Orly Olbum
STAT 2261

HOMEWORK 1

1. #2.8 p.58-59

X = battery life of internal pacemaker

$X \sim$ Pareto ($\theta = 4, \lambda = 5$) $\quad f(x) = \theta\lambda^\theta / x^{\theta+1} = \frac{5 \cdot 4^5}{x^5} = 5 \cdot (\frac{4}{x})^5$

ⓐ $Pr(X \geq 10) = S(10) = \int_{10}^\infty f(t)dt$

$= \int_{10}^\infty 5 \cdot \frac{4^5}{x^5} dx = 5 \cdot 4^5 \int_{10}^\infty x^{-5} dx$

$= 5 \cdot 4^5 [-\frac{1}{4} x^{-4}]_{10}^\infty = 5 \cdot 4^5 [(0) - (-\frac{1}{4}(10)^{-4})]$

$= 5 \cdot 4^5 \cdot \frac{1}{4} \cdot \frac{1}{10^4} = 5 \cdot 4^4 \cdot \frac{1}{10^4} = .128$

ⓑ Mean time to battery failure

$mrl(x) = \int_x^\infty S(t)dt / S(x)$

$S(x) = \frac{\lambda^\theta}{x^\theta} = (\frac{5}{x})^4 = 5^4 \cdot \frac{1}{x^4}$

$\int_x^\infty 5^4 \cdot t^{-4} dt = 5^4 [-\frac{1}{3} t^{-3}]_x^\infty = 5^4 [(0) - (-\frac{1}{3} t^{-3})]$

$= 5^4 \cdot \frac{1}{3} x^{-3}$

$mrl(x) = \frac{\cancel{5^4} \cdot \frac{1}{3} \cdot \frac{1}{x^3}}{\cancel{5^4} \cdot \frac{1}{x^4}} = \frac{1}{3} \cdot \frac{1}{x^3} \cdot \frac{x^4}{1} = \frac{x}{3}$

ⓒ Battery scheduled to be replaced at time t_0, where 99% of batteries haven't failed (ie $Pr(X > t_0) = .99$), find t_0.

$Pr(X > t_0) = S(t_0) = .99$

$.99 = (\frac{5}{t_0})^4$

$\sqrt[4]{.99} = \frac{5}{t_0}$

$t_0 = \frac{5}{\sqrt[4]{.99}} = 5.013$

2. 2.6 (b) (c), p. 58

X = time to death in months

$X \sim$ Gompertz ($\theta = 0.01, \alpha = 0.25$)

$f(x) = \theta e^{\alpha x} \exp\{\frac{\theta}{\alpha}(1 - e^{\alpha x})\} = .01 e^{.25x} \exp[\frac{.01}{.25}(1 - e^{.25x})]$

$S(x) = \exp[\frac{\theta}{\alpha}(1 - e^{\alpha x})] = \exp[\frac{.01}{.25}(1 - e^{.25x})]$

ⓑ prob that randomly chosen mouse will die within the first 6 months

$Pr(X < 6) = S(6) = \exp[\frac{1}{25}(1 - e^{.25(6)})]$

$= .8699$

2, continued

(c) Median time to death

$$S(x_{0.5}) = .5$$

$$S(x) = \exp\left[\frac{\theta}{\alpha}(1-e^{\alpha x})\right]$$

$$.5 = \exp\left[\frac{.01}{.25}(1-e^{.25x})\right]$$

$$\log(.5) = \frac{1}{25}(1-e^{.25x})$$

$$25\log(.5) = 1-e^{.25x}$$

$$e^{.25x} = 1-25\log(.5)$$

$$.25x = \log[1-25\log(.5)]$$

$$x = \log[1-25\log(.5)] / .25 = 11.634 \text{ months}$$

3. 2.10 (a) p. 59

Constant hazard rate model, $\tau_0 = 0$, $\tau_K = \infty$

$$h(x) = \begin{cases} \theta_1, & 0 \le x < \tau_1 \\ \theta_2, & \tau_1 \le x < \tau_2 \\ \vdots & \\ \theta_{K-1}, & \tau_{K-2} \le x < \tau_{K-1} \\ \theta_K, & x \ge \tau_{K-1} \end{cases}$$

(a) Find the survival function for the model.

Since $h(x)$ = constant, we have a piecewise exponential distribution:

$$f(x) = \begin{cases} \theta_1 \exp(-\theta_1 x), & 0 \le x < \tau_1 \\ \theta_2 \exp(-\theta_2 x), & \tau_1 \le x < \tau_2 \\ \vdots & \\ \theta_K \exp(-\theta_K x), & x \ge \tau_{K-1} \end{cases}$$

Which gives us (from $S(x) = \frac{f(x)}{h(x)}$):

$$S(x) = \begin{cases} \exp(-\theta_1 x), & 0 \le x < \tau_1 \\ \exp(-\theta_2 x), & \tau_1 \le x < \tau_2 \\ \vdots & \\ \exp(-\theta_K x), & x \ge \tau_{K-1} \end{cases}$$

Because $H(x) = \theta_i x$, $S(x) = \exp\{-H(x)\} = \exp\{-\theta_i x\}$

4. 2.18 p.61

$X \sim \text{Uniform}[0,100]$ (days)

$f(x) = \begin{cases} \frac{1}{100}, & 0 < x < 100 \\ 0, & \text{ow} \end{cases}$

ⓐ Find $S(25), S(50), S(75)$

$S(x) = \int_x^\infty f(t)\,dt = \int_x^{100} \frac{1}{100}\,dt = \left[\frac{1}{100}t\right]_x^{100} = 1 - \frac{x}{100}$

$S(25) = \int_{25}^{100} \frac{1}{100}\,dt = \left[\frac{1}{100}t\right]_{25}^{100} = 1 - \frac{1}{4} = 3/4$

$S(50) = \int_{50}^{100} \frac{1}{100}\,dt = \left[\frac{1}{100}t\right]_{50}^{100} = 1 - \frac{1}{2} = 1/2$

$S(75) = \int_{75}^{100} \frac{1}{100}\,dt = \left[\frac{1}{100}t\right]_{75}^{100} = 1 - \frac{3}{4} = 1/4$

ⓑ Mean residual lifetime at 25, 50, 75 days

$mrl(x) = \int_x^\infty S(u)\,du \,/ S(x)$

$= \underbrace{\int_x^\infty \left[1 - \frac{u}{100}\right] du}\,/\left[1 - \frac{x}{100}\right]$

$\int_x^\infty \left[1 - \frac{1}{100}u\right] du = \left[u - \frac{1}{200}u^2\right]_x^{100} = \left[(100 - 50) - (x - \frac{1}{200}x^2)\right]$

$mrl(x) = \left(50 - x + \frac{x^2}{200}\right) / \left(1 - \frac{x}{100}\right)$

$mrl(25) = \left(50 - 25 + \frac{625}{200}\right) / \left(1 - \frac{25}{100}\right) = \frac{28.125}{.75} = 37.5$

$mrl(50) = \left(0 + \frac{2500}{200}\right) / \left(1 - \frac{50}{100}\right) = \frac{12.5}{.5} = 25$

$mrl(75) = \left(-25 + \frac{5625}{200}\right) / \left(1 - \frac{75}{100}\right) = \frac{3.125}{.25} = 12.5$

ⓒ Median residual lifetime at 25, 50, 75 days

Because we are looking at uniform distribution, the median will be the ame as the mean from ⓑ