Laboratoire de Glaciologie et Géophysique de l'Environnement

Observatoire des Sciences de l'Univers de Grenoble

CSC

Elmer/Ice Oslo 2016

Shallow models in Elmer/Ice

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Outline

- **Shallow Shelf / Shallow stream Solver**
- **Thickness Solver**
- **Current / planned development**

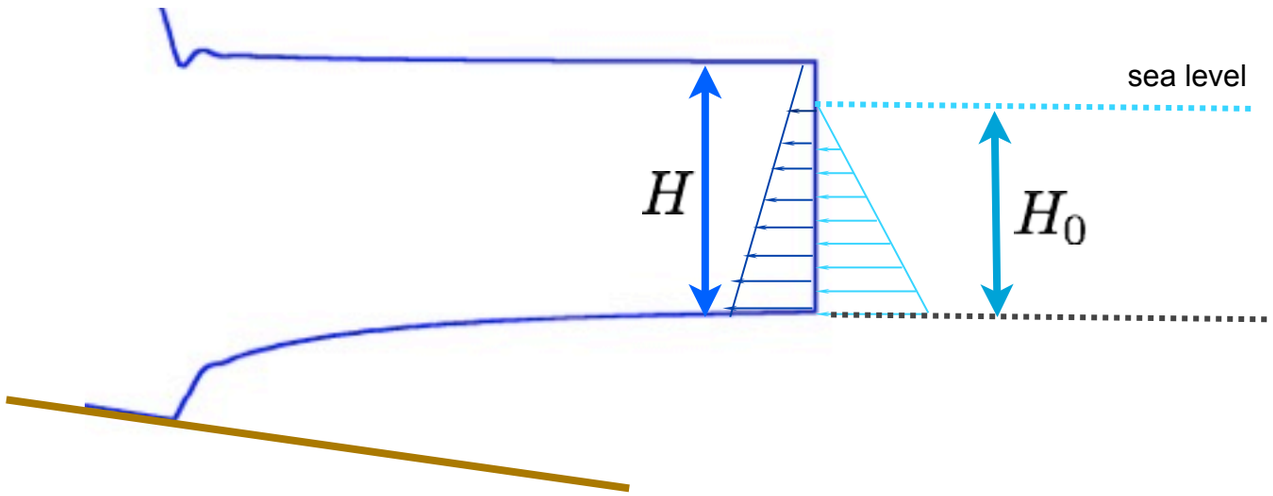
Shallow Shelf Approximation/Shallow Stream Approximation

Field equations:

$$\begin{cases} \frac{\partial}{\partial x}\left(2H\nu\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)-\beta u=\rho g H\frac{\partial z_s}{\partial x} \\ \frac{\partial}{\partial x}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(2H\nu\left(\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}\right)\right)-\beta v=\rho_i g H\frac{\partial z_s}{\partial y} \end{cases}$$

Boundary Conditions:

$$\begin{cases} 4H\nu\frac{\partial u}{\partial x}n_x+2H\nu\frac{\partial v}{\partial y}n_x+H\nu(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x})n_y=(\rho_i g H-\rho_w g H_0)n_x \\ 4H\nu\frac{\partial v}{\partial y}n_y+2H\nu\frac{\partial v}{\partial x}n_y+H\nu(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x})n_x=(\rho_i g H-\rho_w g H_0)n_y \end{cases}$$



Shallow Shelf Approximation/Shallow Stream Approximation

Field equations:

$$\begin{cases} \frac{\partial}{\partial x}\left(2H\nu\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)-\beta u=\rho g H\frac{\partial z_s}{\partial x} \\ \frac{\partial}{\partial x}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(2H\nu\left(\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}\right)\right)-\beta v=\rho_i g H\frac{\partial z_s}{\partial y} \end{cases}$$

$$H = Zs - Zb$$

Elmer/Ice Solvers:

Solver Fortran File: `SSASolver.f90`
Solver Name: `SSABasalSolver`

`Required Output Variable(s):`

- `SSAVelocity`

Required Input Variable(s):

- (1) `Zb`, `Zs` and `Effective Pressure` when using the Coulomb type friction law

The `SSABasalSolver` solve the classical SSA equation, it has been modified in Rev. 6440 to be executed either on a grid of dimension lower than the problem dimension itself (i.e. the top or bottom grid of a 2D or 3D mesh for a SSA 1D or 2D problem), or on a grid of the same dimension of the problem (i.e. 2D mesh for a 2D plane view SSA solution).

It will work on a 3D mesh only if the mesh as been extruded along the vertical direction and if the base line boundary conditions have been preserved (to impose neumann conditions). **Keyword *«Preserve Baseline = Logical True»* in section Simulation**

Shallow Shelf Approximation/Shallow Stream Approximation

Field equations:

$$\begin{cases} \frac{\partial}{\partial x}\left(2H\nu\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)-\beta u=\rho g H\frac{\partial z_s}{\partial x} \\ \frac{\partial}{\partial x}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(2H\nu\left(\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}\right)\right)-\beta v=\rho_i g H\frac{\partial z_s}{\partial y} \end{cases}$$

SIF - Solver Section:

```
Solver 1
  Equation = "SSA"
  Procedure = File "ElmerIceSolvers" "SSABasalSolver"
  Variable = String "SSAVelocity"
  Variable DOFs = 2   ! 1 in SSA 1-D or 2 in SSA-2D

  Linear System Solver = Direct
  Linear System Direct Method = umfpack

  Nonlinear System Max Iterations = 100
  Nonlinear System Convergence Tolerance  = 1.0e-08
  Nonlinear System Newton After Iterations = 5
  Nonlinear System Newton After Tolerance = 1.0e-05

  Nonlinear System Relaxation Factor = 1.00

  Steady State Convergence Tolerance = Real 1.0e-3
End
```

Shallow Shelf Approximation/Shallow Stream Approximation

Field equations:

$$\begin{cases} \frac{\partial}{\partial x}\left(2H\nu\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)-\beta u=\rho g H\frac{\partial z_s}{\partial x} \\ \frac{\partial}{\partial x}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(2H\nu\left(\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}\right)\right)-\beta v=\rho_i g H\frac{\partial z_s}{\partial y} \end{cases}$$

SIF - Material Section:

```
Material 1
! Flow Law
  Viscosity Exponent = Real $1.0/n
  Critical Shear Rate = Real 1.0e-10
  SSA Mean Viscosity = Real $eta
  SSA Mean Density = Real $rhoi

! Friction Law
  ! Which law are we using
  SSA Friction Law = String («linear», «weertman» or «coulomb»)

  ! friction parameter
  SSA Friction Parameter = Real 0.1

! Needed for Weertman and Coulomb
  ! Exponent m
  SSA Friction Exponent = Real $1.0/n

  ! Min velocity for linearisation where ub=0
  SSA Friction Linear Velocity = Real 0.0001

! Needed for Coulomb only
  ! post peak exponent in the Coulomb law (q, in Gagliardini et al., 2007)
  SSA Friction Post-Peak = Real ...
  ! Iken's bound  tau_b/N < C (see Gagliardini et al., 2007)
  SSA Friction Maximum Value = Real ....
  SSA Min Effective Pressure = Real ...
End
```

Friction laws:

- Linear:

$$\tau_b = \beta u$$

- Weertman:

$$\tau_b = \beta |u|^{(m-1)} u$$

- Coulomb:

$$\tau_b = \frac{1}{A_s^{\frac{1}{n}}}\left[\frac{1}{\left(1+\alpha.\chi^q\right)}\right]^{\frac{1}{n}}.u_b^{\frac{1}{n}-1}.u$$

$$\alpha = \frac{\left(q-1\right)^{q-1}}{q^q} \qquad \chi = \frac{u_b}{C^n N^n A_s}$$

Shallow Shelf Approximation/Shallow Stream Approximation

Field equations:

$$\begin{cases} \frac{\partial}{\partial x}\left(2H\nu\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)-\beta u=\rho g H\frac{\partial z_s}{\partial x} \\ \frac{\partial}{\partial x}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(2H\nu\left(\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}\right)\right)-\beta v=\rho_i g H\frac{\partial z_s}{\partial y} \end{cases}$$

Boundary Conditions:

$$\begin{cases} 4H\nu\frac{\partial u}{\partial x}n_x+2H\nu\frac{\partial v}{\partial y}n_x+H\nu(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x})n_y=(\rho_i gH-\rho_w gH_0)n_x \\ 4H\nu\frac{\partial v}{\partial y}n_y+2H\nu\frac{\partial v}{\partial x}n_y+H\nu(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x})n_x=(\rho_i gH-\rho_w gH_0)n_y \end{cases}$$

SIF - Boundary Conditions / Constants / Body Forces:

```
Boundary Condition 1
! Dirichlet condition
  SSAVelocity 1 = Real ...
  SSAVelocity 2 = Real ...
End
Boundary Condition 1
! Neumann Condition
  Calving Front = Logical True
End
```

```
Constants
! Used for Neumann condition
  Water Density = Real ....
  Sea Level = Real ...
End
```

```
Body Force 1
! The gravity from Flow Body Force 2/3 (1D/2D)
    Flow BodyForce 3 = Real $gravity
End
```

Computing mean values

SSA uses mean viscosity and density:

$$\nu(x,y) = \frac{1}{H}\int_{z_b}^{z_s} \mu(x,y,z)dz \longrightarrow \text{coupling with : } \textbf{Temperature, Damage}$$

$$\bar{\rho}(x,y) = \frac{1}{H}\int_{z_b}^{z_s} \rho(x,y,z)dz \longrightarrow \text{coupling with : } \textbf{Porous solver}$$

You can use:

Elmer/Ice solver : *GetMeanValueSolver*

- **unstructured** meshes in the vertical direction

```
Solver 1
  Equation = "SSA-IntValue"
  Procedure = File "ElmerIceSolvers" "GetMeanValueSolver"
  Variable = -nooutput String "Integrated variable"
  Variable DOFs = 1

  Exported Variable 1 = String "Mean Viscosity"
  Exported Variable 1 DOFs = 1
  Exported Variable 2 = String "Mean Density"
  Exported Variable 2 DOFs = 1

  Linear System Solver = Direct
  Linear System Direct Method = umfpack

  Steady State Convergence Tolerance = Real 1.0e-3
End


!!! Upper free surface
Boundary Condition 1
  Depth = Real 0.0
  Mean Viscosity = Real 0.0
  Mean Density = real 0.0
End
```

Elmer solver : *StructuredProjectToPlane*

- **structured** meshes in the vertical direction

```
Solver 1
  Equation = "HeightDepth"
  Procedure = "StructuredProjectToPlane" "StructuredProjectToPlane"
  Active Coordinate = Integer 3

  Operator 1 = depth
  Operator 2 = height
  Operator 3 = thickness

  !! compute the integrated horizontal Viscosity and Density
  Variable 4 = Viscosity
  Operator 4 = int

  Variable 5 = Density
  Operator 5 = int
End

Material 1
  SSA Mean Viscosity = Variable "int Viscosity", thickness
       REAL MATC "tx(0)/tx(1)"
  SSA Mean Density = Variable "int Density", thickness
       REAL MATC "tx(0)/tx(1)"
End
```

=> J. Brondex (LGGE) is working on new solutions for this step and to compute the 3D velocity field (=> coupling with damage and temperature)

Outline

- Shallow Shelf / Shallow stream Solver
- **Thickness Solver**
- Current / planned development

Thickness Solver

Field equations:

$$\frac{\partial H}{\partial v} + \nabla(\bar{u}H) = a_s + a_b$$

Elmer/Ice Solvers:

- **Solver Fortran File:** `ThicknessSolver.f90`
- **Solver Name:** `ThicknessSolver`
- **Required Output Variable(s):** `H`
- **Required Input Variable(s):** `H residual`
- **Optional Output Variable(s):** `dhdt`
- **Optional Input Variable(s):** `FlowSolution`

• This solver is based on the FreeSurfaceSolver and use a **SUPG stabilsation** scheme by **default** (***residual free bubble stabilization*** can be use instead).

• As for the FreeSurfaceSolver ***Min* and *Max* limiters** can be used.

• As for the Free surface solver **only a Dirichlet boundary condition** can be imposed.

• This solver can be used on a mesh of the same dimension as the problem (e.g. solve on the bottom or top boundary of a 3D mesh to solve the 2D thickness field) or on a mesh of lower dimension (e.g. can be use in a 2D plane view mesh with the SSA Solver solver for example)

Thickness Solver

Field equations:

$$\frac{\partial H}{\partial v} + \nabla(\bar{u}H) = a_s + a_b$$

SIF:

```
Solver 1
   Equation = "Thickness"
   Variable = -dofs 1 "H"

   Exported Variable 1 = -dofs 1 "H Residual"

!! To compute dh/dt
   Exported Variable 2 = -dofs 1 "dHdt"
   Compute dHdT = Logical True

  Procedure = "ElmerIceSolvers" "ThicknessSolver"
!   Before Linsolve = "EliminateDirichlet" "EliminateDirichlet"

   Linear System Solver = Direct
   Linear System Direct Method = umfpack
   Linear System Convergence Tolerance = Real 1.0e-12

! equation is linear if no min/max
   Nonlinear System Max Iterations = 50
   Nonlinear System Convergence Tolerance  = 1.0e-6
   Nonlinear System Relaxation Factor = 1.00

! stabilisation method: [stabilized\bubbles]
  Stabilization Method = stabilized

!! to apply Min/Max limiters
  Apply Dirichlet = Logical True

!! to use horizontal ALE formulation
   ALE Formulation = Logical True

!! To get the mean horizontal velocity
!!  either give the name of the variable
     Flow Solution Name = String "SSAVelocity"
!!!!! or give the dimension of the problem using:
!    Convection Dimension = Integer
End
```

```
Body Force 1
!! Mass balance
  Top Surface Accumulation = Real ....
  Bottom Surface Accumulation = Real ....


!! if the convection velocity is not directly given by a variable
!! Then give //Convection Dimension = Integer// in the solver section
!! and the Mean velocity here:
  Convection Velocity 1 = Variable int Velocity 1, thickness
     REAL MATC "tx(0)/tx(1)"
  Convection Velocity 2 = Variable int Velocity 2, thickness
     REAL MATC "tx(0)/tx(1)"

End
```

```
Boundary Condition 1
! Dirichlet condition only
  H = Real ...
End
```

```
Material 1
!! Limiters
  Min H = Real ....
  Max H = Real ....

End
```

Coupling SSA solver / Thickness solver

SSASolver uses Zs and Zb (H=Zs-Zb)
=> requires an intermediate step between *ThicknessSolver* and *SSASolver*

Do it yourself:

```
Initial Condition 1
  H = Real ....
End

Body Force 1
! to update Zb and Zs according to H evolution
  Zb = Real ...
  Zs = Variable Zb , H
     REAL MATC "tx(0)+tx(1)"
End

Solver 1
   Equation = "UpdateExport"
   Procedure = "ElmerIceSolvers" "UpdateExport"
   Variable = -nooutput "dummy"

    Exported Variable 1 = -dofs 1 "Zb"
    Exported Variable 2 = -dofs 1 "Zs"
End

Solver 2
  Equation = "SSA"
  Procedure = File "ElmerIceSolvers" "SSABasalSolver"
  Variable = String "SSAVelocity"
  Variable DOFs = 2   ! 1 in SSA 1-D
End

Solver 3
   Equation = "Thickness"
   Variable = -dofs 1 "H"
End
```

you can write a User Function to apply flotation to Zb and Zs=Zb+H

1. From H compute Zb and Zs
look for definition of Exported variables in «Body Force»

2. From Zb and Zs compute u

3. From u compute H

Coupling SSA solver / Thickness solver

SSASolver uses Zs and Zb (H=Zs-Zb)
=> requires an intermediate step between *ThicknessSolver* and *SSASolver*

Do it yourself:

```
Initial Condition 1
  H = Real ....
End

Body Force 1
! to update Zb and Zs according to H evolution
  Zb = Real ...
  Zs = Variable Zb , H
     REAL MATC "tx(0)+tx(1)"
End

Solver 1
   Equation = "UpdateExport"
   Procedure = "ElmerIceSolvers" "UpdateExport"
   Variable = -nooutput "dumy"

    Exported Variable 1 = -dofs 1 "Zb"
    Exported Variable 2 = -dofs 1 "Zs"
End

Solver 2
  Equation = "SSA"
  Procedure = File "ElmerIceSolvers" "SSABasalSolver"
  Variable = String "SSAVelocity"
  Variable DOFs = 2   ! 1 in SSA 1-D
End

Solver 3
   Equation = "Thickness"
   Variable = -dofs 1 "H"
End
```

I will put a *FlotationSolver* in the distrib soon:

- From H apply Flotation to compute Zb
- If bedrock is given check if floating or grounded
- compute grounded mask (-1: floating, +1: grounded, 0: groundig line)
- Zs = H + Zb
- *optionally:* compute dZs/dt and dZb/dt

Check volume and fluxes using SaveScalars

```
Solver X
  Exec Solver = After Timestep

  Equation = "Save Scalars"
  Procedure = File "SaveData" "SaveScalars"

  Filename = File "Scalars_"$name$".dat"

  Variable 1 = "Time"

! int H = Volume
  Variable 2 = "H"
  Operator 2 = "int"

! int dh/dt = dVolume/dt
 Variable 3 = "dhdt"
 Operator 3 = "int"

! int SMB
  Variable 4 = "smb"
  Operator 4 = "int"

! SMB_H=Artificial additionnal Mass flux due to limits on H
  Variable 5 = "h residual"
  Operator 5 = "sum"

! OUT Flow
  Variable 6 = "SSAVelocity"
  Operator 6 = "convective flux"
  Coefficient 6 = "Flux"

!=> Dvolume/dt ~ SMB + SMB_H - OUT
End
```

Not with bubbles stabilisation

```
Material 1
!! For Save scalar to compute mass flux (=H*SSA_UV)
   Flux = Equals H
End

Boundary Condition 1
  Target Boundaries = 1

  Save Scalars = Logical True

  Calving Front = Logical True

End
```

Examples

Friction Laws:

ismip diagnostic test cases

[ELMER_TRUNK]/elmerice/Tests/SSA_Coulomb
[ELMER_TRUNK]/elmerice/Tests/SSA_Weertman

Coupling SSA/Thickness:

[ELMER_TRUNK]/elmerice/Tests/SSA_IceSheet
[ELMER_TRUNK]/elmerice/examples/Test_SSA → ismip prognostic test:

- 1D (2D mesh)
- 2D (2D mesh)
- 2D (3D mesh; use *StructuredProjectToPlane* to compute mean values))

Coupling Stokes/Thickness:

ismip prognostic test:

*[ELMER_TRUNK]/elmerice/*Tests/ThicknessSolver

Current/planned developments

Inverse methods:

- AdjointSolver for SSA => constrain friction, mean viscosity, Zb, Zs from observation
  - ***Fürst et al.*, Assimilation of Antarctic velocity observations provides evidence for uncharted pinning points, *The Cryosphere*, 2015**
  - ***Fürst et al.,* Passive shelf ice: the safety band of Antarctic ice 1shelves, *Nature Climate Change*, accepted**
- AdjointSolver for Thickness => constrain u,smb from observations of H

(see Morlighem *et al.*, 2011, a mass consservation approach for mapping glacier ice thickness)

SSA*:

- modify viscosity to take into account vertical shearing

(see Cornford *et al.*, 2013, adaptative mesh, finite volume modeling of marine ice sheets)

Sub-Element parameterisation at GL:

- sub-element parameterisation of friction in the GL vicinity (test flotation at IPs; increased number of IPS in firts floating elements; see Seroussi *et al.* (2014))

Efficient hybrid model SSA+SIA

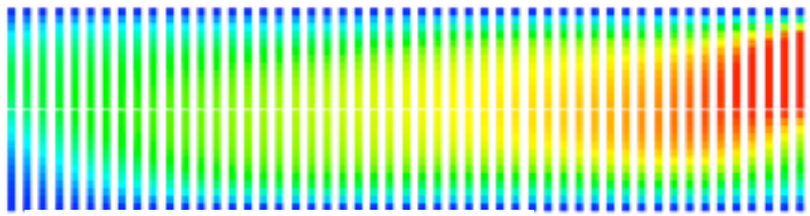
Efficient coupling with Temperature and Damage

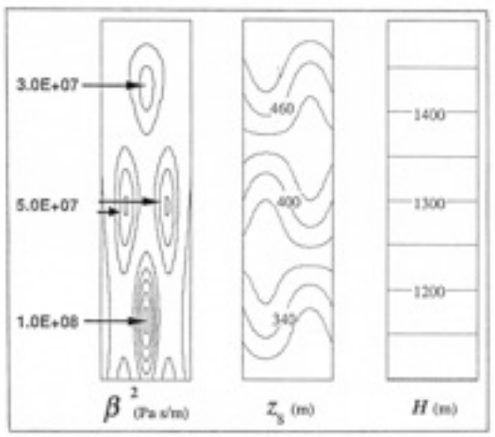
Anisotropic mesh adaptation

Laboratoire de Glaciologie et Géophysique de l'Environnement

Elmer/Ice Oslo 2016

Inverse methods in Elmer/Ice Applications with the SSA





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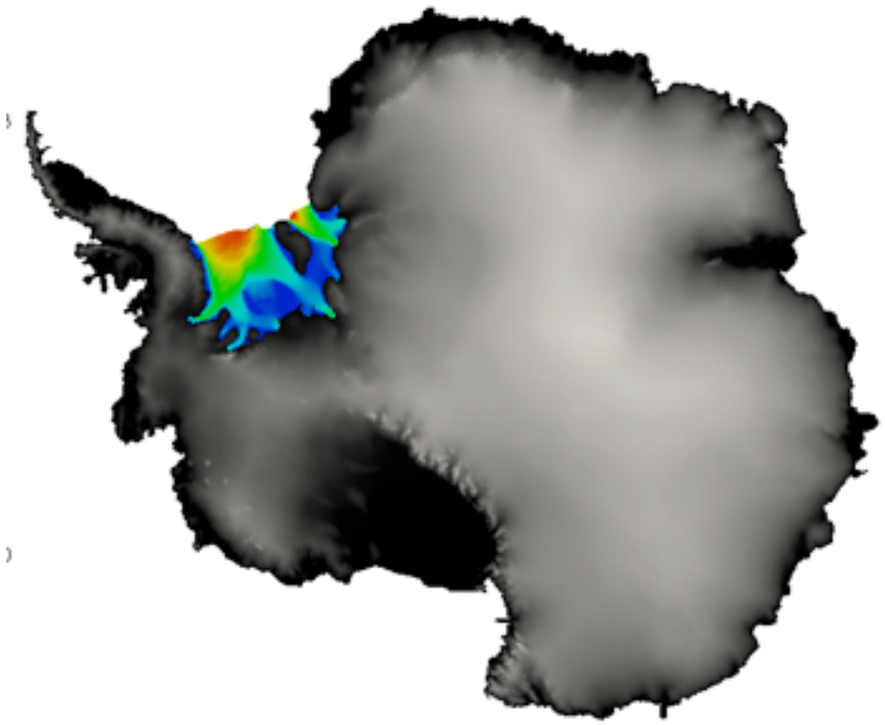
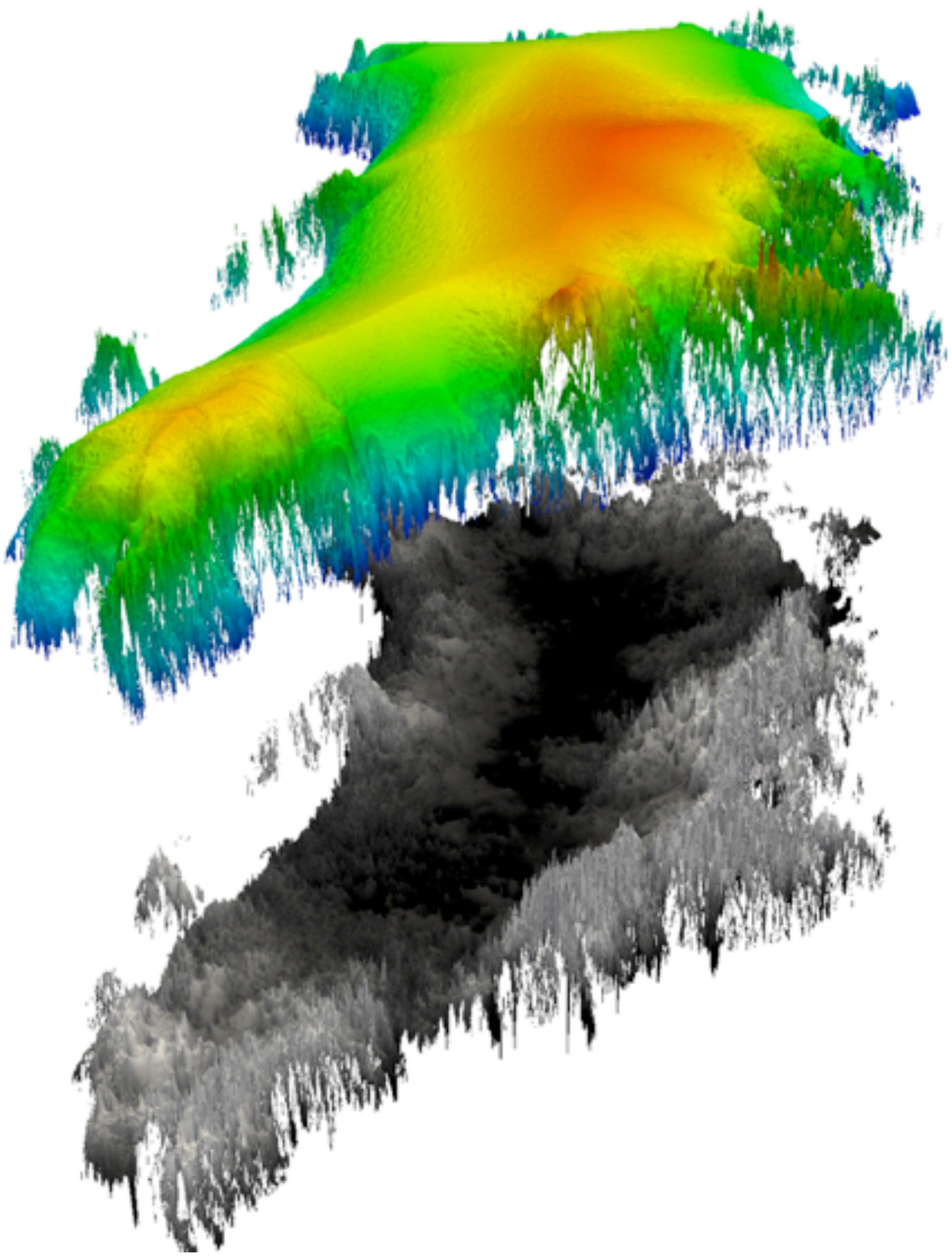
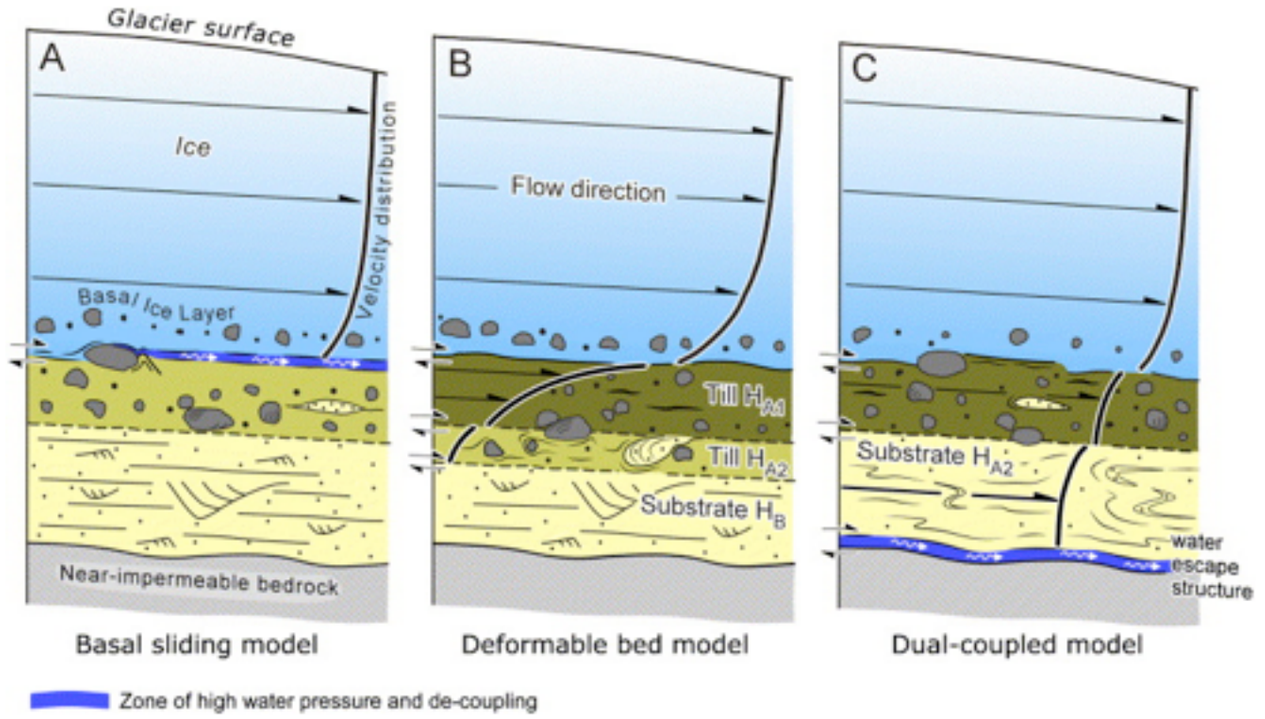
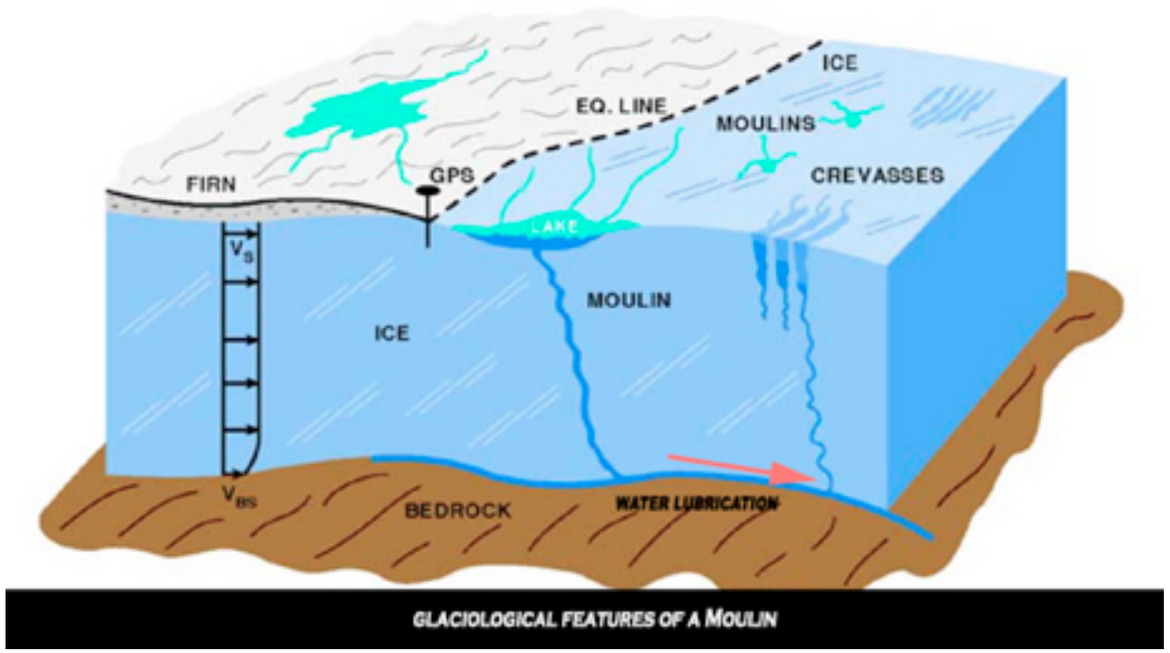


Uncertain parameterisations

e.g. friction of the ice on the bedrock highly variable in space and time
Usually prescribed as a friction law Tau=f(u)

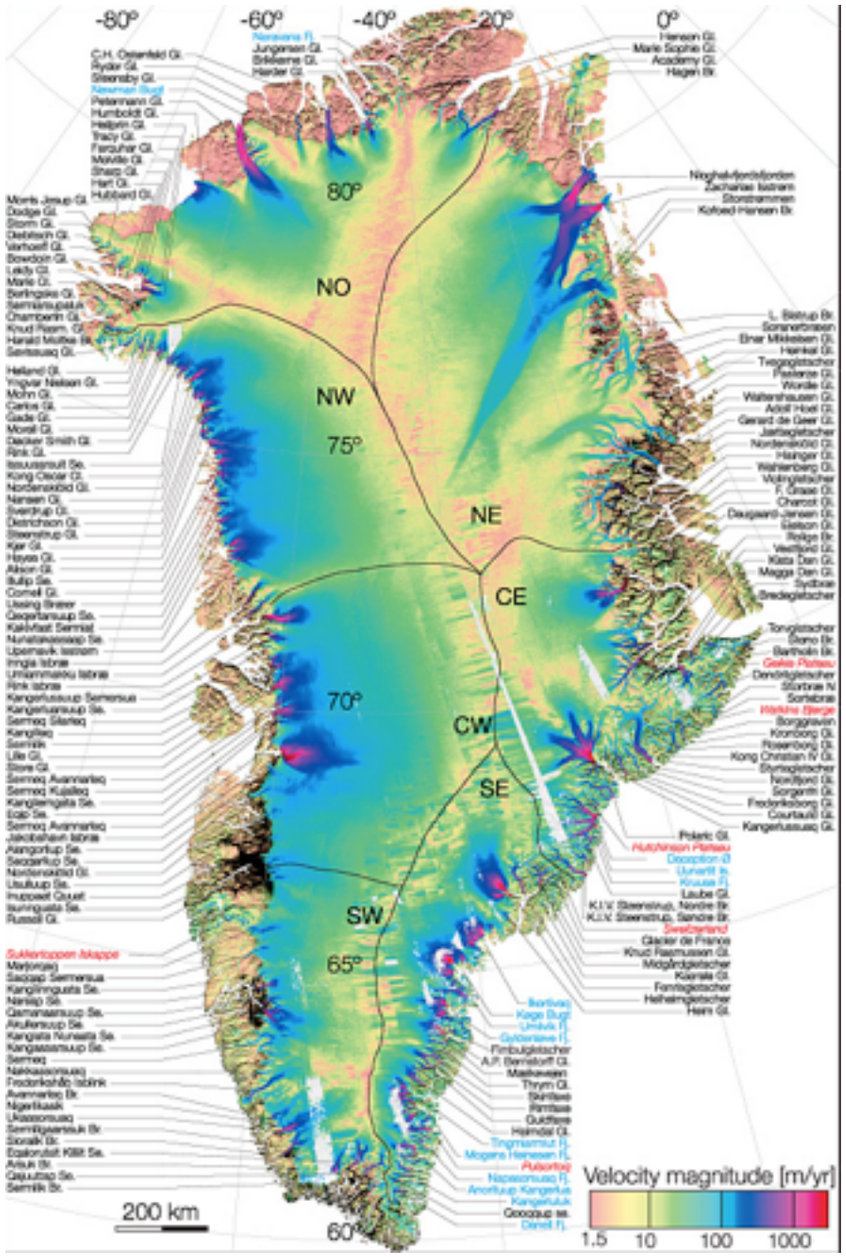




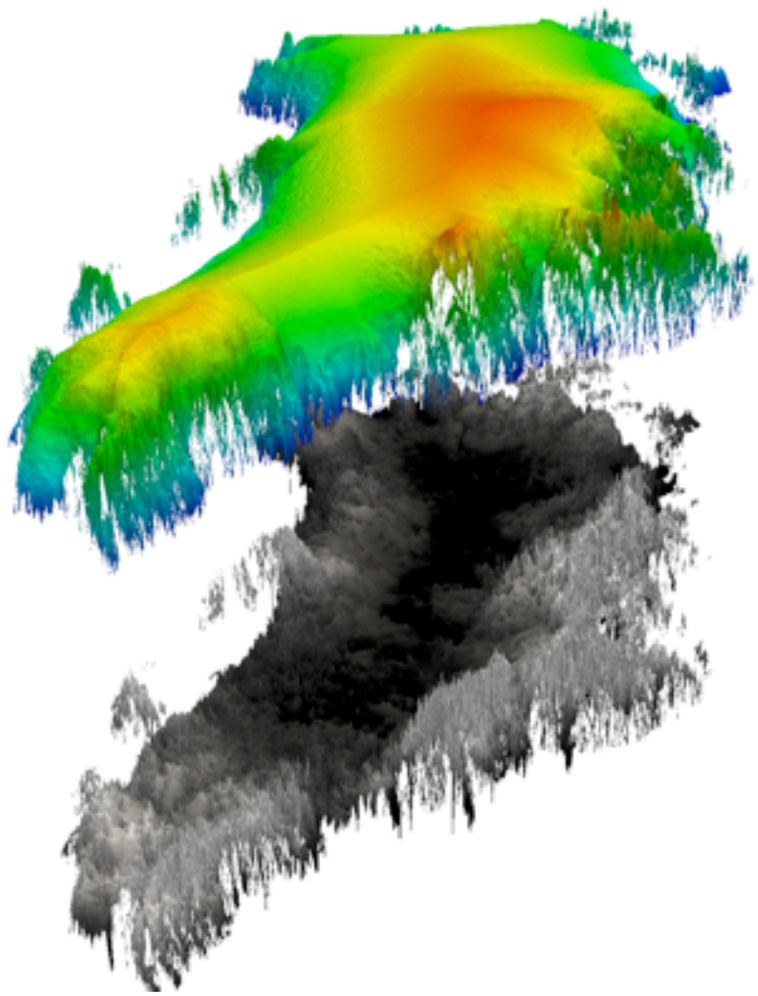


More and more available observations

Surface velocities



Topography



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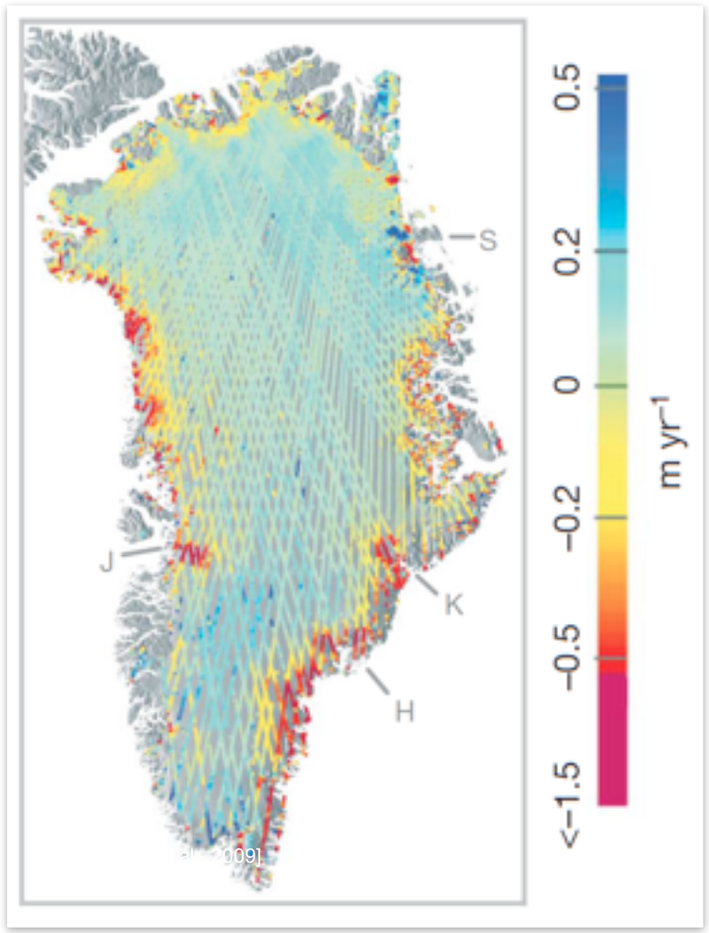
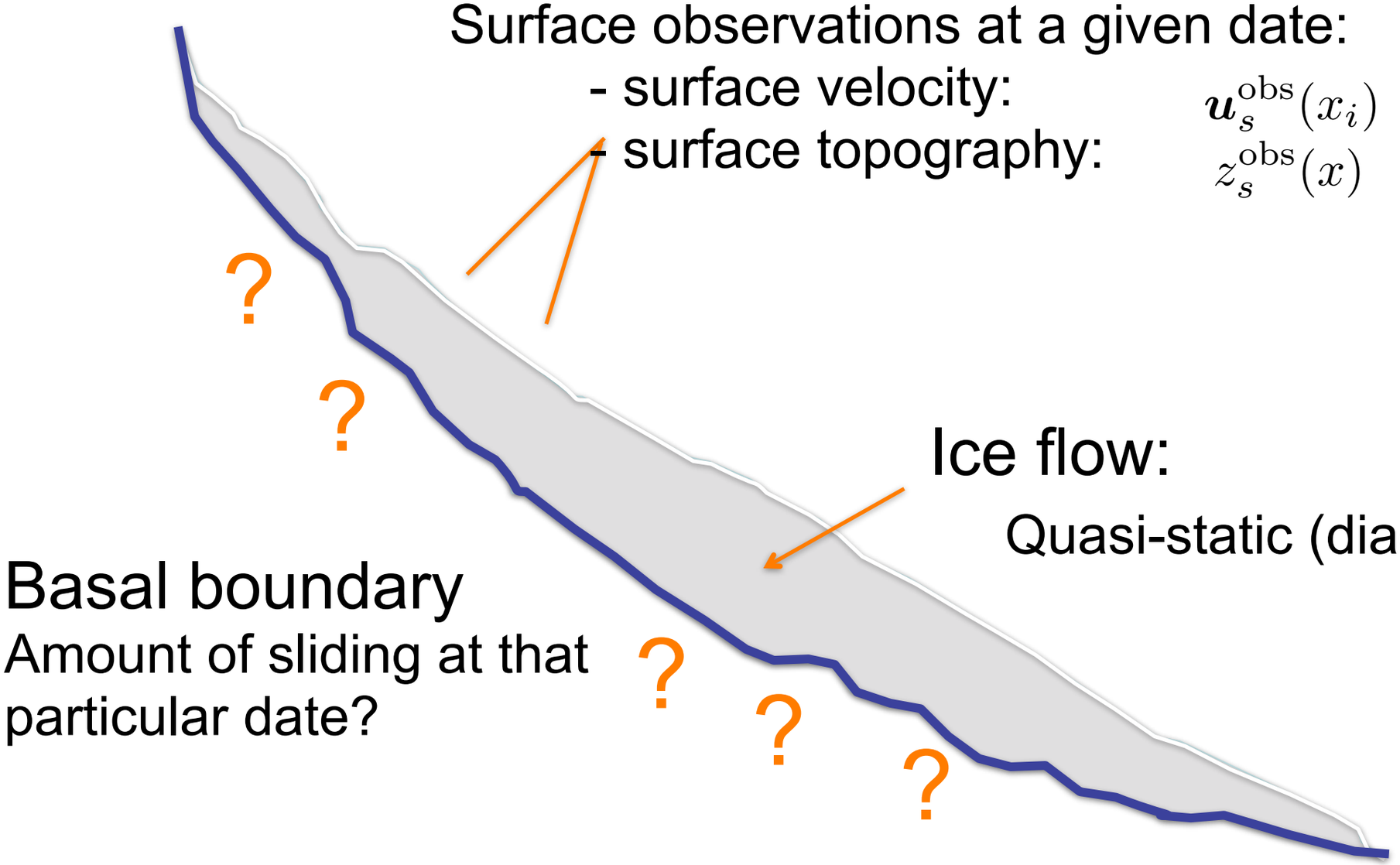


Specificity of ice flow

Very low Reynolds -> no history in the velocity

Surface observations at a given date:

- surface velocity: $\boldsymbol{u}_s^{\mathrm{obs}}(x_i)$
- surface topography: $z_s^{\mathrm{obs}}(x)$



Ice flow:

Quasi-static (diagnostic)

Basal boundary

Amount of sliding at that particular date?

Reconstruction of the basal conditions from surface measurements

Inverse methods in Elmer

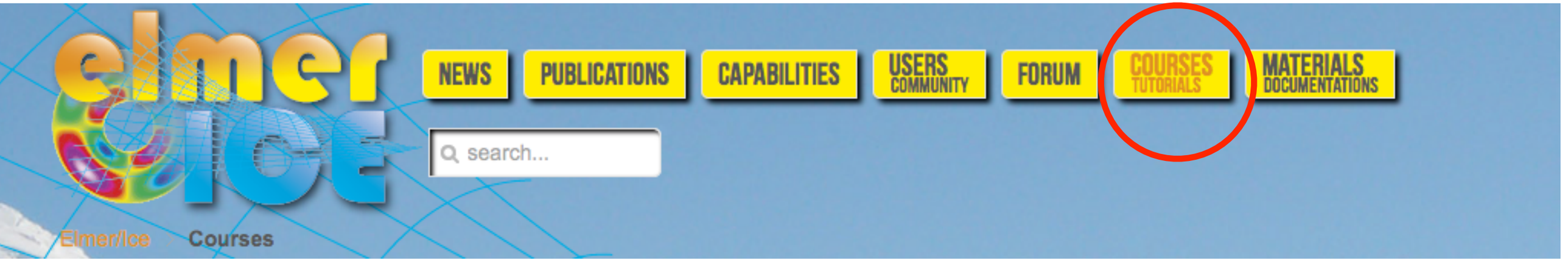
- **STOKES: 2 inverse methods** implemented in Elmer/Ice:
  - **Robin inverse method** (arthern and Gudmundsson, 2010)
  - **Adjoint method** (Mac Ayeal, 1993; Morlighem et al., 2010; Petra et al., 2012)

  Characteristics:
  => restricted to **diagnostic** (no time evolution)
  => **slip coefficient** (Linear sliding law)
  => **ice viscosity**
  => could also do Neumann and Dirichlet BC (Adjoint method)

- **SSA:**
  - **Adjoint Method: in elmerice branch since last week**
  - **will be documented on the wiki soon**
  - **sources are included in the material**

- **Efficient minimisation library** (quasi-Newton algorithm)

Stokes: Nothing new since the CSC - 2013 Advanced Course



CSC - Espoo - 4-6 November 2013

A 3-day Elmer/Ice advanced workshop was organised at CSC (Espoo, Finland) from the 4th to the 6th of November 2013. The course was held by Fabien Gillet-Chaulet (LGGE), Mika Malinen (CSC), Peter Råback (CSC) and Thomas Zwinger (CSC).

Title	Presentation	Material
Introduction to Elmer	pdf	-
Elmer Glaciological Modelling	pdf	-
Simple Hydro Toymodel	pdf	tar file
Structured Meshes	pdf	tar file
Enhanced pre-processing	pdf	USB stick
Block pre-conditioner	pdf	USB stick
Enhanced post-processing	pdf	ZIP archive
Make-file for YAMS on Ubuntu 64bit	-	tar archive
Mesh Adaptation using YAMS (see also these notes)	pdf	tar archive
Inverse methods	pdf	tar archive

See also the Elmer/Ice reference paper

Geosci. Model Dev., 6, 1299–1318, 2013
www.geosci-model-dev.net/6/1299/2013/
doi:10.5194/gmd-6-1299-2013

Geoscientific Model Development
Open Access

Capabilities and performance of Elmer/Ice, a new-generation ice sheet model

O. Gagliardini[1,2], T. Zwinger[3], F. Gillet-Chaulet[1], G. Durand[1], L. Favier[1], B. de Fleurian[1], R. Greve[4], M. Malinen[3], C. Martín[5], P. Råback[3], J. Ruokolainen[3], M. Sacchettini[1], M. Schäfer[6], H. Seddik[4], and J. Thies[7]

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See pioneer paper from Mac Ayeal!!

Journal of Glaciology, Vol. 39, No. 131, 1993

A tutorial on the use of control methods in ice-sheet modeling

DOUGLAS R. MACAYEAL
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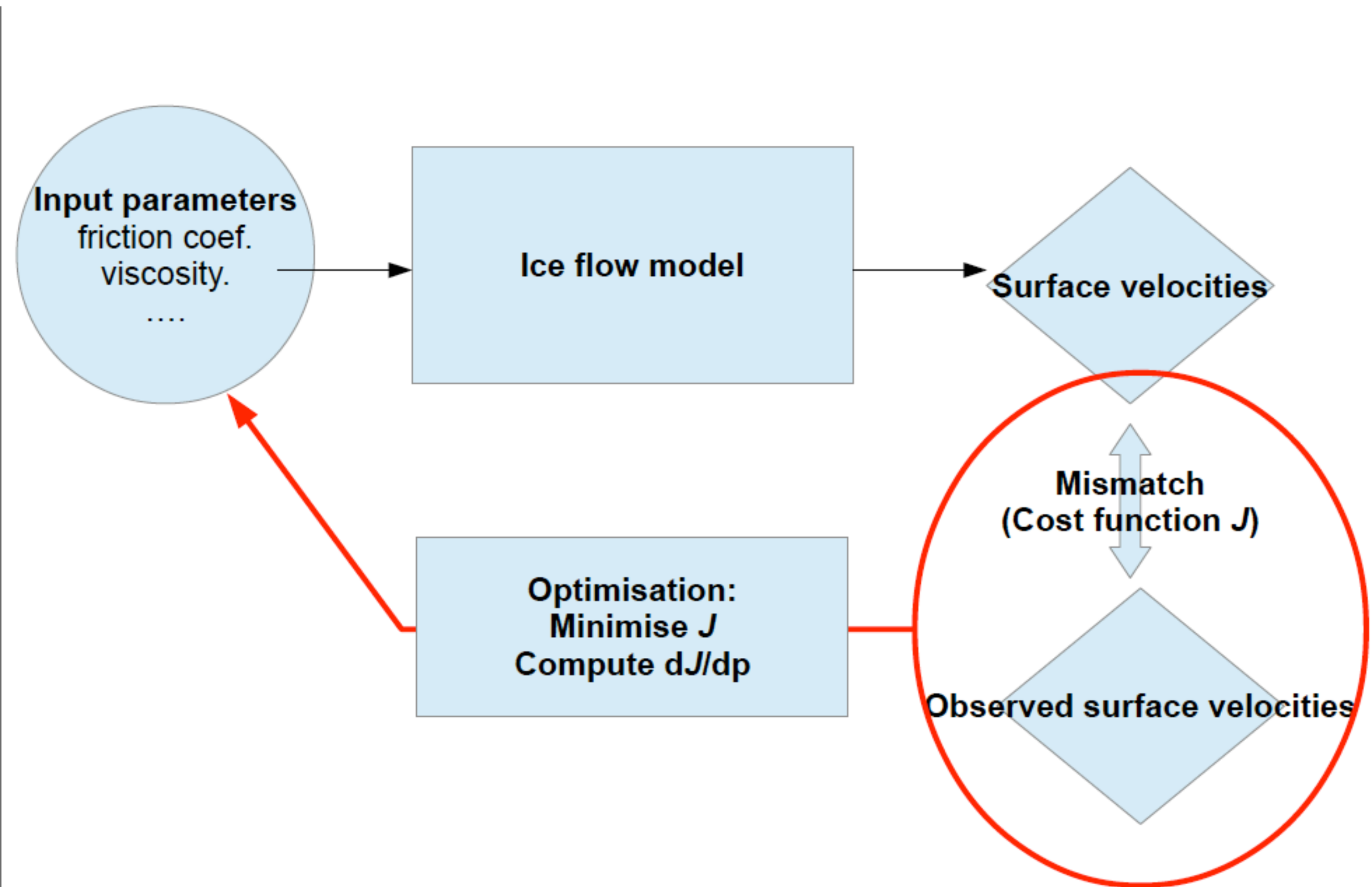
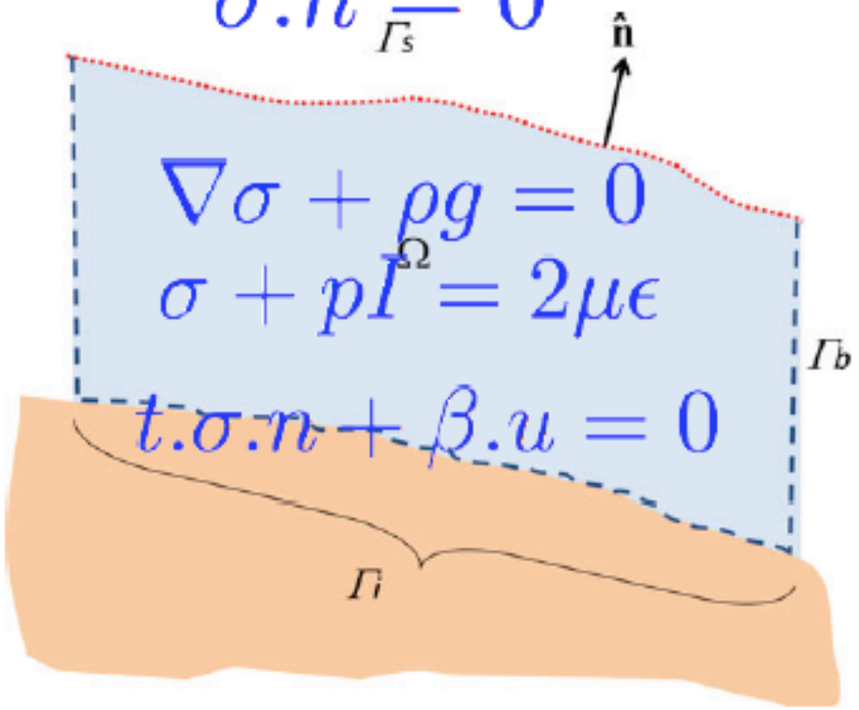
Variational data assimilation



Adjoint method *(Mac Ayeal, 1993)*

Direct problem

$$\sigma.n = 0 \quad \text{on } \Gamma_S$$



1. Define a cost function

$$J = f(u)$$

$$e.g. \quad J = \int_{\Gamma_S} \frac{1}{2}(u - u^{obs})^2 d\Gamma$$

2. Insure that *u* is solution of your problem

$$J' = J(u) + \Lambda(\nabla\sigma + \rho g)$$

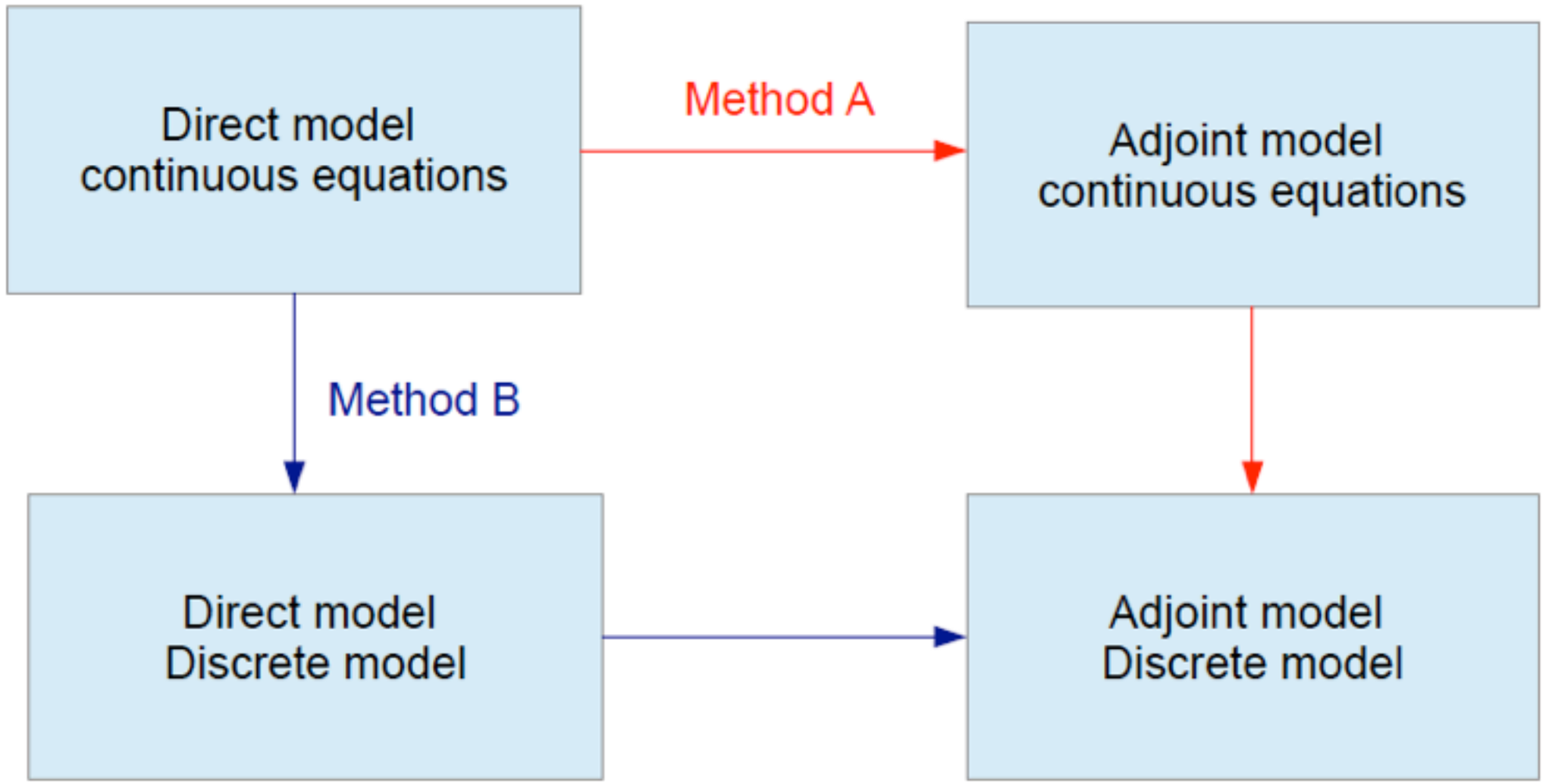
3. Minimisation of *J'* requires that all variations are 0

$$d_\Lambda J' = 0 \quad \text{=> direct problem equation is satisfied}$$

$$d_u J' = 0 \quad \text{=> adjoint equations}$$

=> gradient of *J* w.r. To input parameters *p* $\quad d_p J = f(\Lambda, u)$

Getting the adjoint



Usually Method A ≠ Method B

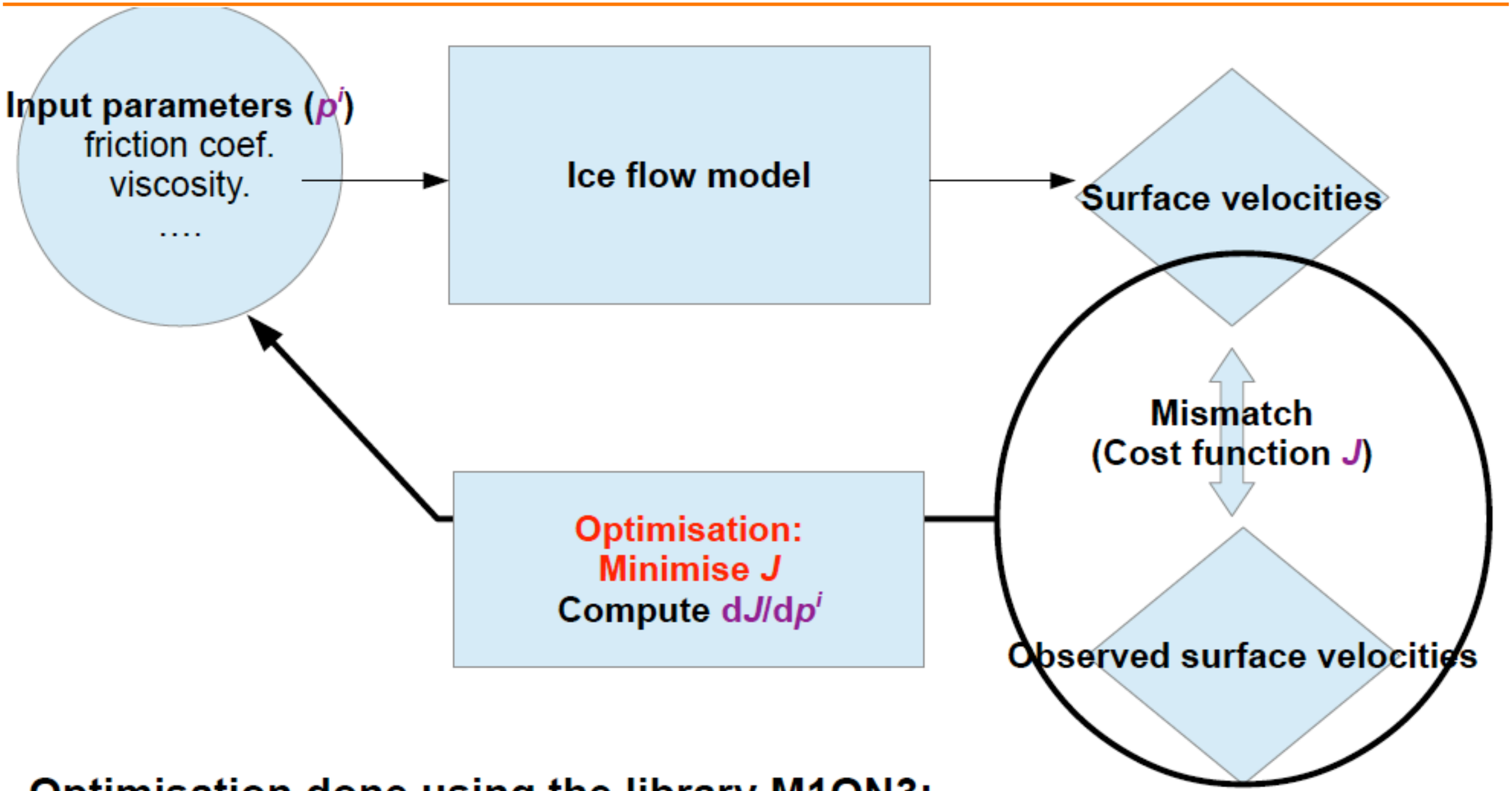
Method B should be preferred

Can be done using automatic differentiation

Pointer arrays
not yet supported

=> crucial parts have been derived by hand

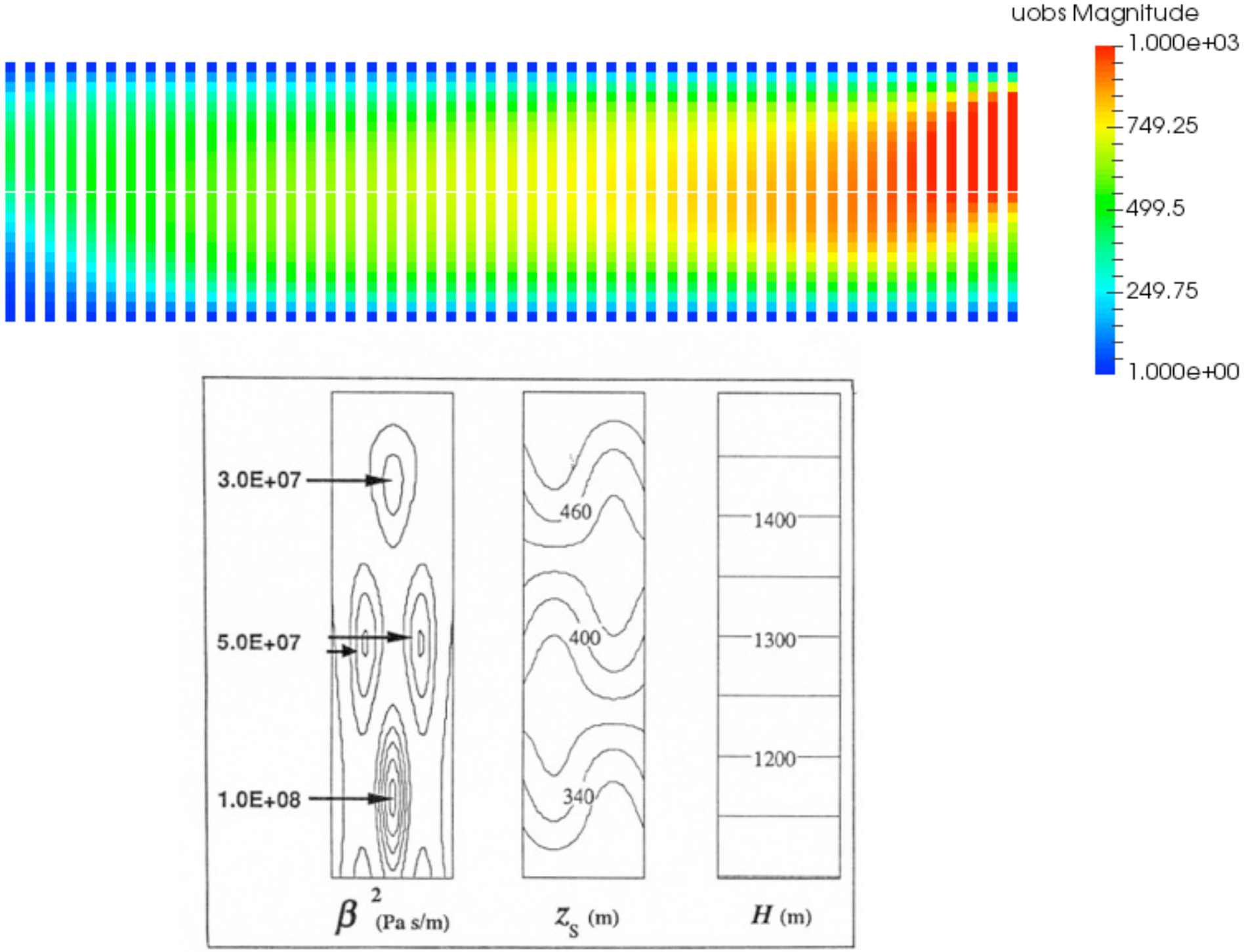
Optimisation algorithm: M1QN3



Optimisation done using the library M1QN3:

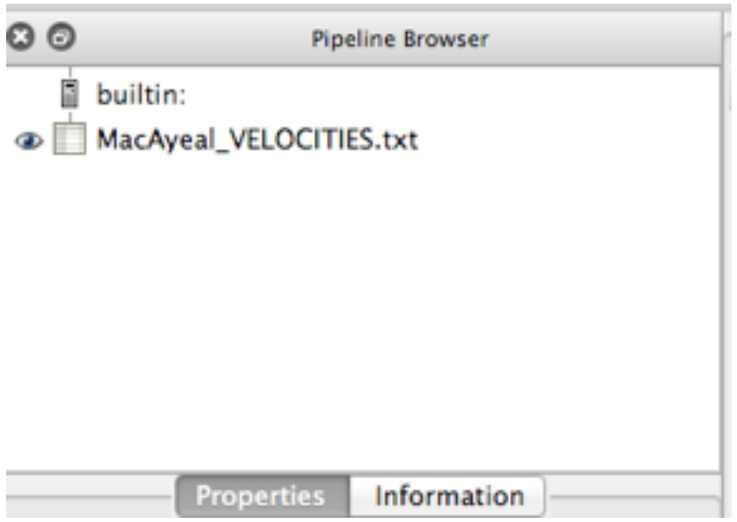
- Limited memory quasi-newton algorithm
- Implemented in reverse communication (i.e. called by Elmer within a solver)
- Iterative procedure: Input: p^i, J^i, dJ/dp^i – Output p^{i+1}
- https://who.rocq.inria.fr/Jean-Charles.Gilbert/modulopt/optimization-routines/m1qn3/m1qn3.html

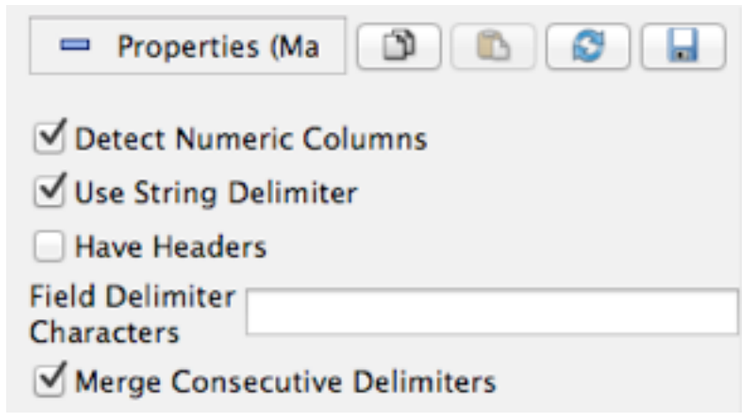
Repeat Mac Ayeal Twin Experiment



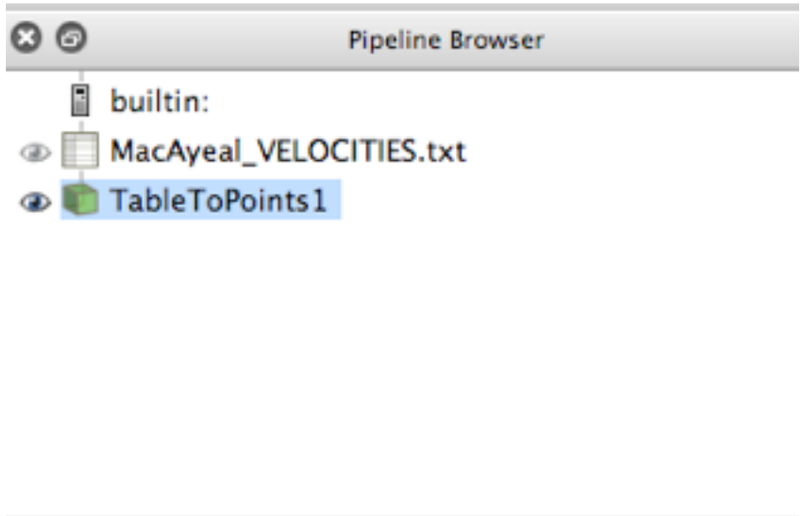
View discret data with parabiew

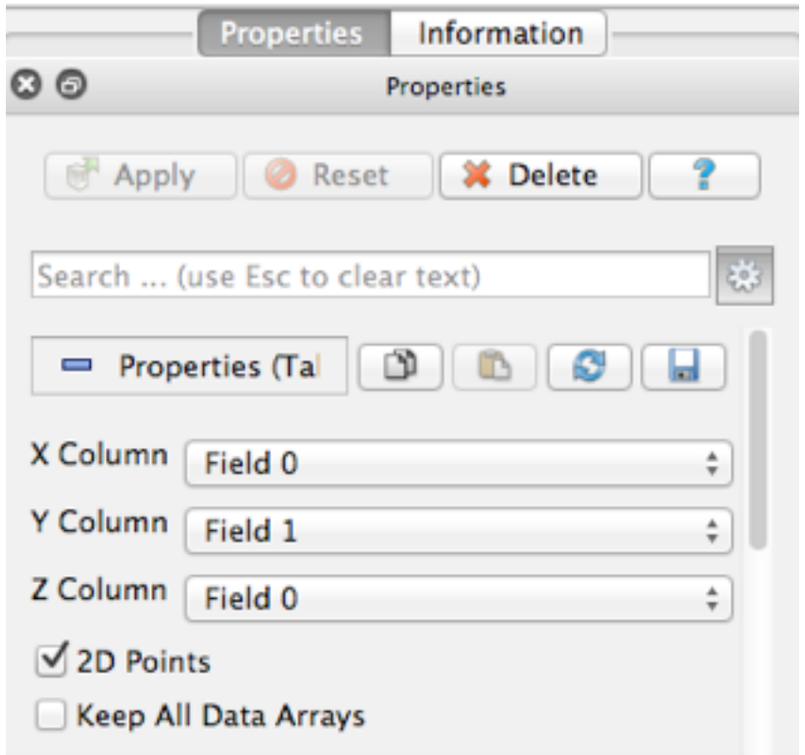
Open File



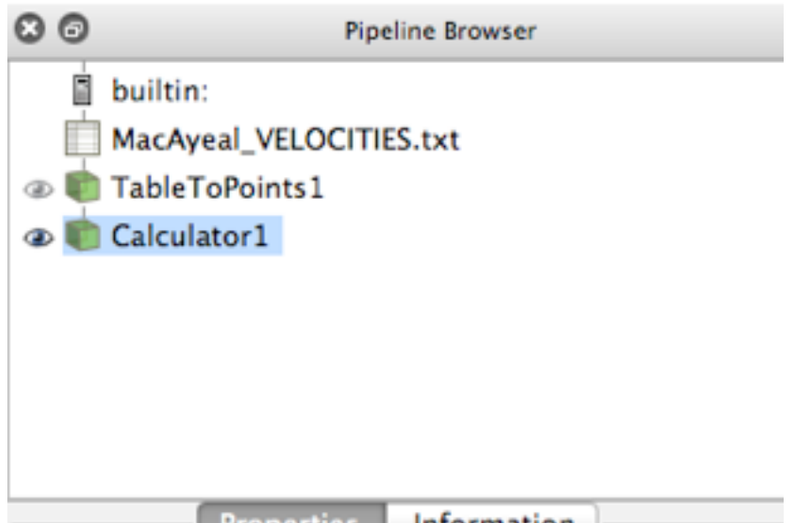


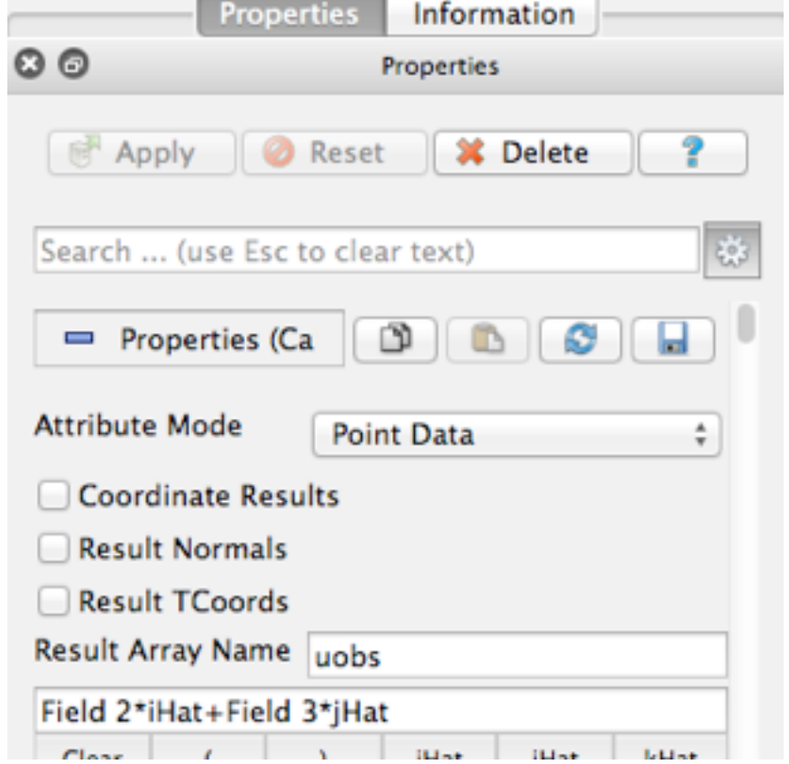
Convert to points



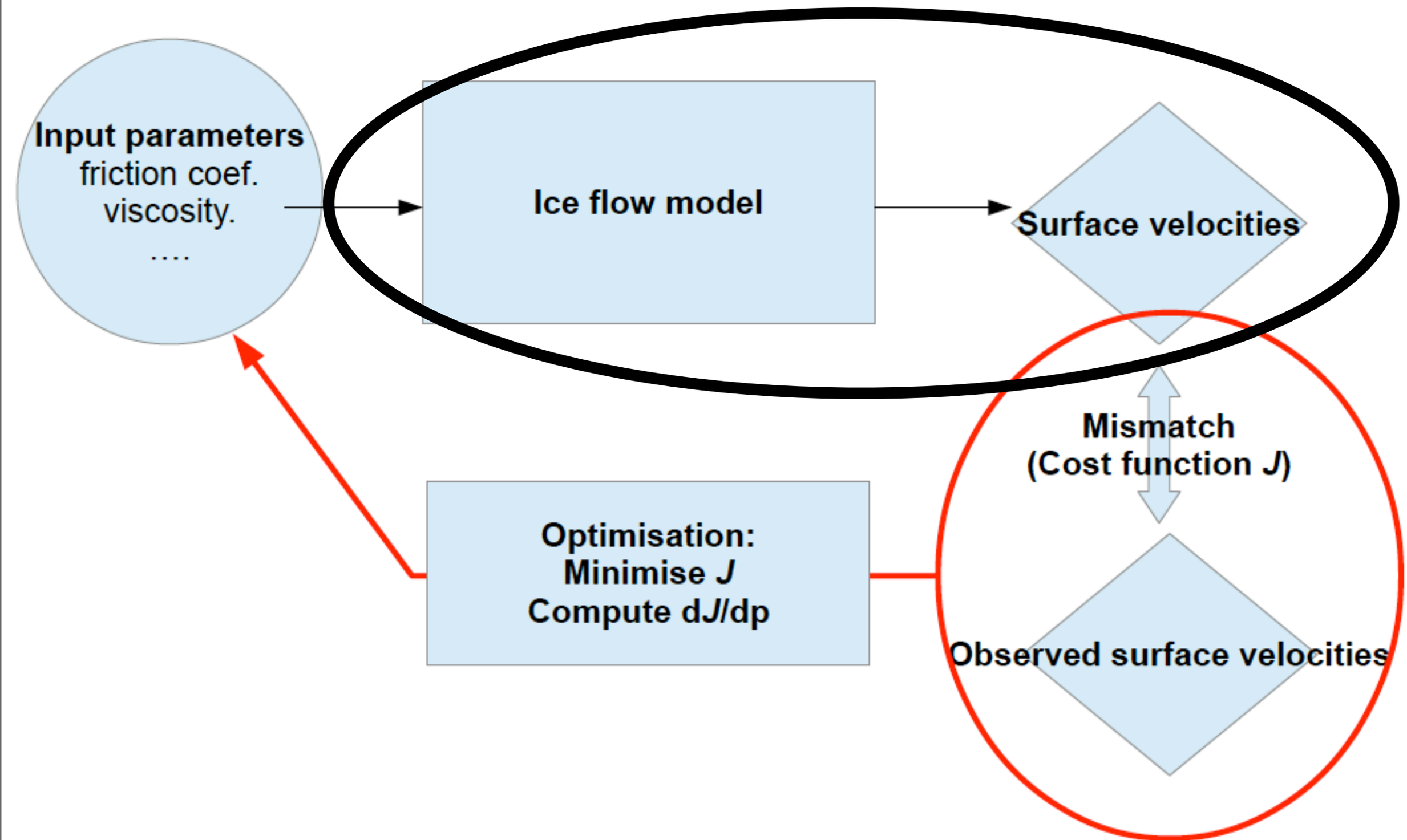


Make velocity vector





Variational data assimilation



Initial guess

the guess (square root of)

```
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
Initial Condition 1
 BetaS = Variable coordinate 1, Coordinate 2
      REAL MATC "betaSquare(tx)"
! alpha is the optimised variable
 alpha = Real 1.0e-3

 Zb = Variable coordinate 1, Coordinate 2
      REAL MATC "zb(tx)"

 Zs = Variable coordinate 1, Coordinate 2
      REAL MATC "zs(tx)"

 SSAVelocity 1 = Real 0.0
 SSAVelocity 2 = Real 0.0
End
```

the truth

```
$ function betaSquare(tx) {\
  Lx = 200.0e3;\
  Ly = 50.0e03;\
  yearinsec = 365.25*24*60*60;\
  F1=sin(3.0*pi*tx(0)/Lx)*sin(pi*tx(1)/Ly);\
  F2=sin(pi*tx(0)/(2.0*Lx))*cos(4.0*pi*tx(1)/Ly);\
  beta=5.0e3*F1+5.0e03*F2;\
  _betaSquare=beta*beta/(1.0e06*yearinsec);\
}
```

Compute model velocity

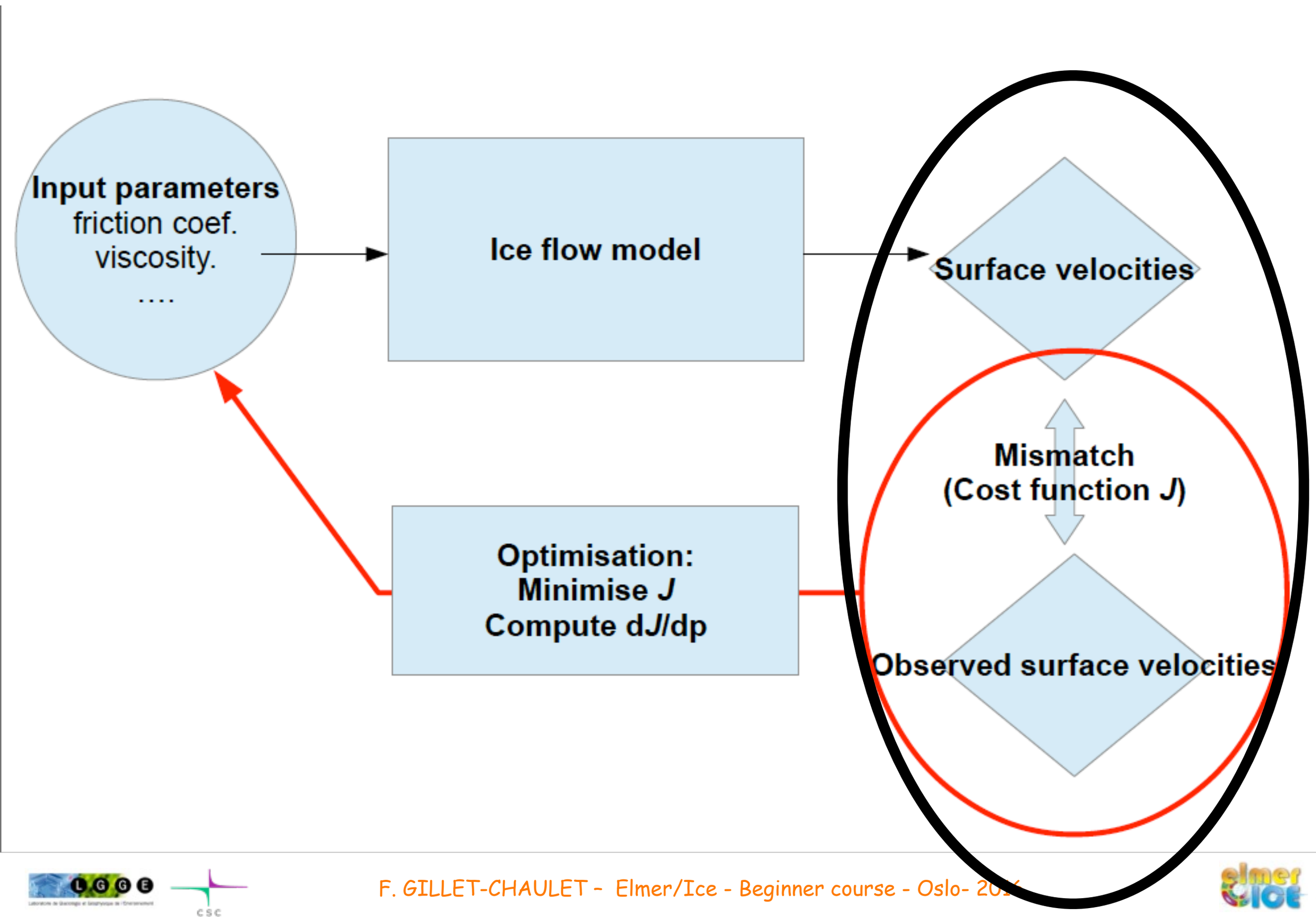
```
----------------------------------------
Material 1

  Viscosity Exponent = Real $1.0e00/3.0e00
  Critical Shear Rate = Real 1.0e-10

  SSA Mean Density = Real $rhoi
  SSA Mean Viscosity = Real $ 1.8e8*1.0e-6*(2.0*yearinsec)^(-1.0/3.0)

  SSA Friction Law = String "linear"
  ! The friction parameter is the square of the optimised variable to insure > 0
  SSA Friction Parameter = Variable alpha
      REAL MATC "tx*tx"
End
```

```
----------------------------------------
Solver 1
  Equation = "SSA"
  Variable = -dofs 2 "SSAVelocity"

  Procedure = "AdjointSSASolvers" "AdjointSSA_SSASolver"

!! Mandatory for the adjoint
  Calculate Loads = Logical True

  Linear System Solver = Direct
  Linear System Direct Method = mumps

  Nonlinear System Max Iterations = 50
  Nonlinear System Convergence Tolerance  = 1.0e-10
  Nonlinear System Newton After Iterations = 40
  Nonlinear System Newton After Tolerance = 1.0e-06
  Nonlinear System Relaxation Factor = 1.00

  Steady State Convergence Tolerance = Real 1.0e-12

  Exported Variable 1 = Zb
  Exported Variable 2 = Zs
  Exported Variable 3 = BetaS
  Exported Variable 4 = CostValue
  Exported Variable 5 = DJDBeta
  Exported Variable 6 = -dofs 2 "Velocityb"
End
```

The direct solver used to make the adjoint (may not be up-to-date wr the latest ssa solver)

For the accuracy of the adjoint use Newton method for non-linear iterations

Variational data assimilation

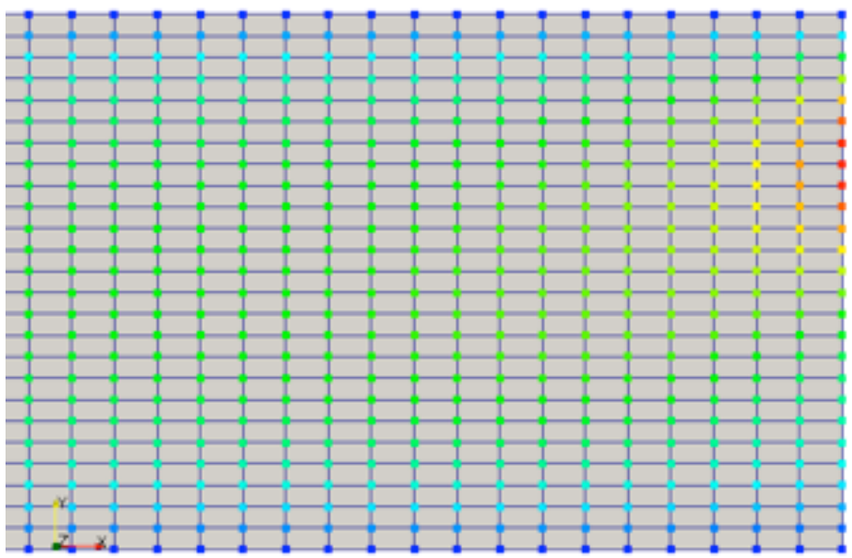


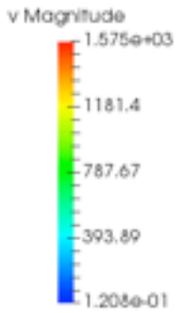
Compute the cost function

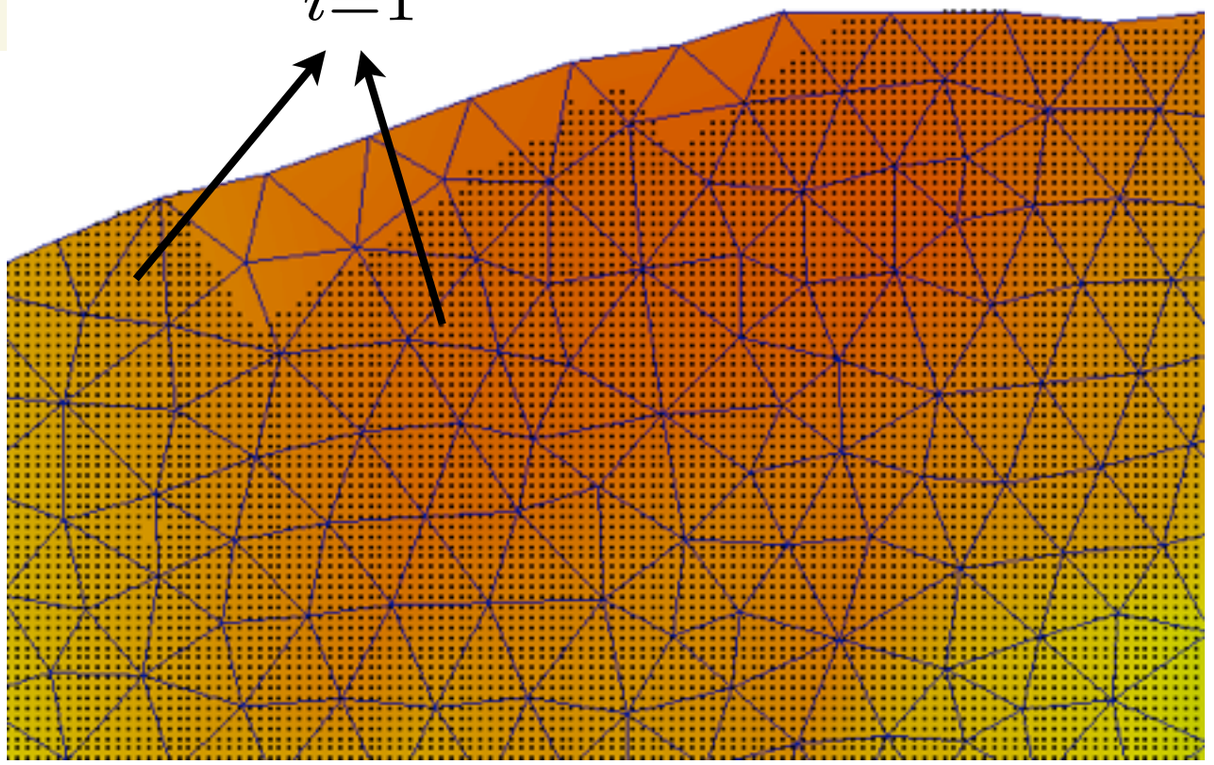
```
Solver 2
  Equation = "Cost"

  !!  Solver need to be associated => Define dumy variable
    Variable = -nooutput "CostV"
    Variable DOFs = 1

    procedure = "AdjointSSASolvers" "AdjointSSA_CostDiscSolver"

   Problem Dimension = Integer 2 !2D mesh and 2D SSA Solution
   Cost Variable Name = String "CostValue"  ! Name of Cost Variable
   Lambda = Real 1.0
 ! save the cost as a function of iterations (iterations,Cost,rms=sqrt(2*Cost/Ndata)
   Cost Filename = File "Cost_$name$.dat"

   Observed Variable Name = String "SSAVelocity"
 ! ASCII File with data: x,y,u,v
   Observation File Name = File "../DATA/MacAyeal_VELOCITIES.txt"

end
```

```
#date,time,1/3/2016,10:35:52
#lambda, 0.10000000E+01
# iter, Lambda*J0, rms
  0.10000E+01   0.31721506E+06   0.21590507E+02
  0.20000E+01   0.89979177E+05   0.11498917E+02
  0.30000E+01   0.15123345E+05   0.47142201E+01
  0.40000E+01   0.79399313E+04   0.34158147E+01
  0.50000E+01   0.59819521E+04   0.29648819E+01
  0.60000E+01   0.37836248E+04   0.23579792E+01
  0.70000E+01   0.26658066E+04   0.19792482E+01
```

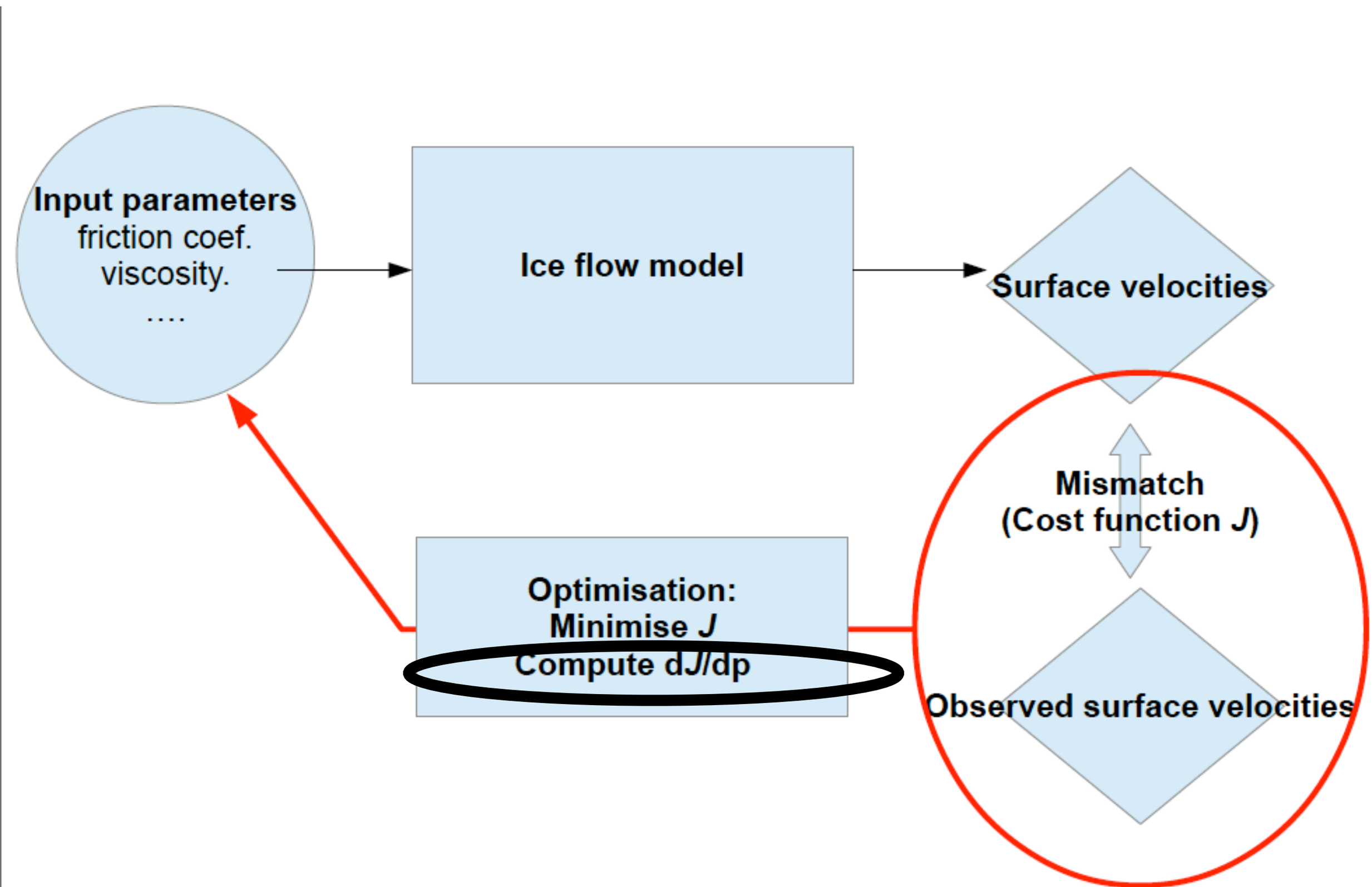
$$\sum_{i=1}^{obs} \|\boldsymbol{u}^{mod} - \boldsymbol{u}^{obs}\|^2 \Big|_{obs}$$







Variational data assimilation



Compute the gradient

```
!!!!  Adjoint Solution
Solver 3
  Equation = "Adjoint"
  Variable = Adjoint
  Variable Dofs = 2

  procedure = "AdjointSSASolvers" "AdjointSSA_AdjointSolver"

!Name of the flow solution solver
  Flow Solution Equation Name = string "SSA"

   Linear System Solver = Direct
   Linear System Direct Method = mumps
End

!!!!!  Compute Derivative of Cost function / Beta
Solver 4
  Equation = "DJDBeta"

  !!  Solver need to be associated => Define dumy variable
    Variable = -nooutput "DJDB"
    Variable DOFs = 1

    procedure = "AdjointSSASolvers" "AdjointSSA_GradientSolver"

    Flow Solution Name = String "SSAVelocity"
    Adjoint Solution Name = String "Adjoint"
    Compute DJDBeta = Logical True   ! Derivative with respect to the Friction parameter
    DJDBeta Name = String "DJDBeta"
end
```

Boundary conditions

```
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
Boundary Condition 1
  Name = "Side Walls"
  Target Boundaries(2) = 1 3

  SSAVelocity 1 = Real 0.0
  SSAVelocity 2 = Real 0.0

  Adjoint 1 = Real 0.0
  Adjoint 2 = Real 0.0
End

Boundary Condition 2
  Name = "Inflow"
  Target Boundaries = 4

   SSAVelocity 1 = Variable Coordinate 2
      REAL MATC "4.753e-6*yearinsec*(sin(2.0*pi*(Ly-tx)/Ly)+2.5*sin(pi*(Ly-tx)/Ly))"
   SSAVelocity 2 = Real 0.0

  Adjoint 1 = Real 0.0
  Adjoint 2 = Real 0.0
End

Boundary Condition 3
  Name = "OutFlow"
  Target Boundaries = 2

   SSAVelocity 1 = Variable Coordinate 2
      REAL MATC "1.584e-5*yearinsec*(sin(2.0*pi*(Ly-tx)/Ly)+2.5*sin(pi*(Ly-tx)/Ly)+0.5*sin(3.0*pi*(Ly-tx)/Ly))"
   SSAVelocity 2 = Real 0.0

  Adjoint 1 = Real 0.0
  Adjoint 2 = Real 0.0
End
```

Change of variable

We have computed the gradient wr to the friction parameter; physically this parameter should remain positive => make change of variable $\beta = \alpha^2$ or $\beta = 10^\alpha$

```
Solver 5
  Equation = "UpdateExport"
    Procedure = File "ElmerIceSolvers" "UpdateExport"
  Variable = -nooutput "dumy"

!used here to update DJDalpha from DJDbeta (see correponding line in Body Force section)
  Exported Variable 1 = -dofs 1 alpha
  Exported Variable 2 = -dofs 1 DJDalpha
End
```

```
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
Body Force 1
  Flow BodyForce 1 = Real 0.0
  Flow BodyForce 2 = Real 0.0
  Flow BodyForce 3 = Real $gravity

! change of variable; DJDBeta is the derivative of the Cost fn w.r.t. the slip coeff.
!                      as slip coeff=alpha^2 => DJDalpha=DJDBeta * DBeta / Dalpha
  DJDalpha = Variable DJDBeta , alpha
     REAL MATC "2.0*tx(0)*tx(1)"
End
```

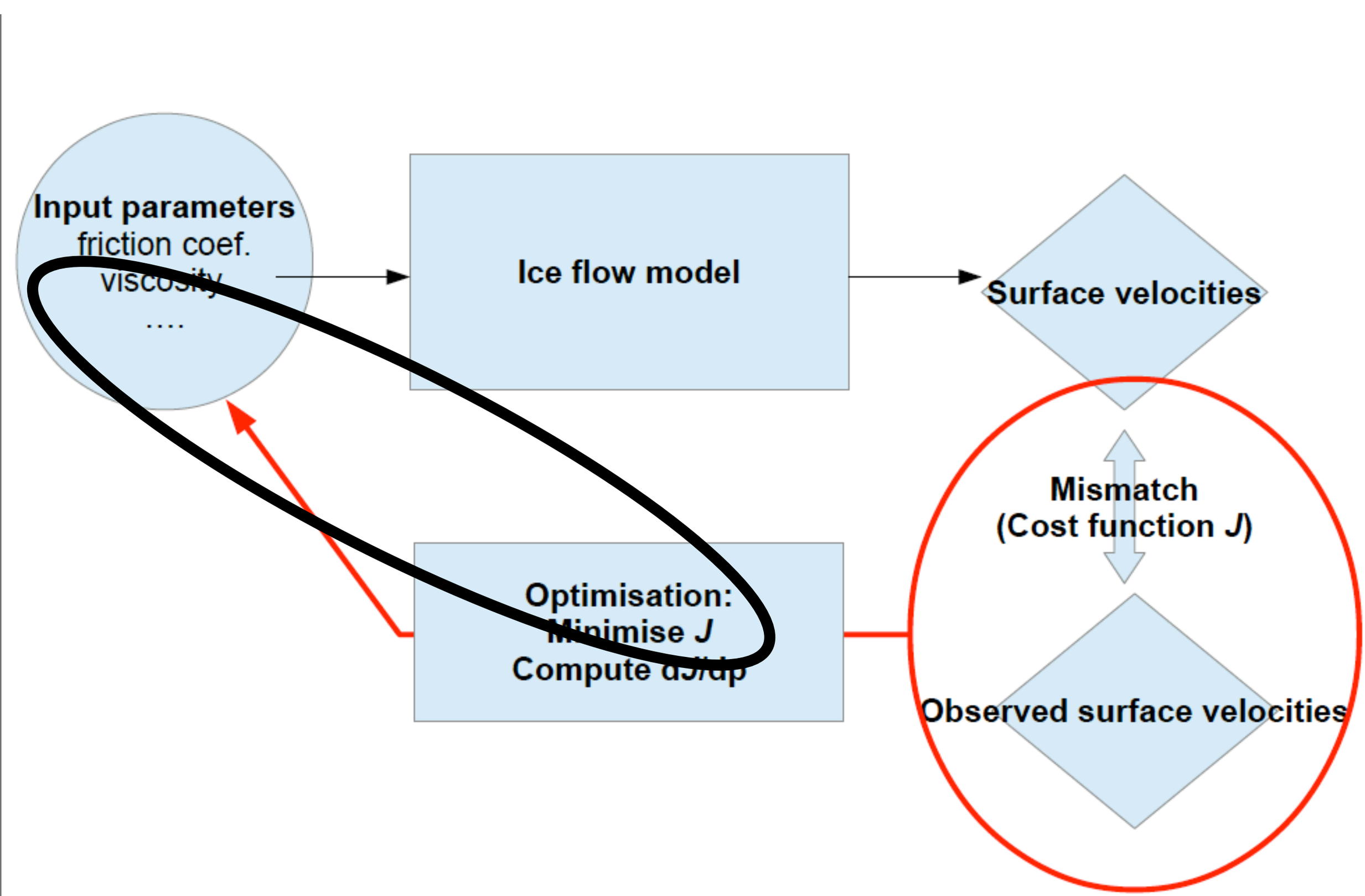
Regularisation

```
!!!!!  Compute Regularistaion term
!   Regularisation by default is: Lambda * int_{Pb dimension} 0.5 * (d(var)/dx)**2
!   A priori regularisation can also be used ( A priori Regularisation=True) :
!                                Lambda * int_{Pb dimension} 0.5 *(1/sigma**2)*(var-var{a_priori})**2
!
!     OUTPUT are : J and DJDvar
Solver 6
  Equation = "DJDBeta_Reg"

  !!  Solver need to be associated => Define dumy variable
    Variable = -nooutput "DJDBReg"
    Variable DOFs = 1

    procedure = "AdjointSSASolvers" "AdjointSSA_CostRegSolver"

    Problem Dimension=Integer 2
    Cost Filename=File "CostReg_$name$.dat"
    Optimized Variable Name= String "alpha"
    Gradient Variable Name= String "DJDalpha"
    Cost Variable Name= String "CostValue"
    Lambda= Real $Lambda
    Reset Cost Value= Logical False  !=> DJDapha already initialized in solver DJDBeta; switch off initialisation to 0 at the
beginning of this solver
    A priori Regularisation= Logical False
end
```

Most real inverse problem are ill-posed
=> add a-priori knowldedge to regularise the problem
=> not needed here as the obs. are perfect!

```
#date,time,1/3/2016,10:35:52
#lambda, 0.00000000E+00
# iter, Jreg
  0.10000E+01   0.96510928E-35
  0.20000E+01   0.26697707E-06
  0.30000E+01   0.91582437E-06
  0.40000E+01   0.12325799E-05
  0.50000E+01   0.14043708E-05
  0.60000E+01   0.16939676E-05
  0.70000E+01   0.22054452E-05
```

Variational data assimilation

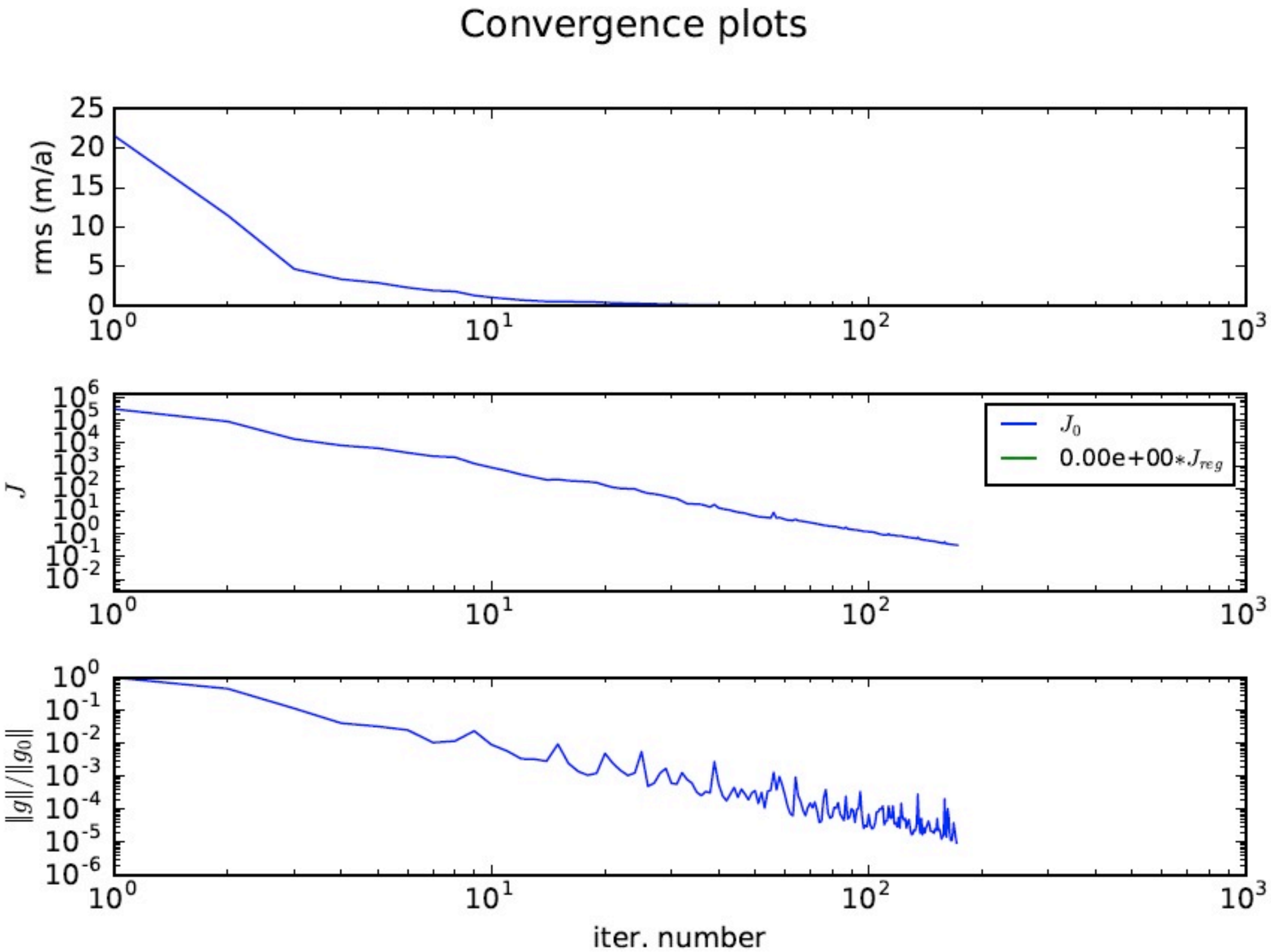


Optimisation

```
!!!!! Optimization procedure : Parallel only
Solver 7
  Equation = "Optimize_m1qn3"
  !!  Solver need to be associated => Define dumy variable
    Variable = -nooutput "UB"
    Variable DOFs = 1

  procedure = "ElmerIceSolvers" "Optimize_m1qn3Parallel"

  Cost Variable Name = String "CostValue"
  Optimized Variable Name = String "alpha"
  Gradient Variable Name = String "DJDalpha"
  gradient Norm File = File "GradientNormAdjoint_$name$.dat"


 ! M1QN3 Parameters
  M1QN3 dxmin = Real 1.0e-10
  M1QN3 epsg = Real  1.e-5
  M1QN3 niter = Integer 200
  M1QN3 nsim = Integer 200
  M1QN3 impres = Integer 5
  M1QN3 DIS Mode = Logical False
  M1QN3 df1 = Real 0.5
  M1QN3 normtype = String "dfn"
  M1QN3 OutputFile = File  "M1QN3_$name$.out"
  M1QN3 ndz = Integer 20

end
```

```
#10/31/2016      10:35:52
   0.10000E+01   0.84790499E+08
   0.20000E+01   0.38736142E+08
   0.30000E+01   0.97311488E+07
   0.40000E+01   0.34591176E+07
   0.50000E+01   0.27595369E+07
   0.60000E+01   0.21367979E+07
   0.70000E+01   0.89028275E+06
   0.80000E+01   0.99144675E+06
   0.90000E+01   0.20111705E+07
```

Check the convergence!!



Output information

M1QN3 last 7 lines

```
m1qn3: output mode is  1
   number of iterations:          158
   number of simulations:         171
   realized relative precision on g:  9.33D-06
   f           =  3.24456286D-01
   dfn-norm of g =  7.90817523D+02
```

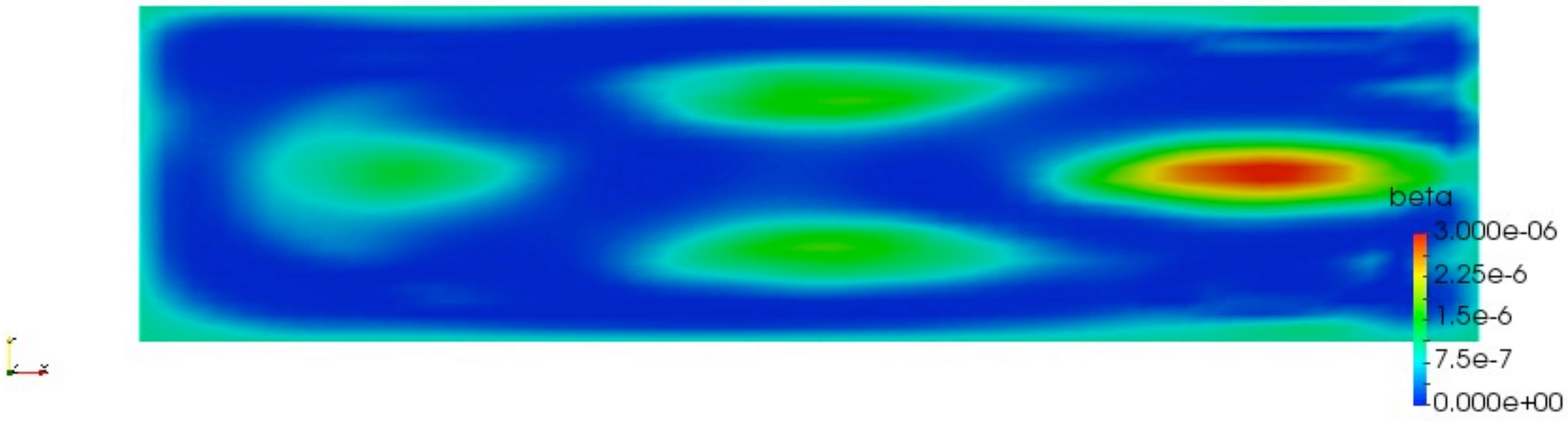
M1QN3 documentation

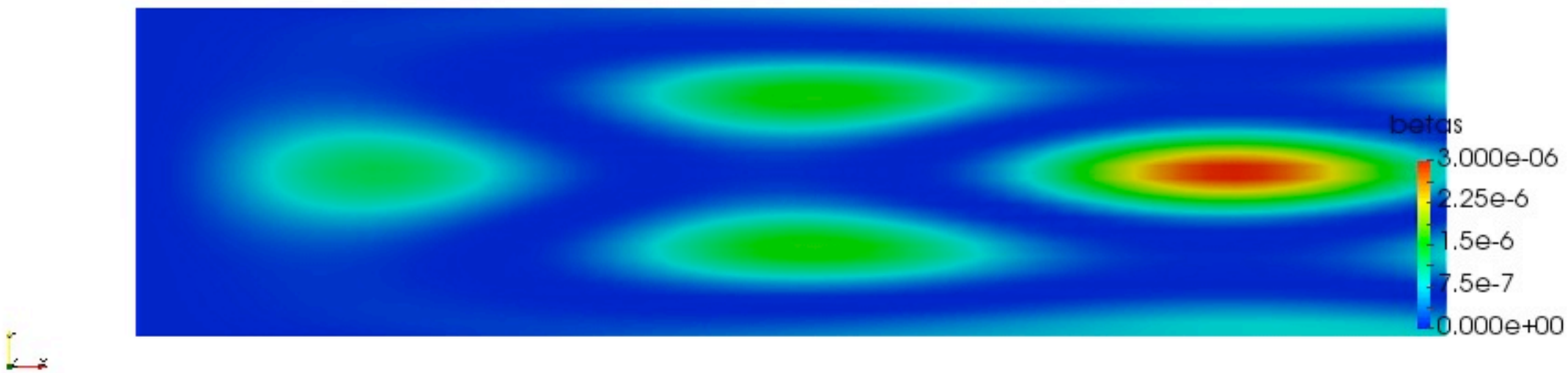
`omode` (O): `Integer` variable that specifies the output mode of `m1qn3`. The following values are meaningful.

- = 0: The simulator asks to stop by returning the value `indic` = 0.
- = 1: This is the normal way of stopping for `m1qn3`: the test on the gradient is satisfied (see the meaning of `epsg`).
- = 2: One of the input arguments is not well initialized. This can be:
  - `n` ≤ 0, `niter` ≤ 0, `nsim` ≤ 0, `dxmin` ≤ 0.0 or `epsg` $\notin]0, 1[$,
  - `ndz` $< 5n + 1$ (in SIS mode) or `ndz` $< 6n + 1$ (in DIS mode): not enough storage in memory,
  - the contents of `iz` is not correct for a warm restart,
  - the starting point is almost optimal (the norm of the initial gradient is less than 10^{-20}).
- = 3: The line-search is blocked on `tmax` = 10^{20} (see section 4.4 and the documentation on `mlis3` in MODULOPT library).
- = 4: The maximal number of iterations is reached.
- = 5: The maximal number of simulations is reached.
- = 6: Stop on `dxmin` during the line-search (see section 4.4).
- = 7: Either $\langle g, d \rangle$ is nonnegative or $\langle y, s \rangle$ is nonpositive (see section 4.4).

For additional information and comments, see section 4.

Look at the results



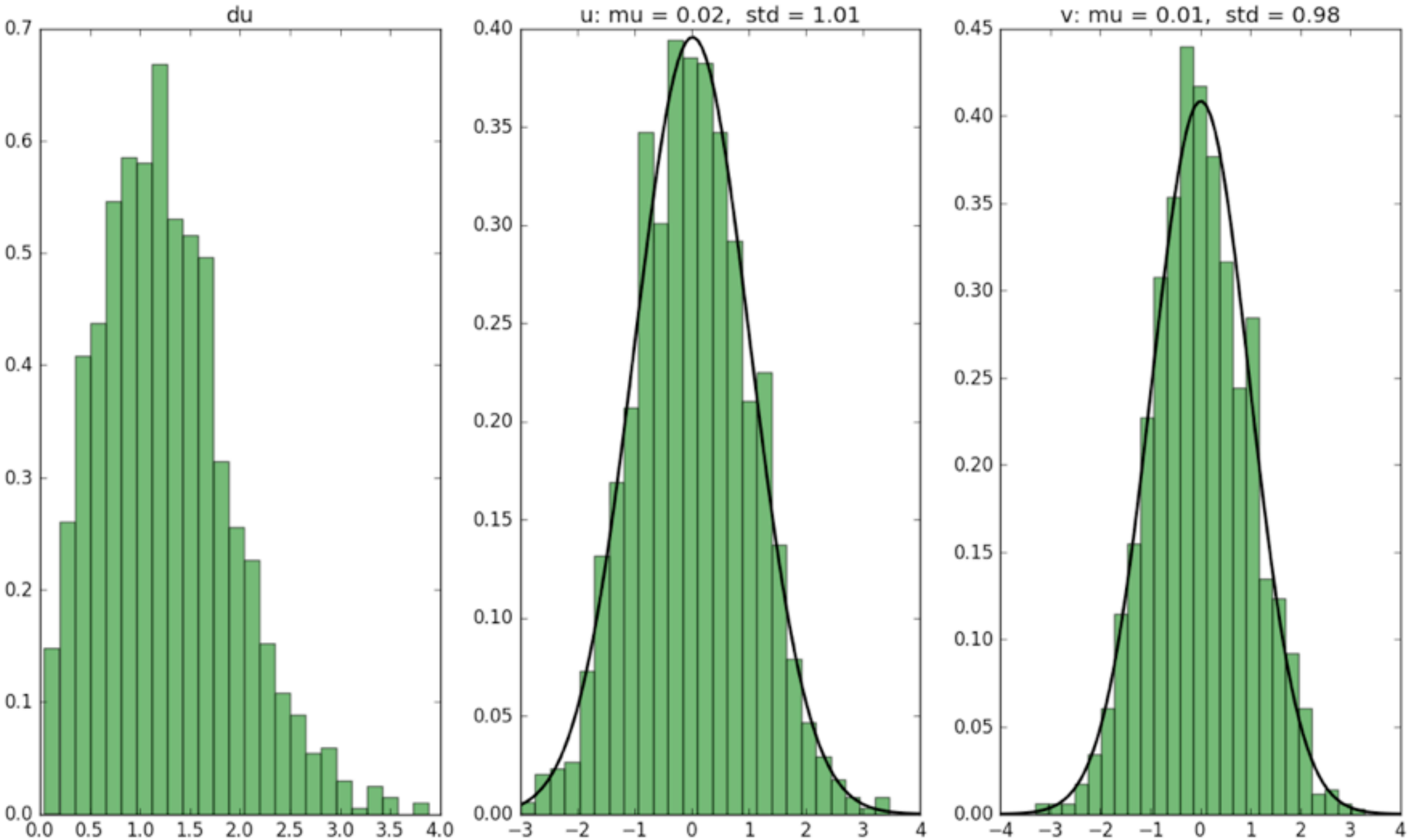


Do it yourself

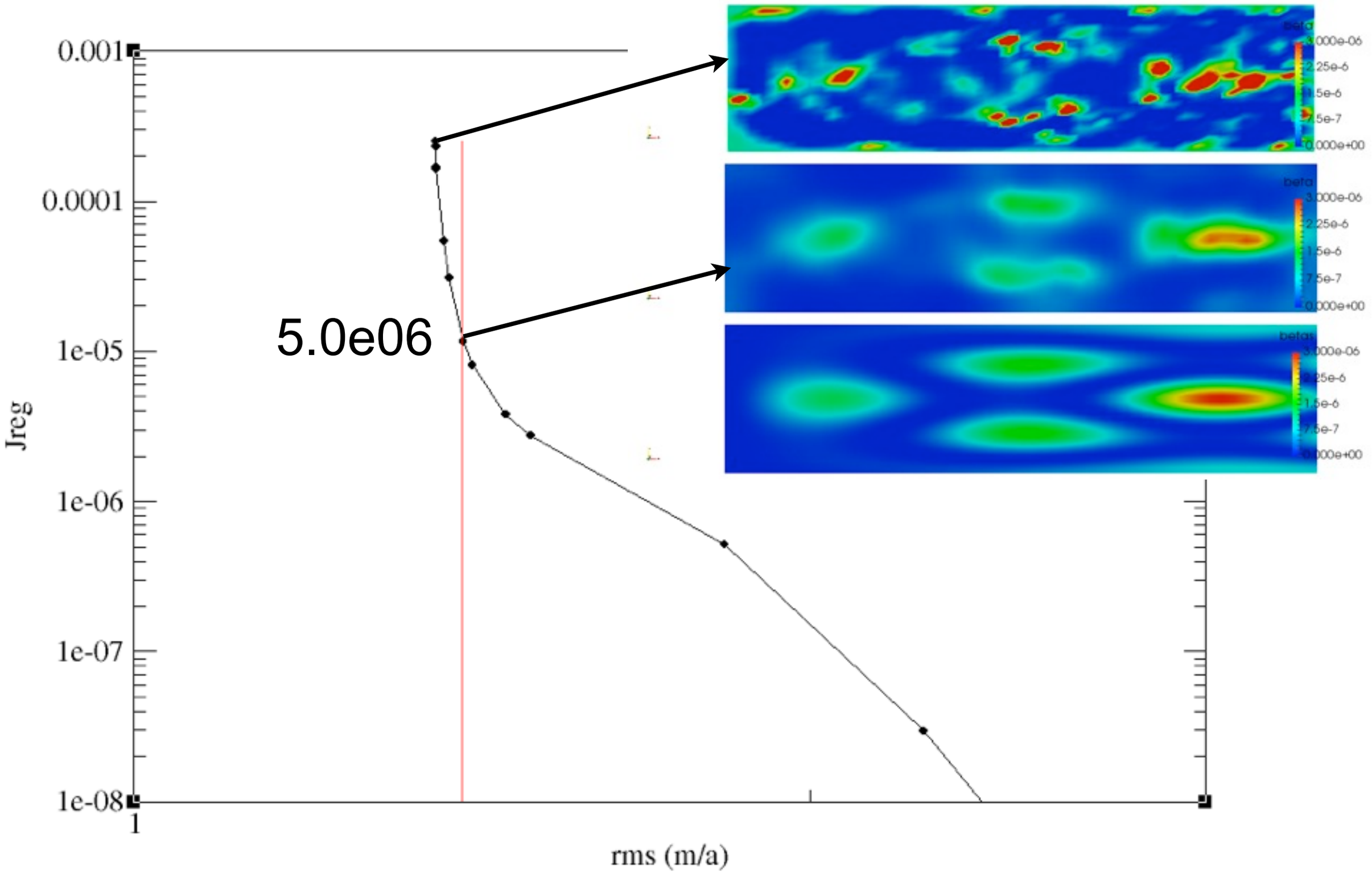
- Play with M1QN3 parameters

- New change of variable:
  friction parameter = 10^alpha

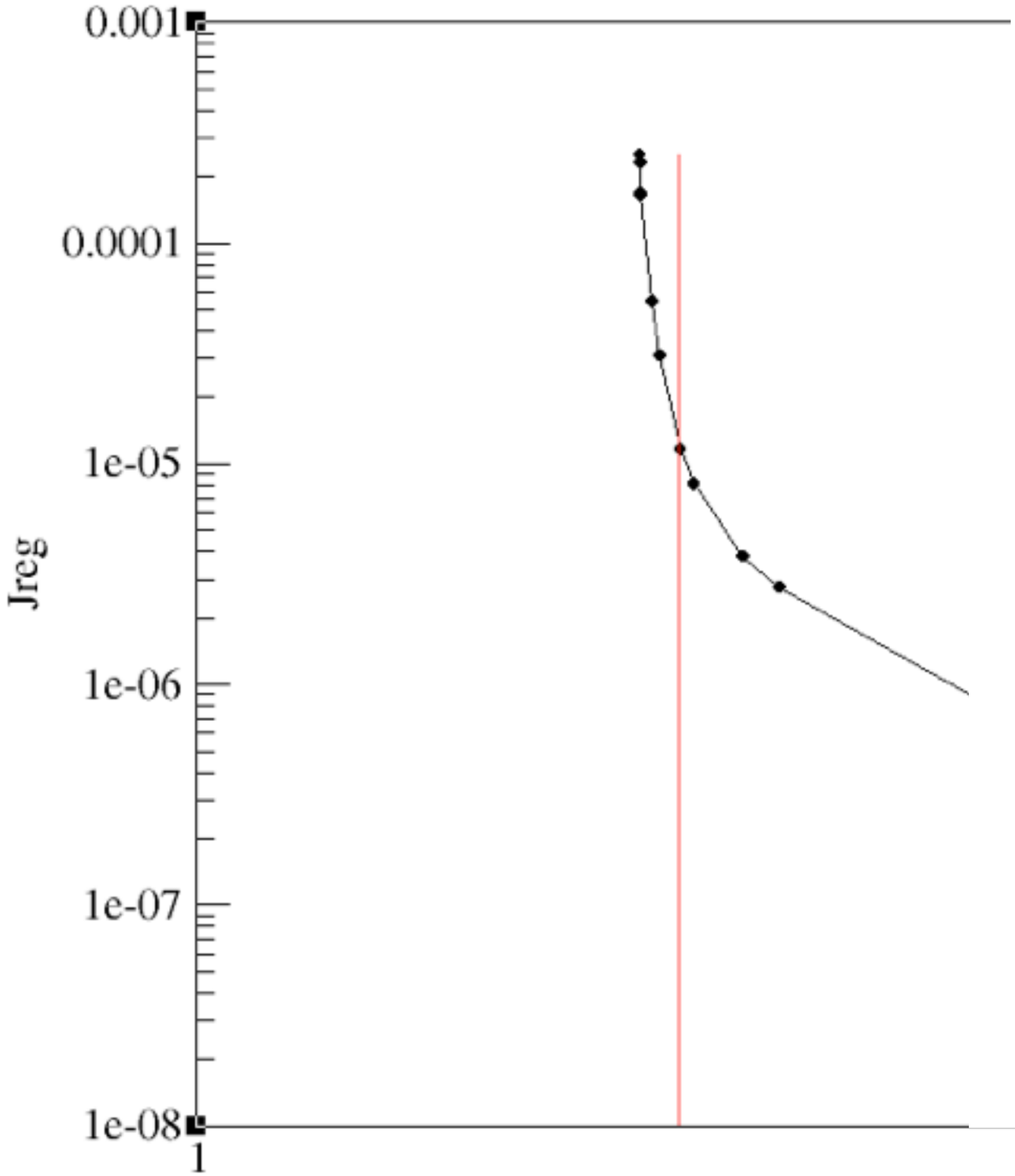
Add noise in the data

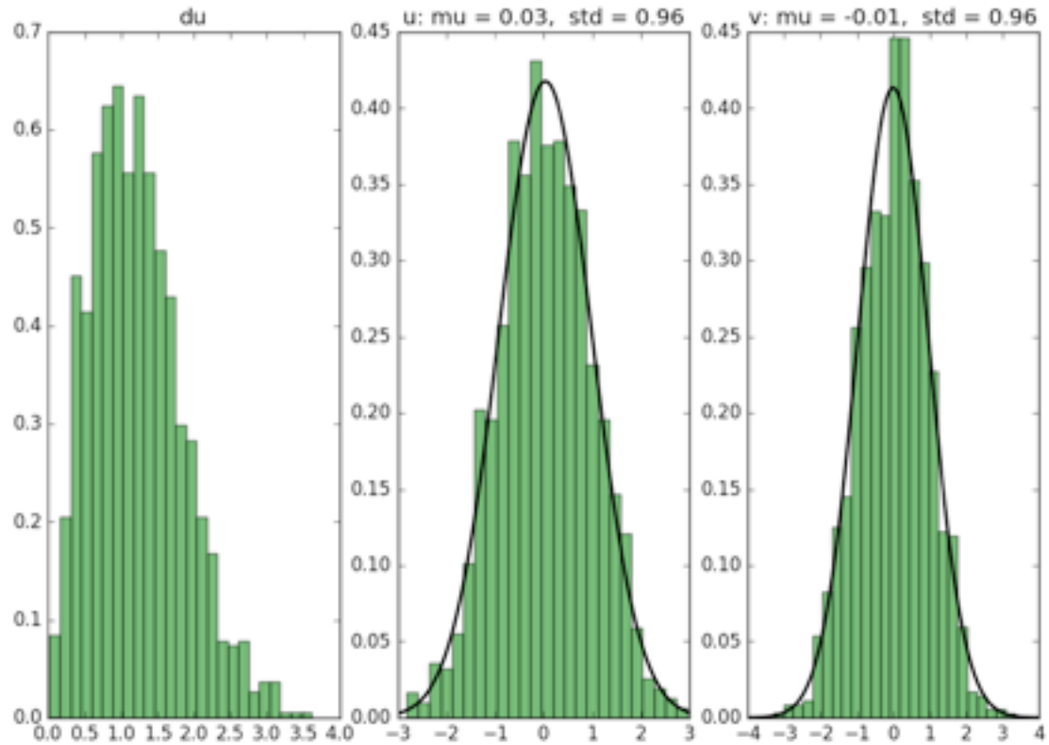


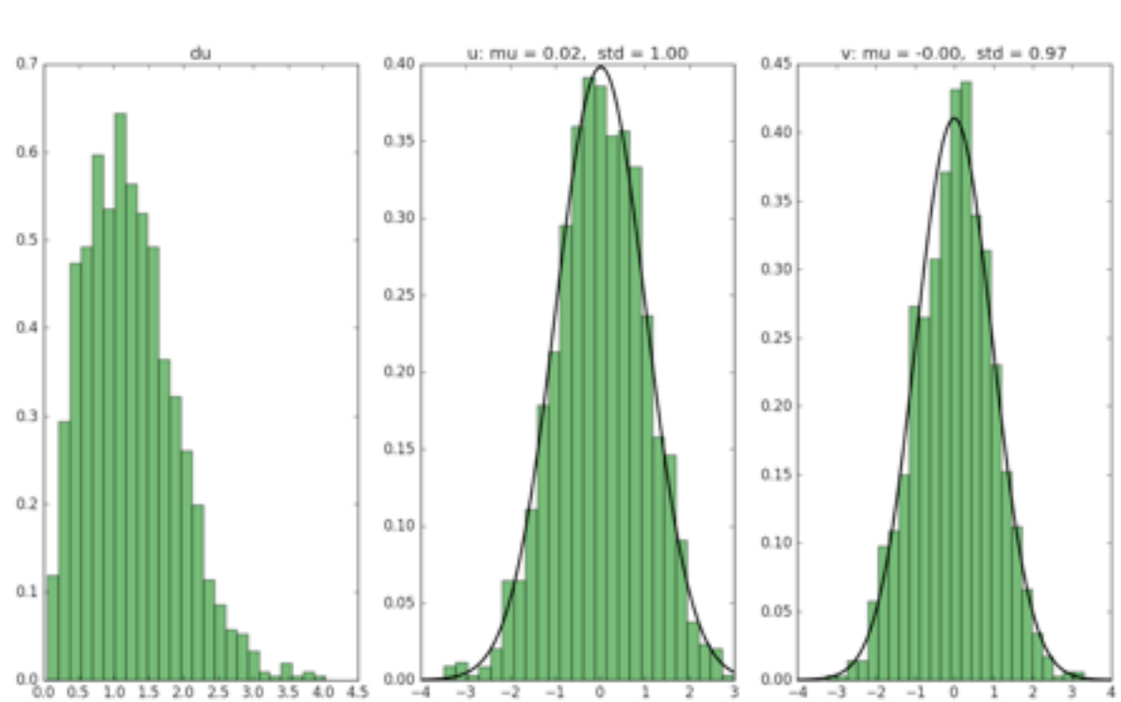
Plot a L-Curve



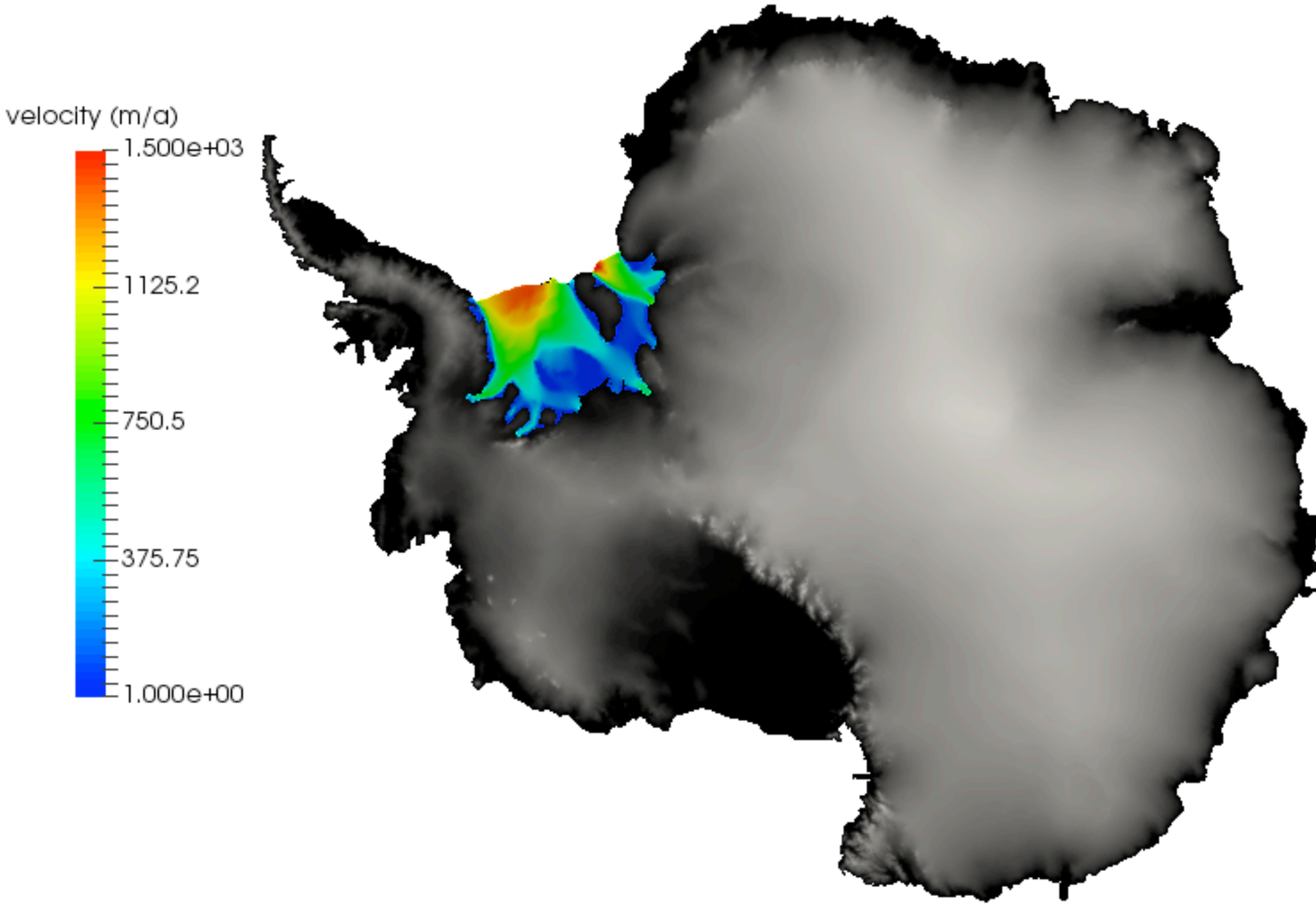
Plot a L-Curve







Model Ronne-Filchner Ice Shelf



Initialisation

```
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
Simulation
  Coordinate System  = Cartesian
  Simulation Type = Steady State

  Steady State Min Iterations = 1
  Steady State Max Iterations = 200

  Post File = "OPTIM_$name$.vtu"

  Restart File = "IMPORT.result"
  Restart Before Initial Conditions = logical True

  max output level = 3
End
```

I have already imported data required for the computation

```
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
Initial Condition 1
! alpha is the optimised variable
 alpha = Variable Mu
    REAL MATC "sqrt(tx)"
End
```

Optimise mean viscosity

```
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
Material 1

  Viscosity Exponent = Real $1.0e00/3.0e00
  Critical Shear Rate = Real 1.0e-10

  SSA Mean Density = Real $rhoi
  SSA Mean Viscosity = Variable alpha
      REAL procedure "Adjoint_USFs" "Asquare"
  SSA Friction Law = String "linear"
  SSA Friction Parameter = Real 0.0
End
```

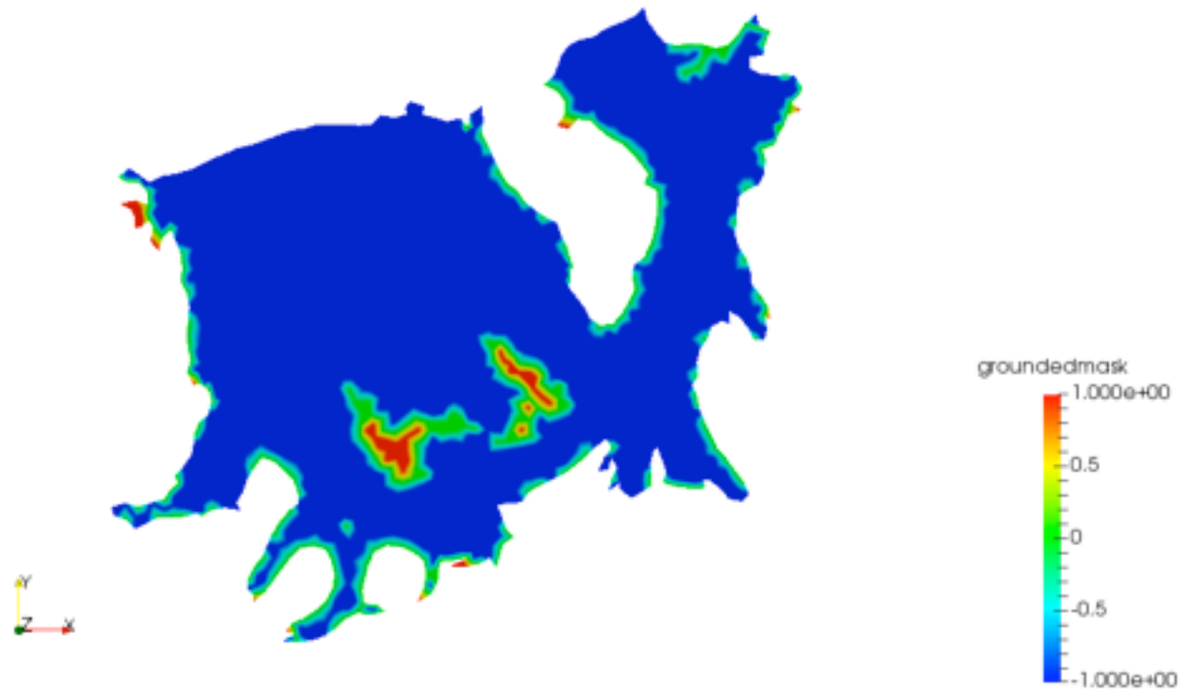
```
!!!!!  Compute Derivative of Cost function / Beta
Solver 4
  Equation = "DJDEta"

  !!  Solver need to be associated => Define dumy variable
    Variable = -nooutput "DJDB"
    Variable DOFs = 1

    procedure = "AdjointSSASolvers" "AdjointSSA_GradientSolver"

    Flow Solution Name = String "SSAVelocity"
    Adjoint Solution Name = String "Adjoint"
    Compute DJDEta = Logical True   ! Derivative with respect to the SSA Mean Viscosity

end
```

Boundary conditions

```
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
Boundary Condition 1
  Target Boundaries(2) = 1 3

  SSAVelocity 1 = Equals Uobs 1
  SSAVelocity 2 = Equals Uobs 2

  Adjoint 1 = Real 0.0
  Adjoint 2 = Real 0.0
End

Boundary Condition 2
  Name = "Ice Front"
  Target Boundaries(2) = 2 4

  calving front = logical true
End
```

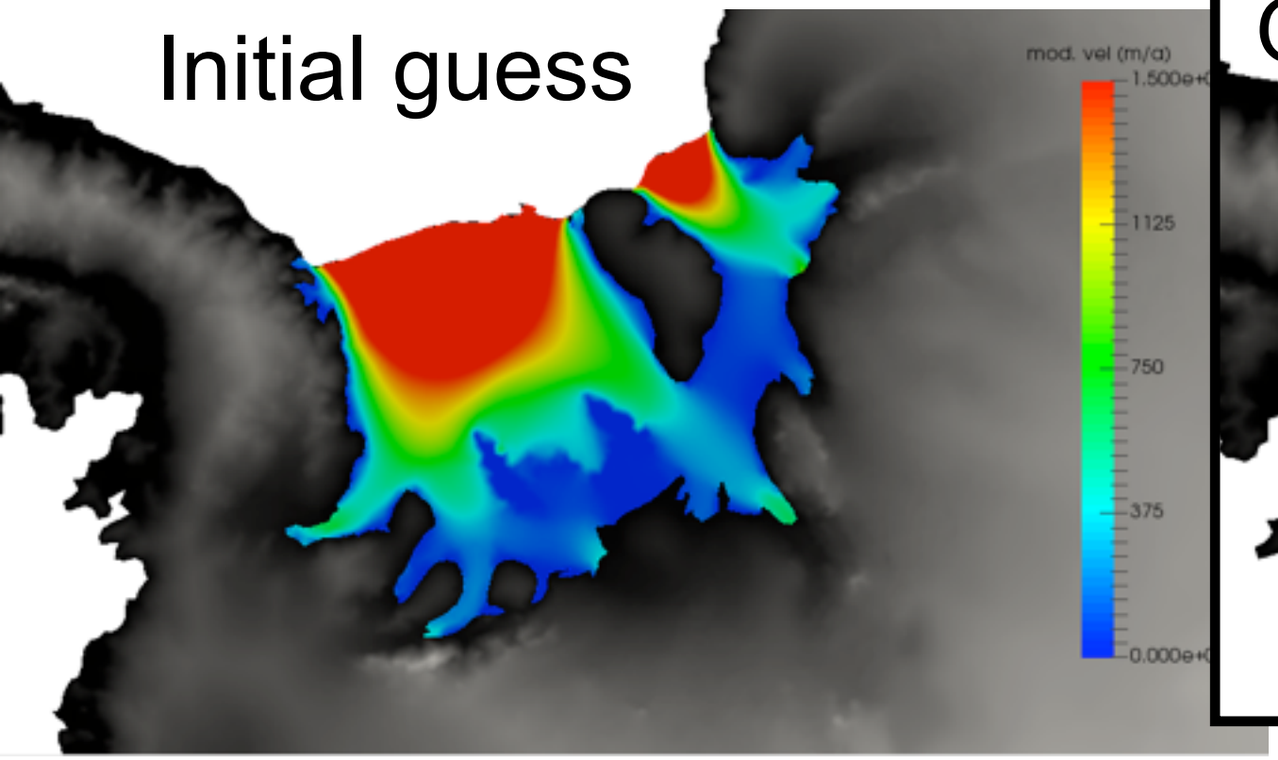
Apply Dirichlet condition in grounded parts

```
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
Body Force 1
  Flow BodyForce 1 = Real 0.0
  Flow BodyForce 2 = Real 0.0
  Flow BodyForce 3 = Real $gravity

  DJDalpha = Variable DJDEta , alpha
     REAL procedure "Adjoint_USFs" "Derivative_Asquare"

  SSAVelocity 1 = Equals Uobs 1
  SSAVelocity 2 = Equals Uobs 2
  SSAVelocity 1 Condition = Variable GroundedMask
    Real procedure "USFs_RonneFilchner" "GM_CONDITION"
  SSAVelocity 2 Condition = Variable GroundedMask
    Real procedure "USFs_RonneFilchner" "GM_CONDITION"

  Adjoint 1 = Real 0.0
  Adjoint 2 = Real 0.0
  Adjoint 1 Condition = Variable GroundedMask
    Real procedure "USFs_RonneFilchner" "GM_CONDITION"
  Adjoint 2 Condition = Variable GroundedMask
    Real procedure "USFs_RonneFilchner" "GM_CONDITION"

End
```

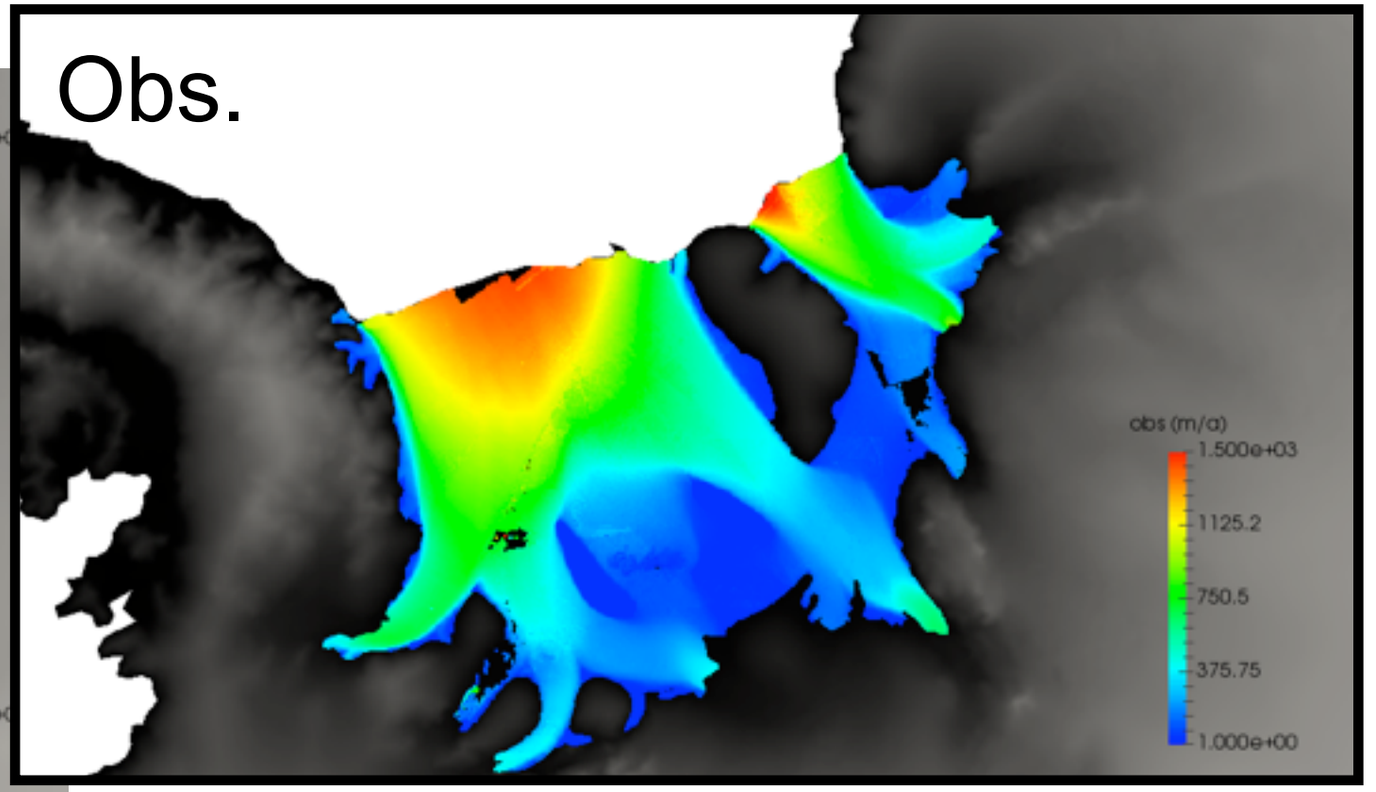


Results

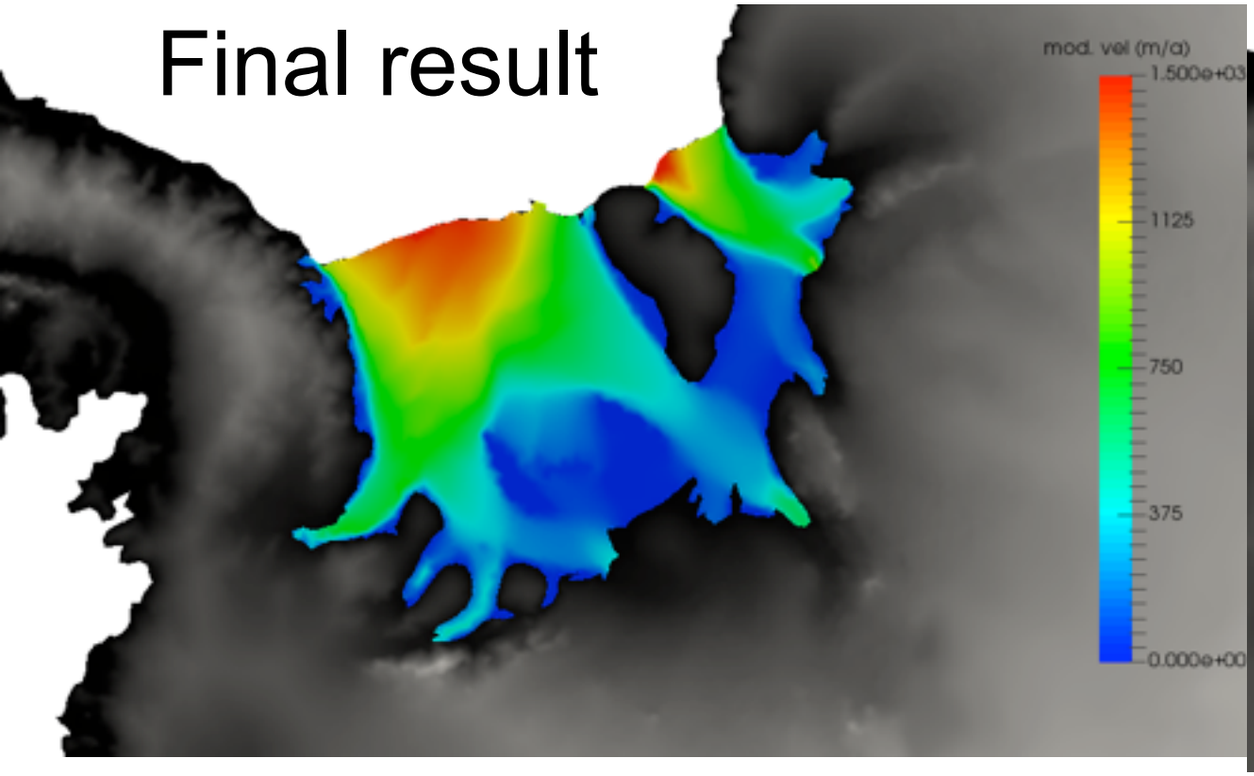
Initial guess



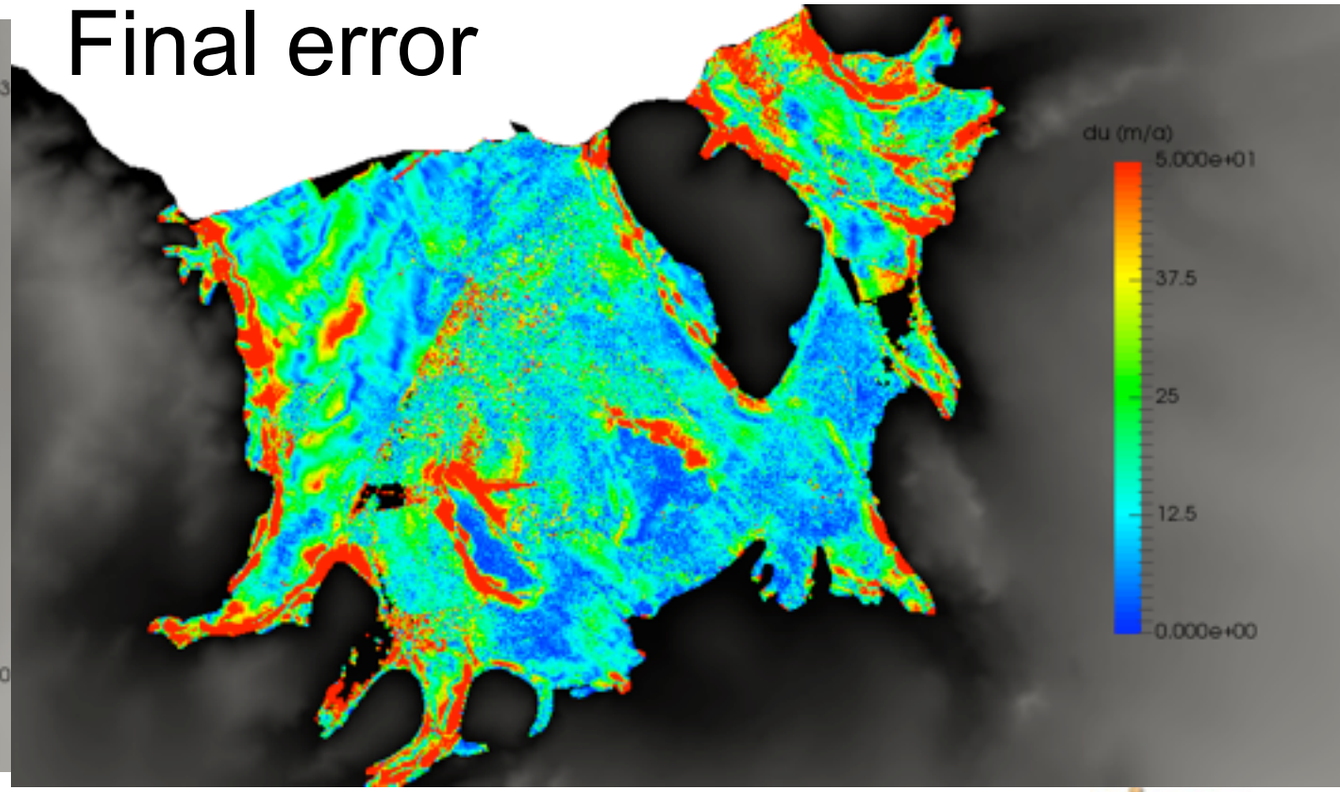
Obs.



Final result

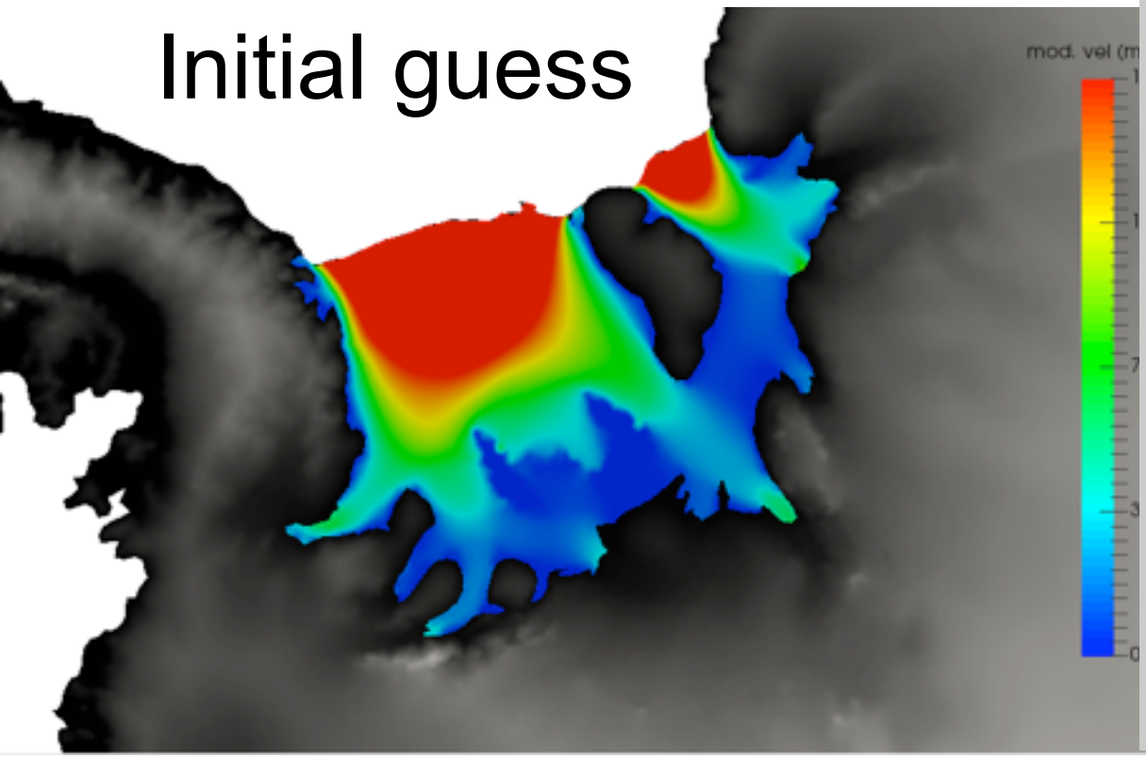


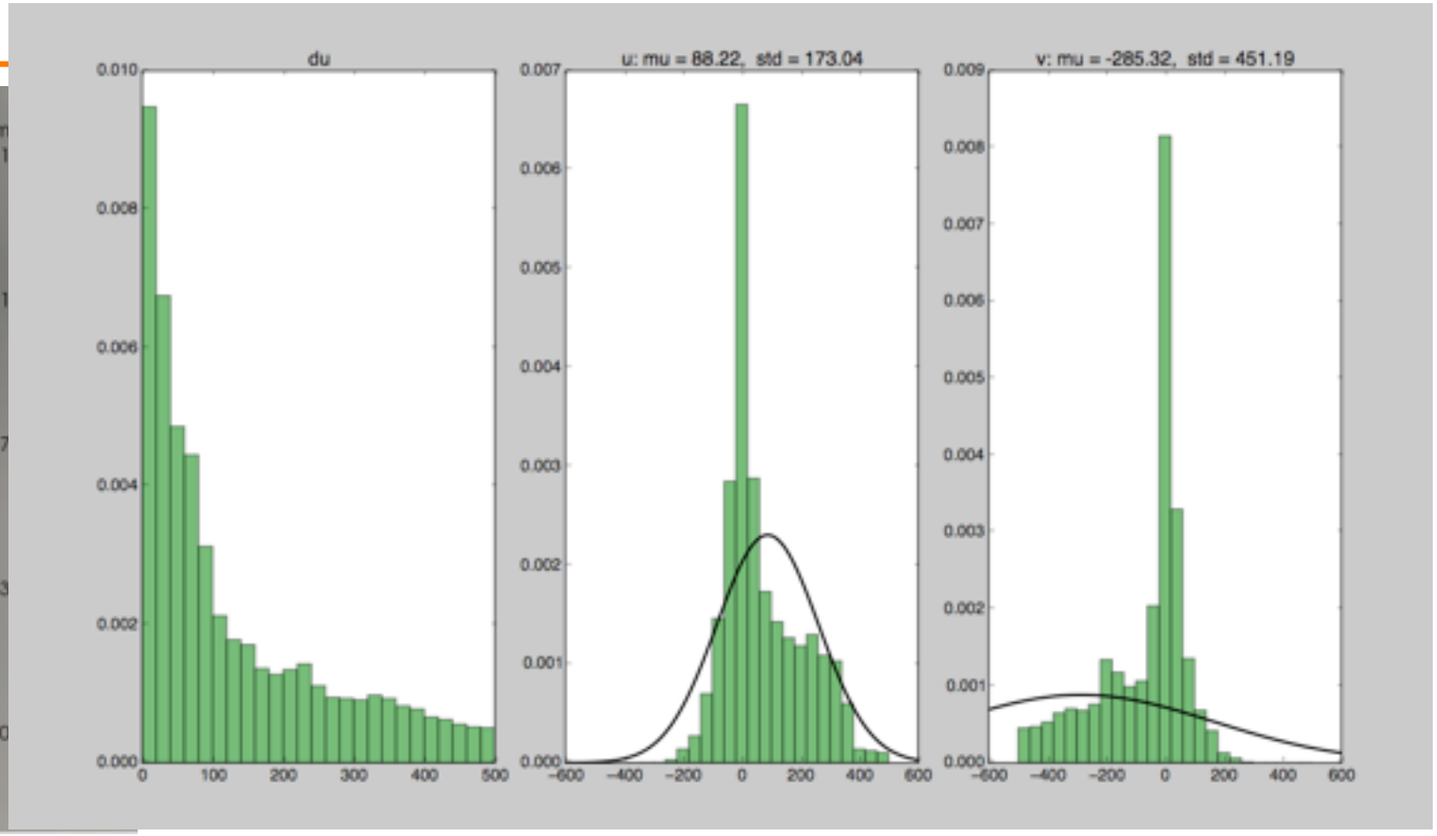
Final error



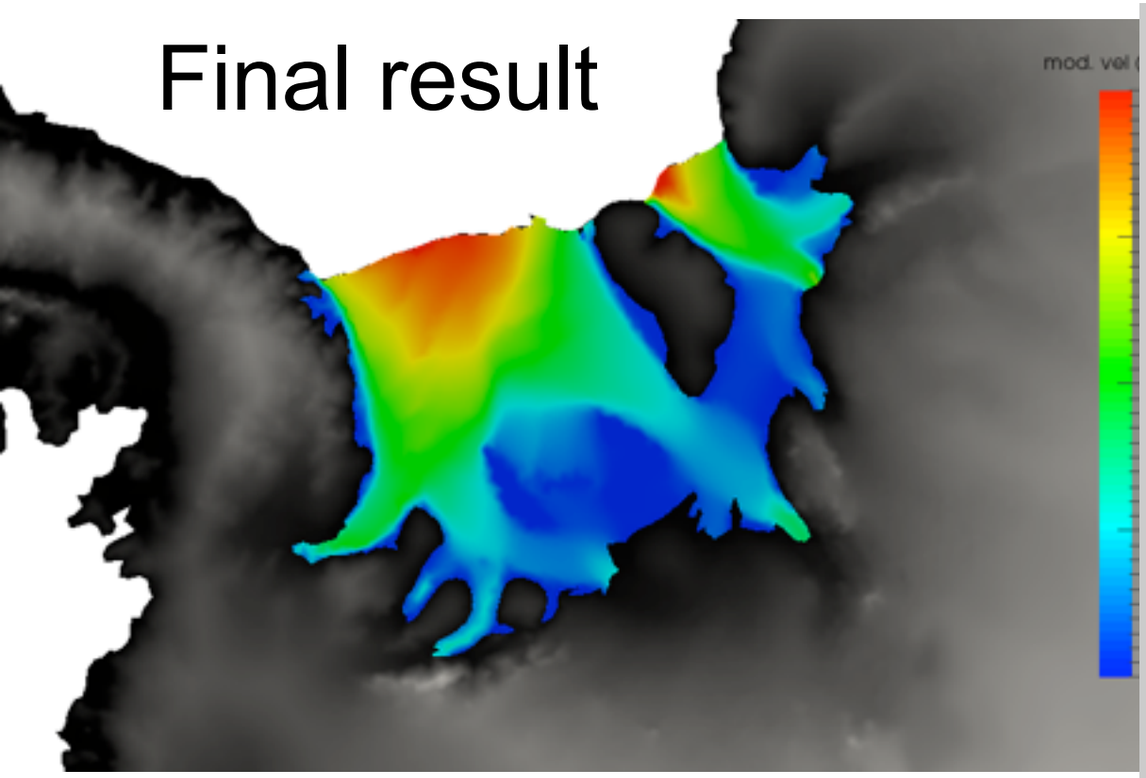
Results

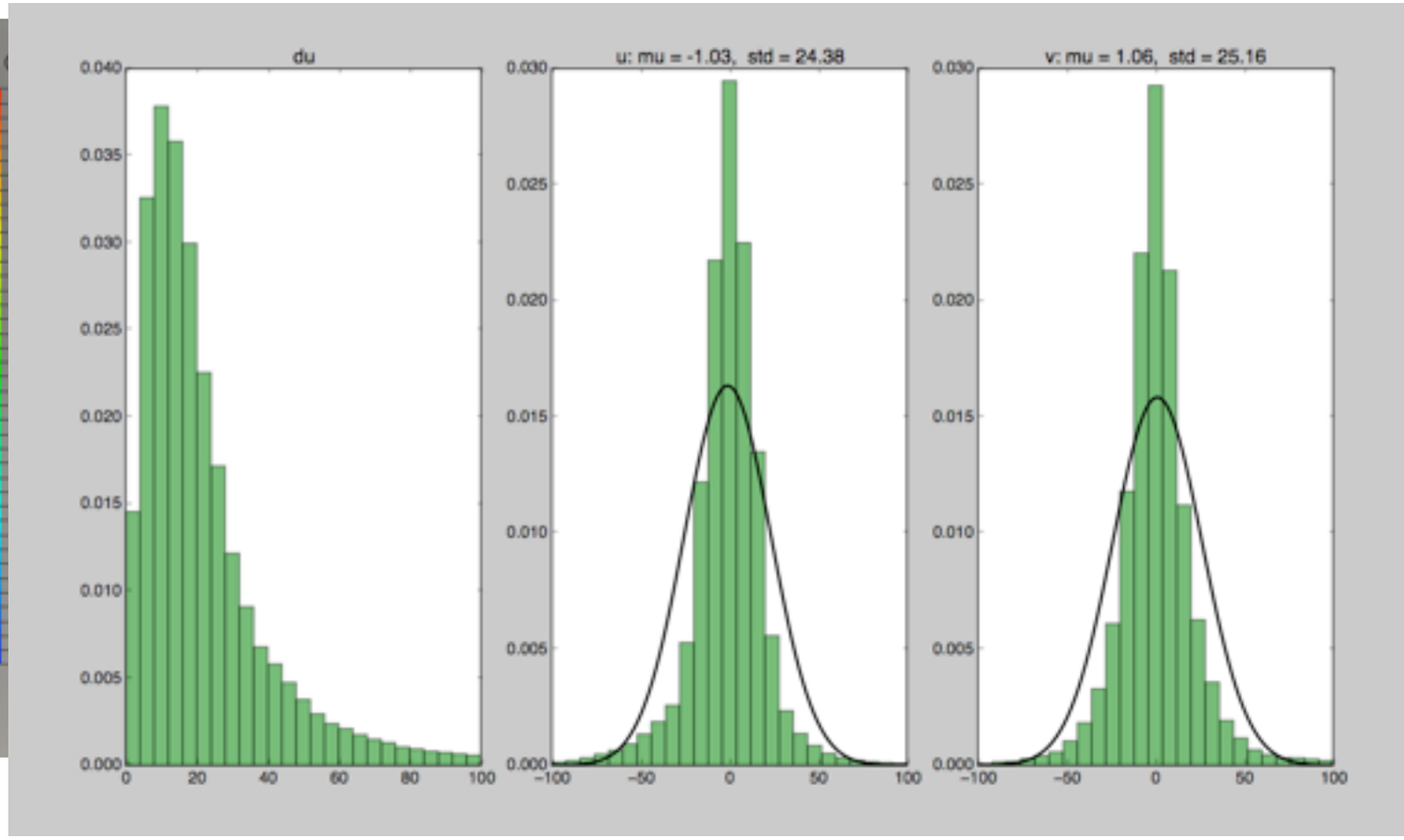
Initial guess



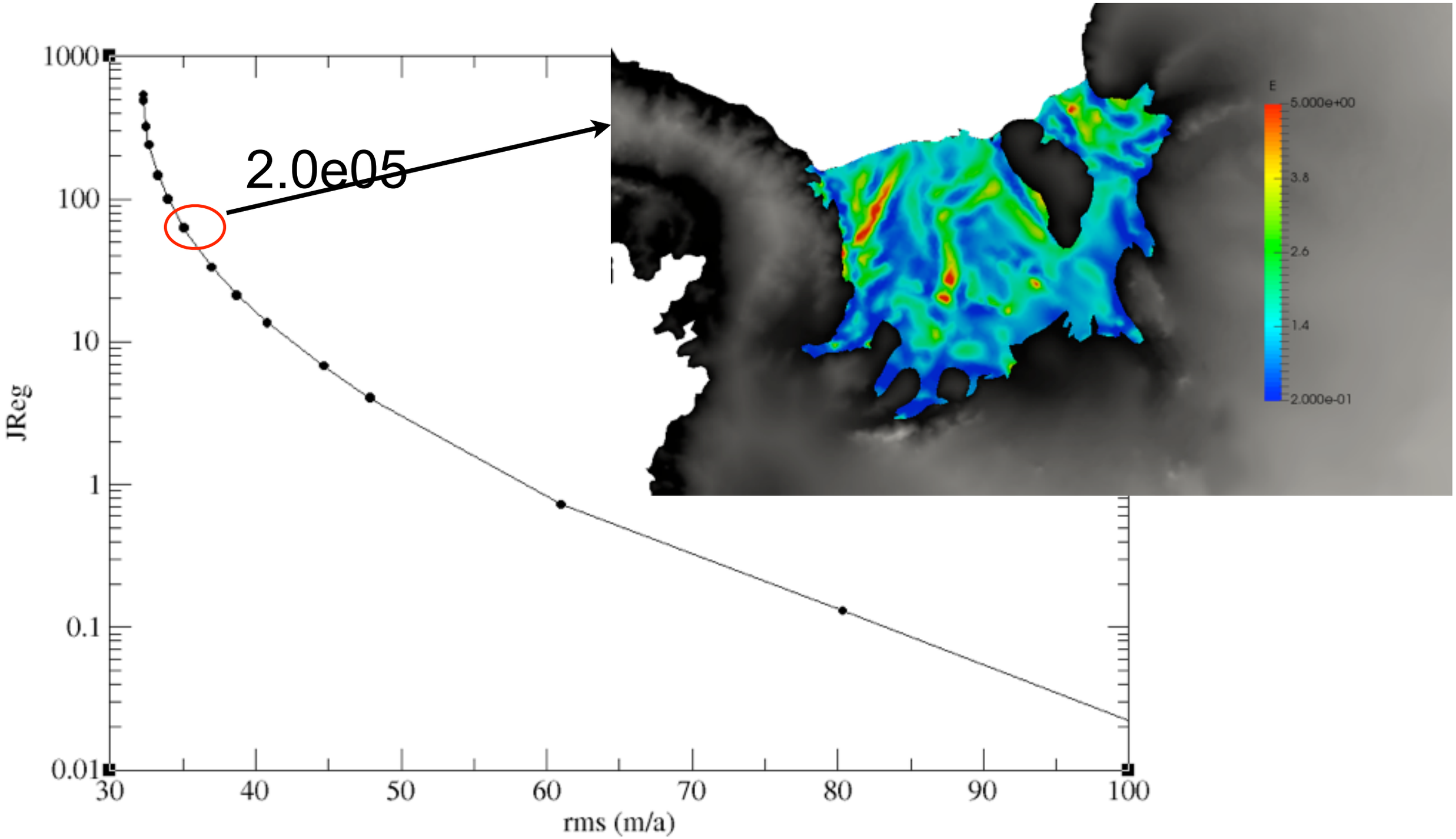


Final result





Results


1000
100
10
1
0.1
0.01
JReg
30
40
50
60
70
80
90
100
rms (m/a)
2.0e05
E
5.000e+00
3.8
2.6
1.4
2.000e-01