Finite-time limit theorems for chaotic dynamics via Stein method

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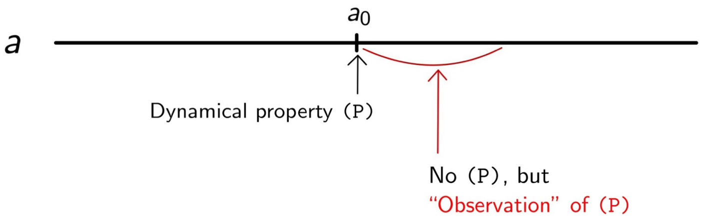
HSI: Computational Methods in Dynamical Systems 2025

29 August 2025

Motivation for this talk

$(f_a : X \to X)_a$



- Spectral approach of "some" transfer op. works in toy models
- Stein method: "beyond" spectral gap

Limit theorems in dynamical systems

Let

- X: measurable space with a "natural" probability measure m (e.g. cpt Riem. manifold with the normalized Lebesgue measure)
- $f : X \to X$: measurable

Limit theorems in dynamical systems

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Q. What is the behavior of $S_n\varphi := \varphi + \varphi \circ f + \cdots + \varphi \circ f^{n-1}$ as $n \to \infty$?

$\exists$ probability measure μ s.t.

$$\lim_{n\to\infty} \frac{S_n\varphi(x)}{n} = \int \varphi \, d\mu \quad m\text{-a.e. } x?$$

and other limit theorems: e.g. $\exists\ \sigma_\varphi^2 \in (0,\infty)$, $\forall a \in \mathbb{R}$,

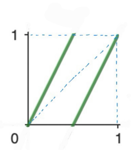
$$m\left(\left\{x : \frac{S_n\varphi(x) - n\int \varphi d\mu}{\sqrt{n}} \leq a\right\}\right) \to \mathrm{Prob}(N(0,\sigma_\varphi^2) \leq a)?$$

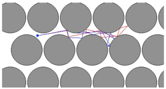
Limit theorems in dynamical systems

A. Yes, for “chaotic” dynamics

like (generic) piecewise expanding maps (loosely, $\operatorname{ess\,inf}_x |f'(x)| > 1$)

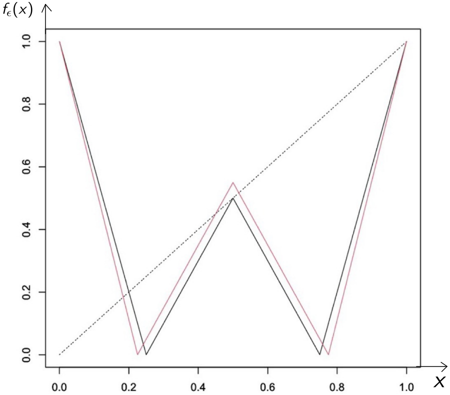
- E.g. $f(x) = 2x \mod 1$ on $X = \mathbb{R}/\mathbb{Z}$
- Related: Lorentz gas, geodesic flow on hyperbolic mfd ...





Finite-time limit theorems

- Blank–Keller's W map (1997)

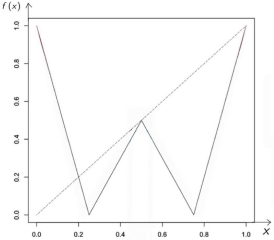


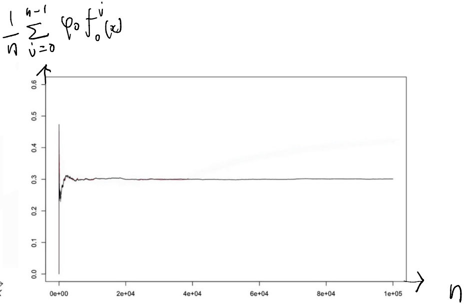
$f_\epsilon : [0,1] \to [0,1]$: piecewise linear

$f_\epsilon(\frac{1}{2}) = \frac{1}{2} + \epsilon$, $f_\epsilon(\frac{1}{2} \pm \epsilon) = \frac{1}{2} - \epsilon$

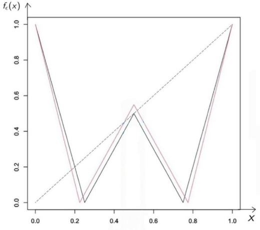
$\epsilon = 0$ (black)
$\epsilon > 0$ (red)

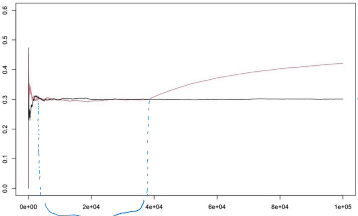
Finite-time limit theorems





Finite-time limit theorems





$\varepsilon = 0$

$\varepsilon > 0$

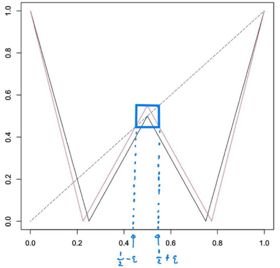
1

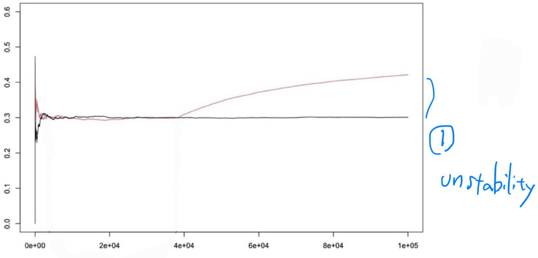
unstability (BK)

2

finite-time stability ?

Finite-time limit theorems





For observation 1: Statistically unstable (Blank–Keller, 1997):

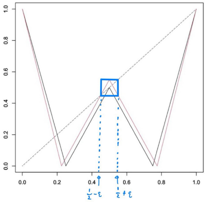
$\forall \epsilon \in [0, \frac{1}{4}]$, $\exists!$ (a.c. invariant) probability measure μ_ϵ,

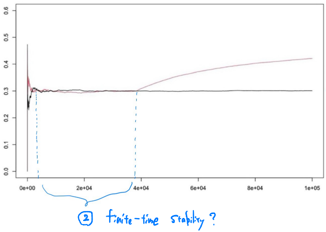
$$\lim_{n\to\infty} \frac{S_{n,\epsilon}\varphi(x)}{n} = \int \varphi \, d\mu_\epsilon \quad \text{Leb-a.e. } x$$

for all $\varphi \in C^1([0,1])$, where $S_{n,\epsilon}\varphi := \sum_{j=0}^{n-1} \varphi \circ f_\epsilon^j$, while

$$\int \varphi \, d\mu_\epsilon \not\to \int \varphi \, d\mu_0 \quad (\epsilon \to 0)$$

Finite-time limit theorems





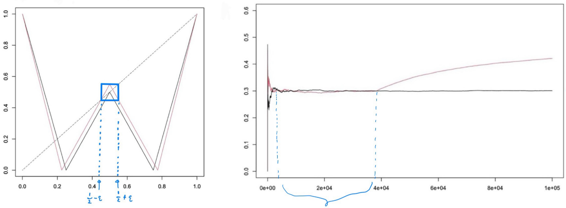
Idea for observation 2:

If $f_\epsilon^j(x) \notin H_\epsilon := [1/2-\epsilon, 1/2+\epsilon]$ for all $j \in [0,n]$ with large n,

$$\frac{S_{n,\epsilon}\varphi(x)}{n} \sim \frac{S_{n,0}\varphi(x)}{n}?$$

(Warning: $\mathrm{Leb}(\{x : f_\epsilon^j(x) \notin H_\epsilon$ for all $j \geq 0\}) = 0$)

Finite-time limit theorems



② finite-time stability ?

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$\rightarrow$ conditional probability (& escape time)

$$\mathrm{Leb}(A \,|\, X_{n,\epsilon}) := \frac{\mathrm{Leb}(A \cap X_{n,\epsilon})}{\mathrm{Leb}(X_{n,\epsilon})}, \quad X_{n,\epsilon} := \{x : f_\epsilon^j(x) \notin H_\epsilon \quad \forall j \in [0, n]\}$$

Conditional WLLN

Thm. (with J. Atnip, G. Froyland, C. González-Tokman, S. Vaienti)
$\forall \epsilon \in [0, 1/4]$, $\exists !$ probability measure $\tilde{\mu}_\epsilon$

(1) $\forall \delta > 0$, $\forall \varphi \in C^1([0,1])$,

$$\mathrm{Leb}\left(\left\{x : \left|\frac{S_{n,\epsilon}\varphi(x)}{n} - \int \varphi \, d\tilde{\mu}_\epsilon\right| > \delta\right\} \,\middle|\, X_{n,\epsilon}\right) \to 0$$

(2) $\forall \varphi \in C^1([0,1])$,

$$\int \varphi \, d\tilde{\mu}_\epsilon \to \int \varphi \, d\mu_0$$

(3) $\mathrm{Leb}(X_{n,\epsilon}) \geq \lambda_\epsilon^n (1 + O(\epsilon)) - \kappa^n$ with $\lambda_\epsilon = 1 - \frac{3}{2}\epsilon + o(\epsilon)$, $\kappa \in (0,1)$

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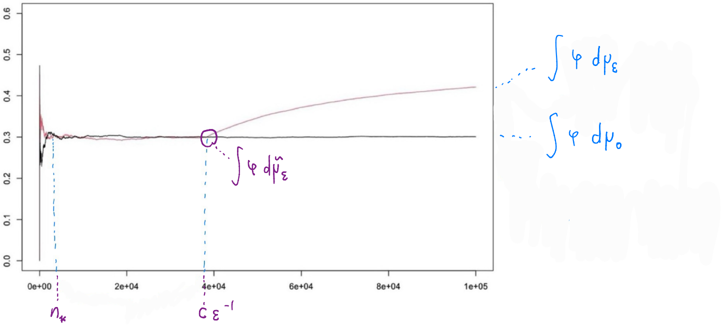
- By (3), $\mathrm{Leb}(X_{n,\epsilon}) \geq 0.99$ whenever $n_\kappa \leq n \leq C\epsilon^{-1}$

  Namely: With 99 % of x,

$$\frac{S_{n,\epsilon}\varphi(x)}{n} \overset{\delta}{\sim} \begin{cases} \int \varphi \, d\tilde{\mu}_\epsilon \sim \int \varphi \, d\mu_0 & (n_\kappa \leq n \leq C\epsilon^{-1}) \\ \int \varphi \, d\mu_\epsilon \not\sim \int \varphi \, d\mu_0 & (n \gg C\epsilon^{-1}) \end{cases}$$

Conditional WLLN



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Conditional CLT

Thm. (with J. Atnip, G. Froyland, C. González-Tokman, S. Vaienti)

(4) for generic $\varphi \in C^1([0,1])$, $\exists\ \sigma_\varphi^2 \in (0,\infty)$, $\forall a \in \mathbb{R}$,

$$\mathrm{Leb}\left(\left\{x : \frac{S_{n,\epsilon}\varphi(x) - n\int \varphi d\tilde{\mu}_\epsilon}{\sqrt{\sigma^2 n}} \leq a\right\} \Big| X_{n,\epsilon}\right) \to \mathrm{Prob}(\mathcal{N} \leq a)$$

where $\mathcal{N}$ is the standard normal distribution.

- Generalization 1: **all** statistically unstable 1D piecewise expanding
- Generalization 2: BV density, **conformal measures** instead of Leb
- Generalization 3: **quenched random** 1D piecewise expanding maps
- Generalization 4: some deterministic **higher-D** piecewise expanding maps

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Q. How fast? Functional CLT? Generalization beyond spectral gap?
$\to$ **Stein method**

Another formulation of finite-time limit theorems

Cor. $\forall \delta > 0$, $\forall \varphi \in C^1([0,1])$, $\forall \; \{\epsilon(n)\}_{n\geq 1}$ satisfying $\epsilon(n) = o(n^{-1})$,

$$\mathrm{Leb}\left(\left\{x : \left|\frac{S_{n,\epsilon(n)}\varphi(x)}{n} - \int \varphi \, d\mu_0\right| > \delta\right\}\right) \to 0$$

- Namely: for Leb-a.e. x,

$$\frac{S_{n,\epsilon(n)}\varphi(x)}{n} \to \int \varphi \, d\mu_0$$

$$\frac{S_{n,\epsilon}\varphi(x)}{n} \to \int \varphi \, d\mu_\epsilon \not\to \int \varphi \, d\mu_0$$

($\{n : \epsilon(n) > \epsilon\} \sim [0, C\epsilon^{-1}]$ is a metastable period)

- No $\tilde{\mu}_\epsilon$ or $X_{n,\epsilon}$ (this formulation can extend to properties without holes; e.g. finite-time bifurcation in a modified SIR model)

Related works

1. Estimate of escape **time** for SDE (Kramers 1940)
2. Similar but different definitions for some Markov chains $(\chi_n)_{n\geq 1}$ with an absorbing state H, e.g. **for all** $x \in X$

$$E_x\left[\varphi(\chi_n) \,|\, \chi_j \notin H \quad \text{for all } j \leq n\right] \to \int \varphi \, d\mu$$

   (Yaglom 1947), where $E_x[\cdot]$ means the expectation under $\chi_1 = x$
3. Escape **time** in terms of conditionally invariant measure for (random) piecewise expanding maps: Pianigiani–Yorke (1979), Demers–Young (2005), Keller–Liverani (2008), etc
4. **Conditional limit theorems** for dynamics: Exceptional work by Collet–Martínez (1999) on some CLT for subshifts of finite type

Idea of the proof: Spectral gap

CLT for iid r.v.:
For iid $\{\phi_n\}$ of 0-mean 1-variance, with $\Delta_n := \frac{\phi_1+\cdots+\phi_n}{\sqrt{n}}$,

$$E[e^{i\theta\Delta_n}] = \left\{\lambda\left(\frac{\theta}{\sqrt{n}}\right)\right\}^n \quad \text{with} \quad \lambda(t) := E[e^{it\phi_1}] = 1+\underbrace{E[\phi_1]}_{=0}\, it+\underbrace{E[\phi_1^2]}_{=1}\,\frac{(it)^2}{2}+o(t^2)$$

(so $E[e^{i\theta\Delta_n}] \to e^{-\frac{\theta^2}{2}} = E[e^{i\theta\mathcal{N}}]$)

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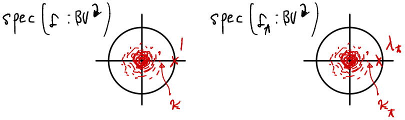
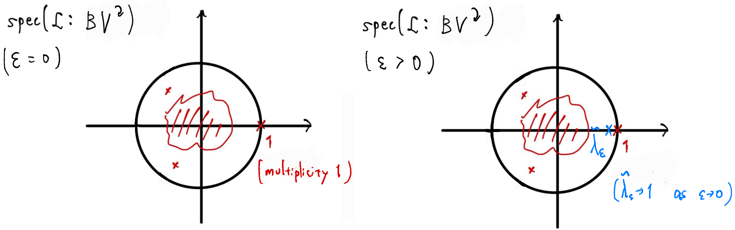


CLT under spectral gap (classical):
For $\phi_n = \varphi \circ f^n$ with spectral gap for $\mathcal{L}$, with $\mathcal{L}_t\psi := \mathcal{L}(e^{it\varphi}\psi)$,

$$\int e^{i\theta\Delta_n}\, d\mathrm{Leb} = \int \mathcal{L}^n_{\frac{\theta}{\sqrt{n}}} 1\, d\mathrm{Leb} = \left\{\lambda\left(\frac{\theta}{\sqrt{n}}\right)\right\}^n + O(\kappa^n) \quad \text{where} \quad \mathcal{L}_t h_t = \lambda(t) h_t$$

while $\lambda(t) = 1 + E_\mu[\varphi]it + (V_\mu[\varphi] - E_\mu[\varphi]^2)\frac{(it)^2}{2} + o(t^2)$
("because" $\mathcal{L}_t\psi = \mathcal{L}\psi + t\mathcal{L}(\varphi\psi) + \frac{t^2}{2}\mathcal{L}(\varphi^2\psi) + o(t^2)$)

Idea of the proof: Spectral gap



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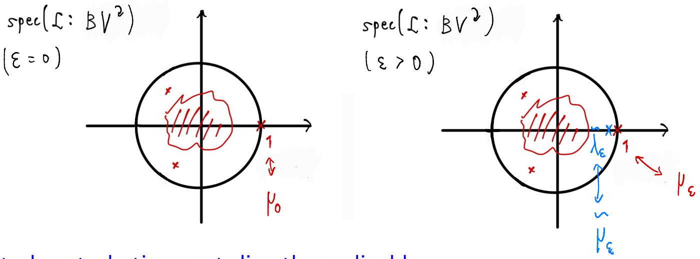


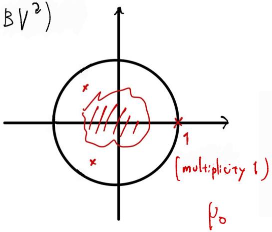
spectral perturbation: not directly applicable

$(\mu_\varepsilon \not\to \mu_0)$

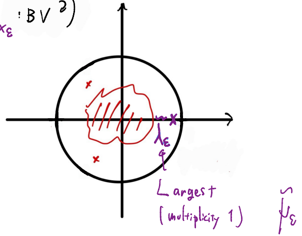
Idea of the proof: Spectral gap

$\left(X_\varepsilon = [0,1] \setminus H_\varepsilon \right)$

$\mathrm{spec}(\mathcal{L} : BV^2)$
$(\varepsilon = 0)$



$\mathrm{spec}(\mathcal{L} 1_{X_\varepsilon} : BV^2)$
$(\varepsilon > 0)$

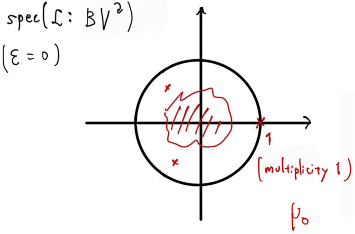


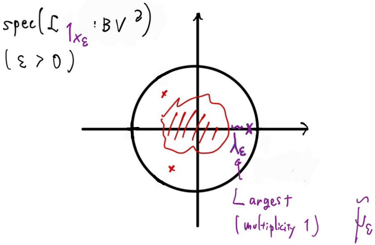
Consider

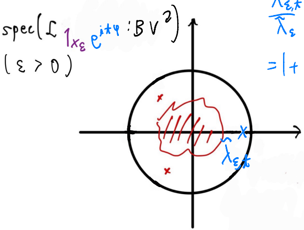
$$\mathcal{L}^n(1_{X_{n-1,\epsilon}} \cdot) = (\mathcal{L}1_{X_\epsilon})^n, \qquad \left(\text{so} \quad \hat{\mu}_\varepsilon \to \mu_0 \right)$$

spectrally close to $(\mathcal{L}|_{\epsilon=0})^n$ (by etimate $||(\mathcal{L}1_{X_{\epsilon(n)}})^n - (\mathcal{L}|_{\epsilon=0})^n|| \leq \lambda^n$, $\lambda \in (0,1)$)

Idea of the proof: Spectral gap







$$\frac{\tilde{\lambda}_{\varepsilon,t}}{\tilde{\lambda}_\varepsilon} = 1 + \int \varphi \, d\tilde{\mu}_\varepsilon \, t + \frac{\sigma_\varphi^2(\tilde{\mu}_\varepsilon)}{2} t^2 + o(t^2)$$

Idea of the proof: Spectral gap

$\mathrm{spec}(\mathcal{L} : BV^2)$

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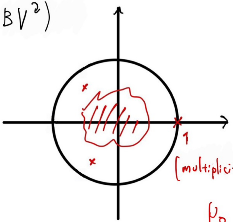

$\mathrm{spec}(\mathcal{L}\, 1_{X_\varepsilon} : BV^2)$

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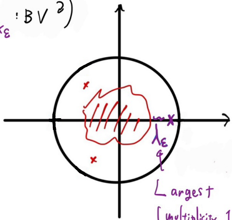
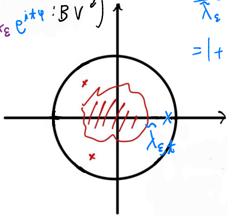


$\tilde{\mu}_\varepsilon$

$\mathrm{spec}(\mathcal{L}\, 1_{X_\varepsilon} e^{it\varphi} : BV^2)$

$(\varepsilon > 0)$



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Stein Method

Stein method: estimate the “distance” between probability distributions using a characteristic operator of the target distribution.

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(1) With $\mathcal{A}\gamma(x) := -x\gamma'(x) + \gamma''(x)$, for any r.v. Δ,

$$d_{\mathrm{Wasserstein}}(\Delta, \mathcal{N}) \leq \sup_{\gamma \in C^{2+\mathrm{Lip}}} E[\mathcal{A}\gamma(\Delta)]$$

(note: $E[\mathcal{A}\gamma(\mathcal{N})] = 0$ for all $\gamma \in C^{2+\mathrm{Lip}}$; $\mathcal{A}$: generator of diffusion)

$$d_{\mathrm{Wasserstein}}(\Delta, \mathcal{N}) := \sup_{\varphi \in \mathrm{Lip}^1} \left| E[\varphi(\Delta)] - E[\varphi(\mathcal{N})] \right|$$

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(2) For iid $\{\phi_n\}$ with 0-mean and 1-variance, with $\Delta_n := \frac{\phi_1 + \cdots + \phi_n}{\sqrt{n}}$ and its leave-one-out version $\Delta_n^{(1)} := \Delta_n - \frac{\phi_1}{\sqrt{n}} = \frac{\phi_2 + \cdots + \phi_n}{\sqrt{n}}$

$$\begin{aligned}
E[\underline{\Delta_n \gamma'(\Delta_n)}] &= \sqrt{n} E[\phi_1 \gamma'(\Delta_n)] \\
&= \sqrt{n}\Big(E[\phi_1 \gamma'(\Delta_n^{(1)})] + E\Big[\phi_1 \frac{\phi_1}{\sqrt{n}} \gamma''(\Delta_n^{(1)})\Big] + O\Big(\frac{1}{n}\Big)\Big) \\
\text{independence} \quad &= \sqrt{n}\, \underbrace{E[\phi_1]}_{=0} E[\gamma'(\Delta_n^{(1)})] + \underbrace{E[\phi_1^2]}_{=1} E[\gamma''(\Delta_n^{(1)})] + O\Big(\frac{1}{\sqrt{n}}\Big) \\
&= 0 + E[\underline{\gamma''(\Delta_n)}] + \mathrm{Lip}(\gamma'') O\Big(\frac{E[|\phi_1|]}{\sqrt{n}}\Big) + O\Big(\frac{1}{\sqrt{n}}\Big)
\end{aligned}$$

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$$\begin{aligned}
E[\Delta_n \gamma'(\Delta_n)] &= \sqrt{n} E[\phi_1 \gamma'(\Delta_n)] \\
&= \sqrt{n}\Big(E[\phi_1 \gamma'(\Delta_n^{(1)})] + E\Big[\phi_1 \frac{\phi_1}{\sqrt{n}} \gamma''(\Delta_n^{(1)})\Big] + O\Big(\frac{1}{n}\Big)\Big) \\
&= \sqrt{n}\, \underbrace{E[\phi_1]}_{=0} E[\gamma'(\Delta_n^{(1)})] + \underbrace{E[\phi_1^2]}_{=1} E[\gamma''(\Delta_n^{(1)})] + O\Big(\frac{1}{\sqrt{n}}\Big) \\
&= 0 + E[\gamma''(\Delta_n)] + \mathrm{Lip}(\gamma'') O\Big(\frac{E[|\phi_1|]}{\sqrt{n}}\Big) + O\Big(\frac{1}{\sqrt{n}}\Big)
\end{aligned}$$

$\rightarrow$ **Function Correlation Bounds**: generalization of independence

Functional Correlation Bound

Let

- $\{(X_n, \zeta_n)\}_{n=0}^{\infty}$: sequence of probability spaces
- $\{\phi_{n,j}\}_{j=0}^{n-1}$: random variables defined on (X_n, ζ_n)

such that

$$\int \phi_{n,j} \, d\zeta_n = 0 \quad \text{for all } 0 \leq j < n$$

E.g.

- Closed: $(X_n, \zeta_n) = ([0,1], \mathrm{Leb})$ and

$$\phi_{n,j} = \varphi \circ f_0^j - \int \varphi \circ f_0^j \, d\mathrm{Leb}$$

- Open: $(X_n, \zeta_n) = (X_{n,\epsilon}, \mathrm{Leb}(\cdot|X_{n,\epsilon}))$ and

$$\phi_{n,j} = \varphi \circ f_\epsilon^j - \int \varphi \circ f_\epsilon^j \, d\mathrm{Leb}(\cdot|X_{n,\epsilon})$$

Functional Correlation Bound

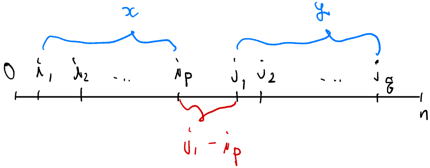
Def. $(\phi_{n,j})_{n\ge 0, 0\le j<n}$ satisfies a **functional correlation bound** (**FCB**) **with rate function** $R : \{1,2,\dots\} \to \mathbb{R}_+$ (with $\lim_{k\to\infty} R(k) = 0$)

if $\exists C_* > 0$, $\forall 0 \le i_1 < \cdots < i_p < j_1 < \cdots < j_q < n$, $\forall G \in \mathrm{Lip}(\mathbb{R}^{p+q})$

$$\left| \int \widetilde{G}(x,x)\, d\zeta_n(x) - \int \widetilde{G}(x,y)\, d\zeta_n(x) d\zeta_n(y) \right| \le C_* \|G\|_{\mathrm{Lip}} R(j_1 - i_p),$$

where

$$\widetilde{G}(x,y) = G(\phi_{n,i_1}(x), \dots, \phi_{n,i_p}(x), \phi_{n,j_1}(y), \dots, \phi_{n,j_q}(y)).$$



Correlation between x's & y's

Functional Correlation Bound

Def. $(\phi_{n,j})_{n\geq 0, 0\leq j<n}$ satisfies a **functional correlation bound** (**FCB**) **with rate function** $R:\{1,2,\ldots\}\to\mathbb{R}_+$ (with $\lim_{k\to\infty} R(k)=0$)

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- Introduced in [Leppänen, 2017] "in the closed case"
- $\{\phi_{n,j}\}_{0\leq j<n}$: independent in $(X_n, \eta_n) \Rightarrow$ (LHS) $= 0$
- "Spectral gap implies" that, when $p = q = 1$ and $G(x,y) = xy$,

$$\text{(LHS)} \leq C_* ||\varphi \circ f^i||_\infty \|\varphi \circ f^i\|_{\mathrm{Lip}} \kappa^{j-i}$$

with some $\kappa \in (0,1)$. **Bad:** $\|\varphi \circ f^i\|_{\mathrm{Lip}} \to \infty$ in general

Functional Correlation Bound

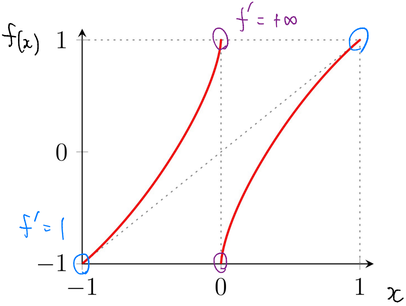
Advantage of (FCB) is that R can decay slowly:

Examples in the closed case ($X_n = X$: a cpt Riem. mfd, $\zeta_n = \mathrm{Leb}$, $\phi_{n,j} = \varphi \circ f^j - \int \varphi \circ f^j \, d\mathrm{Leb}$ with a map $f : X \to X$):

- Pikovsky map $f : [-1, 1] \to [-1, 1]$: defined by the relations

$$x = \begin{cases} \frac{1}{2\gamma}(1 + f(x))^{\gamma}, & 0 \le x \le \frac{1}{2\gamma}, \\ f(x) + \frac{1}{2\gamma}(1 - f(x))^{\gamma}, & \frac{1}{2\gamma} \le x \le 1, \end{cases}$$

and by letting $f(x) = -f(-x)$ for $x \in [-1, 0]$.



$f'(x) \sim C x^{\frac{1}{\gamma}}$ (near $x = -1$)

Functional Correlation Bound

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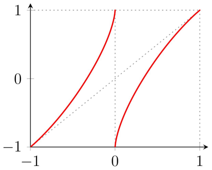

(FCB) holds with $R(k) = k^{-\frac{1}{\gamma - 1}}$ whenever $1 < \gamma < \frac{3}{2}$

with J. Leppänen, Y. Nakajima

Functional Correlation Bound

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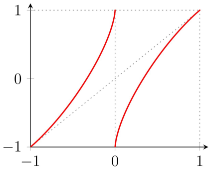

  (FCB) holds with $R(k) = k^{-\frac{1}{\gamma-1}}$ whenever $1 < \gamma < \frac{3}{2}$

- (Random) piecewise expanding maps (etc): $R(k) = \kappa^k$ with a $\kappa \in (0,1)$
- (Random) MP maps $f(x) = x + x^\omega \mod 1$ ($\omega \le \gamma < \frac{3}{2}$): $R(k) \sim k^{1-\frac{1}{\gamma-1}}$

Functional Correlation Bound

Returning to the Blank–Keller W map f_ϵ in a general setting: Let

- $f : [0,1] \to [0,1]$: piecewise expanding map
- $H \subset [0,1]$: measurable such that $\mathrm{Leb}(H)$ is small enough
- $X_n := \{x \in [0,1] : f^j(x) \notin H \quad \text{for all } j \in [0,n]\}$

Thm. (with J. Leppänen, Y. Nakajima)
For any $\varphi \in C^1([0,1])$, the random variables $\{\phi_{n,j}\}$ defined by

$$\phi_{n,j} := \varphi \circ f^j - \int \varphi \, d\mathrm{Leb}(\cdot|X_n) \quad \text{on } (X_n, \mathrm{Leb}(\cdot|X_n)) \quad (0 \le j < n)$$

satisfy (FCB) with $R(k) = \kappa^k$ with some $\kappa \in (0,1)$

- This also holds in (quenched) random environments

CLT with error bound under FCB

Denote by $\mathscr{L}_\zeta(\phi) := \zeta(\{x : \phi(x) \in \cdot\})$ for a m'able ϕ on a prob. sp (X, ζ)

Thm. (with J. Leppänen, Y. Nakajima)
Assume that $\{\phi_{n,j}\}$ on $\{(X_n, \zeta_n)\}$ satisfy (FCB) with $\sum_k R(k) < \infty$, and $\inf_{n\geq 0} \frac{\sigma_n^2}{n} > 0$ where $\sigma_n^2 := \int (\sum_{j=0}^{n-1} \phi_{n,j})^2 \, d\zeta_n$. Then,

(1) with $W_n(x) := \frac{\sum_{j=0}^{n-1} \phi_{n,j}(x)}{\sqrt{\sigma_n^2}}$

$$\sup_{a\in\mathbb{R}} |\mathscr{L}_{\zeta_n}(W_n)((-\infty, a]) - \mathscr{L}(\mathcal{N})((-\infty, a])| \leq \frac{C \log n}{\sqrt{n}},$$

where $\mathcal{N}$ is the standard normal distribution;

(2)

$$\sup_{\gamma\in\mathrm{Lip}^1(\mathbb{R})} \left| \int \gamma \, d\mathscr{L}_{\zeta_n}(W_n) - \int \gamma \, d\mathscr{L}(\mathcal{N}) \right| \leq \frac{C}{\sqrt{n}}$$

(due to [Leppänen–Stenlund, 2020]; optimal error bound);

- CLT: LHS of (1) $\to 0$
- (1)/(2): Kolmogorov/Wasserstein distance of W_n and $\mathcal{N}$

FCLT with error bound under FCB

Let

- D: set of càdlàg functions on $[0,1]$ (right-continuous with left limits; processes with jumps) equipped with the sup norm
- $\mathcal{S}(D)$: set of $C^{2+\mathrm{Lip}}$ functions on D (& "smoothness condition")

(continued)

(3) defining D-valued r.v. $\mathcal{W}_n(x)$ by $\mathcal{W}_n(x)(t) := \frac{\sum_{j=0}^{\lfloor nt \rfloor - 1} \phi_{n,j}(x)}{\sqrt{\sigma_n^2}}$,

$$\sup_{\Gamma \in \mathcal{S}(D)} \left| \int \Gamma \, d\mathscr{L}_{\zeta_n}(\mathcal{W}_n) - \int \Gamma d\mathscr{L}(\mathcal{Z}) \right| \leq \frac{C}{\sqrt{n}}$$

where $\mathcal{Z}$ is the standard Brownian motion.

- $\mathcal{S}(D)$ is large enough in the sense that (2) implies $\mathcal{W}_n \xrightarrow{\mathrm{d}} \mathcal{Z}$.

Cor: Conditional FCLT with error bound $\frac{1}{\sqrt{n}}$ holds for the BK W map

Summary

Motivation: How to formulate and prove limit theorems observed over finite times in chaotic systems?

Approach:

- Introduce **conditional limit theorems**, considering only the trajectories that do not enter a hole (set of no-return).
  - and formulation by time-dependent parameters
- Utilize **Stein method** for its excellence in flexibility and error estimation. The key is to establish a **Functional Correlation Bound** in this conditional setting (arXiv:2501.13498).