Hall Evolution

Pablo Marchant Campos

October 25, 2011

I will consider a magnetic field that evolves according to the eq.

$$\frac{\mathrm{d}\,\boldsymbol{B}}{\mathrm{d}\,t} = -\nabla\times\left(\frac{c}{4\pi ne}[\nabla\times\boldsymbol{B}]\times\boldsymbol{B}+\eta\nabla\times\boldsymbol{B}\right), \tag{1}$$

where n is the electron density and $\eta \equiv c^2/(4\pi\sigma)$ (where σ is the electrical conductivity) is the magnetic diffusivity. For simplicity i will consider both of these quantities to be radial functions, and time independent.

Restricting this to an axisymmetric field, $\boldsymbol{B}$ can be written in terms of two scalar functions as

$$\boldsymbol{B} = \nabla\alpha(r,\theta)\times\nabla\phi+\beta\nabla\phi, \tag{2}$$

where ϕ is the azimuth in spherical coordinates. Thus, $\alpha(r,\theta)$ describes the poloidal field and $\beta(r,\theta)$ the toroidal field. The curl of $\boldsymbol{B}$ can be expressed in terms of these functions,

$$\nabla\times\boldsymbol{B} = -\Delta\alpha\nabla\phi+\nabla\beta\times\nabla\phi., \quad \Delta \equiv= \varpi^2\nabla\cdot(\varpi^{-2}\nabla) = \partial_r^2+\frac{\sin\theta}{r^2}\partial_\theta\left(\frac{\partial_\theta}{\sin\theta}\right) \tag{3}$$

Using this, and defining $\chi \equiv c/(4\pi en\varpi^2)$, where $\varpi \equiv r\sin\theta$ is the radial coordinate, eq. (1) gives

$$\nabla\left(\frac{\partial\alpha}{\partial t}\right)\times\nabla\phi+\frac{\partial\beta}{\partial t}\nabla\phi = \nabla\times\left(\varpi^2\chi[\Delta\alpha\nabla\phi-\nabla\beta\times\nabla\phi]\times[\nabla\alpha\times\nabla\phi+\beta\nabla\phi]-\eta[-\Delta\alpha\nabla\phi+\nabla\beta\times\nabla\phi]\right). \tag{4}$$

After expanding the cross product inside the curl, each term in this equation is either poloidal or toroidal, and taking this into account, i write two equations which together are equivalent to the previous one,

$$\begin{aligned} \nabla\left(\frac{\partial\alpha}{\partial t}\right)\times\nabla\phi =& \nabla\times\left(\varpi^2\chi[\nabla\beta\times\nabla\phi]\times[\nabla\alpha\times\nabla\phi]+\eta\Delta\alpha\nabla\phi\right), \\ \frac{\partial\beta}{\partial t}\nabla\phi =& \nabla\times\left(\varpi^2\chi\Delta\alpha\nabla\phi\times[\nabla\alpha\times\nabla\phi]+\varpi^2\chi\beta\nabla\phi\times[\nabla\beta\times\nabla\phi]-\eta\nabla\beta\times\nabla\phi\right). \end{aligned} \tag{5}$$

With a couple of changes, these equations can be transformed into equations for the scalar functions α and β,

$$\begin{aligned} \frac{\partial\alpha}{\partial t} =& \varpi^2\chi[\nabla\alpha\times\nabla\beta]\cdot\nabla\phi+\eta\Delta\alpha \\ \frac{\partial\beta}{\partial t} =& \varpi^2(\nabla[\chi\Delta\alpha]\times\nabla\alpha+\beta\nabla\chi\times\nabla\beta)\cdot\nabla\phi+\varpi^2\nabla\cdot\left(\frac{\eta\nabla\beta}{\varpi^2}\right). \end{aligned} \tag{6}$$

I will consider an adimensional form of this equation, in which each quantity is expressed in terms of characteristic values B_0 for the magnetic field, χ_0 for χ and the radius of the star R for the radial coordinate r. These characteristic values define two timescales

$$t_h \equiv (\chi_0 B_0)^{-1}, \quad t_d \equiv \frac{R^2}{\eta} \tag{7}$$

and if I consider my unit of time to be t_h, the equations for α and β (considering all the quantities in their adimensional versions) are given by

$$\begin{aligned} \frac{\partial \alpha}{\partial t} =& \varpi^2 \chi [\nabla \alpha \times \nabla \beta] \cdot \nabla \phi + \left(\frac{t_h}{t_d}\right) \eta \Delta \alpha \\ \frac{\partial \beta}{\partial t} =& \varpi^2 (\nabla [\chi \Delta \alpha] \times \nabla \alpha + \beta \nabla \chi \times \nabla \beta) \cdot \nabla \phi + \left(\frac{t_h}{t_d}\right) \varpi^2 \nabla \cdot \left(\frac{\eta \nabla \beta}{\varpi^2}\right). \end{aligned} \tag{8}$$

1 Toroidal Field

To begin, I will consider a toroidal field. It is clear from eq. (8) that a purely toroidal field will remain toroidal. Although in this section I take $\alpha = 0$, I will show first that the eq. for β in (8) can be written as a conservative eq.,

$$\frac{1}{\varpi^2} \frac{\partial \beta}{\partial t} = \nabla \times (\chi \Delta \alpha \nabla \alpha + \chi \beta \nabla \beta) \cdot \phi + \left(\tfrac{t_h}{t_d}\right) \nabla \cdot \left(\tfrac{\eta \nabla \beta}{\varpi^2}\right) \tag{9}$$

$$\frac{1}{\varpi^2} \frac{\partial \beta}{\partial t} = \nabla \cdot (\nabla \phi \times [\chi \Delta \alpha \nabla \alpha + \chi \beta \nabla \beta]) + \left(\tfrac{t_h}{t_d}\right) \nabla \cdot \left(\tfrac{\eta \nabla \beta}{\varpi^2}\right), \tag{10}$$

so I expect the quantity

$$I_1 = \int_V \frac{\beta}{\varpi^2} \, \mathrm{d}\, V \tag{11}$$

to remain constant unless the "fluxes" in eq. (10) are not zero in the boundaries. Since $\beta/\varpi = B_{tor}$, where B_{tor} is the toroidal component of the magnetic field, I_1 can also be written as

$$I_1 = 2\pi \int_0^\pi \int_0^R B_{tor} r \, \mathrm{d}\, r \, \mathrm{d}\, \theta, \tag{12}$$

which is simply the magnetic flux that crosses a half cut of the star that crosses the symmetry axis. So, when considering the evolution of the system under axial symmetry, the toroidal flux is a conserved quantity.

Now, let us consider the case $\alpha = 0$. Expressing eq. (10) in terms of spherical coordinates gives

$$\frac{\partial}{\partial t} \left(\frac{\beta}{\varpi^2}\right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{\chi \beta}{\sin\theta} \frac{\partial \beta}{\partial \theta} + \left(\frac{t_h}{t_d}\right) \frac{\eta r^2}{\varpi^2} \frac{\partial \beta}{\partial r}\right) + \frac{1}{\varpi} \frac{\partial}{\partial \theta} \left(-\frac{\chi \beta}{r} \frac{\partial \beta}{\partial r} + \left(\frac{t_h}{t_d}\right) \frac{\sin\theta \eta}{r \varpi^2} \frac{\partial \beta}{\partial \theta}\right), \tag{13}$$

which after multiplication by $r^2 \sin\theta$ gives an equation for which a conservative code can be readily constructed,

$$\frac{\partial}{\partial t} \left(\frac{\beta}{\sin\theta}\right) = \frac{\partial}{\partial r} \left(\chi \beta \frac{\partial \beta}{\partial \theta} + \left(\frac{t_h}{t_d}\right) \frac{\eta}{\sin\theta} \frac{\partial \beta}{\partial r}\right) + \frac{\partial}{\partial \theta} \left(-\chi \beta \frac{\partial \beta}{\partial r} + \left(\frac{t_h}{t_d}\right) \frac{\eta}{r^2 \sin\theta} \frac{\partial \beta}{\partial \theta}\right). \tag{14}$$

In order to perform the simulation, I will consider a discretization on a grid with spacing Δr in the radial coordinate and $\Delta\theta$ in the angular coordinate, and a timestep Δt. In this way, I will have $\beta(r,\theta,t) \to \beta^n_{i,j}$, where the indexes i, j, n represent steps in radius, angle and time. The conservative scheme will then be

$$\begin{aligned}\frac{\beta^{n+1}_{i,j} - \beta^n_{i,j}}{\Delta t \sin(j\Delta)\theta} =& \frac{1}{\Delta r}\left[\left(\chi\beta\frac{\partial\beta}{\partial\theta} + \left(\frac{t_h}{t_d}\right)\frac{\eta}{\sin\theta}\frac{\partial\beta}{\partial r}\right)_{i+1/2,j} - \left(\chi\beta\frac{\partial\beta}{\partial\theta} + \left(\frac{t_h}{t_d}\right)\frac{\eta}{\sin\theta}\frac{\partial\beta}{\partial r}\right)_{i-1/2,j}\right] \\ &+ \frac{1}{\Delta\theta}\left[\left(-\chi\beta\frac{\partial\beta}{\partial r} + \left(\frac{t_h}{t_d}\right)\frac{\eta}{r^2\sin\theta}\frac{\partial\beta}{\partial\theta}\right)_{i,j+1/2} - \left(-\chi\beta\frac{\partial\beta}{\partial r} + \left(\frac{t_h}{t_d}\right)\frac{\eta}{r^2\sin\theta}\frac{\partial\beta}{\partial\theta}\right)_{i,j-1/2}\right]\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\beta^{n+1}_{i,j} - \beta^n_{i,j}}{\Delta t \sin(j\Delta)\theta} = \frac{1}{\Delta r}\Bigg[&\frac{1}{8\Delta\theta}(\beta_{i+1,j} + \beta_{i,j})(\beta_{i,j+1} + \beta_{i+1,j+1} - \beta_{i,j-1} - \beta_{i+1,j-1})\chi([i+1/2]\Delta r, j\Delta\theta) \\ &- \frac{1}{8\Delta\theta}(\beta_{i,j} + \beta_{i-1,j})(\beta_{i,j+1} + \beta_{i-1,j+1} - \beta_{i,j-1} - \beta_{i-1,j-1})\chi([i-1/2]\Delta r, j\Delta\theta) \\ &+ \left(\frac{t_h}{t_d}\right)\frac{1}{\Delta r\sin(j\Delta\theta)}\eta([i+1/2]\Delta r, j\Delta\theta)(\beta_{i+1,j} - \beta_{i,j}) \\ &- \left(\frac{t_h}{t_d}\right)\frac{1}{\Delta r\sin(j\Delta\theta)}\eta([i-1/2]\Delta r, j\Delta\theta)(\beta_{i,j} - \beta_{i-1,j})\Bigg] \\ + \frac{1}{\Delta\theta}\Bigg[&-\frac{1}{8\Delta r}(\beta_{i,j+1} + \beta_{i,j})(\beta_{i+1,j+1} + \beta_{i+1,j} - \beta_{i-1,j} - \beta_{i-1,j+1})\chi(i\Delta r, [j+1/2]\Delta\theta) \\ &+ \frac{1}{8\Delta r}(\beta_{i,j} + \beta_{i,j-1})(\beta_{i+1,j-1} + \beta_{i+1,j} - \beta_{i-1,j} - \beta_{i-1,j-1})\chi(i\Delta r, [j-1/2]\Delta\theta) \\ &+ \left(\frac{t_h}{t_d}\right)\frac{1}{\Delta\theta(i\Delta r)^2}\frac{\eta(i\Delta r, [j+1/2]\Delta\theta)(\beta_{i,j+1} - \beta_{i,j})}{\sin([j+1/2]\Delta\theta)} \\ &- \left(\frac{t_h}{t_d}\right)\frac{1}{\Delta\theta(i\Delta r)^2}\frac{\eta(i\Delta r, [j-1/2]\Delta\theta)(\beta_{i,j} - \beta_{i,j-1})}{\sin([j-1/2]\Delta\theta)}\Bigg].\end{aligned} \tag{16}$$

This scheme is implemented in the code as_toroidal.cpp, using as boundary conditions $\beta = 0$ in the axis, the inner radius, and the surface.

2 Toroidal plus poloidal field

A purely poloidal field, unless it is in precise equilibrium, will produce a toroidal component. Thus, it is not possible to treat the evolution of a purely poloidal field without including the toroidal component.

The modified form of eq. (14), including the effect of the poloidal field, is

$$\frac{\partial}{\partial t}\left(\frac{\beta}{\sin\theta}\right) = \frac{\partial}{\partial r}\left(\chi\Delta\alpha\frac{\partial\alpha}{\partial\theta} + \chi\beta\frac{\partial\beta}{\partial\theta} + \left(\frac{t_h}{t_d}\right)\frac{\eta}{\sin\theta}\frac{\partial\beta}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(-\chi\Delta\alpha\frac{\partial\alpha}{\partial r} - \chi\beta\frac{\partial\beta}{\partial r} + \left(\frac{t_h}{t_d}\right)\frac{\eta}{r^2\sin\theta}\frac{\partial\beta}{\partial\theta}\right), \tag{17}$$

where the evaluation of the term $\Delta\alpha$ at midpoints of the grid is not a trivial affair.

Regarding the equation for α, it can be written in a form appropriate to be numerically solved by a backwards "characteristics" method[1],

$$\frac{\partial \alpha}{\partial t} + \left(-\sin\theta\chi\frac{\partial \beta}{\partial \theta}\right)\frac{\partial \alpha}{\partial r} + \left(\sin\theta\chi\frac{\partial \beta}{\partial r}\right)\frac{\partial \alpha}{\partial \theta} = \eta\Delta\alpha \tag{18}$$

where α will be advected in the $r-\theta$ grid with a velocity

$$\boldsymbol{v} = \left(-\sin\theta\chi\frac{\partial \beta}{\partial \theta}, \sin\theta\chi\frac{\partial \beta}{\partial r}\right). \tag{19}$$

The use of this method will then impose a necessary stability condition, given by (with an abuse of the $<$ operator)

$$\boldsymbol{v}\Delta t < (\Delta r, \Delta\theta). \tag{20}$$

[1] I believe this is not the appropriate translation...Is it finite moving element?