Drag coeff conversion

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1 Method

Coulomb sliding function, taken straight from the Elmer/Ice wiki:

$$\tau_b = C \cdot N \left[\frac{\chi \cdot u_b^{-n}}{(1 + a \cdot \chi^q)} \right]^{\frac{1}{n}} \cdot u_b \quad (1)$$

$$a = \frac{(q-1)^{q-1}}{q^q} \quad (2)$$

$$\chi = \frac{u_b}{C^n N^n A_s} \quad (3)$$

Standard linear Weertman as used in Elmer/Ice inversions:

$$\tau_b = 10^\beta u_b \quad (4)$$

We know sliding velocity, u_b, and the inverted coefficient, β, from the completed inversions. We want to set the basal resistance to be equal in both sliding functions, so we need to substitute equation 4 into equation 1. This gives:

$$10^\beta u_b = C \cdot N \left[\frac{\chi \cdot u_b^{-n}}{(1 + a \cdot \chi^q)} \right]^{\frac{1}{n}} \cdot u_b \quad (5)$$

$$10^\beta = C \cdot N \left[\frac{\chi \cdot u_b^{-n}}{(1 + a \cdot \chi^q)} \right]^{\frac{1}{n}} \quad (6)$$

where τ_b is basal resistance (aka shear stress), N is effective pressure at the bed, n is a flow exponent (typically set to Glen's 3), q is a "post-peak" exponent (typically set to 1 unless you want a double-valued sliding law, which can give pretty crazy results), and C and A_s are two different friction coefficients that affect the behaviour in different ways.

One problem now is that χ makes this problem intractably non-linear for q not equal to 1. Further, there are clearly more degrees of freedom in the Coulomb sliding law than in the linear Weertman sliding law. Fortunately, $q = 1$ is pretty standard. So let's assume that $q = 1$ and $n = 3$. Let's also fix one of C or A_s. I think fixing C might be more restrictive than fixing A_s, to let's fix A_s.

We can make the usual assumption about N, based on connection to the ocean: $N = Hg\rho_i - zg\rho_o$, where H is ice thickness, g is acceleration due to gravity, ρ_i is ice density, ρ_o is ocean water density, and z is height relative to sea level. Alternatively (and more correctly in a Stokes model), we can use the normal stress instead of $Hg\rho_i$.

So, with the above assumptions, and carrying on from equation 6,

$$
\begin{aligned}
10^{\beta} &= C \cdot N \left[\frac{\chi \cdot u_b^{-3}}{(1+\chi)} \right]^{\frac{1}{3}} && (7) \\
10^{3\beta} &= C^3 \cdot N^3 \frac{\chi \cdot u_b^{-3}}{(1+\chi)} && (8) \\
(1+\chi)\, 10^{3\beta} &= C^3 \cdot N^3 \frac{\chi}{u_b^3} && (9) \\
\left(1 + \frac{u_b}{C^3 N^3 A_s}\right) 10^{3\beta} &= C^3 \cdot N^3 \frac{u_b}{C^3 N^3 A_s u_b^3} && (10) \\
C^3 N^3 A_s 10^{3\beta} + u_b 10^{3\beta} &= \frac{C^3 \cdot N^3}{u_b^2} && (11) \\
C^3 N^3 A_s 10^{3\beta} u_b^2 + u_b^3 10^{3\beta} &= C^3 N^3 && (12) \\
C^3 N^3 \left(A_s 10^{3\beta} u_b^2 - 1\right) &= -u_b^3 10^{3\beta} && (13)
\end{aligned}
$$

So I reckon you could use equation 13 with the above assumptions to calculate C to use in the Coulomb sliding function ...

But this seems likely to give negative values for C, which seems a bit odd to me. One could probably ensure that C is generally positive by choosing a very small value for A_s ...

2 Liyun's method

Again, start with the equations on the wiki (equations 1, 2 and 3 above).

Make the following assumptions:

$$
\begin{aligned}
q &= 1, && (14) \\
n &= 3, && (15) \\
C &= 1. && (16)
\end{aligned}
$$

Then equation 1 becomes

$$\tau_b = N \left[\frac{\chi u_b^{-3}}{1+\chi} \right]^{1/3} u_b, \qquad (17)$$

which can be rearranged to give

$$\chi = \frac{1}{\left(\frac{N}{\tau_b}\right)^3 - 1}. \qquad (18)$$

Also, equation 3 becomes

$$\chi = \frac{u_b}{N^3 A_s}. \tag{19}$$

Then equations 18 and 19 can be combined to give

$$A_s = \frac{u_b}{\tau_b^3} - \frac{u_b}{N^3}. \tag{20}$$

Then we can substitute the expression for τ_b used in the inversions to give

$$A_s = \frac{1}{10^{3\beta} u_b^2} - \frac{u_b}{N^3}, \tag{21}$$

where β is the optimised variable from the basal drag inversion. This is the equation used to calculate A_s. β and u_b come from the inversion. N can be based on the hydrostatic assumption or Elmer's computed normal stress can be used, but in either case the largest uncertainty certainly comes from whatever assumption is made to calculate water pressure (probably depth relative to sea level).

3 Large and small χ limits

Equation 1 can be rearranged in its non-dimensionalized form as

$$\left(\frac{\tau_b}{C \cdot N}\right)^n = \frac{\chi}{1 + a \cdot \chi^q}. \tag{22}$$

We now can identify the limits for large and small χ.

3.1 Weertman limit of regularized Coulomb law

For small values, i.e., $\chi \to 0$, the r.h.s. of (22) gets

$$\frac{\chi}{1 + a \cdot \chi^q} \to \chi. \tag{23}$$

Consequently, (22) becomes

$$\left(\frac{\tau_b}{C \cdot N}\right)^n = \chi, \tag{24}$$

which by rearrangement and evaluation of χ reveals (non-linear) Weertman's law

$$u_b = A_s \cdot \tau_b^n. \tag{25}$$

In other words, for small χ we are in the limit of Weertman sliding. When is χ small? This applies to situations with both, small sliding velocities and large values of the effective pressure, N.

3.2 Coulomb limit of regularized Coulomb law

Again, we can identify the limit of the r.h.s. of (22), but now for large values of $\chi \gg 1$

$$\frac{\chi}{1 + a \cdot \chi^q} \to a^{-1} \cdot \chi^{1-q}, \tag{26}$$

which in case of $q = 1$ reduces to unity. In this case (22) reduces to

$$\left(\frac{\tau_b}{C \cdot N}\right)^n = 1, \tag{27}$$

or, if rearranged,

$$\tau_b = C \cdot N. \tag{28}$$

In other words, for large values of χ we hit the limit of a pure Coulomb friction law (with C being the tangens of a friction angle), that describes a constant relation between shear and normal stress (here in form of effective pressure). When is χ large? This applies to situations with both, large sliding velocities and low effective pressure (i.e., close to floating condition).

3.3 Suggested conversion of linear sliding to regularized Coulomb law

We suggest to pick characteristic values for both parameters, A_s and C. For the latter, we can stick to unity value ($C_0 = 1$) as a referrence. For the Weertman parameter, A_s we suggest to interpret the inverted linear friction coefficient everywhere in terms of a non-linear Weertman coefficient, using the following relation:

$$A_s = u_b^{1-n} \cdot 10^{-n\beta}. \tag{29}$$

The reference value, A_{s0}, then can be taken as the average value over the whole domain, perhaps excluding the fastest regions (streams, shelves). With this given values, a local reference value of χ_0 can be computed using the sliding velocity and the effective pressure from the inversion

$$\chi_0 = \frac{u_b}{N^n A_{s0}}. \tag{30}$$

A visualization of relation (22) as shown in Fig. 1, suggests to use pure non-linear Weertman conversion at $\chi_0 \approx 0.5$ – it has to be made sure that Iken's bound is not exceeded (see later). This means, for regions with $\chi_0 < 0.5$ we exactly use Equation 29, for regions above we set A_{s0} and compute C by some means. In the very large limit for χ we can stick to a pure Coulomb interpretation

$$C = u_b \cdot 10^{\beta} \cdot N^{-1}. \tag{31}$$

Mind, that the Weertman part does not guarantee that the upper value of $\tau_b/(N\,C) \leq 1$ (i.e., Iken's bound) is not exceeded – one has to take care that to not pick a too high threshold for χ_0. We also cannot switch directly between

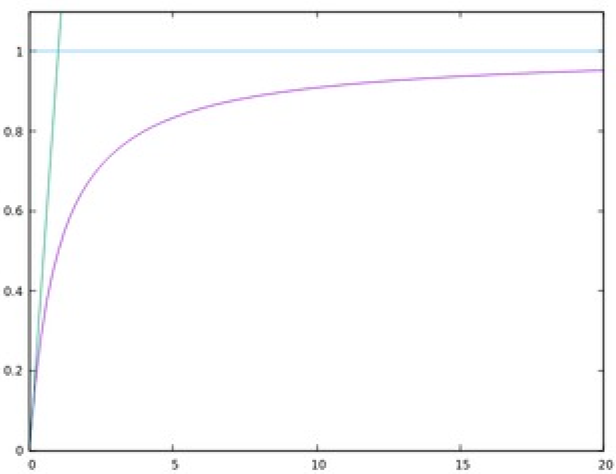

Figure 1: Non-dimensionalized regularized Coulomb law, $(\tau_b/(N\,C))^n$ vs. χ for $q = 1$. The lines depicting the Weertman and the Coulomb limit are included.

(29) to (31), e.g. at $(\tau_b/(N\,C))^n = 1$, as effectively, this would be like applying a Tsai-law. We could use Liyun's formula (21) instead of (29) to a higher χ_0 threshold value and then switch to (31), just to take out the edge of too insane values for A_s in case of large local χ_0.

4 December 2022 update

Minor note: Section 3.2 talks about the $\chi \gg 1$ limit. This is equivalent to A_s approaching zero.

4.1 Potential inconsistency with existing approach

Existing conversion code uses eqns 21 and 31. Note that Eqn 21 assumes $C = 1$ and 31 assumes $A_s = 0$. These two assumptions are not precisely consistent with the approach of using a β threshold for switching between the two equations. Specifically, eqn 21 does not imply $A_s = 0$. The inconsistency arises because eqn 31 makes the full Coulomb assumption whereas the regularised Coulomb law transitions smoothly and so still has some small dependence on A_s even toward the Coulomb end of the regime.

The above might be a possible cause for inconsistency between sliding velocities from the linear Weertman inversions and the restarted regularised Coulomb simulations. However, even if this is fixed, we still may have spatial jumps in both fields across the β threshold value.

4.2 Less discontinuous (i.e. smoother) approach

Here's a possible alternative to provide a smoother field. Use pure Weertman sliding to convert β to A_s. Use a tanh function (of something like sliding velocity or effective pressure or a function of both) to scale A_s to zero toward the Coulomb regime. Use a modified (need to go back to the full regularised Coulomb law without assumptions) version of 31 to calculate C as a function of A_s.

4.3 Tasks:

Provide more general version of eqn 31 (including dependence on A_s).

$$C^n = \frac{u_b \tau_b^n}{N^n(u_b - \tau_b^n A_s)} \tag{32}$$

Leading to...

$$C^n = \frac{u_b^n 10^{n\beta}}{N^n(1 - 10^{n\beta} u_b^{n-1} A_s)} \tag{33}$$

$$C = u_b \cdot 10^\beta \cdot N^{-1} \cdot (1 - 10^{n\beta} u_b^{n-1} A_s)^{-1/n} \tag{34}$$

This simplifies to eqn 31 for $A_s = 0$.

For $n = 3$...

$$C = u_b \cdot 10^\beta \cdot N^{-1} \cdot (1 - 10^{3\beta} u_b^2 A_s)^{-1/3} \tag{35}$$

Provide pure Weertman equation to calculate A_s.
(actually this is done; it is eqn 29)

Figure out tanh function.
Currently using simply tanh(2*effective pressure). Other things involving sliding velocity could be tested.

Test these equations, probably in a Paraview calculator.
See the following figures (2, 3 and 4) from Paraview. These are based on an old set of vtu output files I had for Totten. May differ from latest outputs, but should be adequate for illustrating and testing the approach. In particular the last figure compares basal resistances reconstructed using the linear weertman and regularised coulomb sliding laws with the coefficients described above. The differences are zero to about 16 significant figures.

Synchronise code (user function vs solver: need to keep only one under version control)

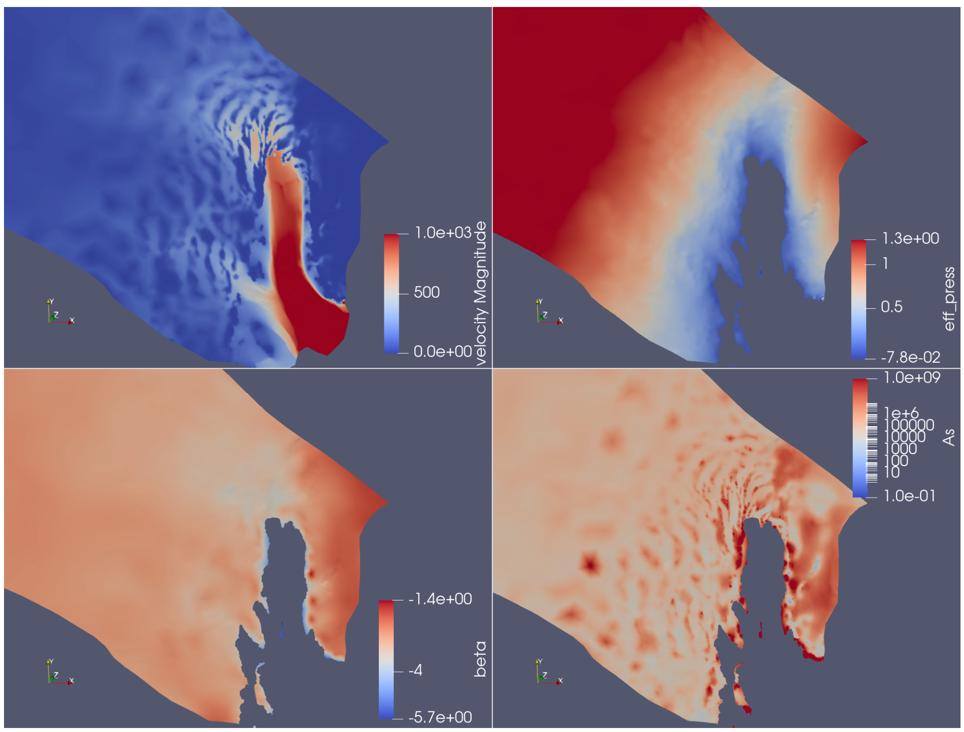

Figure 2: Sliding velocity (top left), effective pressure based on full ocean connectivity (top right), linear weertman sliding coefficient (bottom left) and coulomb As coefficient (bottom right).

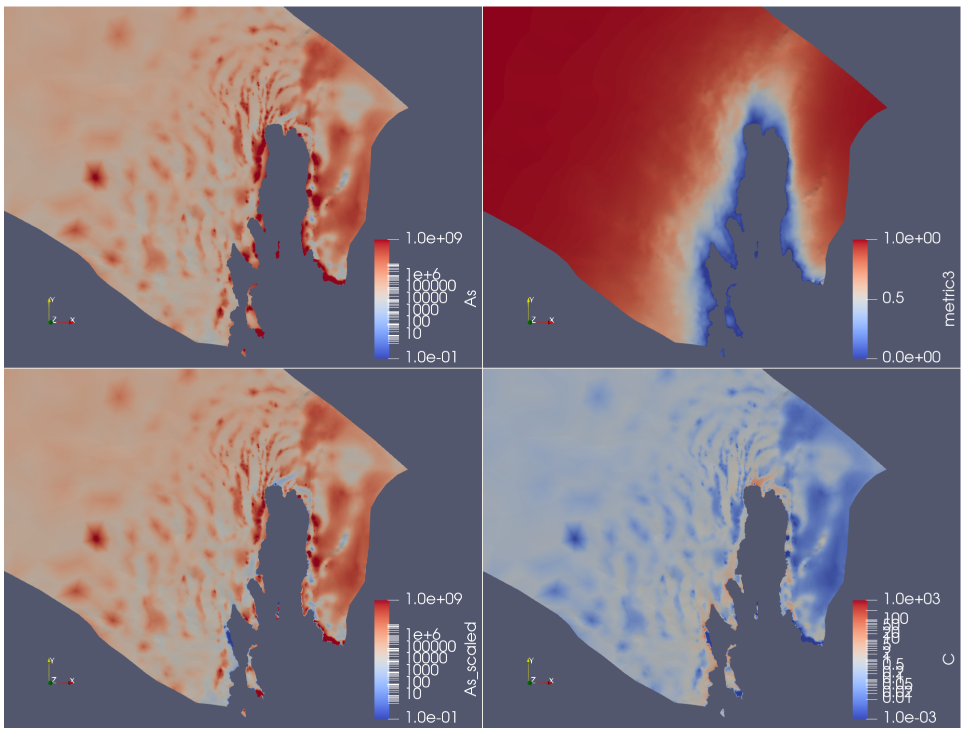

Figure 3: As coefficient (top left), scaling metric tanh(2*effective pressure) (top right), scaled As coefficient (bottom left) and C (max slope) coefficient.

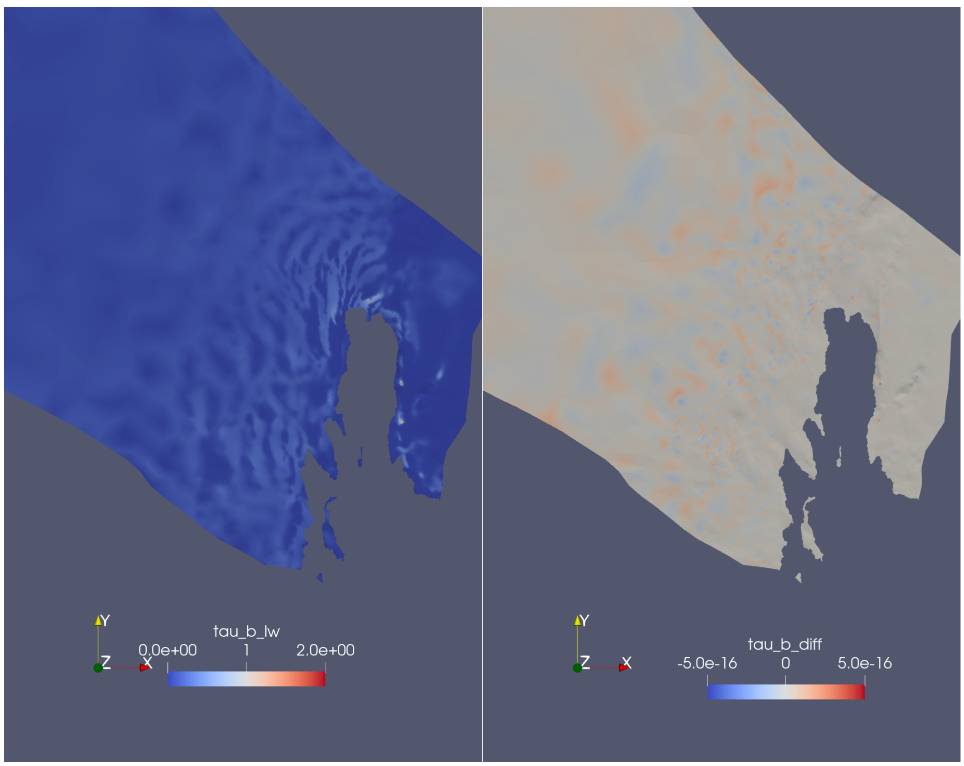

Figure 4: reconstructed linear weertman basal resistance in MPa (left) and differnce between reconstructed regularised coulomb basal resistance and linear weertman basal resistance (right).

Using a separate solver now; weertman2coulomb deleted from USF.

Try this in the Weertman2Coulomb solver section of the sif: Conversion mode = String Smooth

Implement the above equations in the synchronised code base.

Done!