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Simultaneous moving horizon estimation and control for nonlinear systems subject to bounded disturbances

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Summary

In this work, we address the output–feedback control problem for nonlinear systems under bounded disturbances using a moving horizon approach. The controller is posed as an optimisation-based problem that simultaneously estimates the state trajectory and computes future control inputs. It minimizes a criterion that involves finite backward and forward horizons with respect to the unknown initial state, measurement noises and control input variables. The main novelty of this work relies on linking the lengths of the forward and backward windows with the closed-loop stability, assuming detectability and decoding sufficient conditions to assure system stabilizability. It leads to a formulation that does not require to be a Control Lyapunov Function for the terminal cost of the controller. Simulation examples are carried out to compare the performance of solving simultaneously and independently the estimation and control problems. Furthermore, the examples show how the controller influences the length of the estimation window through its gain.

KEYWORDS

nonlinear systems, output feedback, receding horizon control and estimation, robust stability

1 | INTRODUCTION

One of the most popular control techniques both in academia and industry is Model Predictive Control (*MPC*) due to its ability to explicitly accommodate hard state and input constraints.[1] Thereon, much effort has been made to develop a stability theory for *MPC*.[1,2] An overview of recent developments can be found in Raković and Levine.[3] *MPC* involves the solution of an open–loop optimal control problem at each sampling time with the current state as the initial condition. Each of these optimizations provide sequences of future control actions and states. The first element of the control action sequence is applied to the system and the optimization problem is solved again at the next sampling time after updating the initial condition with the system state. *MPC* keeps the computational burden constant by optimizing the system behavior within a finite length window. The system behavior beyond the window is summarized in a term known as *cost–to–go*.

MPC is often formulated assuming that the system state can be measured. However, in many practical cases, the only information available are noisy measurements of system output, so the use of independent algorithms for state estimation (including observers, filters, and estimators) becomes necessary.[4] From all of these methods, Moving Horizon Estimation (*MHE*) is especially engaging for its use with *MPC* because it can be formulated as a similar online optimization problem. Solving the *MHE* problem produces an estimated state that is compatible with a set of past measurements that recedes

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as current time advances.[5-7] This estimate is optimal in the sense that it maximizes a criterion that captures the likelihood of the measurements. Despite the fact that relevant results on *MPC* were developed, research works on *MHE* have just begun. The works of Rao et al.[6,7] provide overviews of linear and nonlinear *MHE*. Recent results regarding *MHE* for nonlinear systems are given for robust stability and estimate convergence properties by Alessandri et al.,[8,9] Garcia-Tirado et al.[10] and Sanchez et al.,[11] among others. In recent years, several results have been obtained for different *MHE* formulations, advancing from idealistic assumptions, like observability and vanishing disturbances, to realistic situations like detectability and bounded disturbances.[12-14] Zou et al.[15] are concerned with the MHE problem for networked linear systems with unknown inputs under dynamic quantization effects. The authors present a two-step MHE design strategy, which they name the decoupling and convergence steps. They provide sufficient conditions under which the estimation error is bounded and illustrate the effectiveness of the proposed method with two simulation examples.

When disturbances, model uncertainties and system constraints can be neglected, state and control sequences can be independently computed.[16-18] However, in practical applications, these conditions are very difficult to fulfil, that is, process disturbances and measurement noise are usually present, as well as model uncertainty. In this context, it becomes necessary to use an approach that includes this information into the controller design. State-feedback *MPC* is a mature field with results that consider model uncertainty, input disturbances, and noises.[19] However, these works did not consider robustness with respect to errors in state estimation. Fewer results are available for output-feedback *MPC*. An overview of nonlinear output-feedback *MPC* is given by Findeisen et al.[20] and the references therein. Many of these approaches involve designing a separate estimator and controller.[21-23] Results on robust output-feedback *MPC* for constrained, linear, discrete-time systems with bounded disturbances and measurement noise can be found in Mayne et al.[24,25] and Voelker et al.[26,27] These approaches first solve the estimation problem and prove the convergence of the estimated state to a bounded set, and then take the uncertainty of the estimation into account when solving the *MPC* problem.

All these works rely on the assumption that estimation and control problems can be solved separately. However, in a context where process disturbances and noisy measurements are present, solving both problems separately may lead to suboptimal solutions.[28] Moreover, stability analysis would be a complicated task when disturbances are not Gaussian. The problem becomes more challenging when the system is nonlinear, constraints on states and/or inputs are present, and the model is not accurate enough.[18,29] Since most of these conditions are usual in real-world applications, analyzing and solving the estimation and control problems in a simultaneous approach would seem the most suitable.

The approach of solving simultaneously *MHE–MPC* was originally introduced by Copp and Hespanha[30] and later developed.[31] In the first paper, Copp and Hespanha[30] proposed an output feedback controller that combines state estimation and control into a single *min-max* optimization problem that, under observability and controllability assumptions, guarantees the boundedness of state and tracking errors. Finally, in their last work, Copp and Hespanha[31] established the conditions for guaranteeing the boundedness of error for trajectory tracking problems. They also introduced a primal–dual interior point method that can be used to efficiently solve the *min-max* optimization problem. The criterion used in these works involves finite forward and backward horizons that are minimized with respect to feedback control policies and maximized with respect to the unknown parameters in order to guaranty robustness in the worst-case scenario. The former results in simultaneous estimation and control have seen to overcome the difficulties that appear in real-world applications. However, physical interpretations are missed or hidden behind the *min-max* formulation.

In this article, we introduce an output–feedback controller for nonlinear systems subject to bounded disturbances using a simultaneous moving horizon estimation and control (SMHEC) approach. This concept implies that we are solving at each iteration an estimation and control problem at the same time. The most common approach found in the literature solves the estimation problem first and then, the control one. In our work, the resulting optimization problem minimizes a criterion that involves finite backward and forward horizons with respect to the unknown initial state, measurement noise and control input variables while it is maximized with respect to the unknown future disturbance variables. The optimization problem is stated as a minimization of an objective function that takes into account the notions of detectability and controllability. The former is a system's property that the estimation window takes advantage of for estimating the optimal state. The latter relates to the ability of the controller to steer the system to the desired operation zone, given a set of controls and constraints over a finite-time forward window whilst exogenous and unknown inputs disturb the system. We show that the proposed controller results in closed–loop trajectories along which the states remain bounded. These results rely on two assumptions: the first assumption requires that the optimization criterion includes the adaptive arrival cost proposed by Sanchez et al.[11] This assumption allows us to ensure the boundedness of the state estimate and to obtain a bound for the estimation error set if the parameters of the estimation problem are properly chosen.[14] The second assumption requires that the backward (estimation) and forward (control) horizons are sufficiently large so that enough information is obtained to find state estimates and control inputs compatible with the system dynamics, noises and constraints. This assumption is satisfied if the system is detectable, stabilizable and the parameters in the cost function (weights and horizons) are chosen appropriately.

This paper is organised as follows: the estimation and control problem–including notation, definitions, and properties that will be used throughout the paper–are introduced in Section 2. The analysis of closed-loop stability of the proposed algorithm is presented in Section 3. In Section 4, two examples to illustrate the concepts presented in this work are given. The first example uses a simple nonlinear model to analyze the consequences of simultaneously solving the estimation and control problems. The second example compares the performance obtained by the simultaneous and independent approaches applied to the regulation problem of the Van der Pol oscillator for two operational conditions. Finally, conclusions and future work are discussed in Section 5.

2 | PRELIMINARIES AND SETUP

2.1 | Notation

Let $\mathbb{Z}$ denote the integer numbers, $\mathbb{Z}_{[a,b]}$ denotes the set of integers in the interval $\left[a,b\right]$, with $b>a$ and $\mathbb{Z}_{\geq a}$ denotes the set of integers greater or equal to a. Boldface symbols denote sequences of finite ($\boldsymbol{w} := \{w_1, \dots, w_p\}$) or infinite ($\boldsymbol{w} := \{w_1, \dots, w_p, \dots\}$) length. We denote $\hat{x}_{j|k}$ as the state at time j estimated at time k. By $|x|$ we denote the euclidean norm of a vector $x \in \mathbb{R}^{n_x}$. Let $\|\boldsymbol{x}\| := \sup_{k\in\mathbb{Z}_{\geq 0}} |x_k|$ denote the supreme norm of the sequence $\boldsymbol{x}$ and $\|\boldsymbol{x}\|_{[a,b]} := \sup_{k\in\mathbb{Z}_{[a,b]}} |x_k|$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if γ is continuous, strictly increasing and $\gamma(0)=0$. If γ is also unbounded, it is of class $\mathcal{K}_\infty$. A function $\zeta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ if ζ is continuous, non increasing and $\lim_{t\to\infty} \zeta(t) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, k)$ is of class $\mathcal{K}$ for each fixed $k \in \mathbb{Z}_{\geq 0}$, and $\beta(r, \cdot)$ of class $\mathcal{L}$ for each fixed $r \in \mathbb{R}_{\geq 0}$. In the following sections, we will use the notation $\Psi_{p,t,l}$ to reference the cost incurred solving the problem p at time t with a horizon length l, while $\Psi_{p,t,l}(x)$ will be used to indicate the cost at the solution x, with x belonging to a consistent domain with the cost function $\Psi_{p,t,l}$. When necessary, we will use the notation $x^{(1)}_{i,k}$ and $x^{(2)}_{i,k}$ to differentiate i–th component of the state vector of two different discrete-time trajectories of the system, with $i \in \mathbb{Z}_{[1,n]}$. Moreover, $x^{(1)}_k(x^{(1)}_0, \boldsymbol{w}^{(1)})$ will denote a trajectory with initial condition $x^{(1)}_0$ and perturbed by the sequence $\boldsymbol{w}^{(1)}$. A similar notation is used for the case of continuous time systems, where t is used instead k to denote continuous time.

2.2 | Problem statement

Let us consider a discrete-time nonlinear system whose behavior is given

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + w_k, \quad \forall\, k \in \mathbb{Z}_{\geq 0}, \\ y_k &= h(x_k) + v_k, \end{aligned} \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^{n_x}$ is the system's state, $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the system's input and $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$ is the unmeasured process disturbance posed as an additive input. The output of the system is $y \in \mathcal{Y} \subset \mathbb{R}^{n_y}$ and $v \in \mathcal{V} \subset \mathbb{R}^{n_v}$ is the measurement noise. The function $f(\cdot,\cdot)$ is assumed to be at least locally Lipschitz in its arguments, and the function $h(\cdot)$ is known to be a continuous function. The sets $\mathcal{X}$, $\mathcal{U}$, $\mathcal{W}$, $\mathcal{V}$, and $\mathcal{Y}$ are assumed to be convex, containing the origin in its interior. The estimation and control problem attempts to simultaneously find the optimal state $\hat{x}_{k|k}$ and the optimal sequence of control inputs $\hat{\boldsymbol{u}}$ which steer the system to the desired operation zone. It is in an infinite-horizon optimization problem given by

$$\begin{aligned} \min_{\hat{x}_{0k}\hat{\boldsymbol{w}}\hat{\boldsymbol{u}}} \Psi_{EC,k,\infty} &:= \sum_{j=0}^{k} \ell_e\left(\hat{w}_{j|k}, \hat{v}_{j|k}\right) + \sum_{j=k}^{\infty}\left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) \\ \text{s.t.} &\begin{cases} \hat{x}_{j+1|k} = f\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) + \hat{w}_{j|k}, \\ y_j = h\left(\hat{x}_{j|k}\right) + \hat{v}_{j|k}, \\ \hat{x}_{j|k} \in \mathcal{X},\ \hat{u}_{j|k} \in \mathcal{U}, \hat{w}_{j|k} \in \mathcal{W},\ \hat{v}_{j|k} \in \mathcal{V}. \end{cases} \end{aligned} \tag{2}$$

where functions $\ell_e(\hat{w}_{j|k}, \hat{v}_{j|k})$, $\ell_c(\hat{x}_{j|k}, \hat{u}_{j|k})$, and $\ell_{w_c}(\hat{w}_{j|k})$ are norms. Therefore, it is clear that they penalise large values of their arguments within their windows ($\hat{w}_{j|k}, \hat{v}_{j|k}\ \forall j \le k$ and $\hat{x}_{j|k}, \hat{u}_{j|k}\ \forall j \ge k$). However, the effect of $\ell_{w_c}(\hat{w}_{j|k})$ on the objective function $\Psi_{EC,k,\infty}$ is different: since $\ell_{w_c}(\hat{w}_{j|k})$ is substracting, it will maximize the value of $\hat{w}_{j|k}\ \forall j \ge k$ given that $\hat{w}_{j|k} \in \mathcal{W}$. In this way, $\Psi_{EC,k,\infty}$ encodes a measure of robustness since, as it will be shown later, $\ell_c(\hat{x}_{j|k}, \hat{u}_{j|k})$ and $\ell_{w_c}(\hat{w}_{j|k})$ are related to the ability of the system to mitigate deviations due to disturbances (actual and future ones) within the forward window.

Problem (2) is valuable from a theoretical point of view, since it guarantees the boundedness of estimates $\hat{x}_{j|k}$ and control inputs $\hat{u}_{j|k}$ provided that the cost function is bounded, $\Psi_{EC,k,\infty} < \gamma\ \forall k \in \mathbb{Z}_{\ge 0}, \gamma \in \mathbb{R}_{\ge 0}$. In fact, if functions $\ell_e(\cdot,\cdot)$, $\ell_c(\cdot,\cdot)$ and $\ell_{w_c}(\cdot)$ are defined using a norm–ℓ_p, then Problem (2) would guarantee that x_k and u_k are ℓ_p, provided that noises w_k and v_k are also ℓ_p. This would mean that the closed-loop system has a finite ℓ_p–induced gain and therefore stable. In this way, problem (2) becomes a benchmark for any simultaneous estimation and control problem formulated for practical situations. Since this problem uses an infinite–horizon, it is intractable from a computational point of view. Thus, it can be reformulated into a receding horizon problem with finite windows and extra terms that summarize the information outside the estimation and control windows. Following these concepts, problem (2) can be rewritten as follows

$$
\begin{aligned}
\min_{\hat{x}_{kN_ek}\hat{\boldsymbol{w}}\hat{\boldsymbol{u}}} \Psi_{EC,k,N_e+N_c} := {} & \Gamma_{k-N_e}(\chi) + \sum_{j=k-N_e}^{k} \ell_e\left(\hat{w}_{j|k}, \hat{v}_{j|k}\right) + \\
& \sum_{j=k}^{k+N_c-1} \left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) + \Upsilon_{k+N_c}(\Xi)
\end{aligned}
$$

$$
\text{s.t.} \begin{cases}
\chi = \hat{x}_{k-N_e|k} - \bar{x}_{k-N_e}, \\
\Xi = \hat{x}_{k+N_c|k}, \\
\hat{x}_{j+1|k} = f\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) + \hat{w}_{j|k}, \\
y_j = h\left(\hat{x}_{j|k}\right) + \hat{v}_{j|k}, \\
\hat{x}_{j|k} \in \mathcal{X},\ \hat{u}_{j|k} \in \mathcal{U}, \hat{w}_{j|k} \in \mathcal{W},\ \hat{v}_{j|k} \in \mathcal{V}.
\end{cases} \tag{3}
$$

In this new formulation, the infinite horizon cost function $\Psi_{EC,k,\infty}$ has been replaced by the finite horizon cost function Ψ_{EC,k,N_e+N_c}. It is composed by two finite windows; one backward and one forward associated with the estimation (Ψ_{E,k,N_e}) and control (Ψ_{C,k,N_c}) problems, respectively. Each of these windows include a finite number of terms to measure the performance of the corresponding problem and an extra term to summarize the information outside these windows. The backward window

$$
\Psi_{E,k,N_e} = \Gamma_{k-N_e}(\chi) + \sum_{j=k-N_e}^{k} \ell_e\left(\hat{w}_{j|k}, \hat{v}_{j|k}\right) \tag{4}
$$

includes N_e terms backward in time corresponding to the estimation stage-cost $\ell_e(\hat{w}_{j|k}, \hat{v}_{j|k})$ and the arrival cost $\Gamma_{k-N_e}(\chi)$. The arrival cost summarises the information left behind the estimation window by penalizing the uncertainty in the initial state $\hat{x}_{k-N_e|k}$.[6,7] The forward window

$$
\Psi_{C,k,N_c} = \sum_{j=k}^{k+N_c-1} \left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) + \Upsilon_{k+N_c}(\Xi) \tag{5}
$$

includes N_c terms forward in time corresponding to the *controller stage-cost*, $\ell_c(\hat{x}_{j|k}, \hat{u}_{j|k})$, the future disturbances stage-cost $\ell_{w_c}(\hat{w}_{j|k})$, and the *cost-to-go* $\Upsilon_{k+N_c}(\Xi)$. The *cost-to-go* summarizes the behavior of the system beyond the control window penalizing the deviation of the final state $\hat{x}_{k+N_c|k}$ while remaining within the set of terminal constraints $\mathcal{X}_f$, as is commonly used in *MPC*.[1]

The goal of problem (3) is to estimate the initial state $\hat{x}_{k-N_e|k}$ and disturbances $\hat{w}_{j|k}\ j \in \mathbb{Z}_{[k-N_e,k-1]}$ such that an estimate $\hat{x}_{k|k}$ is obtained to compute the control inputs $\hat{u}_{j|k}\ j \in \mathbb{Z}_{[k,k+N_c-1]}$ that drive the system states to the desired region. Therefore, there is no point on penalizing the control cost $\ell_c(\cdot,\cdot)$ along the estimation window. The variables $\hat{v}_{j|k}$ are uniquely determined by the remaining optimization variables and the output equation

$$
\hat{v}_{j|k} = y_j - h(\hat{x}_{j|k}), \qquad j \in \mathbb{Z}_{[k-N_e,k]}. \tag{6}
$$

Since future system outputs y_j $\forall j \in \mathbb{Z}_{>k}$ are not available at sample k, then the measurement noise $\hat{v}_{j|k}$ $\forall j \in \mathbb{Z}_{>k}$ will not be included into the control function Ψ_{C,k,N_c}. On the other hand, disturbances $\hat{w}_{j|k}$ must be included in both windows (Ψ_{E,k,N_e} and Ψ_{C,k,N_c}), because they are needed to solve problem (3): **past disturbances** $\hat{w}_{j|k}$ with $j \in \mathbb{Z}_{[k-N_e,k]}$ are required to compute $\hat{x}_{k|k}$, which is used as initial condition of the control problem, while the **future disturbances** $\hat{w}_{j|k}$ with $j \in \mathbb{Z}_{[k,k+N_c]}$ are required to compute the control sequence $\mathbf{\hat{u}_{j|k}}$ that drive the system states to $\mathcal{X}_f$. Future disturbances are frequently unknown, thus they must be computed in a way that closed-loop stability and feasibility are guaranteed for bounded disturbances.

There are many ways of computing future disturbances.[1] One way is computing $\hat{w}_{j|k}$ with $j \in \mathbb{Z}_{[k,k+N_c]}$ for the worst case scenario, in this case by maximizing $\hat{w}_{j|k}$ for each sample time. Other way of computing future disturbances is through model augmentation.[32] In this approach the disturbance is computed recursively through the system model, and it is removed from the optimization problem. In this work, we will keep the disturbance term $\ell_{w_c}(\hat{w}_{j|k})$ within the control cost function Ψ_{C,k,N_c} along the theoretical analysis to include the effects of disturbances on stability. For implementation, the future disturbances will be computed through model augmentation and estimation in order to reduce the computational burden of the optimization problem.

Remark 1. The sequence of process disturbances $\hat{w}_{j|k}$ is minimized within the estimation window, that is, $j \in \left[k - N_e - 1, k - 1\right]$, and it is maximized within the control window, $j \in \left[k, k + N_c - 1\right]$.

2.3 | Relationship with MHE and MPC

The criterion Ψ_{EC,k,N_e+N_c} can be rewritten as follows

$$\Psi_{EC,k,N_e+N_c} := \varphi\Psi_{E,k,N_e} + (1-\varphi)\Psi_{C,k,N_c}, \quad \varphi \in \mathbb{R}_{[0,1]}, \tag{7}$$

which corresponds to a multi-objective formulation of Ψ_{EC,k,N_e+N_c}, where φ controls the influence of Ψ_{E,k,N_e} and Ψ_{C,k,N_c}. When $\varphi = 0$, $\Psi_{EC,k,N_e+N_c} := \Psi_{C,k,N_c}$ and problem (3) becomes a *model predictive control* problem with terminal cost.[33] On the other case, when $\varphi = 1$, $\Psi_{EC,k,N_e+N_c} := \Psi_{E,k,N_e}$ and problem (3) becomes a *moving horizon estimation* problem.[10,12,14,34] In these cases, the optimization problem (3) has only one objective function and the separation principle needs to be applied since the estimator and the controller are implemented independently.

When $0 < \varphi < 1$, Ψ_{E,k,N_e} and Ψ_{C,k,N_c} are simultaneously considered by Ψ_{EC,k,N_e+N_c} and the optimization problem (3) becomes a multi-objective one. The relevance of Ψ_{E,k,N_e}, and therefore of Ψ_{C,k,N_c}, is defined by φ emphasising or deemphasizing the influence of the estimation problem on the solution. In the case of $\varphi = 0.5$, Ψ_{E,k,N_e} and Ψ_{C,k,N_c} have a similar influence on the solution of (3) and it becomes the problem proposed by Copp and Hespanha.[31]

Definition 1. Given the points $z_E \in \mathcal{Z}_E := \mathbb{R}^{n_w N_e} \times \mathbb{R}^{n_v(N_e+1)} \times \mathbb{R}^{n_x(N_e+1)}$ and $z_C \in \mathcal{Z}_C := \mathbb{R}^{n_w N_c} \times \mathbb{R}^{n_u N_c} \times \mathbb{R}^{n_x(N_c+1)}$ such that $z \in \mathcal{Z} := \mathcal{Z}_E \times \mathcal{Z}_C$. A point $z^o \in \mathcal{Z}$ is Pareto optimal if and only if there does not exist another point $z \in \mathcal{Z}$ such that $\Psi_{EC,N_e+N_c,k}(z) \leq \Psi_{EC,N_e+N_c,k}(z^o)$ and $\Psi_{E,N_e,k}(z_E) < \Psi_{E,N_e,k}(z_E^o)$, $\Psi_{C,N_c,k}(z_C) < \Psi_{C,N_c,k}(z_C^o)$.[35]

According to Definition 1, problem (3) looks for solutions in a sense that neither Ψ_{E,k,N_e} nor Ψ_{C,k,N_c} can be improved without deteriorating one of them. Any optimal solution of problem (3) with $0 < \varphi < 1$ is Pareto optimal,[35] therefore, it has an optimal trade-off between Ψ_{E,k,N_e} and Ψ_{C,k,N_c}. When $\varphi = 0$ or $\varphi = 1$ the solutions of problem (3) are optimal in the sense of the selected objective function (Ψ_{C,k,N_e} or Ψ_{C,k,N_c}, respectively). In both cases, the solutions obtained are not Pareto optimal and, therefore the overall system performance can be poorer than the one provided by the multi-objective problem.

From a practical point of view, φ can be used to improve the numerical properties of the optimisation problem (3). This fact allows to improve the convergence properties of numerical algorithms employed to solve it (see Example 4.2). For example, if $N_e \ll N_c$ and the stage costs $\ell_e(\cdot)$, $\ell_c(\cdot)$ and $\ell_{w_c}(\cdot)$ have similar values, the optimisation problem will improve Ψ_{C,k,N_c} at the expense of Ψ_{E,k,N_e} (because $\Psi_{C,k,N_c} \gg \Psi_{E,k,N_e}$), deteriorating the estimation of $\hat{x}_{k|k}$ and, thus, producing potentially ill conditioned Jacobian and Hessian matrices of Ψ_{EC,k,N_e+N_c}. This numerical problems can lead to an increment of the computational cost of the optimisation problem. A similar situation can happen when $N_e \gg N_c$.

3 ROBUST STABILITY UNDER BOUNDED DISTURBANCES

In this section, we introduce the results regarding feasibility and stability of the proposed algorithm. Firstly, the properties of *MHE* and *MPC* are analyzed and then the results for the simultaneous *MHE–MPC* are given. In addition, feasibility conditions for the existence of a solution to problem (3) and minimum horizon lengths required to achieve the desired estimation and control performances are also analyzed.

3.1 Estimation problem

The proposed algorithm relies on the backward window to formulate an optimisation problem of fixed length N_e to compute the optimal state estimate $\hat{x}_{k|k}$, which is used by the forward window (control problem) to compute $\hat{u}_{j|k}$. The estimation problem—and therefore problem (3)—has a feasible solution if and only if the system is detectable. For non-linear systems, *incremental input-output-to-state stability-i-IOSS-* is a popular characterization of detectability[36]

$$|x_k(x_0^{(1)}, \boldsymbol{w}^{(1)}) - x_k(x_0^{(2)}, \boldsymbol{w}^{(2)})| \leq \beta\left(|x_0^{(1)} - x_0^{(2)}|, k\right) + \gamma_1\left(\|\boldsymbol{w}^{(1)} - \boldsymbol{w}^{(2)}\|\right) + \gamma_2\left(\|h\left(\boldsymbol{x}^{(1)}\right) - h\left(\boldsymbol{x}^{(2)}\right)\|\right), \tag{8}$$

where $\beta(\cdot,\ \cdot) \in \mathcal{KL}, \gamma_1(\cdot),\ \gamma_2(\cdot) \in \mathcal{K}$. *i-IOSS* not only guarantees the detectability of the system, but also means the system admits a robust stable state estimator.[37] Note that inequality (8) only includes the process disturbance as input to the system, however it can be extended to include control inputs. Moreover, as will be shown in Example 4.1, the resulting control law have not only affects on the forward window but also in the backward window, influencing the estimation process.

Previous results on robust output-feedback *MPC* with bounded disturbances the estimation problem is solved in the first place, showing the convergence of estimated states to a bounded set, and then take into account the uncertainty of estimation when solving the *MPC* problem.[24,25] The key idea in these works was to consider the estimation error as an additional, unknown but bounded uncertainty that must be considered in order to guarantee the stability and feasibility of the resulting closed–loop system. Let us define the robust estimable set

$$\mathcal{E}_{N_e}\left(\hat{x}_{k|k}, \varepsilon_e(k)\right) := \left\{x_k\ :\ |x_k - \hat{x}_{k|k}| \leq \varepsilon_e(k), \forall\ k\right\}, \tag{9}$$

where x_k is the real state, $\hat{x}_{k|k}$ is the best estimate of x_k and $\varepsilon_e(k)$ is the estimation error at time bounded by

$$\varepsilon_e(k) \leq \overline{\Phi}\left(|x_0 - \overline{x}_0|, k\right) + \pi_w\left(\|\boldsymbol{w}\|\right) + \pi_v\left(\|\boldsymbol{v}\|\right). \tag{10}$$

This is a property of *i-IOSS* systems,[36] where the parameters and structure of functions $\overline{\Phi}(\cdot)$, $\pi_w(\cdot)$ and $\pi_v(\cdot)$ depend on the stage costs $\ell_e(\hat{w}_{j|k}, \hat{v}_{j|k})$ and arrival cost $\Gamma_{k-N_e}(\mathcal{X})$ employed in problem (3), and they should be derived for each choice of $\ell_e(\cdot)$ and $\Gamma_{k-N_e}(\cdot)$.[12-14] In this work, we use the adaptive arrival cost proposed by Sanchez et al.[11]

$$\Gamma_{k-N_e}(\mathcal{X}) = \|\hat{x}_{k-Ne|k} - \hat{x}_{k-Ne+1|k-1}\|_{P^{-1}_{k-N_e|k}} \tag{11}$$

where $P^{-1}_{k-N_e|k}$ is updated using a recursive least square algorithm with variable forgetting factor.[11] This choice of terminal cost leads to the following functions[14]

$$\overline{\Phi}\left(|x_0 - \overline{x}_0|, k\right) := \theta^i |x_0 - \overline{x}_0|^{\zeta} \frac{\mathbb{N}_e}{N_e}\left(\left(\frac{\overline{\lambda}_{P^{-1}}}{\underline{\lambda}_{P^{-1}}}\right)^{\rho}\left(c_\beta 18^p + \underline{\lambda}_{P^{-1}}^{\alpha_1}\left(P^{-1}_{k-N_e}\right)\left(c_1\ 3^{\alpha_1} + c_2\ 3^{\alpha_2}\right)\right) + c_\beta\ 2^p\right), \tag{12}$$

$$\pi_w\left(\|\boldsymbol{w}\|\right) := 2\left(1+\mu\right)\left(\frac{c_\beta\ 18^p}{\underline{\lambda}_{P^{-1}}}\ \overline{\gamma}_w^{\frac{p}{a}}\left(\|\boldsymbol{w}\|\right) + c_2\ 3^{\alpha_2}\overline{\gamma}_w^{\alpha_2}\left(\|\boldsymbol{w}\|\right) + \gamma_1\left(6\|\boldsymbol{w}\|\right) + \gamma_1\left(6\underline{\gamma}_w^{-1}\left(3\overline{\gamma}_w\left(\|\boldsymbol{w}\|\right)\right)\right)\right), \tag{13}$$

$$\pi_v\left(\|\boldsymbol{v}\|\right) := 2\left(1+\mu\right)\left(\frac{c_\beta\ 18^p}{\underline{\lambda}_{P^{-1}}}\ \overline{\gamma}_v^{\frac{p}{a}}\left(\|\boldsymbol{v}\|\right) + c_1\ 3^{\alpha_1}\overline{\gamma}_v^{\alpha_1}\left(\|\boldsymbol{v}\|\right) + \gamma_2\left(6\|\boldsymbol{v}\|\right) + \gamma_2\left(6\underline{\gamma}_v^{-1}\left(3\overline{\gamma}_v\left(\|\boldsymbol{v}\|\right)\right)\right)\right), \tag{14}$$

where $\theta = \frac{2+\mu}{2(1+\mu)} < 1$, $\mu \in \mathbb{R}_{\geq 0}$, $i = \lfloor \frac{k}{N_e} \rfloor$, $\rho = max\{p, \alpha_1, \alpha_2\}$, $\underline{\lambda}_{P^{-1}}$ and $\overline{\lambda}_{P^{-1}}$ are the minimal and maximal eigenvalues of P^{-1}. The remaining parameters of Equations (12) to (14) are related to stage cost $\ell_e(\hat{w}_{j|k}, \hat{v}_{j|k})$ used in Ψ_{E,k,N_e} and functions

$\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ of *i-IOSS* definition. In fact, these are functions that bound the behavior of $\ell_e(\hat{w}_{j|k}, \hat{v}_{j|k})$ over time, thus $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are related to $\ell_e(\hat{w}_{j|k}, \hat{v}_{j|k})$.[14] The key reason to use the arrival cost given by (11) is the fact that the observer gains π_w and π_v only depend on the stage cost $\ell_e(\hat{w}_{j|k}, \hat{v}_{j|k})$ and the properties of w and v.

Assuming an additive cost structure for the estimator

$$\ell_e(\hat{w}_{j/k}, \hat{v}_{j/k}) = \ell_{w_e}(\hat{w}_{j/k}) + \ell_{v_e}(\hat{v}_{j/k}), \tag{15}$$

and quadratic functions for them, the functions involved in π_w and π_v get the following form

$$\gamma_1(\cdot) \leq c_1 w^{\alpha_1} \qquad \overline{\gamma}_w(\cdot) \leq \overline{\lambda}_{Q_E} w^2 \tag{16}$$

$$\gamma_2(\cdot) \leq c_2 v^{\alpha_2} \qquad \overline{\gamma}_v(\cdot) \leq \overline{\lambda}_{R_E} v^2 \tag{17}$$

where $c_1, c_2, \alpha_1, \alpha_2 \in \mathbb{R}_{\geq 0}$, $\overline{\lambda}_{Q_E}, \overline{\lambda}_{R_E}$ are the maximum and minimum eigenvalues of matrices Q_E and R_E respectively. and N_e is the length of the backward window. $\mathbb{N}_e$ is the minimum length of the backward window required to guarantee the boundness of the estimation error, which is given by

$$\mathbb{N}_e = \left\lceil \left(2^{\rho} e_{max}^{\rho-1} \overline{c}_{\beta}\right)^{\frac{1}{\eta}} \right\rceil, \tag{18}$$

where $\eta \in \mathbb{R}_{[0,1/p]]}$ and $e_{\max}$ denotes the maximal error on the prior estimate of the initial condition. In the following, we will assume that $N_e \geq \mathbb{N}_e$.

At each sampling time, the measurements available along the backward window are used to obtain $\hat{x}_{k|k}$. Whenever $N_e \geq \mathbb{N}_e$, the estimation error will decrease until it reaches an invariant space whose volume depends on the process and measurement noises as well as the stage-cost and the system itself. The behavior of the system is forecasted from the estimate $\hat{x}_{k|k}$, whereas x_k remains within $\mathcal{E}_{N_e}$.

3.2 | Control problem

The forward window corresponds to the receding horizon problem, which computes the optimal control input sequence $\hat{\boldsymbol{u}}$ using the optimal estimate $\hat{x}_{k|k}$, computed by the backward window, as the initial condition of the control problem. Its feasibility depends on the fact that its initial condition x_k must belong to the *robust N_c-steps controllable set* $\mathcal{R}_{N_c}(\Omega, \mathbb{T})$[38]

$$\mathcal{R}_{N_c}(\Omega, \mathbb{T}) := \left\{ x_0 \in \Omega | \exists\, u_j \in \mathcal{U} \ : \left\{ x_j \in \Omega, x_{N_c} \in \mathbb{T} \right\} \quad \forall j \in \mathbb{Z}_{[0,N_c-1]} \right\}. \tag{19}$$

where Ω is the input admissible set and $\mathbb{T} = \mathcal{X}_f$ is the terminal set of the controller. Since $x_k \in \mathcal{E}_{N_e}\left(\hat{x}_{k|k}, \varepsilon_e\right) \forall k$ the feasibility of the control problem is guaranteed if

$$\mathcal{E}_{N_e}\left(\hat{x}_{k|k}, \varepsilon_e\right) \subseteq \mathcal{R}_{N_c}(\Omega, \mathbb{T}) \quad \forall k \in \mathbb{R}_{\geq 0}. \tag{20}$$

Note that this feasibility condition is not only necessary for the simultaneous *MHE–MPC*, but also for independent *MHE* and *MPC*.[24,25] Let us state this condition in the following assumption.

Assumption 1. The robust estimable set $\mathcal{E}_{N_e}$ belongs to the robust controllable set $\mathcal{R}_{N_c}(\Omega, \mathbb{T})$ in N_c steps $\forall k \in \mathbb{R}_{\geq 0}$, $\mathcal{E}_{N_e} \subseteq \Omega$, $\mathcal{X}_f \subseteq \mathbb{T}$, then, we are interested in the following robust controllable set: $\mathcal{R}_{N_c}\left(\mathcal{E}_{N_e}, \mathcal{X}_f\right)$ $\forall k \in \mathbb{R}_{\geq 0}$.

Remark 2. This assumption formalises the compatibility notion between the estimator and the controller. Since the true system's state x_k is not known perfectly, we require that it belongs to a neighborhood of estimated one, $\hat{x}_{k|k}$, such that the control u_k of the sequence $\boldsymbol{u}_{[k,k+N_c-1]}$ computed from $\hat{x}_{k|k}$ and applied to $f(x_k, u_k)$ will lead to the state x_{x+1} not too far from the desired reference. Moreover, the volume of the robust estimable set decrease faster with longer backward windows and the size of the robust controllable set can be enlarged by mean of larger forward window and with the appropriate design of the set $\mathcal{U}$.

The satisfaction of Assumption 1 can be easily verified since its depends on the adequate selection of problem (3) parameters. This fact implies the selection of the estimation and control stage costs parameters (Q_E, R_E, Q_C, R_C for quadratic cost functions), and the estimation and control horizons (N_e, N_c).

Regarding stability along the forward window, a common approach to guarantee the stability of *MPC* is by mean of the inclusion of a terminal constraint set, which is generally a level set of a control Lyapunov function.[39] This set is an artificial constraint set but it guarantees stability.[40] In this work we will analyse the stability of the controller following a similar approach as in Tuna, Messina and Teel,[40] where the analysis is carried out as a function of the length of the forward window, taking into account the effect of the process disturbances and the estimation errors. A pseudo measure of the system controllability property will be introduced and the minimum forward window length which guarantees the stability of the simultaneous *MHE–MPC* is given, without imposing neither extra terminal constraints nor forcing the cost-to-go to be a closed–loop Lyapunov function (*CLF*). In this sense, let us state the following assumption.

Assumption 2. There exist a constant $\delta \in \mathbb{R}_{\geq 0}$ such that the cost-to-go and the stage cost satisfy the following relation:

$$\Upsilon_{k+N_c}\left(f\left(x,u\right)\right) - \Upsilon_{k+N_c}\left(x\right)\left(1+\delta\right) \leq -\ell_c\left(x,u\right) + \ell_{w_c}\left(w\right). \tag{21}$$

Remark 3. A similar assumption was already used in Tuna, Messina and Teel,[40] where the constant δ is introduced in order to relax the requirement on function $\Upsilon_{k+N_c}(\cdot)$ to be a CLF for the nominal case.

Despite using a different notation for the cost-to-go term $\Upsilon_{k+N_c}(\Xi)$, this function can take the same behavior as the stage-cost, i.e., $\Upsilon_{k+N_c}(\Xi) = \ell_c(\Xi, 0)$. Here we extend it to a more general case where process disturbances are affecting the system, and it will lead, as it will be shown later, in longer control windows. Regarding the elements of the optimization problem corresponding to the control problem, we will assume that the stage-cost is lower bounded.

Assumption 3. The stage cost $\ell_c(x,u)$ is lower bounded by a function $\sigma(x) \in \mathcal{K}_\infty$, such that $\sigma(x) \leq \ell_c(x,u)\ \forall\, x \in \mathcal{X}, u \in \mathcal{U}$.

Remark 4. Note that for a quadratic stage-cost, that is, $\ell_c(x,u) = x^T Q_C x + u^T R_C u$, with $Q_C \succcurlyeq 0$ and $R_C \succ 0$, Assumption 3 holds with $\sigma(x) = \underline{\lambda}_{Q_C} |x|^2$, where $\underline{\lambda}_{Q_C}$ denotes the minimal eigenvalue of matrix Q_C.

Furthermore, we will assume that there is an increasing sequence that relates the function $\sigma(x)$ with the cost of the control problem $\Psi_{C,k,i}$, where i represents different lengths of the forward window. Finally, let us define the following quantity

$$\Delta_c^w := \max\left\{\min_{\hat{u}_{k|k}} \frac{\ell_{w_c}\left(\hat{w}_{k|k}\right)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)}\right\},\ \forall\ \hat{x}_{k|k} \in \mathcal{X},\ \forall\ \hat{w}_{k|k} \in \mathcal{W}. \tag{22}$$

It encodes a pseudo–measure of the system controllability relating the capability of control actions to compensate process disturbances. The term pseudo–measure is used here because the relation Δ_c^w is given via the penalization functions $\ell_{w_c}(\cdot)$ and $\ell_c(\cdot,\cdot)$. In the following, we will assume that the system is controllable from this point of view.

Assumption 4. The system's actuator can be designed such that the following relation can be always verified

$$\Delta_c^w < 1 \tag{23}$$

Remark 5. It entails that the system's actuator is able to compensate for deviation of the system's output due to disturbances. Roughly speaking, the actuator is able to generate control actions able to compensate the effect of the disturbances. This fact is equivalent to require a system bandwidth wider than the disturbance ones.

Finally, let us state the following common assumption that establish lower and upper bounds on the cost function

Assumption 5. There exists an increasing sequence $\boldsymbol{L} := [1, L_1, \ldots, L_i]$, $L_i \in \mathbb{R}$, $1 \leq L_i \leq L$, $i \in \mathbb{Z}_{\geq 0}$ that verifies

$$\sigma\left(x\right) \leq \Psi_{C,k,i} \leq L_i\ \sigma\left(x\right). \tag{24}$$

Remark 6. Note that the behavior of the sequence $\boldsymbol{L}$ unveils the stability (or instability) associated with sequence $\Psi_{C,k,i}$ $i \in \mathbb{Z}_{\geq 0}$. If the increasing sequence $\boldsymbol{L}$ is bounded, then Ψ_{C,k,N_c} will be bounded and the resulting closed-loop system stable.

3.3 | Simultaneous estimation and control

With all the elements introduced in the previous section, we are ready to derive the main result: the stability of the resulting closed-loop system of the proposed output-feedback controller with estimation horizon $\mathbb{N}_e$ and control horizon $\mathbb{N}_c$ for nonlinear detectable and controllable systems under bounded disturbances.

Theorem 1. *Given the i-IOSS nonlinear system (1) with a prior estimate $\overline{x}_0 \in \mathcal{X}_0$ of its unknown initial condition x_0, bounded disturbances $\boldsymbol{w} \in \mathcal{W}(w_{\max})$, $\boldsymbol{v} \in \mathcal{V}(v_{\max})$, Assumptions 1 to 4 are fulfilled and the estimation and control horizons verify that*

$$N_e \geq \mathbb{N}_e = \left\lceil \left(2^{\zeta} e_{max}^{\zeta-1} \overline{c}_{\beta}\right)^{\frac{1}{\eta}} \right\rceil, \qquad N_c \geq \mathbb{N}_c = \left\lceil 1 + \frac{\ln\left(\frac{\delta(L-1)}{1-\Delta_c^w}\right)}{\ln\left(\frac{L}{L-1}\right)} \right\rceil, \tag{25}$$

then the simultaneous estimation and control algorithm given by problem (3) will be robustly stable.

Proof. See Appendix A. ■

Theorem 1 links the robust stability of the SMHEC with the length of the estimation and control windows without requiring the MPC's cost-to-go to be a CLF. It is worth noting that robust stability is guaranteed under mild practical assumptions. It can be stated as follows: *given an i-IOSS system with control actions large enough to mitigate the deviation due to process disturbances, then, a SMHEC with estimation and control windows long enough will be robust and stable.* Theorem 1 establishes how long the backward and forward windows should be.

As expected, a larger uncertainty $e_{\max}$ on the initial state of the system involves a longer estimation window. However, MHE technique would enable the possibility of shortening the backward window after the initial uncertainty is reduced. To the best of the authors' knowledge, this topic deserves further research since it could have a deep practical meaning. In the same way, the length of the control window increases as the requirement on the cost-to-go of being a CLF is relaxed. Moreover, when the amplitude of control actions are comparable with those of the disturbances, the length of the forward window needs to be enlarged too.

4 | EXAMPLES

In this section, we discuss two examples to illustrate the results presented previously and the performance of the proposed framework is analyzed. The first example applies the ideas introduced in previous sections to a nonlinear scalar system. The emphasis is placed in the effect of constraints and disturbances on closed-loop stability and performance. The second example discusses the simulations results for a van der Pol oscillator using the SMHEC. The discussion is focused on the effect of N_e and N_c on the performance and computational time.

4.1 | Example 1

Let us consider the continuous–time nonlinear system

$$\begin{aligned} \dot{x} &= a x_t^3 + w_t + u_t, \quad a \in \mathbb{R}_{>0} \\ y_t &= x_t + v_t. \end{aligned} \tag{26}$$

Firstly, we show its detectability, that is, the existence of an estimate with a structure like Equation (8). Let assume two arbitrary and feasible trajectories $x_t^{(1)}$ and $x_t^{(2)}$ such that $\Delta x := x_t^{(1)} - x_t^{(2)}$ and $p_t := |\Delta x|$; then $\dot{p}_t$ can be written as follows

$$\dot{p}_t = \frac{\Delta x}{|\Delta x|}\left(\dot{x}_t^{(1)} - \dot{x}_t^{(2)}\right). \tag{27}$$

Assuming a *LTV* control law $u_t = -K_t x_t$, we obtain

$$\dot{p}_t = \frac{\Delta x}{|\Delta x|}\left(a\Delta x\left(x_t^{(1)^2} + x_t^{(1)}x_t^{(2)} + x_t^{(2)^2}\right) - K_t\Delta x + \Delta w_t\right), \tag{28}$$

which is upper bounded by

$$\dot{p}_t \leq -K_t\, p_t + a\, g|\Delta h_t| + |\Delta w_t|, \tag{29}$$

where

$$g := h^2\left(x_0^{(1)}\right) + h\left(x_0^{(1)}\right)h\left(x_0^{(2)}\right) + h^2\left(x_0^{(2)}\right). \tag{30}$$

Solving p_t for initial condition $p_0 = |x_0^{(1)} - x_0^{(2)}|$ we obtain

$$|x_t^{(1)} - x_t^{(2)}| \leq |x_0^{(1)} - x_0^{(2)}|e^{-K_t t} + \frac{\|\Delta w_{0:t}\|}{K_t} + \frac{ag\|\Delta y_{0:t}\|}{K_t}, \tag{31}$$

it follows the fact that system (26) is *i-IOSS* (for all details, the reader can refer to Appendix B).

In the case of *MHE–MPC* controllers with quadratic costs

$$\begin{aligned}\ell_e &:= \hat{w}_{j|k}^2 Q_e + \hat{v}_{j|k}^2 R_e, \Gamma_{k-N_e} := P_{k-N_e}^{-1}\chi^2,\\ \ell_c &:= \hat{x}_{j|k}^2 Q_c + \hat{u}_{j|k}^2 R_c, \Upsilon_{k+N_c} := S_c \Xi^2,\end{aligned} \tag{32}$$

the bound (31) can be written as follows

$$\begin{aligned}|x_k - \hat{x}_{k|k}| \leq{}& |x_0 - \bar{x}_0|\left(\frac{\theta\,\mathbb{N}_e}{2N_e}\right)^i\left(2 + \frac{\left(P_{k-N_e}^{-1}R_e\right)^{1/2} + \left(P_{k-N_e}^{-1}Q_e\right)^{1/2} a\, g}{(Q_eR_e)^{1/2}K_{1|k}^c}\right)\\ &+ 2(1+\mu)\|\boldsymbol{w}\|\left(\frac{2}{K_{1|k}^c} + \frac{(Q_e\, R_e)^{1/2}K_{1|k}^c + \left(P_{k-N_e}^{-1}\, Q_e\right)^{1/2} a\, g}{\left(P_{k-N_e}^{-1}\, Q_e\right)^{1/2}K_{1|k}^c}\right)\\ &+ 2(1+\mu)\|\boldsymbol{v}\|\left(\frac{2\, g}{K_{1|k}^c} + \frac{(R_e\, Q_e)^{1/2}K_{1|k}^c + \left(P_{k-N_e}^{-1}\, R_e\right)^{1/2}}{\left(P_{k-N_e}^{-1}Q_e\right)^{1/2}K_{1|k}^c}\right),\end{aligned} \tag{33}$$

with $\mathbb{N}_e$ given by

$$\mathbb{N}_e = \left\lceil 4\left(2 + \frac{\left(P_{k-N_e}^{-1}\, R_e\right)^{1/2} + \left(P_{k-N_e}^{-1}\, Q_e\right)^{1/2} a\, g}{(Q_e\, R_e)^{1/2}K_{1|k}^c}\right)^2\right\rceil, \tag{34}$$

and $K_{1|k}^c$ is the equivalent controller gain resulting from applying $\hat{u}_{1|k}$.

Equations (33) and (34) show the influence of the controller on the state estimation. Larger controller gains improve estimation error and shorten the convergence time. However, controller gains are bounded by stability conditions and input constraints, which are limiting factors in this potential improvement. This example highlights the relevance of simultaneously solving the estimation and control problems, or at least to take into account the solution of control problem

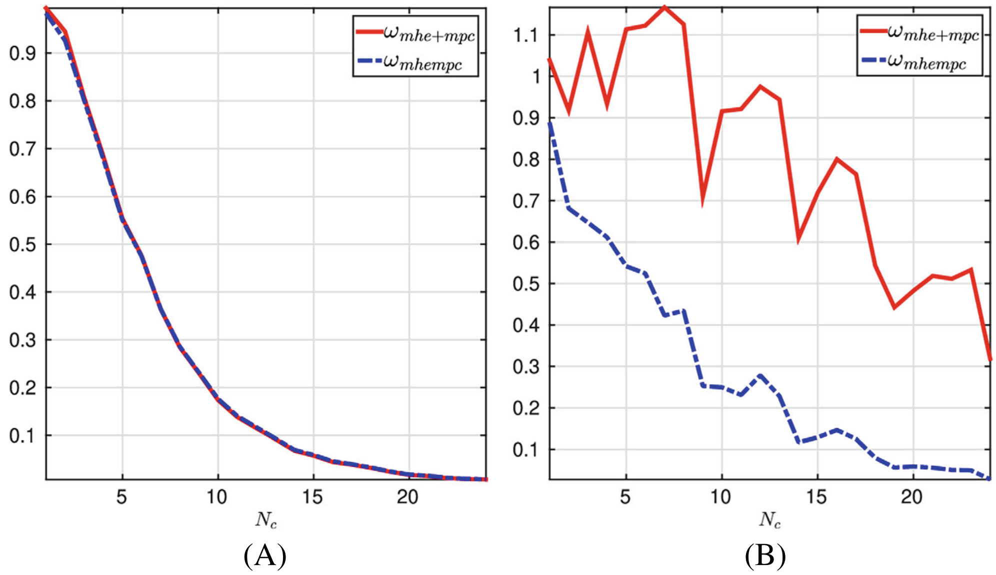

FIGURE 1 Evolution of $\omega(N_c)$ for $\delta = 1$ and $\Delta_c^w = 10^{-1}$: (A) with constraints set (36), and (B) with constraint set (37).

on the estimation one. Since *MPC* gains $K^c_{1|k}$ are time-varying because they are recomputed at every sampling time, a conservative approach can employ its lowest value.

In order to compare the performances of independent and simultaneous *MHE–MPC*, both output–feedback controllers have the same parameters

$$P_0 = 10^5,\ Q_e = 15,\ R_e = 10^3,\ Q_c = 5,\ R_c = 5,\ S_c = Q_c,\ \mu = 0.05, \tag{35}$$

with constraints sets

$$\mathcal{X} := \{x : |x| \le 0.8\} \text{ and } \mathcal{U} := \{u : |u| \le 2.5\}, \tag{36}$$

$w_t \sim U(0,\ 0.01)$, $v_t \sim \mathcal{N}\left(0,\ 0.02^2\right)$, $a = 1$, $g = 3x^3_{0\ \max}$ and $\varphi = 0.5$ such that both controllers implement the same optimisation criterion.

The control problems of both controllers are configured without terminal constraints. The process disturbance is not taken into account to compute $\hat{u}_{j|k}$, but it will be considered in the computation of N_c. It can be computed directly from Equation (25) once the values of δ, L and Δ_c^w had been established. Another approach, employed in this example, consists of computing ω through simulations. In this example, we set the initial condition that maximizes the controller costs and then computes the values of L and ω. The process is repeated until the maximal value of N_c is reached.

Figure 1 shows the computed values of ω as a function of N_c ($\omega(N_c)$) for the same Δ_c^w and different set of constraints and distributions for process and measurement noises. In this figure, the effect of constraints on $\omega(N_c)$ can be seen: They increase $\omega(N_c)$, for the same N_c, depending how the controller is implemented. This change is smaller for the simultaneous *MHE–MPC* approach than the independent one. When constraints are no relevant (constraints set (36)), both controllers have similar values (see Figure 1A), and the control problem of both can use the same N_c. However when constraints (37) are considered, the way of solving the estimation and control problems has a direct effect on $\omega(N_c)$ (see Figure 1B), and the control problem of both controllers must use different N_c in order to ensure stability, affecting the computational requirements of the implementation. Since we are using constraints set (36) we choose $N_c = 10$ for both controllers (Figure 1A). Finally, the minimum estimation horizon $\mathbb{N}_e$ is computed from (18) using the parameters listed in (35), leading to $\mathbb{N}_e = 27$ for both controllers. We choose $N_e = 30$ for both controllers.

Figure 2 shows the system responses and the corresponding *i-IOSS* bound for the regulation problem. Figure 2A shows two trajectories generated by both controllers from different initial condition ($x_0^{(1)} = 0.766$ and $x_0^{(2)} = -0.766$) with the same prior guess ($\bar{x}_0 = -2.5$). Figure 2B shows the difference between the trajectories and its *i-IOSS* bound, for the minimum controller gain along the simulation ($K^c_{1|k} = 0.7326$). From this figure it can be seen the decreasing behavior

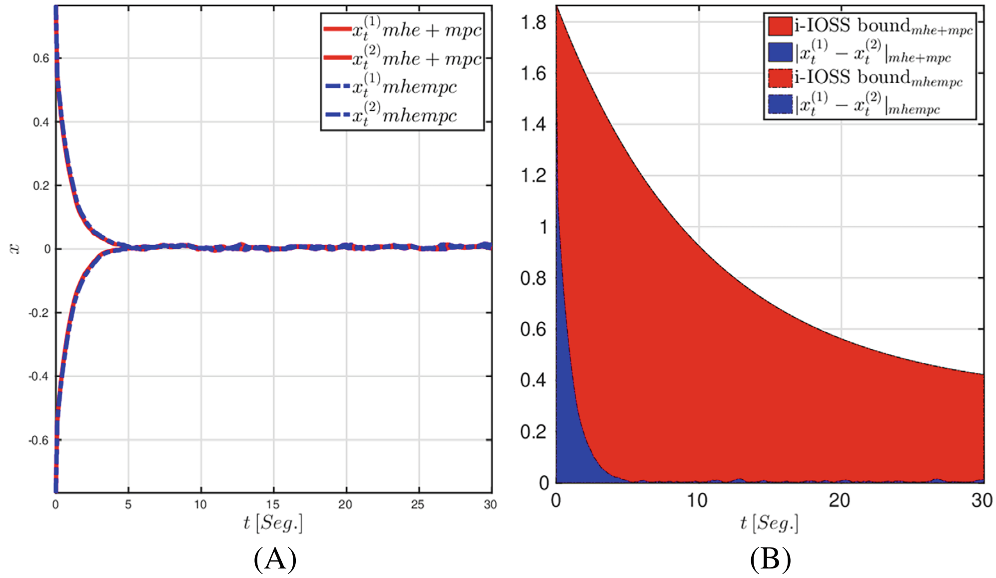

FIGURE 2 Evolution of system output for different initial conditions (A) and difference between trajectories and *i-IOSS* bound (B).

of the estimation error bound, as expected from Equation (10) for the general case and (33) for this particular example. Despite the fact that the value of μ is small ($\mu = 0.05$), the bound (33) is quite conservative. In these figures, we can also see that both controllers provide a similar response, since constraints and disturbances have not relevant effect on the system behavior, and therefore the separation principle can be applied.

Now let us compare the performance in a more challenging setup. In the following, we will assume the next constraints set

$$\mathcal{U} := \{u : |u| \leq 0.6\}, \mathcal{W} := \{w : |w| \leq 0.4\} \text{ and } \mathcal{V} := \{v : |v| \leq 0.8\}. \tag{37}$$

The controls $\hat{u}_{j|k}$ have been tightened and the estimates $\hat{w}$ and $\hat{v}$ have been constrained to the sets $\mathcal{W}$ and $\mathcal{V}$, respectively. Disturbances w_t and v_t are now given by $w_t \sim U(0,\ 0.1)$ and $v_t \sim \mathcal{N}\left(0,\ 0.2^2\right)$, respectively.

Under this new operational conditions $\mathbb{N}_e$ is recomputed, obtaining $\mathbb{N}_e = 98$ for the independent *MHE* and *MPC*, and $\mathbb{N}_e = 52$ for the simultaneous *MHE–MPC*. This is the effect of constraints set (37) on the estimator parameters, while the effect on the controller is shown in Figure 1B. This figure shows that the independent *MHE* and *MPC* approach is more sensitive to disturbances, requiring conservative values of N_c to guarantee the closed–loop stability.

Finally, Figure 3 shows the system responses for regulation problem for different realisations of w_t and v_t and different N_c, for $N_e = 30$. The independent *MHE* and *MPC* strategy fails to regulate the system states for some noise realisations, even though it regulates few of them. On the other hand, the simultaneous *MHE–MPC* controller manages to regulate the system states for all noise realizations. This problem is caused by the failure of the independent *MHE* and *MPC* to satisfy Assumption 1. In fact, its design procedure applies the *separation principle*, which entails the automatic satisfaction of Assumption 1 and it does not include the constraints information in the selection of N_e and N_c, while the simultaneous *MHE–MPC* controller does.

4.2 | Example 2

Let us consider the van der Pol oscillator whose dynamic is described by

$$\begin{aligned} \dot{x}_t &= \begin{bmatrix} \epsilon\left(1 - x_{2,t}^2\right)x_{1,t} - 2x_{2,t} + u_t + w_{1,t} \\ 2x_{1,t} + w_{2,t} \end{bmatrix} \quad \epsilon \in \mathbb{R}_{\geq 0}, \\ y_t &= \frac{1}{2}\left(x_{1,t} + x_{2,t}\right) + v_{1,t}. \end{aligned} \tag{38}$$

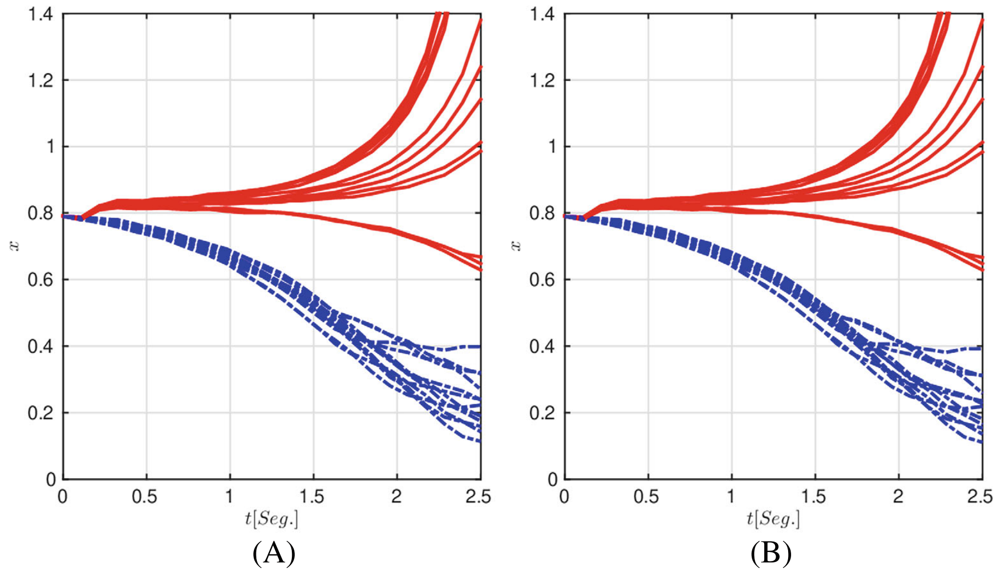

FIGURE 3 Evolution of system output for $N_c = 20$ (A) and $N_c = 70$ (B), with $N_e = 30$ for independent *MHE* and *MPC* (continuous red line) and simultaneous *MHE–MPC* (blue dashed line).

It is known to be *i-IOSS*, and a proof of this property can be made using the averaging lemma.[41] In this example we will focus the analysis on the system performance under different set of parameters. The independent and simultaneous *MHE–MPC* controllers have the same parameters to allow a direct comparison of their performances. All the stage costs have a quadratic structure (Equation 32) and their parameters are

$$P_0 = 10^5,\ Q_e = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix},\ R_e = 150,\ Q_c = \begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix},\ S = Q_c,\ R_u = 10^{-2}, \tag{39}$$

with constraints sets given by

$$\mathcal{X} := \{x : |x_1| \leq 5, |x_2| \leq 5\},\ \mathcal{U} := \{u : |u| \leq 5, |\Delta u_k| \leq 2\}, \tag{40}$$

$w_t \sim U(0, 0.25)$ and $v_t \sim U(0, 0.025)$, instead of zero mean normal distribution, as it is common in the literature.

The minimum estimation ($\mathbb{N}_e$) and control ($\mathbb{N}_c$) horizons were computed using equations (25) and parameters given by (39), leading to $\mathbb{N}_e = 10$ and $\mathbb{N}_c = 15$. However, different values of N_e and N_c were chosen

$$N_e := \{2, 5, 10, 20\},\ N_c := \{5, 10, 35\}. \tag{41}$$

to analyse their effect on the closed-loop performance.

Since the difference between N_e and N_c can lead to unbalanced cost functions (emphasizing the control cost over the estimation one), which can deteriorate the overall closed-loop performance. To avoid this problem, φ is used to improve the closed-loop performance. It takes the following values $\varphi := \{0.95, 0.95, 0.85, 0.65\}$ for the corresponding N_e value.

Figure 4 summarize the mean square error (*MSE*) obtained by both controllers along 100 simulations for $\epsilon = 0.1$ and $\epsilon = 3$ respectively. These figures show the superior performance of the simultaneous *MHE–MPC* for any combination of $N_e - N_c$ and scenario. In general, there are no meaningful changes of *MSE* with N_c, however closed-loop performance varies with N_e. Figure 4A shows the results for $\epsilon = 0.1$. In this case the independent *MHE* and *MPC* performance improves with N_e, while the simultaneous *MHE–MPC* ones remains similar (a deviation lower than 8% from the average) for any combination of $N_e - N_c$. For this value of ϵ, the system (38) behaves like a harmonic oscillator, therefore the closed-loop performance depends on the estimation error (see Figure 5), which decreases for larger values of N_e. Figure 4B shows the results for $\epsilon = 3$. In this condition, the performance of both controllers deteriorates with N_e, because for this value of ϵ the system (38) behaves like a non-linear dampened oscillator and the state estimates take longer to converge to the estimation invariant set (see Figure 5A,B).

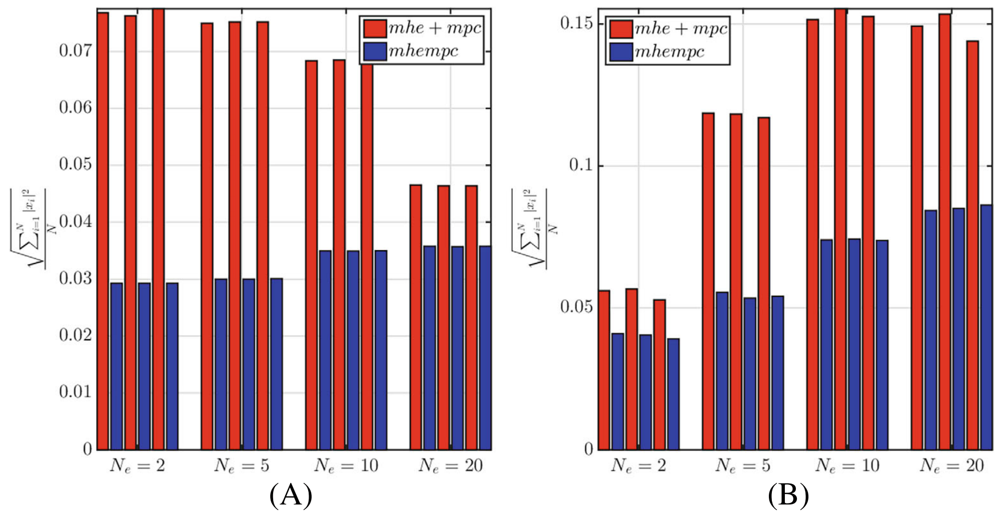

FIGURE 4 *MSE* of 100 simulations for different values of N_e, N_c and ϵ.

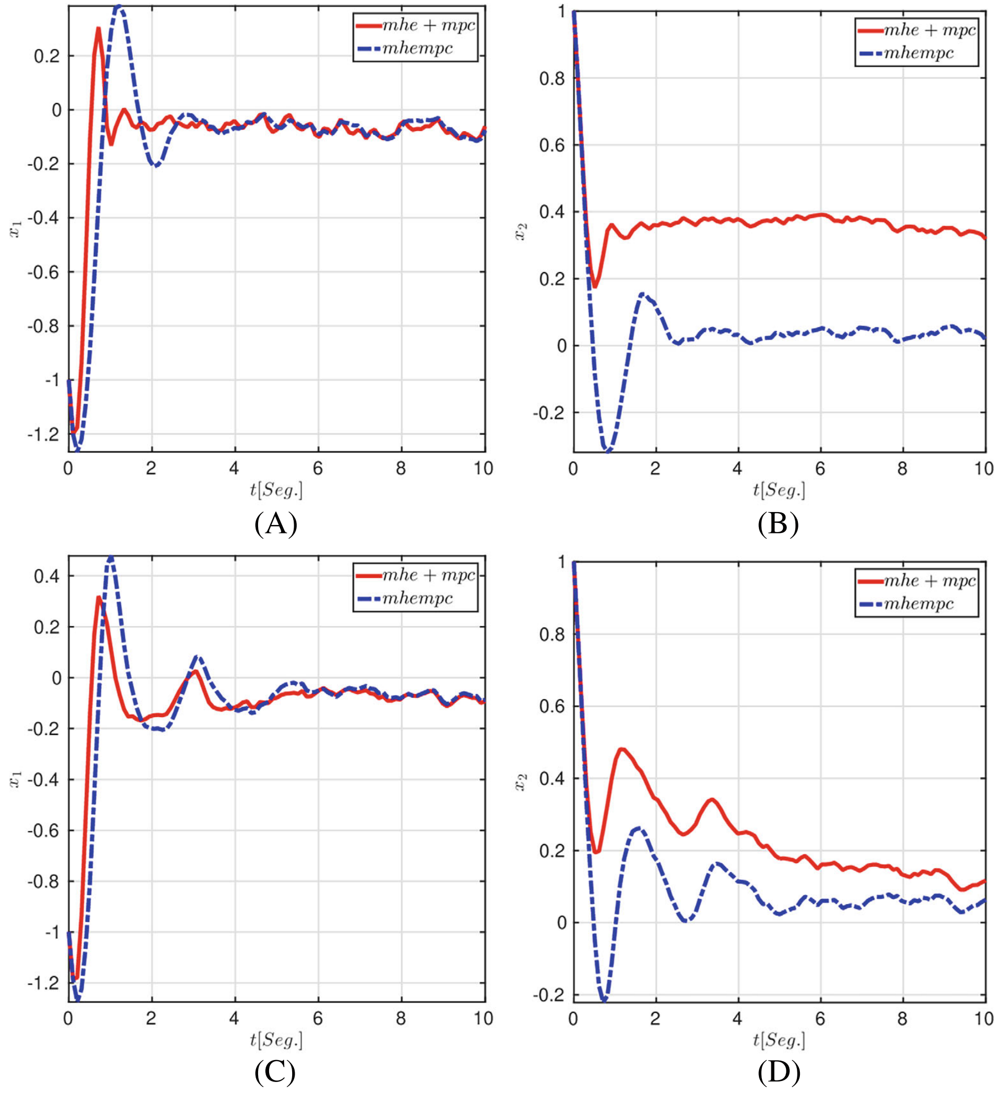

FIGURE 5 Two realizations of x_1 and x_2 for $\epsilon = 0.1$, $N_e = 2$ (A,B), $N_e = 20$ (C,D) and $N_c = 35$.

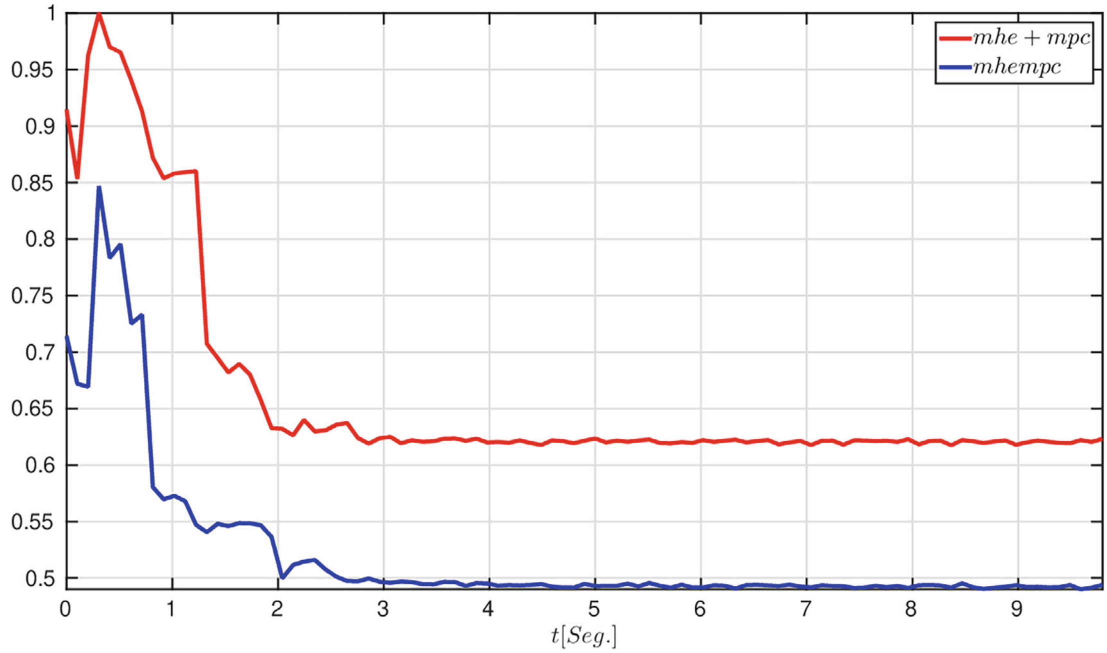

FIGURE 6 Average execution time in seconds over 100 trials for $N_e = N_c = 10$.

Figures 5 show the simulations resulting from two noise realizations for $N_c = 35$, $N_e = 2$ and $N_e = 20$, respectively. They show that the SMHEC manages to regulate both states and it achieves a better performance than the independent one. While Figure 5A,C show that independent *MHE* and *MPC* achieves a better performance than the simultaneous one for state x_1, Figure 5B,D show how it fails to regulate state x_2 for short estimation horizons. Under this condition, x_2 has an offset that it is not compensated by the controller. Only large values of N_e allow the independent *MHE* and *MPC* to regulate x_2 (Figure 5D). On the other hand, the SMHEC regulates both states and it takes shorter times than the independent one to regulate both states.

The computational burden of the simultaneous *MHE–MPC* is lower than the independent one, as can be seen in Figure 6. The execution times were averaged over 100 trials. The lower time, in the beginning, is due to the backward window corresponding to the estimation has not achieved yet its full length.

5 | CONCLUSIONS

We presented an output-feedback approach for nonlinear systems subject to bounded disturbances using *MHE–MPC*. The proposed approach combines the state estimation and control problems into a single optimisation, which is solved at each sampling time. Theorem 1 states the necessary conditions to guaranty the feasibility and stability of the optimization problem, and therefore the boundedness of system states, as a function of the windows lengths N_e and N_c. This result requires the compatibility between the robust estimated and controllable sets (Assumption 1) and the existence of a relaxed closed–loop Lyapunov function for the disturbed system (Assumption 21). These conditions imply forward (N_c) and backward (N_e) horizons to find state estimates and control actions that are consistent with the system dynamics, constraints and disturbances.

Even though the stability and performance of the simultaneous approach are seen to outperform the *MHE* and *MPC* solved independently, the simultaneous method has shown a higher sensibility to the parameters of the backward and forward windows, requiring a more detailed and precise tuning process. The stage-cost matrices on both windows need to be chosen carefully. The parameter φ can help in the tuning procedure. However, a high level of expertise is required in order to succeed with the adjustment of matrices.

Future work may involve the design of the forward window with properties that allow the improvement of the estimation process and the design of an adaptive law to compute φ such that the estimation and control problems keep balanced and the overall system performance and numerical properties are improved.

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CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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APPENDIX A. PROOF THEOREM 1

In the following we will analyze the stability of the simultaneous *MHE–MPC* algorithm by means of the difference in costs at two consecutive sampling time

$$\Delta\Psi = \Psi_{EC,k+1,N_e+N_c} - \Psi_{EC,k,N_e+N_c}. \tag{A1}$$

Evaluating $\Psi_{EC,k+1,N_e+N_c}$ with the tail of the solution computed at time k, with $\hat{u}_{k+N_c} = 0$ and $\hat{x}_{k+N_c+1} = f(\Xi, \hat{u}_{k+N_c})$, we obtain

$$\begin{aligned}
\Delta\Psi = {} & \Gamma_{k-N_e+1}\left(\chi_{k-N_e+1}\right) + \sum_{j=k-N_e+1}^{k} \ell_{w_e}\left(\hat{w}_{j|k+1}\right) + \sum_{j=k-N_e+1}^{k+1} \ell_{v_e}\left(\hat{v}_{j|k+1}\right) \\
& + \sum_{j=k+1}^{k+N_c} \left(\ell_c\left(\hat{x}_{j|k+1}, \hat{u}_{j|k+1}\right) - \ell_{w_c}\left(\hat{w}_{j|k+1}\right)\right) + \Upsilon_{k+N_c+1}\left(f(\Xi, \hat{u}_{k+N_c})\right) \\
& - \left(\Gamma_{k-N_e}(\chi) + \sum_{j=k-N_e}^{k-1} \ell_{w_e}\left(\hat{w}_{j|k}\right) + \sum_{j=k-N_e}^{k} \ell_{v_e}\left(\hat{v}_{j|k}\right)\right. \\
& \left. + \sum_{j=k}^{k+N_c-1} \left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) + \Upsilon_{k+N_c}(\Xi)\right).
\end{aligned} \tag{A2}$$

Since $\chi_{k-N_e+1} = \hat{x}_{k-N_e+1|k+1} - \overline{x}_{k-N_e+1}$ and

$$\begin{aligned}
\overline{x}_{k-N_e+1} &= \hat{x}_{k-N_e+1|k}, \\
\hat{x}_{k-N_e+1|k+1} &= \hat{x}_{k-N_e+1|k},
\end{aligned} \tag{A3}$$

then $\Gamma_{k-N_e+1}(\chi_{k-N_e+1}) = 0$. Using inequality (21) and Assumption 5, $\Delta\Psi$ can be rewritten as follows

$$\begin{aligned}
\Delta\Psi \leq & -\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)\left(1-\delta\left(\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)}+\frac{1}{\delta}\frac{\ell_{w_c}\left(\hat{w}_{k|k}\right)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)}\right)\right) \\
& -\Gamma_{k-N_e}(\chi)+\ell_{w_e}\left(\hat{w}_{k|k+1}\right)-\ell_{w_e}\left(\hat{w}_{k-N_e|k}\right)-\ell_{v_e}\left(\hat{v}_{k-N_e|k}\right), \\
\leq & -\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)\left(1-\delta\left(\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)}+\frac{1}{\delta}\,\Delta_c^w\right)\right) \\
& -\Gamma_{k-N_e}(\chi)+\ell_{w_e}\left(\hat{w}_{k|k}\right)-\ell_e\left(\hat{w}_{k-N_e|k},\hat{v}_{k-N_e|k}\right).
\end{aligned} \tag{A4}$$

for $\delta \in \mathbb{R}_{>0}$. Defining functions ω and π_E as follows

$$\begin{aligned}
\omega &:= \frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)}+\frac{1}{\delta}\,\Delta_c^w, \\
\pi_E &:= -\Gamma_{k-N_e}(\chi)+\ell_{w_e}\left(\hat{w}_{k|k}\right)-\ell_e\left(\hat{w}_{k-N_e|k},\hat{v}_{k-N_e|k}\right),
\end{aligned} \tag{A5}$$

the Equation (A4) can be written in a compact way

$$\Delta\Psi \leq -\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)(1-\delta\omega)+\pi_E. \tag{A6}$$

The term ω quantifies the improvements in the control cost (through the ratio between the *cost-to-go* $\Upsilon_{k+N_c}(\cdot)$ and the control stage cost $\ell_c(\cdot,\cdot)$ at time k) and the disturbance controllability (the ratio between the control stage costs $\ell_{w_c}(\cdot)$ and $\ell_c(\cdot,\cdot)$ at time k).

The term π_E quantifies the changes in the estimation cost by measuring the amount of information left behind the estimation window (the *arrival–cost* $\Gamma_{k-N_e}(\cdot)$). Since $\hat{w}_{k|k}$ was computed within the control window (maximised), it tends to take larger values than $\hat{w}_{k-N_e|k}$ which was computed within the estimation window (minimised). Therefore, when state estimation is precise (i.e., $\Gamma_{k-N_e}(\chi)$ remains low), the term π_E will tend to take positive values, whereas if a major correction is made on the initial condition $\hat{x}_{k-N_e|k}$ (i.e., $\Gamma_{k-N_e}(\chi)$ will take big values), the improvement in the estimated trajectory will lead a decreasing cost with sharper slope.

Since

$$\begin{aligned}
\pi_E &= -\Gamma_{k-N_e}(\chi)+\ell_{w_e}\left(\hat{w}_{k|k}\right)-\ell_e\left(\hat{w}_{k-N_e|k},\hat{v}_{k-N_e|k}\right), \\
&\leq \ell_{w_e}\left(\hat{w}_{k|k}\right), \\
&\leq \overline{\gamma}_{w_e}\left(|\hat{w}_{k|k}|\right), \\
&\leq \overline{\gamma}_{w_e}(\|\hat{\boldsymbol{w}}\|),
\end{aligned} \tag{A7}$$

which can be written in term of $\mathcal{K}$ functions as follows[14]

$$\pi_E \leq \overline{\gamma}_w(\|\hat{\boldsymbol{w}}\|) \leq \overline{\pi}_E := \overline{\gamma}_w\left(\underline{\gamma}_w^{-1}\left(\frac{\overline{\gamma}_p(\chi)}{N_e}+\overline{\gamma}_w(\|\boldsymbol{w}\|)+\overline{\gamma}_v(\|\boldsymbol{v}\|)\right)\right), \tag{A8}$$

Restating (A6) with $\overline{\pi}_E$, $\Delta\Psi$ can be posed as

$$\Delta\Psi \leq -\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)(1-\delta\omega)+\overline{\pi}_E, \tag{A9}$$

From the first term in the right hand side of (A9), one can see that if

$$0 \leq \delta\omega < 1 \tag{A10}$$

then, for large values of $\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)$ so that it becomes dominating in (A9), the sequence of cost will present a contractive behavior until $\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)(1-\delta\omega)$ reaches the value of $\overline{\pi}_E$, that is, the decreasing behavior in the costs is determined by the estimation error. Moreover, one is interested in guaranteeing that $0 \leq \delta\omega < 1$. In order to relate the quantity $\delta\omega$ with design parameter of the SMHEC framework, let us recall that

$$\omega = \frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \frac{1}{\delta}\,\Delta_c^w,$$

or

$$\delta\omega = \frac{\delta\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \Delta_c^w,$$

Therefore, the following needs to be guaranteed

$$0 \leq \frac{\delta\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \Delta_c^w, < 1 \tag{A11}$$

The non negativity holds trivially since all quantities involved are positive. Therefore, the following is the inequality to fulfil

$$\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} < \frac{1-\Delta_c^w}{\delta} \tag{A12}$$

Since $0 < \Delta_c^w < 1$ by Assumption 4, right hand side of inequality (A12) will be positive. The problem consists now in to find a value of N_c such that (A12) be verified. In order to relate (A12) with N_c, let us note that

$$\begin{aligned}
\Psi_{C,k,N_c} &= \sum_{j=k}^{k+N_c-1}\left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) + \Upsilon_{k+N_c}(\Xi),\\
&= \ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)\sum_{j=k}^{k+N_c-1}\frac{\left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)},\\
&\leq \ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)\sum_{j=k}^{k+N_c-1}\frac{\left(\ell_c \hat{x}_{j|k}, \hat{u}_{j|k}\right)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)}.
\end{aligned} \tag{A13}$$

Then, the term $\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c(\hat{x}_{k|k}, \hat{u}_{k|k})}$ can be upper bounded as follows

$$\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} = \frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c(x_{k+N_c-1}, u_{k+N_c-1})}\,\frac{\ell_c(x_{k+N_c-1}, u_{k+N_c-1})}{\ell_c(x_{k+N_c-2}, u_{k+N_c-2})}\cdots\frac{\ell_c(x_{k+1}, u_{k+1})}{\ell_c(x_k), u_k}, \tag{A14}$$

The term $\Upsilon_{k+N_c}(\Xi)$ can be upper as follows

$$\Upsilon_{k+N_c}(\Xi) = \Psi_{C,k,N_c} - \sum_{j=0}^{N_c-1}\ell_c(x_j, u_j), \tag{A15}$$

$$\leq L_{N_c}\sigma(x) - \sigma(x), \tag{A16}$$

$$= \sigma(x)(L_{N_c} - 1), \tag{A17}$$

Then, then term $\ell_c(x_{k+N_c-1}, u_{k+N_c-1})$ is upper bounded as

$$\ell_c(x_{k+N_c-1}, u_{k+N_c-1}) = \Psi_{C,k,N_c} - \left(\sum_{j=0}^{N_c-2}\ell_c(x_j, u_j) + \Upsilon_{k+N_c}(\Xi)\right), \tag{A18}$$

$$\leq L_{N_c}\sigma(x) - \sigma(x), \tag{A19}$$

$$= \sigma(x)(L_{N_c} - 1), \tag{A20}$$

The term $\ell_c(x_{k+N_c-2}, u_{k+N_c-2})$ is bounded by

$$\ell_c(x_{k+N_c-2}, u_{k+N_c-2}) = \Psi_{C,k,N_c-1} - \left(\sum_{j=0}^{N_c-3} \ell_c(x_j, u_j) - \Upsilon_{k+N_c-1}\left(x_{k+N_c-1}\right)\right), \tag{A21}$$

$$\leq L_{N_c-1}\sigma(x) - \sigma(x), \tag{A22}$$

$$= \sigma(x)(L_{N_c-1} - 1), \tag{A23}$$

The remaining terms are bounded similarly. Then, by noting that the terms in the denominators in (A14) delayed one sampling time respect the terms in the numerators, (A14) can be bounded as

$$\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} \leq \prod_{j=1}^{N_c} \frac{L_i - 1}{L_{i-1}} \leq \prod_{j=1}^{N_c} \frac{L-1}{L} = \left(\frac{L-1}{L}\right)^{N_c} = \frac{(L-1)}{L}\left(\frac{L-1}{L}\right)^{N_c-1}, \tag{A24}$$

$$\leq (L-1)\left(\frac{L-1}{L}\right)^{N_c-1}, \tag{A25}$$

Then, Inequality (A12) can be stated as

$$\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} \leq (L-1)\left(\frac{L-1}{L}\right)^{N_c-1} \leq \frac{1-\Delta_c^w}{\delta}, \tag{A26}$$

Then, by algebraic manipulations, one can verify that if the length of the forward window is selected as

$$N_c = \left\lceil \frac{\ln\left(\frac{\delta(L-1)}{1-\overline{\Delta}_c^w}\right)}{\ln\left(\frac{L}{L-1}\right)} + 1 \right\rceil. \tag{A27}$$

Then, the stability condition (A12) is verified, and $\delta\omega < 1$.

APPENDIX B. DERIVATION OF $\dot{P}_T$

$$\begin{aligned}
\dot{p}_t &= \frac{\Delta x}{|\Delta x|}\left(\dot{x}_t^{(1)} - \dot{x}_t^{(2)}\right), \\
&= \frac{\Delta x}{|\Delta x|}\left(a x_t^{(1)^3} + w_t^{(1)} + u_t^{(1)} - a x_t^{(2)^3} - w_t^{(2)} - u_t^{(2)}\right), \\
&= \frac{\Delta x}{|\Delta x|}\left(a\left(x_t^{(1)^3} - x_t^{(2)^3}\right) - K\Delta x + \Delta w_t\right), \\
&= \frac{\Delta x}{|\Delta x|}\left(a\Delta x\left(x_t^{(1)^2} + x_t^{(1)}x_t^{(2)} + x_t^{(2)^2}\right) - K\Delta x + \Delta w_t\right), \\
&\leq -K|\Delta x| + |\Delta x| a\left(x_t^{(1)^2} + x_t^{(1)}x_t^{(2)} + x_t^{(2)^2}\right) + |\Delta w_t|, \\
&\leq -K|\Delta x| + |\Delta x| a\frac{\left(x^{(1)^3} - x^{(2)^3}\right)}{\Delta x} + |\Delta w_t|, \\
&\leq -Kp_t + a\left(\left(y_t^{(1)} - v_t^{(1)}\right)^3 - \left(y_t^{(2)} - v_t^{(2)}\right)^3\right) + |\Delta w_t|, \\
&\leq -Kp_t + a|h^3\left(x_t^{(1)}\right) - h^3\left(x_t^{(2)}\right)| + |\Delta w_t|, \\
&\leq -Kp_t + ag|\Delta h_t| + |\Delta w_t|.
\end{aligned}$$