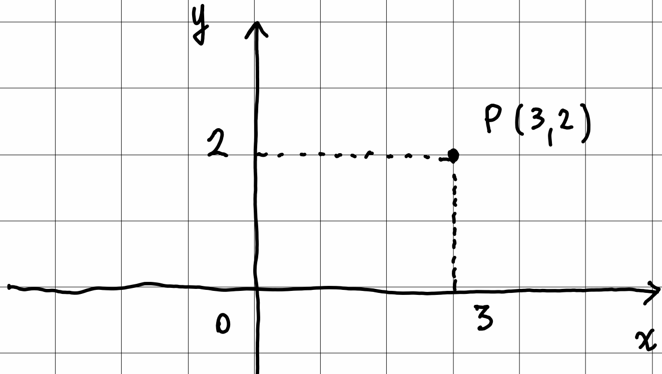
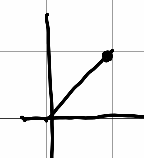
PRECORSO DI MATEMATICA 2021/2022 – CLASSE N

GEOMETRIA ANALITICA

LEZ. 4
02/10/2021

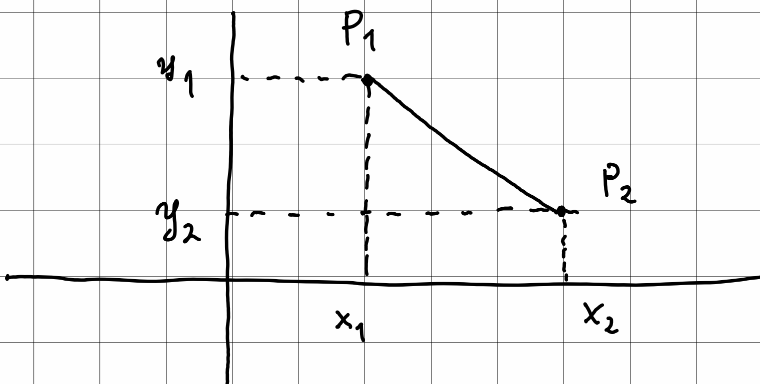


$P = (x, y)$



$P_1 = (x_1, y_1)$

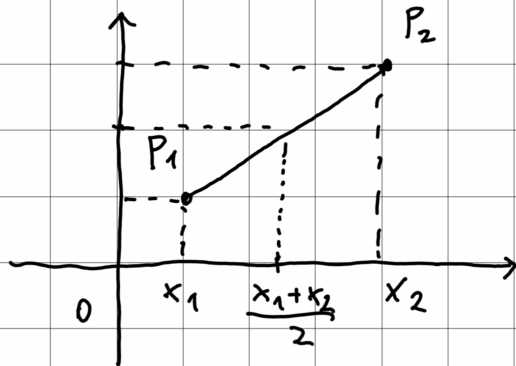
$P_2 = (x_2, y_2)$



$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$P_1 = (0,0) \qquad P_2 = (1,1)$

$d(P_1, P_2) = \sqrt{2}$



$$M = \left(\frac{x_1+x_2}{2} , \frac{y_1+y_2}{2} \right)$$

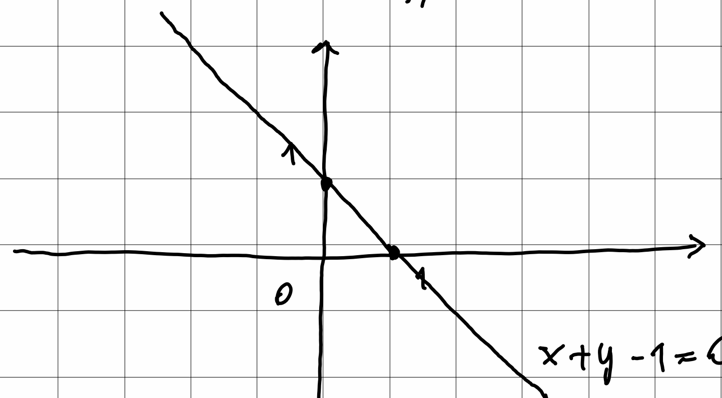
punto medio

Rette

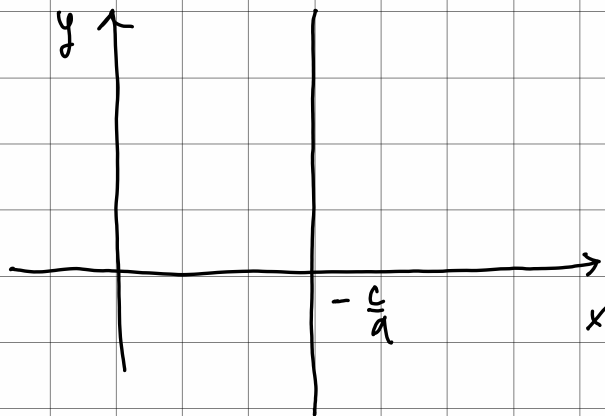
(Forma implicita) $\quad ax + by + c = 0$

$a, b, c \in \mathbb{R} \quad a \neq 0$ oppure $b \neq 0$

Esempio: $\quad x + y - 1 = 0$



Se $b = 0$ $\qquad ax + c = 0 \qquad x = -\frac{c}{a}$



Se $b \neq 0$ si può scrivere l'eq. della retta in forma esplicita

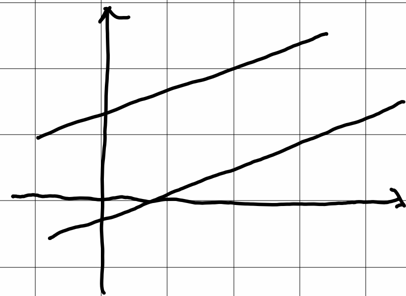
$$ax + by + c = 0 \iff by = -ax - c \iff y = \underbrace{-\frac{a}{b}}_{m} x \underbrace{-\frac{c}{a}}_{q}$$

$$y = mx + q$$

Intersezione tra due rette:

$$\begin{cases} ax+by+c=0 \\ a'x+b'y+c'=0 \end{cases}$$

Se $a'b = ab'$ le due rette sono parallele e non si intersecano.



Supponiamo, ad esempio, che $b' \neq 0$

$$a = \underbrace{\frac{b}{b'}}_{\lambda} a' = \lambda a'$$

$$\lambda = \frac{b}{b'} \Rightarrow b = \lambda b'$$

$$\begin{cases} \lambda a'x + \lambda b'y + c = 0 \\ a'x + b'y + c' = 0 \end{cases}$$

$$\begin{cases} \lambda a'x + \lambda b'y + c = 0 \\ \lambda a'x + \lambda b'y + \lambda c' = 0 \end{cases}$$

Retta passante per due punti:

$$P_1 = (x_1, y_1) \qquad P_2 = (x_2, y_2)$$

$$ax + by + c = 0$$

$$\begin{cases} a x_1 + b y_1 + c = 0 \\ a x_2 + b y_2 + c = 0 \end{cases} \qquad \begin{cases} a(x_2 - x_1) + b(y_2 - y_1) = 0 \\ \text{—} \end{cases}$$

$$a(x_2 - x_1) = -b(y_2 - y_1)$$

$$a = -(y_2 - y_1)$$
$$b = (x_2 - x_1)$$

$$c = -a x_1 - b y_1 = +(y_2 - y_1) x_1 - (x_2 - x_1) y_1$$

$$-(y_2 - y_1) x + (x_2 - x_1) y + (y_2 - y_1) x_1 - (x_2 - x_1) y_1 = 0$$

$$-(y_2-y_1)(x-x_1)+(x_2-x_1)(y-y_1)=0$$

Se $x_2 \neq x_1$

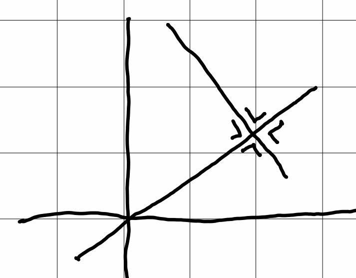
$$y = \underbrace{\frac{y_2-y_1}{x_2-x_1}}_{m}(x-x_1)+y_1$$

Rette perpendicolari

$$ax+by+c=0 \quad \text{e} \quad a'x+b'y+c'=0$$

sono perpendicolari se

$$aa'+bb'=0$$



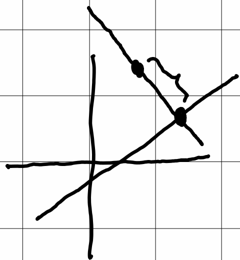
Parallele $\quad a'b = ab'$

$$m = -\frac{a}{b} \qquad m' = -\frac{a'}{b'}$$

$$\frac{a'}{b'} = \frac{a}{b} \qquad -\frac{a'}{b'} = -\frac{a}{b} \qquad m' = m$$

Perpendicolari $\quad aa' + bb' = 0 \Leftrightarrow aa' = -bb'$

$$mm' = \left(-\frac{a}{b}\right)\left(-\frac{a'}{b'}\right) = \frac{aa'}{bb'} = -1$$

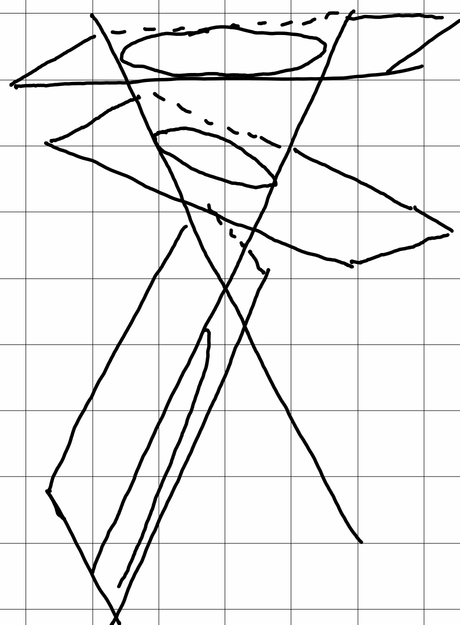


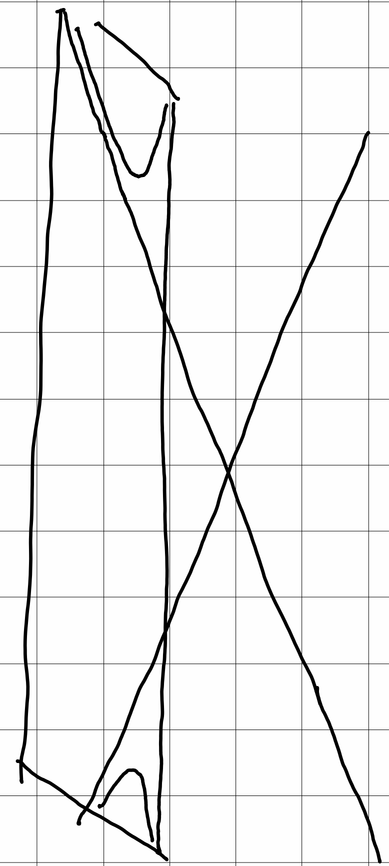
Distanza di un punto da una retta

$$P_0 = (x_0, y_0) \qquad r: \ ax + by + c = 0$$

$$d(P_0, r) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

CONICHE

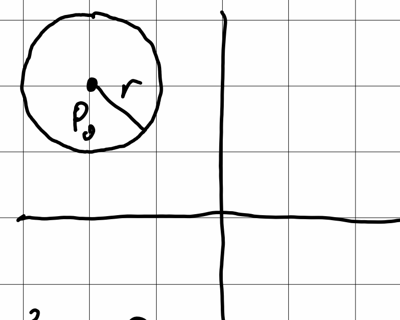




<u>Circonferenza</u>: luogo geometrico dei punti del piano equidistanti da un punto fissato detto centro.

(raggio = distanza dei punti dal centro)

$P_0 = (x_0, y_0) \qquad P = (x, y) \qquad r > 0$



$$\sqrt{(x-x_0)^2 + (y-y_0)^2} = r$$

$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

$$x^2 \underbrace{-\,2x_0}_{a}\, x + x_0^2 + y^2 \underbrace{-\,2y_0}\, y + y_0^2 - r^2 = 0$$

$$x^2 + y^2 + ax + by + c = 0$$

$$ax = \frac{2}{2}ax \qquad + \frac{a^2}{4} - \frac{a^2}{4}$$

Esercizio: Equazione della circonferenza che passa per $(1,2)$ e ha centro $(-1,3)$

$$(x+1)^2+(y-3)^2=r^2$$

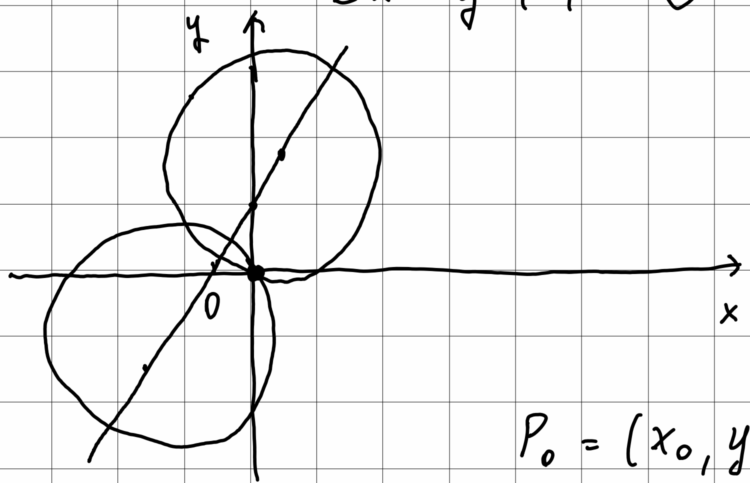
$$r=\sqrt{(1+1)^2+(2-3)^2}=\sqrt{4+1}=\sqrt{5} \qquad r^2=5$$

$$(x+1)^2+(y-3)^2=5$$

$$x^2+2x+1+y^2-6y+9-5=0$$

$$x^2+y^2+2x-6y+5=0$$

Esercizio: Trovare le circonferenze di raggio $\sqrt{2}$ con centro sulla retta di equazione $2x - y + 1 = 0$ e passante per l'origine



$P_0 = (x_0, y_0)$

Il centro soddisfa all'equazione:

$$2x_0 - y_0 + 1 = 0$$

$$y_0 = 2x_0 + 1$$

$r = \sqrt{2}$ $\quad$ $r^2 = 2$

$$(x-x_0)^2+(y-y_0)^2=r^2$$

$$(x-x_0)^2+(y-2x_0-1)^2=2$$

Il punto $(0,0)$ appartiene alla circonferenza, quindi

$$(-x_0)^2+(-2x_0-1)^2=2$$

$$x_0^2+4x_0^2+4x_0+1-2=0$$

$$5x_0^2+4x_0-1=0$$

$$5x_0^2+5x_0-x_0-1=0$$

$$5x_0(x_0+1)-(x_0+1)=0$$

$$(x_0+1)(5x_0-1)=0$$

$x_0=-1$ oppure $x_0=\frac{1}{5}$

$y_0=2\cdot(-1)+1=-1$ o $y_0=\frac{2}{5}+1=\frac{7}{5}$

$$(x+1)^2+(y+1)^2=2$$

oppure

$$\left(x-\frac{1}{5}\right)^2+\left(y-\frac{7}{5}\right)^2=2$$

<u>Esercizio</u>: Trovare il punto di intersezione delle rette $2x+y+1=0$ e $x-3y+2=0$

$$\begin{cases} 2x+y+1=0 \\ x-3y+2=0 \end{cases} \qquad \begin{cases} 2x+y+1=0 \\ 2x-6y+4=0 \end{cases}$$

$$\begin{cases} 2x+y+1=0 \\ 7y-3=0 \end{cases} \qquad \begin{cases} 2x=-\frac{3}{7}-1=-\frac{10}{7} \\ y=\frac{3}{7} \end{cases} \qquad \begin{cases} x=-\frac{5}{7} \\ y=\frac{3}{7} \end{cases}$$

$$\left(-\frac{5}{7}, \frac{3}{7}\right)$$

<u>Esercizio</u>: Trovare la retta passante per $(-1,5)$ e perpendicolare alla retta di eq.

$$y = -x + 7$$

$$x + y - 7 = 0$$

$$ax + by + c = 0$$

$$a(x - x_0) + b(y - y_0) = 0$$

$$a(x+1) + b(y-5) = 0$$

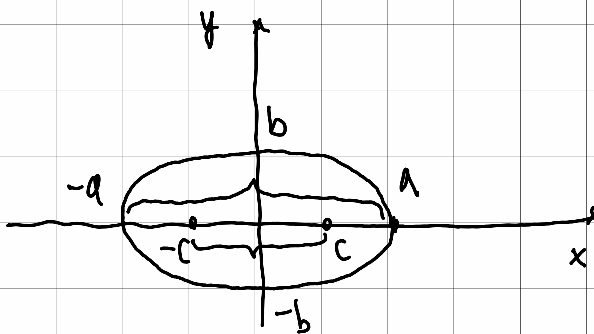
$$a \cdot 1 + b \cdot 1 = 0 \qquad a = -b$$

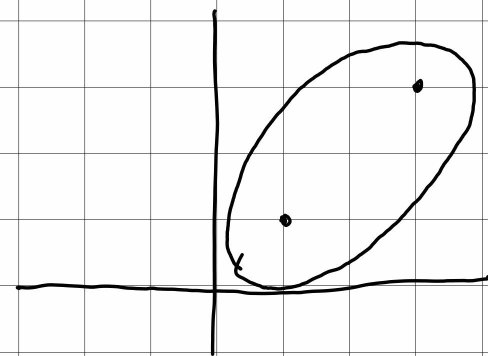
$$a(x+1) - a(y-5) = 0$$

$$x + 1 - y + 5 = 0 \qquad \underline{x - y + 6 = 0}$$

Ellisse: Luogo dei punti per cui è costante la somma delle distanze da due punti detti fuochi.

$F, F' \qquad 2a \qquad 2a > d(F, F')$





$F = (c, 0) \qquad F' = (-c, 0)$

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2+y^2 = 4a^2+(x+c)^2+y^2-4a\sqrt{(x+c)^2+y^2}$$

$$\cancel{x^2}-\underbrace{2xc}+\cancel{c^2}+\cancel{y^2} = 4a^2+\cancel{x^2}+\underbrace{2xc}+\cancel{c^2}+\cancel{y^2}-$$
$$-4a\sqrt{(x+c)^2+y^2}$$

$$\cancel{4}a\sqrt{(x+c)^2+y^2} = \cancel{4}xc+\cancel{4}a^2$$

$$a^2(x+c)^2+a^2y^2 = x^2c^2+a^4+2a^2xc$$

$$\underbrace{a^2x^2}+\cancel{2a^2xc}+\underbrace{a^2c^2}+\underbrace{a^2y^2} = \underbrace{x^2c^2}+\underbrace{a^4}+2\cancel{a^2xc}$$

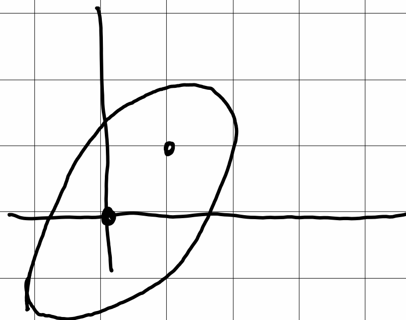
$$\underbrace{(a^2-c^2)}_{b^2}x^2+a^2y^2+a^2(c^2-a^2)=0$$

$$b^2=a^2-c^2 \iff c^2=a^2-b^2$$

$$b^2x^2+a^2y^2-a^2b^2=0$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

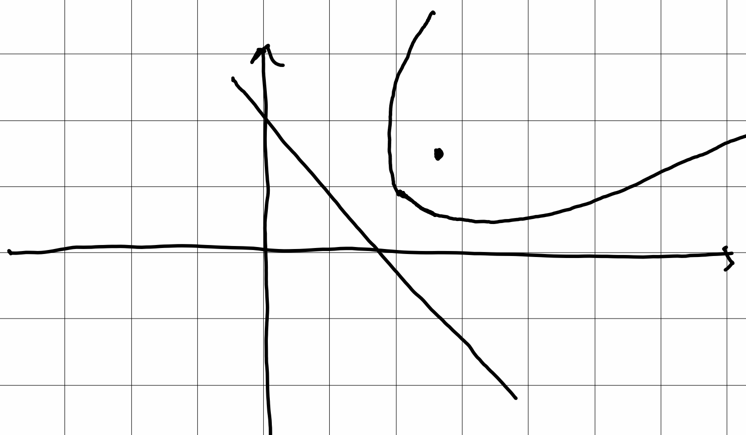
Esercizio: Ellisse con fuochi $(0,0)$, $(1,1)$ e $a=1$

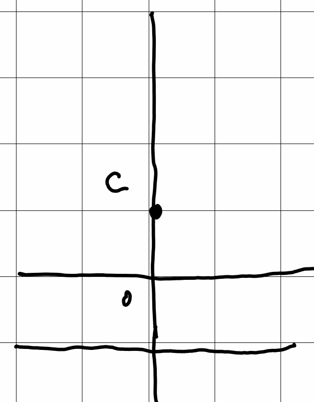


$$\sqrt{x^2+y^2}+\sqrt{(x-1)^2+(y-1)^2}=2$$

$$3x^2+3y^2-2xy-2x-2y-1=0$$

<u>Parabola</u>: Fissata una retta r (direttrice) e un punto $F \notin r$ (fuoco), si dice parabola il luogo dei punti che hanno uguale distanza da F e da r





$F = (0, c)$

$y = -c \quad , \quad y + c = 0$

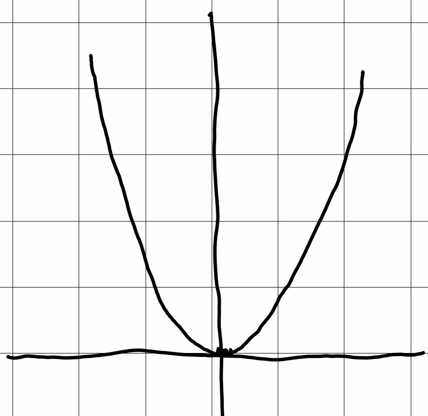
$$\sqrt{x^2 + (y-c)^2} = \frac{|y+c|}{\sqrt{1}}$$

$$x^2 + (y-c)^2 = (y+e)^2$$

$$x^2 + \cancel{y^2} - 2cy + \cancel{c^2} = \cancel{y^2} + 2cy + \cancel{c^2}$$

$$x^2 = 4cy$$

$$y = \underbrace{\frac{1}{4c}}_{a} x^2 = a x^2$$

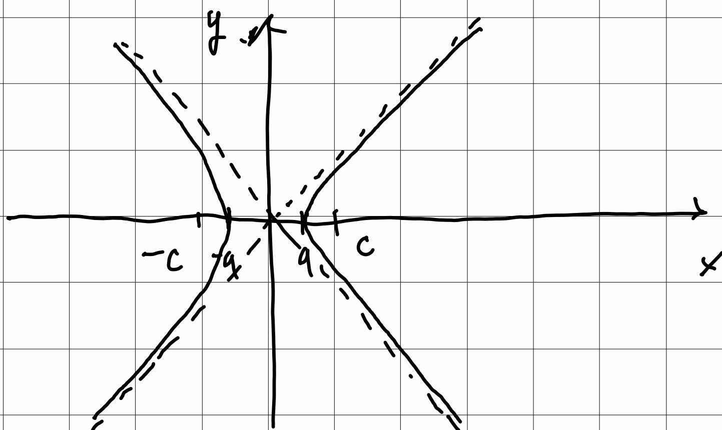


$F = (x_0, y_0)$ $\quad r: ax + by + c = 0$

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} = \frac{|ax+by+c|}{\sqrt{a^2+b^2}}$$

Iperbole Dato $a > 0$ e due punti F, F' (fuochi) con $2a < dist(F, F')$, si dice iperbole il luogo geometrico dei punti per cui è costante $(= 2a)$ la differenza delle distanze dai fuochi.

$F(c, 0)$ $F(-c, 0)$ $2a$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$b^2 = c^2 - a^2$$

$F(0, c)$ $F(0, -c)$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Coniche

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$\Delta = B^2 - 4AC$$

$$x - 4 \leq \sqrt{x^2 - 5x + 6}$$

Definizione: $x^2 - 5x + 6 \geq 0$

$(x-3)(x-2) \geq 0$

$x \leq 2 \,,\, x \geq 3$ $\qquad (-\infty, 2] \cup [3, +\infty)$

I CASO: $x < 4$ sempre vera

$\underline{(-\infty, 2] \cup [3, 4)}$

II CASO: $\underline{4 \leq x}$ $\qquad \underline{x \in [4, +\infty)}$

$\cancel{x^2} + 16 - 8x \leq \cancel{x^2} - 5x + 6$

$10 \leq 3x$ $\qquad \frac{10}{3} \leq x$ ✓ sempre vera

$\frac{10}{3} < 4$

Soluzione $(-\infty, 2] \cup [3, +\infty)$

$$y = -x$$

$$(x - x_0)^2 + (y + x_0)^2 = r^2$$

$$\begin{cases} (-3 - x_0)^2 + (4 + x_0)^2 = r^2 \\ (2 - x_0)^2 + (3 + x_0)^2 = r^2 \end{cases}$$

$$(4 + x_0)^2 - (2 - x_0)^2 = 0$$

$$4 + x_0 = 2 - x_0 \Rightarrow 2x_0 = -2 \Rightarrow x_0 = -1$$

$$4 + x_0 = -2 + x_0 \Rightarrow \text{niente} \qquad y_0 = 1$$

$$(x+1)^2 + (y-1)^2 = r^2$$

Per $x = 2$ e $y = 3$

$$3^2 + 2^2 = r^2 \Rightarrow r^2 = 9 + 4$$

$$x^2 + 2x + 1 + y^2 - 2y + 1 - 13 = 0$$

$$3^{2x+1} - 2 \cdot 3^x - 1 \geq 0$$

$$y = 3^x$$

$$3 \cdot y^2 - 2 \cdot y - 1 \geq 0$$

$$3y^2 - 3y + y - 1 \geq 0$$

$$(3y+1)(y-1) \geq 0 \qquad y \leq -\frac{1}{3} \quad , \quad y \geq 1$$

$$3^x \geq 1 \qquad x \geq 0$$

$$\frac{(|x-2|-5)^2}{x^2+2x-2|x-2|} \geq 0$$

Osservo che $(|x-2|-5)^2 = 0$ per $|x-2| = 5$ cioè

$x-2 = \pm 5$ cioè $x = 8$ o $x = -3$

Devo tenerne conto per $= 0$.

Altrimenti il numeratore è sempre > 0.

Vediamo il segno del denominatore, vogliamo sia > 0.

$x^2 + 2x - 2|x-2| > 0$

I caso: $x \geq 2$ $\quad x \in [2, +\infty)$

$x^2 + 2x - 2x + 4 = x^2 + 4 > 0$ sempre

II caso: $x < 2$ $\quad x \in (-\infty, 2)$

$x^2 + 2x + 2x - 4 = x^2 + 4x - 4$

$\Delta = b^2 - 4ac = 16 + 16 = 32$

Radici: $\dfrac{-4 \pm \sqrt{32}}{2} = \dfrac{-4 \pm \sqrt{2 \cdot 2^4}}{2} = \dfrac{-4 \pm 4\sqrt{2}}{2} =$

$= -2 \pm 2\sqrt{2}$

Mi chiedo se $-2 + 2\sqrt{2} \overset{?}{<} 2$

$2\sqrt{2} \overset{?}{<} 4$

$\sqrt{2} \overset{?}{<} 2$ sì

$\left(x - (-2-\sqrt{2})\right)\left(x - (-2+\sqrt{2})\right) > 0$

$x < -2-\sqrt{2}$ o $x > -2+\sqrt{2}$

$x \in (-\infty, -2-\sqrt{2}) \cup (-2+\sqrt{2}, 2)$

Unisco tutte le soluzioni:

$$x \in (-\infty, -2-2\sqrt{2}) \cup (-2+2\sqrt{2}, +\infty)$$

$$-2+2\sqrt{2} > 0 > -3$$

Mi chiedo $-2-2\sqrt{2} < -3$?

$1 < 2\sqrt{2}$ sì!

Allora dobbiamo aggiungere anche -3!

$$x \in (-\infty, -2-2\sqrt{2}) \cup \{-3\} \cup (-2+2\sqrt{2}, +\infty)$$

$$\frac{|x+1|(x-2)}{x+1} \leq 0$$

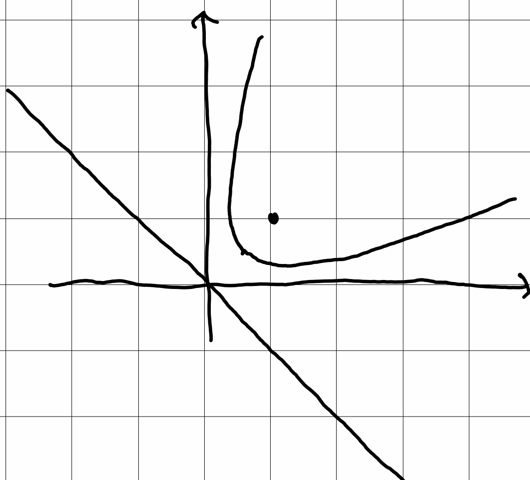
Ben definita per $x \neq -1$, quindi $|x+1| > 0$

Allora devo guardare solo il segno di $(x-2)$ e $(x+1)$.

	-1		2	
$x-2$	$-$	$-$		$+$
$x+1$	$-$	$+$		$+$
	$+$	$-$		$+$

$x \in (-1, 2]$

Esercizio: Trovare la parabola con fuoco $(1,1)$ e direttrice $x+y=0$



$$\sqrt{(x-1)^2+(y-1)^2} = \frac{|x+y|}{\sqrt{1+1}}$$

$$(x-1)^2+(y-1)^2 = \frac{(x+y)^2}{2}$$

$$2x^2-4x+2+2y^2-4y+2 = x^2+y^2+2xy$$

$$x^2+y^2-2xy-4x-4y+4=0$$

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$