STAT 2132
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HOMEWORK 6

1. one-way ANOVA model

$$Y_{ij} = \mu_i + \epsilon_{ij}, \quad i=1,\dots,r, \quad j=1,\dots,n_i$$

$$\operatorname{cov}(\epsilon_{ij}, \epsilon_{i'j'}) = \sigma^2 \delta_{ii'} \delta_{jj'}$$

$$Y = (Y_{11},\dots,Y_{1n_1},\dots,Y_{r1},\dots,Y_{rn_r})^T$$

$$\epsilon = (\epsilon_{11},\dots,\epsilon_{rn_r})^T$$

$$\beta = (\mu_1,\dots,\mu_r)^T$$

ⓐ what is the design matrix X?

The design matrix X will have rank r, with $\Sigma n_i = n_T$ rows and r columns

Say $r=3$ and $n_i = 2$

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & 1 & 0 \cdots & 0 \\ & & \ddots & 1 \\ 0 & & 0 \cdots & 1 \end{bmatrix}$$

where entries are 1 for presence and 0 for none

ⓑ compute corresponding hat matrix $H = X(X^TX)^{-1}X^T$

$$X^TX = \begin{bmatrix} n_1 & & \\ & \ddots & \\ & & n_r \end{bmatrix}, \quad r \times r \text{ matrix}$$

$$(X^TX)^{-1} = \begin{bmatrix} \frac{1}{n_1} & & \\ & \ddots & \\ & & \frac{1}{n_r} \end{bmatrix}, \quad r \times r \text{ matrix}$$

$$X(X^TX)^{-1} = \begin{bmatrix} \frac{1}{n_1} & 0 & \cdots & 0 \\ \vdots\frac{1}{n_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{n_r} \\ \vdots 0 & \cdots & 0 & \frac{1}{n_r} \end{bmatrix}, \quad r \times \Sigma n_i \text{ matrix}$$

$$H = X(X^TX)^{-1}X^T = \begin{bmatrix} \frac{1}{n_1} & \cdots & \frac{1}{n_1} & 0 & \cdots & 0 \\ \frac{1}{n_1} & \cdots & \frac{1}{n_1} & 0 & \cdots & 0 \\ 0 & & 0 & \vdots & \vdots & \frac{1}{n_r} \\ \vdots 0 & \cdots & \vdots & \vdots & & \vdots \\ & & 0 & & & \frac{1}{n_r} \end{bmatrix}, \quad \Sigma n_i \times \Sigma n_i \text{ matrix}$$

1, continued

(c) Show that $HY = (\bar{Y}_{1\cdot} 1_{n_1}, \ldots, \bar{Y}_{r\cdot} 1_{n_r})^T$

HY (H is $n \times n$, Y is $n \times 1$) where $n = \sum n_i$

$$HY = \begin{bmatrix} \frac{1}{n_1} & \cdots & \frac{1}{n_1} & 0 & \cdots & & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{n_1} & \cdots & \frac{1}{n_1} & 0 & \cdots & & 0 \\ 0 & \cdots & 0 & & \cdots & \frac{1}{n_r} & \frac{1}{n_r} \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & & \frac{1}{n_r} & \frac{1}{n_r} \end{bmatrix} \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1n_1} \\ \vdots \\ Y_{rn_r} \end{bmatrix}$$

$$= \left[\frac{1}{n_1}(Y_{11} + \ldots + Y_{1n_1}) \cdots \frac{1}{n_1}(Y_{r_1} + \ldots + Y_{rn_r}) \cdots \frac{1}{n_r}(Y_{11} + \cdots + Y_{1n_1}) \cdots \frac{1}{n_r}(Y_{r1} + \ldots + Y_{rn_r}) \right]^T$$

$$= [\bar{Y}_{1\cdot} \cdots \bar{Y}_{1\cdot} \cdots \bar{Y}_{n\cdot} \cdots \bar{Y}_{n\cdot}]^T$$

$$= [\bar{Y}_{1\cdot} 1_{n_1}, \ldots, \bar{Y}_{r\cdot} 1_{n_r}]^T$$

↳ each average repeated n_i times

(d) Let $n_T = \sum_{i=1}^r n_i$, $L = n_T^{-1} 1_{n_T} 1_{n_T}^{-1}$, $\bar{Y}_{\cdot\cdot} = n_T^{-1} \sum_{i=1}^r \sum_{j=1}^{n_i} Y_{ij}$

(i) Show that

$$SSTR = \sum_{i=1}^r n_i(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 = Y^T(H-L)Y$$

where $H-L$ is itself an orthogonal projection operator (ie, symmetric and idempotent).

$$\sum_{i=1}^r n_i(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 = \sum_{i=1}^r \{ n_i(\bar{Y}_{i\cdot}^2 - 2\bar{Y}_{i\cdot}\bar{Y}_{\cdot\cdot} + \bar{Y}_{\cdot\cdot}^2) \}$$

$$= \sum_{i=1}^r n_i \bar{Y}_{i\cdot}^2 - 2\sum_{i=1}^r n_i \bar{Y}_{i\cdot}\bar{Y}_{\cdot\cdot} + \sum_{i=1}^r n_i \bar{Y}_{\cdot\cdot}^2$$

$$= n_T \sum_{i=1}^r \bar{Y}_{i\cdot}^2 - 2n_T \sum_{i=1}^r \bar{Y}_{i\cdot}\bar{Y}_{\cdot\cdot} + n_T \sum_{i=1}^r \bar{Y}_{\cdot\cdot}^2$$

$$= n_T \sum_{i=1}^r \bar{Y}_{i\cdot}^2 - 2n_T \bar{Y}_{\cdot\cdot} \sum_{i=1}^r \bar{Y}_{i\cdot} + n_T \bar{Y}_{\cdot\cdot}^2$$

$$\sum_{i=1}^r \bar{Y}_{i\cdot} = n\bar{Y}_{\cdot\cdot}$$

$$= n_T \sum_{i=1}^r \bar{Y}_{i\cdot}^2 - n_T \bar{Y}_{\cdot\cdot}^2 = \underbrace{Y^T H Y - Y^T L Y}_{\downarrow} = Y^T(H-L)Y$$

$$Y^T(H-L)Y = Y^T H Y - Y^T L Y$$

$$Y^T H Y = [Y_{11} \cdots Y_{rn_r}]^T [\bar{Y}_{1\cdot} 1_{n_1} \cdots \bar{Y}_{r\cdot} 1_{n_r}]$$

$$= Y_{11}\bar{Y}_{1\cdot} + \cdots + Y_{rn_r}\bar{Y}_{r\cdot} = n_T \sum_{i=1}^r Y_{i\cdot}^2$$

$$Y^T L Y = [Y_{11} \cdots Y_{rn_r}]^T [\bar{Y}_{\cdot\cdot} \cdots \bar{Y}_{\cdot\cdot}]$$

$$= Y_{11}\bar{Y}_{\cdot\cdot} + \cdots + Y_{rn_r}\bar{Y}_{\cdot\cdot} = n_T \bar{Y}_{\cdot\cdot}^2$$

1 ⓓ continued

(i, continued)

- we know from 2131 that H is symmetric and idempotent

$(HH = H,\ H^T = H)$

- $L^T = n_T^{-1}(1_{n_T}^{\ T})^T 1_n^{\ T} = n_T^{-1} 1_{n_T} 1_{n_T}^{-1} = L$
- $LL = n_T^{-1} 1_{n_T} 1_{n_T}^{\ T} n_T^{-1} 1_{n_T} 1_{n_T}^{\ T}$

$= n_T^{-2} 1_{n_T} 1_{n_T}^{\ T} 1_{n_T} 1_{n_T}^{\ T} = n_T^{-2} J_{n_T}^{\ 2}$

where J_{n_T} is $n_T \times n_T$ of 1's, and $J_{n_T}^{K} = n^{K-1} J_{n_T}$

$= n_T^{-2} \cdot n_T \cdot J_{n_T}$

$= n_T^{-1} J_{n_T} = n_T^{-1} 1_{n_T} 1_{n_T}^{\ T} = L$

- so L is symmetric and idempotent, and so is $(H-L)$

(ii) $\text{rank}(H-L) = \text{rank}(H) - \text{rank}(n_T^{-1} 1_{n_T} 1_{n_T}^{\ T})$

$= r-1$

(iii) Show that $Im(H-L)$ is orthogonal to $Im(I_n - H)$

Let $\vec{a} \in Im(H-L)$

$\vec{a} = (H-L)\vec{v_1}$, v_1 is $n_T \times 1$

Let $\vec{b} \in Im(I_n - H)$

$\vec{b} = (I_n - H)\vec{v_2}$, v_2 is $n_T \times 1$

$\vec{a}^T \vec{b} = \vec{v_1}^T (H-L)^T (I_n - H)\vec{v_2}$

$= \vec{v_1}^T (H-L)(I_n - H)\vec{v_2}$ {by (i)}

$= \vec{v_1}^T (H - H - L + LH)\vec{v_2}$ → and $HL = L$

$$\begin{bmatrix} \frac{1}{n_1} \cdots \frac{1}{n_1} & 0 & \cdots & 0 \\ 0 & & & \vdots \\ 0 & 0 & 0 & \frac{1}{n_r} \end{bmatrix} \times \begin{bmatrix} \frac{1}{n_T} & \cdots & \frac{1}{n_T} \\ \vdots & & \vdots \\ \frac{1}{n_T} & \cdots & \frac{1}{n_T} \end{bmatrix} = \begin{bmatrix} \frac{1}{n_1} \cdot n_1 \cdot \frac{1}{n_T} & \cdots \\ & n_r \frac{1}{n_r} \frac{1}{n_T} \end{bmatrix} = L$$

and if $(H-L)(I_n - H) = 0$,

$Im(H-L)$ contains $Im(I_n - H)$

$= HI_n - HH - LI_n + LH$

$= H - H - L + L = 0_{n \times n}$

so $Im(H-L)$ is orthogonal to $Im(I_n - H)$.

1 ⓓ, continued

(iv) If $Y \sim N$, show that SSTR is independent of SSE

If $Y \sim N$, $E(Y) = X\beta$ and

$$\hat{\beta}^{OLS} = (X^TX)^{-1}X^TY$$

which gives us

$$SSE = (Y - X\hat{\beta})^T(Y - X\hat{\beta})$$

$$= (Y - X(X^TX)^{-1}X^TY)^T(Y - X(X^TX)^{-1}X^TY)$$

$$= Y^T(I - X(X^TX)^{-1}X^T)^T(I - X(X^TX)^{-1}X^T)Y$$

and

$$SSR = Y^TX(X^TX)^{-1}X^TY$$

we need

$\{X(X^TX)^{-1}X^T\}$ and $\{I - X(X^TX)^{-1}X^T\}$ to be idempotent

$\rightarrow \{X(X^TX)^{-1}X^T\}\{I - X(X^TX)^{-1}X^T\} = 0$

$\rightarrow X(X^TX)^{-1}X^T + I - X(X^TX)^{-1}X^T = I$

so by Cochran's Theorem,

SSE & SSTR are independent

Because both H and L are symmetric/idempotent and so is (H - L), then we have proved Cochran's theorem in the parts before this one and the solution holds.

$\rightarrow$

2. $Z \in \mathbb{R}^n$ is a random vector, $E(Z) = \mu$ and $var(Z) = V$

ⓐ show that for any non-random matrix $A \in \mathbb{R}^{n\times n}$, $E(Z^TAZ) = Tr(AV) + \mu^TA\mu$

$E(Z^TAZ) = E\{tr(Z^TAZ)\}$

$Z^TAZ = (Z-\mu)^TA(Z-\mu) + \mu^TAZ + Z^TA\mu - \mu^TA\mu$

$E[(Z-\mu)^TA(Z-\mu)] = E\{tr[(Z-\mu)^TA(Z-\mu)]\}$

$= E\{tr[A(Z-\mu)(Z-\mu)^T]\}$

$= tr\{E[A(Z-\mu)(Z-\mu)^T]\} = tr\{AE[(Z-\mu)(Z-\mu)^T]\}$

$= tr(AV)$

$E[\mu^TAZ + Z^TA\mu - \mu^TA\mu] = \mu^TA\mu$

So, $E(Z^TAZ) = tr(AV) + \mu^TA\mu$

ⓑ using notation from problem 1, show that $E(MSE) = \sigma^2$ and $E(MSTR) = \sigma^2 + \frac{\sum_{i=1}^{r} n_i(\mu_i - \mu_.)^2}{r-1}$

where $\mu_. = \sum_{i=1}^{r} n_i\mu_i / \sum_{i=1}^{r} n_i$

If $SSE = \sum_i\sum_j (Y_{ij} - \bar{Y}_{i.})^2$,

$MSE = \frac{1}{n_T - r}\sum_i\sum_j (Y_{ij} - Y_{i.})^2 = \frac{1}{n_T - r}\sum_i\left[(n_j - 1)\underbrace{\frac{\sum_j (Y_{ij} - \bar{Y}_{i.})^2}{n_j - 1}}_{\text{sample variance} = S_i^2}\right]$

$= \frac{1}{n_T - r}\sum_i (n_j - 1) S_i^2$

$E(MSE) = E\left\{\frac{1}{n_T - r}\sum_j (n_j - 1)S_i^2\right\} = \frac{1}{n_T - r}\sum_j (n_j - 1) E\{S_i^2\}$

$= \frac{1}{n_T - r}\underbrace{(n_T - r)}_{\sum_j n_j = n_T,\ \sum_j 1 = r}\sigma^2 = \sigma^2$

$SSTR = \sum_i n_i(\bar{Y}_{i.} - \bar{Y}_{..})^2$

$MSTR = \frac{1}{r-1}\sum_i n_i(\bar{Y}_{i.} - \bar{Y}_{..})^2$

we have $Y_{ij} = \mu_i + \epsilon_{ij}$

$\bar{Y}_{i.} = \mu_i + \bar{\epsilon}_{i.}$

$\bar{Y}_{..} = \mu_. + \bar{\epsilon}_{..} \rightarrow \mu_. = \frac{n_i \sum \mu_i}{n_i r}$

$\bar{Y}_{..} = \frac{\sum \bar{Y}_{i.}}{r}$, $\bar{\epsilon}_{..} = \frac{\sum \bar{\epsilon}_{i.}}{r}$

$\bar{Y}_{i.} - \bar{Y}_{..} = (\mu_i + \bar{\epsilon}_{i.}) - (\mu_. + \bar{\epsilon}_{..})$

$= (\mu_i - \mu_.) - (\bar{\epsilon}_{i.} - \bar{\epsilon}_{..})$

2 ⓑ, continued

square both sides and sum

$\Sigma_i(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 = \Sigma_i(\mu_i - \mu_\cdot)^2 + \Sigma_i(\bar{\epsilon}_{i\cdot} - \bar{\epsilon}_{\cdot\cdot})^2 - 2\Sigma_i(\mu_i - \mu_\cdot)(\bar{\epsilon}_{i\cdot} - \bar{\epsilon}_{\cdot\cdot})$

we want $E\{\Sigma_i(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2\}$, so we can take expectation of each element on the right

$E\{\Sigma_i(\mu_i - \mu_\cdot)^2\} = \Sigma_i(\mu_i - \mu_\cdot)^2$

$E\{\Sigma_i(\bar{\epsilon}_{i\cdot} - \bar{\epsilon}_{\cdot\cdot})^2\}$

$\hookrightarrow \sigma^2(\bar{\epsilon}_{i\cdot}) = \frac{\sigma^2(\epsilon_{ij})}{n} = \frac{\sigma^2}{n}$

$\hookrightarrow = \frac{\sigma^2}{n}(r-1)$

$E\{\Sigma_i(\mu_i - \mu_\cdot)(\bar{\epsilon}_{i\cdot} - \bar{\epsilon}_{\cdot\cdot})\} = 0$ because $E(\bar{\epsilon}_{i\cdot}) = E(\bar{\epsilon}_{\cdot\cdot}) = 0$

So, $E(MSTR) = \Sigma_i(\mu_i - \mu_\cdot)^2 + \frac{\sigma^2}{n}(r-1)$

$$= \sigma^2 + \frac{n\Sigma_i(\mu_i - \mu_\cdot)^2}{r-1}$$