

Hall Evolution

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I will consider a magnetic field that evolves according to the eq.

$$\frac{d\mathbf{B}}{dt} = -\nabla \times \left(\frac{c}{4\pi ne} [\nabla \times \mathbf{B}] \times \mathbf{B} + \eta \nabla \times \mathbf{B} \right), \quad (1)$$

where n is the electron density and $\eta \equiv c^2/(4\pi\sigma)$ (where σ is the electrical conductivity) is the magnetic diffusivity. For simplicity i will consider both of these quantities to be radial functions, and time independent.

Restricting this to an axisymmetric field, \mathbf{B} can be written in terms of two scalar functions as

$$\mathbf{B} = \nabla\alpha(r, \theta) \times \nabla\phi + \beta\nabla\phi, \quad (2)$$

where ϕ is the azimuth in spherical coordinates. Thus, $\alpha(r, \theta)$ describes the poloidal field and $\beta(r, \theta)$ the toroidal field. The curl of \mathbf{B} can be expressed in terms of these functions,

$$\nabla \times \mathbf{B} = -\Delta\alpha\nabla\phi + \nabla\beta \times \nabla\phi, \quad \Delta \equiv \varpi^2 \nabla \cdot (\varpi^{-2} \nabla) = \partial_r^2 + \frac{\sin\theta}{r^2} \partial_\theta \left(\frac{\partial_\theta}{\sin\theta} \right) \quad (3)$$

Using this, and defining $\chi \equiv c/(4\pi en\varpi^2)$, where $\varpi \equiv r \sin\theta$ is the radial coordinate, eq. (1) gives

$$\nabla \left(\frac{\partial\alpha}{\partial t} \right) \times \nabla\phi + \frac{\partial\beta}{\partial t} \nabla\phi = \nabla \times (\varpi^2 \chi [\Delta\alpha\nabla\phi - \nabla\beta \times \nabla\phi] \times [\nabla\alpha \times \nabla\phi + \beta\nabla\phi] - \eta [-\Delta\alpha\nabla\phi + \nabla\beta \times \nabla\phi]). \quad (4)$$

After expanding the cross product inside the curl, each term in this equation is either poloidal or toroidal, and taking this into account, i write two equations which together are equivalent to the previous one,

$$\begin{aligned} \nabla \left(\frac{\partial\alpha}{\partial t} \right) \times \nabla\phi &= \nabla \times (\varpi^2 \chi [\nabla\beta \times \nabla\phi] \times [\nabla\alpha \times \nabla\phi] + \eta \Delta\alpha \nabla\phi), \\ \frac{\partial\beta}{\partial t} \nabla\phi &= \nabla \times (\varpi^2 \chi \Delta\alpha \nabla\phi \times [\nabla\alpha \times \nabla\phi] + \varpi^2 \chi \beta \nabla\phi \times [\nabla\beta \times \nabla\phi] - \eta \nabla\beta \times \nabla\phi). \end{aligned} \quad (5)$$

With a couple of changes, these equations can be transformed into equations for the scalar functions α and β ,

$$\begin{aligned} \frac{\partial\alpha}{\partial t} &= \varpi^2 \chi [\nabla\alpha \times \nabla\beta] \cdot \nabla\phi + \eta \Delta\alpha \\ \frac{\partial\beta}{\partial t} &= \varpi^2 (\nabla[\chi \Delta\alpha] \times \nabla\alpha + \beta \nabla\chi \times \nabla\beta) \cdot \nabla\phi + \varpi^2 \nabla \cdot \left(\frac{\eta \nabla\beta}{\varpi^2} \right). \end{aligned} \quad (6)$$

I will consider an adimensional form of this equation, in which each quantity is expressed in terms of characteristic values B_0 for the magnetic field, χ_0 for χ and the radius of the star R for the radial coordinate r . These characteristic values define two timescales

$$t_h \equiv (\chi_0 B_0)^{-1}, \quad t_d \equiv \frac{R^2}{\eta} \quad (7)$$

and if I consider my unit of time to be t_h , the equations for α and β (considering all the quantities in their adimensional versions) are given by

$$\begin{aligned} \frac{\partial \alpha}{\partial t} &= \varpi^2 \chi [\nabla \alpha \times \nabla \beta] \cdot \nabla \phi + \left(\frac{t_h}{t_d} \right) \eta \Delta \alpha \\ \frac{\partial \beta}{\partial t} &= \varpi^2 (\nabla [\chi \Delta \alpha] \times \nabla \alpha + \beta \nabla \chi \times \nabla \beta) \cdot \nabla \phi + \left(\frac{t_h}{t_d} \right) \varpi^2 \nabla \cdot \left(\frac{\eta \nabla \beta}{\varpi^2} \right). \end{aligned} \quad (8)$$

1 Toroidal Field

To begin, I will consider a toroidal field. It is clear from eq. (8) that a purely toroidal field will remain toroidal. Although in this section I take $\alpha = 0$, I will show first that the eq. for β in (8) can be written as a conservative eq.,

$$\frac{1}{\varpi^2} \frac{\partial \beta}{\partial t} = \nabla \times (\chi \Delta \alpha \nabla \alpha + \chi \beta \nabla \beta) \cdot \phi + \left(\frac{t_h}{t_d} \right) \nabla \cdot \left(\frac{\eta \nabla \beta}{\varpi^2} \right) \quad (9)$$

$$\frac{1}{\varpi^2} \frac{\partial \beta}{\partial t} = \nabla \cdot (\nabla \phi \times [\chi \Delta \alpha \nabla \alpha + \chi \beta \nabla \beta]) + \left(\frac{t_h}{t_d} \right) \nabla \cdot \left(\frac{\eta \nabla \beta}{\varpi^2} \right), \quad (10)$$

so I expect the quantity

$$I_1 = \int_V \frac{\beta}{\varpi^2} dV \quad (11)$$

to remain constant unless the "fluxes" in eq. (10) are not zero in the boundaries. Since $\beta/\varpi = B_{tor}$, where B_{tor} is the toroidal component of the magnetic field, I_1 can also be written as

$$I_1 = 2\pi \int_0^\pi \int_0^R B_{tor} r dr d\theta, \quad (12)$$

which is simply the magnetic flux that crosses a half cut of the star that crosses the symmetry axis. So, when considering the evolution of the system under axial symmetry, the toroidal flux is a conserved quantity.

Now, let us consider the case $\alpha = 0$. Expressing eq. (10) in terms of spherical coordinates gives

$$\frac{\partial}{\partial t} \left(\frac{\beta}{\varpi^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\chi \beta \frac{\partial \beta}{\partial \theta} + \left(\frac{t_h}{t_d} \right) \frac{\eta r^2}{\varpi^2} \frac{\partial \beta}{\partial r} \right) + \frac{1}{\varpi} \frac{\partial}{\partial \theta} \left(-\frac{\chi \beta}{r} \frac{\partial \beta}{\partial r} + \left(\frac{t_h}{t_d} \right) \frac{\sin \theta \eta}{r \varpi^2} \frac{\partial \beta}{\partial \theta} \right), \quad (13)$$

which after multiplication by $r^2 \sin \theta$ gives an equation for which a conservative code can be readily constructed,

$$\frac{\partial}{\partial t} \left(\frac{\beta}{\sin \theta} \right) = \frac{\partial}{\partial r} \left(\chi \beta \frac{\partial \beta}{\partial \theta} + \left(\frac{t_h}{t_d} \right) \frac{\eta}{\sin \theta} \frac{\partial \beta}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(-\chi \beta \frac{\partial \beta}{\partial r} + \left(\frac{t_h}{t_d} \right) \frac{\eta}{r^2 \sin \theta} \frac{\partial \beta}{\partial \theta} \right). \quad (14)$$

In order to perform the simulation, I will consider a discretization on a grid with spacing Δr in the radial coordinate and $\Delta\theta$ in the angular coordinate, and a timestep Δt . In this way, I will have $\beta(r, \theta, t) \rightarrow \beta_{i,j}^n$, where the indexes i, j, n represent steps in radius, angle and time. The conservative scheme will then be

$$\begin{aligned} \frac{\beta_{i,j}^{n+1} - \beta_{i,j}^n}{\Delta t \sin(j\Delta)\theta} = & \frac{1}{\Delta r} \left[\left(\chi\beta \frac{\partial\beta}{\partial\theta} + \left(\frac{t_h}{t_d} \right) \frac{\eta}{\sin\theta} \frac{\partial\beta}{\partial r} \right)_{i+1/2,j} - \left(\chi\beta \frac{\partial\beta}{\partial\theta} + \left(\frac{t_h}{t_d} \right) \frac{\eta}{\sin\theta} \frac{\partial\beta}{\partial r} \right)_{i-1/2,j} \right] \\ & + \frac{1}{\Delta\theta} \left[\left(-\chi\beta \frac{\partial\beta}{\partial r} + \left(\frac{t_h}{t_d} \right) \frac{\eta}{r^2 \sin\theta} \frac{\partial\beta}{\partial\theta} \right)_{i,j+1/2} - \left(-\chi\beta \frac{\partial\beta}{\partial r} + \left(\frac{t_h}{t_d} \right) \frac{\eta}{r^2 \sin\theta} \frac{\partial\beta}{\partial\theta} \right)_{i,j-1/2} \right] \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{\beta_{i,j}^{n+1} - \beta_{i,j}^n}{\Delta t \sin(j\Delta)\theta} = & \frac{1}{\Delta r} \left[\frac{1}{8\Delta\theta} (\beta_{i+1,j} + \beta_{i,j})(\beta_{i,j+1} + \beta_{i+1,j+1} - \beta_{i,j-1} - \beta_{i+1,j-1}) \chi([i+1/2]\Delta r, j\Delta\theta) \right. \\ & - \frac{1}{8\Delta\theta} (\beta_{i,j} + \beta_{i-1,j})(\beta_{i,j+1} + \beta_{i-1,j+1} - \beta_{i,j-1} - \beta_{i-1,j-1}) \chi([i-1/2]\Delta r, j\Delta\theta) \\ & + \left(\frac{t_h}{t_d} \right) \frac{1}{\Delta r \sin(j\Delta\theta)} \eta([i+1/2]\Delta r, j\Delta\theta) (\beta_{i+1,j} - \beta_{i,j}) \\ & \left. - \left(\frac{t_h}{t_d} \right) \frac{1}{\Delta r \sin(j\Delta\theta)} \eta([i-1/2]\Delta r, j\Delta\theta) (\beta_{i,j} - \beta_{i-1,j}) \right] \\ & + \frac{1}{\Delta\theta} \left[-\frac{1}{8\Delta r} (\beta_{i,j+1} + \beta_{i,j})(\beta_{i+1,j+1} + \beta_{i+1,j} - \beta_{i-1,j} - \beta_{i-1,j+1}) \chi(i\Delta r, [j+1/2]\Delta\theta) \right. \\ & + \frac{1}{8\Delta r} (\beta_{i,j} + \beta_{i,j-1})(\beta_{i+1,j-1} + \beta_{i+1,j} - \beta_{i-1,j} - \beta_{i-1,j-1}) \chi(i\Delta r, [j-1/2]\Delta\theta) \\ & + \left(\frac{t_h}{t_d} \right) \frac{1}{\Delta\theta (i\Delta r)^2} \frac{\eta(i\Delta r, [j+1/2]\Delta\theta) (\beta_{i,j+1} - \beta_{i,j})}{\sin([j+1/2]\Delta\theta)} \\ & \left. - \left(\frac{t_h}{t_d} \right) \frac{1}{\Delta\theta (i\Delta r)^2} \frac{\eta(i\Delta r, [j-1/2]\Delta\theta) (\beta_{i,j} - \beta_{i,j-1})}{\sin([j-1/2]\Delta\theta)} \right]. \end{aligned} \quad (16)$$

This scheme is implemented in the code `as_toroidal.cpp`, using as boundary conditions $\beta = 0$ in the axis, the inner radius, and the surface.

2 Toroidal plus poloidal field

A purely poloidal field, unless it is in precise equilibrium, will produce a toroidal component. Thus, it is not possible to treat the evolution of a purely poloidal field without including the toroidal component.

The modified form of eq. (14), including the effect of the poloidal field, is

$$\frac{\partial}{\partial t} \left(\frac{\beta}{\sin\theta} \right) = \frac{\partial}{\partial r} \left(\chi\Delta\alpha \frac{\partial\alpha}{\partial\theta} + \chi\beta \frac{\partial\beta}{\partial\theta} + \left(\frac{t_h}{t_d} \right) \frac{\eta}{\sin\theta} \frac{\partial\beta}{\partial r} \right) + \frac{\partial}{\partial\theta} \left(-\chi\Delta\alpha \frac{\partial\alpha}{\partial\theta} - \chi\beta \frac{\partial\beta}{\partial r} + \left(\frac{t_h}{t_d} \right) \frac{\eta}{r^2 \sin\theta} \frac{\partial\beta}{\partial\theta} \right), \quad (17)$$

where the evaluation of the term $\Delta\alpha$ at midpoints of the grid is not a trivial affair.

Regarding the equation for α , it can be written in a form appropriate to be numerically solved by a

backwards "characteristics" method¹,

$$\frac{\partial \alpha}{\partial t} + \left(-\sin \theta \chi \frac{\partial \beta}{\partial \theta} \right) \frac{\partial \alpha}{\partial r} + \left(\sin \theta \chi \frac{\partial \beta}{\partial r} \right) \frac{\partial \alpha}{\partial \theta} = \eta \Delta \alpha \quad (18)$$

where α will be advected in the $r - \theta$ grid with a velocity

$$\mathbf{v} = \left(-\sin \theta \chi \frac{\partial \beta}{\partial \theta}, \sin \theta \chi \frac{\partial \beta}{\partial r} \right). \quad (19)$$

The use of this method will then impose a necessary stability condition, given by (with an abuse of the $<$ operator)

$$\mathbf{v} \Delta t < (\Delta r, \Delta \theta). \quad (20)$$

¹I believe this is not the appropriate translation...Is it finite moving element?