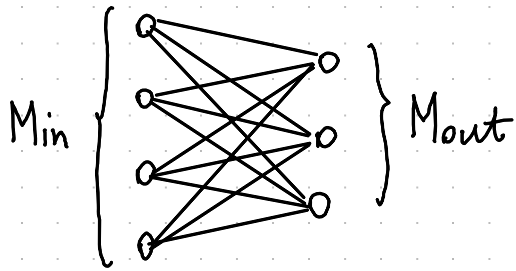
LINEAR LAYER

N data points



x input

y output

$$y = xW + b$$

- $x \in \mathbb{R}^{N \times M_{in}}$

  N = size of dataset

  Min = size of input layer

- $y \in \mathbb{R}^{N \times M_{out}}$

  size of output layer

- $W \in \mathbb{R}^{M_{in} \times M_{out}}$

- $b \in \mathbb{R}^{1 \times M_{out}}$

Aim: Assuming that we know

$$\frac{\partial L}{\partial y} \in \mathbb{R}^{M_{out} \times N}$$

compute

$\boxed{1}$ $\nabla_W L \in \mathbb{R}^{M_{in} \times M_{out}}$

$\boxed{2}$. $\nabla_b L \in \mathbb{R}^{1 \times M_{out}}$

$\boxed{3}$ $\frac{\partial L}{\partial x} \in \mathbb{R}^{M_{in} \times N}$

$\boxed{1}$ Let compute the entries:

$$\frac{\partial L}{\partial W_{hk}} = \sum_{i=0}^{N-1} \sum_{j=0}^{M_{out}-1} \frac{\partial L}{\partial y_{ij}} \underbrace{\frac{\partial y_{ij}}{\partial W_{hk}}}$$

$$y = xW + b$$

$$y_{ij} = \sum_{l=0}^{M_{in}-1} x_{il} W_{lj} + b_j$$

$$\frac{\partial y_{ij}}{\partial W_{hk}} = \underbrace{Id_{kj}\, x_{ih}}$$

$$\frac{\partial L}{\partial W_{hk}} = \sum_{i=0}^{N-1} \sum_{j=0}^{M_{out}-1} \frac{\partial L}{\partial y_{ij}} Id_{kj} \cdot x_{ih} =$$

$$= \sum_{i=0}^{N-1} \frac{\partial L}{\partial y_{ik}} x_{ih} = \left(\frac{\partial L}{\partial y} \quad x \right)_{kh}$$

$$\underbrace{\frac{\partial L}{\partial W}}_{\in \mathbb{R}^{M_{out} \times M_{in}}} = \underbrace{\frac{\partial L}{\partial y}}_{\in \mathbb{R}^{M_{out} \times N}} \cdot \underbrace{x}_{\in \mathbb{R}^{N \times M_{in}}}$$

$$\nabla_W L = \left(\frac{\partial L}{\partial W}\right)^T \in \underbrace{\mathbb{R}^{M_{in} \times M_{out}}}_{\text{same shape of } W}$$

$\boxed{2}$ $$\frac{\partial L}{\partial b_k} = \sum_{i=0}^{N-1} \sum_{j=0}^{M_{out}-1} \frac{\partial L}{\partial y_{ij}} \underbrace{\frac{\partial y_{ij}}{\partial b_k}}$$

$$y_{ij} = \sum_{l=0}^{M_{in}-1} x_{il} W_{lj} + b_j$$

$$\frac{\partial y_{ij}}{\partial b_k} = \underbrace{Id_{kj}}$$

$$\frac{\partial L}{\partial b_k} = \sum_{i=0}^{N-1} \sum_{j=0}^{M_{out}-1} \frac{\partial L}{\partial y_{ij}} Id_{kj} =$$

$$= \sum_{i=0}^{N-1} \frac{\partial L}{\partial y_{ik}} = \Bigg(\underbrace{\frac{\partial L}{\partial y}}_{\in \mathbb{R}^{M_{out} \times N}} \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{\in \mathbb{R}^{N \times 1}} \Bigg)_k$$

$\nabla_b L \in \mathbb{R}^{1 \times M_{out}}$

$\boxed{3}$ $$\frac{\partial L}{\partial x_{lk}} = \sum_{i=0}^{N-1} \sum_{j=0}^{M_{in}-1} \frac{\partial L}{\partial y_{ij}} \cdot \underbrace{\frac{\partial y_{ij}}{\partial x_{lk}}}$$

$$y_{ij} = \sum_{h=0}^{M_{in}-1} x_{ih} W_{hj} + b_j$$

$$\frac{\partial y_{ij}}{\partial x_{lk}} = \underbrace{Id_{il} W_{kj}}$$

$$\frac{\partial L}{\partial x_{lk}} = \sum_{i=0}^{N-1} \sum_{j=0}^{M_{in}-1} \frac{\partial L}{\partial y_{ij}} Id_{il} W_{kj} =$$

$$= \sum_{j=0}^{M_{in}-1} \frac{\partial L}{\partial y_{lj}} W_{kj} = \Big(\underbrace{W}_{\in \mathbb{R}^{M_{in} \times M_{out}}} \cdot \underbrace{\frac{\partial L}{\partial y}}_{\in \mathbb{R}^{M_{out} \times N}} \Big)_{kl}$$