STAT 2321
Orly Olbum

Ch.2 Part 2

2.6 Is global temp data from Example 1.2 stationary or non-stationary?
While there are deviations from a general upward trend of both land and sea temps from 1880 to 2017, the upward trend itself would indicate stationarity, but separate chunks of the timeline change, and especially the drastic uptick in the last 40 years would indicate a changing trend, indicating <u>non-stationarity</u> (or trend stationarity).

2.7 Periodic time series:

$$x_t = U_1 \sin(2\pi\omega_0 t) + U_2 \cos(2\pi\omega_0 t)$$

where $U_1 \perp\!\!\!\perp U_2$, rV's, means 0 & $E(U_1^2) = E(U_2^2) = \sigma^2$
constant ω_0 determines period of time to complete a cycle. Show that x_t is weakly stationary with acv function

$$\gamma(h) = \sigma^2 \cos(2\pi\omega_0 h)$$

Say $\lambda = 2\pi\omega_0$

$\gamma(h) = \gamma(h,h) = \text{var}(x_h)$

$\hookrightarrow \gamma(h) = \text{var}(U_1 \sin(\lambda h) + U_2 \cos(\lambda h))$

$= \text{cov}[(U_1 \sin(\lambda h) + U_2 \cos(\lambda h)), (U_1 \sin(\lambda h) + U_2 \cos(\lambda h))]$

$= \text{cov}[U_1 \sin(\lambda h), U_1 \sin(\lambda h)] + \text{cov}[U_1 \sin(\lambda h), U_2 \cos(\lambda h)]$
$+ \text{cov}[U_2 \cos(\lambda h), U_1 \sin(\lambda h)] + \text{cov}[U_2 \cos(\lambda h), U_2 \cos(\lambda h)]$

$= \sigma^2 \sin^2(\lambda h) + 0 + 0 + \sigma^2 \cos^2(\lambda h)$

$= \sigma^2 [\sin^2(\lambda h) + \cos^2(\lambda h)]$

$= \sigma^2 \cos(\lambda h) = \sigma^2 \cos(2\pi\omega_0 h)$

The series is weakly stationary because the ACV depends not on time t, but on time difference (lag) h, making it trend, or weakly, stationary.

2.8 Consider

$$x_t = w_t$$
$$y_t = w_t - \theta w_{t-1} + u_t$$

w_t, u_t are independent white noise σ_w^2 & σ_u^2 and θ is a constant

(a) Express ACF $\rho_y(h)$ for $h = 0, \pm 1, \pm 2, \ldots$ of y_t as a function of σ_w^2, σ_u^2, and θ

$$\rho_y(h) = \frac{\gamma(h)}{\gamma(0)}$$

$$\gamma_y(h) = \gamma(y_{t+h}, y_t) = cov(y_{t+h}, y_t)$$
$$= cov\{(w_{t+h} - \theta w_{t+h-1} + u_{t+h}), (w_t - \theta w_{t-1} + u_t)\}$$

$$\gamma_y(0) = cov\{(w_t - \theta w_{t-1} + u_t), (w_t - \theta w_{t-1} + u_t)\}$$
$$= var(w_t) + \theta^2 var(w_{t-1}) + var(u_t)$$
$$= \sigma_w^2 + \theta^2 \sigma_w^2 + \sigma_u^2 = \sigma_w^2(1+\theta^2) + \sigma_u^2$$

$$\gamma_y(1) = \gamma_y(-1) = cov\{(w_{t+1} - \theta w_t + u_{t+1}), (w_t - \theta w_{t-1} + u_t)\}$$
$$= -\theta\, var(w_t) = -\theta \sigma_w^2$$

$$\gamma_y(2) = \gamma_y(-2) = cov\{(w_{t+2} - \theta w_{t+1} + u_{t+2}), (w_t - \theta w_{t-1} + u_t)\}$$
$$= 0$$

$$\gamma_y(h) = \begin{cases} \sigma_w^2(1+\theta^2) + \sigma_u^2, & h = 0 \\ -\theta\sigma_w^2, & h = \pm 1 \\ 0, & |h| > 1 \end{cases}$$

$$\rho_y(h) = \frac{\gamma(h)}{\gamma(0)}$$

$$\rho_y(0) = 1$$
$$\rho_y(1) = \frac{-\theta\sigma_w^2}{\sigma_w^2(1+\theta^2) + \sigma_u^2} = \frac{-\theta\sigma_w^2}{\sigma_w^2 + \theta^2\sigma_w^2 + \sigma_u^2}$$
$$\rho_y(2) = 0$$

$$\rho_y(h) = \begin{cases} 1, & h = 0 \\ \dfrac{-\theta\sigma_w^2}{\sigma_w^2(1+\theta^2) + \sigma_u^2}, & h = \pm 1 \\ 0, & |h| > 1 \end{cases}$$

next page →

(2.8, continued)

(b) CCF $\rho_{xy}(h)$ relating to x_t & y_t

$x_t = w_t$

$y_t = w_t - \theta w_{t-1} + u_t$

$\rho_{xy}(h) = \gamma_{xy}(h) / \gamma_{xy}(0)$

$\gamma_{xy}(h) = \text{cov}(x_t, y_t) = \text{cov}\{w_t, w_t - \theta w_{t-1} + u_t\}$

$\gamma_{xy}(0) = \text{cov}\{w_t, w_t - \theta w_{t-1} + u_t\}$

$= \text{var}(w_t) = \sigma_w^2$

$\gamma_{xy}(-1) = \text{cov}\{w_{t-1}, w_t - \theta w_{t-1} + u_t\}$

$= -\theta \text{var}(w_{t-1}) = -\theta\sigma_w^2$

$\gamma_{xy}(2) = \text{cov}\{w_{t-2}, w_t - \theta w_{t-1} + u_t\} = 0$

$$\gamma_{xy}(h) = \begin{cases} \sigma_w^2, & h=0 \\ -\theta\sigma_w^2, & h=-1 \quad , 0 \quad h=1 \\ 0 & , |h|>1 \end{cases}$$

$\rho_{xy}(h) = \gamma_{xy}(h) / \sqrt{\gamma_x(0)\gamma_y(0)}$

$\rho_{xy}(0) = \sigma_w^2 / \sqrt{(\sigma_w^2)(\sigma_w^2(1+\theta^2) + \sigma_u^2)} = \sigma_w^2 / \sigma_w^2[(1+\theta^2) + \sigma_u^2]$

$\rho_{xy}(1) = -\theta / \sqrt{(\sigma_w^2)(\sigma_w^2(1+\theta^2) + \sigma_u^2)} = -\theta / \sigma_w^2[(1+\theta^2) + \sigma_u^2]$

$\rho_{xy}(2) = 0 / \sqrt{(\sigma_w^2)(\sigma_w^2(1+\theta^2) + \sigma_u^2)} = 0$

$$\rho_{xy}(h) = \begin{cases} \sigma_w^2 / \sigma_w^2[(1+\theta^2) + \sigma_u^2] & , h=0 \\ -\theta / \sigma_w^2[(1+\theta^2) + \sigma_u^2] & , h=\pm 1 \\ 0, |h|>1 \end{cases}$$

(c) Show that x_t and y_t are jointly stationary

x_t and y_t are stationary if

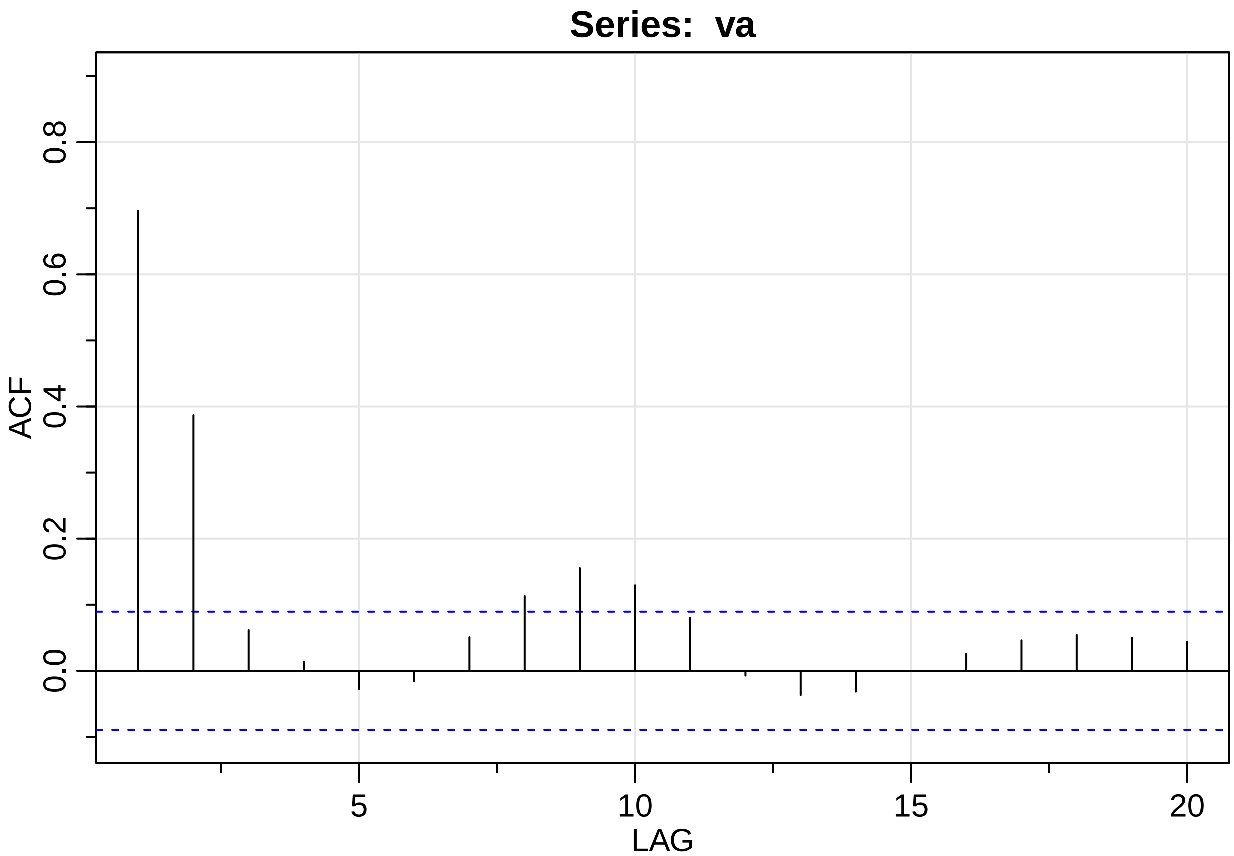
$\rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0)\gamma_x(y)}}$ is a function of h only

we have $\gamma_{xy}(h)$ from (b), and see that it satisfies the requirements of joint stationarity. Both of their γ_x, γ_y are stationary (no dependence on time), and is only a function of lag h.

2.11

(a) Simulate a series of n = 500 Gaussian white noise observations as in Example 1.7 and compute the sample ACF, p(h), to lag 20. Compare the sample ACF you obtain to the actual ACF, p(h). [Recall Example 2.17.]

```
wa = rnorm(500)
va = filter(wa, side = 2, filter = rep(1/3, 3))
acf1(va, 20)
```

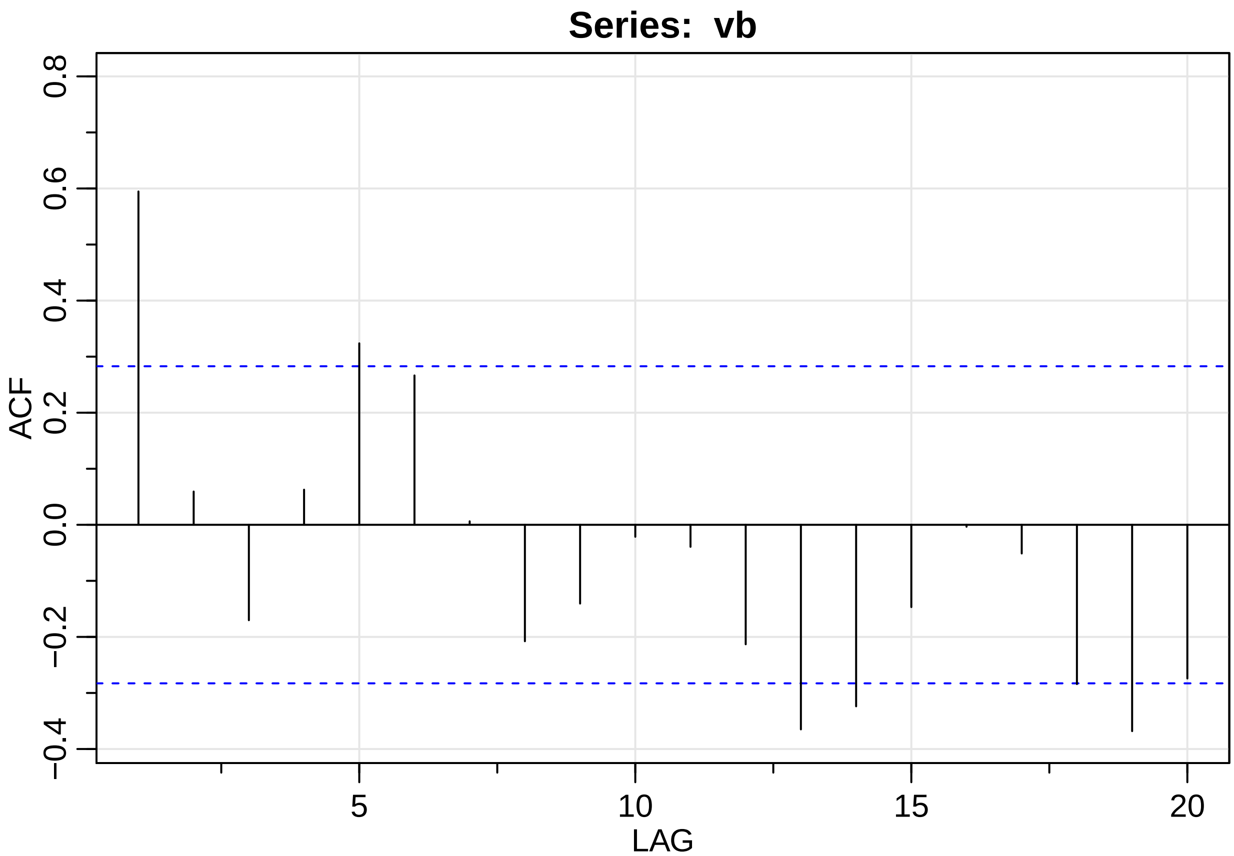



```
##  [1]  0.70  0.39  0.06  0.01 -0.03 -0.02  0.05  0.11  0.16  0.13  0.08 -0.01
## [13] -0.04 -0.03  0.00  0.03  0.05  0.05  0.05  0.04
```

Compare sample ACF and true ACF: Theoretically should be zero (actual ACF, calculated in the book), but some fluctuation around 0. Plotted, there are some residual values. Almost all spikes are within bounds so we can consider them white noise.

(b) Repeat part (a) using only n = 50. How does changing n affect the results?

```
wb = rnorm(50)
vb = filter(wb, side = 2, filter = rep(1/3, 3))
acf1(vb, 20)
```



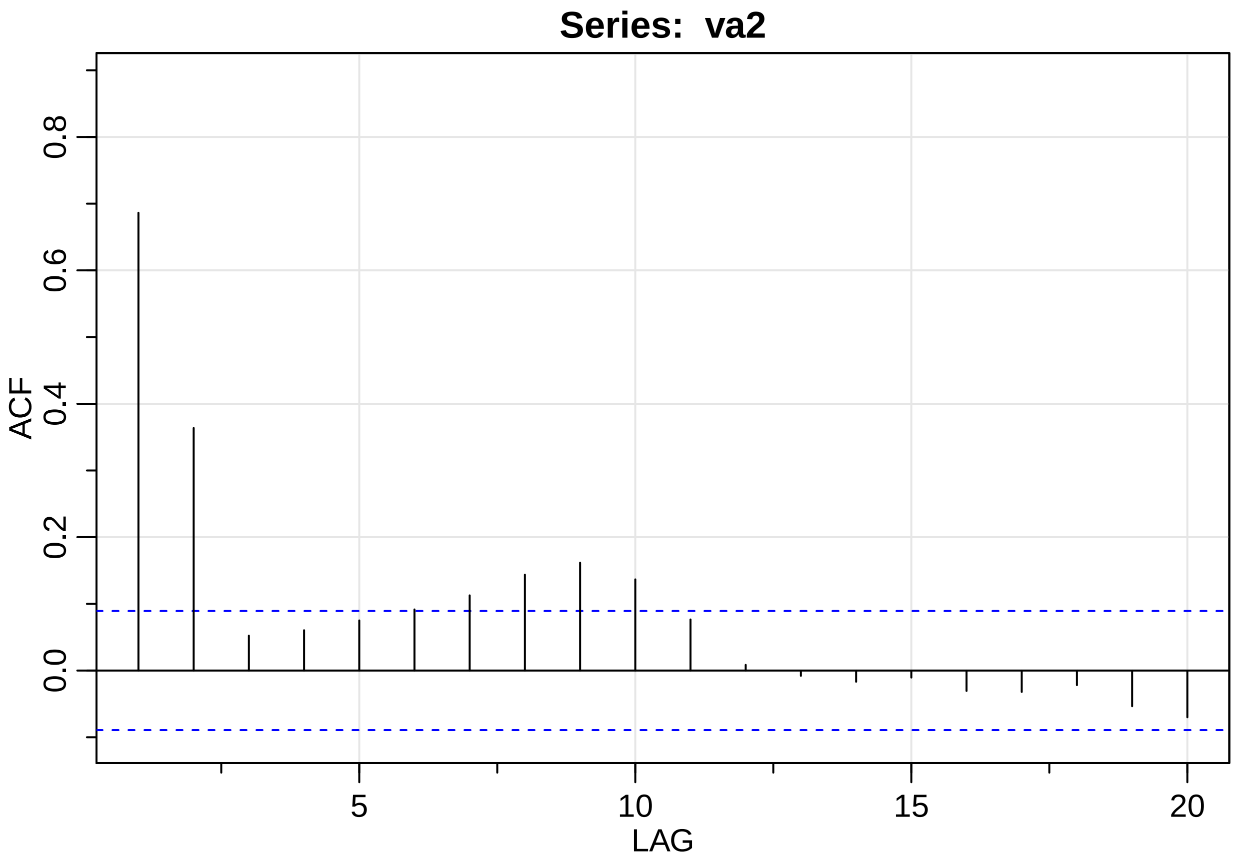

```
##  [1]  0.59  0.06 -0.17  0.06  0.32  0.27  0.01 -0.21 -0.14 -0.02 -0.04 -0.21
## [13] -0.37 -0.32 -0.15  0.00 -0.05 -0.28 -0.37 -0.27
```

This changes results: There is more variability because we have fewer observations, and we see a few more spikes beyond the bounds. We expect about 95% of observations to remain within the bounds.

2.12

(a) Simulate a series of n = 500 moving average observations as in Example 1.8 and compute the sample ACF, p(h), to lag 20. Compare the sample ACF you obtain to the actual ACF, p(h). [Recall Example 2.18 .]

```
wa2 = rnorm(502, 0, 1)
va2 = filter(wa2, rep(1/3, 3))
acf1(va2, 20)
```

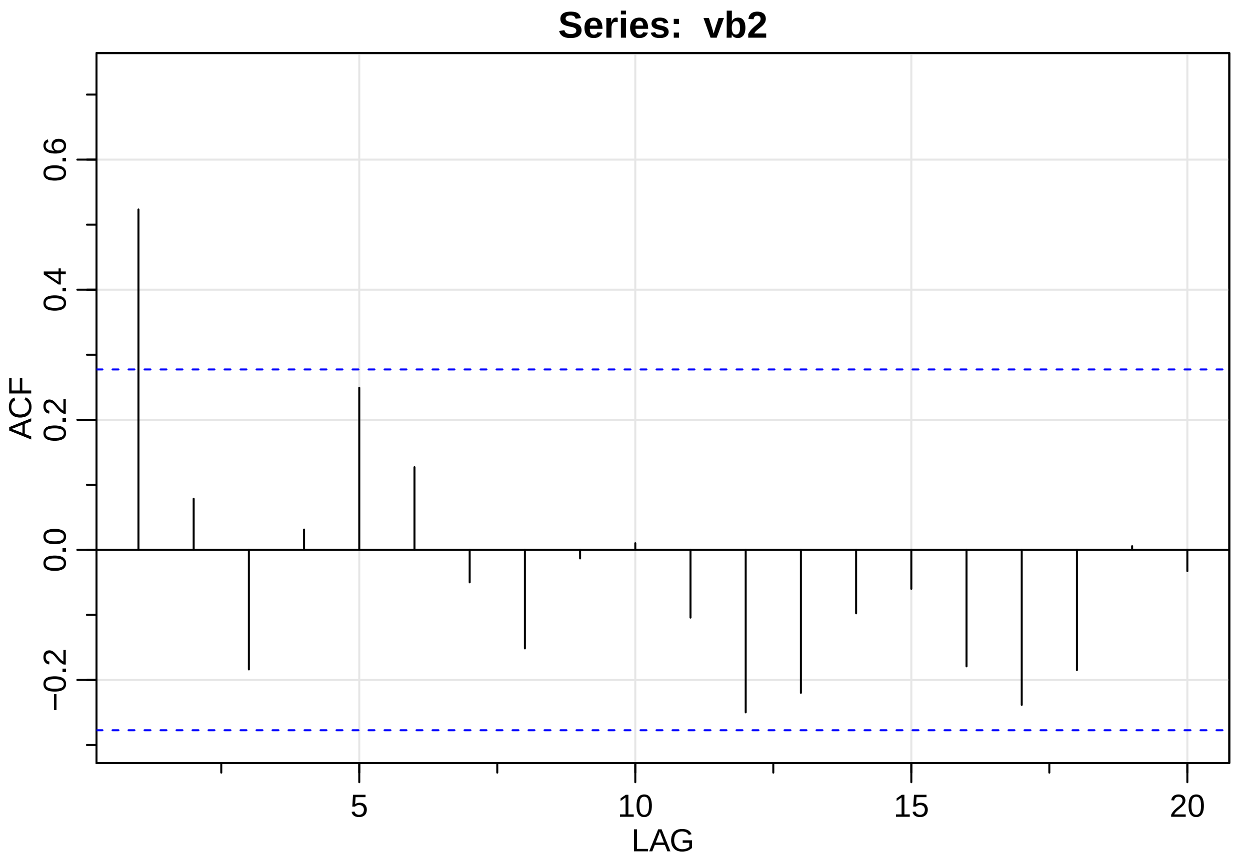



```
##  [1]  0.69  0.36  0.05  0.06  0.08  0.09  0.11  0.14  0.16  0.14  0.08  0.01
## [13] -0.01 -0.02 -0.01 -0.03 -0.03 -0.02 -0.05 -0.07
```

There are only two spikes beyond the bounds, at 1 and 2, otherwise they are within the bounds, whereas our actual ACF will be zero.

(b) Repeat part (a) using only n = 50. How does changing n affect the results?

```
wb2 = rnorm(52, 0, 1)
vb2 = filter(wb2, rep(1/3, 3))
acf1(vb2, 20)
```




```
##  [1]  0.52  0.08 -0.18  0.03  0.25  0.13 -0.05 -0.15 -0.01  0.01 -0.10 -0.25
## [13] -0.22 -0.10 -0.06 -0.18 -0.24 -0.18  0.01 -0.03
```

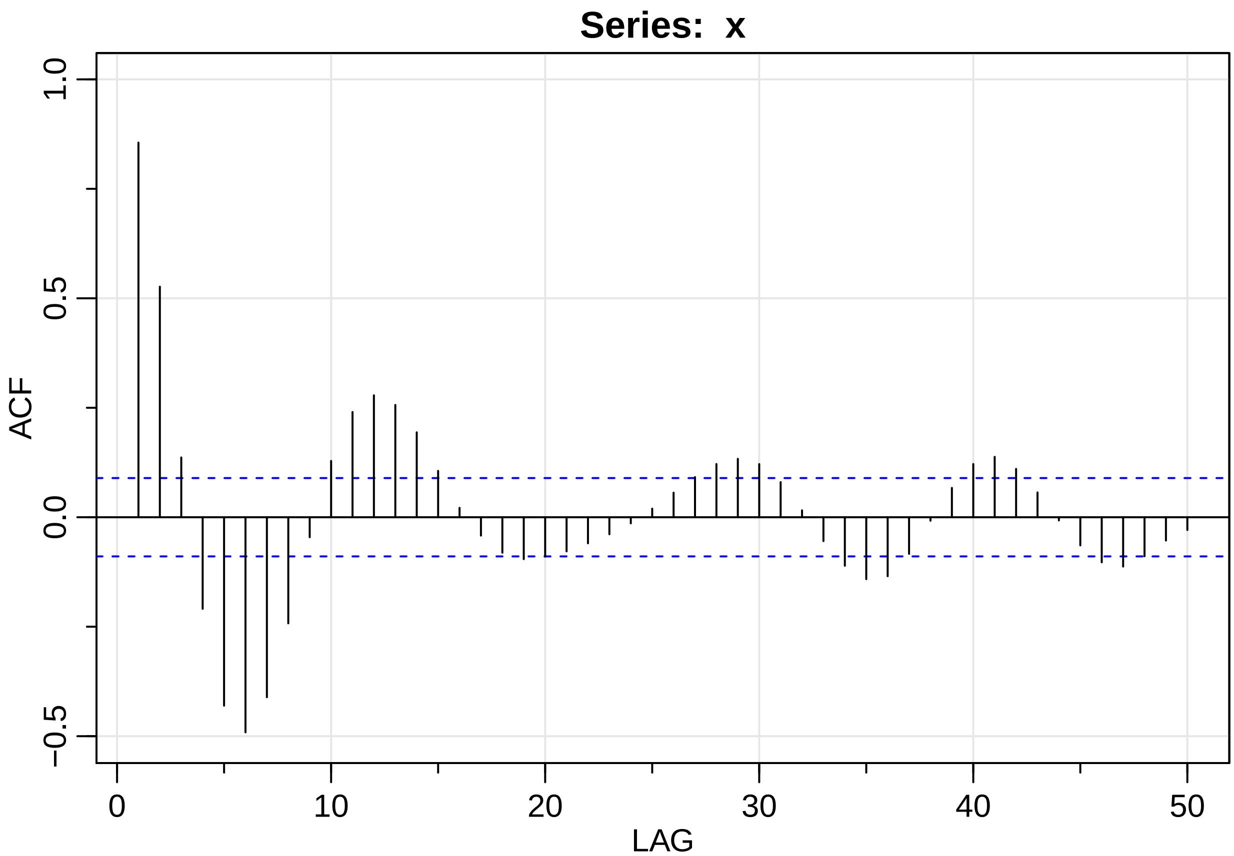
This changes the results: There is less variability with fewer observations generated, but we still have most observations within bounds leading our sample ACF to be approximately equal to the actual ACF.

2.13

Simulate 500 observations from the AR model specified in Example 1.9 and then plot the sample ACF to lag 50. What does the sample ACF tell you about the approximate cyclic behavior of the data? Hint: Recall Example 2.32.

xt = 1.5xt-1 - .75-x2 + wt

```
set.seed(90210)
w = rnorm(500 + 50)
x = filter(w, filter = c(1.5, -.75), method = "recursive")[-(1:50)]
acf1(x, 50)
```




```
## [1]  0.86  0.53  0.14 -0.21 -0.43 -0.49 -0.41 -0.24 -0.05  0.13  0.24  0.28
## [13]  0.26  0.19  0.11  0.02 -0.04 -0.08 -0.10 -0.09 -0.08 -0.06 -0.04 -0.01
## [25]  0.02  0.06  0.09  0.12  0.13  0.12  0.08  0.02 -0.05 -0.11 -0.14 -0.13
## [37] -0.08 -0.01  0.07  0.12  0.14  0.11  0.06 -0.01 -0.06 -0.10 -0.11 -0.09
## [49] -0.05 -0.03
```

The sample ACF of the generated data with lag 50 shows cyclical behavior about every 10 units, with a positive autocorrelation exhibited on 5 units and negative autocorrelation every 5 units in a cyclical manner.

2.15 For y_t in Ex 2.29 $(y_t = 5 + x_t - .5x_{t-1})$
verify stated result that
$\rho_y(1) = -.4$
and $\rho_y(h) = 0$ for $h > 1$

$\rho_y(h) = \frac{\gamma_y(h)}{\gamma_y(0)}$

$\gamma_y(h) = cov(y_{t+h}, y_t)$
$= cov\{(5 + x_{t+h} - .5x_{t+h-1}), (5 + x_t - .5x_{t-1})\}$

$\gamma(0) = cov(y_t, y_t) = var(y_t)$
$= var(x_t) + .25\, var(x_{t-1})$
$= (1 + .25)\sigma^2$

$\gamma(1) = cov(y_{t+1}, y_t) = cov\{(x_{t+1} - .5x_t), (x_t - .5x_{t-1})\}$
$= -.5\, var(x_t) = -.5\sigma^2$

$\gamma(2) = cov\{(x_{t+2} - .5x_{t+1}), (x_t - .5x_{t-1})\} = 0$

so $\rho_y(1) = {-.5}/{(1+.25)} = \frac{-.5}{1.25} = -.4$
and $\rho_y(h) = 0$ for $h > 1$