Newton–Kantorovich hypotheses in the mixed weak–strong setting (finite-rank case)

1 Goal and setting

Let $\mathcal{B}_s \subset \mathcal{B}_w$ be Banach spaces with continuous embedding ("strong" vs "weak"), so $\|u\|_w \leq \|u\|_s$ for $u \in \mathcal{B}_s$. Fix a bounded linear functional $\ell \in (\mathcal{B}_w)^*$ and consider the normalized fixed point problem:

$$T(f) = f, \qquad \ell(f) = 1,$$

for a (possibly nonlinear) map $T : U \subset \mathcal{B}_w \to \mathcal{B}_w$.

We rewrite it as a zero problem on $\mathcal{B}_w \times \mathbb{C}$:

$$F(f) := (T(f) - f, \ \ell(f) - 1).$$

Newton's method is most convenient if we keep the normalization $\ell(f) = 1$ at each step: write $f_{k+1} = f_k + h_k$ with $h_k \in \ker \ell$, and solve

$$(\mathrm{Id} - DT(f_k))h_k \ = \ T(f_k) - f_k \qquad \text{in } \ker \ell.$$

2 Case (1): discretize T and differentiate the discretization

Let $\pi_n : \mathcal{B}_w \to X_n \subset \mathcal{B}_s$ be a finite-rank projection and define the finite-rank surrogate

$$T_n(f) := \pi_n \, T(\pi_n f).$$

Then T_n takes values in X_n and its derivative is finite rank:

$$DT_n(f) = \pi_n \, DT(\pi_n f) \, \pi_n.$$

Assumption 1 (Projection stability and approximation). *There are constants $C_\pi \geq 1$ and $a_n \geq 0$ such that:*

1. *(**Weak stability**) $\|\pi_n\|_{w \to w} \leq C_\pi$.*

2. *(**Strong→weak approximation**) for all $g \in \mathcal{B}_s$,*

$$\|(I - \pi_n)g\|_w \leq a_n \|g\|_s.$$

3 Computable residual and computable inverse bound

Residual

Let $\tilde{f} \in X_n$ be the numerically computed candidate, with $\ell(\tilde{f}) = 1$.

Define the finite-dimensional defect

$$r_n := \left\| T_n(\tilde{f}) - \tilde{f} \right\|_w,$$

and the (infinite-dimensional) consistency error

$$e_n(\tilde{f}) := \left\| T(\tilde{f}) - T_n(\tilde{f}) \right\|_w.$$

Then the true residual satisfies

$$\delta := \left\| T(\tilde{f}) - \tilde{f} \right\|_w \leq e_n(\tilde{f}) + r_n.$$

If $\tilde{f}$ is an exact fixed point of T_n (within the finite-dimensional space), then $r_n = 0$.

Inverse bound via finite rank (Neumann argument)

Let $J(f) := \mathrm{Id} - DT(f)$ acting on $\ker \ell$ (with domain $\ker \ell \cap \mathcal{B}_s$), and define

$$J_n(f) := \mathrm{Id} - DT_n(f) = \mathrm{Id} - \pi_n DT(\pi_n f)\pi_n$$

acting on $X_n \cap \ker \ell$.

Assume you can compute in finite dimension the norm

$$\widetilde{M} := \left\| J_n(\tilde{f})^{-1} \right\|_{w \to w; \, X_n \cap \ker \ell},$$

and you can bound the Jacobian discretization error

$$\varepsilon_J := \left\| J(\tilde{f}) - J_n(\tilde{f}) \right\|_{w \to w; \, \ker \ell}.$$

Lemma 1 (Certified inverse bound). *If $\widetilde{M}\,\varepsilon_J < 1$, then $J(\tilde{f})$ is invertible on $\ker \ell$ and*

$$M := \left\| J(\tilde{f})^{-1} \right\|_{w \to w; \, \ker \ell} \leq \frac{\widetilde{M}}{1 - \widetilde{M}\,\varepsilon_J}.$$

Proof. Write $J(\tilde{f}) = J_n(\tilde{f})(\mathrm{Id} + J_n(\tilde{f})^{-1}(J(\tilde{f}) - J_n(\tilde{f})))$. If $\|J_n^{-1}(J - J_n)\| < 1$, the second factor is invertible by a Neumann series, and the norm bound follows. $\square$

4 Lipschitz control of the differential (mixed norm)

Assumption 2 (Mixed Lipschitz control of the differential). *There exist $\gamma \geq 0$ and $R > 0$ such that $B_R(\tilde{f}) \subset U$ and for all $f, g \in B_R(\tilde{f})$,*

$$\|DT(f) - DT(g)\|_{s \to w} \leq \gamma \, \|f - g\|_w.$$

5 Newton–Kantorovich hypothesis checklist (case 1)

Theorem 1 (Newton–Kantorovich (case 1, mixed weak–strong)). *Assume:*

1. *(Scale) $\mathcal{B}_s \subset \mathcal{B}_w$ continuously and $\ell \in (\mathcal{B}_w)^*$ is fixed.*

2. *(Finite rank) π_n satisfies Assumption 1 and $T_n(f) = \pi_n T(\pi_n f)$.*

3. *(Candidate) $\tilde{f} \in X_n$ with $\ell(\tilde{f}) = 1$.*

4. *(Residual bound) $\delta \le e_n(\tilde{f}) + r_n$ where $r_n = \|T_n(\tilde{f}) - \tilde{f}\|_w$ and $e_n(\tilde{f}) = \|T(\tilde{f}) - T_n(\tilde{f})\|_w$.*

5. *(Inverse bound) There exist $\widetilde{M}$ and ε_J with $\widetilde{M} \varepsilon_J < 1$, so that $J(\tilde{f}) = \mathrm{Id} - DT(\tilde{f})$ is invertible on $\ker \ell$ and*
$$M \le \frac{\widetilde{M}}{1 - \widetilde{M}\, \varepsilon_J} \qquad \text{(Lemma 1)}.$$

6. *(Differential Lipschitz) Assumption 2 holds with constants γ, R.*

7. *(Kantorovich smallness) The quantity $2M^2 \gamma\, \delta \le 1$ and the radius*
$$r := \frac{1 - \sqrt{1 - 2M^2 \gamma\, \delta}}{M\gamma} \qquad \text{(interpret } r := M\delta \text{ if } \gamma = 0\text{)}$$
satisfies $r \le R$.

Then there exists a unique normalized fixed point $f^* \in \mathcal{B}_w$ such that
$$T(f^*) = f^*, \qquad \ell(f^*) = 1, \qquad \left\| f^* - \tilde{f} \right\|_w \le r.$$

Moreover, Newton's method on the constraint hyperplane $\ell(f) = 1$, i.e. solving $(\mathrm{Id} - DT(f_k)) h_k = T(f_k) - f_k$ in $\ker \ell$, is well-defined starting from $\tilde{f}$ and converges to f^*.

Remark 1. In practice, the checklist amounts to: compute r_n and $\widetilde{M}$ in finite dimension, and bound $e_n(\tilde{f})$, ε_J, and γ analytically using strong→weak estimates (e.g. Lasota–Yorke) together with the approximation properties of π_n.

6 Gaussian noise coupled maps

6.1 Analytic strong norm and Fourier truncation

Definition 1 (Analytic strip norm). Fix $\tau > 0$. For $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$ define
$$\|f\|_{\mathcal{A}_\tau}^2 := \sum_{k \in \mathbb{Z}} e^{4\pi \tau |k|} |\hat{f}(k)|^2.$$

Let $\mathcal{A}_\tau(\mathbb{T}) := \{ f \in L^2(\mathbb{T}) : \|f\|_{\mathcal{A}_\tau} < \infty \}$.

Lemma 2 (Exponential tail for Fourier truncation). *Let Π_N be the L^2-orthogonal projection onto Fourier modes $|k| \le N$. Then for all $f \in \mathcal{A}_\tau$,*
$$\|(I - \Pi_N) f\|_{L^2} \le e^{-2\pi \tau N} \|f\|_{\mathcal{A}_\tau}.$$

6.2 Noisy transfer operator with additive Gaussian noise

Definition 2 (Periodized Gaussian kernel). Let $\bar{\rho}_\sigma$ be the periodized Gaussian on $\mathbb{T}$ with Fourier coefficients

$$\widehat{\bar{\rho}_\sigma}(k) = e^{-2\pi^2\sigma^2 k^2}.$$

Define the convolution (noise) operator $C_\sigma u := \bar{\rho}_\sigma * u$.

Lemma 3 (Gaussian smoothing into analytic strip). *For every $\tau > 0$ and $\sigma > 0$, $C_\sigma : L^2 \to \mathcal{A}_\tau$ is bounded with*

$$\|C_\sigma u\|_{\mathcal{A}_\tau} \le S_{\tau,\sigma}\|u\|_2, \qquad S_{\tau,\sigma} := \sup_{k\in\mathbb{Z}} \exp\bigl(2\pi\tau|k| - 2\pi^2\sigma^2 k^2\bigr).$$

In particular, $S_{\tau,\sigma} \le \exp(\tau^2/(2\sigma^2))$ (crude but explicit).

Definition 3 (Noisy operator with shift). Fix a measurable map $T_0 : \mathbb{T} \to \mathbb{T}$ (e.g. a quadratic map modulo 1) and define, for $c \in$,

$$(P_c u)(x) := \int_{\mathbb{T}} \bar{\rho}_\sigma\bigl(x - T_0(y) - c\bigr)\, u(y)\, dy.$$

Define $Q_c := \partial_c P_c$ (derivative w.r.t. the shift).

Lemma 4 (Mixed bounds for P_c). *For every $c \in$ and $u \in L^2(\mathbb{T})$,*

$$\|P_c u\|_{\mathcal{A}_\tau} \le S_{\tau,\sigma}\|u\|_2, \qquad \|P_c u\|_2 \le \|u\|_2.$$

Definition 4 (Shift-derivative operator). Let $Q_c := \partial_c P_c$. Then

$$(Q_c u)(x) = -\int_{\mathbb{T}} \bar{\rho}'_\sigma\bigl(x - T_0(y) - c\bigr)\, u(y)\, dy,$$

and in Fourier:

$$\widehat{Q_c u}(k) = (-2\pi i k)\widehat{\bar{\rho}_\sigma}(k) e^{-2\pi i k c} \int_{\mathbb{T}} u(y) e^{-2\pi i k T_0(y)}\, dy.$$

Define

$$S^{(1)}_{\tau,\sigma} := \sup_{k\in\mathbb{Z}}(2\pi|k|)\exp\bigl(2\pi\tau|k| - 2\pi^2\sigma^2 k^2\bigr), \qquad S^{(1)}_{0,\sigma} := \sup_{k\in\mathbb{Z}}(2\pi|k|)e^{-2\pi^2\sigma^2 k^2}.$$

Lemma 5 (Mixed bounds for Q_c). *For every $c \in$ and $u \in L^2(\mathbb{T})$,*

$$\|Q_c u\|_{\mathcal{A}_\tau} \le S^{(1)}_{\tau,\sigma}\|u\|_2, \qquad \|Q_c u\|_2 \le S^{(1)}_{0,\sigma}\|u\|_2.$$

6.3 Self-consistent coupling with analytic observable

Assumption 3 (Coupling functional and regularity). *Let $\phi \in L^2(\mathbb{T})$ (in applications $\phi \in \mathcal{A}_\tau$) and define*

$$m(f) := \langle \phi, f \rangle_{L^2}.$$

Let $G :\to$ be C^1 on a relevant interval, and set

$$c(f) := \delta\, G(m(f)), \qquad c'(f)[h] = \delta\, G'(m(f))\, \langle \phi, h \rangle_{L^2}.$$

Denote

$$L_G := \sup |G'| \quad \textit{on the relevant interval}, \qquad \mathrm{Lip}(G) := \sup_{x\ne y} \frac{|G(x) - G(y)|}{|x - y|} \quad \textit{on that interval}.$$

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Definition 5 (Self-consistent map). Define

$$T(f) := P_{c(f)}f.$$

6.4 Two Fourier discretizations and the parameter mismatch

Definition 6 (Two choices of T_N). Let $N \in \mathbb{N}$. Define

$$\textbf{(A) ``semi-discrete coupling'':} \quad T_N^A(f) := \Pi_N \, P_{c(f)} \, \Pi_N f,$$
$$\textbf{(B) ``fully discrete coupling'':} \quad T_N^B(f) := \Pi_N \, P_{c(\Pi_N f)} \, \Pi_N f.$$

Remark 2 (Trig polynomial case as a special simplification). If ϕ is a trigonometric polynomial of degree K and $N \geq K$, then $\langle \phi, \Pi_N f \rangle = \langle \phi, f \rangle$ for all $f \in L^2$. Hence $c(\Pi_N f) = c(f)$ and therefore $T_N^A = T_N^B$ exactly (and likewise for their differentials). In this case all "parameter mismatch" terms below vanish.

Lemma 6 (Mismatch in m and in c for analytic ϕ). *Assume $\phi \in \mathcal{A}_\tau$ and $f \in L^2$. Then*

$$|m(f) - m(\Pi_N f)| = |\langle (I - \Pi_N)\phi, f \rangle| \leq \|(I - \Pi_N)\phi\|_2 \, \|f\|_2 \leq e^{-2\pi\tau N} \, \|\phi\|_{\mathcal{A}_\tau} \, \|f\|_2.$$

Consequently,
$$|c(f) - c(\Pi_N f)| \leq |\delta| \operatorname{Lip}(G) \, e^{-2\pi\tau N} \, \|\phi\|_{\mathcal{A}_\tau} \, \|f\|_2.$$

Remark 3 (Why T_N^B is convenient in practice). T_N^B is the genuinely finite-dimensional self-consistent map: all ingredients depend only on $\Pi_N f$. This is typically what you implement if you cannot evaluate the true residual $T(f) - f$. When ϕ is analytic (not a polynomial), T_N^A and T_N^B differ by an exponentially small mismatch term controlled by Lemma 6.

6.5 Exponential discretization bounds for T and the mismatch term

Lemma 7 (Exponential discretization error for T vs. T_N^A). *For all $f \in \mathcal{A}_\tau$,*

$$\|T(f) - T_N^A(f)\|_{L^2} \leq e^{-2\pi\tau N} \, (S_{\tau,\sigma} + 1) \, \|f\|_{\mathcal{A}_\tau}.$$

Equivalently,
$$\|T - T_N^A\|_{\mathcal{A}_\tau \to L^2} \leq e^{-2\pi\tau N} \, (S_{\tau,\sigma} + 1).$$

Proof. Split
$$T(f) - T_N^A(f) = (I - \Pi_N)P_{c(f)}f + \Pi_N P_{c(f)}(I - \Pi_N)f.$$

Use Lemma 2 and Lemma 4 exactly as in the polynomial case. $\qquad \square$

Lemma 8 (Mismatch error between T_N^A and T_N^B). *Assume $\phi \in \mathcal{A}_\tau$ and $f \in L^2$, and Assumption 3. Then*
$$\|T_N^A(f) - T_N^B(f)\|_2 \leq |c(f) - c(\Pi_N f)| \, S_{0,\sigma}^{(1)} \, \|f\|_2,$$

and hence, using Lemma 6,

$$\|T_N^A(f) - T_N^B(f)\|_2 \leq |\delta| \operatorname{Lip}(G) \, e^{-2\pi\tau N} \, \|\phi\|_{\mathcal{A}_\tau} \, S_{0,\sigma}^{(1)} \, \|f\|_2^2.$$

Proof. Write $c_A := c(f)$ and $c_B := c(\Pi_N f)$. Then

$$T_N^A(f) - T_N^B(f) = \Pi_N(P_{c_A} - P_{c_B})\Pi_N f.$$

By the one-dimensional mean value theorem in the parameter c, for some $\theta \in (0,1)$,

$$P_{c_A} - P_{c_B} = (c_A - c_B)\, Q_{c_B + \theta(c_A - c_B)}.$$

Therefore,

$$\|T_N^A(f) - T_N^B(f)\|_2 \le |c_A - c_B|\, \|Q.\Pi_N f\|_2 \le |c(f) - c(\Pi_N f)|\, S_{0,\sigma}^{(1)}\, \|f\|_2,$$

and the final bound follows from Lemma 6. $\qquad\square$

6.6 Differentials DT, DT_N and exponential bounds

Lemma 9 (Fréchet derivative of T). *Assume Assumption 3. Then $T : \mathcal{A}_\tau \to L^2$ is Fréchet differentiable and*

$$DT(f)[h] = P_{c(f)}h + c'(f)[h]\, Q_{c(f)}f, \qquad c'(f)[h] = \delta\, G'(m(f))\, \langle\phi, h\rangle.$$

Moreover,

$$|c'(f)[h]| \le |\delta|\, L_G\, \|\phi\|_2\, \|h\|_{\mathcal{A}_\tau}.$$

Lemma 10 (Fréchet derivatives of T_N^A and T_N^B). *Assume Assumption 3. Then $T_N^A, T_N^B : \mathcal{A}_\tau \to L^2$ are Fréchet differentiable and*

$$DT_N^A(f)[h] = \Pi_N P_{c(f)} \Pi_N h \;+\; c'(f)[h]\, \Pi_N Q_{c(f)} \Pi_N f,$$
$$DT_N^B(f)[h] = \Pi_N P_{c(\Pi_N f)} \Pi_N h \;+\; c'(\Pi_N f)[\Pi_N h]\, \Pi_N Q_{c(\Pi_N f)} \Pi_N f.$$

Remark 4 (Polynomial case). If ϕ is a trigonometric polynomial of degree K and $N \ge K$, then $c(\Pi_N f) = c(f)$ and $c'(\Pi_N f)[\Pi_N h] = c'(f)[h]$, hence $DT_N^A(f) = DT_N^B(f)$ exactly.

Lemma 11 (Exponential bound for $DT - DT_N^A$). *Assume Assumption 3 and let $f \in \mathcal{A}_\tau$. Then for all $h \in \mathcal{A}_\tau$,*

$$\|(DT(f) - DT_N^A(f))[h]\|_2 \le e^{-2\pi\tau N}(S_{\tau,\sigma}+1)\|h\|_{\mathcal{A}_\tau} + |\delta|L_G\|\phi\|_2\, e^{-2\pi\tau N}(S_{\tau,\sigma}^{(1)}+S_{0,\sigma}^{(1)})\, \|f\|_{\mathcal{A}_\tau}\, \|h\|_{\mathcal{A}_\tau}.$$

In particular, on $\{f : \|f\|_{\mathcal{A}_\tau} \le K\}$,

$$\sup_{\|f\|_{\mathcal{A}_\tau} \le K} \|DT(f) - DT_N^A(f)\|_{\mathcal{A}_\tau \to L^2} \le e^{-2\pi\tau N}(S_{\tau,\sigma}+1) + |\delta|L_G\|\phi\|_2\, K\, e^{-2\pi\tau N}(S_{\tau,\sigma}^{(1)}+S_{0,\sigma}^{(1)}).$$

Proof. **Step 0: remark about the choice of T_N.** We use $T_N(f) = \Pi_N P_{c(f)}\Pi_N f$ ("semi-discrete coupling"). If instead one uses the fully discrete coupling $T_N^B(f) = \Pi_N P_{c(\Pi_N f)}\Pi_N f$, then since ψ has degree K and $N \ge K$, we have $\langle\psi, \Pi_N f\rangle = \langle\psi, f\rangle$ and thus $c(\Pi_N f) = c(f)$; hence $T_N^B = T_N$ and the same formulas apply.

Step 1: Fréchet derivative of T. Write $c = c(f)$ and $c_h = c(f + h)$. Then

$$T(f + h) - T(f) = P_{c_h}(f + h) - P_c f = P_c h + (P_{c_h} - P_c)f + (P_{c_h} - P_c)h.$$

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We claim that
$$(P_{c_h} - P_c)f = (c_h - c)\, Q_c f + o(\|h\|_{\mathcal{A}_\tau}) \quad \text{in } L^2.$$

Indeed, for fixed f, the map $c \mapsto P_c f$ is C^1 (differentiate under the integral, or use the Fourier formula), with derivative $Q_c f$, so by the mean value theorem in one real variable there exists $\theta \in (0,1)$ such that
$$P_{c_h} f - P_c f = (c_h - c)\, Q_{c+\theta(c_h - c)} f.$$

Hence
$$\|(P_{c_h} - P_c)f - (c_h - c)Q_c f\|_2 \le |c_h - c| \, \|(Q_{c+\theta(c_h - c)} - Q_c)f\|_2 = o(|c_h - c|),$$

as $c_h \to c$. Since $c(\cdot)$ is Fréchet differentiable and thus $|c_h - c - c'(f)[h]| = o(\|h\|_{\mathcal{A}_\tau})$, we obtain
$$(P_{c_h} - P_c)f = c'(f)[h]\, Q_c f + o(\|h\|_{\mathcal{A}_\tau}).$$

Finally, $(P_{c_h} - P_c)h$ is a remainder: using the (local) Lipschitz continuity in c of P_c as an operator $L^2 \to L^2$,
$$\|(P_{c_h} - P_c)h\|_2 \le C|c_h - c| \, \|h\|_2 = o(\|h\|_{\mathcal{A}_\tau}),$$

because $|c_h - c| = O(\|h\|_2) \le O(\|h\|_{\mathcal{A}_\tau})$. Collecting terms yields the derivative formula
$$DT(f)[h] = P_{c(f)}h + c'(f)[h]\, Q_{c(f)}f.$$

Step 2: Fréchet derivative of T_N. Since $T_N(f) = \Pi_N P_{c(f)} \Pi_N f$ is a composition of bounded linear maps with the scalar C^1 functional $c(\cdot)$, the chain rule gives
$$DT_N(f)[h] = \Pi_N P_{c(f)} \Pi_N h + c'(f)[h]\, \Pi_N Q_{c(f)} \Pi_N f.$$

Step 3: Split $DT - DT_N$. Let $c = c(f)$. Using the two formulas,
$$(DT(f) - DT_N(f))[h] = \underbrace{\left(P_c - \Pi_N P_c \Pi_N\right)h}_{\text{(I)}} + c'(f)[h] \underbrace{\left(Q_c - \Pi_N Q_c \Pi_N\right)f}_{\text{(II)}}.$$

Step 4: Bound term (I). Decompose
$$(P_c - \Pi_N P_c \Pi_N)h = (I - \Pi_N)P_c h \; + \; \Pi_N P_c (I - \Pi_N)h.$$

For the first part, apply the exponential tail with $g = P_c h$ and then the mixed bound:
$$\|(I - \Pi_N)P_c h\|_2 \le e^{-2\pi\tau N}\|P_c h\|_{\mathcal{A}_\tau} \le e^{-2\pi\tau N} S_{\tau,\sigma}\|h\|_2 \le e^{-2\pi\tau N} S_{\tau,\sigma}\|h\|_{\mathcal{A}_\tau}.$$

For the second part, use $\|\Pi_N\|_{2\to2} = 1$ and the L^2 bound for P_c:
$$\|\Pi_N P_c (I - \Pi_N)h\|_2 \le \|P_c(I - \Pi_N)h\|_2 \le \|(I - \Pi_N)h\|_2 \le e^{-2\pi\tau N}\|h\|_{\mathcal{A}_\tau}.$$

Thus
$$\|(\text{I})\|_2 \le e^{-2\pi\tau N}(S_{\tau,\sigma} + 1)\|h\|_{\mathcal{A}_\tau}.$$

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Step 5: Bound term (II). Similarly,

$$(Q_c - \Pi_N Q_c \Pi_N)f = (I - \Pi_N)Q_c f + \Pi_N Q_c (I - \Pi_N)f.$$

For the first part,

$$\|(I - \Pi_N)Q_c f\|_2 \le e^{-2\pi\tau N}\|Q_c f\|_{\mathcal{A}_\tau} \le e^{-2\pi\tau N} S_{\tau,\sigma}^{(1)}\|f\|_2 \le e^{-2\pi\tau N} S_{\tau,\sigma}^{(1)}\|f\|_{\mathcal{A}_\tau}.$$

For the second part,

$$\|\Pi_N Q_c(I - \Pi_N)f\|_2 \le \|Q_c(I - \Pi_N)f\|_2 \le S_{0,\sigma}^{(1)}\|(I - \Pi_N)f\|_2 \le e^{-2\pi\tau N} S_{0,\sigma}^{(1)}\|f\|_{\mathcal{A}_\tau}.$$

Hence

$$\|(\mathrm{II})\|_2 \le e^{-2\pi\tau N}(S_{\tau,\sigma}^{(1)} + S_{0,\sigma}^{(1)})\|f\|_{\mathcal{A}_\tau}.$$

Step 6: Control of $|c'(f)[h]|$. By Cauchy–Schwarz,

$$|\langle \psi, h \rangle| \le \|\psi\|_2 \|h\|_2 \le \|\psi\|_2 \|h\|_{\mathcal{A}_\tau}.$$

Therefore

$$|c'(f)[h]| = |\delta|\,|G'(\langle \psi, f\rangle)|\,|\langle \psi, h \rangle| \le |\delta|\,L_G\,\|\psi\|_2\,\|h\|_{\mathcal{A}_\tau}.$$

Step 7: Combine. Using the split in Step 3, the bound on (I), and the bound on (II) multiplied by $|c'(f)[h]|$, we obtain exactly:

$$\|(DT(f) - DT_N(f))[h]\|_2 \le e^{-2\pi\tau N}(S_{\tau,\sigma}+1)\|h\|_{\mathcal{A}_\tau} + |\delta|L_G\|\psi\|_2\,e^{-2\pi\tau N}(S_{\tau,\sigma}^{(1)}+S_{0,\sigma}^{(1)})\,\|f\|_{\mathcal{A}_\tau}\|h\|_{\mathcal{A}_\tau}.$$

Taking the supremum over $\|h\|_{\mathcal{A}_\tau} = 1$ yields the operator norm bound, and then restricting to $\|f\|_{\mathcal{A}_\tau} \le K$ gives the stated uniform estimate. $\square$

Remark 5 (From $DT - DT_N^A$ to $DT - DT_N^B$). To estimate $DT - DT_N^B$, add the mismatch term

$$DT_N^A(f) - DT_N^B(f),$$

which is controlled by $|c(f) - c(\Pi_N f)|$ and $|c'(f)[h] - c'(\Pi_N f)[\Pi_N h]|$. When ϕ is a trig polynomial and $N \ge \deg(\phi)$ this mismatch vanishes exactly. When $\phi \in \mathcal{A}_\tau$, the mismatch is exponentially small by Lemma 6.

Lemma 12 (Explicit mismatch bound for $DT_N^A - DT_N^B$). *Assume $\phi \in \mathcal{A}_\tau$ and define*

$$m(f) := \langle \phi, f \rangle_{L^2}, \qquad c(f) := \delta\, G(m(f)), \qquad c'(f)[h] = \delta\, G'(m(f))\,\langle \phi, h \rangle.$$

Assume $G \in C^2$ on the relevant range and set

$$L_G := \sup|G'|, \qquad \mathrm{Lip}(G) := \sup_{x \ne y} \frac{|G(x) - G(y)|}{|x - y|}, \qquad L_{G'} := \sup|G''|.$$

Let $T_N^A(f) = \Pi_N P_{c(f)}\Pi_N f$ and $T_N^B(f) = \Pi_N P_{c(\Pi_N f)}\Pi_N f$. Let $Q_c = \partial_c P_c$ and $R_c := \partial_c Q_c = \partial_c^2 P_c$ and assume the L^2-bounds

$$\|Q_c u\|_2 \le S_{0,\sigma}^{(1)}\|u\|_2, \qquad \|R_c u\|_2 \le S_{0,\sigma}^{(2)}\|u\|_2 \quad \text{uniformly in } c \in .$$

Then for every $f \in L^2(\mathbb{T})$ and every $h \in \mathcal{A}_\tau$,

$$\left\| (DT_N^A(f) - DT_N^B(f))[h] \right\|_2 \ \leq \ e^{-2\pi\tau N} \, \mathcal{C}(f) \, \|h\|_{\mathcal{A}_\tau},$$

where

$$\mathcal{C}(f) := |\delta| \, \|\phi\|_{\mathcal{A}_\tau} \Big[\mathrm{Lip}(G) \, S_{0,\sigma}^{(1)} \, \|f\|_2 + L_G \, S_{0,\sigma}^{(1)} \, \|f\|_2 + L_{G'} \, \|\phi\|_2 \, S_{0,\sigma}^{(1)} \, \|f\|_2^2 \Big] + |\delta|^2 \, L_G \, \mathrm{Lip}(G) \, \|\phi\|_2 \, \|\phi\|_{\mathcal{A}_\tau} \, S_{0,\sigma}^{(2)} \, \|f\|_2^2.$$

Consequently, on the ball $\{f : \|f\|_{\mathcal{A}_\tau} \leq K\}$ (hence $\|f\|_2 \leq K$),

$$\sup_{\|f\|_{\mathcal{A}_\tau} \leq K} \|DT_N^A(f) - DT_N^B(f)\|_{\mathcal{A}_\tau \to L^2} \ \leq \ e^{-2\pi\tau N} \Big(|\delta| \, \|\phi\|_{\mathcal{A}_\tau} \big[(\mathrm{Lip}(G) + L_G) S_{0,\sigma}^{(1)} K + L_{G'} \|\phi\|_2 S_{0,\sigma}^{(1)} K^2 \big] + |\delta|^2 L_G \mathrm{Lip}(G)$$

Polynomial case. *If ϕ is a trigonometric polynomial of degree K_0 and $N \geq K_0$, then $m(\Pi_N f) = m(f)$ for all f, so $c(\Pi_N f) = c(f)$ and $c'(\Pi_N f)[\Pi_N h] = c'(f)[h]$. Hence $DT_N^A(f) = DT_N^B(f)$ exactly for all f, h.*

Proof. Write $c_A := c(f)$ and $c_B := c(\Pi_N f)$ and

$$\alpha := c'(f)[h] = \delta \, G'(m(f))\langle \phi, h \rangle, \qquad \beta := c'(\Pi_N f)[\Pi_N h] = \delta \, G'(m(\Pi_N f))\langle \phi, \Pi_N h \rangle.$$

Using the formulas for the Fréchet derivatives,

$$DT_N^A(f)[h] = \Pi_N P_{c_A} \Pi_N h + \alpha \, \Pi_N Q_{c_A} \Pi_N f, \qquad DT_N^B(f)[h] = \Pi_N P_{c_B} \Pi_N h + \beta \, \Pi_N Q_{c_B} \Pi_N f.$$

Hence

$$(DT_N^A - DT_N^B)[h] = \Pi_N(P_{c_A} - P_{c_B})\Pi_N h + (\alpha - \beta) \, \Pi_N Q_{c_A} \Pi_N f + \beta \, \Pi_N(Q_{c_A} - Q_{c_B})\Pi_N f.$$

We bound the three terms separately in L^2 (using $\|\Pi_N\|_{2\to2} = 1$).

(I) The P-mismatch. By the one-dimensional mean value theorem in the parameter c,

$$P_{c_A} - P_{c_B} = (c_A - c_B) \, Q_{c_\theta} \quad \text{for some } c_\theta = c_B + \theta(c_A - c_B).$$

Thus

$$\|\Pi_N(P_{c_A} - P_{c_B})\Pi_N h\|_2 \leq |c_A - c_B| \, \|Q_{c_\theta}\Pi_N h\|_2 \leq |c_A - c_B| S_{0,\sigma}^{(1)} \|h\|_2 \leq |c_A - c_B| S_{0,\sigma}^{(1)} \|h\|_{\mathcal{A}_\tau}.$$

(II) The c'-mismatch. We estimate

$$|\alpha - \beta| \leq |\delta| \, |G'(m(f))| \, |\langle \phi, h - \Pi_N h \rangle| + |\delta| \, |\langle \phi, \Pi_N h \rangle| \, |G'(m(f)) - G'(m(\Pi_N f))|.$$

Since $\phi \in \mathcal{A}_\tau$,

$$|\langle \phi, h - \Pi_N h \rangle| = |\langle (I - \Pi_N)\phi, h \rangle| \leq \|(I - \Pi_N)\phi\|_2 \|h\|_2 \leq e^{-2\pi\tau N}\|\phi\|_{\mathcal{A}_\tau} \|h\|_{\mathcal{A}_\tau}.$$

Also $|\langle \phi, \Pi_N h \rangle| \leq \|\phi\|_2 \|h\|_2 \leq \|\phi\|_2 \|h\|_{\mathcal{A}_\tau}$. Finally, by Lipschitz continuity of G' (i.e. bounded G''),

$$|G'(m(f)) - G'(m(\Pi_N f))| \leq L_{G'} \, |m(f) - m(\Pi_N f)| = L_{G'} \, |\langle (I - \Pi_N)\phi, f \rangle| \leq L_{G'} \, e^{-2\pi\tau N}\|\phi\|_{\mathcal{A}_\tau} \|f\|_2.$$

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Combining gives

$$|\alpha - \beta| \le e^{-2\pi\tau N} |\delta| \, \|\phi\|_{\mathcal{A}_\tau} \Big[L_G + L_{G'} \|\phi\|_2 \, \|f\|_2 \Big] \|h\|_{\mathcal{A}_\tau}.$$

Therefore

$$\|(\alpha-\beta)\Pi_N Q_{c_A}\Pi_N f\|_2 \le |\alpha-\beta| \, S_{0,\sigma}^{(1)} \|f\|_2 \le e^{-2\pi\tau N} |\delta| \, \|\phi\|_{\mathcal{A}_\tau} \Big[L_G \, S_{0,\sigma}^{(1)} \|f\|_2 + L_{G'} \|\phi\|_2 \, S_{0,\sigma}^{(1)} \|f\|_2^2 \Big] \|h\|_{\mathcal{A}_\tau}.$$

(III) The Q-mismatch. Again by the mean value theorem,

$$Q_{c_A} - Q_{c_B} = (c_A - c_B) \, R_{c_{\theta'}}.$$

Hence

$$\|\beta \, \Pi_N (Q_{c_A} - Q_{c_B})\Pi_N f\|_2 \le |\beta| \, |c_A - c_B| \, \|R_{c_{\theta'}}\Pi_N f\|_2 \le |\beta| \, |c_A - c_B| \, S_{0,\sigma}^{(2)} \|f\|_2.$$

Moreover $|\beta| \le |\delta| \, L_G \, \|\phi\|_2 \, \|h\|_2 \le |\delta| L_G \|\phi\|_2 \|h\|_{\mathcal{A}_\tau}$. And (by Lipschitz continuity of G and the same tail estimate)

$$|c_A - c_B| = |\delta| \, |G(m(f)) - G(m(\Pi_N f))| \le |\delta| \, \mathrm{Lip}(G) \, |m(f) - m(\Pi_N f)| \le |\delta| \, \mathrm{Lip}(G) \, e^{-2\pi\tau N} \|\phi\|_{\mathcal{A}_\tau} \|f\|_2.$$

Thus

$$\|\beta \, \Pi_N (Q_{c_A} - Q_{c_B})\Pi_N f\|_2 \le e^{-2\pi\tau N} |\delta|^2 \, L_G \, \mathrm{Lip}(G) \, \|\phi\|_2 \, \|\phi\|_{\mathcal{A}_\tau} \, S_{0,\sigma}^{(2)} \|f\|_2^2 \|h\|_{\mathcal{A}_\tau}.$$

Conclusion. Summing (I)–(III) gives the displayed bound with $\mathcal{C}(f)$. The uniform bound on $\|f\|_{\mathcal{A}_\tau} \le K$ follows from $\|f\|_2 \le K$. Finally, in the polynomial case with $\deg(\phi) \le N$ we have $m(\Pi_N f) = m(f)$ and $\langle \phi, \Pi_N h \rangle = \langle \phi, h \rangle$, hence $c_A = c_B$ and $\alpha = \beta$, so the difference is 0. $\qquad\square$

Corollary 1 (Mixed Newton–Kantorovich for T via Fourier surrogate T_N). *Fix $\tau > 0$, and set $\mathcal{B}_s := \mathcal{A}_\tau(\mathbb{T})$ and $\mathcal{B}_w := L^2(\mathbb{T})$ (with $\|u\|_2 \le \|u\|_{\mathcal{A}_\tau}$). Let Π_N be the L^2-orthogonal projection onto Fourier modes $|k| \le N$.*
   Let $\phi \in \mathcal{A}_\tau$ and define the coupling

$$m(f) := \langle \phi, f \rangle_{L^2}, \qquad c(f) := \delta \, G(m(f)),$$

where $G \in C^2$ on the relevant range. Define the true self-consistent operator

$$T(f) := P_{c(f)} f,$$

where P_c is the noisy (periodized Gaussian) shift-kernel operator, and set

$$T_N(f) := T_N^B(f) := \Pi_N \, P_{c(\Pi_N f)} \, \Pi_N f \qquad \text{(fully discrete self-consistent map)}.$$

Let $\ell \in (\mathcal{B}_w)^*$ be the normalization functional and consider the constrained fixed-point problem $T(f) = f$, $\ell(f) = 1$; Newton is performed on $\ker \ell$.

Assume the following uniform bounds ("LY-w" / "LY-s" style hypotheses):

$$\text{(P-w)} \quad \|P_c u\|_2 \le \|u\|_2,$$

$$\text{(P-s)} \quad \|P_c u\|_{\mathcal{A}_\tau} \le S_{\tau,\sigma} \|u\|_2,$$

$$\text{(Q-w)} \quad \|Q_c u\|_2 \le S_{0,\sigma}^{(1)} \|u\|_2, \qquad Q_c := \partial_c P_c,$$

$$\text{(R-w)} \quad \|R_c u\|_2 \le S_{0,\sigma}^{(2)} \|u\|_2, \qquad R_c := \partial_c^2 P_c,$$

and the analytic Fourier tail estimate

$$\|(I - \Pi_N)g\|_2 \le e^{-2\pi\tau N} \|g\|_{\mathcal{A}_\tau}.$$

Denote the coupling constants on the relevant range

$$L_G := \sup |G'|, \qquad \mathrm{Lip}(G) := \sup_{x \ne y} \frac{|G(x) - G(y)|}{|x - y|}, \qquad L_{G'} := \sup |G''|.$$

Let $\tilde{f} \in \mathrm{Ran}(\Pi_N)$ be a candidate with $\ell(\tilde{f}) = 1$. Define the computable *quantities*

$$\delta_N := \|T_N(\tilde{f}) - \tilde{f}\|_2, \qquad J_N := \left(I - DT_N(\tilde{f})\right)\big|_{\mathrm{Ran}(\Pi_N) \cap \ker \ell}, \qquad M_N := \|J_N^{-1}\|_{2\to 2},$$

and the a priori size bounds

$$K_2 := \|\tilde{f}\|_2, \qquad K_\tau := \|\tilde{f}\|_{\mathcal{A}_\tau}.$$

(1) Explicit residual bound for the true map. *Set the* true residual

$$\Delta := \|T(\tilde{f}) - \tilde{f}\|_2.$$

Then

$$\Delta \le \delta_N + e_T + e_{\mathrm{mis}},$$

where

$$e_T := e^{-2\pi\tau N}(S_{\tau,\sigma} + 1) K_\tau, \qquad e_{\mathrm{mis}} := |\delta| \, \mathrm{Lip}(G) \, e^{-2\pi\tau N} \|\phi\|_{\mathcal{A}_\tau} S_{0,\sigma}^{(1)} K_2^2.$$

(In particular, if ϕ is a trigonometric polynomial of degree $\le N$, then $m(\Pi_N f) = m(f)$ and $e_{\mathrm{mis}} = 0$.)

(2) Last explicit bound: Jacobian discretization error. *Define the Jacobian mismatch*

$$\varepsilon_J := \|DT(\tilde{f}) - DT_N(\tilde{f})\|_{\mathcal{A}_\tau \to L^2}.$$

Then one may take

$$\varepsilon_J \le e^{-2\pi\tau N}(S_{\tau,\sigma} + 1) + |\delta| L_G \|\phi\|_2 K_\tau \, e^{-2\pi\tau N}\left(S_{\tau,\sigma}^{(1)} + S_{0,\sigma}^{(1)}\right) + e^{-2\pi\tau N} C_{\mathrm{mis}}(K_2),$$

where the (explicit) mismatch constant can be chosen as

$$C_{\mathrm{mis}}(K_2) := |\delta| \, \|\phi\|_{\mathcal{A}_\tau} \left[(\mathrm{Lip}(G) + L_G) S_{0,\sigma}^{(1)} K_2 + L_{G'} \|\phi\|_2 S_{0,\sigma}^{(1)} K_2^2\right] + |\delta|^2 L_G \mathrm{Lip}(G) \|\phi\|_2 \|\phi\|_{\mathcal{A}_\tau} S_{0,\sigma}^{(2)} K_2^2.$$

(Again, if $\deg(\phi) \le N$ then $C_{\mathrm{mis}}(K_2) = 0$.)

(3) Certified inverse bound for the true Jacobian on $\ker \ell$**.** If

$$M_N \, \varepsilon_J < 1,$$

then $J(\tilde{f}) := (I - DT(\tilde{f}))|_{\ker \ell}$ is invertible and

$$M := \|J(\tilde{f})^{-1}\|_{2 \to 2; \ker \ell} \; \le \; \frac{M_N}{1 - M_N \varepsilon_J}.$$

(4) Lipschitz constant for the differential (mixed norm). On any L^2-ball $\{f : \|f\|_2 \le K\}$, one may take the (explicit) Lipschitz constant

$$\|DT(f) - DT(g)\|_{2 \to 2} \; \le \; \gamma(K) \, \|f - g\|_2,$$

with

$$\gamma(K) := |\delta| \, \|\phi\|_2 \, S_{0,\sigma}^{(1)} (\mathrm{Lip}(G) + L_G) \; + \; |\delta| \, L_{G'} \|\phi\|_2^2 \, S_{0,\sigma}^{(1)} \, K \; + \; |\delta|^2 \, L_G \mathrm{Lip}(G) \, \|\phi\|_2^2 \, S_{0,\sigma}^{(2)} \, K.$$

(For the Kantorovich test below you may take any $K \ge \|\tilde{f}\|_2 + r$; in practice one often starts with $K := \|\tilde{f}\|_2$ and enlarges if needed.)

(5) Kantorovich smallness and conclusion. Let $K \ge \|\tilde{f}\|_2$ and define

$$h := M^2 \, \gamma(K) \, \Delta.$$

Assume

$$h \le \frac{1}{2}.$$

Then there exists a *unique normalized fixed point* f^* of T with $\ell(f^*) = 1$ in the L^2-ball $B_r(\tilde{f})$, and

$$\|f^* - \tilde{f}\|_2 \le r, \qquad r := \frac{1 - \sqrt{1 - 2h}}{M \, \gamma(K)} \quad \textit{(interpreting } r := M\Delta \textit{ if } \gamma(K) = 0\textit{).}$$

Moreover, the Newton map on $\ell(f) = 1$ is a contraction on $B_r(\tilde{f})$ with contraction constant

$$\kappa := \frac{1 - \sqrt{1 - 2h}}{h} \; < \; 1.$$

What must be computed numerically. To apply the corollary you need only:

1. the Fourier candidate $\tilde{f} \in \mathrm{Ran}(\Pi_N)$ with $\ell(\tilde{f}) = 1$;

2. the finite-dimensional residual $\delta_N = \|T_N(\tilde{f}) - \tilde{f}\|_2$;

3. the finite-dimensional Jacobian $J_N = (I - DT_N(\tilde{f}))$ on $\mathrm{Ran}(\Pi_N) \cap \ker \ell$ and a certified bound $M_N = \|J_N^{-1}\|_{2 \to 2}$;

4. the norms $K_2 = \|\tilde{f}\|_2$ and $K_\tau = \|\tilde{f}\|_{\mathcal{A}_\tau}$ (both explicit from Fourier coefficients).

All remaining constants are explicit from $(\sigma, \tau, \delta, G, \phi)$ and the Gaussian bounds $S_{\tau,\sigma}$, $S_{0,\sigma}^{(1)}$, $S_{0,\sigma}^{(2)}$.

Lemma 13 (Explicit Gaussian smoothing constants). *Let $\bar{\rho}_\sigma$ be the periodized Gaussian on $\mathbb{T}$ with*

$$\widehat{\rho_\sigma}(k) = e^{-2\pi^2\sigma^2 k^2}, \qquad k \in \mathbb{Z}.$$

Define, for $\tau \geq 0$,

$$S_{\tau,\sigma} := \sup_{k \in \mathbb{Z}} \exp(2\pi\tau|k| - 2\pi^2\sigma^2 k^2),$$

$$S_{\tau,\sigma}^{(1)} := \sup_{k \in \mathbb{Z}} (2\pi|k|) \exp(2\pi\tau|k| - 2\pi^2\sigma^2 k^2),$$

$$S_{\tau,\sigma}^{(2)} := \sup_{k \in \mathbb{Z}} (2\pi|k|)^2 \exp(2\pi\tau|k| - 2\pi^2\sigma^2 k^2).$$

Then:

1. (***Exact computability***) *Each supremum is attained at a finite k and can be computed by checking k in a small neighborhood of the continuous maximizer:*

$$k_0 := \left\lfloor \frac{\tau}{2\pi\sigma^2} \right\rceil \quad \implies \quad S_{\tau,\sigma} = \max\{f(k_0-1), f(k_0), f(k_0+1)\},$$

where $f(k) = \exp(2\pi\tau|k| - 2\pi^2\sigma^2 k^2)$. Similarly $S_{\tau,\sigma}^{(1)}$ and $S_{\tau,\sigma}^{(2)}$ are attained near the continuous maximizers of $k \mapsto (2\pi k)^j e^{2\pi\tau k - 2\pi^2\sigma^2 k^2}$.

2. (***Fully explicit bounds***) *One has the closed-form upper bounds*

$$S_{\tau,\sigma} \leq \exp\left(\frac{\tau^2}{2\sigma^2}\right),$$

$$S_{\tau,\sigma}^{(1)} \leq \exp\left(\frac{\tau^2}{2\sigma^2}\right)\left(\frac{\tau}{\sigma^2} + \frac{1}{\sigma\sqrt{e}}\right),$$

$$S_{\tau,\sigma}^{(2)} \leq \exp\left(\frac{\tau^2}{2\sigma^2}\right)\left(\frac{\tau^2}{\sigma^4} + \frac{2\tau}{\sigma^3\sqrt{e}} + \frac{2}{\sigma^2 e}\right).$$

In particular (take $\tau = 0$),

$$S_{0,\sigma}^{(1)} \leq \frac{1}{\sigma\sqrt{e}}, \qquad S_{0,\sigma}^{(2)} \leq \frac{2}{\sigma^2 e}.$$

Proof. For $S_{\tau,\sigma}$, complete the square:

$$2\pi\tau k - 2\pi^2\sigma^2 k^2 = -2\pi^2\sigma^2\left(k - \frac{\tau}{2\pi\sigma^2}\right)^2 + \frac{\tau^2}{2\sigma^2},$$

so $\sup_k e^{2\pi\tau k - 2\pi^2\sigma^2 k^2} \leq e^{\tau^2/(2\sigma^2)}$. For $S_{\tau,\sigma}^{(1)}$ write $k = b+y$ with $b = \tau/(2\pi\sigma^2)$ and use $k \leq b+|y|$ together with

$$\sup_{y \in \mathbb{R}} |y| e^{-ay^2} = \frac{1}{\sqrt{2ae}}, \qquad a = 2\pi^2\sigma^2,$$

which yields the stated bound after multiplying by $2\pi e^{\tau^2/(2\sigma^2)}$. For $S_{\tau,\sigma}^{(2)}$ use $k^2 \leq b^2 + 2b|y| + y^2$ and

$$\sup_{y \in \mathbb{R}} y^2 e^{-ay^2} = \frac{1}{ae}.$$

The "exact computability" follows because all these sequences are unimodal for $k \geq 0$ and decay super-exponentially in k. $\qquad\square$

[Fourier coordinates on (Π_N)] Let $\mathcal{V}_N := (\Pi_N) = \operatorname{span}\{e_k(x) := e^{2\pi ikx}, \ |k| \leq N\}$. Write $f = \sum_{|m|\leq N} \hat{f}_m e_m$ and identify f with the vector $\hat{f} = (\hat{f}_m)_{|m|\leq N} \in \mathbb{C}^{2N+1}$. Let $\Phi = \sum_{m\in\mathbb{Z}} \hat{\Phi}_m e_m$ be the coupling observable and define

$$m(f) := \langle \Phi, f \rangle_{L^2} = \sum_{|m|\leq N} \overline{\hat{\Phi}_m}\, \hat{f}_m \quad \text{(if } \Phi \text{ is a trig. polynomial of degree } \leq N, \text{ this is exact)}.$$

Set $c(f) := \delta\, G(m(f))$ and $c'(f)[h] = \delta\, G'(m(f))\langle \Phi, h \rangle$.

Lemma 14 (Fourier matrix for P_c and rank-one structure of DT_N). *Let $\bar{\rho}_\sigma$ be the periodized Gaussian with $\widehat{\bar{\rho}_\sigma}(k) = e^{-2\pi^2\sigma^2 k^2}$, and let $T_0 : \mathbb{T} \to \mathbb{T}$ be measurable. Define, for $c \in \mathbb{R}$,*

$$(P_c u)(x) := \int_{\mathbb{T}} \bar{\rho}_\sigma(x - T_0(y) - c)u(y)\, dy, \qquad Q_c := \partial_c P_c.$$

For each $k \in \mathbb{Z}$ define the Fourier coefficients

$$V_k(m) := \int_{\mathbb{T}} e^{-2\pi ikT_0(y)}\, e^{2\pi imy}\, dy \qquad (m \in \mathbb{Z}),$$

so that $V_k(m) = \widehat{e^{-2\pi ikT_0(\cdot)}}(-m)$.

Then, restricted to $\mathcal{V}_N$, the operator $\Pi_N P_c \Pi_N$ is represented by the matrix $A(c) \in \mathbb{C}^{(2N+1)\times(2N+1)}$ with entries

$$A(c)_{k,m} = \widehat{\bar{\rho}_\sigma}(k)\, e^{-2\pi ikc}\, V_k(m), \qquad |k| \leq N, \ |m| \leq N.$$

Similarly, $\Pi_N Q_c \Pi_N$ is represented by

$$B(c)_{k,m} = (-2\pi ik)\, \widehat{\bar{\rho}_\sigma}(k)\, e^{-2\pi ikc}\, V_k(m).$$

Consider the fully discrete self-consistent map

$$T_N(f) := \Pi_N P_{c(\Pi_N f)} \Pi_N f \quad \text{on } \mathcal{V}_N.$$

Then T_N is Fréchet differentiable and for $f \in \mathcal{V}_N$, the derivative $DT_N(f) : \mathcal{V}_N \to \mathcal{V}_N$ is the rank-one update

$$DT_N(f) = A(c_N) + b_N \otimes a_N^*,$$

where

$$c_N := c(f) = \delta\, G\Big(\sum_{|m|\leq N} \overline{\hat{\Phi}_m}\, \hat{f}_m \Big), \qquad a_N := \delta\, G'(m(f))\, (\hat{\Phi}_m)_{|m|\leq N} \in \mathbb{C}^{2N+1},$$

and

$$b_N := B(c_N)\hat{f} \in \mathbb{C}^{2N+1}, \qquad (a_N^* \hat{h}) = \delta\, G'(m(f)) \sum_{|m|\leq N} \overline{\hat{\Phi}_m}\, \hat{h}_m = \delta\, G'(m(f))\langle \Phi, \ h \rangle.$$

Equivalently, in coordinates,

$$\widehat{DT_N(f)}[h] = A(c_N)\hat{h} + b_N\, (a_N^* \hat{h}).$$