STAT 2321
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Homework 3: Chapter 2 part 1

2.1 Stationarity is important because we need to assume regularity (ie, that statistical properties hold over time) exists in the behavior over the entire time series.

2.2 $x_t = \beta_0 + \beta_1 t + w_t$

β_0 & β_1 are regression coefficients, w_t is white noise σ_w^2

(a) Determine if x_t is stationary.

- Conditions of stationarity: mean does not rely on t, and $\gamma(s,t)$ (acv function) only depends on times t & s through their difference

$E(x_t) = E(\beta_0 + \beta_1 t + w_t) = \beta_0 + \beta_1 t$, and since the mean depends on t, x_t is not stationary

(b) Show that $y_t = x_t - x_{t-1}$ is stationary.

$y_t = (\cancel{\beta_0} + \beta_1 t + w_t) - (\cancel{\beta_0} + \beta_1(t-1) + w_{t-1})$

$= \cancel{\beta_1 t} + w_t - \cancel{\beta_1 t} + \beta_1 - w_{t-1}$

$= \beta_1 + w_t - w_{t-1}$

$E(y_t) = \beta_1$ checks out

$\gamma(t+h, t) = \text{cov}(y_{t+h}, y_t) = E[(y_{t+h} - E(y_{t+h}))(y_t - \overbrace{\beta_1}^{E(y_t)})]$

$[E(y_{t+h}) = E(\beta_1 + w_{t+h} - w_{t+h-1}) = \beta_1]$

$= E[(y_{t+h} - \beta_1)(y_t - \beta_1)] = E[(y_{t+h}y_t - \beta_1 y_{t+h} - \beta_1 y_t + \beta_1^2)]$

$= E(y_{t+h}y_t) - \beta_1 E(y_{t+h}) - \beta_1 E(y_t) + \beta_1^2$

$= E(y_{t+h}y_t) - \beta_1(\beta_1) - \beta_1(\beta_1) + \beta_1^2$

$= E(y_{t+h}y_t) - 2\beta_1^2 + \beta_1^2 = E(y_{t+h}y_t) - \beta_1^2$

And if $y_t \perp\!\!\!\perp y_{t+h}$, $E(y_t y_{t+h}) = E(y_t)E(y_{t+h}) = \beta_1^2$

So, $\gamma(t+h, t) = 0$

making $y_t = x_t - x_{t-1}$ stationary

→

2.2, continued $\quad X_t = \beta_0 + \beta_1 t + w_t$

(a) show mean of

$$V_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1}) \quad \text{is} \quad \beta_0 + \beta_1 t$$

$X_{t-1} = \beta_0 + \beta_1(t-1) + w_{t-1} = \beta_0 + \beta_1 t - \beta_1 + w_{t-1}$

$X_t = \beta_0 + \beta_1 t + w_t$

$X_{t+1} = \beta_0 + \beta_1(t+1) + w_{t+1} = \beta_0 + \beta_1 t + \beta_1 + w_{t+1}$

$E(V_t) = \frac{1}{3} E[X_{t-1} + X_t + X_{t-1}]$

$= \frac{1}{3} E[\cancel{\beta_0} + \cancel{\beta_1 t} - \beta_1 + w_{t-1} + \cancel{\beta_0} + \cancel{\beta_1 t} + w_t + \cancel{\beta_0} + \cancel{\beta_1 t} + \beta_1 + w_{t+1}]$

$= \frac{1}{3} E[3\beta_0 + 3\beta_1 t - \cancel{\beta_1} + \cancel{\beta_1} + w_{t-1} + w_t + w_{t+1}]$

$= \frac{1}{3}[3\beta_0 + 3\beta_1 t] = \boxed{\beta_0 + \beta_1 t}$

2.3 Consider $X_t = \frac{1}{4}(w_{t-1} + 2w_t + w_{t+1})$

w_t iid, $(0, \sigma_w^2)$

Find acv and acf as a function of lag h, sketch ACF as a function of h

Autocovariance Function:

$\gamma(t+h, t) = \text{cov}(X_{t+h}, X_t) = \text{cov}\{\frac{1}{4}(w_{t+h-1} + 2w_{t+h} + w_{t+h+1}), \frac{1}{4}(w_{t-1} + 2w_t + w_{t+1})\}$

$= \frac{1}{16}\text{cov}\{(w_{t+h-1} + 2w_{t+h} + w_{t+h+1}), (w_{t-1} + 2w_t + w_{t+1})\}$

for $h=0$:

$\gamma(0) = \frac{1}{16}\text{cov}\{w_{t-1} + 2w_t + w_{t+1}), (w_{t-1} + 2w_t + w_{t+1})\}$

$= \frac{1}{16}\{\text{var}(w_{t-1}) + \text{var}(2w_t) + \text{var}(w_{t+1})\}$

$= \frac{1}{16}(\sigma_w^2 + 4\sigma_w^2 + \sigma_w^2) = \frac{6}{16}\sigma_w^2 = \frac{3}{8}\sigma_w^2$

for $h = 1, -1$ $\quad (\gamma(h) = \gamma(-h))$

$\gamma(1) = \frac{1}{16}\text{cov}\{(w_t + 2w_{t+1} + \cancel{w_{t+2}}), (\cancel{w_{t-1}} + 2w_t + w_{t+1})\}$

$= \frac{1}{16}\{2\text{var}(w_t) + 2\text{var}(w_{t+1})\} = \frac{1}{16}\{4\sigma_w^2\} = \frac{1}{4}\sigma_w^2$

for $h = 2, -2$

$\gamma(2) = \frac{1}{16}\text{cov}\{(\cancel{w_{t-3}} + \cancel{2w_{t-2}} + w_{t-1}), (w_{t-1} + \cancel{2w_t} + \cancel{w_{t+1}})\}$

$= \frac{1}{16}\text{var}(w_{t-1}) = \frac{1}{16}\sigma_w^2$

$$\gamma(h) = \left.\begin{cases} \frac{3}{8}\sigma_w^2, & h=0 \\ \frac{1}{4}\sigma_w^2, & |h|=1 \\ \frac{1}{16}\sigma_w^2, & |h|=2 \\ 0, & |h|>2 \end{cases}\right\} \text{ and } E(X_t) = 0$$

(2.3, continued)

Autocorrelation Function (ACF)

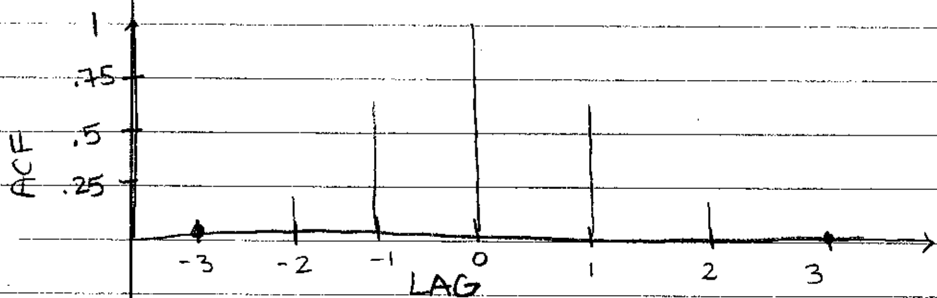
$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$

$\rho_v(0) = 1$

$\rho_v(1) = \frac{1/4\,\sigma_w^2}{3/8\,\sigma_w^2} = \frac{1}{4} \cdot \frac{8}{3} = \frac{2}{3}$

$\rho_v(2) = \frac{1/16\,\sigma_w^2}{3/8\,\sigma_w^2} = \frac{1}{16} \cdot \frac{8}{3} = \frac{1}{6}$

So, $\rho_v(h) = \begin{cases} 1, & h = 0 \\ \frac{2}{3}, & |h| = 1 \\ \frac{1}{6}, & |h| = 2 \\ 0, & |h| > 2 \end{cases}$



2.4 $X_t = \phi X_{t-1} + w_t$, $w_t \sim wn(0,1)$, ϕ is a constant

Assume X_t is stationary, X_{t-1} is uncorrelated with w_t

ⓐ Show that mean function X_t is $\mu_{X_t} = 0$

$E(X_t) = E(\phi X_{t-1} + w_t) = \phi E(X_{t-1}) + E(w_t)$

We know $E(X_t) = \mu_t$ is constant,

$\phi E(X_{t-1}) + \underbrace{E(w_t)}_{0} = \phi E(X_{t-1})$ has to mean

$\phi\mu + 0 = \mu$, so $\mu_{X_t} = 0$

ϕ cannot be 1, otherwise X_t will be random walk

ⓑ Show $\gamma_x(0) = var(X_t) = \frac{1}{1-\phi}$

$\gamma_x(0) = var(X_t) = var(\phi X_{t-1} + w_t) = \phi^2 var(X_{t-1}) + 1$

$= \phi^2 \gamma_x(0) + 1$

$\gamma_x(0) = \phi^2\gamma_x(0) + 1 \rightarrow \gamma_x(0)(1 - \phi^2) = 1$

So, $\gamma_x(0) = \frac{1}{(1-\phi^2)}$

→

2.4, continued

(c) For which values of ϕ does (b) make sense?
→ we need $|\phi|<1$ for (b) to work,
with finite, positive variance

(d) Find $\rho_x(1)$
$\rho_x(1) = \gamma_x(1)/\gamma_x(0)$ $\quad \gamma_x(0) = \frac{1}{1-\phi}$
$\gamma_x(1) = cov(X_t, X_{t-1}) = cov(\phi X_{t-1} + w_t, X_{t-1})$
$= cov(\phi X_{t-1}, X_{t-1}) = \phi\, cov(X_{t-1}, X_{t-1}) = \phi\gamma_x(0) = \frac{\phi}{1-\phi}$
So, $\rho_x(1) = \left(\frac{\phi}{1-\phi}\right) / \left(\frac{1}{1-\phi}\right) = \left(\frac{\phi}{\cancel{1-\phi}}\right)\left(\frac{\cancel{1-\phi}}{1}\right) = \phi$

2.5 Random walk with drift model:
$$X_t = \delta + X_{t-1} + w_t$$
for $t=1,2,\ldots$ with $X_0=0$, w_t is white noise σ_w^2
(a) Write model as $X_t = \delta t + \sum_{k=1}^{t} w_k$
Go backwards:
$X_t = \delta + (\delta + X_{t-2} + w_{t-1}) + w_t$
$= 2\delta + X_{t-2} + w_{t-1} + w_t$
$= 2\delta + (\delta + X_{t-3} + w_{t-2}) + w_{t-1} + w_t$
$= \underbrace{3\delta}_{t=3} + X_{t-3} + w_{t-2} + w_{t-1} + w_t \rightarrow \Sigma w_t$
And $X_1 = \delta + w_t$
→ For $t-1$, above is
$= \delta(t-1) + \sum_{k=1}^{t-1} w_k$ and for t,
$X_t = \delta + X_{t-1} + w_t = \delta + \delta(t-1) + \sum_{k=1}^{t-1} w_k + w_t$
$= \delta t + \sum_{k=1}^{t} w_k$

(b) Find mean & acv functions for X_t
Because w_t is uncorrelated, and δ is a constant,
$E(X_t) = E\left[\delta t + \sum_{k=1}^{t} w_k\right] = \delta t \sum_{k=1}^{t} E(w_k) = \delta t$

→

$$X_t = \delta + X_{t-1} + w_t$$
$$= \delta t + \sum_{k=1}^{t} w_t$$

2.5 (b), continued $E(X_t) = \delta t$

Autocovariance function:

$\gamma_X(s,t) = cov(X_s, X_t) = E[(X_s - E(X_s))(X_t - E(X_t))]$

$cov(X_t, X_{t-1}) = cov(\delta + X_{t-1} + w_t, X_{t-1}) = var(X_{t-1})$

$var(X_{t-1}) = t\sigma_w^2$

(c) X_t is not stationary → we have already seen the first requirement broken because the mean function $\mu_{X_t} = E(X_t) = \delta t$, depends on time t, so X_t is not stationary

(d) Show $\rho_X(t-1,t) = \sqrt{\frac{t-1}{t}} \to 1$ as $t \to \infty$ (& what is the implication?)

$$\rho_X(t-1,t) = \frac{\gamma(t-1,t)}{\sqrt{\gamma_X(t-1,t-1)\gamma_X(t,t)}}$$
$$= \frac{var(t-1)}{\sqrt{var(t-1)var(t)}}$$
$$= \frac{(t-1)\sigma_w^2}{\sqrt{[(t-1)\sigma_w^2 \cdot t\sigma_w^2]}}$$
$$= \frac{(t-1)\cancel{\sigma_w^2}}{\sqrt{t(t-1)}\cancel{\sigma_w^2}} = \frac{t-1}{\sqrt{t(t-1)}} = \sqrt{\frac{t-1}{t}}$$

As $t \to \infty$, $t-1$ & t are approximately equal, leading $\rho_X(t-1,t)$ to approach 1

This implies that as time goes on $(\to\infty)$, the ACF loses its power

(e) To make the series stationary we might try differencing: $y_t = X_t - X_{t-1}$

$y_t = X_t - X_{t-1} = \delta + X_{t-1} + w_t - X_{t-1} = \delta + w_t$

Check stationarity:

$E(y_t) = E(\delta + w_t) = \delta$ (not dependent on t!)

ACV: $\gamma(t+h,t) = cov(X_{t+h}, X_t) = cov(\delta + w_{t+h}, \delta + w_t) = \sigma^2$, when $h=0$

so we satisfy both stationarity conditions, y_t is stationary