Set Theory - exercise 10

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Question 3:

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Proposition: Let α be an ordinal and define $\alpha + 1 = \alpha \cup \{\alpha\}$, then $\alpha + 1$ is an ordinal and there is no ordinal β such that $\alpha \in \beta \in \alpha + 1$

Proof:

$\alpha + 1$ is an ordinal:

We need to prove that $\alpha + 1$ is transitive and that $\langle \alpha + 1, \in \rangle$ is well ordered.

$\alpha + 1$ is transitive:

For $c \in b \in \alpha + 1$, if $b \in a$ then from α transitivity, we get $c \in \alpha + 1$. If $b = \alpha$,then $c \in \alpha \Rightarrow c \in \alpha + 1$.

The relation $\langle \alpha + 1, \in \rangle$ is well ordered:

i. $\langle \alpha + 1, \in \rangle$ is a (strict) total order:

a. $\langle \alpha + 1, \in \rangle$ is transitive: For $a, b, c \in \alpha + 1$ s.t. $a \in b$ and $b \in c$.

if $c \in \alpha \Rightarrow$ since $\langle \alpha, \in \rangle$ is a transitive order, we get $a \in c$.

if $c = \alpha$ from transitivity of α we get $a \in c$.

b. $\langle \alpha + 1, \in \rangle$ is a-symmetric: For $b, c \in \alpha + 1$ s.t. $b \in c$, if we assume $c \in b$, then we get $b \in c \in b$ and from from transitivity of $\langle \alpha + 1, \in \rangle$, we get $b \in b$ - contradiction to the axiom of foundation.

c. For $b, c \in \alpha + 1$ s.t. $b \neq c$, it can't be that $b = \alpha = c$,then at least one of b and c is an element of α. If both $b, c \in \alpha$, we get $b \in c$ or $c \in b$ because α is an ordinal.

w.l.o.g. $b = \alpha$ then $c \in \alpha = b$.

ii. $\alpha + 1$ is well founded:

For $B \neq \emptyset$, $B \subseteq \alpha+1$. If $B = \{\alpha\}$,then $B \setminus \{\alpha\} = \emptyset$, otherwise $\emptyset \neq B \setminus \{\alpha\} \subseteq \alpha$, and since α is well founded, we get that $B \setminus \{\alpha\}$ has a minimal element c.

$c \in B \setminus \{\alpha\} \subseteq \alpha \Rightarrow c \in \alpha$. We get that c is a minimal element of B.

Assuming that there is ordinal β s.t $\alpha \in \beta \in \alpha + 1$:

$\beta \in \alpha + 1 \Rightarrow \beta = \alpha$ or $\beta \in \alpha$.

If $\beta = \alpha$ we get $\alpha \in \alpha$ - contradiction to foundation axiom.

If $\beta \in \alpha$, we get $\alpha \in \beta \in \alpha$ and from transitivity of α,we get $\alpha \in \alpha$ - contradiction.