STAT. 2131
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Homework 1

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1. $Tr(AB) = Tr(BA)$

we know that $tr(A) = \sum_{i=1}^{n} a_{ii}$, so

$tr(AB) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ji}$

$= \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji} a_{ij}$

$= tr(BA)$

2. Let $A \in \mathbb{R}^{m \times n}$, $im(A) = \{y \in \mathbb{R}^m \mid y = Ax \in \mathbb{R}^n$ for some $x \in \mathbb{R}^n\}$,

$r(A) = \dim\{im(A)\}$, $Ker(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$

$im(A) \in \mathbb{R}^m$ and $ker(A) \in \mathbb{R}^n$ are vector subspaces

ⓐ For any $G \subseteq \mathbb{R}^{n \times q}$, show that $im(AG) \subseteq im(A)$ and $ker(G) \subseteq ker(AG)$

say $y \in AG \rightarrow y = (AG)(x)$, $x \in \mathbb{R}^n$

so $y = (AG)x = A(Gx)$, $x \in \mathbb{R}^n$

and $y = Ax$ is $im(A)$, so $im(AG) \subseteq im(A)$

say $z \in AG \rightarrow z = AGx$, $x \in \mathbb{R}^n$

and if $Ax = 0$, $z = (Ax)G = 0$, so $Ker(G) \subseteq Ker(AG)$

ⓑ G is surjective; show $im(AG) = im(A)$

we know $im(AG) \subseteq im(A)$. If $im(A) \subseteq im(AG)$, this is true

If $w \in im(A)$, then some $v \in \mathbb{R}^q$ so $w = A(v)$

Since $G: \mathbb{R}^n \rightarrow \mathbb{R}^q$ is surjective, some $u \in \mathbb{R}^n$ so $v = G(u)$.

So, $w = A(G(u)) = A \cdot G(u)$, or $w \in im(AG)$,

so $im(AG) \subseteq im(A)$

and $im(AG) = im(A)$

ⓒ Show $r(A+B) \leq r(A) + r(B)$

If column space $(A) \rightarrow \{a_1, a_2, \ldots, a_n\}$

and col space $(B) \rightarrow \{b_1, b_2, \ldots, b_n\}$

and all are linearly independent,

then $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ span columns of $A+B$,

so $r(A+B) \leq r(A) + r(B)$

(some columns between could be dependent!)

3. $W \subset \mathbb{R}^n$; $W^\perp = \{v \in \mathbb{R}^n : v^T w = 0 \text{ for } w \in W\}$

(a) show ① $W \cap W^\perp = \{0\}$ and ② $(W^\perp)^\perp = W$

① let $x = W \cap W^\perp$

$x \in W$ & $x \in W^\perp$, so $x^T x = 0$ satisfies (x is ort. to itself)

so, $W \cap W^\perp = \{0\}$

② use dim to show $\dim(W) = \dim(W^\perp)$ and $W \subseteq (W^\perp)^\perp$

$\dim(W) + \dim(W^\perp) = n$, when $W \in \mathbb{R}^n$

and $(W^\perp)^\perp$ and $W^\perp$ are orthogonal

so, $\dim((W^\perp)^\perp) + \dim(W^\perp) = n$ as well

$\dim((W^\perp)^\perp) + \dim(W^\perp) = \dim(W) + \dim(W^\perp)$

$\rightarrow \dim((W^\perp)^\perp) = \dim(W)$

so, $W = (W^\perp)^\perp$

(b) ① $\ker(A^T) = \operatorname{im}(A)^\perp$

let $y \in \operatorname{im}(A)$ $\quad y^T = (Ax)^T = A^T x^T$

let $z \in \ker(A^T)$ $\quad A^T z = 0 \rightarrow y^T z = 0$

so $x^T(0) = 0$ and $y^T z = 0$

and $y \in \operatorname{im}(A)$, $z \in \ker(A^T)$, so $\ker(A^T) = \operatorname{im}(A)$

② $\operatorname{im}(A^T) = \ker(A)^\perp$

If we replace A with A^T, we have $\operatorname{im}(A) = \ker(A^T)^\perp$, which is the same as: $\operatorname{im}(A)^\perp = \ker(A^T)$. $A^T A = I$.

(c) ① $\operatorname{im}(A^T) \cap \ker(A) = 0$

→ we can conclude this to be true through

(a) (if two matrices are compliments, $\operatorname{im}(A^T) \cap \ker(A) = 0$) and (b) (these two are orthogonal) → same for $\operatorname{im}(A) \cap \ker(A^T)$ as shown in (b)

② $\operatorname{im}(A^T) \cup \ker(A) = \mathbb{R}^n$ & $\operatorname{im}(A) \cup \ker(A^T) = \mathbb{R}^m$

if $\operatorname{im}(A^T)$ and $\ker(A)$ are orthogonal, replace A^T with A

(see next page)

3c, continued

so $im(A^T) \cup Ker(A)$

$\Rightarrow \dim(Ker(A) + im(A^T)) + \underbrace{\dim(Ker(A) \cap im(A^T))}^{\{0\}}$

$= \dim(Ker(A) + \dim(im(A))$

$= \dim(\mathbb{R}^n)$

so $\dim(Ker(A) + im(A^T)) = \dim(\mathbb{R}^n)$

and $Ker(A) + im(A) = \mathbb{R}^n$

(or $Ker(A) \cup im(A^T) = \mathbb{R}^n$)

→ similar for $im(A) \cup Ker(A^T) = \mathbb{R}^m$

3(d) For all $x \in \mathbb{R}^m$, $x_0 \in im(A)$, $x_1 \in Ker(A^T)$ so

$x = x_0 + x_1$ and $x_0^T x_1 = 0$. conclude $\|x\|_2^2 = \|x_0\|_2^2 + \|x_1\|_2^2$

$x = x_0 + x_1$ when $x_0 \in im(A)$, $x_1 \in Ker(A^T)$

because $Ker(A^T) + im(A) = \mathbb{R}^m$

and $\dim(Ker(A^T)) + \dim(im(A)) = m$

and the basis for $\mathbb{R}^m$ is m linearly ind. vectors

any vector in $\mathbb{R}^m$ can be written as a combination of linearly independent vectors: $x = x_0 + x_1$

$\|x\|^2 = \|x_0\|^2 + \|x_1\|^2$

$= (x_0 + x_1)^T (x_0 + x_1)$

and since $x_1^T x_0 = x_0^T x_1 = 0$,

$= x_0^T x_0 + x_1^T x_1 = \|x_0\|^2 + \|x_1\|^2$

4. Population = 7.594 billion, female = 49.6%
sex ~ independent

ⓐ **Null**: P(female) = .50
Alt: P(female) ≠ .50

Testing proportions, so $Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$

where $p_0 = .5$ and $\hat{p} = .496$ and $n = 7.594$ billion
so, test stat $Z = -697.1485$ $Z = (-1.96, 1.96)$
with $\alpha = .05$ (standard), we can reject the null hypothesis and conclude that the chance of a child being born female is NOT .50

ⓑ Calculate **95%** CI for the probability that a child is born female → $\hat{p} \pm Z \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
(.49599, .49601) $= .496 \pm -.00001$

ⓒ You can use a confidence interval to test a hypothesis because, when based on the same alpha α, we fail to reject Ho when p_0 falls inside the CI, and we reject Ho when p_0 falls outside the CI

5. $T_\nu \sim t_\nu$ with $\nu > 0$ df and $\chi^2_\nu \sim \chi^2(\nu)$

ⓐ Using $T_\nu \overset{D}{=} \frac{N(0,1)}{\sqrt{\chi^2_\nu/\nu}}$, show

$$P(T_\nu \leq -t \text{ or } T_\nu \geq t) = P(F_{1,\nu} \geq t^2)$$

for all $t > 0$, $F_{1,\nu} \sim F(1,\nu)$

Since $T_\nu \overset{D}{=} \frac{N(0,1)}{\sqrt{\chi^2_\nu/\nu}}$, Replace T_ν and get

$P\left(\frac{N(0,1)}{\sqrt{\chi^2_\nu/\nu}} \geq t \text{ or } \leq t\right)$

$N(0,1)$ is standard normal $\rightarrow \frac{x_i - \mu}{\sigma^2} \sim N(0,1)$

when we square, $N^2(0,1) \sim \chi^2_1$

So, $P\left(\frac{N^2(0,1)}{\chi^2_\nu/\nu} \geq t^2\right)$

And $F_{1,\nu} \overset{D}{=} \frac{\chi^2_1/1}{\chi^2_\nu/\nu}$,

so $P\left(\frac{N^2(0,1)}{\chi^2_\nu/\nu} \geq t^2\right) = P\left(\frac{\chi^2_1/1}{\chi^2_\nu/\nu} \geq t^2\right) = P(F_{1,\nu} \geq t^2)$

and $P(F_{1,\nu} \geq t^2) = P(F_{1,\nu} \geq t^2)$

ⓑ Show that $n^{-1}\chi^2_n \xrightarrow{a.s.} 1$ as $n \to \infty$

to conclude $\lim_{n\to\infty} P(T_n \leq t) = P\{N(0,1) \leq t\}$

$n^{-1}\chi^2_n \Rightarrow \frac{\chi^2_n}{n} \xrightarrow{a.s.} 1$ as $n \to \infty$

If $X^2 \sim \chi^2_n$, we write X^2 as $X^2 = Z_1^2 + \dots + Z_n^2$

and Z_i iid $\sim N(0,1)$

By SLoLN, (Strong) $\lim_{n\to\infty} \frac{X^2}{n} \equiv \frac{\chi^2_n}{n} \to 1$

If $T_n = \frac{N(0,1)}{\sqrt{\chi^2_n/n}}$, $P(T_n \leq t) = P\left(\frac{N(0,1)}{\sqrt{\chi^2_n/n}} \leq t\right)$

and by the first proof, as $n \to \infty$,

$\sqrt{\chi^2_n/n} \to 1$, so

$\lim_{n\to\infty} P\left(\frac{N(0,1)}{\sqrt{\chi^2_n/n}} \leq t\right) = P\left(\frac{N(0,1)}{1} \leq t\right) = P\{N(0,1) \leq t\}$