CHAPTER 5: $\mathbb{S}^1$-VALUED MAPS

In this last part of the course we shall use Cartesian currents to study the area of $\mathbb{S}^1$-valued maps.

For $u \in C^1(\Omega; \mathbb{S}^1)$, the area is

Note: it works for $\Omega \subset \mathbb{R}^n$, we study here $\Omega \subset \mathbb{R}^2$.

$$\mathcal{A}(u) := \int_\Omega \sqrt{1+|\nabla u|^2}\, dx$$

and we can extend $\mathcal{A}$ to $+\infty$ if u is not C^1.

As for the previous cases, the relaxed area functional is

$$\overline{\mathcal{A}}(u) = \inf \left\{ \liminf_j \mathcal{A}(u_j) : u_j \to u \text{ in } L^1(\Omega; \mathbb{R}^2) \right\}.$$

- We shall see that BV functions with values in $\mathbb{S}^1$ are the domain of $\overline{\mathcal{A}}$, but the expression of $\overline{\mathcal{A}}$ does not only depend on u!
- It is an example where the topology of the target manifold may create some obstructions. Cartesian currents help in detecting these obstructions.

If $\overline{\mathcal{A}}(u) < +\infty$, there exists a sequence $u_j \to u$ with $\mathcal{A}(u_j) \leq C < +\infty$.

Then u_j is equibounded in $W^{1,1} \Rightarrow u \in BV(\Omega; \mathbb{S}^1)$, i.e., $u \in BV(\Omega; \mathbb{R}^2)$ and $|u| = 1$ a.e.

In general, $\overline{\mathcal{A}}(u) \neq \int_\Omega \sqrt{1+|\nabla u|^2}\, dx$. For example, for $u = \frac{x}{|x|}$ we cannot have $\overline{\mathcal{A}}(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\, dx$. Otherwise, there would exist a sequence $u_j \to u$ s.t. $\mathcal{A}(u_j) \to M([G_u])$. We saw that this is not possible, since $[G_{u_j}] \to T \in \operatorname{cart}(\Omega \times \mathbb{R}^2)$, $T = [G_{\frac{x}{|x|}}] + S$ and $M(S) \neq 0$!

This means that

$$M([G_{\frac{x}{|x|}}]) < \liminf_j M([G_{u_j}]) \quad \forall\, u_j \to u$$

$\uparrow$

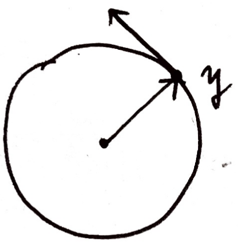
there is a gap

To study this problem we need to intoduce the class of Cartesian currents that "live" on $\Omega \times \mathbb{S}^1$.

We need to recall some facts about forms on manifolds before defining a current on a manifold.

We will mainly use 2-forms on $\Omega\times\mathbb{S}^1$ as test forms. A 2-form on $\Omega\times\mathbb{S}^1$ is a smooth section of the bundle $\Lambda^2 T(\Omega\times\mathbb{S}^1)$. This means that it is an object ω which to each point $(x,y)\in\Omega\times\mathbb{S}^1$ assigns a 2-covector $\omega(x,y)\in\Lambda^2 \mathrm{Tan}(\Omega\times\mathbb{S}^1,(x,y)) = \Lambda^2(\mathbb{R}^2\oplus\mathrm{Tan}(\mathbb{S}^1,y))$. The tangent space to $\mathbb{S}^1$ is a 1-dimensional subspace of $\mathbb{R}^2$, where $\mathbb{S}^1$ is embedded.

If $y\in\mathbb{S}^1$, then $\mathrm{Tan}(\mathbb{S}^1,y)=\mathrm{span}\{-y^2e_1+y^1e_2\}$.



~~2-covectors $\Lambda_2(\mathbb{R}^2\oplus\mathrm{Tan}(\mathbb{S}^1,y))$ have components in the basis $\{e_1\wedge e_2, e_1\wedge\omega_{\mathbb{S}^1}, e_2\wedge\omega_{\mathbb{S}^1}\}$, where $\omega_{\mathbb{S}^1}$ is the 1-form dual to the basis of $\mathrm{Tan}(\mathbb{S}^1,y)$, the so-called volume form. $\omega_{\mathbb{S}^1}(y)=-y^2dy^1+y^1dy^2$, so that $\langle\omega_{\mathbb{S}^1}(y),-y^2e_1+y^1e_2\rangle=1$.~~

2-covectors $\Lambda_2(\mathbb{R}^2\oplus\mathrm{Tan}(\mathbb{S}^1,y))$ have components in the basis ~~$e_1\wedge e_2$~~ $\{dx^1\wedge dx^2, dx^1\wedge\omega_{\mathbb{S}^1}(y), dx^2\wedge\omega_{\mathbb{S}^1}(y)\}$, where $\omega_{\mathbb{S}^1}(y)$ is the 1-covector dual to the basis of $\mathrm{Tan}(\mathbb{S}^1,y)$, i.e., $\omega_{\mathbb{S}^1}(y)=-y^2dy^1+y^1dy^2$. $\omega_{\mathbb{S}^1}$ is the so-called volume form on $\mathbb{S}^1$.

To define currents on an embedded manifold, in this case $\Omega\times\mathbb{S}^1$, let us consider an open set that contains $\mathbb{S}^1$, e.g., $A:=B_2\setminus\overline{B}_{1/2}$. Let $i:\Omega\times\mathbb{S}^1\to\Omega\times A$ be the inclusion map. Then $i^\#:\mathcal{D}^2(\Omega\times A)\to\mathcal{D}^2(\Omega\times\mathbb{S}^1)$. We say that a form $\omega\in\mathcal{D}^2(\Omega\times A)$ is null on $\Omega\times\mathbb{S}^1$ if $i^\#\omega=0$.

<u>Example</u>: The form $y^1dy^1+y^2dy^2$ is null on $\Omega\times\mathbb{S}^1$.

A current $T\in\mathcal{D}_2(\Omega\times A)$ is a current on $\Omega\times\mathbb{S}^1$, $T\in\mathcal{D}_2(\Omega\times\mathbb{S}^1)$, if $T(\omega)=0$ $\forall\,\omega\in\mathcal{D}^2(\Omega\times A)$ null on $\Omega\times\mathbb{S}^1$.

<u>Remark</u>: if $T\in\mathcal{D}_2(\Omega\times\mathbb{S}^1)$, then $\mathrm{supp}\,T\subset\overline{\Omega}\times\mathbb{S}^1$. Indeed, let $\omega\in\mathcal{D}^2(\mathbb{R}^2\times A)$ be such that $\mathrm{supp}\,\omega\subset(\mathbb{R}^2\times A)\setminus(\overline{\Omega}\times\mathbb{S}^1)$. Then $T(\omega)=0$, since $i^\#\omega=0$.

Let $u_j \in C^1(\Omega; S^1)$ be a sequence s.t. $[G_{u_j}] \rightharpoonup T \in \mathrm{cart}(\Omega \times \mathbb{R}^2)$. Then $T \in \mathrm{cart}(\Omega \times S^1)$. Indeed, $[G_{u_j}] \in \mathrm{cart}(\Omega \times S^1)$ and the condition $T(\omega) = 0$ for $i^\# \omega = 0$ is closed under weak convergence. To check that $[G_{u_j}] \in \mathrm{cart}(\Omega \times S^1)$ we do it for $u \in C^1(\Omega; S^1)$ (not a sequence).

Let $u \in C^1(\Omega; S^1)$. Let us fix $\omega \in \mathcal{D}^2(\Omega \times A)$, $i^\# \omega = 0$, and let us prove that $[G_u](\omega) = 0$. ω has components

$$\phi(x,y)\, dx, \quad \phi(x,y)\, dx^i \wedge dy^j, \quad \phi(x,y)\, dy.$$

- $[G_u](\phi(x,y)\, dx) = \int_\Omega \phi(x, u(x))\, dx = 0$

  since $0 = i^\#(\phi(x,y) dx) = \phi(x,y) dx$ just means that $\phi(x,y) = 0$ for $(x,y) \in \Omega \times S^1$

- $[G_u](\phi(x,y)\, dy) = 0$ because $[G_u]$ has no component corresponding to the determinant.

- $[G_u](\phi_1(x,y)\, dx^1 \wedge dy^1 + \phi_2(x,y)\, dx^1 \wedge dy^2) =$

$$= \int_\Omega \phi_1(x, u(x))\, \partial_{x_2} u^1(x) + \phi_2(x, u(x))\, \cancel{dx}\, \partial_{x_2} u^2(x)\, dx$$

$$0 = i^\#(\phi_1 dx^1 \wedge dy^1 + \phi_2 dx^1 \wedge dy^2)(x,y) =$$

$$= \phi_1(x,y)\, dx^1 \wedge dy^1 + \phi_2(x,y)\, dx^1 \wedge dy^2$$

$(x,y) \in \Omega \times S^1$

it means that if you compute this 2-covector in duality with a 2-vector of the type $e_1 \wedge v$ with $v \in \mathrm{Tan}(S^1, y)$ it is $= 0$. Thus

$$\phi_1(x,y)\, v^1 + \phi_2(x,y)\, v^2 = 0.$$

But $\partial_{x_2} u(x) \in \mathrm{Tan}(S^1, u(x))$.

Indeed, from $(u^1)^2 + (u^2)^2 = 1$ we get $u^1 \partial_{x_2} u^1 + u^2 \partial_{x_2} u^2 = 0$

$$\Rightarrow u(x) \cdot \partial_{x_2} u(x) = 0.$$

Def: $\mathrm{cart}(\Omega \times S^1) := \{ T \in \mathrm{cart}(\Omega \times \mathbb{R}^2) : T \in \mathcal{D}_2(\Omega \times S^1) \}$.

There is an easy way to produce a current in $cart(\Omega \times \mathbb{S}^1)$.
Let us consider the covering map

$$\chi : \Omega \times \mathbb{R} \to \Omega \times \mathbb{S}^1$$
$$(x, \theta) \mapsto (x, \cos\theta, \sin\theta)$$

Then

$$\chi^\# : \mathcal{D}^2(\Omega \times \mathbb{S}^1) \to \{\text{2-forms in } \Omega \times \mathbb{R} \text{ with support in } \tilde{\Omega} \times \mathbb{R},\ \tilde{\Omega} \subset\subset \Omega, \text{ with bounded coefficients, } 2\pi\text{-periodic in } \theta\} =: \mathcal{B}^2_{2\pi}(\Omega \times \mathbb{R}).$$

Proposition: Let $\varphi \in BV(\Omega; \mathbb{R})$ and let $G_\varphi := (-1)^2 \partial[\mathcal{S}_\varphi] \in cart(\Omega \times \mathbb{R})$.
Then $T := \chi_\# G_\varphi \in cart(\Omega \times \mathbb{S}^1)$.

Proof: $\chi_\# G_\varphi$ is well-defined, since G_φ has finite mass and we can compute it on $\chi^\# \omega$, for $\omega \in \mathcal{D}^2(\Omega \times \mathbb{S}^1)$, which has bounded coefficients.

We have to be more precise in the definition of T, since we have to test it with $\omega \in \mathcal{D}^2(\Omega \times A)$.
Let $i : \Omega \times \mathbb{S}^1 \to \Omega \times A$ be the inclusion map.
For $\omega \in \mathcal{D}^2(\Omega \times A)$ we define:

$$T(\omega) := i_\# \chi_\# G_\varphi(\omega) = G_\varphi(\chi^\# i^\# \omega).$$

T satisfies all the assumptions:

- $T \in \mathcal{D}_2(\Omega \times \mathbb{S}^1)$. If $i^\# \omega = 0$, by definition, $T(\omega) = 0$.
- T is rectifiable with integer multiplicity. This is a general fact (not trivial): it is the image through $\chi_\#$ of a rectifiable current.
- $M(T) < +\infty$. Let $\omega \in \mathcal{D}^2(\Omega \times A)$ with $\|\omega\|_\infty \leq 1$. Then

  $|\omega| = |\chi^\# i^\# \omega|$ and thus $M(T) \leq M(G_\varphi) < +\infty$.

  use $\chi^\# \omega_{\mathbb{S}^1} = d\theta$
- $T^h \geq 0$.

  $$T(\phi(x,y)\,dx) = G_\varphi(\chi^\# i^\# \phi(x,y)\,dx) \geq 0.$$
- $\|T\|_1 < +\infty$; $T(\phi(x,y) y^i dx) = G_\varphi(\chi^\# i^\# (\phi(x,y) y^i dx))$

- $\pi_\# T = \pi_\# i_\# \chi_\# G_\varphi = \pi_\# G_\varphi = [\Omega]$.
- $\partial T(\omega) = T(d\omega) = i_\# \chi_\# G_\varphi(d\omega) = G_\varphi(\chi^\# i^\# d\omega) = G_\varphi(d\,\chi^\# i^\# \omega) = 0$.

$\square$

We shall see that every $T \in \operatorname{cart}(\Omega\times S^1)$ is locally of this form. This allows us to prove a structure theorem for $\operatorname{cart}(\Omega\times S^1)$.

More precisely, assume $T = \chi_\# G_\varphi$. Then

$$T(\phi(x,y)\,d\hat{x}^i\wedge dy^2) = \chi_\# G_\varphi(\phi(x,y)\,d\hat{x}^i\wedge dy^2) = G_\varphi(\chi^\#(\phi\, d\hat{x}^i\wedge dy^2)(x,\theta)) =$$

$$= G_\varphi(\phi(x,\cos\theta,\sin\theta)\,d\hat{x}^i\wedge d(\sin\theta)) = \partial[\mathcal{G}_\varphi](\phi(x,\cos\theta,\sin\theta)\cos\theta\, d\hat{x}^i\wedge d\theta) =$$

$$= ([\mathcal{G}_\varphi] + G^c_\varphi + G^j_\varphi)(\underline{\quad}) =$$

$$= (-1)^i \int_\Omega \phi(x,\cos\varphi(x),\sin\varphi(x))\cos\varphi(x)\,\partial^a_{x_i}\varphi(x)\,dx \;+$$

$$+ (-1)^i \int_\Omega \phi(x,\cos\tilde\varphi(x),\sin\tilde\varphi(x))\cos\tilde\varphi(x)\,d D^c_{x_i}\varphi(x) \;+$$

$$+ (-1)^i \int_{J_\varphi}\left(\int_{\varphi^-(x)}^{\varphi^+(x)} \phi(x,\cos\theta,\sin\theta)\cos\theta\,d\theta\right)\nu^i_\varphi(x)\,d\mathcal{H}^1(x) =$$

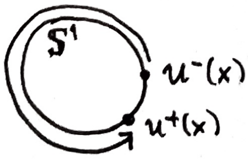
Chain rule:
$u(x) = (\cos\varphi(x), \sin\varphi(x))$ is BV and
$\partial^a_{x_i} u^2(x) = \cos\varphi(x)\,\partial^a_{x_i}\varphi(x)$
$D^c_{x_i} u^2 = \cos\tilde\varphi(x)\,D^c_{x_i}\varphi$

$$= (-1)^i \int_\Omega \phi(x,u(x))\,\partial^a_{x_i} u(x)\,dx + (-1)^i \int_\Omega \phi(x,\tilde u(x))\,dD^c_{x_i} u(x) \;+$$

$$+ (-1)^i \int_{J_\varphi}\left[\left(\int_{geo(u^-(x),u^+(x))} \phi(x,y)\,dy^2\right) + \left(K(x)\int_{S^1}\phi(x,y)\,dy^2\right)\right]\nu^i_\varphi(x)\,d\mathcal{H}^1(x)$$

where u jumps, $\nu_\varphi(x) = \nu_u(x)$ and $J_u \subset J_\varphi$.

$u^\pm(x) = (\cos\varphi^\pm(x), \sin\varphi^\pm(x))$
$u^+(x), u^-(x) \in S^1$. The angles $\varphi^-(x)$ and $\varphi^+(x)$ may be far apart and the image of $[\varphi^-(x),\varphi^+(x)]$ (the interval) is an arc on S^1 which is not necessarily the geodesic arc connecting $u^-(x)$ and $u^+(x)$. It may also be an arc which covers S^1 more than once (if $|\varphi^+(x)-\varphi^-(x)| > 2\pi$). Then we can decompose this arc as the geodesic arc connecting $u^-(x)$ and $u^+(x)$ plus an integer number of turns around S^1.



In the end:

$$T = [\mathcal{G}_u] + G^c_u + G^j_u + L\times[S^1]$$

where L is the i.m. 1-rectifiable current in Ω concentrated on J_φ, with multiplicity K, oriented by $\hat\tau^i = (-1)^i \nu^i_\varphi$, i.e., τ = normal to ν_φ.

To show that every $T \in \mathrm{cart}(\Omega\times \mathbb{S}^1)$ has this structure, we have to show that (locally) $T = \chi_\# G_\varphi$. We need to construct φ in some sense from "nowhere". The idea is the following: in the scalar case we can identify the function φ by taking the boundary of the current $[\mathcal{Y}_\varphi]$, which is a current in the top dimension. If $T = \chi_\# G_\varphi$, then

$$T = \chi_\#(\partial[\mathcal{Y}_\varphi]) = \chi_\#(\partial[\mathcal{Y}_\varphi] - \partial[\mathcal{Y}_0] + \partial[\mathcal{Y}_0]) = \chi_\#(\partial\underbrace{([\mathcal{Y}_\varphi]-[\mathcal{Y}_0])}_{\text{this now has finite mass}}) + \chi_\# \partial[\mathcal{Y}_0] =$$

we want to exchange these two

We have to pay attention: $[\mathcal{Y}_\varphi]$ has not finite mass, we cannot integrate it on forms $\chi^\#\omega$ which only have bounded coefficients!

$$= \partial \chi_\#([\mathcal{Y}_\varphi]-[\mathcal{Y}_0]) + \underbrace{G_{e_1}}_{[G_{e_1}]} \quad \Rightarrow \quad T - G_{e_1} = \partial\Sigma \text{ with } \Sigma \in \mathcal{D}_3(\Omega\times\mathbb{S}^1).$$

Theorem: Assume that $T\in \mathrm{cart}(\Omega\times\mathbb{S}^1)$ and that there exists a $\Sigma\in\mathcal{D}_3(\Omega\times\mathbb{S}^1)$ with finite mass such that $T - G_{e_1} = \partial\Sigma$. Then T can be lifted, i.e., there exists a $\varphi\in BV(\Omega;\mathbb{R})$ such that $\chi_\# G_\varphi = T$.

~~Example: $\frac{x}{|x|}$ cannot be lifted to a $W^{1,1}$ function, but $T = [G_{\frac{x}{|x|}}] - [L]\times[\mathbb{S}^1]$ $\in \mathrm{cart}(\Omega\times\mathbb{S}^1)$~~

The proof of the previous result is based on the following observation: $\Omega\times\mathbb{S}^1$ is a manifold of dimension 3, Σ is a 3-current, normal. Then Σ is a BV-function.

This is based on a general result about currents called the constancy theorem, which we prove here.

Theorem (Constancy theorem): Let $U\subset\mathbb{R}^k$ be a connected open set and let $S\in\mathcal{D}_k(U)$ be such that $\partial S = 0$.
Then $\exists\, c\in\mathbb{R}$ s.t. $S = c[U]$.
(there exists, up to a multiplicative constant, a unique boundaryless current in the top dimension!)

Proof: Let us consider the distribution $\Lambda\in\mathcal{D}'(U)$ defined for every $\phi\in C_c^\infty(U)$ by

$$\Lambda(\phi) := S(\phi\, dx).$$

We can relate the distributional derivative of Λ with the boundary of the current S. Let $\omega = \phi\, d\hat{x}^i \in \mathcal{D}^{k-1}(U)$.

$$0 = \partial S(\phi\, d\hat{x}^i) = S(d(\phi\, d\hat{x}^i)) = (-1)^{i-1} S(\partial_{x_i}\phi\, dx) = (-1)^{i-1}\Lambda(\partial_{x_i}\phi) =$$
$$= (-1)^i \partial_{x_i}\Lambda(\phi).$$

This means that the distributional derivative of Λ is zero in the connected open set $U \Rightarrow \Lambda$ is a constant (proven, e.g., by convolution!).
Then

$$S(\phi\,dx) = \Lambda(\phi) = c\int_U \phi\,dx = c[U](\phi\,dx).$$

$\square$

In the proof we found a precise relation between Λ and S. This allows us to prove this more general result:

Theorem: Let $U \subset \mathbb{R}^k$ be an open set. Let $S \in \mathcal{D}_k(U)$ be a normal current. Then there exists a $m \in BV_{loc}(U)$ with $|Dm|(U) < +\infty$ s.t. $S = m[U]$.

Proof: $(-1)^i \partial_{x_i}\Lambda = \partial S(\cdot\, d\hat{x}^i) \leftarrow$ a measure. Hence the distributional derivative $D\Lambda$ is a measure.
Let $\Lambda_\varepsilon = \rho_\varepsilon * \Lambda$. Let $\phi \in C_c^\infty(U)$ with $\int_U \phi\,dx \neq 0$. Then, by a Poincaré-inequality-type argument:

$$\|\Lambda_\varepsilon\|_{L^{1^*}} \leq \left(\|D\Lambda_\varepsilon\|_{L^1} + |\langle \Lambda_\varepsilon, \phi\rangle|\right) C \quad \text{for some constant } C.$$

$\Rightarrow \Lambda$ is a function in $L^{1^*}_{loc}$! $\square$

The previous result is generalized to the case of currents on manifolds by decomposing the manifold with charts. The version on manifolds is the one that we apply to our problem.

Proof (of lifting of Cartesian currents in $\Omega \times \mathbb{S}^1$):
Let us assume that $T - G_{e_1} = \partial\Sigma$ with $\Sigma \in \mathcal{D}_3(\Omega \times \mathbb{S}^1)$.
There exists a $m \in BV(\Omega \times \mathbb{S}^1; \mathbb{R})$ s.t. $\Sigma = m[\Omega \times \mathbb{S}^1]$. Since Σ is with integer multiplicity, up to a translation by a real number, m takes values in $\mathbb{Z}$. (if m has to be translated by $c_0 \in \mathbb{R}$, work with $\Sigma = (m - c_0)[\Omega \times \mathbb{S}^1] + c_0[\Omega \times \mathbb{S}^1]$ and note that $\partial[\Omega \times \mathbb{S}^1] = 0$.)
By the structure theorem we know that $T = [G_{u_T}] + S$, $u_T \in BV(\Omega; \mathbb{S}^1)$.

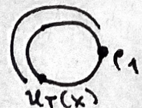
$$T(\phi(x,y)\,dx) = \int_\Omega \phi(x, u_T(x))\,dx.$$

$u_T(x) \in \mathbb{S}^1 \Rightarrow u_T(x) = (\cos\varphi_T(x), \sin\varphi_T(x))$, with $\varphi_T(x) \in [0, 2\pi)$.
There is a relation between $\varphi_T(x)$ and $m(x,y)$.
Let $\bar{m}(x,\theta)$ be such that $\bar{m}(x,\theta) = m(x, \cos\theta, \sin\theta)$, $\bar{m}$ 2π-periodic in θ.
On the one hand:

$\Phi(x,\theta) = \phi(x, \cos\theta, \sin\theta)$

$$(T - G_{e_1})(\phi(x,y)\,dx) = \int_\Omega \phi(x, u_T(x)) - \phi(x, e_1)\,dx =$$

$$= \int_\Omega \int_{arc(e_1, u_T(x))} \nabla_T^{\mathbb{S}^1}\phi(x,y)\,\omega_{\mathbb{S}^1}(y)\,dx = \int_\Omega \int_0^{\varphi_T(x)} \partial_\theta \Phi(x,\theta)\,d\theta\,dx$$


e_1
$u_T(x)$

On the other hand:

$$(T-G_{e_1})(\phi(x,y)dx) = \partial\Sigma(\phi(x,y)dx) = \Sigma(\partial_{y^1}\phi\, dx\wedge dy^1 + \partial_{y^2}\phi\, dx\wedge dy^2) =$$

$$= \Sigma((-y^2\partial_{y^1}\phi \cancel{dx} + y^1\partial_{y^2}\phi)\, dx\wedge\omega_{S^1}) = \Sigma(\nabla_\tau^{S^1}\phi\, dx\wedge\omega_{S^1}) =$$

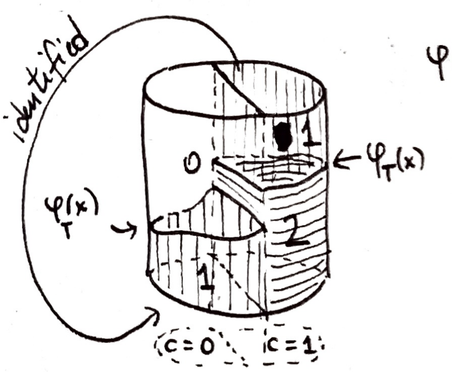
$$= \int_\Omega\int_{S^1}\nabla_\tau^{S^1}\phi(x,y)\, m(x,y)\,\omega_{S^1}(y)\, dx = \int_\Omega\int_0^{2\pi}\partial_\theta\Phi(x,\theta)\cancel{d\theta}\,\bar{m}(x,\theta)\,d\theta\, dx$$

Thus

$$\int_\Omega\int_0^{2\pi}\partial_\theta\Phi(x,\theta)\left(\bar{m}(x,\theta) - \mathbb{1}_{[0,\varphi_T(x)]}(\theta)\right)d\theta\, dx = 0 \quad \forall\,\Phi$$

$$\Rightarrow \bar{m}(x,\theta) - \mathbb{1}_{[0,\varphi_T(x)]}(\theta) = c(x)\in\mathbb{Z}$$

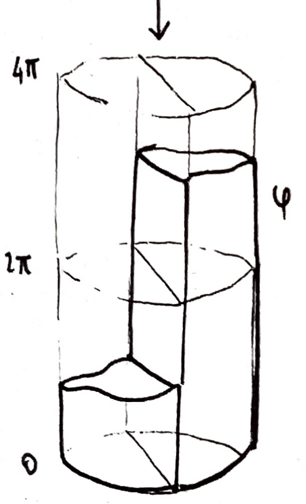
$$\Rightarrow \bar{m}(x,\theta) = \mathbb{1}_{[0,\varphi_T(x)]}(\theta) + c(x).$$



$$\varphi(x) := 2\pi c(x) + \varphi_T(x) = \int_0^{2\pi}\bar{m}(x,\theta)\,d\theta = \int_{S^1} m(x,y)\,\omega_{S^1}(y)$$

Now we can prove that $\Sigma = \chi_\#([\mathcal{S}_\varphi]-[\mathcal{S}_0])$

assuming $m\in\mathbb{Z}$, otherwise add $c_0[\Omega\times S^1]$.



If we prove that $\Sigma = \chi_\#([\mathcal{S}_\varphi]-[\mathcal{S}_0])$, then

$$T - G_{e_1} = \partial\Sigma = \partial\chi_\#([\mathcal{S}_\varphi]-[\mathcal{S}_0]) = \chi_\#\partial([\mathcal{S}_\varphi]-[\mathcal{S}_0]) =$$

$$= \chi_\# G_\varphi - G_{e_1} \Rightarrow T = \chi_\# G_\varphi.$$

To prove that $\Sigma = \chi_\#([\mathcal{S}_\varphi]-[\mathcal{S}_0])$:

$$\Sigma(\phi(x,y)dx\wedge\omega_{S^1}) = \int_\Omega\int_{S^1} m(x,y)\,\phi(x,y)\,\omega_{S^1}(y)\,dx =$$

$$= \int_\Omega\int_0^{2\pi}\bar{m}(x,\theta)\,\Phi(x,\theta)\,d\theta\,dx.$$

$$\chi_\#([\mathcal{S}_\varphi]-[\mathcal{S}_0])(\phi(x,y)dx\wedge\omega_{S^1}) = ([\mathcal{S}_\varphi]-[\mathcal{S}_0])(\Phi(x,\theta)\,dx\wedge d\theta) =$$

$$= \int_\Omega\int_0^{\varphi(x)}\Phi(x,\theta)\,d\theta\,dx = \int_\Omega\left(c(x)\int_0^{2\pi}\Phi(x,\theta)d\theta + \int_0^{\varphi_T(x)}\Phi(x,\theta)d\theta\right)dx =$$

$$= \int_\Omega\int_0^{2\pi}\left(c(x) + \mathbb{1}_{[0,\varphi_T(x)]}(\theta)\right)\Phi(x,\theta)\,d\theta\,dx = \int_\Omega\int_0^{2\pi}\bar{m}(x,\theta)\,\Phi(x,\theta)\,d\theta\,dx.$$

$\square$

The problem of lifting $T \in \mathrm{cart}(\Omega \times \mathbb{S}^1)$ then reduces to the problem of finding $\Sigma \in \mathcal{D}_3(\Omega \times \mathbb{S}^1)$ s.t. $T - G_{e_1} = \partial \Sigma$.

To define Σ given T, we try to understand which forms are enough to completely determine T.

It's not enough to have $T(\phi(x,y)\,dx) = 0 \ \forall \phi$ to conclude that $T = 0$ (in contrast to the case ~~T~~ of $\mathrm{cart}(\Omega \times \mathbb{R})$!). We have to check that $T = 0$ on other forms which are also vertical. Think about $T = [G_{\frac{x}{|x|}}] + [L] \times [\mathbb{S}^1]$: the vertical part is not determined by the horizontal part. However, we have basically to check the behaviour of T on vertical circles everywhere in Ω:

Proposition: Let $T \in$ ~~$\mathrm{cart}(\Omega \times \mathbb{S}^1)$~~ $\mathcal{D}_2(\Omega \times \mathbb{S}^1)$ with $\partial T = 0$. Then $T = 0$ if and only if:

- $T(\phi(x,y)\,dx) = 0 \quad \forall \phi \in C^\infty_c(\Omega \times \mathbb{S}^1)$.
- $T(\alpha \wedge \omega_{\mathbb{S}^1}) = 0 \quad \forall \alpha \in \mathcal{D}^1(\Omega)$.

Proof: (idea): Let $\phi(x,y)\, d\hat{x}^i \wedge \omega_{S^1}$ be a vertical form.

$$\bar{\Phi}(x) := \fint_{S^1} \phi(x,y)\, \omega_{S^1}(y). \qquad \phi(x,y) d\hat{x}^i \wedge \omega_{S^1} = \bar{\Phi}(x) d\hat{x}^i \wedge \omega_{S^1} + (\phi(x,y) - \bar{\Phi}(x))\, d\hat{x}^i \wedge \omega_{S^1}$$

$T(\phi(x) d\hat{x}^i \wedge \omega_{S^1}) = 0$ by assumption.

To show that $T((\phi(x,y) - \bar{\Phi}(x))\, d\hat{x}^i \wedge \omega_{S^1}) = 0$ we write this form as a differential.

$$-\eta(x,y) := \left(\int_{\mathrm{arc}(e_1, y)} (\phi(x,z) - \bar{\Phi}(x))\, \omega_{S^1}(z) \right) d\hat{x}^i \Rightarrow d^{S^1}_y \eta(x,y) = (\phi(x,y) - \bar{\Phi}(x))\, d\hat{x}^i \wedge \omega_{S^1}$$

$d_x \eta$ is horizontal. Thus $\partial T(\eta) = 0 \Rightarrow T(d^{S^1}_y \eta) = 0$

$$d_y \eta = -\partial_{y^1}\eta\, d\hat{x}^i \wedge dy^1 - \partial_{y^2}\eta\, d\hat{x}^i \wedge dy^2 = (+y^2 \partial_{y^1}\eta - y^1 \partial_{y^2}\eta)\, d\hat{x}^i \wedge \omega_{S^1} = -d^{S^1}_y \eta$$

$\square$

Based on this we can define Σ on these forms in such a way that

1) $\partial\Sigma(\phi(x,y)\,dx) = T(\phi(x,y)dx) - G_{e_1}(\phi(x,y)dx)$

2) $\partial\Sigma(\alpha\wedge\omega_{S^1}) = T(\alpha\wedge\omega_{S^1}) \quad \forall\ \alpha\in\mathcal{D}^1(\Omega)$

For 1) we need:

$$\Sigma(\nabla_y^{S^1}\phi(x,y)\,dx\wedge\omega_{S^1}) = T(\phi(x,y)dx) - G_{e_1}(\phi(x,y)dx) =$$
$$= T(\phi(x,y)dx) - \int_\Omega \phi(x,e_1)\,dx =$$
$$= T((\phi(x,y)-\phi(x,e_1))dx)$$

i.e., $\Sigma(d_y^{S^1}\eta) = T(\eta) \quad \forall\ \eta\in\mathcal{D}^2(\Omega\times S^1)$ with $\eta(x,e_1)=0$.

For 2) we need:

$$\Sigma(d\alpha\wedge\omega_{S^1}) = T(\alpha\wedge\omega_{S^1}).$$

A form $\omega\in\mathcal{D}^3(\Omega\times S^1)$ can be split:

$$\omega = \overline{\omega}\wedge\omega_{S^1} + d_y^{S^1}\eta$$

where $\overline{\omega}\in\mathcal{D}^2(\Omega)$, $\eta\in\mathcal{D}^2(\Omega\times S^1)$ with $\eta(x,e_1)=0$.

If $\overline{\omega}$ where a differential, then $\overline{\omega}=d\alpha$ and we could define

$$\Sigma(\omega) := \Sigma(d\alpha\wedge\omega_{S^1} + d_y^{S^1}\eta) := T(\alpha\wedge\omega_{S^1}+\eta)$$

so that 1) and 2) are satisfied.

But, in general, $\overline{\omega}$ is not exact. Indeed, a necessary condition for being the differential of a form with compact support is

$$\int_\Omega \overline{\omega} = \int_\Omega d\alpha = \int_{\partial\Omega}\alpha = 0 \quad \text{if } \alpha\in\mathcal{D}^1(\Omega),$$

i.e., $\overline{\omega}$ has average zero. This might not be true. However, one can prove that by removing the average $\overline{\omega}_\Omega = \int_\Omega\overline{\omega}$ there exists a 1-form α of class C^1 in Ω with $\alpha=0$ on $\partial\Omega$ and such that $d\alpha = \overline{\omega} - \overline{\omega}_\Omega dx$. Then we can define

$$\Sigma(\omega) := \Sigma((\overline{\omega}-\overline{\omega}_\Omega dx)\wedge\omega_{S^1} + \overline{\omega}_\Omega dx\wedge\omega_{S^1} + d_y^{S^1}\eta) :=$$
$$:= T(\alpha\wedge\omega_{S^1} + \overline{\omega}_\Omega\, x^1\, d\hat{x}^1\wedge\omega_{S^1} + \eta).$$

<u>If Ω is simply connected</u>, this definition is well-posed. Indeed, let α' be another primitive of ~~α~~ $\overline{\omega}-\overline{\omega}_\Omega dx$. Then $d(\alpha-\alpha')=0$. By simply connectedness, $\alpha-\alpha'=d\beta$ for a function β. $T((\alpha-\alpha')\wedge\omega_{S^1}) = T(d\beta\wedge\omega_{S^1}) =$
$= \partial T(\beta\,\omega_{S^1}) = 0$.

In conclusion:

Theorem: Assume Ω simply connected in $\mathbb{R}^2$ and let $T \in \mathrm{cart}(\Omega \times \mathbb{S}^1)$. Then there exists $\varphi \in BV(\Omega; \mathbb{R})$ s.t. $\chi_\# G_\varphi = T$.

As already explained, applying the previous result locally we obtain the structure of $\mathrm{cart}(\Omega \times \mathbb{S}^1)$: $T = G_u + L \times [\mathbb{S}^1]$, where G_u denotes the extended graph of a function in $BV(\Omega; \mathbb{S}^1)$:
$G_u = G_u^a + G_u^c + G_u^j$, $G_u^a = [g_u]$, and G_u^j "connects" the jumps $u^-(x)$ and $u^+(x)$ by means of a geodesic arc between $u^-(x)$ and $u^+(x)$ in $\mathbb{S}^1$.

We can express the mass of a current $T = G_u + L \times [\mathbb{S}^1]$. At every $\mathcal{H}^1$-a.e. point x where the 1-rectifiable current L is concentrated we can associate a curve which connects $u^-(x)$, $u^+(x)$ in $\mathbb{S}^1$. It may happen that $u^-(x) = u^+(x)$ if u does not jump. But the curve is in general not the geodesic arc. Let $\gamma_T(x)$ be this curve. Then

$$\mathbb{M}(T) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + |D^c u|(\Omega) + \int_{\mathcal{L}} \cancel{\mathcal{H}^1(\gamma_T(x))\,d\mathcal{H}^1} \underbrace{\mathrm{length}(\gamma_T(x))}\, d\mathcal{H}^1(x)$$

← this has length $2 \cdot 2\pi$. →

not the length of the support of the curve, but the length of the parametrized curve with multiplicity

Let $u_j \in C^1(\Omega; \mathbb{S}^1)$ be a sequence converging in L^1 to u and such that $\mathcal{A}(u_j) \leq C$. Then $[g_{u_j}] \rightharpoonup T = G_u + L \times [\mathbb{S}^1]$ and

$$\underbrace{\int_\Omega \sqrt{1+|\nabla u|^2}\, dx + |D^c u|(\Omega) + \int_{\mathcal{L}} \mathrm{length}(\gamma_T(x))\, d\mathcal{H}^1(x)}_{\mathbb{M}(T)} \leq \liminf_j \mathcal{A}(u_j).$$

Then we can write

$$\inf\{\mathbb{M}(T) : T \text{ s.t. } u_T = u\} \leq \liminf_j \mathcal{A}(u_j)$$

↑ in $\mathrm{cart}(\Omega \times \mathbb{S}^1)$

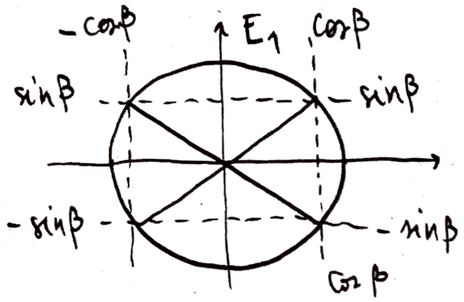
$$\Rightarrow \inf\{\mathbb{M}(T) : T \in \mathrm{cart}(\Omega \times \mathbb{S}^1);\ u_T = u\} \leq \overline{\mathcal{A}}(u).$$

What about the converse inequality?

Proposition: Let $u \in BV(\Omega; S^1)$. Then $\exists\ T \in cart(\Omega \times S^1)$ s.t. $u_T = u$.

Proof: The problem is that if u makes a "vortex" we cannot lift it. So one can split the domain in 4 regions:

$\beta \in (0, \frac{\pi}{2})$



$E_1 := \{x : u^2(x) \geq \sin\beta\}$
$E_2 := \{x : u^1(x) < -\cos\beta\}$
$E_3 := \{x : u^2(x) \leq -\sin\beta\}$
$E_4 := \{x : u^1(x) > \cos\beta\}$

- If $x \in E_1$, we can invert the function cos and find $\varphi(x) \in [\beta, \pi - \beta]$ s.t. $\cos\varphi(x) = u^1(x)$ (and then $\sin\varphi(x) = u^2(x)$).
- $x \in E_2 \Rightarrow \varphi(x) \in (\pi - \beta, \pi + \beta)$ s.t. $\sin\varphi(x) = u^2(x)$
- $x \in E_3 \Rightarrow \varphi(x) \in [\pi + \beta, 2\pi - \beta]$ s.t. $\cos\varphi(x) = u^1(x)$
- $x \in E_4 \Rightarrow \varphi(x) \in [2\pi - \beta, 2\pi + \beta]$ s.t. $\sin\varphi(x) = u^2(x)$.

In this way $(\cos\varphi(x), \sin\varphi(x)) = u(x)$ a.e., so if we define $\chi_\# G_\varphi =: T$ we have $u_T = u$.

We have only to be careful with the definition of φ on these sets. They have to be sets of finite perimeter to guarantee that $\varphi \in BV$. But thanks to the area formula (they are super level sets) for a.e. β they are indeed sets of finite perimeter. □

Theorem: Let $T \in cart(\Omega \times S^1)$. Then there exists a sequence $u_j \in C^1(\Omega; S^1)$ such that $[G_{u_j}] \to T$ in $\mathcal{D}_2(\Omega \times S^1)$ and $M([G_{u_j}]) \to M(T)$.

Proof: If Ω is simply connected: lift T and approximate the BV function. If Ω is not simply connected: careful covering. □

In conclusion:

$$\bar{A}(u) = \inf\{M(T) : T \in cart(\Omega \times S^1);\ u_T = u\}$$

- It is not an integral of the function u
- The term given by $\text{length}(\gamma_T(x))$ taken into account the energy needed by a sequence of smooth functions to generate a "vortex".

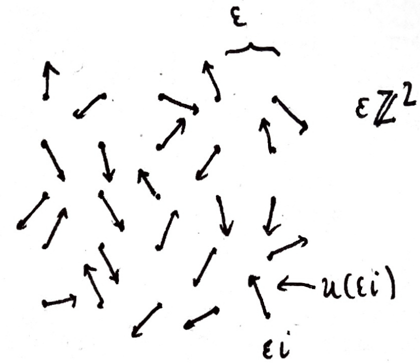
CHAPTER 6: AN APPLICATION TO DISCRETE SPIN SYSTEMS

Let us consider a discrete lattice, for example $\mathbb{Z}^2$. Let $\varepsilon > 0$ be the lattice spacing.

To each point of the lattice εi, $i \in \mathbb{Z}^2$, we associate a vector of the unit circle $u(\varepsilon i)$. Thus we have a map

$$u : \varepsilon\mathbb{Z}^2 \longrightarrow \mathbb{S}^1.$$

This map u is usually called spin field.



In statistical physics one is interested in studying spin systems basing on the temperature of the system. When the temperature is very high, it is very likely that the spins will be oriented in a disordered way. When the temperature is almost zero, the spins will sit on a ground state which depends on the Hamiltonian that governs the system.

An important model in this theory is the XY-model. In this model, the ground state at temperature 0 is given by a configuration of parallel vectors (and all possible rotations).

The energy for the XY-model is thus given by:

$$-\sum_{\langle i,j\rangle} u(\varepsilon i)\cdot u(\varepsilon j)$$

where $\langle i,j\rangle$ means $i,j \in \mathbb{Z}^2$, $|i-j| = 1$. With this energy, two neighboring spins are happy if they are parallel.

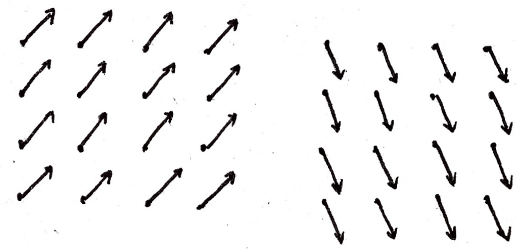
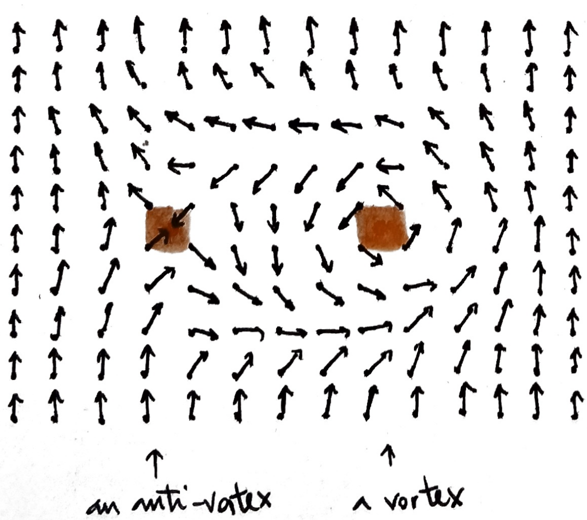


Fig.: two possible ground states for the XY-model.

Berezinskii and then Kosterlitz-Thoulen (Nobel prizes) discovered that this model has another phase transition in between the phase where all the spins are parallel and that where they are oriented in a disordered way.



In this phase vortices appear.

The presence of these vortices and the way they interact one with each other explains many phenomena in superconductivity, for example.

In our analysis we will not consider the temperature of the system. We will do a variational analysis to understand the effective energy when the lattice space $\varepsilon \longrightarrow 0$.

For the variational analysis we will fix an open set $\Omega \subset \mathbb{R}^2$ and we will avarage the energy with the number of points, i.e.,

$$-\sum_{|i-j|=1} \varepsilon^2 u(\varepsilon i)\cdot u(\varepsilon j)$$

And study the asymptotic behaviour of this energy as $\varepsilon \to 0$.
Actually, we shall consider in this course an approximation of the XY-model usually used in simulations. It consists in sampling the unit circle with N points.

$$\mathcal{Y}_N = \{ \exp(\imath \tfrac{2\pi}{N} k) : k = 0, \dots, N-1 \}.$$

$$u : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathcal{Y}_N \subset \mathbb{S}^1$$

This model is usually referred to as the N-clock model (or Potts model).
More precisely, we will consider a $N = N_\varepsilon$ depending on ε and $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We will try to understand which rate of divergence $N_\varepsilon \to +\infty$ really gives an approximation of the XY-model. We start by understanding the two extemal cases.

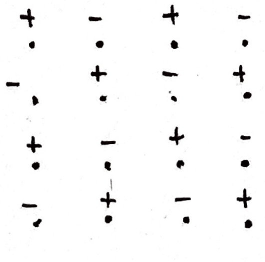
N = 2 : THE ISING MODEL [Alicandro - Braides - Cicalese]

For $N=2$, u takes values in $\{(1,0),(-1,0)\}$ and the scalar product $u(\varepsilon i)\cdot u(\varepsilon j)$ is either 1 or -1. This means that we can think of u as a scalar variable $u : \varepsilon\mathbb{Z}^2 \cap \Omega \to \{1,-1\}$.
Studying the asymptotic behaviour of

removing the minimal energy so that ≥ 0

$$-\sum_{|i-j|=1} \varepsilon^2 u(\varepsilon i)\cdot u(\varepsilon j) \quad \text{or} \quad \sum_{|i-j|=1} \varepsilon^2 (1 - u(\varepsilon i)\cdot u(\varepsilon j))$$

is not that interesting. Let us understand why. Let us consider the sequence:



we associate a piecewise constant function

+1	-1	+1	-1
-1	+1	-1	+1
+1	-1	+1	-1
-1	+1	-1	+1

The energy of this sequence is

$$+\sum_{\substack{|i-j|=1 \\ \varepsilon i, \varepsilon j \in \Omega}} \varepsilon^2 \sim |\Omega| \quad \text{bounded.}$$

This wild sequence which converges w^*-L^∞ to 0 has bounded energy.

When this happens one tries to find a new scaling of the energy which allows for "less wild" sequences. So one looks for an $\eta_\varepsilon \to 0$ and asks that

$$E_\varepsilon(u) = \sum_{|i-j|=1} \varepsilon^2 (1 - u(\varepsilon i)\cdot u(\varepsilon j)) \leq C\eta_\varepsilon .$$

It is often possible to find an interesting η_ε. In this case we notice that a sequence like this

$$\varepsilon\Big\{ \begin{array}{|c|c|c|c|} \hline -1 & -1 & +1 & +1 \\ \hline -1 & -1 & +1 & +1 \\ \hline -1 & -1 & +1 & +1 \\ \hline -1 & -1 & +1 & +1 \\ \hline \end{array} \quad \updownarrow 1$$

pays:

$$\underbrace{\frac{1}{\varepsilon}}_{\substack{\text{no. of} \\ \text{vertical} \\ \text{lines where} \\ \text{there is interaction}}} \overbrace{\varepsilon^2}^{\text{in the energy}} (1 - \underbrace{(-1)}_{(-1)\cdot(+1)}) \sim \varepsilon$$

$\Rightarrow$ The scaling $\eta_\varepsilon = \varepsilon$, usually called surface scaling should be such that ~~energy~~ sequences u_ε with energy

$$E_\varepsilon(u_\varepsilon) \leq C\varepsilon$$

only manifest this behaviour: the separation between the two regions where $u_\varepsilon = -1$ and $u_\varepsilon = +1$ happens on lines.

To prove this fact, let $u_\varepsilon : \varepsilon\mathbb{Z}^2 \cap \Omega \to \{+1,-1\}$ be a sequence with

$$\frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon) \leq C.$$

We identify u_ε with its piecewise constant interpolation.

$$\frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon) = \sum_{|i-j|=1} \varepsilon \frac{1}{2} |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 \underset{\substack{\uparrow \\ |u(\varepsilon i)-u(\varepsilon j)|=2 \\ \text{if } \neq 0}}{=} \sum_{|i-j|=1} \varepsilon |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| = \int_{J_{u_\varepsilon}} |u_\varepsilon^+ - u_\varepsilon^-| \, d\mathcal{H}^1 =$$

$$= |Du_\varepsilon|(\Omega)$$

Thus $u_\varepsilon \to u$ w*-BV. Moreover $u_\varepsilon \to u$ s-L^1 $\Rightarrow$ $u \in \{+1,-1\}$ a.e.

By lower semicontinuity of the total variation:

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon) \geq |Du|(\Omega) = \int_{J_u} 2 \, d\mathcal{H}^1 = 2 \, \mathrm{Per}(\{u=1\};\Omega).$$

This lower bound is too rough! It means that given $u \in BV(\Omega;\{+1,-1\})$ we cannot always find a sequence u_ε s.t. $\lim_\varepsilon \frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon) = 2\,\mathrm{Per}(\{u=1\};\Omega)$.

The problem is that we forgot that the discrete structure has an underlying anisotropy.

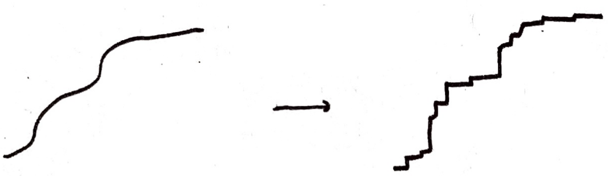
$$\frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon) = \sum_\varepsilon \varepsilon \, |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| = \int_{J_{u_\varepsilon}} |u_\varepsilon^+ - u_\varepsilon^-| \, \underbrace{|\nu_{u_\varepsilon}|_1}\, d\mathcal{H}^1 = 2 \, \mathrm{Per}_1(\{u_\varepsilon = 1\}; \Omega)$$

this is equal to the euclidean norm of the normal for the discrete functions, but in general it is bigger!

$$\liminf_\varepsilon \frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon) \geq 2 \, \mathrm{Per}_1(\{u=1\}; \Omega)$$

$\nwarrow$ anisotropic perimeter $= \int_{\partial^* \{u=1\}} |\nu_u|_1 \, d\mathcal{H}^1$

This is optimal! The proof goes more or less like this: you approximate the set of finite perimeter $\{u=1\}$ with smooth sets, you approximate the smooth set using only $-$ and Γ, and then you discretize on the lattice to define u_ε s.t.

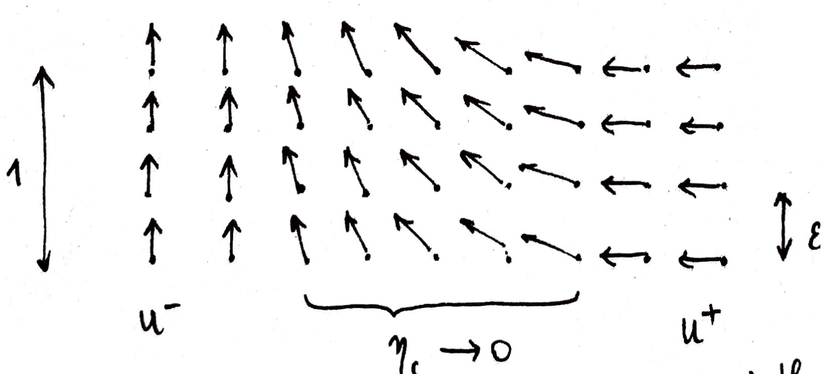


$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon) = 2 \, \mathrm{Per}_1(\{u=1\}; \Omega).$$

This means that the interfacial (anisotropic) energy is the effective energy for $\frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon)$ as $\varepsilon \to 0$.

"N = ∞": THE XY-MODEL

For the XY-model it's not possible to see interfaces, because the spin is free to rotate. Assume that we want to construct a sequence of u_ε which makes a transition from a vector $u^- = \exp(i\varphi^-)$, $u^+ = \exp(i\varphi^+)$.



$u_\varepsilon(\varepsilon i)$ and $u_\varepsilon(\varepsilon j)$ have angles that differ of: $\dfrac{|\varphi^+ - \varphi^-|}{\eta_\varepsilon / \varepsilon}$

$$E_\varepsilon(u_\varepsilon) = \frac{1}{2} \sum \varepsilon^2 \, |u(\varepsilon i) - u(\varepsilon j)|^2 \sim \underbrace{\frac{1}{\varepsilon}}_{\substack{\text{no. of} \\ \text{vertical} \\ \text{inter.}}} \; \underbrace{\frac{\eta_\varepsilon}{\varepsilon} \, 1}_{\substack{\text{horiz.} \\ \text{interactions}}} \; \underbrace{\overbrace{\varepsilon^2}^{\text{in the energy}} \frac{|\varphi^+ - \varphi^-|^2}{\eta_\varepsilon^2} \cdot \varepsilon^2}_{\substack{\text{energy paid} \\ \text{for each inter.}}} \sim |\varphi^+ - \varphi^-|^2 \frac{\varepsilon^2}{\eta_\varepsilon}$$

$$\big(1 - u(\varepsilon i) \cdot u(\varepsilon j)\big)$$

Let us try to rescale the energy by $k_\varepsilon \to 0$.

Then, on this sequence

$$\frac{1}{K_\varepsilon} E_\varepsilon(u_\varepsilon) \sim \frac{\varepsilon^2}{K_\varepsilon \eta_\varepsilon}$$

For every K_ε s.t. $\frac{\varepsilon^2}{K_\varepsilon} \to 0$ we can find a η_ε s.t. $\frac{\varepsilon^2}{K_\varepsilon \eta_\varepsilon} \to 0$.

$$\varepsilon^2 << K_\varepsilon \Rightarrow \frac{\varepsilon^2}{K_\varepsilon} << 1 \Rightarrow \frac{\varepsilon^2}{K_\varepsilon} \ll \eta_\varepsilon << 1.$$

This means that for every scaling K_ε close to ε^2, $\frac{1}{K_\varepsilon} E_\varepsilon$ is asymptotically 0 for a transition on an hypersurface.

It turns out [Alicandro - Cicalese] that an interesting scaling for this energy is $\varepsilon^2 |\log \varepsilon|$.

To understand the behaviour of this energy $\frac{1}{\varepsilon^2 |\lg \varepsilon|} E_\varepsilon$, we make a comparison with a continuous model.

First of all, we notice that

$$\frac{1}{2} \sum \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2 = \frac{1}{2} \varepsilon^2 \sum \varepsilon^2 \left| \frac{u(\varepsilon i) - u(\varepsilon j)}{\varepsilon} \right|^2 =$$

$$\xrightarrow{\text{every int. counted twice}} \simeq \cancel{\tfrac{1}{2}}\, \varepsilon^2 \int_\Omega |\nabla \hat{u}_\varepsilon|^2$$

where $\hat{u}_\varepsilon$ is the piecewise affine interpolation: 

We can estimate how much the piecewise affine interpolation is far from S^1. In a triangle 

$$\hat{u}_\varepsilon(x) = u_\varepsilon(\varepsilon i) + \frac{u_\varepsilon(\varepsilon(i+e_1)) - u_\varepsilon(\varepsilon i)}{\varepsilon}(x_1 - \varepsilon i) + \frac{u_\varepsilon(\varepsilon(i+e_2)) - u_\varepsilon(\varepsilon i)}{\varepsilon}(x_2 - \varepsilon i)$$

Thus $\quad (1 - |\hat{u}_\varepsilon(x)|^2)^2 = |(u_\varepsilon(\varepsilon i) - \hat{u}_\varepsilon(x)) \cdot (u_\varepsilon(\varepsilon i) + u(\varepsilon i))|^2 \leq$

$$\leq C |\nabla \hat{u}_\varepsilon|^2 \varepsilon^2$$

This means that

$$(1+k\sigma) \frac{1}{\varepsilon^2 |\lg \varepsilon|} E_\varepsilon(u_\varepsilon) \geq \frac{1}{|\lg \varepsilon|} \int_\Omega |\nabla \hat{u}_\varepsilon|^2 dx + \frac{\sigma}{\varepsilon^2 |\lg \varepsilon|} \int_\Omega (1 - |\hat{u}_\varepsilon(x)|^2)^2 dx =$$

$$= \frac{1}{|\lg \varepsilon|} GL_\varepsilon(\hat{u}_\varepsilon)$$

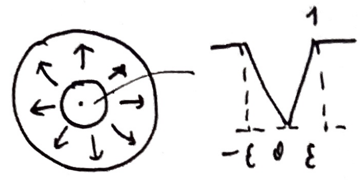
The functional $GL_\varepsilon(u) := \int_\Omega |\nabla u|^2 + \frac{\sigma}{\varepsilon^2} \int_\Omega (1 - |u|^2)^2 dx \quad , u \in H^1(\Omega; \mathbb{R}^2)$

is the Ginzburg - Landau functional.

<u>Ref</u>: [Bethuel, Brezis, Helein], [Sandier, Serfaty], [Jerrard], [Alberti, Baldo, Orlandi], [Alicandro, Ponsiglione], [Jerrard, Soner], ...

For the Ginzburg-Landau, the energy concentrates around vortices. To understand this, let us compute its energy on the sequence

$$u_\varepsilon(x) = \begin{cases} \frac{x}{|x|} & \text{for } x \in B_1 \setminus \overline{B}_\varepsilon \\ \frac{x}{\varepsilon} & \text{for } x \in B_\varepsilon \end{cases}$$



$$\int_\Omega |\nabla u_\varepsilon|^2 dx = \int_{B_1 \setminus \overline{B}_\varepsilon} + \int_{B_\varepsilon} = \int_\varepsilon^1 \int_0^{2\pi} \frac{1}{\rho^2} \rho \, d\theta \, d\rho + \int_0^\varepsilon \int_0^{2\pi} \frac{1}{\varepsilon^2} \rho \, d\theta \, d\rho =$$

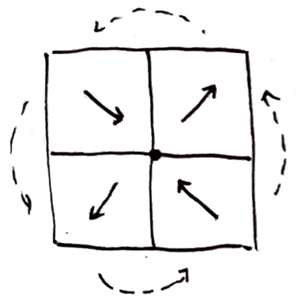
$$\left|\nabla \frac{x}{|x|}\right|^2 = \frac{1}{|x|^2}$$

$$= |\log \varepsilon| \cdot 2\pi + o(1)$$

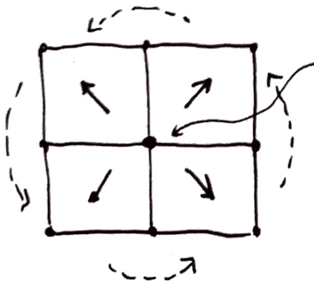
$$\frac{1}{\varepsilon^2} \int_{B_1} (1 - |u_\varepsilon|^2)^2 = \frac{1}{\varepsilon^2} \int_{B_\varepsilon} \left(1 - \left|\frac{x}{\varepsilon}\right|^2\right)^2 \le C.$$

Then $\frac{1}{|\log \varepsilon|} GL_\varepsilon(u_\varepsilon) \to 2\pi$.

In the discrete, i.e., for the XY-model, it's easier to explain what is the relevant object for the asymptotic analysis: it's the discrete vorticity. Let $u_\varepsilon : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{S}^1$ be a spin field. We define a measure which describes the discrete vorticity of u_ε as follows:



vortex with -1



vortex with $+1$

there is a Dirac delta here with $\pm$ sign if, summing the angles ~~of~~ in $[-\pi, \pi]$ between the adjacent vectors in a complete counterclockwise circle, we get $\pm 2\pi$.

The discrete vorticity is the measure μ_{u_ε} given by the sum of all these Dirac deltas.

The measures μ_{u_ε} do not have, in general, equibounded mass, so they do not converge weakly* to a measure. Understanding the topology with respect to which a sequence μ_{u_ε} with $XY(u_\varepsilon) \le C |\log \varepsilon| \varepsilon^2$ is compact is not trivial at all. The point is that many dipoles made by vortex-antivortex $\overset{+1}{\cdot}\ \overset{-1}{\cdot}$ (very close) may appear with equibounded energy and the right topology should identify these dipoles as close to 0. It turns out that the appropriate topology is the flat convergence, for which μ_{u_ε} are interpreted as boundaries of 1-currents.

Then, on this sequence

$$\frac{1}{\kappa_\varepsilon} E_\varepsilon(u_\varepsilon) \sim \frac{\varepsilon^2}{\kappa_\varepsilon \eta_\varepsilon}$$

For every κ_ε s.t. $\frac{\varepsilon^2}{\kappa_\varepsilon} \to 0$ we can find a η_ε s.t. $\frac{\varepsilon^2}{\kappa_\varepsilon \eta_\varepsilon} \to 0$.

$$\varepsilon^2 << \kappa_\varepsilon \Rightarrow \frac{\varepsilon^2}{\kappa_\varepsilon} << 1 \Rightarrow \frac{\varepsilon^2}{\kappa_\varepsilon} \ll \eta_\varepsilon << 1.$$

This means that for every scaling κ_ε close to ε^2, $\frac{1}{\kappa_\varepsilon} E_\varepsilon$ is asymptotically 0 for a transition on an hypersurface.

It turns out [Alicandro - Cicalese] that an interesting scaling for this energy is $\varepsilon^2 |\log \varepsilon|$.

To understand the behaviour of this energy $\frac{1}{\varepsilon^2 |\lg \varepsilon|} E_\varepsilon$, we make a comparison with a continuous model.

First of all, we notice that

$$\frac{1}{2} \sum \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2 = \frac{1}{2} \varepsilon^2 \sum \varepsilon^2 \left| \frac{u(\varepsilon i) - u(\varepsilon j)}{\varepsilon} \right|^2 =$$

every int. counted twice

$$\approx \varepsilon^2 \int_\Omega |\nabla \hat{u}_\varepsilon|^2$$

where $\hat{u}_\varepsilon$ is the piecewise affine interpolation: 

We can estimate how much the piecewise affine interpolation is far from $\mathbb{S}^1$. In a triangle 

$$\hat{u}_\varepsilon(x) = u_\varepsilon(\varepsilon i) + \frac{u_\varepsilon(\varepsilon(i+e_1)) - u_\varepsilon(\varepsilon i)}{\varepsilon}(x_1 - \varepsilon i) + \frac{u_\varepsilon(\varepsilon(i+e_2)) - u_\varepsilon(\varepsilon i)}{\varepsilon}(x_2 - \varepsilon i)$$

Thus

$$(1 - |\hat{u}_\varepsilon(x)|^2)^2 = |(u_\varepsilon(\varepsilon i) - \hat{u}_\varepsilon(x)) \cdot (u_\varepsilon(\varepsilon i) + u(\varepsilon i))|^2 \leq$$

$$\leq C |\nabla \hat{u}_\varepsilon|^2 \varepsilon^2$$

This means that

$$(1 + C\sigma) \frac{1}{\varepsilon^2 |\lg \varepsilon|} E_\varepsilon(u_\varepsilon) \geqslant \frac{1}{|\lg \varepsilon|} \int_\Omega |\nabla \hat{u}_\varepsilon|^2 dx + \frac{\sigma}{\varepsilon^2 |\lg \varepsilon|} \int_\Omega (1 - |\hat{u}_\varepsilon(x)|^2)^2 dx =$$

$$= \frac{1}{|\lg \varepsilon|} GL_\varepsilon(\hat{u}_\varepsilon)$$

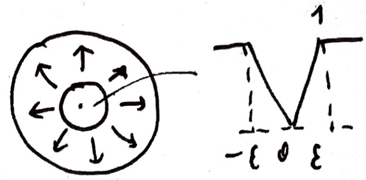
The functional $GL_\varepsilon(u) := \int_\Omega |\nabla u|^2 + \frac{\sigma}{\varepsilon^2} \int_\Omega (1 - |u|^2)^2 dx$, $u \in H^1(\Omega; \mathbb{R}^2)$

is the Ginzburg - Landau functional.

Ref: [Bethuel, Brezis, Helein], [Sandier, Serfaty], [Jerrard], [Alberti, Baldo, Orlandi], [Alicandro, Ponsiglione], [Jerrard Soner] ...

For the Ginzburg-Landau, the energy concentrates around vortices. To understand this, let us compute its energy on the sequence

$$u_\varepsilon(x) = \begin{cases} \dfrac{x}{|x|} & \text{for } x \in B_1 \setminus \overline{B_\varepsilon} \\[2mm] \dfrac{x}{\varepsilon} & \text{for } x \in B_\varepsilon \end{cases}$$



$$\int_\Omega |\nabla u_\varepsilon|^2\,dx = \int_{B_1\setminus \overline{B_\varepsilon}} + \int_{B_\varepsilon} = \int_\varepsilon^1 \int_0^{2\pi} \frac{1}{\rho^2}\,\rho\, d\theta\, d\rho + \int_0^\varepsilon \int_0^{2\pi} \frac{1}{\varepsilon^2}\,\rho\, d\theta\, d\rho =$$

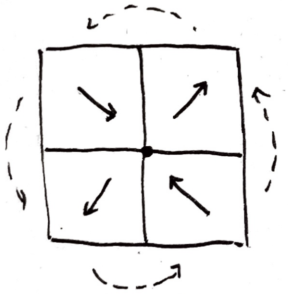
$$\left|\nabla \frac{x}{|x|}\right|^2 = \frac{1}{|x|^2}$$

$$= |\log \varepsilon| \cdot 2\pi + O(1)$$

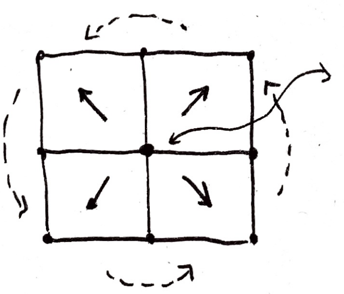
$$\frac{1}{\varepsilon^2}\int_{B_1} (1-|u_\varepsilon|^2)^2 = \frac{1}{\varepsilon^2}\int_{B_\varepsilon} \left(1-\left|\frac{x}{\varepsilon}\right|^2\right)^2 \le C.$$

Then $\dfrac{1}{|\log \varepsilon|} GL_\varepsilon(u_\varepsilon) \to 2\pi$.

In the discrete, i.e., for the XY-model, it's easier to explain what is the relevant object for the asymptotic analysis: it's the discrete vorticity. Let $u_\varepsilon : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{S}^1$ be a spin field. We define a measure which describes the discrete vorticity of u_ε as follows:



vortex with -1



vortex with $+1$

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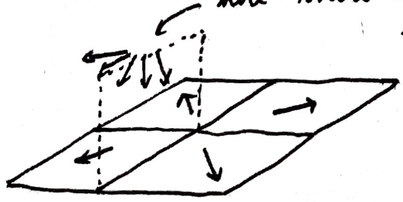
Given $\mu \in \mathcal{D}_0(\Omega)$, its flat norm is $\inf\{ \mathbb{M}(S) : S \in \mathcal{D}_1(\Omega), \partial S = \mu \}$.
We say that $\mu_j \xrightarrow{f} \mu$ if the flat norm of $\mu_j - \mu$ goes to zero.
In this way $\overset{-1}{\cdot}\ \overset{+1}{\cdot}$ is very close in flat norm to 0.
A technique called ball construction allows us to prove that a sequence of discrete vorticities μ_{u_ε} with $\frac{1}{\varepsilon^2 |\lg \varepsilon|} E_\varepsilon(u_\varepsilon) \leq C$ is pre-compact with respect to the flat convergence, i.e., up to ~~of~~ a subsequence $\mu_{u_\varepsilon} \xrightarrow{f} \mu$, where μ is a boundary. If $|\mu|(\Omega) < +\infty \Rightarrow \mu = \sum_{i=1}^{N} d_i \delta_{x_i}$, $d_i \in \mathbb{Z}$.

__Remark__: If $\mu_{u_\varepsilon} \xrightarrow{f} \mu$, then $\mu_{u_\varepsilon}(\varphi) \to \mu(\varphi)$ for every $\varphi \in C^1_c(\Omega)$ (with $\|d\varphi\|_\infty \leq 1$). Indeed, let $\delta > 0$. For $\varepsilon > 0$ small enough, the inf in the definition of flat norm is $< \delta$, i.e., there exists $S_{\delta,\varepsilon} \in \mathcal{D}_1(\Omega)$, $\partial S_{\delta,\varepsilon} = \mu_{u_\varepsilon} - \mu$ s.t. $\mathbb{M}(S_{\delta,\varepsilon}) < \delta$.
Then $|\mu_{u_\varepsilon}(\varphi) - \mu(\varphi)| = |(\mu_{u_\varepsilon} - \mu)(\varphi)| = |\partial S_{\delta,\varepsilon}(\varphi)| = |S_{\delta,\varepsilon}(d\varphi)| \leq$
$\leq \mathbb{M}(S_{\delta,\varepsilon}) \|d\varphi\|_\infty < \delta \|d\varphi\|_\infty$.
If $\|d\varphi\|_\infty$ is not bounded, we do not have convergence!

The previous discussion shows compactness. The following lower bound holds true. Assume u_ε is such that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$; then

$$\liminf_{\varepsilon} \frac{1}{\varepsilon^2 |\lg \varepsilon|} E_\varepsilon(u_\varepsilon) \geq 2\pi |\mu|(\Omega).$$

This lower bound is also optimal.
With the language of cartesian currents we have an alternative way of interpreting the discrete vorticity. Given 4 spins in 4 squares of the lattice, we can extend the piecewise constant function to a current in the product space $\Omega \times \mathbb{S}^1$ by connecting the jumps with the geodesic arc in $\mathbb{S}^1$ which connects the two traces.



here there is a vertical part of the current concentrated on the geodesic arc that connects u_ε^- and u_ε^+.

We denote this current by G_{u_ε}. It is decomposed in two parts:

$$G_{u_\varepsilon} = \underbrace{[g_{u_\varepsilon}]}_{\text{graph of the piecewise constant function } u_\varepsilon} + \underbrace{G^J_{u_\varepsilon}}_{\text{jump part, concentrated on } J_{u_\varepsilon}\text{, vertical, given by geodesic arc in } \mathbb{S}^1 \text{ that connects } u_\varepsilon^- \text{ and } u_\varepsilon^+}$$

With an approximation argument one can see that these currents behave like the graph of $\frac{x}{|x|}$ where there is a vorticity with degree $+1$.

Then

$$\partial G_{u_\varepsilon} = -\mu_{u_\varepsilon} \times [\mathbb{S}^1]$$

Since $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ we have that $\partial G_{u_\varepsilon} \to -\mu \times [\mathbb{S}^1]$.

This alternative way of seeing the discrete vorticity will be useful later.

THE N-CLOCK MODEL

From the previous discussion it is clear that N_ε (the number of points of the discretization of $\mathbb{S}^1$) might affect the behaviour of the system. Therefore, understanding for which regime of N_ε and ε the N-clock ~~prob~~ model really approximates the XY-model is a sensible question.

The first thing that we notice is that the construction of the transition from u^- to u^+ on a slow scale is not anymore possible if the spin field is constrained to lie in a discretization $\mathcal{S}_\varepsilon$ of $\mathbb{S}^1$. Let $\theta_\varepsilon = 2\pi/N_\varepsilon$, i.e., the smallest non-zero angle achievable by two nearest-neighboring vectors. Then η_ε has a maximal length! $\eta_\varepsilon \sim \frac{|\varphi^+-\varphi^-|}{\theta_\varepsilon}\varepsilon$ because $\theta_\varepsilon \sim \frac{|\varphi^+-\varphi^-|}{\eta_\varepsilon/\varepsilon}$. The energy for this construction becomes:

$$E_\varepsilon(u_\varepsilon) \sim \underbrace{\frac{1}{\varepsilon}}_{\text{vertical interactions}} \underbrace{\frac{\eta_\varepsilon}{\varepsilon}}_{\text{no. of horiz. inter}} \overbrace{\varepsilon^2}^{\text{in the energy}} \theta_\varepsilon^2 \sim \varepsilon\theta_\varepsilon |\varphi^+-\varphi^-| = \varepsilon\theta_\varepsilon\, d_{\mathbb{S}^1}(u^+,u^-).$$

This means that if we scale the energy by $\varepsilon\theta_\varepsilon$ we see a finite energy paid for a transition between u^- and u^+ on a 1-dimensional interface, i.e., $\frac{1}{\varepsilon\theta_\varepsilon}E_\varepsilon$ is able to see transitions on interfaces!

There is something different from the case where N is a fixed number: now u^- and u^+ can be also very close vectors and the energy paid for a transition between u^- and u^+ is proportional to their distance in $\mathbb{S}^1$. This is a way of measuring the total variation of a piecewise constant function ~~in~~ with values in $\mathbb{S}^1$. Imagining that with piecewise constant functions we can also create functions with diffuse part, we can expect that the limit functional which describes the effective energy is finite on $BV(\Omega;\mathbb{S}^1)$.

$$TV^{\mathbb{S}^1}(u) = \int_\Omega |\nabla u|\,dx + |D^c u|(\Omega) + \int_{J_u} d_{\mathbb{S}^1}(u^+,u^-)\,d\mathcal{H}^1 \quad \text{for } u\in BV(\Omega;\mathbb{S}^1)$$

Note that:

$$\int_{J_u} d_{\mathbb{S}^1}(u^+,u^-)\,d\mathcal{H}^1 > \int_{J_u} |u^+-u^-|\,d\mathcal{H}^1 = |D^j u|(\Omega), \text{ thus}$$

$$TV^{\mathbb{S}^1}(u) \geq |Du|(\Omega).$$

Proposition: Let $u_\varepsilon : \varepsilon\mathbb{Z}^2\cap\Omega \to \mathcal{S}_\varepsilon$ and assume that $\frac{1}{\varepsilon\theta_\varepsilon}E_\varepsilon(u_\varepsilon)\leq C$. Then, up to a subsequence, $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ with $u\in BV(\Omega;\mathbb{S}^1)$.

Proof: Let $A\subset\subset\Omega$.

$$C \geq \frac{1}{\varepsilon\theta_\varepsilon}E_\varepsilon(u_\varepsilon) = \frac{1}{2}\frac{1}{\varepsilon\theta_\varepsilon}\sum_{|i-j|=1}\varepsilon^2|u_\varepsilon(\varepsilon i)-u_\varepsilon(\varepsilon j)|^2 \geq$$

if $u_\varepsilon(\varepsilon i)\neq u_\varepsilon(\varepsilon j)$, then

θ_ε

$|u_\varepsilon(\varepsilon i)-u_\varepsilon(\varepsilon j)|\geq 2\sin\left(\frac{\theta_\varepsilon}{2}\right)$

$$\geq \frac{1}{2}\frac{1}{\varepsilon\theta_\varepsilon}\cdot 2\sin\left(\frac{\theta_\varepsilon}{2}\right)\sum_{\substack{|i-j|=1\\ \varepsilon i,\varepsilon j\in A}}\varepsilon^2|u_\varepsilon(\varepsilon i)-u_\varepsilon(\varepsilon j)|$$

$$= \frac{1}{2}\,\frac{2\sin\left(\frac{\theta_\varepsilon}{2}\right)}{\theta_\varepsilon}\sum_{\substack{|i-j|=1\\ \varepsilon i,\varepsilon j\in A}}\varepsilon\,|u_\varepsilon(\varepsilon i)-u_\varepsilon(\varepsilon j)| =$$

$$= \frac{2\sin\left(\frac{\theta_\varepsilon}{2}\right)}{\theta_\varepsilon}\int_{J_{u_\varepsilon}\cap A}|u_\varepsilon^+-u_\varepsilon^-|\,d\mathcal{H}^1 = \frac{2\sin\left(\frac{\theta_\varepsilon}{2}\right)}{\theta_\varepsilon}|Du_\varepsilon|(A).$$

By a diagonal argument we find a subsequence such that

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad w^*\text{-}BV(A;\mathbb{R}^2)\ \forall A\subset\subset\Omega, \quad u_\varepsilon\to u \text{ a.e. in } \Omega$$

Then $u_\varepsilon \to u$ s-L^1 because $\|u_\varepsilon\|_\infty \leq 1$. $\square$

The lower bound in the previous proof can be improved.

Proposition: Let $\sigma\in(0,1)$. Then for ε small enough:

$$|u_\varepsilon(\varepsilon i)-u_\varepsilon(\varepsilon j)|^2 \geq (1-\sigma)\,\theta_\varepsilon\, d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i),u_\varepsilon(\varepsilon j)).$$

Proof: It's a trigonometric computation. $\square$

This means that

$$\frac{1}{\varepsilon\theta_\varepsilon}E_\varepsilon(u_\varepsilon) \geq (1-\sigma)\,\frac{1}{2}\sum_{|i-j|=1}\varepsilon\, d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i),u_\varepsilon(\varepsilon j)) =$$

$$= (1-\sigma)\int_{J_{u_\varepsilon}} d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i),u_\varepsilon(\varepsilon j))\,|\nu_{u_\varepsilon}|_1\,d\mathcal{H}^1 =$$

$$= (1-\sigma)\,TV_1^{\mathbb{S}^1}(u_\varepsilon)$$

anisotropic

From this one gets for $u \in BV(\Omega;\mathbb{S}^1)$.

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \geq TV_1^{\mathbb{S}^1}(u) =$$

$$= \int_\Omega |\nabla u|_{2,1}\,dx + |D^c u|_{2,1}(\Omega) + \int_{J_u} d_{\mathbb{S}}(u^-,u^+)|\nu_u|_1 d\mathcal{H}^1.$$

This lower bound is <u>not</u> optimal for every rate $\theta_\varepsilon/\varepsilon$!

To understand why, we make this observation:

$$C \geq \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) = \frac{\varepsilon^2|\log\varepsilon|}{\varepsilon\theta_\varepsilon} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon) \Rightarrow \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon) \leq C \frac{\theta_\varepsilon}{\varepsilon|\log\varepsilon|}$$

If $\theta_\varepsilon \ll \varepsilon|\log\varepsilon|$, this is telling us that $\mu_{u_\varepsilon} \xrightarrow{f} 0$, i.e., disappear. $\partial G_{u_\varepsilon} \to 0$. There might be some energy due to the fact that vortices γ

Remembering the definition of G_{u_ε}, we notice that the mass of G_{u_ε} is controlled by $TV^{\mathbb{S}^1}(u_\varepsilon)$. Then $G_{u_\varepsilon} \to T$ and $T \in \mathrm{cart}(\Omega\times\mathbb{S}^1)$. Then the optimal lower bound if $\theta_\varepsilon \ll \varepsilon|\log\varepsilon|$ is an anisotropic version of the mass of $T = G_u + L\times[\mathbb{S}^1]$

$$\int_\Omega |\nabla u|_{2,1}\,dx + |D^c u|_{2,1}(\Omega) + \int_{\mathcal{L}} \mathrm{length}(\gamma_x^T)\,d\mathcal{H}^1(x) \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon).$$

The proof of the optimality of the lower bound is very technical. Moreover, for $\theta_\varepsilon \ll \varepsilon$, this lower bound is not optimal.

If $\theta_\varepsilon \sim \varepsilon|\log\varepsilon|$, $\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$ does not exclude the possible formation of vortices. Indeed, $\mu_\varepsilon \xrightarrow{f} \mu$ and $\partial G_{u_\varepsilon} \to -\mu\times[\mathbb{S}^1]$. Therefore, the limit current T is not cartesian anymore, but $\partial T = -\mu\times[\mathbb{S}^1]$ and the lower bound is:

$$2\pi|\mu|(\Omega) + \int_\Omega |\nabla u|_{2,1} + |D^c u|_{2,1} + \int_\Omega \mathrm{length}(\gamma_x^T)\,d\mathcal{H}^1 \leq \liminf_\varepsilon \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(\theta_\varepsilon).$$

If $\theta_\varepsilon \ll \varepsilon|\log\varepsilon|$, $\frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon) \leq C$ implies $\mu_\varepsilon \xrightarrow{f} \mu$. We can remove the energy needed to create μ with $|\mu|(\Omega) = M$ and rescale the functional:

$$\left(\frac{1}{\varepsilon|\log\varepsilon|} E_\varepsilon(u_\varepsilon) - 2\pi M\right)\frac{\varepsilon^2|\log\varepsilon|}{\varepsilon\theta_\varepsilon} = \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) - 2\pi M \frac{\varepsilon|\log\varepsilon|}{\theta_\varepsilon}.$$

Again $G_{u_\varepsilon} \to T$ with $\partial T = -\mu\times[\mathbb{S}^1]$ and

$$\liminf_{\varepsilon}\left(\frac{1}{\varepsilon\theta_\varepsilon}E_\varepsilon(u_\varepsilon)-2\pi M\,\frac{\varepsilon|\log\varepsilon|}{\theta_\varepsilon}\right)\geqslant$$ anisotropic energy of T with $\partial T=-\mu\times[S^1]$