CARTESIAN CURRENTS

We find now necessary conditions for a current to be the limit of graphs of smooth functions.

Let us fix $T \in \mathcal{D}_K(\Omega \times \mathbb{R}^n)$ and let us assume that $u_j \in C^1(\Omega; \mathbb{R}^n)$ are such that $[\mathcal{G}_{u_j}] \rightharpoonup T$ in $\mathcal{D}_K(\Omega \times \mathbb{R}^n)$, $\|u_j\|_1 \leq C$, $\mathbb{M}([\mathcal{G}_{u_j}]) \leq C$.

$\boxed{1}$ Since $\partial[\mathcal{G}_{u_j}] = 0 \ \forall j$, we have $\partial T = 0$ in $\mathcal{D}_{K-1}(\Omega \times \mathbb{R}^n)$.

$\boxed{2}$ Each current $[\mathcal{G}_{u_j}]$ is a K-rectifiable current with integer multiplicity (equal to 1, by definition). Moreover $\mathbb{M}([\mathcal{G}_{u_j}])$ is equibounded by assumption and $\mathbb{M}(\partial[\mathcal{G}_{u_j}]) = 0$. By Federer & Fleming closure theorem we have that T is a rectifiable current with integer multiplicity with $\mathbb{M}(T) < +\infty$.

$\boxed{3}$ Let $\omega = \phi(x,y)\, dx$, $\phi \geq 0$. (a nonnegative horizontal form). Then

$$[\mathcal{G}_{u_j}](\omega) = \int_\Omega \langle \phi(x, u_j(x))\, dx, e_1 \wedge \dots \wedge e_K \rangle\, dx = \int_\Omega \phi(x, u_j(x))\, dx \geq 0$$

Passing to the limit we get $T(\omega) \geq 0$. Let $T^h(\phi) := T(\omega)$. Then $T^h \geq 0$. (In some sense, T does not "switch" the orientation).

$\boxed{4}$ Let us define:

this extends to all functions with bounded coefficients

$$\|T\|_1 := \sup \left\{ T(\phi(x,y)|y|\, dx) : \phi \in C_c^0(\Omega \times \mathbb{R}^n), |\phi| \leq 1 \right\}.$$

Note that:

$$\|[\mathcal{G}_{u_j}]\|_1 = \sup \left\{ \int_\Omega \phi(x, u_j(x))\, |u_j(x)|\, dx : \phi \in C_c^0(\Omega \times \mathbb{R}^n), |\phi| \leq 1 \right\} =$$

$$= \|u_j\|_1 .$$

Then $\|T\|_1 < +\infty$.

$\boxed{5}$ Let $\pi : \Omega \times \mathbb{R}^n \to \Omega$ be the projection on the first component. Given $\omega = \phi(x)\, dx \in \mathcal{D}^K(\Omega)$ we consider the pull-back $\pi^\# \omega$, which is a K-form in $\Omega \times \mathbb{R}^n$ with smooth coefficients but not with compact support:

$$\pi^\#(\phi(x,y)\, dx) = \phi(\pi(x,y))\, d\pi = \phi(x)\, dx$$

independent of y!

Since T has finite mass, we can however compute it on forms $\omega(x,y)$ with bounded coefficients. To do so, one fixes a cutoff function χ_R such that $0 \leq \chi_R \leq 1$, $\chi_R \equiv 0$ outside B_{2R} and $\chi_R \equiv 1$ on $\overline{B}_R$.

Then $T(\chi_R(y)\omega(x,y))$ is a Cauchy sequence: for $R > R'$:

$$|T(\chi_R(y)\omega(x,y)) - T(\chi_{R'}(y)\omega(x,y))| = |T((\chi_R(y) - \chi_{R'}(y))\omega(x,y))| \leq$$

$$\leq |T|(B_{2R} \setminus B_{R'}) \, \|\omega\|_\infty \longrightarrow 0 \quad \text{as } R, R' \longrightarrow +\infty.$$

The limit as $R \to +\infty$ does not depend on the choice of χ_R.

Then $T(\omega) := \lim_{R \to \infty} T(\chi_R(y)\omega)$.

$$T(\phi(x)\,dx) = \lim_{R \to \infty} T(\chi_R(y)\phi(x)\,dx).$$

For $[G_{u_j}]$ we have:

$$[G_{u_j}](\phi(x)\,dx) = \lim_{R \to +\infty} [G_{u_j}](\phi(x)\chi_R(y)\,dx) =$$

$$= \lim_{R \to \infty} \int_\Omega \chi_R(u_j(x))\phi(x)\,dx = \int_\Omega \phi(x)\,dx$$

i.e., $\pi_\#[G_{u_j}] = [\Omega]$.

This condition passes to the limit.

$$|\pi_\#[G_{u_j}](\phi(x)\,dx) - \pi_\# T(\phi(x)\,dx)| \leq$$

$$\leq |[G_{u_j}](\chi_R(y)\phi(x)\,dx) - T(\chi_R(y)\phi(x)\,dx)| +$$

$$+ |[G_{u_j}]((1-\chi_R(y))\phi(x)\,dx) - T((1-\chi_R(y))\phi(x)\,dx)|$$

The first term $\to 0$ because $[G_{u_j}] \to T$ in $\mathcal{D}_K(\Omega \times \mathbb{R}^n)$.

For the second term, we divide and multiply by $|y|$ (we are far from 0):

$$\left|[G_{u_j}]\left(\frac{(1-\chi_R(y))}{|y|}\phi(x)|y|\,dx\right)\right| + \left|T\left(\frac{(1-\chi_R(y))}{|y|}\phi(x)|y|\,dx\right)\right| \leq$$

$$\leq \left(\|[G_{u_j}]\|_1 + \|T\|_1\right)\frac{\|\phi\|_\infty}{R} \longrightarrow 0 \quad \text{as } R \to +\infty.$$

We can finally give the definition of Cartesian currents.

Def: (Giaquinta - Modica - Souček): The class of Cartesian currents is defined by:

$$\mathrm{cart}(\Omega\times\mathbb{R}^n) := \{ T\in\mathcal{D}_K(\Omega\times\mathbb{R}^n) : \begin{array}{l} \partial T = 0 \text{ in } \mathcal{D}_{K-1}(\Omega\times\mathbb{R}^n), \\ T \text{ is } K\text{-rectifiable with i.m.}, \\ \mathbb{M}(T) < +\infty, \\ T^h \geq 0, \\ \|T\|_1 < +\infty, \\ \pi_\# T = [\Omega] \end{array} \}.$$

The class of cartesian currents is closed with respect to the weak convergence in the sense of currents with equibounded masses (the proof is basically the same done to deduce the properties in the definition).

The general problem of characterizing all the currents which are limits of smooth graphs with equibounded masses is still an open question in its full generality. Unfortunately, there are elements of $\mathrm{cart}(\Omega\times\mathbb{R}^n)$ which are not limits of smooth graphs ~~was~~ with equibounded masses.

[the example is: $T = G_0 + S \in \mathcal{D}_2(B\times\mathbb{R}^2)$, $B\subset\mathbb{R}^2$ unit ball, $S = \sum_{i=1}^{\infty} [\partial B_{r_i}(x_i)]\times[\partial B_1(y_i)]$ with $\sum_i r_i < +\infty$ and $\sum |y_i| = +\infty$.]

However, Cartesian currents have a very nice structure. We shall see that there exists a BV function $u_T \in BV(\Omega;\mathbb{R}^n)$ such that $T = [G_{u_T}] + S$, where S is a vertical current.

BV FUNCTIONS

Def: A function $u\in L^1(\Omega)$ is a function of bounded variation (in symbols $u\in BV(\Omega)$) if its distributional derivative Du can be represented by a bounded measure, i.e., $\exists\ \mu\in\mathcal{M}_b(\Omega;\mathbb{R}^k)$ s.t.

$$\int_\Omega \varphi\cdot d\mu = -\int_\Omega \mathrm{div}\,\varphi\, u\, dx \quad \forall \varphi\in C_c^\infty(\Omega;\mathbb{R}^k).$$

The total variation is the sup of among all $\varphi\in C_c^\infty(\Omega;\mathbb{R}^k)$ with $\|\varphi\|_\infty\le 1$ and is denoted by $|Du|(\Omega)$. It is lower semicontinuous w.r.t. the L^1-convergence of u.

Proposition: Let $u\in BV(\Omega)$. Then there exists a sequence $u_j \in C^\infty(\Omega)\cap W^{1,1}(\Omega)$ s.t. $u_j \to u$ in $L^1(\Omega)$ and $\int_\Omega |\nabla u_j|\,dx \to |Du|(\Omega)$.

[Proof: convolution + partition of unity]

For example, through approximations one can deduce that the embeddings for $W^{1,1}$ also hold true for BV: if Ω is bounded, open, Lipschitz, then

$$BV(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega), \quad BV(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega) \text{ for } p \in [1, \tfrac{n}{n-1}).$$

Du is a measure and we can decompose it as

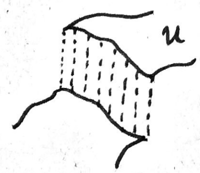
$$Du = D^a u + D^s u$$

where $D^a u \ll \mathcal{L}^k$ and $D^s u \perp \mathcal{L}^k$. We already discussed about the absolutely continuous part previously:

Theorem (Calderón-Zygmund): Let $u \in BV(\Omega)$. Then u is approximately differentiable a.e. and the approximate gradient ∇u is given a.e. by the density of $D^a u$ w.r.t. $\mathcal{L}^k$, i.e., $\nabla u(x) = \frac{dD^a u}{d\mathcal{L}^k}(x)$ for $\mathcal{L}^k$-a.e. x.

In conclusion

$$Du = \nabla u \, \mathcal{L}^k + D^s u.$$



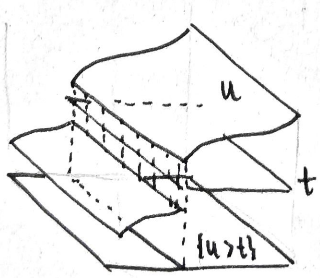
The singular part can, e.g., appear because of concentration effects.

Def: A measurable set $E \subset \mathbb{R}^k$ is a set of finite perimeter if $D\mathbb{1}_E$ is a finite measure (if $|E| < +\infty$, this is equivalent to ask $\chi_E \in BV(\mathbb{R}^k)$).
If $E \subset \Omega$, it is a set of finite perimeter in Ω if $D\mathbb{1}_E \in \mathcal{M}_b(\Omega; \mathbb{R}^k)$.
$P(E;\Omega) := |D\mathbb{1}_E|(\Omega)$ is the perimeter of E in Ω.

Theorem (Coarea Formula - Fleming, Rishel): Let $\Omega \subset \mathbb{R}^k$ open. Let $u \in L^1_{loc}(\Omega)$. Then

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} P(\{x \in \Omega : u(x) > t\}; \Omega)\, dt.$$

If $u \in BV(\Omega)$, the set $\{u > t\}$ has finite perimeter for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$.



Def: Let E be a set of finite perimeter in Ω. The reduced boundary $\partial^* E$ is the set of points $x \in \operatorname{supp} |D\mathbb{1}_E| \cap \Omega$ s.t. $\exists \nu_E(x) = \lim_{r \to 0} \frac{D\mathbb{1}_E(B_r(x))}{|D\mathbb{1}_E(B_r(x))|}$ and $|\nu_E(x)| = 1$.

Theorem (De Giorgi): Let E be a set of finite perimeter. Then $\partial^* E$ is $(\mathcal{H}^{k-1}, k-1)$-countably rectifiable, $D\mathbb{1}_E = \nu_E \, \mathcal{H}^{k-1} \llcorner \partial^* E$ and $|D\mathbb{1}_E| = \mathcal{H}^{k-1} \llcorner \partial^* E$. Moreover $\mathrm{Tan}(\partial^* E, x) = \nu_E^\perp(x)$ for $x \in \partial^* E$.

The generalization of the Gauss-Green formula is true:

$$\int_E \mathrm{div}\,\varphi \, dx = -\int_{\partial^* E} \varphi \, \nu_E(x) \, d\mathcal{H}^{k-1}(x)$$

If $u \in BV(\Omega)$, the coarea formula reads:

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} \mathcal{H}^{k-1}(\partial^* \{u > t\} \cap \Omega) \, dt$$

Def: Let $u \in L^1_{loc}(\mathbb{R}^k)$ and $x \in \Omega$. We say that x is a *jump point* for u if there exist $u^+(x), u^-(x) \in \mathbb{R}$ and $\nu_u(x) \in \mathbb{S}^{k-1}$ such that $u^+(x) \neq u^-(x)$ and

$$\fint_{B^+_\nu(x,r)} |u(y) - u^+(x)| \, dy \longrightarrow 0 \quad \text{and} \quad \fint_{B^-_\nu(x,r)} |u(y) - u^-(x)| \, dy \longrightarrow 0 .$$

(the triplet $(u^+(x), u^-(x), \nu_u(x))$ is unique up to the change $(u^-(x), u^+(x), -\nu_u(x))$).

Theorem: Let $u \in BV(\Omega)$. Then:

- $\mathcal{H}^{k-1}(S_u \setminus J_u) = 0$
- S_u is $(\mathcal{H}^{k-1}, k-1)$-countably rectifiable
- $\mathrm{Tan}(J_u, x) = \nu_u(x)^\perp$ for $\mathcal{H}^{k-1}$-a.e. $x \in J_u$
- $Du \llcorner J_u = (u^+ - u^-) \nu_u \, \mathcal{H}^{k-1} \llcorner J_u$.

We can split:

$$Du = \nabla u \, \mathcal{L}^k + D^c u + (u^+ - u^-) \nu_u \, \mathcal{H}^{k-1} \llcorner J_u$$

By the coarea formula

$$|Du|(B) = 0 \quad \text{for every Borel set } B \text{ with } \mathcal{H}^{k-1}(B) = 0.$$

$$|D^c u|(B) = 0 \quad \text{for every } B \text{ with } \mathcal{H}^{k-1}(B) < +\infty \text{ but } B \cap S_u = \emptyset.$$

STRUCTURE OF CARTESIAN CURRENTS

We show here that every $T \in \text{cart}(\Omega \times \mathbb{R}^n)$ is of the form

$$T = [G_u] + S$$

where u is a BV function and S is a vertical current.

Let us start by observing that if $T = [G_u]$ for some smooth function u, then

$$T(\phi(x)\, y^j dx) = [G_u](\phi(x)\, y^j dx) = \int_\Omega \phi(x)\, u^j(x)\, dx \quad \text{for } \phi \in C^0_c(\Omega).$$

This means that testing a graph with horizontal forms of the type $\phi(x)\, y^j dx$ allows us to identify the components of the function underlying the graph. Luckily, if $T \in \text{cart}(\Omega \times \mathbb{R}^n)$ is a generic cartesian current, $\|T\|_1 < +\infty$, and thus we can always test it with forms $\phi(x)\, y^j dx$ (actually, this is the reason why $\|T\|_1 < +\infty$ is included in the definition).

Given $T \in \text{cart}(\Omega \times \mathbb{R}^n)$ we define the measures $\mu^j_T \in \mathcal{M}_b(\Omega)$ by

$$\int_\Omega \phi(x)\, d\mu^j_T(x) := T(\phi(x)\, y^j dx) \quad \text{for } \phi \in C^0_c(\Omega).$$

By definition, T is a rectifiable current with i.m., thus

$$T = [M, \tau, m], \quad M \text{ } k\text{-rectifiable}, \quad |\tau| = 1, \quad m \in L^1_{\mathcal{H}^k}(M; \mathbb{Z}).$$

Then

$$\int_\Omega \phi(x)\, d\mu^j_T(x) = T(\phi(x)\, y^j dx) = \int_M \langle \phi(x)\, y^j dx, \tau(x,y) \rangle\, m(x,y)\, d\mathcal{H}^k(x,y) =$$

$$= \int_M \phi(x)\, y^j\, \tau^h(x,y)\, m(x,y)\, d\mathcal{H}^k(x,y)$$

where $\tau^h(x,y)$ is the horizontal component of $\tau(x,y)$, i.e., the component corresponding to the vector of the basis $e_1 \wedge e_2 \wedge \dots \wedge e_k$.

The idea is the following: the parts of M where M can be represented as the graph of a function are the parts which are not completely vertical. More precisely, M admits $\mathcal{H}^k$-a.e. an approximate tangent space $\text{Tan}(M, (x,y))$. Removing from M an $\mathcal{H}^k$-negligible set, the definition of T does not change, but we can assume that the approximate tangent space exists at every point.

We can also assume that $m \geq 0$, by changing the orientation τ, and we can remove, without changing the current T, the set where $m=0$. Thus, we can assume that $m \geq 1$.

We remark that $\mathrm{Tan}(M,(x,y))$ contains no vertical vector if and only if $\tau^h(x,y) > 0$.

First of all, we have $\tau^h(x,y) \geq 0$ everywhere. This follows from the fact that the cartesian current T does not "switch" the orientation of Ω.

More precisely, by definition $T^h \geq 0$. This means that for $\phi(x,y) \geq 0$:

$$0 \leq T^h(\phi(x,y)\,dx) = T(\phi(x,y)\,dx) = \int_M \langle \phi(x,y)\,dx, \tau(x,y)\rangle\, m(x,y)\, d\mathcal{H}^k(x,y) =$$

$$= \int_M \phi(x,y)\, \tau^h(x,y)\, m(x,y)\, d\mathcal{H}^k(x,y).$$

This is true for every ϕ, therefore $\tau^h(x,y) \geq 0$ $\mathcal{H}^k$-a.e. (we used $m \geq 1$).

Now assume that $\mathrm{Tan}(M,(x,y))$ contains a vertical vector z_1 and complete it to a basis $(z_1,\dots,z_k)$. Split each $z_i = v_i + w_i$ with $v_i \in \mathbb{R}^k$ and $w_i \in \mathbb{R}^n$. Since z_1 is vertical, $v_1 = 0$. Both $\tau(x,y)$ and $(z_1,\dots,z_k)$ span $\mathrm{Tan}(M,(x,y))$, we deduce

$$\tau(x,y) = c\, z_1 \wedge \dots \wedge z_k = c(v_1+w_1)\wedge\dots\wedge(v_k+w_k) \quad \text{for some } c \in \mathbb{R}\setminus\{0\}.$$

$$\tau^h(x,y) = \langle dx, \tau(x,y)\rangle = \langle dx, c(v_1+w_1)\wedge\dots\wedge(v_k+w_k)\rangle = c\langle dx, v_1\wedge\dots\wedge v_k\rangle = 0.$$

Conversely, assume that $\mathrm{Tan}(M,(x,y))$ has no vertical vectors and let again $(v_1+w_1,\dots,v_k+w_k)$ be a basis. Then $\tau_h = c\langle dx, v_1\wedge\dots\wedge v_k\rangle$, but $v_1,\dots,v_k$ are linearly independent. Indeed:

$$\sum_i \lambda_i v_i = 0 \Rightarrow \sum_i \lambda_i z_i = \sum_i \lambda_i w_i \text{ vertical!}$$

Thus $\tau^h(x,y) > 0$.

We define $M_+ := \{(x,y) \in M : \tau_h(x,y) > 0\}$.

<u>Theorem</u>: Let $T = [M,\tau,m] \in \mathrm{cart}(\Omega \times \mathbb{R}^n)$. Then:

1) $\mu_T \ll \mathcal{L}^k$. Let $\mu_T = u_T \mathcal{L}^k$ with $u_T \in L^1(\Omega;\mathbb{R}^k)$

2) $u_T \in BV(\Omega;\mathbb{R}^n)$

3) $T \llcorner M_+ = [G_{u_T}, \tau_{u_T}, 1]$.

Moreover, the current $S := T - [G_{u_T}]$ is vertical, i.e., $\pi_\# S = 0$.

projection on horizontal component

this follows from the fact that $\pi_\#[G_{u_T}] = [\Omega]$.

Lemma: Let V be a vector subspace of $\mathbb{R}^k \times \mathbb{R}^n$ oriented by a k-vector τ. Then $\tau^h = J\pi_{|V}$, where $\pi: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k$ is the projection on the first component.

Proof: Case 1: $\tau_h = 0$. Then V contains vertical vectors $\Rightarrow \pi_{|V}$ is not injective $\Rightarrow$
$\Rightarrow J\pi_{|V} = 0$.

Case 2: $\tau^h > 0$. Then a basis of V can be written as $z_i = e_i + w_i$, $w_i \in \mathbb{R}^n$, where $(e_1, \dots, e_k)$ the standard basis of $\mathbb{R}^k$ (because V has no vertical vectors). Then $\pi_{|V}(z_i) = e_i$. This means that if $L: \mathbb{R}^k \to \mathbb{R}^n$ denotes the linear map defined by $L(e_i) = w_i$, then $(id \times L)(e_i) = e_i + w_i = z_i$, and $\pi_{|V}$ is the inverse map to $id \times L$. The linear space $(id \times L)(\mathbb{R}^k)$ (which is equal to V) is oriented by $z_1 \wedge \dots \wedge z_k$ and $|z_1 \wedge \dots \wedge z_k| = |\det(id \times L)|$.
Also τ orients $V \Rightarrow \tau = \dfrac{z_1 \wedge \dots \wedge z_k}{|z_1 \wedge \dots \wedge z_k|} = \dfrac{z_1 \wedge \dots \wedge z_k}{|\det(id \times L)|}$. Applying dx to τ we get $\tau^h = \dfrac{1}{|\det(id \times L)|} = |\det(d\pi_{|V})| = J\pi_{|V}$. $\square$

We are now in a position to prove the structure theorem.

Proof: We start from the computation we did before stating the theorem:

$$\int_\Omega \phi(x)\, d\mu^j_T(x) = \int_M \phi(x) y^j \tau^h(x,y) m(x,y)\, d\mathcal{H}^k(x,y) = \int_{M^+} \underbrace{\phi(x) y^j \overbrace{J\pi_{|Tan(M,(x,y))}}^{m(x,y)}}_{\psi(x,y)} d\mathcal{H}^k(x,y)$$

$$= \cancel{\int_{\pi(M^+)} \phi(x)} = \int_{\pi(M^+)} \sum_{(x,y) \in \pi^{-1}(x) \cap M_+} \psi(x,y)\, d\mathcal{L}^k(x) =$$

$$= \int_{\pi(M_+)} \phi(x) \underbrace{\left(\sum_{(x,y) \in \pi^{-1}(x) \cap M_+} y^j m(x,y) \right)}_{\overset{\shortparallel}{u^j_T(x)} \ \leftarrow \text{ measurable}} dx$$

To prove that $\mu^j_T \ll \mathcal{L}^k$ we have to prove that $\mathcal{L}^k(\Omega \setminus \pi(M_+)) = 0$.
Let us compute:

$$\int_\Omega \phi(x)\, dx = [\Omega](\phi(x)\,dx) = \pi_\# T(\phi(x)\,dx) = T(\pi^\#(\phi(x)\,dx)) =$$

$$= \int_M \langle \phi(x)\,dx, \tau(x,y) \rangle m(x,y)\, d\mathcal{H}^k(x,y) =$$

$$= \int_{M_+} \phi(x) \tau^h(x,y) m(x,y)\, d\mathcal{H}^k(x,y) = \int_{\pi(M_+)} \phi(x) \overbrace{\sum_{y \in \pi^{-1}(y) \cap M_+} m(x,y)}^{N(x)} dx =$$

$$= \int_{\pi(M_+)} \phi(x) N(x)\, dx .$$

This is true for every $\phi \in C_c^\infty(\Omega)$. It implies that $\mathcal{L}^k(\Omega \setminus \pi(M_+)) = 0$ and $N(x) = 1$ a.e. in Ω.

In conclusion

$$\int_\Omega \phi(x)\, d\mu_T^j(x) = \int_\Omega \phi(x)\, u_T^j(x)\, dx \qquad \forall \phi \in C_c^0(\Omega)$$

i.e., $\mu_T^j = u_T^j \mathcal{L}^k$.

Note that for $\mathcal{L}^k$-a.e. $x \in \Omega$: $N(x) = 1$. If $N(x) = 1$, we have $\sum_{(x,y) \in \pi^{-1}(x) \cap M_+} m(x,y) = 1$. Since $m(x,y) \geq 1$, we get that there exists only one $y =: \bar{u}(x)$ ~~xxxxx~~, $(x, \bar{u}(x)) \in \pi^{-1}(x) \cap M_+ \Rightarrow (x, \bar{u}(x)) \in M_+$.

Then

$$u_T^j(x) = \sum_{(x,y) \in \pi^{-1}(x) \cap M_+} y^j\, m(x,y) = \bar{u}^j(x)$$ for such points, i.e., for $\mathcal{L}^k$-a.e. $x \in \Omega$.

We have proved that $\mu_T^j \ll \mathcal{L}^k$, $\mu_T^j = u_T^j \mathcal{L}^k$, and for $\mathcal{L}^k$-a.e. $x \in \Omega$ $(x, u_T(x)) \in M_+$.

2) To check that $u_T \in BV(\Omega; \mathbb{R}^n)$ we consider ~~xxx~~ a function $\phi \in C_c^\infty(\Omega)$ and we consider the $(k-1)$-form:

$$\omega(x,y) = y^j \phi(x)\, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^k = y^j \phi(x)\, \widehat{dx^i}$$

We have:

$$d\omega(x,y) = (-1)^{i-1} y^j \partial_i \phi(x)\, dx + (-1)^{k-1} \phi(x)\, \widehat{dx^i} \wedge dy^j .$$

Since $\partial T = 0$:

$$0 = T(d\omega) = (-1)^{i-1} T(y^j \partial_i \phi(x) dx) + (-1)^{k-1} T(\phi(x) \widehat{dx^i} \wedge dy^j) =$$

$$= (-1)^{i-1} \int_\Omega u_T^j(x) \partial_i \phi(x)\, dx + (-1)^{k-1} T(\phi(x) \widehat{dx^i} \wedge dy^j)$$

$$\Rightarrow -\int_\Omega u_T^j(x) \partial_i \phi(x)\, dx = (-1)^{k-i} \underbrace{T(\phi(x) \widehat{dx^i} \wedge dy^j)}_{\text{This is a distribution of order } 0 \text{, thus a measure}}$$

$$\Rightarrow D_i u_T^j = (-1)^{k-i} T(\cdot\ \widehat{dx^i} \wedge dy^j) \in \mathcal{M}_b(\Omega).$$

3) Let us prove that $\mathcal{H}^k(M_+ \triangle \mathcal{G}_{u_T}) = 0$.

Let $\Omega_1 \subset \Omega$ be the set of points x such that $u_T(x)$ is the unique value s.t. $(x, u_T(x)) \in M_+$. We recall by 1) that $\mathcal{L}^k(\Omega \setminus \Omega_1) = 0$.

The graph is defined in such a way that

$$\mathcal{H}^k(\mathcal{G}_{u_T} \setminus (id \times u_T)(R_{u_T} \cap \Omega_1)) = 0 \quad , \text{ ~~mucho~~ since } \mathcal{L}^k(\Omega \setminus \Omega_1) = 0 .$$

But if $x \in R_{u_T} \cap \Omega_1 \subset \Omega_1$, then $(x, u_T(x)) \in M_+$, thus $(id \times u_T)(R_{u_T} \cap \Omega_1) \subset M_+$. Up to a $\mathcal{H}^k$-null set $\mathcal{G}_{u_T}$ is contained in M_+.

We only have to check $\mathcal{H}^k(M_+ \setminus \mathcal{G}_{u_T}) = 0$.

First of all, notice that

$$+\infty > \mathbb{M}(T) \geq \mathcal{H}^k(M_+) = \int_{M_+} \frac{1}{\tau^h(x,y)} \overbrace{\tau^h(x,y)}^{J_k \pi | Tan(M_+,(x,y))} d\mathcal{H}^k(x,y) = \quad \text{Area}$$

$\tau^h > 0$ on M_+

$$= \int_{\underbrace{\pi(M_+)}_{\mathcal{L}^k\text{-a.e.} = \Omega}} \underbrace{\sum_{(x,y) \in \pi^{-1}(x) \cap M_+} \frac{1}{\tau^h(x,y)}}_{= K(x)} dx = \int_\Omega K(x) dx \Rightarrow K \in L^1$$

Then

$$\mathcal{H}^k(M_+ \setminus \mathcal{G}_{u_T}) \overset{=}{\leq} \mathcal{H}^k(M_+ \setminus (id \times u_T)(R_{u_T} \cap \Omega_1)) .$$

If $(x,y) \in M_+ \setminus (id \times u_T)(R_{u_T} \cap \Omega_1)$, then $(x,y) \in M_+ \cap (\Omega \setminus (R_{u_T} \cap \Omega_1)) \times \mathbb{R}^n$.

Indeed, if by contradiction ~~$(x,y) \in M_+$~~ $(x,y) \in (R_{u_T} \cap \Omega_1) \times \mathbb{R}^n$, then $x \in \Omega_1 \Rightarrow$ there is only one vector $u_T(x)$ s.t. $(x, u_T(x)) \in M_+$.

Then $(x,y) = (x, u_T(x)) \in (id \times u_T)(R_{u_T} \cap \Omega_1)$!

We can continue the estimate above

$$\mathcal{H}^k(M_+ \setminus \mathcal{G}_{u_T}) \leq \mathcal{H}^k(M_+ \cap \underbrace{(\Omega \setminus (R_{u_T} \cap \Omega_1))}_{N} \times \mathbb{R}^n) =$$

$$= \mathcal{H}^k(M_+ \cap (N \times \mathbb{R}^n)) = \int_{M_+ \cap (N \times \mathbb{R}^n)} \frac{1}{\tau^h} \tau^h d\mathcal{H}^k \leq \int_N K(x) dx = 0 , \text{ since } \mathcal{L}^k(N) = 0 .$$

$\square$

<u>Remark</u>: What about the vertical part S? For forms $\omega = \phi(x) d\hat{x}^i \wedge dy^j$

$$S(\phi(x) d\hat{x}^i \wedge dy^j) = T(\phi(x) d\hat{x}^i \wedge dy^j) - [\mathcal{G}_{u_T}](\phi(x) d\hat{x}^i \wedge dy^j) =$$

$$= (-1)^{k-i} \int_\Omega \phi(x) dD_i u_T^j(x) - (-1)^{k-i} \int_\Omega \phi(x) \underbrace{\partial_{x_i} u_T^j(x)}_{\text{appr. grad}} dx =$$

$$= (-1)^{k-i} \int_\Omega \phi(x) dD_i^S u_T^j(x)$$

Remark: In the proof we found out that

$$D_i u_T^j = (-1)^{k-i}\, T(\cdot\, d\hat{x}^i \wedge dy^j) \quad \text{in } \mathcal{M}_b(\Omega).$$

In general, this relation does not allow us to recover completely T just from u_T. The example that we saw is

$$T = \underbrace{[\mathcal{G}_{\frac{x}{|x|}}]}_{\substack{\text{This is the}\\ \text{BV (actually}\\ W^{1,1}\text{) function}\\ \text{associated to } T}} - \underbrace{[L]\times[S^1]}_{\substack{\text{This can be}\\ \text{changed by any}\\ \text{other radius. This}\\ \text{changes the current but}\\ \text{not the function } \frac{x}{|x|}\,!}} \in \operatorname{cart}(B\times\mathbb{R}^2)$$

it is only tested with $\phi(x)\,d\hat{x}^i\wedge dy^j$

we saw that it is the limit of Lipschitz graphs.

CARTESIAN CURRENTS IN CODIMENSION 1

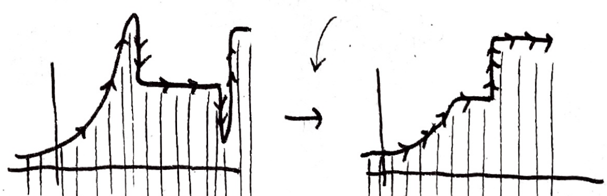
We study now the special case $T\in \operatorname{cart}(\Omega\times\mathbb{R})$. We shall see that they are completely determined by the underlying BV function.

Theorem: Let $T\in\operatorname{cart}(\Omega\times\mathbb{R})$. Let $u_T\in BV(\Omega)$ be the function associated to T and let $\mathcal{S}_{u_T}$ be the subgraph

$$\mathcal{S}_{u_T} := \{(x,y)\in\Omega\times\mathbb{R} : y < u_T(x)\}.$$

Then $T = (-1)^k\,\partial[\mathcal{S}_{u_T}]$.

concentration effects "compensate"!



Lemma: Let $T\in\mathcal{D}_k(\Omega\times\mathbb{R})$ with $\partial T=0$. Then $T=0$ if and only if $T(\phi(x,y)\,dx)=0$ $\forall\,\phi\in C_c^\infty(\Omega\times\mathbb{R})$.

Proof: We have to check that $T(\omega)=0$ $\forall\,\omega\in\mathcal{D}^2(\Omega\times\mathbb{R})$. A 2-form in $\Omega\times\mathbb{R}$ has components of the type

$$\phi(x,y)\,dx \quad , \quad \phi(x,y)\,d\hat{x}^i\wedge dy$$

For these, we already know that the current is $=0$.

To show that $T(\phi(x,y)\,d\hat{x}^i\wedge dy)=0$ we write $\phi(x,y)\,d\hat{x}^i\wedge dy$ as a differential (exploiting the property $\partial T=0$) plus a form in dx.

More precisely, let us find $\eta \in \mathcal{D}^{k-1}(\Omega\times\mathbb{R})$ and $\psi \in C^\infty(\Omega\times\mathbb{R})$ s.t.

$$d\eta = \psi(x,y)\,dx + \underbrace{\phi(x,y)}_{\partial_y\alpha(x,y)\cdot\cancel{(-1)^{k-1}}}\, d\hat{x}^i \wedge dy$$

$$(-1)^{k-1}\alpha(x,y) = \int_{-\infty}^{y} \phi(x,z)\,dz \quad \leftarrow \text{this has bounded coefficients.}$$

$$\eta(x,y) := \alpha(x,y)\,d\hat{x}_i \;\Rightarrow\; d\eta = (-1)^{i-1}\partial_{x_i}\alpha(x,y)\,dx + \phi(x,y)\,d\hat{x}^i\wedge dy$$

$\uparrow$ $(k-1)$-form with bounded coefficients in y. Needs to be fixed with cut-off function $\chi_R \equiv 1$ on $[-R,R]$ and $\chi_R \equiv 0$ outside $[-R-1,R+1]$.

$$\partial T(\eta) = \lim_{R\to\infty}\partial T(\chi_R\eta) = 0$$

$$\partial T(\chi_R\eta) = T(d\chi_R\,\eta + \chi_R\,d\eta) = T(\underbrace{d\chi_R\,\eta}_{\substack{\text{supported}\\ \text{in } \Omega\times[R,R+1]\\ \cup[-R-1,-R]}}) + \overbrace{T(\chi_R d\eta)}^{T(d\eta)}$$

$\to$ Use finiteness of mass.

$$0 = \partial T(\eta) = T(d\eta) = \underbrace{T(\psi(x,y)\,dx)}_{=0} + T(\phi(x,y)\,d\hat{x}^i\wedge dy).$$

$\hookrightarrow$ Use later:
$$T(\phi(x,y)\,d\hat{x}^i\wedge dy) = -T(\psi\,dx) = -T\left((-1)^{k+i}\int_{-\infty}^{y}\partial_{x_i}\phi\,dx\right)$$
$\square$

<u>Proof</u> (of theorem):

On the one hand, we have

$$T(\phi(x,y)\,dx) = ([\mathcal{G}_{u_T}] + S)(\phi(x,y)\,dx) = [\mathcal{G}_{u_T}](\phi(x,y)\,dx) = \int_\Omega \phi(x,u_T(x))\,dx$$

On the other hand,

$$(-1)^k\partial[\mathcal{S}_{u_T}](\phi(x,y)\,dx) = (-1)^k[\mathcal{S}_{u_T}](d(\phi(x,y)\,dx)) =$$
$$= (-1)^k[\mathcal{S}_{u_T}](\partial_y\phi(x,y)\,dy\wedge dx) = [\mathcal{S}_{u_T}](\partial_y\phi(x,y)\,dx\wedge dy) =$$
$$= \int_{\{y<u_T(x)\}} \partial_y\phi(x,y)\,dx\,dy = \int_\Omega\left(\int_{-\infty}^{u_T(x)}\partial_y\phi(x,y)\,dy\right)dx =$$
$$= \int_\Omega \phi(x,u_T(x))\,dx.$$

This concludes the proof thanks to the previous lemma.

Proposition: Let $u \in BV(\Omega)$. Then $\mathcal{G}_u$ is a set of finite perimeter in $\Omega\times\mathbb{R}$.

Proof: Assume u is smooth. Fix $\phi \in C_c^\infty(\Omega\times\mathbb{R})$. Then

$$\int_{\Omega\times\mathbb{R}} \phi(x,y)\, dD_{x_i} 1\!\!1_{\mathcal{G}_u}(x,y) = -\int_{\Omega\times\mathbb{R}} \partial_{x_i}\phi(x,y)\, 1\!\!1_{\mathcal{G}_u}(x,y)\, d\mathcal{L}^n(x,y) = -\int_\Omega \int_{-\infty}^{u(x)} \partial_{x_i}\phi(x,y)\, dy\, dx =$$

$$= -\int_\Omega \partial_{x_i}\left(\int_{-\infty}^{u(x)} \phi(x,y)\, dy\right) - \phi(x,u(x))\, D_{x_i}u(x)\, dx = \int_\Omega \phi(x,u(x))\, D_{x_i}u(x)\, dx$$

which implies:

$$|D_{x_i} 1\!\!1_{\mathcal{G}_u}|(\Omega\times\mathbb{R}) \le |D_{x_i}u|(\Omega).$$

$$\int_\Omega \phi(x,y)\, dD_y 1\!\!1_{\mathcal{G}_u}(x,y) = -\int_{\Omega\times\mathbb{R}} \partial_y\phi(x,y)\, 1\!\!1_{\mathcal{G}_u}(x,y)\, d\mathcal{L}^k(x,y) = -\int_\Omega \int_{-\infty}^{u(x)} \partial_y\phi(x,y)\, dy\, dx =$$

$$= -\int_\Omega \phi(x,u(x))\, dx$$

which implies:

$$|D_y 1\!\!1_{\mathcal{G}_u}|(\Omega\times\mathbb{R}) \le \mathcal{L}^k(\Omega).$$

If u is BV, let u_j be a sequence of smooth functions s.t. $u_j \to u$ in L^1 and $|Du_j|(\Omega) \to |Du|(\Omega)$. Then the claim follows from the l.s.c. of total variation. $\square$

Remark: If $E \subset \mathbb{R}^n$ is a set of finite perimeter, then $\partial[E]$ is a $\mathcal{D}_{n-1}(\mathbb{R}^n)$ current of finite mass. Indeed, let $\omega \in \mathcal{D}^{n-1}(\mathbb{R}^n)$. ~~If~~ It has components of the form $\omega = \phi(z)\, d\hat{z}^i$. Then

$$\partial[E](\phi(z)\, d\hat{z}^i) = [E]((-1)^{i-1}\partial_{z_i}\phi(z)\, dz) = (-1)^{i-1}\int_{\mathbb{R}^n} \partial_{z_i}\phi(z)\, 1\!\!1_E(z)\, dz$$

$$= (-1)^i \int_{\mathbb{R}^n} \phi(z)\, dD_{z_i} 1\!\!1_E(z) = (-1)^i \int_{\mathbb{R}^n} \phi(z)\, \nu_E^i\, d\mathcal{H}^{n-1} \llcorner \partial^* E =$$

$$= \int_{\partial^* E} \langle \phi(z)\, d\hat{z}^i, \tau_E(z)\rangle\, d\mathcal{H}^n(z)$$

where $\tau_E = \sum_i (-1)^i \nu_E^i \hat{e}_i$ ← this is the $(n-1)$-vector s.t. $(-\nu_E) \wedge \tau_E = e_1 \wedge \dots \wedge e_n$.

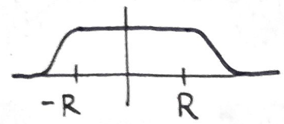
Thus it is also a rectifiable current with integer multiplicity.

Applying this to $(-1)^k\partial[\mathcal{G}_u]$, we get that $(-1)^k\partial[\mathcal{G}_u]$ is a boundaryless rectifiable current with i.m. and finite mass. It is not difficult to check that the horizontal component is ≥ 0, $\|\partial[\mathcal{G}_u]\|_1 < +\infty$, and $\pi_\#(-1)^k\partial[\mathcal{G}_u] = [\Omega]$. Thus $(-1)^k\partial[\mathcal{G}_u] \in \mathrm{cart}(\Omega\times\mathbb{R})$.

We can easily guess what is the BV function associated to the Cartesian current $(-1)^k \partial[\mathcal{G}_u] \in \mathrm{cart}(\Omega\times\mathbb{R})$... To compute it, remember the structure theorem!

$$(-1)^k \partial[\mathcal{G}_u](\phi(x)\, y\, dx) = \lim_{R\to+\infty} (-1)^k \partial[\mathcal{G}_u](\phi(x)\chi_R(y)\, y\, dx) =$$

the form has only bounded coefficients in $\Omega\times\mathbb{R}$



$$= \lim_{R\to+\infty} \mathcal{G}[\mathcal{G}_u](\phi(x)\,\partial_y(\chi_R(y)y)\, dx\wedge dy) =$$

$$= \lim_{R\to+\infty} \int_\Omega \int_{-\infty}^{u(x)} \phi(x)\,\partial_y(\chi_R(y)y)\, dy\, dx = \lim_{R\to+\infty} \int_\Omega \phi(x)\chi_R(u(x))u(x)\,dx =$$

$$= \int_\Omega \phi(x)\,u(x)\,dx$$

Thus $u\in BV(\Omega)$ is the BV ~~un~~ function underlying $(-1)^k\partial[\mathcal{G}_u] \in \mathrm{cart}(\Omega\times\mathbb{R})$.

From the structure theorem:

$$(-1)^k \partial[\mathcal{G}_u] \llcorner \underbrace{(\partial^*\mathcal{G}_u)_+}_{} = [\mathcal{G}_u]$$

the current is concentrated on the k-rectifiable set $\partial^*\mathcal{G}_u$; here we take the part with no vertical vectors

graph of u obtained by taking the approximate differentiability points.

$$(-1)^k\partial[\mathcal{G}_u] = [\mathcal{G}_u] + S, \text{ where } S \text{ is a vertical current, } \pi_\# S = 0.$$

It is possible to compute the vertical part S.

A formal computation that shows how is its structure:

$$(-1)^k \partial[\mathcal{G}_u](\phi(x,y)\,d\hat{x}^i\wedge dy) = (-1)^k(-1)^{k+i+1}\partial[\mathcal{G}_u]\left(\left(\int_{-\infty}^y \partial_{x_i}\phi(x,z)\,dz\right)dx\right)$$

$$= (-1)^{k-i+1}[\mathcal{G}_u](\partial_{x_i}\phi(x,y)\, dx\wedge dy) = (-1)^{k-i}\int_{\Omega\times\mathbb{R}} \phi(x,y)\, dD_{x_i}1\!\!1_{\mathcal{G}_u}(x,y) =$$

$$= (-1)^{k-i}\int_\Omega \phi(x,u(x))\underbrace{\partial^a_{x_i}u(x)}_{\text{appr. grad}}\,dx + (-1)^{k-i}\int_\Omega \phi(x,\tilde{u}(x))\,D^c_{x_i}u(x)\,dx +$$

Chain rule $G(x,t) := \int_{-\infty}^t \phi(x,z)\,dz$, $D_{x_i}(G(x,u(x))) = \dots$

$$+ (-1)^{k-i}\int_{J_u}\left(\int_{u^-(x)}^{u^+(x)}\phi(x,z)\,dz\right)\nu_u^i(x)\,d\mathcal{H}^{k-1}(x) = [\mathcal{G}_u] + S$$

$$S = (-1)^{k-i}\left[\int_\Omega \phi(x,\tilde{u}(x))\,D^c_{x_i}u(x)\,dx + \int_{J_u}\left(\int_{u^-(x)}^{u^+(x)}\phi(x,z)\,dz\right)\nu_u^i(x)\,d\mathcal{H}^{k-1}(x)\right]$$

If $T = (-1)^k \partial[\mathcal{Y}_u] \in cart(\Omega \times \mathbb{R})$, then

$$\underbrace{M(T)}_{\substack{[G_u] \text{ and } S \\ \text{are mutually} \\ \text{singular}}} = M([G_u]) + \underbrace{M(S)}_{\text{vertical part}} \geq \underbrace{\int_\Omega \sqrt{1+|\nabla u|^2}}_{\substack{\text{def of area} \\ \text{of graph}}} + |D^s u|(\Omega).$$

← (actually this is an equality, see formula for S)

Let now $u_j \in C^1(\Omega; \mathbb{R})$ be a sequence such that $Area(G_{u_j}) = \mathcal{H}^k(G_{u_j}) = M([G_{u_j}]) \leq C$ and such that $u_j \to u$ in $L^1(\Omega)$.
The currents $[G_{u_j}]$ converge, up to a subsequence, to a current T, which is necessarily cartesian. To T we associate a function $u_T \in BV(\Omega)$ with the structure theorem. Then

$$\int_\Omega \phi(x, u_j(x))\,dx = [G_{u_j}](\phi(x,y)\,dx) \to T(\phi(x,y)\,dx) = [G_{u_T}](\phi(x,y)\,dx) =$$

$$= \int_\Omega \phi(x, u_T(x))\,dx \Rightarrow$$

on the other hand $\downarrow$

$$\int_\Omega \phi(x, u(x))\,d \qquad \text{Thus } u_T = u \text{ a.e.}$$

Then

$$\liminf_j M([G_{u_j}]) \geq M(T) = \int_\Omega \sqrt{1+|\nabla u|^2} + |D^s u|(\Omega).$$

↑ This inequality is actually optimal.

<u>Proposition</u>: Let $T \in cart(\Omega \times \mathbb{R})$. Then $\exists\, u_j \in C^1(\Omega; \mathbb{R})$ such that $[G_{u_j}] \to T$ in $\mathcal{D}_k(\Omega \times \mathbb{R})$ and $M([G_{u_j}]) \to M(T)$.

(Proof based on the fact that T is identified with a BV function u that can be approximated by $u_j \to u$ in such a way that $\int_\Omega \sqrt{1+|\nabla u_j|^2} \to \int_\Omega \sqrt{1+|\nabla u|^2} + |D^s u|(\Omega)$.)

This means that:

$$\int_\Omega \sqrt{1+|\nabla u|^2}\,dx + |D^c u|(\Omega) + \int_{J_u} |u^+ - u^-|\,d\mathcal{H}^{n-1}, \quad u \in BV(\Omega)$$

is the natural generalization of the area functional for scalar-valued functions.

AREA OF VECTOR-VALUED MAPS

Let $u \in C^1(\Omega;\mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$, and let us consider the area of the graph:

$$\mathbb{M}([G_u]) = \int_\Omega \sqrt{1+|\nabla u|^2 + (\det \nabla u)^2}\,dx =: \mathcal{A}(u).$$

$$\mathcal{A}(u) = \begin{cases} \int_\Omega \sqrt{1+|\nabla u|^2 + (\det \nabla u)^2} & \text{for } C^1 \\ +\infty & \text{otherwise} \end{cases}$$

We can define $\mathcal{A}$ to be this quantity for $u \in C^1(\Omega;\mathbb{R}^2)$ with the minors in L^1 and extended to $+\infty$ outside in $L^1(\Omega;\mathbb{R}^2)$.

A natural way to extend the area of the graph of a function in L^1 which is not in C^1 is to consider the relaxation:

$$\bar{\mathcal{A}}(u) := \inf\left\{ \liminf_j \mathcal{A}(u_j) : u_j \in C^1(\Omega;\mathbb{R}^2),\ u_j \to u \text{ in } L^1(\Omega;\mathbb{R}^2)\right\}$$

The first question that one wants to understand is: what is $\mathrm{Dom}(\bar{\mathcal{A}})$, the space of functions where the relaxed area is finite?

(for scalar-valued functions we understood that it's $BV(\Omega;\mathbb{R})$).

If ~~u_j~~ $u \in L^1(\Omega;\mathbb{R}^2)$ is such that $\bar{\mathcal{A}}(u) < +\infty$, then there is a sequence $u_j \in C^1(\Omega;\mathbb{R}^2)$ s.t. $\lim_j \mathcal{A}(u_j) = \bar{\mathcal{A}}(u)$ and $u_j \to u$ in $L^1(\Omega;\mathbb{R}^2)$.

But $\mathcal{A}(u_j) \geq \int_\Omega |\nabla u_j|\,dx \Rightarrow u_j$ is equibounded in $BV(\Omega;\mathbb{R}^2) \Rightarrow$

$\Rightarrow u \in BV(\Omega;\mathbb{R}^2)$. This means that $\mathrm{Dom}(\bar{\mathcal{A}}) \subset BV(\Omega;\mathbb{R}^2)$.

This inclusion is strict. To show this fact, we start with a proposition.

Proposition: Let $u \in BV(\Omega;\mathbb{R}^2)$. Then:

$$\bar{\mathcal{A}}(u) \geq \int_\Omega \sqrt{1+|\nabla u|^2 + |\det \nabla u|^2}\,dx + |D^s u|(\Omega).$$

if $\bar{\mathcal{A}}(u) = +\infty$, this tells nothing. If $\bar{\mathcal{A}}(u) < +\infty$, this functional is finite.

In particular, $\bar{\mathcal{A}}(u) = \mathcal{A}(u)$ if $u \in C^1(\Omega;\mathbb{R}^2)$. it's a "good" extension

Proof: ~~[If $\bar{\mathcal{A}}(u) = +\infty$, we have nothing to prove. Let us assume $\bar{\mathcal{A}}(u) < +\infty$.]~~

Let $u_j \in C^1(\Omega;\mathbb{R}^2)$ be a sequence such that $u_j \to u$ strongly in $L^1(\Omega;\mathbb{R}^2)$. ~~Then~~ Up to extracting a subsequence, we can assume that $\liminf_j \mathcal{A}(u_j) = \lim_j \mathcal{A}(u_j)$. We want to prove that

$$\int_\Omega \sqrt{1+|\nabla u|^2 + |\det \nabla u|^2}\,dx + |D^s u|(\Omega) \leq \lim_j \mathcal{A}(u_j).$$

If $\lim_j \mathcal{A}(u_j) = +\infty$, nothing to prove. We can assume that $\mathcal{A}(u_j)$ is a bounded sequence. $\mathbb{M}([G_{u_j}]) = \mathcal{A}(u_j) \leq C$. A sequence of C^1 functions with equibounded masses converges, up to a subsequence, to a Cartesian current T, $[G_{u_j}] \rightharpoonup T$ in $\mathcal{D}_2(\Omega \times \mathbb{R}^2)$.

We know, by the structure theorem, that there exists a $u_T \in BV(\Omega;\mathbb{R}^2)$ such that $T = [\mathcal{G}_{u_T}] + S$, with S vertical.
Then

$$T(\phi(x,y)\,dx) = [\mathcal{G}_{u_T}](\phi(x,y)dx) = \int_\Omega \phi(x, u_T(x))\,dx$$
$$\parallel$$
$$\lim_j [\mathcal{G}_{u_j}](\phi(x,y)\,dx) = \lim_j \int_\Omega \phi(x, u_j(x))\,dx = \int_\Omega \phi(x, u(x))\,dx.$$

Since $\int_\Omega \phi(x, u_T(x))\,dx = \int_\Omega \phi(x, u(x))\,dx \quad \forall \phi$, we conclude that $u_T = u$ a.e. in Ω. Thus $T = [\mathcal{G}_u] + S$.
By the lower semicontinuity of the mass:

$$\lim_j \mathbb{M}([\mathcal{G}_{u_j}]) \geq \mathbb{M}(T) = \mathbb{M}([\mathcal{G}_u]) + \mathbb{M}(S) \geq$$
$$\geq \int_\Omega \sqrt{1 + |\nabla u|^2 + |\det \nabla u|^2}\,dx + |D^s u|(\Omega).$$

If additionally $u \in C^1(\Omega;\mathbb{R}^2)$, we have, by definition $\bar{\mathcal{A}}(u) \leq \mathcal{A}(u)$.
Instead, by the previous inequality:

$$\mathcal{A}(u) = \int_\Omega \sqrt{1+|\nabla u|^2 + |\det \nabla u|^2}\,dx \leq \bar{\mathcal{A}}(u).$$

$\square$

Example: $\mathrm{Dom}(\bar{\mathcal{A}}) \subsetneq BV(\Omega;\mathbb{R}^2)$. Consider the open set $\Omega = B_1((1,0))$

and the function $u(x) = \frac{x}{|x|^{3/2}}$. Then $\bar{\mathcal{A}}(u) = \mathcal{A}(u)$ because $u \in C^1(\Omega;\mathbb{R}^2)$. But $\mathcal{A}(u) = +\infty$: indeed $u \in W^{1,1}(\Omega;\mathbb{R}^2)$, but $|\det \nabla u| \notin L^1$.

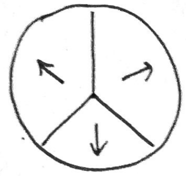
The lower bound proven in the previous proposition is, unfortunately, too low. To provide an example, let us consider the function $u(x) = \frac{x}{|x|}$ defined in the unit ball B. Let us argue by contradiction, assuming

$$\bar{\mathcal{A}}(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\,dx.$$

This implies that there exists a sequence $u_j \in C^1(\Omega;\mathbb{R}^2)$ s.t. $u_j \to u$ in $L^1(\Omega;\mathbb{R}^2)$ and $\mathcal{A}(u_j) \to \int_\Omega \sqrt{1+|\nabla u|^2}$. Since $\mathbb{M}([\mathcal{G}_{u_j}]) = \mathcal{A}(u_j)$, we get that, up to a subsequence, $[\mathcal{G}_{u_j}] \rightharpoonup T$ in $\mathcal{D}_2(\Omega\times\mathbb{R}^2)$. Moreover, $T \in \mathrm{cart}(\Omega\times\mathbb{R}^2)$. By the structure theorem, $T = [\mathcal{G}_{u_T}] + S$, with $u_T \in BV(\Omega;\mathbb{R}^2)$. But $u = u_T$ a.e. in Ω, since $u_j \to u$ in L^1. Then $T = [\mathcal{G}_u] + S$. We get $\mathbb{M}([\mathcal{G}_u]) \leq \mathbb{M}(T) \leq \liminf_j \mathbb{M}([\mathcal{G}_{u_j}]) = \int_\Omega \sqrt{1+|\nabla u|^2}$
This chain of inequalities is then made of equalities and $\mathbb{M}(T) = \mathbb{M}([\mathcal{G}_u])$.

Then $S = 0$, since $\mathbb{M}(S) = 0$, and $[G_u] = T \in \mathrm{cart}(\Omega \times \mathbb{R}^2)$. This is a contradiction, since $\partial [G_u] = -\delta_0 \times [S^1] \neq 0$.

The situation is worse than this. Not only this is not the expression of the functional, but $\bar{\mathcal{A}}$ cannot be written as an integral of some integrand depending on u and ∇u. De Giorgi conjectured that the functional $\bar{\mathcal{A}}(u, \cdot)$ seen as a function of the set Ω is not a measure because it's not subadditive in the set variable. He suggested to study the case of the triple junction, i.e., the BV function given by



Acerbi and Dal Maso proved that indeed $\bar{\mathcal{A}}(u,\cdot)$ is not subadditive for u given by this function, i.e., there exist $\Omega_1, \Omega_2, \Omega_3$ with $\Omega_3 \subset\subset \Omega_1 \cup \Omega_2$ but $\bar{\mathcal{A}}(u, \Omega_3) > \bar{\mathcal{A}}(u, \Omega_1) + \bar{\mathcal{A}}(u, \Omega_2)$.
Bellettini and Paolini provided an upper bound for this triple junction and Scala recently proved that the upper bound is optimal.

On the book by Giaquinta, Modica, and Souček there is an example of $u \in W^{1,1}(\Omega; \mathbb{R}^2)$, $\mathbb{M}([G_u]) < +\infty$, $\partial [G_u] = 0$, but there exists no sequence $u_j \in C^1(\Omega; \mathbb{R}^2)$ s.t. $[G_{u_j}] \to [G_u]$ and $\mathbb{M}([G_{u_j}]) \to \mathbb{M}([G_u])$.