Hyperfactorial

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1 Hyperfactorial

The hyperfactorial (Sloane and Plouffe 1995) is the function defined by

$$H(n) = K(n+1) = \prod_{k=1}^{n} k^k, \tag{1}$$

where $K(n)$ is the K-function.
The hyperfactorial is implemented in the Wolfram Language as Hyperfactorial[n].
For integer values n=1, 2, ... are 1, 4, 108, 27648, 86400000, 4031078400000, 3319766398771200000, ... (OEIS A002109).
The hyperfactorial can also be generalized to complex numbers, as illustrated above.
The Barnes G-function and hyperfactorial $H(z)$ satisfy the relation

$$H(z-1)G(z) = e^{(z-1)\log\Gamma(z)} \tag{2}$$

for all complex z.
The hyperfactorial is given by the integral

$$H(z) = (2\pi)^{-\frac{z}{2}} e^{(z+1)/2+\int_0^z \ln(t!)dt} \tag{3}$$

and the closed-form expression

$$K(z) = e^{\zeta^{'}(-1,z+1)-\zeta^{'}(-1)} \tag{4}$$

for $R[z] > 0$, where $\zeta(z)$ is the Riemann zeta function, $\zeta^{'}(z)$ its derivative, $\zeta(a,z)$ is the Hurwitz zeta function, and

$$\zeta^{'}(a,z) = \left[\frac{\partial\zeta(s,z)}{\partial s}\right]_{s=a} \tag{5}$$

$H(z)$ also has a Stirling-like series

$$H(z) \sim Ae^{-\frac{z^2}{4}} z^{\frac{z(z+1)}{2}+\frac{1}{12}} \times (1 + \frac{1}{720z^2} - \frac{1433}{7257600z^4} + \dots) \tag{6}$$

(OEIS A143475 and A143476).
$H(-1/2)$ has the special value

$$H(-1/2) = e^{-\frac{[\ln 2/3+12\zeta^{'}(-1)]}{8}} \tag{7}$$

$$= 2^{\frac{1}{12}}\pi^{\frac{1}{8}}e^{\frac{[\gamma-1-\zeta^{'}(2)/\zeta(2)]}{8}} \tag{8}$$

$$= \frac{A^{3/2}}{2^{1/24}e^{1/8}}, \tag{9}$$

where γ is the Euler-Mascheroni constant and A is the Glaisher-Kinkelin constant.
The derivative is given by

$$\frac{dH(x)}{dx} = H(x)\left(\frac{1}{2}\left[1-\ln(2\pi)\right]+\ln(\Gamma(x+1))+x\right) \tag{10}$$