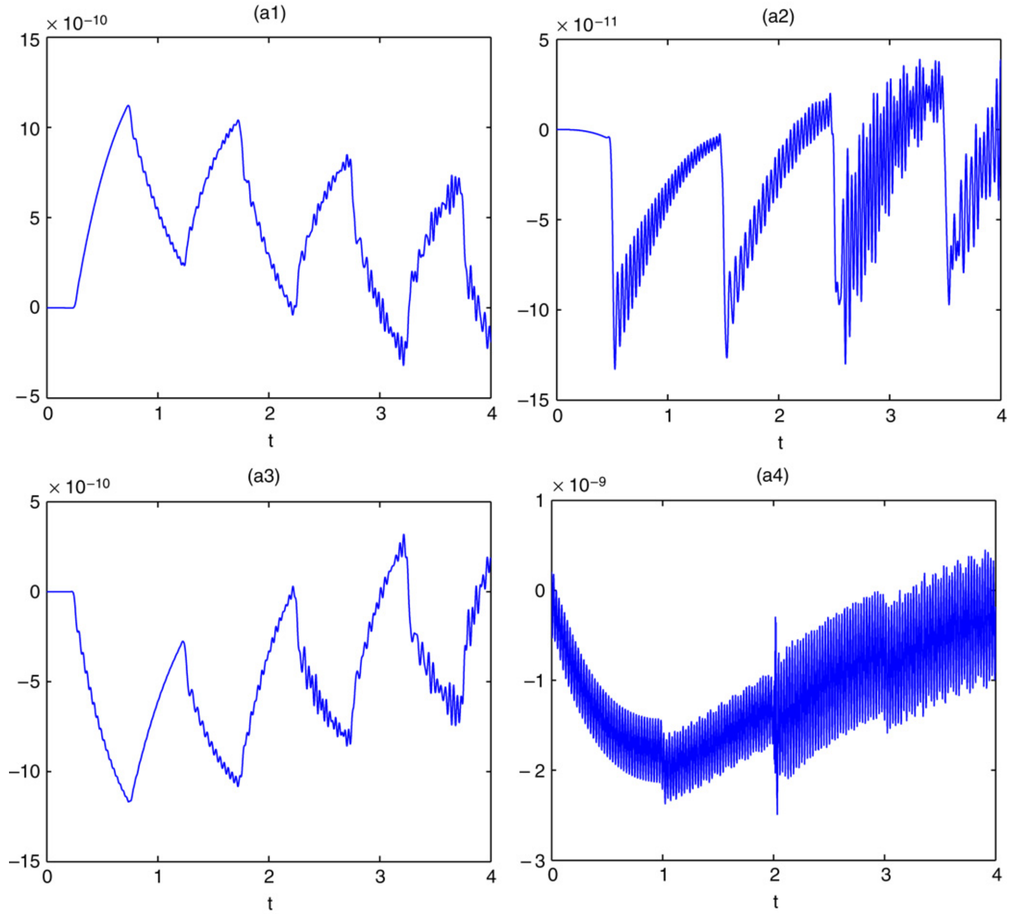

Fig. 10. Plot of the $v(x_i, t) - v_i(t)$ for (a1) $x_i = \frac{1}{4}$, (a2) $x_i = \frac{1}{2}$, (a3) $x_i = \frac{3}{4}$, (a4) $x_i = 1$ in Example 3.

Table 7
The norm $\|v_{\text{exact}} - v_{\text{MOL}}\|_{x_i,\infty}$ in Example 3

x_i	MOL I	MOL II
0.10	$8.790697957105 \times 10^{-7}$	$1.3567209786181 \times 10^{-9}$
0.20	$1.2140191431882 \times 10^{-6}$	$1.2268780713587 \times 10^{-9}$
0.30	$1.1133469118643 \times 10^{-6}$	$9.826361113685 \times 10^{-10}$
0.40	$7.043162150097 \times 10^{-7}$	$6.474974688364 \times 10^{-10}$
0.50	$4.225513185602 \times 10^{-7}$	$1.327798704320 \times 10^{-10}$
0.60	$7.709645756804 \times 10^{-7}$	$7.196475637627 \times 10^{-10}$
0.70	$1.0072074030434 \times 10^{-6}$	$1.0519506238316 \times 10^{-9}$
0.80	$1.2092548875610 \times 10^{-6}$	$1.2563831364165 \times 10^{-9}$
0.90	$1.1058150297394 \times 10^{-6}$	$1.3559769695970 \times 10^{-9}$
1.00	$1.3828775877189 \times 10^{-6}$	$2.4905782158857 \times 10^{-9}$

$v(x, t) = xt\exp(-(x+t))$ is the exact solution of this problem which can be readily verified. Such as the previous examples, the results are shown in Tables 7 and 8 and Figs. 8–10.

3.4. Test 4

In the fourth example, we solve the problem (1.1)–(1.5) with $l = 1, T = 5$,

$$q(x, t) = 2x^5 + 2x^3 - 2x^2 - (20x^3 + 6x - 2)(t^2 - t),$$

$$f_1(x) = 0, \qquad f_2(x) = -x^5 - x^3 + x^2,$$

$$g_1(t) = 0, \qquad g_2(t) = \frac{t(t-1)}{12}.$$

Table 8
The norm $\|v_{\text{exact}} - v_{\text{MOL}}\|_\infty$ in Example 3

MOL I	MOL II
$1.3828775877189 \times 10^{-6}$	$2.4905782158857 \times 10^{-9}$

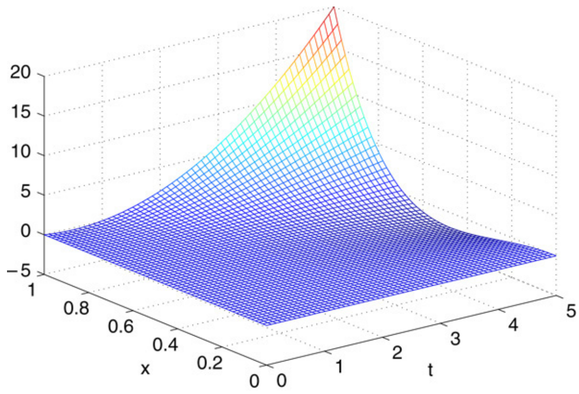

Fig. 11. Plot of the exact solution in Example 4.

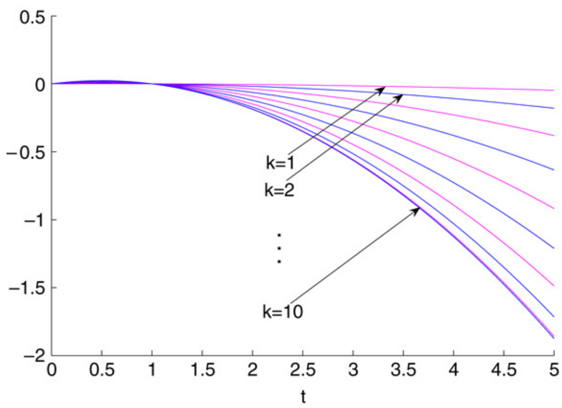

Fig. 12. Plot of the $v_i(t)$ for $i = 5k, k = 1, \ldots, 10$ in Example 4.

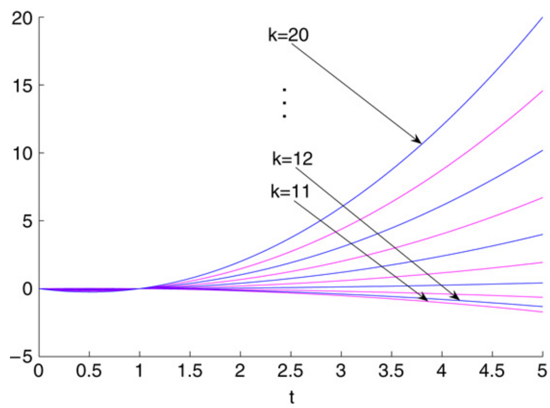

Fig. 13. Plot of the $v_i(t)$ for $i = 5k, k = 11, \ldots, 20$ in Example 4.

The exact solution of this equation is

$$v(x, t) = (x^5 + x^3 - x^2)(t^2 - t).$$

Some numerical results are summarized in Tables 9 and 10 and Figs. 11–14.

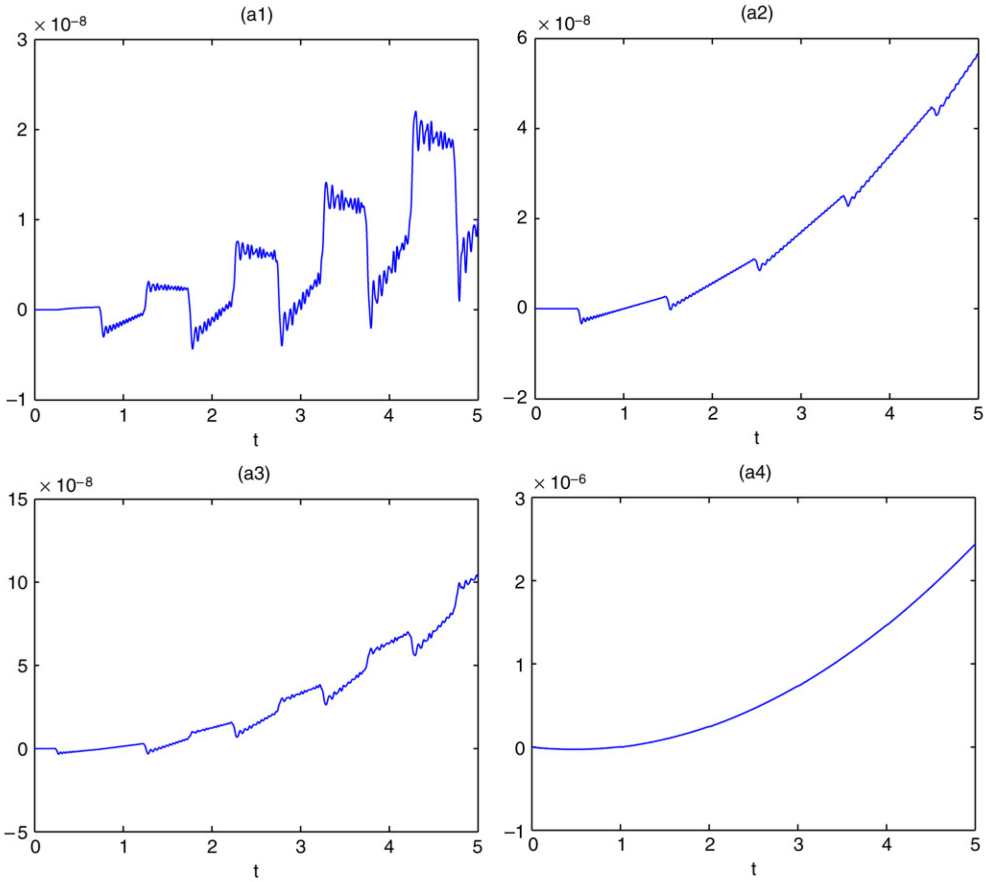

Fig. 14. Plot of the $v(x_i, t) - v_i(t)$ for (a1) $x_i = \frac{1}{4}$, (a2) $x_i = \frac{1}{2}$, (a3) $x_i = \frac{3}{4}$, (a4) $x_i = 1$ in Example 4.

Table 9
The norm $\|v_{\text{exact}} - v_{\text{MOL}}\|_{x_i,\infty}$ in Example 4

x_i	MOL I	MOL II
0.10	$1.4589452381 \times 10^{-4}$	$2.27205158021 \times 10^{-8}$
0.20	$2.7172977818 \times 10^{-4}$	$1.59301172808 \times 10^{-8}$
0.30	$3.7488135133 \times 10^{-4}$	$2.98653940467 \times 10^{-8}$
0.40	$4.3322433196 \times 10^{-4}$	$4.39071972114 \times 10^{-8}$
0.50	$4.1659346493 \times 10^{-4}$	$5.65441311551 \times 10^{-8}$
0.60	$2.9994415140 \times 10^{-4}$	$7.65543484160 \times 10^{-8}$
0.70	$5.996250296 \times 10^{-5}$	$9.44377208656 \times 10^{-8}$
0.80	$3.3337581084 \times 10^{-4}$	$1.134464433505 \times 10^{-7}$
0.90	$8.9980978039 \times 10^{-4}$	$1.326242955457 \times 10^{-7}$
1.00	$1.66653416461 \times 10^{-3}$	$2.4346786631213 \times 10^{-6}$

Table 10
The norm $\|v_{\text{exact}} - v_{\text{MOL}}\|_{\infty}$ in Example 4

MOL I	MOL II
$1.66653416461 \times 10^{-3}$	$2.4346786631213 \times 10^{-6}$

The MOL approach usually enables us to solve quite general and complicated partial differential equations relatively easily and with acceptable efficiency. It is applicable to a wide range of problems in many areas [47]. The reader can see the Appendix of [22] for some problems in physics, fluid dynamics, reactor models, automatic control, and more.

4. Conclusion

The method of lines is generally recognized as a comprehensive and powerful approach to the numerical solution of time-dependent partial differential equations. This method proceeds in two separate steps: Spatial derivatives are first replaced with finite difference, finite volume, finite element or other algebraic approximations and then the resulting system of ordinary differential equations which is usually stiff, is integrated in time. The success of this method is explained by the availability of high-quality numerical algorithms for the solution of stiff systems of ODEs. This paper investigated MOL approach for solving the one-dimensional hyperbolic equation with an integral condition [1,48]. Also the new algorithms were tested on several problems. The computational results confirmed the efficiency, reliability and accuracy of this procedure. It is worth pointing out that the solutions for MOL 1 and MOL 2 (second- and fourth-order finite differences, respectively) clearly demonstrate the improved accuracy of the higher-order (MOL 2) method. Also, this superior performance is achieved with very little increased computational effort (since the fourth-order finite difference approximation of MOL 2 requires only five terms in a weighted sum compared with three terms in MOL 1). Thus, clearly the use of higher-order finite differences in the MOL solution of the wave equation, and generally for other partial differential equations, is recommended. Finally we would like to mention that one issue of future work is to develop similar method to solve the parabolic inverse problems investigated in [49]. Other task is to apply the technique proposed in the current research to find the solutions of the problems studied in [50–53].

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