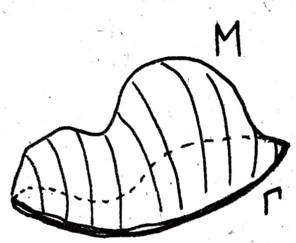
# AN INTRODUCTION TO THE THEORY OF CURRENTS

Tor Vergata, PhD Course I Semester

This course will be divided in two parts. In the first part we shall give an introduction to the theory of currents.

Currents are a generalization of the notion of oriented manifolds and it was developed in the form we are going to study by Federer & Fleming. One of the main problems that lead to the notion of currents is the **Plateau problem**:



Given a curve $\Gamma$ in the space, find a surface $M$ which minimizes the surface area among all surfaces with boundary given by $\Gamma$. (P)

As we usually do, a convenient way to approach the problem is to weaken the notion of surfaces and consider them as objects in a larger space with good compactness properties. Think about the Laplace equation, i.e., the problem of finding a smooth function $u : \Omega \to \mathbb{R}$ such that

$$(L) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

A way to solve this problem is to minimize the Dirichlet energy $\int_\Omega |\nabla u|^2 dx$ among all functions $u$ s.t. $u = g$ on $\partial\Omega$. The space of $C^2$ functions has very bad compactness/closure properties. Thus one defines the energy on $H^1(\Omega)$ functions. Sequences with bounded energy are converging weakly in $H^1(\Omega)$ and the direct method in the Calculus of Variations allows to prove that $\int_\Omega |\nabla u|^2 dx$ has a minimum in $H^1(\Omega)$ among functions with trace $g$ on $\partial\Omega$. These minima (actually unique) satisfy (L) in a weak sense. A posteriori one proves regularity of weak solutions and finds a classical solution of (L).

The approach to solve the classical Plateau problem (P) is similar: the class of manifolds has bad compactness properties (with equibounded surface area). The idea is to consider them as elements of a larger class with better compactness properties.

The idea behind currents is the following. Given a manifold $M$ of dimension $k$ embedded in $\mathbb{R}^n$ and oriented, we can define the integral

$$\int_M \omega$$

where $\omega$ is a $k$-form in $\mathbb{R}^n$. Thus manifolds can be seen as linear functionals on forms!

It is clear that in this course we will need some notation concerning forms. We will start today very slowly with:

- Preliminaries in Multilinear algebra.

Then the plan is to start with the theory of currents.

- Definition of currents and of mass. The mass of a current is the generalization of the area of a manifold.
- Boundary of currents and normal currents.
- Rectifiable currents.
- Federer & Fleming closure theorem for integral currents.
- Morgan

<u>REFERENCES</u>:

- Simon: Introduction / Lecture notes on GMT
- Alberti: Lecture notes on Currents (← WONDERFUL NOTES!)
- Federer: GMT
- Krantz-Parks: Geometric Integration Theory

In the second part of the course we will study Cartesian currents. Cartesian currents are a subclass of currents introduced to understand the behaviour of limits of graphs of functions. I will focus in particular on how they help in studying concentration effects in the limit. The plan of the second part is roughly the following:

- Definition of Cartesian currents
- Understanding the relation between Cartesian currents and BV functions
- The area functional for maps $\mathbb{R}^2 \to \mathbb{R}$, $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbb{R}^2 \to \mathbb{S}^1$.

If time permits I will show an application of Cartesian currents to a discrete-to-continuum problem.

<u>REFERENCES</u>:

- Giaquinta, Modica, Souček: Cartesian currents in the CV.

# CHAPTER 1 : MULTILINEAR ALGEBRA

Let $V$ be a vector space.

**Def:** A **k-covector** is a k-linear alternating form, i.e., a function $\alpha : \underbrace{V \times \dots \times V}_{k\text{-times}} \longrightarrow \mathbb{R}$ such that

- $\alpha$ is linear in each argument;
- $\alpha(v_1, \dots, v_k) = \text{sign}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$ for every $v_1, \dots, v_k \in V$ and every permutation $\sigma$ of $\{1, \dots, k\}$.

We denote the space of k-covectors by $\Lambda^k(V)$ and we set $\Lambda^0(V) := \mathbb{R}$.

**Def:** Given $\alpha \in \Lambda^h(V)$ and $\beta \in \Lambda^k(V)$, we define their **exterior product** by

$$(\alpha \wedge \beta)(v_1, \dots, v_{h+k}) := \frac{1}{h!\,k!} \sum_{\substack{\sigma \text{ perm} \\ \text{of } \{1,\dots,h+k\}}} \text{sign}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(h)})\, \beta(v_{\sigma(h+1)}, \dots, v_{\sigma(h+k)})$$

Then $\alpha \wedge \beta \in \Lambda^{h+k}(V)$.

With the exterior product at hand, we can construct a basis of $\Lambda^k(V)$. Let $\{e_1, \dots, e_n\}$ be a basis of $V$, let $\{e_1^*, \dots, e_n^*\}$ be the dual basis, i.e., $e_j^*(e_i) = \delta_{ij}$.

We denote by $I_{n,k} := \{ \text{multiindices } (i_1, \dots, i_k),\ 1 \le i_1 < \dots < i_k \le n \}$.

Usually, we shall denote a multiindex $(i_1, \dots, i_k)$ simply by $\underline{i}$.

We set $e_{\underline{i}}^* := e_{i_1}^* \wedge \dots \wedge e_{i_k}^* \in \Lambda^k(V)$. Then $(e_{\underline{i}}^*)_{\underline{i} \in I_{n,k}}$ is a basis of $\Lambda^k(V)$.

Proof: Let us fix $\alpha \in \Lambda^k(V)$. We define the form

$$\beta := \alpha - \sum_{\underline{i} \in I_{n,k}} \alpha(e_{i_1}, \dots, e_{i_k})\, e_{\underline{i}}^*$$

and we want to prove that it is the zero form.

First of all, let us compute for $\underline{j} \in I_{n,k}$

$$\beta(e_{j_1}, \dots, e_{j_k}) = \alpha(e_{j_1}, \dots, e_{j_k}) - \sum_{\underline{i} \in I_{n,k}} \alpha(e_{i_1}, \dots, e_{i_k})\, e_{\underline{i}}^*(e_{j_1}, \dots, e_{j_k}) =$$

$$= \alpha(e_{j_1}, \dots, e_{j_k}) - \alpha(e_{j_1}, \dots, e_{j_k}) = 0.$$

By linearity we get $\beta(v, e_{j_2}, \dots, e_{j_k}) = 0 \ \forall\, v \in V$. Then it's easy to conclude that $\beta = 0$. Then $\alpha = \sum_{\underline{i} \in I_{n,k}} \alpha(e_{i_1}, \dots, e_{i_k})\, e_{\underline{i}}^*$ and the decomposition is unique because those must be the coeff. $\square$

In particular, the dimension of $\Lambda^k(V)$ is the number of combinations of $k$ elements of $\{1,\dots,n\}$ without repetitions, i.e.,

$$\dim \Lambda^k(V) = \begin{cases} \binom{n}{k} & k \le n \\ 0 & k > n \end{cases}$$

Recalling that there is a canonical isomorphism between $V$ and $V^{**}$, we define:

**Def**: The space of **k-vectors** $\Lambda_k(V)$ is the space $\Lambda^k(V^*)$.

If $v_1,\dots,v_k \in V$, then $v_1 \wedge \dots \wedge v_k \in \Lambda_k(V)$ is called **simple k-vector**. The previous definition makes sense by interpreting $v_i$ as an element of $V^{**}$.

A basis of $V^{**}$ is $\{e_1^*,\dots,e_n^*\}$, thus, by the previous discussion, $(e_{\underline{i}}^{**})_{\underline{i}\in I_{n,k}}$ is a basis of $\Lambda_k(V)$. With the identification $e_i \cong e_i^{**}$ we say that $(e_{\underline{i}})_{\underline{i}\in I_{n,k}}$ is a basis of $\Lambda_k(V)$.

We can define a duality pairing between k-covectors and k-vectors.

We define:

$$\langle e_{\underline{i}}^*, e_{\underline{j}} \rangle := \delta_{\underline{i}\,\underline{j}} \quad \text{for } \underline{i},\underline{j} \in I_{n,k}$$

and we extend this by linearity to $\Lambda^k(V) \times \Lambda_k(V)$.

In particular, for $\alpha \in \Lambda^k(V)$:

$$\langle \alpha, e_{\underline{j}} \rangle = \langle \sum_{\underline{i}\in I_{n,k}} \alpha(e_{\underline{i}})\, e_{\underline{i}}^*, e_{\underline{j}} \rangle = \alpha(e_{\underline{j}}).$$

$$\langle \alpha, v_1 \wedge \dots \wedge v_k \rangle = \alpha(v_1,\dots,v_k).$$

## GEOMETRIC MEANING OF SIMPLE k-VECTORS

We shall now recall that simple k-vectors describe oriented k-planes.

**Proposition**: We have the following:

(i) $v_1 \wedge \dots \wedge v_k = 0$ iff $v_1,\dots,v_k$ are linearly dependent

(ii) If $v_1 \wedge \dots \wedge v_k = v_1' \wedge \dots \wedge v_k' \neq 0$, then $\operatorname{span}(v_1,\dots,v_k) = \operatorname{span}(v_1',\dots,v_k')$ and the matrix of change of basis from $(v_1,\dots,v_k)$ to $(v_1',\dots,v_k')$ has $\det = 1$.

*Proof*: (i) Assume that $v_1,\dots,v_k$ are linearly dependent. Then one of the vectors is a linear combination of the others $v_j = \sum_{i\neq j} a_i v_i$.

Then

$$v_1 \wedge \ldots \wedge v_k = v_1 \wedge \ldots \wedge \Big( \sum_{i \neq j} a_i v_i \Big) \wedge \ldots \wedge v_k =$$

$$= \sum_{i \neq j} a_i \, v_1 \wedge \ldots \wedge v_i \wedge \ldots \wedge v_i \wedge \ldots \wedge v_k = 0$$

since $v_i \wedge v_i = - v_i \wedge v_i$ !

Conversely, assume that $v_1, \ldots, v_k$ are linearly independent. We construct a $\alpha \in \Lambda^k(V)$ such that $\alpha(v_1, \ldots, v_k) \neq 0$. To do so, we complete $v_1, \ldots, v_k$ to a basis $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ and we consider the dual basis $v_1^*, \ldots, v_n^*$. We put $\alpha := v_1^* \wedge \ldots \wedge v_k^*$. Then $\alpha(v_1, \ldots, v_k) = (v_1^* \wedge \ldots \wedge v_k^*)(v_1, \ldots, v_k) = \det(v_i^*(v_j)) = 1 \neq 0$.

To prove this:

**Lemma:** $\alpha_1, \ldots, \alpha_k \in V^*$, $v_1, \ldots, v_k \in V$, then
$$\alpha_1 \wedge \ldots \wedge \alpha_k (v_1, \ldots, v_k) = \det(\alpha_i(v_j)).$$

Proof: By definition:

$$\alpha_1 \wedge \ldots \wedge \alpha_k (v_1, \ldots, v_k) = \frac{1}{(k-1)!} \sum_\sigma \operatorname{sign}(\sigma) \alpha_1(v_{\sigma(1)}) \cdot \alpha_2 \wedge \ldots \wedge \alpha_k (v_{\sigma(2)}, \ldots, v_{\sigma(k)})$$

induction

$$\overset{\downarrow}{=} \frac{1}{(k-1)!} \sum_\sigma \operatorname{sign}(\sigma) \alpha_1(v_{\sigma(1)}) \det\big(\alpha_j(v_{\sigma(h)})_{j,h=1}^{k}\big) =$$

Sum over $\sigma$ that satisfy $\sigma(1) = i$. They map $(2, \ldots, k)$ to a $\tau(1, \ldots, \hat{i}, \ldots, k)$, $\operatorname{sign}\sigma = (-1)^{i-1} \operatorname{sign}\tau$

$$= \frac{1}{(k-1)!} \sum_{i=1}^{k} \sum_{\substack{\tau \text{ perm} \\ (1, \ldots, \hat{i}, \ldots, k)}} \operatorname{sign}(\tau)(-1)^{i-1} \alpha_1(v_i) \underbrace{\det(\alpha_j(v_{\tau(h)}))}_{\substack{j = 2, \ldots, k \\ h = 1, \ldots, \hat{i}, \ldots, k}}$$

$$= \sum_{i=1}^{k} (-1)^{i-1} \alpha_1(v_i) M_{i2} = \det(\alpha_i(v_j)). \quad \square$$

(ii) Let $W = \operatorname{span}(v_1, \ldots, v_k)$ and $W' = \operatorname{span}(v'_1, \ldots, v'_k)$ and assume, by contradiction, that $W \neq W'$. Then one of the $v'_i$ is linearly independent from $v_1, \ldots, v_k$. Let us denote $v_{k+1} := v'_i$ and let us complete $(v_1, \ldots, v_{k+1})$ to a basis $(v_1, \ldots, v_n)$ of $V$. Let us consider $\alpha := v_1^* \wedge \ldots \wedge v_k^* \in \Lambda^k(V)$. Then $\alpha(v_1, \ldots, v_k) = 1$, as before. However, $\alpha(v'_1, \ldots, v'_k) = \det(v_i^*(v'_j)) = 0$.

because $v_j^*(v_{k+1}) = 0$ for $j = 1, \ldots, k$.

The same computation also shows that the matrix of the change of basis has det $= 1$.

$\square$

For the last observation let us assume that $V$ has a scalar product $\cdot$. Then this naturally induces a scalar product on $\Lambda_k(V)$ and $\Lambda^k(V)$ that makes the bases $(e_{\underline{i}})_{\underline{i}\in I_{n,k}}$ and $(e^*_{\underline{i}})_{\underline{i}\in I_{n,k}}$ orthonormal, respectively, i.e.,

$$e_{\underline{i}} \cdot e_{\underline{j}} = \delta_{\underline{i}\,\underline{j}} \quad , \quad e^*_{\underline{i}} \cdot e^*_{\underline{j}} = \delta_{\underline{i}\,\underline{j}}$$

(if $(e_1, \dots, e_n)$ is a orthonormal basis of $V$).

Let us compute the norm of a simple $k$-vector with this notion of scalar product:

$$|v_1 \wedge \dots \wedge v_k|^2 = (v_1 \wedge \dots \wedge v_k) \cdot (v_1 \wedge \dots \wedge v_k)$$

Let us decompose $v_1 \wedge \dots \wedge v_k$ in the basis $(e_{\underline{i}})_{\underline{i}\in I_{n,k}}$:

$$v_1 \wedge \dots \wedge v_k = \sum_{\underline{i}\in I_{n,k}} e^*_{\underline{i}}(v_1, \dots, v_k)\, e_{\underline{i}} = \sum_{\underline{i}\in I_{n,k}} \det\left( (e^*_{i_h}(v_j))_{h,j=1}^{k} \right) e_{\underline{i}}$$

Then $|v_1 \wedge \dots \wedge v_k|^2 = \sum\limits_{\underline{i}\in I_{n,k}} \det(e^*_{i_h}(v_j))^2 = \sum\limits_{\substack{A \\ k\times k \text{ minor}}} \det(A)^2$, where the last sum is taken among all $k \times k$ minors of the $n \times k$ matrix that represents the vectors $(v_1, \dots, v_k)$ in the basis $(e_1, \dots, e_n)$.

We can rewrite this sum over all minors in a more compact way:

**Theorem** (Generalized Binet Identity): Let $M$ be a $n \times k$ matrix and for $\underline{i} \in I_{n,k}$ let $M^{\underline{i}}$ be the $k \times k$ matrix with rows given by the rows of $M$ corresponding to the indices $\underline{i} = (i_1, \dots, i_k)$. Let $N$ be a $n \times k$ matrix. Then

$$\det(N^T M) = \sum_{\underline{i}\in I_{n,k}} \det(N^{\underline{i}}) \det(M^{\underline{i}}).$$

Proof: Let us define $\alpha \in \Lambda^k(\mathbb{R}^n)$ by $\alpha(v_1, \dots, v_k) = \det(N^T V)$, where $V$ is the matrix with columns $(v_1, \dots, v_k)$. We denote by $dx^1, \dots, dx^n$ the dual basis to the standard basis. Then we can decompose:

$$\alpha = \sum_{\underline{i}\in I_{n,k}} \alpha(e_{i_1}, \dots, e_{i_k})\, dx^{\underline{i}}.$$

We have:

$$\alpha(e_{i_1}, \dots, e_{i_k}) = \det(N^T(e_{i_1}, \dots, e_{i_k})) = \det(N^{\underline{i}}).$$

On the other hand

$$dx^{\underline{i}}(v_1, \dots, v_k) = dx^{i_1} \wedge \dots \wedge dx^{i_k}(v_1, \dots, v_k) =$$
$$= \det\left( \left(dx^{i_h}(v_j)\right)_{h,j=1}^{k} \right) = \det(V^{\underline{i}}).$$

This concludes the proof. $\square$

Thanks to the previous theorem we can continue the computation of the norm of simple $k$-vectors. We get:

$$|v_1 \wedge \dots \wedge v_k|^2 = \det(V^T V), \quad \text{where} \quad V = (v_1, \dots, v_k).$$

The latter quantity is the area ($k$-dimensional) of the rectangle in $\mathbb{R}^n$ spanned by $(v_1, \dots, v_k)$.

To check this, we start from the case when $n = k$. Then $|\det V| = \mathcal{L}^k(\text{rectangle}(v_1, \dots, v_k))$. To check this one could simply ortoghonalize $(v_1, \dots, v_k)$ using the Gram-Schmidt algorithm. Let us now consider the general case $k \leq n$.

Let us consider the linear map $L : \mathbb{R}^k \to W = \text{span}(v_1, \dots, v_k) \subset \mathbb{R}^n$ given by $L(e_i) = v_i$, so that $L([0,1]^k)$ is the rectangle spanned by $(v_1, \dots, v_k)$. Let now $M$ be the $n \times k$ matrix that represents $L$ with respect to orthonormal bases of $\mathbb{R}^k$ and $\mathbb{R}^n$, let $\tilde{M}$ be the matrix $k \times k$ that represents $L$ with respect to orthonormal bases of $\mathbb{R}^k$ and $W$. Then the matrices $M^T M$ and $\tilde{M}^T \tilde{M}$ both represent $T^* T : \mathbb{R}^k \to \mathbb{R}^k$ with respect to the orthonormal basis of $\mathbb{R}^k$ and thus $M^T M = \tilde{M}^T \tilde{M}$.

$$\det(M^T M) = \det(\tilde{M}^*)^2 = \det(L)^2 = \mathcal{L}^k(L([0,1]^k)).$$

$\uparrow$ Note that $M$ is exactly the matrix $V$ we introduced before.

We conclude that

$$|v_1 \wedge \dots \wedge v_k| = k\text{-volume of rectangle } (v_1, \dots, v_k).$$

We have a bijection:

$$v_1 \wedge \dots \wedge v_k \mapsto (\text{span}(v_1, \dots, v_k), \text{orientation}, \text{vol}(\text{rect}(v_1, \dots, v_k))).$$

$\uparrow$ two spaces have the same orientation if the change of basis has $\det > 0$.

Let us check injectivity:

Assume that

$(\mathrm{sp}(v_1,\dots,v_k), \text{orientation}, \mathrm{vol}(\mathrm{rect}(v_1,\dots,v_k)))$

$\|$

$(\mathrm{span}(v_1',\dots,v_k'), \text{orientation}, \mathrm{vol}(\mathrm{rect}(v_1',\dots,v_k')))$

and let us prove $v_1\wedge\dots\wedge v_k = v_1'\wedge\dots\wedge v_k'$. Fix $\alpha\in\Lambda^k(V)$.

The space $\Lambda^k(V)$ is spanned by $v_1^*\wedge\dots\wedge v_k^*$, thus $\alpha_{|W\times W}$ is of the form $\alpha_{|W\times W} = c\, v_1^*\wedge\dots\wedge v_k^*$ for some $c\in\mathbb{R}$.

On the one hand $\alpha(v_1,\dots,v_k) = c\det I = 1c = c$.

On the other hand $\alpha(v_1',\dots,v_k') = c\,\underbrace{\det(v_i^*(v_j'))}_{\text{this is the volume of the rectangle with sign.}} = c$.

## INTEGRATION ON MANIFOLDS

We recall that we can integrate differential forms on manifolds with an orientation.

**Def:** A differential k-form is a function $x\in\mathbb{R}^n\mapsto\omega(x)\in\Lambda^k(\mathbb{R}^n)$. We denote by $\mathcal{D}^k(\mathbb{R}^n)$ the space of smooth differential k-forms with compact support.

Let $M$ be a manifold of dimension $k$. A $k$-form on $M$ is a smooth map $p\in M\mapsto\omega(p)\in\Lambda^k(T_pM)$. If $M$ is oriented, we can integrate $k$-forms on $M$. Let us first recall that $M$ is oriented if there exists an atlas $\{(U_i,\varphi_i)\}_i$, $\varphi_i: U_i\subset M\to\varphi_i(U_i)\subset\mathbb{R}^k$ s.t. $\forall\, i,j$ with $U_i\cap U_j\neq 0$ we have $\det\nabla(\varphi_i\circ\varphi_j^{-1})_{|\varphi_j(U_i\cap U_j)}>0$.

To show how to integrate $\omega$, let us first assume that $\mathrm{supp}\,\omega\subset U_i$. Then we can go from $U_i$ to the usual space $\mathbb{R}^k$ with the aid of $\varphi_i$. More precisely, $\varphi_i^{-1}:\varphi_i(U_i)=:V_i\subset\mathbb{R}^k\to U_i\subset M$. This allows us to pull-back the form $\omega$ to $\mathbb{R}^k$: for $v_1,\dots,v_k\in\mathbb{R}^k$

$$(\varphi_i^{-1})^\#\omega(v_1,\dots,v_k) = \omega(d\varphi_i^{-1}(v_1),\dots,d\varphi_i^{-1}(v_k))$$

If not, we use a partition of unity

$(\varphi_i^{-1})^\#\omega$ is a $k$-form on $\mathbb{R}^k$ and thus it is of the type

$$(\varphi_i^{-1})^\#\omega = f\,dx^1\wedge\dots\wedge dx^k \quad\text{for some function } f.$$

Then:

$$\int_M \omega = \int_{U_i} \omega := \int_{\varphi_i(U_i)} f\,dx .$$

This definition does not depend on the chart, and we understand here why orientation is important. Assume that $\operatorname{supp}\omega \subset U_i \cap U_j$.

Then:

$$\int_{\varphi_i(U_i)} f(x)\,dx = \int_{\varphi_i(U_i\cap U_j)} f(x)\,dx \overset{y=\varphi_j(\varphi_i^{-1}(x))}{=} \int_{\varphi_j(U_i\cap U_j)} f(\varphi_i(\varphi_j^{-1}(y)))\, \overbrace{\det \nabla \varphi_i\circ\varphi_j^{-1}(y)}^{\text{this is positive!}}\,dy =$$

$$= \int_{\varphi_j(U_i\cap U_j)} f(\varphi_i(\varphi_j^{-1}(y)))\, d(\varphi_i\circ\varphi_j^{-1})^1 \wedge \ldots \wedge d(\varphi_i\circ\varphi_j^{-1})^k =$$

$$= \int_{\varphi_j(U_i\cap U_j)} f(\varphi_i(\varphi_j^{-1}(y)))\, (\varphi_i\circ\varphi_j^{-1})^\#(dx^1\wedge\ldots\wedge dx^k) =$$

$\boxed{\begin{array}{l}\psi^\# dx^i(v) = dx^i(d\psi(v)) \\ \quad = d\psi^i(v) \\ \text{and } \psi^\# \text{ commutes with} \\ \text{exterior product!}\end{array}}$

$$= \int_{\varphi_j(U_i\cap U_j)} (\varphi_i\circ\varphi_j^{-1})^\#(f\,dx^1\wedge\ldots\wedge dx^k) = \int_{\varphi_j(U_i\cap U_j)} (\varphi_i\circ\varphi_j^{-1})^\#(\varphi_i^{-1})^\#\omega = \int_{\varphi_j(U_j)} (\varphi_j^{-1})^\#\omega .$$

Let us now consider the case of an oriented manifold $M$ embedded in $\mathbb{R}^n$, which is the case we are interested in. First of all, we can integrate $\omega \in \mathcal{D}^k(\mathbb{R}^n)$ on $M$. Indeed, if $i: M \hookrightarrow \mathbb{R}^n$ is the inclusion map, then $i^\#\omega$ is a $k$-form on $M$ and the previous theory applies. Note that $i^\#\omega$ is simply the restriction of $\omega$ to vectors in the tangent space of $M$. Thus

$$\int_M \omega := \int_M i^\#\omega$$

but we use $\omega$ for both $\omega$ and $i^\#\omega$.

If $(U_i, \varphi_i)$ is a chart on $M$, then $\psi_i := \varphi_i^{-1} : \varphi_i(U_i) = V_i \longrightarrow U_i \subset M$ is a parametrization of the manifold and the tangent space can be written in terms of $\psi_i$. More precisely, for $x = \psi_i(t)$, the tangent space $T_xM \subset \mathbb{R}^n$ is spanned by $\frac{\partial \psi_i}{\partial t_1}(t), \ldots, \frac{\partial \psi_i}{\partial t_k}(t) \in \mathbb{R}^n$. In particular, it is oriented by

$$\tau(x) := \left.\frac{\frac{\partial \psi_i}{\partial t_1}(t)\wedge\ldots\wedge\frac{\partial \psi_i}{\partial t_k}(t)}{\left|\frac{\partial \psi_i}{\partial t_1}(t)\wedge\ldots\wedge\frac{\partial \psi_i}{\partial t_k}(t)\right|}\right|_{t=\varphi_i(x)} \in \Lambda^k(\mathbb{R}^n).$$

We can write more explicitly the integral $\int_M \omega$. Assume $\operatorname{supp}\omega \subset U_i$.
Then

$$\int_M \omega = \int_{U_i} \omega = \int_{\varphi_i(U_i)} (\varphi_i^{-1})^{\#}\omega = \int_{\varphi_i(U_i)} \langle (\varphi_i^{-1})^{\#}\omega, e_1 \wedge \dots \wedge e_K \rangle \, d\mathcal{L}^K =$$

$$= \int_{\varphi_i(U_i)} \langle \omega(\varphi_i^{-1}(t)), d_t \varphi_i^{-1}(e_1) \wedge \dots \wedge d_t \varphi_i^{-1}(e_K) \rangle \, d\mathcal{L}^K =$$

$$= \int_{\varphi_i(U_i)} \langle \omega(\psi_i(t)), \frac{\partial \psi_i}{\partial t_1}(t) \wedge \dots \wedge \frac{\partial \psi_i}{\partial t_K}(t) \rangle \, d\mathcal{L}^K =$$

$$= \int_{\varphi_i(U_i)} \langle \omega(\psi_i(t)), \tau(\psi_i(t)) \rangle \, \underbrace{|\det(\nabla\psi_i^T \nabla\psi_i)|^{\frac{1}{2}}}_{J_K d\psi_i(t)} \, d\mathcal{L}^K$$

$$\underset{\text{Area Formula}}{=} \int_{U_i} \langle \omega(x), \tau(x) \rangle \, d\mathcal{H}^K(x) = \int_M \langle \omega, \tau \rangle \, d\mathcal{H}^K$$

## THE AREA FORMULA

If $L: \mathbb{R}^K \to \mathbb{R}^n$ is a linear map, then $\mathcal{H}^K(L(E)) = |\det L| \, \mathcal{L}^K(E)$.
With a technical linearization lemma by Federer one can prove the following:

**Theorem**: (Area Formula) Let $f: \mathbb{R}^K \to \mathbb{R}^n$ be a $C^1$ (or Lipschitz) map, $1 \le K \le n$. Let $E \subset \mathbb{R}^K$ be measurable. Then

$$\int_{f(E)} \#(f^{-1}(y) \cap E) \, d\mathcal{H}^K(y) = \int_E J_K df(x) \, dx$$

where $y \in \mathbb{R}^n \mapsto \# \mathcal{H}^0(f^{-1}(y) \cap E)$ is $\mathcal{H}^K$-measurable.

From this notion of integration of forms in $\mathcal{D}^K(\mathbb{R}^n)$ on $K$-manifolds we can introduce the notion of currents.

# CHAPTER 2: THEORY OF CURRENTS

Currents are a generalization of manifolds in a manner similar to distribution ~~of~~ as generalization of functions.

**Def:** (of currents in the sense of de Rham): A **k-current** is a linear and continuous* functional which acts on smooth k-forms with compact support. More precisely, $T : \mathcal{D}^k(\mathbb{R}^n) \longrightarrow \mathbb{R}$ linear which satisfies the following continuity* condition:
for every compact set $K \subset \mathbb{R}^n$ there exists $C_K < +\infty$ and $N_K \in \mathbb{N}$ s.t. $\forall\, \omega \in \mathcal{D}^k(\mathbb{R}^n)$ with $\operatorname{supp}\omega \subset K$ it holds

$$|T(\omega)| \leq C_K \sup\{ |\partial^\alpha \omega(x)| : x \in K,\ |\alpha| \leq N_K \}.$$

$\mathcal{D}_k(\mathbb{R}^n) := \{k\text{-currents}\}$

* this is the same notion of continuity of distributions. It means continuous w.r.t. the topology of locally convex vector space $\mathcal{D}^k(\mathbb{R}^n)$.

In conclusion, currents are distributions for $\Lambda^k(\mathbb{R}^n)$-valued smooth maps with compact support as test functions. Therefore we can lend some results already known for distributions.

**Def:** (Convergence) A sequence $T_j \in \mathcal{D}_k(\mathbb{R}^n)$ **converges** to $T \in \mathcal{D}_k(\mathbb{R}^n)$ **in the sense of $\mathcal{D}_k(\mathbb{R}^n)$** and we write $T_j \rightarrow T$ in $\mathcal{D}_k(\mathbb{R}^n)$ iff for every $\omega \in \mathcal{D}^k(\mathbb{R}^n)$ we have $T_j(\omega) \rightarrow T(\omega)$.

**Def:** (Mass) We define the **total variation** of a current $T \in \mathcal{D}_k(\mathbb{R}^k)$

$$|T| := \sup\{ T(\omega) : \omega \in \mathcal{D}^k(\mathbb{R}^n),\ |\omega| \leq 1 \}$$ norm introduced on k-covectors

To define the mass of a current, we introduce the comass of a k-covector $\alpha \in \Lambda^k(\mathbb{R}^n)$:

$$\|\alpha\| := \sup_{\substack{v \text{ simple} \\ |v| \leq 1}} \langle \alpha, v \rangle .$$

ONE NOTICES THE DIFFERENCE BETWEEN $|T|$ AND $\mathbb{M}(T)$ ONLY IN APPROXIMATION THEOREMS

Note that $\|\alpha\| \leq |\alpha|$ (with equality if and only if $\alpha$ is simple).

The **mass** of a current $T \in \mathcal{D}_k(\mathbb{R}^n)$ is defined by:

$$\mathbb{M}(T) := \sup\{ T(\omega) : \omega \in \mathcal{D}^k(\mathbb{R}^n),\ \|\omega\| \leq 1 \}$$

Both $|T|$ and $\mathbb{M}(T)$ are l.s.c. with respect to the convergence in $\mathcal{D}_k(\mathbb{R}^n)$

**Example:** Let $M$ be a $k$-manifold in $\mathbb{R}^n$ and $T = [M]$, i.e.,

$$T(\omega) := \int_M \langle \omega, \tau \rangle \, d\mathcal{H}^k = \int_M \omega$$

Then $\mathbb{M}(T) = |T|(\mathbb{R}^n) = \mathcal{H}^k(M)$.

**Example:** Let $\mu \in \mathcal{M}_b(\mathbb{R}^n)$, i.e., a finite measure on $\mathbb{R}^n$. Let $\tau \in L^1_\mu(\mathbb{R}^n; \Lambda^*_k(\mathbb{R}^n))$. Then we can define the current

$$T(\omega) := \int_{\mathbb{R}^n} \langle \omega(x), \tau(x) \rangle \, d\mu(x) \quad , \quad \omega \in \mathcal{D}^k(\mathbb{R}^n).$$

Then $|T(\omega)| \leq \int_{\mathbb{R}^n} |\tau(x)| \, d\mu(x) = \|\tau\|_{L^1_\mu} < +\infty$.

Note that in the example above the measure $\mu$ is in general not related with the algebraic dimension $k$ of the current. For example, $\mu = \mathcal{L}^n$ or $\mu = \delta_0$ and $\tau$ a constant $k$-vector.

However, the last example is exhaustive. Indeed currents with finite mass are exactly of that form.

**Proposition:** Let $T \in \mathcal{D}_k(\mathbb{R}^n)$ with $\mathbb{M}(T) < +\infty$. Then there exists a finite measure $\mu \in \mathcal{M}_b(\mathbb{R}^n)$ and $\tau \in L^1_\mu(\mathbb{R}^n; \Lambda^*_k(\mathbb{R}^n))$ such that $T = \tau\mu$.

**Proof:** $T$ is defined on $\mathcal{D}^k(\mathbb{R}^n)$. We can extended by density to $\mathcal{C}^0_0(\mathbb{R}^n; \Lambda^k(\mathbb{R}^n))$, i.e., the closure of $\mathcal{D}^k(\mathbb{R}^n)$ with respect to the supremum norm. Given a sequence $\omega_j \in \mathcal{D}^k(\mathbb{R}^n)$ s.t. $\omega_j \to \omega$ uniformly, $\omega \in C^0_0(\mathbb{R}^n; \Lambda^k(\mathbb{R}^k))$. Then $T(\omega_j)$ is a Cauchy sequence: $|T(\omega_j) - T(\omega_\ell)| = |T(\omega_j - \omega_\ell)| \leq \mathbb{M}(T)\|\omega_j - \omega_\ell\|$. Then $T(\omega) := \lim_j T(\omega_j)$. $T : C^0_0(\mathbb{R}^n; \Lambda^k(\mathbb{R}^n)) \to \mathbb{R}$ is a linear and continuous functional and therefore it is represented by a finite measure $\tilde{\mu} \in \mathcal{M}_b(\mathbb{R}^n; \Lambda_k(\mathbb{R}^n))$. We then define $\mu := |\tilde{\mu}|$ and $\tau := \frac{d\tilde{\mu}}{d\mu} \in L^1_\mu(\mathbb{R}^n; \Lambda_k(\mathbb{R}^n))$. $\square$

**Theorem** (Compactness for currents with finite mass): Let $T_j \in \mathcal{D}_k(\mathbb{R}^n)$ be such that $\mathbb{M}(T_j) \leq C < +\infty$. Then there exists a subsequence (not relabeled) such that $T_j \rightharpoonup T$ in $\mathcal{D}_k(\mathbb{R}^n)$. Moreover,
$\mathbb{M}(T) \leq \liminf_j \mathbb{M}(T_j)$.

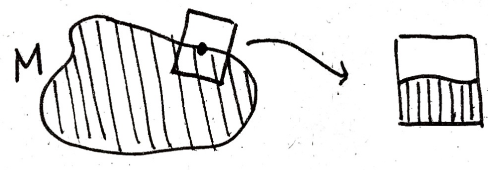
*Proof*: This is just a consequence of the compactness of measures with equibounded mass. $\square$

The theory of currents that we have seen up to now is not much different from the theory of distribution. The next object that we are going to introduce is what really starts to differentiate the theory of currents.

## BOUNDARY OF CURRENTS

To introduce the notion of boundary of a current, we recall Stoke's Theorem. This theorem has to do with the integration on the boundary of manifolds, so we have to recall how the boundary of a manifold is oriented.

A manifold with boundary ~~Kxk~~ has also charts that cover the points on the boundary and are parametrized with relative open sets of $\mathbb{R}^k_+$. Thanks to these charts we can define for every $x \in \partial\Omega$ a vector field $\eta(x)$ ~~in the complex~~ which is "exterior".




Given an orientation $\tau : M \to \Lambda_k M$, $x \mapsto \Lambda_k T_x M$ and assuming that $\tau$ is continuous up to the boundary $\partial M$, an orientation $\tau' : \partial M \to \Lambda_k \partial M$, $x \mapsto \Lambda_k T_x \partial M$ is naturally induced on $\partial M$ through the relation

$$\tau(x) = \eta(x) \wedge \tau'(x) \quad \text{for } x \in \partial M.$$

As a consequence one can integrate $(k-1)$-forms on $\partial M$.

### EXTERIOR DERIVATIVE

There exists a unique operator, the exterior derivative, $d : \mathcal{D}^k(\mathbb{R}^n) \to \mathcal{D}^{k+1}(\mathbb{R}^n)$ satisfying the following characterizing properties:

1) $d : \mathcal{D}^0(\mathbb{R}^n) \to \mathcal{D}^1(\mathbb{R}^n)$ is the differential of functions;
2) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for $\omega \in \mathcal{D}^k(\mathbb{R}^n)$, $\eta \in \mathcal{D}^h(\mathbb{R}^n)$;
3) $d \circ d = 0$.

It is defined for $\omega = \sum_{\underline{i} \in I_{n,k}} \omega_{\underline{i}} \, dx^{\underline{i}}$ by $d\omega = \sum_{\underline{i} \in I_{n,k}} d\omega_{\underline{i}} \wedge dx^{\underline{i}} \in \mathcal{D}^{k+1}(\mathbb{R}^n)$

<u>Remark</u>: The definition above is very general and is also used to define the exterior derivative on a manifold $d : \Lambda^k M \to \Lambda^{k+1} M$. To prove the existence of the exterior derivative it is done working in charts and using the formula above.

## STOKE'S THEOREM

We can finally state Stoke's Theorem:

**Theorem**: Let $M$ be a compact oriented surface of dimension $k$ in $\mathbb{R}^n$ and let $\omega \in \mathcal{D}^{k-1}(\mathbb{R}^n)$. Then:

$$\int_M d\omega = \int_{\partial M} \omega .$$

We exploit this relation to define the boundary of a current.

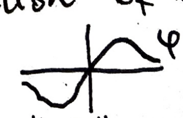
## BOUNDARY OF CURRENTS

**Def**: Given $T \in \mathcal{D}_k(\mathbb{R}^n)$ we define the current $\partial T \in \mathcal{D}_{k-1}(\mathbb{R}^n)$ as follows: for every $\omega \in \mathcal{D}^{k-1}_{\cancel{k+1}}(\mathbb{R}^n)$: $\partial T(\omega) := T(d\omega)$.

## NORMAL CURRENTS

**Def**: A current $T \in \mathcal{D}_k(\mathbb{R}^n)$ is said to be **normal** if both $T$ and $\partial T$ have finite mass.

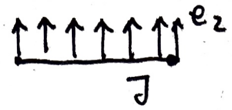
<u>Example</u>: (of a current which is not normal). Let $T := e_1 \delta_0 \in \mathcal{D}_1(\mathbb{R}^2)$. Then of course $\mathbb{M}(T) < +\infty$. Let us compute its boundary. $\partial T \in \mathcal{D}_0(\mathbb{R}^2)$
Let us fix a test function $\varphi \in \mathcal{D}^0(\mathbb{R}^2)$. $\partial T(\varphi) = T(d\varphi) =$
$= \langle d\varphi(0), e_1 \rangle = \partial_{x_1} \varphi(0)$. Thus $\partial T$ looks like a derivative of a Dirac delta, which is an example of distribution of order 1.
Fix $\varphi \in \mathcal{D}^0(\mathbb{R}^2)$ which around zero has $\frac{\partial \varphi}{\partial x_1}(0) = 1$




and put $\varphi_j(x_1, x_2) := \varphi(j x_1, x_2)$. Then $\|\varphi_j\|_\infty \le \|\varphi\|_\infty$ but $\frac{\partial \varphi_j}{\partial x_1}(0) \sim j \to +\infty$.

Example: Let $J = [-1,1] \times \{0\}$ and $T = e_2 \mathcal{H}^1 \llcorner J \in \mathcal{D}_1(\mathbb{R}^2)$.

$T$ :




Let us compute the boundary of $T$: fix $\varphi \in \mathcal{D}^0(\mathbb{R}^2)$ and

$$\partial T(\varphi) = T(d\varphi) = \int_J \langle d\varphi(x_1,x_2), e_2 \rangle \, d\mathcal{H}^1(x_1,x_2) = \int_{-1}^{1} \frac{\partial \varphi}{\partial x_2}(x_1,0)\, dx_1 .$$

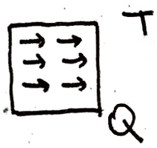
The $\|\cdot\|_\infty$ cannot control the derivative in the vertical direction, thus $\mathbb{M}(\partial T) = +\infty$.

Example: $J = [-1,1]\times\{0\}$, $T = e_1 \mathcal{H}^1 \llcorner J \in \mathcal{D}_1(\mathbb{R}^2)$. Then

$$\partial T(\varphi) = T(d\varphi) = \int_{-1}^{1} \frac{\partial \varphi}{\partial x_1}(x_1,0)\,dx_1 = \varphi(1) - \varphi(-1) = (\delta_1 - \delta_{-1})(\varphi).$$

Thus $T$ is normal!

Example: $Q = [-1,1]^2$ and $T = e_1 \cancel{\mathcal{H}} \mathcal{L}^2 \llcorner Q$.




$$\text{Then } \partial T(\varphi) = T(d\varphi) = \int_Q \frac{\partial \varphi}{\partial x_1}\, dx = \int_{\partial Q} \varphi \nu^1 = \int_{-1}^{1} \varphi(1,x_2)\,dx_2 - \int_{-1}^{1} \varphi(-1,x_2)\,dx_2 =$$

$$= \left(\delta_1 \times (\mathcal{H}^1 \llcorner [-1,1]) - \delta_{-1} \times (\mathcal{H}^1 \llcorner [-1,1])\right)(\varphi).$$

We shall not prove this, but a theorem states that normal $k$-currents are supported on sets with Hausdorff dimension larger than $k$. Thus the fact that the boundary has finite mass gives some geometric structure to the current and starts to give a geometric meaning to the algebraic dimension $k$ of the current.

Most importantly, the notion of boundary allows us to formulate the Plateau problem in the class of normal currents: fix a normal current $T_0 \in \mathcal{D}_k(\mathbb{R}^n)$ and find:

$$\min \{ \mathbb{M}(T) : T \in \mathcal{D}_k(\mathbb{R}^n) \text{ normal}, \ \partial T = \partial T_0 \}.$$

To prove the existence of a solution to this problem we need a compactness result.

Theorem (compactness for normal currents): Let $(T_j)$ be a sequence of normal currents in $\mathcal{D}_k(\mathbb{R}^n)$ such that $\mathbb{M}(T_j), \mathbb{M}(\partial T_j) \le C$. Then, up to a subsequence, $\exists$ a normal current $T \in \mathcal{D}_k(\mathbb{R}^n)$ s.t. $T_j \rightharpoonup T$ in $\mathcal{D}_k(\mathbb{R}^n)$.

Proof: By the compactness of currents of finite mass we have, up to a subsequence,

$$T_j \rightharpoonup T \text{ in } \mathcal{D}_K(\mathbb{R}^n), \quad \partial T_j \rightharpoonup S \text{ in } \mathcal{D}_{K-1}(\mathbb{R}^n).$$

However, the boundary of currents is continuous w.r.t. the weak convergence in the sense of currents:

for $\omega \in \mathcal{D}_{K-1}(\mathbb{R}^n)$

$$S(\omega) = \lim_j \partial T_j(\omega) = \lim_j T_j(d\omega) = T(d\omega) = \partial T(\omega). \quad \square$$

With the previous compactness theorem we easily prove the existence of a solution to the Plateau problem in the class of normal currents. This solution is, in general, too far from being a manifold in the classical sense, cf. the examples of normal currents.
For this reason one introduces a new class of currents which has even stronger geometric properties. Of course, one still wants good compactness properties for this class and the price to pay for a stronger geometric structure is that it is more difficult to prove compactness.

# CHAPTER 3: INTEGRAL CURRENTS

We will now introduce two classes of currents: rectifiable currents, namelly currents that are concentrated ~~at~~ on rectifiable sets of the Hausdorff dimension equal to the algebraic dimension of the current; integral currents, i.e., rectifiable currents with integer multiplicity whose boundary are rectifiable currents with integer multiplicity. The latter class is very close to the class of currents given by manifolds.
To give these definitions we have to recall what rectifiable sets are.

## HAUSDORFF MEASURE

The $K$-dimensional Hausdorff measure is defined using the Caratheodory construction:

- construct an outer measure ($\sigma$-subadditive on all sets);
- show that the outer measure is additive on sets which are far apart;
- then the outer measure is a measure on the $\sigma$-algebra of Borel sets.

**Def:** Let $X$ be a metric space and $\kappa$ a non-negative real number. Let $\delta \in (0,+\infty)$. The Hausdorff $\delta$-premeasure is defined for every $E \subset X$ by:

$$\mathcal{H}^\kappa_\delta(E) := \underbrace{c_\kappa}_{\substack{\downarrow \\ \text{suitable} \\ \text{constant}}} \inf \left\{ \sum_i (\operatorname{diam} E_i)^\kappa : E \subset \bigcup_i E_i ,\ \{E_i\}_i \text{ countable}, \operatorname{diam} E_i \le \delta \right\}.$$

The $\kappa$-dimensional Hausdorff measure is:

$$\mathcal{H}^\kappa(E) := \sup_{\delta > 0} \mathcal{H}^\kappa_\delta(E) = \lim_{\delta \searrow 0} \mathcal{H}^\kappa_\delta(E).$$

**Proposition:** The following facts are true:

- $\mathcal{H}^\kappa$ is a measure on Borel sets;
- $\mathcal{H}^0$ is the counting measure;
- $\mathcal{H}^\kappa$ is translation invariant;
- $\mathcal{H}^n = \mathcal{L}^n$ ($\leftarrow$ choice of $c_\kappa$);
- if $f : X \to X'$ is Lipschitz, then $\mathcal{H}^\kappa(f(E)) \le (\operatorname{Lip}(f))^\kappa \mathcal{H}^\kappa(E)$;
- If $\mathcal{H}^\kappa(E) \in (0,+\infty)$, then

$$\mathcal{H}^\alpha(E) = 0 \quad \text{for } \alpha > \kappa$$
$$\mathcal{H}^\alpha(E) = +\infty \quad \text{for } \alpha < \kappa.$$

The latter property motivates the definition

$$\dim_{\mathcal{H}}(E) := \inf \{ \alpha \ge 0 : \mathcal{H}^\alpha(E) = 0 \}.$$

## RECTIFIABLE SETS

**Def:** Let $\kappa \in \mathbb{N}$ and $E \subset \mathbb{R}^n$ an $\mathcal{H}^\kappa$-measurable set. $E$ is $(\mathcal{H}^\kappa, \kappa)$-countably rectifiable* if $E = \bigcup_{i=0}^{\infty} E_i \cup N$ where $\nwarrow$ countable union

- $\mathcal{H}^\kappa(N) = 0$
- $E_i$ is a $\mathcal{H}^\kappa$-measurable subset of a $\kappa$-manifold of class $C^1$.

**Example:** The image of $\mathbb{R}^\kappa$ through Lipschitz maps is $\kappa$-rectifiable.

The proof of the statement in the example relies on an important property of Lipschitz functions: they satisfy a Lusin property with $C^1$ functions.

* [ this is the notation used in the book by Federer, who introduces many notions of rectifiability. By "rectifiable" we will mean this!

**Theorem** ($C^1$-Lusin property): If $f:\mathbb{R}^K \to \mathbb{R}$ is a.e. differentiable, then for every $\varepsilon>0$ there exists a function $g \in C^1(\mathbb{R}^K)$ s.t. $\mathcal{L}^K(\{f\neq g\}) < \varepsilon$.

The proof of the previous fact is based on the following powerful result.

**Theorem** (Whitney's Extension): Let $K \subset \mathbb{R}^K$ be a compact set, $f: K \to \mathbb{R}$, $F: K \to \mathbb{R}^K$ continuous functions. Assume that

$$\lim_{\delta\to 0} \sup\left\{ \frac{|f(x)-f(y)-F(x)\cdot(y-x)|}{|x-y|} : x,y\in K,\ 0<|x-y|<\delta \right\} = 0.$$

Then there exists a function $g\in C^1(\mathbb{R}^K)$ s.t. $g=f$ and $\nabla g = F$ on $K$.

[No proof].

Proof (of the $C^1$-Lusin property): We know that for a.e. $x\in\mathbb{R}^K$ it holds

$$\lim_{y\to x} \frac{|f(y)-f(x)-\nabla f(x)\cdot(y-x)|}{|y-x|} = 0.$$

The function $\nabla f: \mathbb{R}^K \to \mathbb{R}^K$ is measurable (for Lipschitz functions: it coincides with the distributional derivative, which is in $L^\infty$).

By the classical Lusin theorem for measurable functions we have that there exists a compact set $K$ s.t.

- $\mathcal{L}^K(\operatorname{supp} f \setminus K) < \varepsilon$ (we assume that $f$ has compact support, otherwise we use a locally finite partition of unity).
- $f|K$ is continuous.
- $\nabla f|K$ is continuous.
- $f$ is differentiable at every point of $K$.

In particular, we have

$$\lim_{\substack{y\in K\\ y\to x}} \frac{|f(y)-f(x)-\nabla f(x)\cdot(y-x)|}{|y-x|} = 0, \quad \text{for } x\in K.$$

The last condition is still weaker than the one needed to apply Whitney's extension theorem. However we can define

$$\omega_\delta(x) := \sup\left\{ \frac{|f(y)-f(x)-\nabla f(x)\cdot(y-x)|}{|y-x|} ;\ x,y\in K,\ 0<|x-y|<\delta \right\}.$$

We know that $\omega_\delta(x)\to 0$ for every $x\in K$. By Egorov's Theorem we find a $K'\subset K$ with $\mathcal{L}^K(K\setminus K')<\varepsilon$ s.t. $\omega_\delta \to 0$ uniformly on $K'$. Then we apply Whitney on $K'$! $\square$

**Example**: We can go back to the previous example. Let $f: \mathbb{R}^k \to \mathbb{R}^n$ be a Lipschitz map and let us prove that $f(\mathbb{R}^k)$ is $k$-rectifiable. $E := f(\mathbb{R}^k)$. By the $C^1$-Lusin property ($f$ is a.e. differentiable by Rademacher's theorem) We can find a sequence of open sets $A_j$ satisfying:

- $f = f_j$ on $\mathbb{R}^k \setminus A_j$
- $\mathcal{L}^k(A_j) \to 0$, $A_{j+1} \subset A_j$
- $f_j \in C^1(\mathbb{R}^k; \mathbb{R}^n)$.

Put $A := \bigcap_j A_j$ and decompose $f(\mathbb{R}^k) \subset f(A) \cup \bigcup_j f_j(\mathbb{R}^k)$.

Note that $\mathcal{H}^k(f(A)) \leq (\mathrm{Lip}\, f)^k \mathcal{L}^k(A) = 0$. We are left to show that $E_j = f_j(\mathbb{R}^k)$ is rectifiable. Let $\Omega_j := \{ ~~x~~ t \in \mathbb{R}^k : \operatorname{rank} df_j(t) = k \}$. Then $Jf_j(t) = 0$ for $x \in \mathbb{R}^k \setminus \Omega_j$. By the Area Formula $\mathcal{H}^k(f_j(\mathbb{R}^k \setminus \Omega_j)) = 0$. $\Omega_j$ is an open set of $\mathbb{R}^k$ and $f_j$ is a local parametrization of a $k$-manifold of class $C^1$ by the implicit function theorem. $\square$

Actually, this can be used to give another equivalent definition of $k$-rectifiable sets: an $\mathcal{H}^k$-measurable set $E$ is $k$-rectifiable iff $E = N \cup \bigcup_j E_j$ where $\mathcal{H}^k(N) = 0$ and $E_j$ is a $\mathcal{H}^k$-measurable subset of the image of a Lipschitz map $\mathbb{R}^k \to \mathbb{R}^n$.

## WEAK TANGENT BUNDLE

**Proposition/Def**: Let $E$ be a $k$-rectifiable set in $\mathbb{R}^n$. Then there exists an $\mathcal{H}^k$-measurable map

$$\tau : E \to G(n,k) \leftarrow \text{Grassmannian of vector subspaces of dim } k \text{ in } \mathbb{R}^n$$

such that for every $k$-manifold $M$ of class $C^1$ in $\mathbb{R}^n$ it holds:

$$\tau(x) = \mathrm{Tan}(M, x) \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in E \cap M.$$

Such $\tau$ is unique up to an $\mathcal{H}^k$-negligible set. $\tau$ is called the **weak tangent bundle** of $E$.

To prove the previous proposition we need a basic result in differential geometry.

**Lemma**: Let $M_1$ and $M_2$ be two $C^1$ manifolds of dimension $k$ in $\mathbb{R}^n$. Then $\mathrm{Tan}(M_1,x) = \mathrm{Tan}(M_2,x)$ for $\mathcal{H}^k$-a.e. $x \in M_1 \cap M_2$.

*Proof*: Using local parametrizations of $M_1$ and $M_2$ we can reduce the problem to the following: given $f_1, f_2 : \mathbb{R}^k \to \mathbb{R}$ $C^1$-functions and putting $I := \{x \in \mathbb{R}^k : f_1(x) = f_2(x)\}$, then

$$df_1(x) = df_2(x) \quad \text{for } \mathcal{L}^k\text{-a.e. } x \in I.$$

To prove the last claim, let $f := f_1 - f_2$. The set

$$M := \{x \in \mathbb{R}^k : f(x) = 0,\ df(x) \neq 0\}$$

is a manifold of codimension 1 in $\mathbb{R}^k$ by the Implicit Function Theorem. Thus $df(x) = 0$ for $\mathcal{L}^k$-a.e. $x \in \mathbb{R}^k$. $\square$

*Proof* (of existence of the weak tangent bundle): To construct $\tau$ we use the definition of rectifiable set. $E$ can be decomposed as

$$E = N \cup \bigcup_j E_j \quad , \quad \mathcal{H}^k(N) = 0$$

where $E_j \subset M_j$, $M_j$ a $C^1$ manifold of dim $k$. We define:

$\tilde{E}_j := E_j \setminus \bigcup_{i<j} E_i$. The sets $\tilde{E}_j$ still cover $E$ up to sets of measure 0 (but they are pairwise disjoint).

Then the following map is well defined:

$$\tau(x) := \begin{cases} \mathrm{Tan}(M_j, x) & \text{if } x \in \tilde{E}_j, \\ \text{whatever} & \text{if } x \in N \setminus \bigcup_j E_j. \end{cases}$$

Let us check that $\tau$ satisfies the property that characterizes the weak tangent bundle. Let $M$ be a $k$-manifold of class $C^1$ and let $x \in E \cap M$. If $x \in \tilde{E}_j$ for some $j$, then $x \in M_j$ and we have

$$\underbrace{\tau(x) = \mathrm{Tan}(M_j,x)}_{\text{definition}} \underbrace{=}_{\text{if } x \text{ is chosen properly}} \mathrm{Tan}(M,x)$$

Then $\tau(x) = \mathrm{Tan}(M,x)$ for $\mathcal{H}^k$-a.e. $x \in \tilde{E}_j \cap M$. The $\tilde{E}_j$'s cover $E$ up to $\mathcal{H}^k$-null sets.

Let us prove uniqueness. Let $\tau' : E \to G(n,k)$ be another map which satisfies the property. Let us fix $j$. Since $M_j$ is a $C^1$-manifold, we have $\tau'(x) = \mathrm{Tan}(M_j,x)$ for $\mathcal{H}^k$-a.e. $x \in E \cap M_j$. In particular,

$\tau'(x) = \mathrm{Tan}(M_j, x) = \tau(x)$ for $\mathcal{H}^k$-a.e. $x \in \tilde{E}_j$. The sets $\tilde{E}_j$ cover $E$ and we conclude that $\tau' = \tau$ $\mathcal{H}^k$-a.e. on $E$. $\square$

As for a measurable function it makes no sense to define its pointwise value, we cannot define pointwise a tangent space to a $k$-rectifiable set. However, for measurable functions we can define an approximate limit at a.e. point (for $L^1_{loc}$ functions a.e. point is a Lebesgue point!). This means that the blow-ups of the function at a.e. point converge. In the same way we have the following result for rectifiable sets.

**Theorem/Def:** Let $E$ be an $k$-rectifiable set in $\mathbb{R}^n$ with $\mathcal{H}^k(E) < +\infty$ (or at least locally finite*). For every $x \in \mathbb{R}^n$ and $r > 0$ let $E_{x,r} := \frac{E-x}{r}$. Then for $\mathcal{H}^k$-a.e. $x \in E$ we have

$$\mathcal{H}^k \llcorner E_{x,r} \xrightarrow{*} \mathcal{H}^k \llcorner \tau(x) \quad \text{as } r \to 0$$

$w^*$-$\mathcal{M}_b(B(0,1))$ (i.e., in duality with $C_c(B(0,1))$ maps).

~~Def:~~ If the property above holds true at $x$, we say that $\tau(x)$ is the approximate tangent space of $E$ at $x$ and we denote it by $\mathrm{Tan}(E,x)$.

**Proof:** Let $E = N \cup \bigcup_j E_j$ with $E_j \subset M_j$, $M_j$ manifold of class $C^1$.

We will show that

$$\mathcal{H}^k \llcorner E_{x,r} \xrightarrow{*} \mathcal{H}^k \llcorner \mathrm{Tan}(M_j, x) \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in E_j.$$

To prove the previous convergence, we split the set $E$ in three parts:

$$E = (E \cap M_j) \cup (E \setminus M_j) = (M_j \setminus (M_j \setminus E)) \cup (E \setminus M_j)$$

so that, if we define

$$\lambda_{x,r} := \mathcal{H}^k \llcorner (M_j)_{x,r}, \quad \lambda'_{x,r} := \mathcal{H}^k \llcorner (M_j \setminus E)_{x,r}, \quad \lambda''_{x,r} := \mathcal{H}^k \llcorner (E \setminus M_j)_{x,r}$$

we have

$$\mathcal{H}^k \llcorner E_{x,r} = \lambda_{x,r} - \lambda'_{x,r} + \lambda''_{x,r}.$$

* There are $k$-rectifiable sets with non locally finite measure. Countable unions allow to make crazy constructions like $\bigcup \{y = ax + b\}$, $a, b \in \mathbb{Q}$.

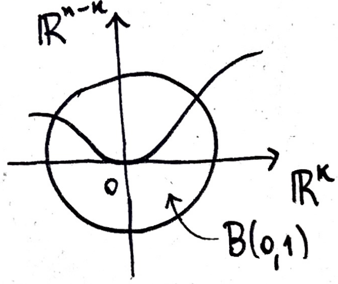
We claim that:

1) $\lambda_{x,r} \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner \mathrm{Tan}(M_j,x)$.

2) $\lambda''_{x,r} \overset{*}{\rightharpoonup} 0$ for $\mathcal{H}^k$-a.e. $x \in E_j$.

3) $\lambda'_{x,r} \overset{*}{\rightharpoonup} 0$ for $\mathcal{H}^k$-a.e. $x \in E_j$.

Claim 1: Let us start with the case when $M_j$ is the graph of a function $f: \mathbb{R}^k \to \mathbb{R}^{n-k}$, $f(0)=0$, $df(0)=0$.




Let $F(t) := (t, f(t)) \in \mathbb{R}^n$ and let $\varphi \in C_c(B(0,1))$.
We are assuming that $x = 0$.

$$y = \frac{z-x}{r} = \frac{z}{r}$$

$\varphi$ has compact support in $B(0,1)$

$$\int_{B(0,1)} \varphi(y)\, d\lambda_{x,r}(y) = \int_{B(0,1)\cap(M_j)_{x,r}} \varphi(y)\, d\mathcal{H}^k(y) \overset{\downarrow}{=} \frac{1}{r^k}\int_{B(0,r)\cap M_j} \varphi\left(\frac{z}{r}\right) d\mathcal{H}^k(z) =$$

$$= \frac{1}{r^k}\int_{M_j} \varphi\left(\frac{z}{r}\right) d\mathcal{H}^k(z) = \frac{1}{r^k}\int_{F(\mathbb{R}^k)} \varphi\left(\frac{z}{r}\right) d\mathcal{H}^k(z) = \frac{1}{r^k}\int_{\mathbb{R}^k} \varphi\left(\frac{F(t)}{r}\right) J_k F(t)\, d\mathcal{L}^k(t)$$

Area Formula

$$\overset{t=rs}{=} \int_{\mathbb{R}^k} \varphi\left(\frac{F(rs)}{r}\right)(J_k dF)(rs)\, d\mathcal{L}^k(s) = \int_{\mathbb{R}^k} \varphi\left(s, \frac{f(rs)}{r}\right)(J_k dF)(rs)\, d\mathcal{L}^k(s)$$

$$\longrightarrow \int_{\mathbb{R}^k} \varphi(s,0)\, \underbrace{J_k dF(0)}_{\substack{\| \\ \sqrt{1 + \text{sum of } \nabla f \ k\times k \text{ minors}} = 1}}\, d\mathcal{L}^k(s) = \int_{\mathrm{Tan}(M_j,x)} \varphi(y)\, d\mathcal{H}^k(y).$$

$$\nabla F = \begin{array}{c} \overbrace{\phantom{Id_k}}^{k} \\ \begin{pmatrix} Id_k \\ \nabla f \end{pmatrix} \end{array} \begin{array}{l} \\ \}\,k \\ \}\,n-k \end{array}$$

In the general case, we use the fact that the manifold $M_j$ can be seen locally as the graph of functions $f: \mathbb{R}^k \to \mathbb{R}^{n-k}$, $f(0)=0$, $df(0)=0$.
[Note: to see this general fact, argue as follows. Up to translation and rotations, $0 \in M$ and $\mathrm{Tan}(M,0) = \mathbb{R}^k$. Let $(U,\phi)$ be a chart around $0$. $\phi: U \subset M \subset \mathbb{R}^n \to \phi(U) =: V \subset \mathbb{R}^k$, $\phi(0) = 0$. Let $\psi := \phi^{-1}: V \subset \mathbb{R}^k \to U \subset \mathbb{R}^n$.
$\mathrm{Tan}(M,0) = \mathbb{R}^k$ is generated by $\partial_{t_1}\psi(0), \dots, \partial_{t_k}\psi(0) \in \mathbb{R}^n$. The map $\pi_1 \circ \psi: V \subset \mathbb{R}^k \to \mathbb{R}^k$ is a local diffeomorphism, $\pi_1: \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$.
Let $\xi: \tilde{U} \subset \mathbb{R}^k \to \xi(\tilde{U}) =: \tilde{V} \subset V$ be the local inverse around $0$.

Then $M$ is locally around $0$ given by the graph $\{(x, \pi_2 \circ \Psi_0 \circ \xi(x)), x \in \tilde{U}\}$ where $\pi_2 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$. ]

Claim 2: $\lambda''_{x,r} \xrightarrow{*} 0$ for $\mathcal{H}^k$-a.e. $x \in E_j$.

$\lambda''_{x,r} = \mathcal{H}^k \llcorner (E \setminus M_j)_{x,r}$.

We prove, in fact, that $\lambda''_{x,r}(B(0,1)) \to 0$. Indeed

$$\lambda''_{x,r}(B(0,1)) = \mathcal{H}^k(B(0,1) \cap (E \setminus M_j)_{x,r}) = \mathcal{H}^k((B(x,r) \cap (E \setminus M_j))_{x,r}) = \frac{\mathcal{H}^k(B(x,r) \cap (E \setminus M_j))}{r^k}$$

But for $\mathcal{H}^k$-a.e. $x \in M_j$ we have

$$\frac{d\mathcal{H}^k \llcorner (E \setminus M_j)}{d\mathcal{H}^k \llcorner M_j}(x) = 0 \quad \text{and} \quad \mathcal{H}^k(M_j \cap B(x,r)) \sim \omega_k r^k$$

as in the proof of Claim 1.

Thus $\lambda''_{x,r}(B(0,1)) \to 0$ for $\mathcal{H}^k$-a.e. $x \in E_j$.

Claim 3: $\lambda'_{x,r} \xrightarrow{*} 0$ for $\mathcal{H}^k$-a.e. $x \in E_j$.

$\lambda'_{x,r} = \mathcal{H}^k \llcorner (M_j \setminus E)_{x,r}$.

$$\lambda'_{x,r}(B(0,1)) = \mathcal{H}^k(B(0,1) \cap (M_j \setminus E)_{x,r}) = \frac{\mathcal{H}^k(B(x,r) \cap (M_j \setminus E))}{r^k} =$$

$$= \underbrace{\frac{\mathcal{H}^k(B(x,r) \cap M_j)}{r^k}}_{\sim \omega_k} - \frac{\mathcal{H}^k(B(x,r) \cap (E \cap M_j))}{r^k}$$

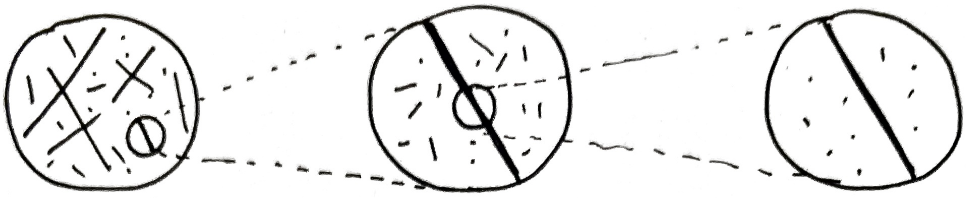
$E \cap M_j \subset M_j \Rightarrow \mathcal{H}^k \llcorner (E \cap M_j) = \mathbb{1}_{E \cap M_j} \mathcal{H}^k \llcorner M_j \Rightarrow$

$\Rightarrow$ for $\mathcal{H}^k$-a.e. $x \in M_j$:

$$\frac{d\mathcal{H}^k \llcorner (E \cap M_j)}{d\mathcal{H}^k \llcorner M_j}(x) = \mathbb{1}_{E \cap M_j}(x)$$

and, in particular, it is equal to $1$ for $\mathcal{H}^k$-a.e. $x \in E_j$.
Again $\mathcal{H}^k \llcorner M_j(B(x,r)) \sim \omega_k r^k$. Thus $\lambda'_{x,r}(B(0,1)) \sim 0$ and we conclude the proof. □

**Example:** Let $E$ be the 1-rectifiable set in $\mathbb{R}^2$ given by the union of a countable family of segments with $\sum \mathcal{H}^1(I_i) < +\infty$ and such that for every disk in $\mathbb{R}^2$ the directions of the segments ~~are~~ found in the disk are dense. Then $E \cap B(x,r)$ always includes parts of segments with many directions, but just one is predominant in terms of measure!



## RECTIFIABILITY CRITERIA

**Proposition:** If $\mu \in \mathcal{M}_b(\mathbb{R}^n)$, $\mu$ positive such that $\mu$ is concentrated on a Borel set $E$ and admits an approximate tangent space with multiplicity $\theta(x) > 0$ for $\mu$-a.e. $x \in E$, then $E$ is $k$-rectifiable and $\mu = \theta \, \mathcal{H}^k \llcorner E$.

**Theorem** (due to Marstrand, Preiss): If $E$ is such that $\mathcal{H}^k(E) < +\infty$ and for $\mathcal{H}^k$-a.e. $x \in E$ it holds

$$\lim_{r \to 0} \frac{\mathcal{H}^k(E \cap B(x,r))}{\omega_k r^k} > 0$$

not necessarily equal to 1!

then $k$ is integer and $E$ is $k$-rectifiable.

**Theorem:** $E$ $\mathcal{H}^k$-measurable with $\mathcal{H}^k(E) < +\infty$. Then $E$ is $k$-rectifiable $\Leftrightarrow$ the approximate tangent space $\mathrm{Tan}(E,x)$ exists for $\mathcal{H}^k$-a.e. $x \in E$.

## RECTIFIABLE CURRENTS

**Def:** Let $E$ be a $k$-rectifiable set in $\mathbb{R}^n$, let $\tau : E \to \Lambda_k(\mathbb{R}^n)$ be an **orientation** of $E$, i.e., an $\mathcal{H}^k$-measurable map that to $\mathcal{H}^k$-a.e. $x \in E$ associates a unit norm simple $k$-vector $\tau(x) = \tau_1(x) \wedge \dots \wedge \tau_k(x)$ such that $\mathrm{span}(\tau_1(x), \dots, \tau_k(x)) = \mathrm{Tan}(E,x)$, where $\mathrm{Tan}(E,x)$ is the approximate tangent space of $E$ at $x$. Let $m \in L^1_{\mathcal{H}^k}(E;\mathbb{R})$ be a **multiplicity**. Then we can define the current

$$[E,\tau,m](\omega) := \int_E m \langle \omega, \tau \rangle \, d\mathcal{H}^k, \quad \omega \in \mathcal{D}^k(\mathbb{R}^n).$$

Currents with the structure $[E,\tau,m]$ are called **$k$-rectifiable currents**.
If $m \in L^1_{\mathcal{H}^k}(E;\mathbb{Z})$, then $[E,\tau,m]$ is a **$k$-rectifiable current with integer multiplicity**.

If $T$ and $\partial T$ are currents with integer multiplicity (rectifiable), then $T$ is called integral current.

**Theorem** (Compactness of integral currents, Federer & Fleming): Let $T_j$ be a sequence of integral $k$-currents with

$$\mathbb{M}(T_j) + \mathbb{M}(\partial T_j) \leq C < +\infty .$$

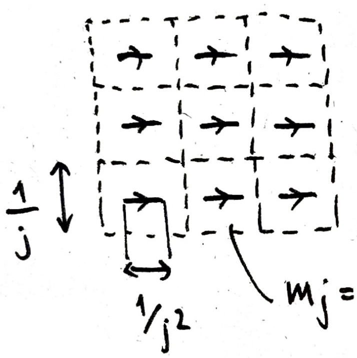
Then, up to a subsequence, $T_j \rightarrow T$ in $\mathcal{D}_k(\mathbb{R}^n)$ where $T$ is an integral $k$-current. [No proof]

**Corollary**: Let $T_0$ be an integral $k$-current in $\mathbb{R}^n$. Then the Plateau problem

$$\min\{\mathbb{M}(T) : T \text{ integral current}, \ \partial T = \partial T_0\}$$

has a solution.

**Example 1** (Importance of $\mathbb{M}(\partial T_j) \leq C < +\infty$):

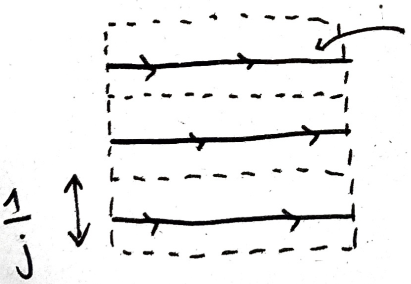


$m_j = 1, \ \tau_j = e_1$

$T_j \rightharpoonup e_1 \mathcal{L}^2 \llcorner [0,1]^2$

$\mathbb{M}(\partial T_j) = 2j^2$

**Example 2** (Importance of integer multiplicity):

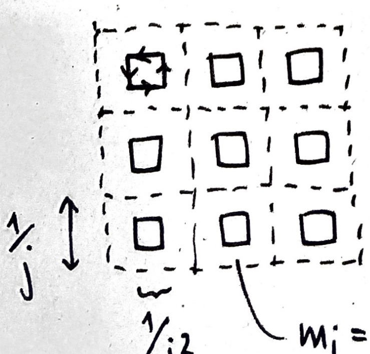


$m_j = \frac{1}{j}$

$e_1 = \tau_j$

$T_j \rightharpoonup e_1 \mathcal{L}^2 \llcorner [0,1]^2$

**Example 3** (Importance of orientation):



$m_j = 1 \quad \tau_j$

$T_j \rightharpoonup 0$ in $\mathcal{D}_1(\mathbb{R}^2)$ and F&F applies!

However $\mathcal{H}^k \llcorner E_j \overset{*}{\rightharpoonup} \mathcal{L}^2 \llcorner [0,1]^2$.

**Theorem** (Boundary rectifiability): Let $T$ be an integer multiplicity rectifiable current with $\mathbb{M}(\partial T) < +\infty$. Then $T$ is an integral current.
[No proof]