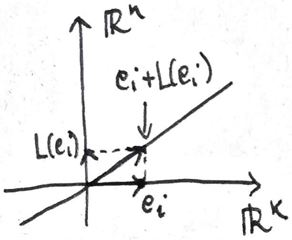
CHAPTER 4: CARTESIAN CURRENTS

GRAPHS AS CURRENTS

A simple example of manifold is the graph of a C^1 function $u: \mathbb{R}^k \to \mathbb{R}^n$, seen as a k-dimensional object in $\mathbb{R}^{k+n}$.

We fix some notation for the product space $\mathbb{R}^k \times \mathbb{R}^n$. We denote by $(x^1, \dots, x^k)$ the coordinates in $\mathbb{R}^k$ and by $(e_1, \dots, e_k)$ the standard basis. For $\mathbb{R}^n$ (the target space) we use the notation $(y^1, \dots, y^n)$ and $(\varepsilon_1, \dots, \varepsilon_n)$.

Let $L: \mathbb{R}^k \to \mathbb{R}^n$ be a linear map. Its graph $\{(x, L(x)) : x \in \mathbb{R}^k\}$ is a linear subspace of $\mathbb{R}^k \times \mathbb{R}^n$ of dimension k. Notice that, even if the image $L(\mathbb{R}^k)$ might be trivial, the graph has always dimension k! We can also find a basis for the graph:



$$(e_1 + L(e_1), \dots, e_k + L(e_k))$$

In particular, the unit simple k-vector in $\mathbb{R}^{k+n}$:

$$\tau = \frac{M(L)}{|M(L)|}, \qquad M(L) = (e_1 + L(e_1)) \wedge \dots \wedge (e_k + L(e_k))$$

orients the graph of L. There is an explicit way for computing the norm of a simple k-vector: $|M(L)|^2$ is given by

$$|M(L)|^2 = \sum_{A\ k\times k \text{ minor}} \det(A)^2$$

, where the minors are taken in the matrix which represents the vectors $e_1 + L(e_1), \dots, e_k + L(e_k)$ in the basis of $\mathbb{R}^{k \times n}$.

This matrix is easy to compute:

$$\begin{array}{c} k \\ \left(\begin{array}{c} Id_k \\ L \end{array} \right) \begin{array}{l} \}\, k \\ \}\, n \end{array} \\ k \end{array}$$

Thus

$$|M(L)| = \sqrt{1 + \sum_{\substack{A \text{ minors of} \\ \text{all orders} \le k \\ \text{of } L}} \det(A)^2} = J_k(id \times L)$$

We use what just said to describe the graph of a C^1-function.

Let $\Omega \subset \mathbb{R}^k$ be an open set and let $u \in C^1(\Omega; \mathbb{R}^k)$. Its *graph* is the subset of $\mathbb{R}^k \times \mathbb{R}^n$ given by:

$$\mathcal{G}_u := \{ (x, u(x)) : x \in \Omega \} = (id \times u)(\Omega).$$

The tangent space to $\mathcal{G}_u$ at $(x, u(x))$ is spanned by the basis:

$$(e_1 + \partial_1 u(x), \dots, e_k + \partial_k u(x))$$

and can be oriented (as in the linear case) by the simple k-vector

$$\tau(x, u(x)) = \frac{M(\nabla u(x))}{|M(\nabla u(x))|}, \quad M(\nabla u(x)) = (e_1 + \partial_1 u(x)) \wedge \dots \wedge (e_k + \partial_k u(x))$$

$$|M(\nabla u(x))| = \sqrt{1 + \sum_{\substack{A \text{ minor} \\ \text{of } \nabla u(x) \\ \text{of order} \le k}} \det(A)^2} = J_k(id \times u)$$

We define the **current associated to the graph of u** as the current $[\mathcal{G}_u, \tau, 1]$. More precisely; $[\mathcal{G}_u, \tau, 1] \in \mathcal{D}_k(\Omega \times \mathbb{R}^n)$ and $\forall\, \omega \in \mathcal{D}^k(\Omega \times \mathbb{R}^n)$:

$$[\mathcal{G}_u, \tau, 1](\omega) = \int_{\Omega \times \mathbb{R}^n} \langle \omega(x,y), \tau(x,y) \rangle \, d\mathcal{H}^k \llcorner \mathcal{G}_u(x,y) =$$

$$= \int_{(id \times u)(\Omega)} \langle \omega(x,y), M(\nabla u(x)) \rangle \, J_k(id \times u)(x) \, d\mathcal{H}^k(x,y) \overset{\text{Area Formula}}{=}$$

$$= \int_{\Omega} \langle \omega(x, u(x)), M(\nabla u(x)) \rangle \, d\mathcal{L}^k(x) =$$

$$= \int_{\Omega} \langle ((id \times u)^{\#} \omega)(x), e_1 \wedge \dots \wedge e_k \rangle \, d\mathcal{L}^k(x) = \int_{\Omega} (id \times u)^{\#} \omega = (id \times u)_{\#} [\Omega, e, 1](\omega)$$

where the symbol $(id \times u)_{\#} [\Omega, e, 1]$ denotes the **push-forward** of the current $[\Omega, e, 1]$ through the map $(id \times u)$. Note that it makes sense, since $(id \times u)^{\#} \omega$ has compact support (it does not have C^∞ coefficients, but $[\Omega, e, 1]$ has zero order so it can be extended to continuous forms with compact support.).

In the next lectures we shall study limits of graphs. As you know, limits of functions might ~~loss~~ lose regularity, e.g., Sobolev functions arise as limits of smooth functions. For this reason we need to extend the notion of graph to less regular functions.

APPROXIMATE DIFFERENTIABILITY

Let $\Omega \subset \mathbb{R}^k$ and $u \in L^1_{loc}(\Omega;\mathbb{R}^n)$.

Def: A Lebesgue point of u is a point $x \in \Omega$ s.t. $\exists\ \tilde{u}(x) \in \mathbb{R}^n$ satisfying:

$$\fint_{B_r(x)} |u(y) - \tilde{u}(x)|\, dy \to 0 .$$

The discontinuity set is the set S_u where the property above does not hold true.

- The function $\tilde{u} : \Omega \setminus S_u \to \mathbb{R}^n$ is Borel.
- $\mathcal{L}^k(S_u) = 0$
- $u = \tilde{u}$ $\mathcal{L}^k$-a.e. in $\Omega \setminus S_u$, i.e., $\tilde{u}$ is a representative of u.

If $u(x) = \tilde{u}(x)$ we say that u is approximately continuous at x.

Def: Let $x \in \Omega \setminus S_u$. We say that u is approximately differentiable at x if there exists a $(n \times k)$-matrix $\nabla u(x)$ such that

$$\fint_{B_r(x)} \frac{|u(y) - \tilde{u}(x) - \nabla u(x)(y-x)|}{|y-x|}\, dy \to 0 \quad \text{as } r \to 0 . \qquad *$$

$\nabla u(x)$ is the approximate gradient of u at x. (Analogously one can define the linear map $du(x)$ approximate differential of u at x).

We denote by $\mathcal{D}_u$ the set of approximate differentiability points.

The set $\mathcal{D}_u$ is Borel and $\nabla u : \mathcal{D}_u \to \mathbb{R}^{n\times k}$ is a Borel map.

[Instead we shall use Du to denote the distributional derivative].

NOTE: The blow-up $u_{x,r}(z) = \frac{u(x+rz) - u(x)}{r}$ converges to the affine function $\nabla u(x) \cdot z$.

THE CALDERÓN-ZYGMUND THEOREM

Theorem (Calderón-Zygmund): Let $u \in W^{1,1}_{loc}(\mathbb{R}^k;\mathbb{R}^n)$. Then u is approximately differentiable at $\mathcal{L}^k$-a.e. point and ∇u coincides with the distributional derivative a.e.

Lemma: Let $u \in W^{1,1}_{loc}(\mathbb{R}^k;\mathbb{R}^n)$ and let $x \in \Omega \setminus S_u$. Then

$$\fint_{B_r(x)} \frac{|u(y) - \tilde{u}(x)|}{|y-x|} \leq \int_0^1 \fint_{B_{tr}(x)} |Du(y)|\, dy\, dt .$$

* if u is differentiable at x, u is appr. diff. at x.

Proof (of Lemma): Let us assume, without loss of generality, that $x=0$, $\tilde{u}(0)=0$.

Let us first assume that u is smooth.

$$|u(y)-u(\rho y)| = \left|\int_\rho^1 Du(ty)\cdot y\,dt\right| \leq |y|\int_\rho^1 |Du(ty)|\,dt$$

Integrate the previous inequality in $B_r(0)$:

$$\int_{B_r(0)} \frac{|u(y)-u(\rho y)|}{|y|}\,dy \leq \int_{B_r(0)}\left(\int_\rho^1 |Du(ty)|\,dt\right)dy =$$

$$= \int_\rho^1\left(\int_{B_r(0)} |Du(ty)|\,dy\right)dt \underset{\substack{\uparrow \\ z=ty}}{=} \int_\rho^1\left(\int_{B_{tr}(0)} |Du(z)|\frac{1}{t^k}\,dz\right)dt.$$

By approximation, the inequality above is true for $W^{1,1}_{loc}(\mathbb{R}^k;\mathbb{R}^n)$ functions

Since $0\notin S_u$ by assumption, we have that $\fint_{B_r(0)} |u(z)|\,dz \to 0$, $r\to 0$.

In particular,

$$\int_{B_r(0)} |u(\rho y)|\,dy \underset{\substack{\uparrow \\ z=\rho y}}{=} \frac{1}{\rho^k}\int_{B_{r\rho}(0)} |u(z)|\,dz \to 0 \quad \text{as } \rho\to 0.$$

Then we can extract a sequence $\rho_j\to 0$ s.t. $u(\rho_j\,\cdot)\to 0$ a.e.

By Fatou's Lemma, we can pass to the limit in the inequality found before:

$$\int_{B_r(0)} \frac{|u(y)|}{|y|}\,dy \leq \int_0^1\left(\int_{B_{tr}(0)} |Du(z)|\frac{1}{t^k}\,dz\right)dt.$$

This concludes the proof. $\square$

Proof (of Calderón-Zygmund): We prove that u is approximate diff. at every $x \in \Omega \setminus (S_u \cup S_{Du})$. We apply the previous Lemma to the function

$$y \longmapsto u(y) - \tilde{u}(x) - \widetilde{Du}(x)(y-x)$$

to get

$$\fint_{B_r(x)} \frac{|u(y)-\tilde{u}(x)-\widetilde{Du}(x)(y-x)|}{|y-x|} \leq \int_0^1 \fint_{B_{tr}(x)} |Du(y)-\widetilde{Du}(x)|\,dy\,dt \leq$$

$$\leq \sup_{t\in(0,1)} \fint_{B_{tr}(x)} |Du(y)-\widetilde{Du}(x)|\,dy \longrightarrow 0 \quad \text{as } r \to 0, \text{ since } x \notin S_{Du}.$$

□

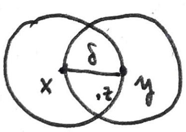
We recall that the maximal function of $f \in L^1_{loc}(\mathbb{R}^k)$ is the function

$$M_f(x) := \sup_{r>0} \fint_{B_r(x)} |f|.$$

Proposition: Let $u \in W^{1,1}_{loc}(\mathbb{R}^k;\mathbb{R}^n)$. Then $\forall\, x,y \in \Omega \setminus S_u$:

$$|\tilde{u}(x) - \tilde{u}(y)| \leq C\,(M_{Du}(x) + M_{Du}(y))\,|x-y|.$$

Proof: Let $x,y \in \Omega \setminus S_u$, let $\delta := |x-y|$ and for $z \in B_\delta(x) \cap B_\delta(y)$ let us compute:



$$|\tilde{u}(x)-\tilde{u}(y)| \leq |u(z)-\tilde{u}(x)| + |u(z)-\tilde{u}(y)|$$

Integrating:

$$|\tilde{u}(x)-\tilde{u}(y)| \leq \fint_{B_\delta(x)\cap B_\delta(y)} |\tilde{u}(x)-u(z)|\,dz + \fint_{B_\delta(x)\cap B_\delta(y)} |u(z)-\tilde{u}(y)|\,dz$$

However:

$$\fint_{B_\delta(x)\cap B_\delta(y)} |u(z)-\tilde{u}(x)|\,dz \leq C \fint_{B_\delta(x)} |u(z)-\tilde{u}(x)|\,dz \leq C\delta \fint_{B_\delta(x)} \frac{|u(z)-\tilde{u}(x)|}{|z-x|}\,dz \overset{\text{Lemma}}{\leq}$$

$$\leq C\delta \int_0^1 \fint_{B_\delta(x)} |Du(z)|\,dz \leq C\delta\, M_{Du}(x) = C\, M_{Du}(x)\cdot|x-y|.$$

□

Remark: The previous results hold true also in the case $u \in BV(\mathbb{R}^k;\mathbb{R}^n)$, i.e., when the distributional derivative Du is a measure.
In particular, a BV function is approximately differentiable a.e. and the approximate gradient ∇u is the density of Du with respect to the Lebesgue measure $\mathcal{L}^k$.

GRAPHS OF SOBOLEV FUNCTIONS

Let $u \in W^{1,1}(\Omega;\mathbb{R}^n)$, $\Omega \subset \mathbb{R}^k$. Let us consider the set of points:

$$R_u := \mathcal{D}_u \cap (\Omega \setminus S_{Du}) \cap \{u = \tilde{u}\} \leftarrow \text{"good points"}$$

We know, in particular, that if $x \in R_u$, then

- $x \in \Omega \setminus S_u$ (by definition $\mathcal{D}_u \subset \Omega \setminus S_u$)
- $M_{Du}(x) < +\infty$ (follows from $x \in \Omega \setminus S_{Du}$)

We define the graph of u as:

$$G_u := (id \times u)(R_u).$$

Proposition: G_u is rectifiable.

Proof: We start by noticing that $R_u \subset \Omega \setminus S_{Du} \subset \{x : M_{Du}(x) < +\infty\}$.

Then we define the sets $C_j := \{x : M_{Du}(x) \leq j\}$. The sets C_j's are relatively closed in Ω (since M_{Du} is lower semicontinuous) and $C_j \subset C_{j+1}$. $\bigcup_j C_j = \{x : M_{Du}(x) < +\infty\}$ and therefore the sets C_j cover R_u.

On $R_u \cap C_j$ we have, by the previous proposition (remember that $R_u \subset \Omega \setminus S_u$):

$$|u(x) - u(y)| \underset{R_u \subset \{u=\tilde{u}\}}{=} |\tilde{u}(x) - \tilde{u}(y)| \underset{\text{Prop.}}{\leq} C\big(M_{Du}(x) + M_{Du}(y)\big)|x-y| \leq$$
$$\leq 2Cj\,|x-y|,$$

that is, $u|_{R_u \cap C_j}$ is a Lipschitz function. Then we can extend it to a Lipschitz function in $\mathbb{R}^k$, say v_j, so that $v_j = u$ on $R_u \cap C_j$.

Then we can decompose:

$$G_u = (id \times u)(R_u) = (id \times u)\Big(\bigcup_j R_u \cap C_j\Big) = \bigcup_j (id \times u)(R_u \cap C_j) =$$
$$= \bigcup_j (id \times v_j)(C_j \cap R_u)$$

$(id \times v_j)(C_j \cap R_u)$ is a subset of $(id \times v_j)(\mathbb{R}^k)$, which is rectifiable since it is the image through a Lipschitz map.

The fact that $(id \times v_j)(C_j \cap R_u)$ is measurable is guaranteed by the Area Formula for Lipschitz maps.

$\square$

Proposition: Let $u, v \in W^{1,1}(\Omega; \mathbb{R}^n)$ be such that $u = v$ a.e. Then $\mathcal{H}^k(\mathcal{G}_u \Delta \mathcal{G}_v) = 0$.

Proof: Since $u = v$ a.e., we have that $\mathcal{L}^k(\mathcal{R}_u \Delta \mathcal{R}_v) = 0$.
Indeed: $\mathcal{L}^k(\mathcal{D}_u \Delta \mathcal{D}_v) = 0$, actually $\mathcal{D}_u = \mathcal{D}_v$, since the definition of approximate differentiability points does not depend on the Lebesgue representative. Analogously $S_{Du} = S_{Dv}$. Finally, $\mathcal{L}^k(\{u = \tilde{u}\} \Delta \{\tilde{u} = v\}) = 0$, since $\tilde{u}$ does not depend on the choice of the representative ($\tilde{u} = \tilde{v}$).

Then:

$$\begin{aligned}
\mathcal{G}_u \setminus \mathcal{G}_v &= (id \times u)(\mathcal{R}_u) \setminus (id \times v)(\mathcal{R}_v) = \\
&= \big((id \times u)(\mathcal{R}_u \cap \mathcal{R}_v) \cup (id \times u)(\mathcal{R}_u \setminus \mathcal{R}_v)\big) \setminus (id \times v)(\mathcal{R}_v) = \\
&\underset{\substack{\uparrow \\ \text{on } \mathcal{R}_u \cap \mathcal{R}_v \\ u = \tilde{u} = v}}{=} \big((id \times v)(\mathcal{R}_u \cap \mathcal{R}_v) \cup (id \times u)(\mathcal{R}_u \setminus \mathcal{R}_v)\big) \setminus (id \times v)(\mathcal{R}_v) = \\
&= (id \times u)(\underbrace{\mathcal{R}_u \setminus \mathcal{R}_v}_{N}) \qquad \text{where } \mathcal{L}^k(N) = 0.
\end{aligned}$$

We claim that $\mathcal{H}^k((id \times u)(N)) = 0$. If we could apply the Area Formula, this would follow immediately. To apply it, let $C_j := \{M_{Du} \le j\}$ be the sets already used in the previous proof.

Then $u|_{\mathcal{R}_u \cap C_j}$ is a Lipschitz function and

$$\mathcal{H}^k((id \times u)(N \cap C_j)) = \int_{N \cap C_j} J_k(id \times u)(x)\, dx = 0 \quad \leftarrow \text{since } \mathcal{L}^k(N) = 0.$$

This shows that $\mathcal{H}^k(\mathcal{G}_u \setminus \mathcal{G}_v) = 0$. Analogously we have $\mathcal{H}^k(\mathcal{G}_v \setminus \mathcal{G}_u) = 0$, which concludes the proof. $\square$

Remark: If $u \in W^{1,1}(\Omega; \mathbb{R}^n)$, the Area Formula holds true:

$$\mathcal{H}^k(\mathcal{G}_u) = \int_\Omega |M(\nabla u)(x)|\, dx \quad \leftarrow (e_1 + \partial_1 u) \wedge \ldots \wedge (e_k + \partial_k u)$$

In particular, $\mathcal{H}^k(\mathcal{G}_u) < +\infty$ if and only if all the minors of ∇u are L^1 functions.

To show the previous formula, use the sets $\tilde{C}_j := C_j \setminus C_{j-1}$ to cover $\mathcal{R}_u$.

Remark: $\mathcal{G}_u$ has an approximate tangent space at $\mathcal{H}^k$-a.e. $(x,y) \in \mathcal{G}_u$, being a rectifiable set, if $\mathcal{H}^k(\mathcal{G}_u) < +\infty$. Given $x_0 \in \mathcal{R}_u$, we can compute, with a blow-up formula, that

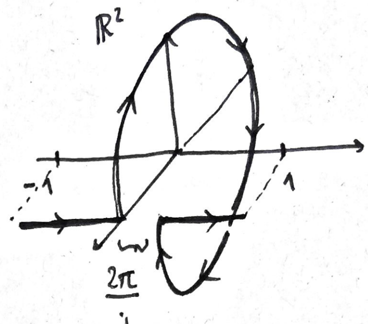
$$Tan(\mathcal{G}_u, (x_0, u(x_0))) = \{(x,y) \in \mathbb{R}^{k+n} : y = \nabla u(x_0) x\}.$$

We can define the current $[\mathcal{G}_u, \tau_u, 1]$.

CONCENTRATION EFFECTS

<u>Example</u>: Let $\Omega = (-1,1)$ and let us consider the sequence $u_j : \Omega \to \mathbb{S}^1$ given by

$$u_j(t) := \begin{cases} (1,0) & \text{otherwise} \\ (\cos(jt), \sin(jt)) & \text{for } t \in (0, \frac{2\pi}{j}) \end{cases}$$



Then $u_j \to u_0 \equiv (1,0)$ strongly in L^1.
(Actually w^* - BV).
The length of the curve is :

$$\mathbb{M}([G_{u_j}]) = \int_{-1}^{1} \sqrt{1+|u_j'(t)|^2}\,dt = \int_{(-1,1)\setminus(0,\frac{2\pi}{j})} \ldots + \int_{(0,\frac{2\pi}{j})} \ldots =$$

$$= 2 - \frac{2\pi}{j} + \int_0^{\frac{2\pi}{j}} \sqrt{1+j^2}\,dt \underset{s=jt}{=} 2 - \frac{2\pi}{j} + \int_0^{2\pi} \frac{\sqrt{1+j^2}}{j}\,ds \xrightarrow{j\to\infty} 2+2\pi$$

Therefore:

$$\mathbb{M}([G_{u_0}]) = 2 < 2+2\pi = \lim_j \mathbb{M}([G_{u_j}]).$$

The reason for this gap is that, due to a concentration effect, the limit of the graphs G_{u_j} is not given by the graph of the limit.

With the language of currents we can identify the limit of the graphs $[G_{u_j}]$.
Let us fix $\omega \in \mathcal{D}^1((-1,1)\times\mathbb{R}^2)$. We can write ω in components :

$$\omega = \omega_0(t,x_1,x_2)\,dt + \omega_1(t,x_1,x_2)\,dx^1 + \omega_2(t,x_1,x_2)\,dx^2.$$

$$[G_{u_j}](\omega) = \int_{-1}^{1} \langle \omega(t,u_j(t)), \tau(t,u_j(t)) \rangle \, d\mathcal{L}^1(t) =$$

$$= \underbrace{\int_{-1}^{1} \omega_0(t,u_j(t))\,dt}_{} + \underbrace{\underbrace{\int_{-1}^{1} \omega_1(t,u_j(t))\,\dot{u}_j^1(t)\,dt}_{} + \underbrace{\int_{-1}^{1} \omega_2(t,u_j(t))\,\dot{u}_j^2(t)\,dt}_{}}_{\text{both are } = 0 \text{ in } (-1,1)\setminus(0,\frac{2\pi}{j})}$$

$$\int_{(-1,1)\setminus(0,\frac{2\pi}{j})} \longrightarrow \int_{(-1,1)} \omega_0(t,u_0)$$

$$\int_{(0,\frac{2\pi}{j})} \longrightarrow 0$$

Let us compute the mass of the graph.

$$\mathbb{M}([G_{u_j}]) = \int_B \sqrt{1+|\nabla u_j|^2 + \underbrace{|\det \nabla u_j|^2}} \, dx = \int_B \sqrt{1+|\nabla u_j|^2} \, dx$$

u_j maps the whole ball B to $\mathbb{S}^1$. By the Area Formula $|\det \nabla u_j| = 0$. $\rightarrow$ This also follows from $|u_j|^2 = 1$, if you want.

It is easier to compute $|\nabla u_j|^2$ in polar coordinates:

$$|\nabla u_j|^2 = |\partial_\rho u_j|^2 + \frac{1}{\rho^2}|\partial_\theta u_j|^2 \Rightarrow \begin{array}{l} |\partial_\rho u_j| \sim j \text{ in } B_{\frac{2}{j}} \text{ and } = 0 \text{ outside } B_{\frac{2}{j}} \\ \frac{1}{\rho}|\partial_\theta u_j| \sim \frac{j}{\rho} \text{ in } S_j \text{ and } \sim \frac{1}{\rho} \text{ outside } S_j \end{array}$$

Then we split:

$$\int_B \sqrt{1+|\nabla u_j|^2} = \int_{B \setminus (B_{\frac{2}{j}} \cup S_j)} \sqrt{1+|\nabla u|^2} \, dx + \int_{B_{\frac{2}{j}} \cap S_j} \_\_ + \int_{B_{\frac{2}{j}} \setminus S_j} \_\_ + \int_{S_j \setminus B_{\frac{2}{j}}} \_\_$$

- In $B_{\frac{2}{j}} \cap S_j$ we have $|\partial_\rho u_j| \sim j$ and $\frac{1}{\rho}|\partial_\theta u_j| \sim \frac{j}{\rho}$. The latter is worse. But:

$$\int_{B_{\frac{2}{j}} \cap S_j} \frac{j}{\rho} \, dx = \int_0^{2/j} \int_{2\pi - \frac{1}{j}}^{2\pi} \frac{j}{\rho} \rho \, d\theta \, d\rho = \frac{2}{j} \cdot \frac{1}{j} \cdot j \sim \frac{1}{j^2} \to 0$$

- In $B_{\frac{2}{j}} \setminus S_j$ we have $|\partial_\rho u_j| \sim j$ and $\frac{1}{\rho}|\partial_\theta u_j| \sim \frac{1}{\rho}$. But

$$\int_{B_{\frac{2}{j}} \setminus S_j} j + \frac{1}{\rho} = \int_0^{2/j} \int_0^{2\pi - \frac{1}{j}} \left(j + \frac{1}{\rho}\right) \rho \, d\theta \, d\rho = \int_0^{2/j} \rho j + 1 \, d\rho \to 0 .$$

- For the last integral, we do a more precise computation: in $S_j \setminus B_{\frac{2}{j}}$

$$|\nabla u_j| = \frac{1}{\rho} |\partial_\theta u_j| = \frac{1}{\rho} |\partial_\theta \varphi_j(\theta)| = \frac{1}{\rho} j \left(2\pi - \frac{1}{j}\right)$$

because it does not depend on ρ / by definition of φ_j.

Then

$$\int_{S_j \setminus B_{\frac{2}{j}}} \sqrt{1 + \frac{1}{\rho^2} j^2 \left(2\pi - \frac{1}{j}\right)^2} = \int_{\frac{2}{j}}^1 \int_{2\pi - \frac{1}{j}}^{2\pi} \sqrt{1 + \frac{1}{\rho^2} j^2 \left(2\pi - \frac{1}{j}\right)^2} \, \rho \, d\theta \, d\rho =$$

$$= \int_{\frac{2}{j}}^1 \frac{1}{j} \sqrt{\rho^2 + j^2 \left(2\pi - \frac{1}{j}\right)^2} \, d\rho \longrightarrow 2\pi$$

$$\left(2\pi - \frac{1}{j}\right) \le \frac{1}{j} \sqrt{\rho^2 + j^2 \left(2\pi - \frac{1}{j}\right)^2} \le \frac{\rho}{j} + \left(2\pi - \frac{1}{j}\right)$$

In conclusion:

$$\mathbb{M}([\mathcal{G}_{u_j}]) \longrightarrow \underbrace{2\pi}_{} + \mathbb{M}([\mathcal{G}_u]) > \mathbb{M}([\mathcal{G}_u])$$

We can interpret this 2π only if we look at the limit of the graphs.

there is a gap. Can we do better and try to approximate $\mathbb{M}([\mathcal{G}_u])$? We will answer this question in the course.

<u>Computation of the limit current</u>:

Let us fix $\omega \in \mathcal{D}^2(B\times\mathbb{R}^2)$ and let us compute the limit of $[\mathcal{G}_{u_j}](\omega)$.

We recall that

$$[\mathcal{G}_{u_j}](\omega) = \int_\Omega \langle \omega(x, u_j(x)), M(\nabla u_j) \rangle \, dx$$

and

$$\begin{aligned} M(\nabla u_j) &= (e_1 + \partial_1 u_j)\wedge(e_2 + \partial_2 u_j) = \\ &= (e_1 + \partial_1 u_j^1 \varepsilon_1 + \partial_1 u_j^2 \varepsilon_2)\wedge(e_2 + \partial_2 u_j^1 \varepsilon_1 + \partial_2 u_j^2 \varepsilon_2) \\ &= e_1\wedge e_2 + \partial_2 u_j^1 \, e_1\wedge\varepsilon_1 + \partial_2 u_j^2 \, e_1\wedge\varepsilon_2 \\ &\quad - \partial_1 u_j^1 \, e_2\wedge\varepsilon_1 - \partial_1 u_j^2 \, e_2\wedge\varepsilon_2 \\ &\quad + \underbrace{\det \nabla u_j}_{=0} \, \varepsilon_1\wedge\varepsilon_2 \end{aligned}$$

A form $\omega\in\mathcal{D}^2(B\times\mathbb{R}^2)$ has components in the basis of $\Lambda^2(\mathbb{R}^2\times\mathbb{R}^2)$:

$$\phi\, dx^1\wedge dx^2, \quad \phi\, dx^i\wedge dy^j, \quad \underbrace{\phi\, dy^1\wedge dy^2}_{\text{When we integrate these components we get } = 0.}$$

$$\begin{aligned} [\mathcal{G}_{u_j}](\phi\, dx^1\wedge dx^2) &= \int_\Omega \langle \phi(x, u_j(x))\, dx^1\wedge dx^2, e_1\wedge e_2\rangle\, dx = \\ &= \int_\Omega \phi(x, u_j(x))\, dx \longrightarrow \int_\Omega \phi(x, u(x))\, dx = \\ &= [\mathcal{G}_u](\phi\, dx^1\wedge dx^2). \end{aligned}$$

Let us compute, for example, $[G_{u_j}](\phi\, dx^1 \wedge dy^1)$.

$$[G_u](\phi\, dx^1\wedge dy^1) = \int_\Omega \langle \phi(x,u_j(x))\, dx^1\wedge dy^1,\ \partial_2 u_j^1\ e_1\wedge \varepsilon_2\rangle\, dx =$$

$$= \int_\Omega \phi(x,u_j(x))\, \partial_2 u_j^1(x)\, dx =$$

in polar coordinates

$$= \int_\Omega \phi(x,u_j(x)) \left(\partial_\rho u_j^1(x) \sin\theta + \frac{1}{\rho}\partial_\theta u_j^1(x)\cos\theta\right) dx$$

The only important part for this integral is $S_j \setminus B_{\frac{2}{j}}$. Indeed in $B_{\frac{2}{j}}$ the whole mass goes to 0. In $B\setminus(S_j\cup B_{\frac{2}{j}})$ the function u_j is equal to u, thus we have:

$$\int_{B\setminus(S_j\cup B_{\frac{1}{j}})} \phi(x,u_j(x))\,\partial_2 u_j^1(x)\,dx = \int_{B\setminus(S_j\cup B_{\frac{1}{j}})} \phi(x,u(x))\,\partial_2 u^1(x)\,dx \to \int_B \phi(x,u(x))\,\partial_2 u^1(x)\,dx$$

Let us compute the "important" part in polar coordinates:

$$\int_{S_j\setminus B_{\frac{2}{j}}} \phi(x,u_j(x))\,\partial_2 u_j^1(x)\,dx = \int_{\frac{2}{j}}^1\int_{2\pi-\frac{1}{j}}^{2\pi} \phi(x,u_j(\rho,\theta))\Big(\underbrace{\partial_\rho u_j^1}_{\substack{=0 \text{ since}\\ u_j \text{ does not depend}\\ \text{on } \rho \text{ in this region}}}\sin\theta + \frac{1}{\rho}\underbrace{\partial_\theta u_j^1}_{=-\sin(\varphi_j(\theta))\cdot\partial_\theta\varphi_j(\theta)}\cos\theta\Big)\rho\, d\theta\, d\rho$$

$$= \int_{\frac{2}{j}}^1\int_{2\pi-\frac{1}{j}}^{2\pi} \phi(\rho,\theta,u_j(\rho,\theta))\left(-\sin(\varphi_j(\theta))\cos\theta\ \varphi_j'(\theta)\right) d\theta\, d\rho =$$

change of variables
$t = \varphi_j(\theta) = j(2\pi-\frac{1}{j})(2\pi-\theta)$
$\Rightarrow \theta = 2\pi - \frac{t}{j(2\pi-\frac{1}{j})}$

$$= \int_{\frac{2}{j}}^1\int_{2\pi-\frac{1}{j}}^{0} \phi\left(\rho, 2\pi-\frac{t}{j(2\pi-\frac{1}{j})}, \cos t, \sin t\right)(-\sin t)\cdot\cos\left(\frac{t}{j(2\pi-\frac{1}{j})}\right) dt\, d\rho \to$$

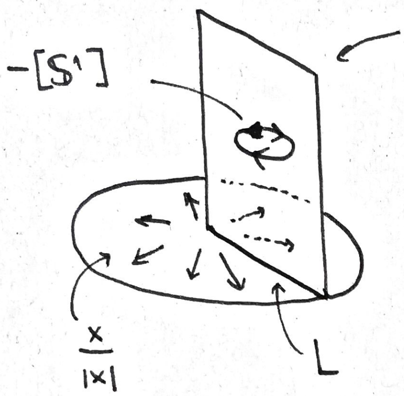
$$\to -\int_0^1\int_0^{2\pi} \phi(\rho, 2\pi, \cos t, \sin t)(-\sin t)\, dt\, d\rho = \int_{[0,1]\times\{2\pi\}}\left(-\int_{S^1}\phi(x,\cdot)\,dy^1\right) d\mathcal{H}^1(x) =$$

$$= -\underbrace{L}_{\text{L is the radius}} \times \underbrace{[S^1]}_{\text{circle oriented counterclockwise}} (\phi\, dx^1\wedge dy^1)$$

oriented like this

With similar computations we get the limit on the other components and we conclude that

$$[G_{u_j}] \to \underbrace{[G_u] - L \times [\![\mathbb{S}^1]\!]}_{\text{This limit current looks like this:}} \text{ in } \mathcal{D}_2(B \times \mathbb{R}^2).$$



There is a "vertical" part in the limit current, given by $-L \times [\mathbb{S}^1]$. This part is non-zero only on forms with components $dx^i \wedge dy^j$ (in contrast to the "horizontal" form $dx^1 \wedge dx^2$).
This vertical part is keeping track in the limit of the concentration effect that is happening on this line.

With the aid of currents we will understand that this is the optimal (in mass) way to approximate the map $\frac{x}{|x|}$ through C^1 maps with values in $\mathbb{S}^1$. More precisely, for every sequence $v_j \longrightarrow \frac{x}{|x|}$ in L^1 we have $\liminf_j \mathbb{M}([G_{v_j}]) \geq 2\pi + \mathbb{M}([G_{\frac{x}{|x|}}])$!

A first suggestion of this fact is the following:

Proposition: Let $u \in C^1(\Omega; \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^k$. Then

$$\partial [G_u] = 0 \quad \text{in } \mathcal{D}_{k-1}(\Omega \times \mathbb{R}^n).$$

Proof: Let us fix $\omega \in \mathcal{D}_{k-1}(\Omega \times \mathbb{R}^n)$. Then:

$$\partial [G_u](\omega) = [G_u](d\omega) = (id \times u)_\# [\Omega](d\omega) = [\Omega]((id \times u)^\# d\omega) =$$
$$= [\Omega](\underbrace{d(id \times u)^\# \omega}) = 0$$

this belongs to $\mathcal{D}^k_c(\Omega \times \mathbb{R}^n)$, i.e., has compact support, indeed

$$(id \times u)^\#(\phi(x,y)\, dx^{\hat{i}} \wedge dy^{\hat{j}}) = \phi(x, u(x))\, dx^{\hat{i}} \wedge du^{\hat{j}}$$

$\square$

Proposition: Let $\Omega \subset \mathbb{R}^k$ be bounded. Let $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, ~~$\nu$~~.
Assume $p \geq \min\{n, k\}$. Then $[G_u] \in \mathcal{D}_k(\Omega \times \mathbb{R}^n)$ and
$\partial [G_u] = 0$ in $\mathcal{D}_{k-1}(\Omega \times \mathbb{R}^n)$.

As a consequence of this fact, if $u_j \in C^1(\Omega;\mathbb{R}^n)$ and $[\mathcal{G}_{u_j}] \to T$ in $\mathcal{D}_K(\Omega\times\mathbb{R}^n)$, then necessarily

$$\partial T = 0 \quad \text{in } \mathcal{D}_{K-1}(\Omega\times\mathbb{R}^n).$$

Indeed, the boundary operator is continuous with respect to the convergence in the sense of currents. (For $\omega \in \mathcal{D}^{K-1}(\Omega\times\mathbb{R}^n)$ we have:
$0 = \partial[\mathcal{G}_{u_j}](\omega) = [\mathcal{G}_{u_j}](d\omega) \to T(d\omega) = \partial T(\omega)$)

<u>Example</u>: Let us go back to our previous example. We found a sequence u_j of Lipschitz maps s.t. $u_j \to \frac{x}{|x|} = u(x)$, $[\mathcal{G}_{u_j}] \to [\mathcal{G}_u] - L\times[\mathbb{S}^1]$ in $\mathcal{D}_2(B\times\mathbb{R}^2)$.

Then

$$0 = \partial[\mathcal{G}_{u_j}] \to \partial[\mathcal{G}_{\frac{x}{|x|}}] - \partial\left(L\times[\mathbb{S}^1]\right) \Rightarrow$$

$$\Rightarrow \partial[\mathcal{G}_{\frac{x}{|x|}}] = \underbrace{\partial(L\times[\mathbb{S}^1])}.$$

EXERCISE: There is no $\varphi \in W^{1,1}(B;\mathbb{R})$ s.t. $\frac{x}{|x|} = \exp(i\varphi(x))$.

To compute this boundary:
fix $\omega \in \mathcal{D}^1(B\times\mathbb{R}^2)$, $\omega = \omega_i dx^i + \omega_j dy^j$.

$$\partial(L\times[\mathbb{S}^1])(\omega) = L\times[\mathbb{S}^1](d\omega) = L\times[\mathbb{S}^1](d\omega_i\wedge dx^i + d\omega_j \wedge dy^j) =$$

$$= L\times[\mathbb{S}^1]\left(-\frac{\partial\omega_i}{\partial y_j} dx^i\wedge dy^j + \frac{\partial \omega_j}{\partial x_i} dx^i \wedge dy^j\right) =$$

$$= L\left([\mathbb{S}^1]\left(-\frac{\partial\omega_i}{\partial y_j} dy^j + \frac{\partial\omega_j}{\partial x_i} dy^j\right) dx^i\right) =$$

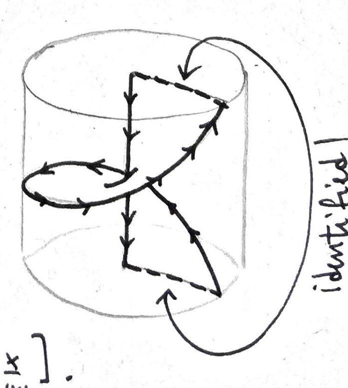
$$= L\left(\left([\mathbb{S}^1](-d_y\omega_i) + \frac{\partial}{\partial x_i}[\mathbb{S}^1](\omega^j dy^j)\right) dx^i\right) =$$

$$= L\left(\frac{\partial}{\partial x_i}[\mathbb{S}^1](\omega^j dy^j)\, dx^i\right) = \partial L\left([\mathbb{S}^1](\omega^j dy^j)\right) =$$

$$= \partial L \times [\mathbb{S}^1](\omega) = -\delta_0 \times [\mathbb{S}^1](\omega)$$

In conclusion,

$$\partial[\mathcal{G}_{\frac{x}{|x|}}] = -\delta_0\times[\mathbb{S}^1]$$



identified!

If $v_j \in C^1(B;\mathbb{R}^2)$ s.t. $v_j \to \frac{x}{|x|}$ in L^1 with $\int_B |\nabla v_j| \le C$, then $[\mathcal{G}_{v_j}] \to T$ but $T \ne [\mathcal{G}_{\frac{x}{|x|}}]$.